CMPE 180-92

Data Structures and Algorithms in C++

November 2 Class Meeting

Department of Computer Engineering
San Jose State University

SILICON VALLEY'S FIRST CHOICE FOR NEW ENGINEERING HIRES
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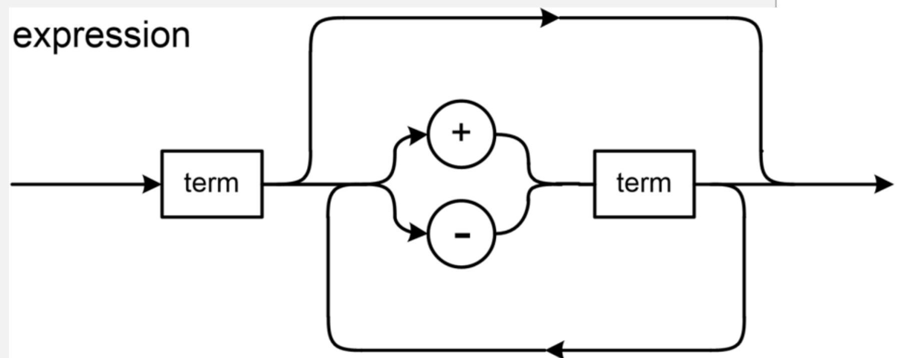
San José State UNIVERSITY

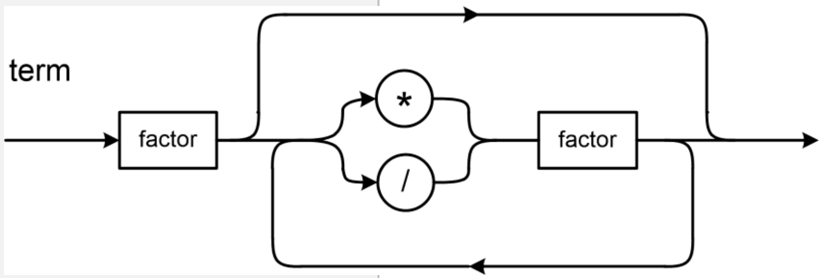
Assignment #10 Sample Solution

```
double Calculator::expression() const throw(string)
{
    double value = term();  // evaluate the first term
    bool done = false;
    char ch;

    do
    {
        cin >> ws;
        ch = cin.peek();

        switch (ch)
        {
            case '+':
            {
                cin >> ch;         // read the +
                value += term();   // evaluate the next term and add its value
                break;
            }

            case '-':
            {
                cin >> ch;         // read the -
                value -= term();   // evaluate the next term and subtract its value
                break;
            }

            default: done = true; // no more terms
        }
    } while (!done);


    return value;  // the expression's value
}
```



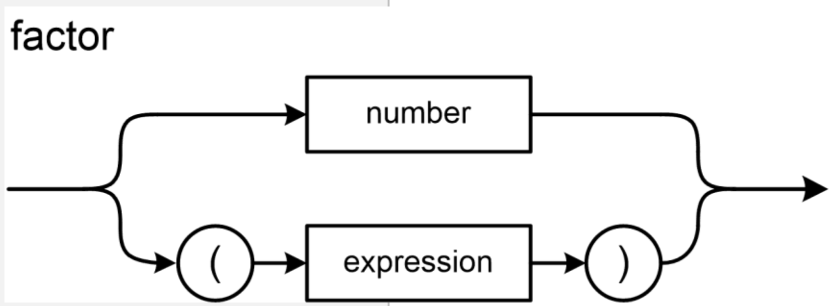
Assignment #10 Sample Solution, *cont'd*

```
double Calculator::term() const throw(string)
{
    double value = factor();  // evaluate the first factor
    bool done = false;
    char ch;

    do
    {
        cin >> ws;
        ch = cin.peek();

        switch (ch)
        {
            case '*':
            {
                cin >> ch;          // read the *
                value *= factor();  // evaluate the next factor and multiply its value
                break;
            }

            case '/':
            {
                cin >> ch;                    // read the /
                double value2 = factor();  // evaluate the next factor

                // Do the division unless the value is zero.
                if (value2 != 0) value /= value2;
                else
                {
                    throw string("Division by zero");
                    return 0;
                }

                break;
            }


            default: done = true;  // no more factors
        }
    } while (!done);


    return value;  // the term's value
}
```



Assignment #10 Sample Solution, *cont'd*

```
double Calculator::factor() const throw(string)
{
    cin >> ws;               // skip blanks
    char ch = cin.peek();    // what's the next input character?

    if (isdigit(ch))
    {
        return number();   // evaluate and return a number's value
    }

    else if (ch == '(')
    {
        cin >> ch;                     // read the (
        double value = expression();   // evaluate the subexpression

        ch = cin.peek();
        if (ch == ')') cin >> ch;      // read the )
        else
        {
            throw string("Missing )");
        }


        return value;  // return the parenthesized subexpression's value
    }

    else
    {
        throw string("Invalid factor");
        return 0;
    }
}
```



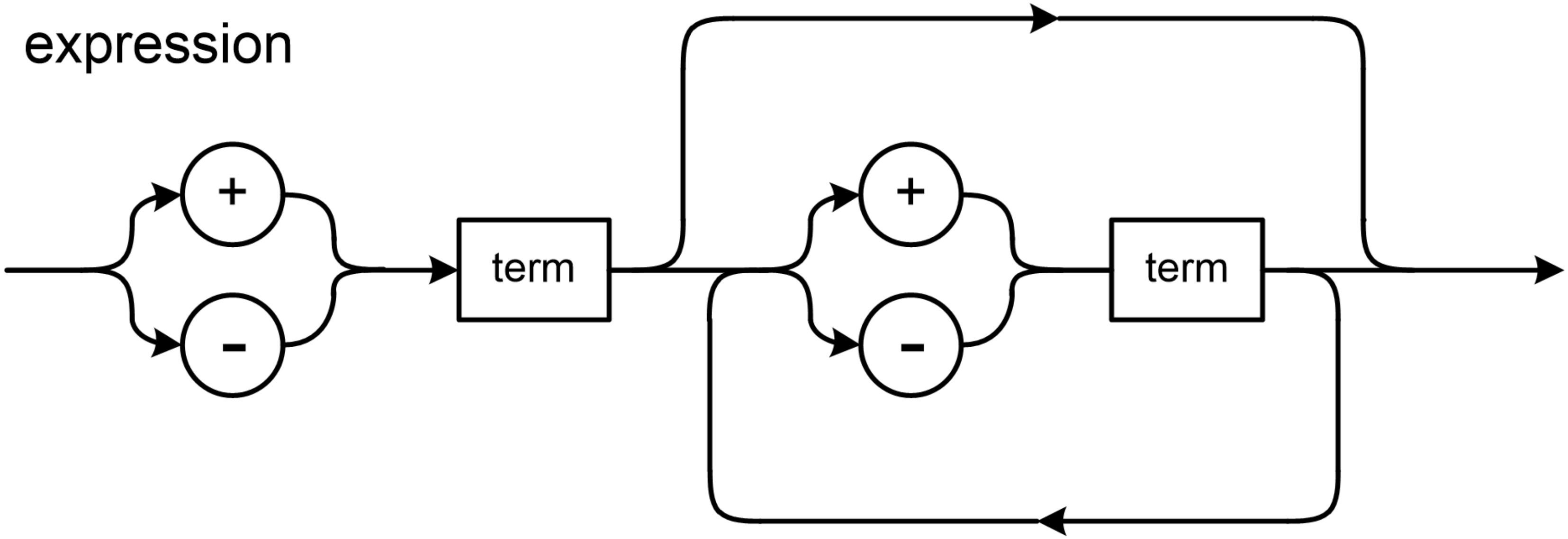
Assignment #10 Sample Solution, *cont'd*

```
double Calculator::number() const throw(string)
{
    double value;
    cin >> value;   // let >> read and evaluate the number
    return value;
}
```

Assignment #10 Extra Credit

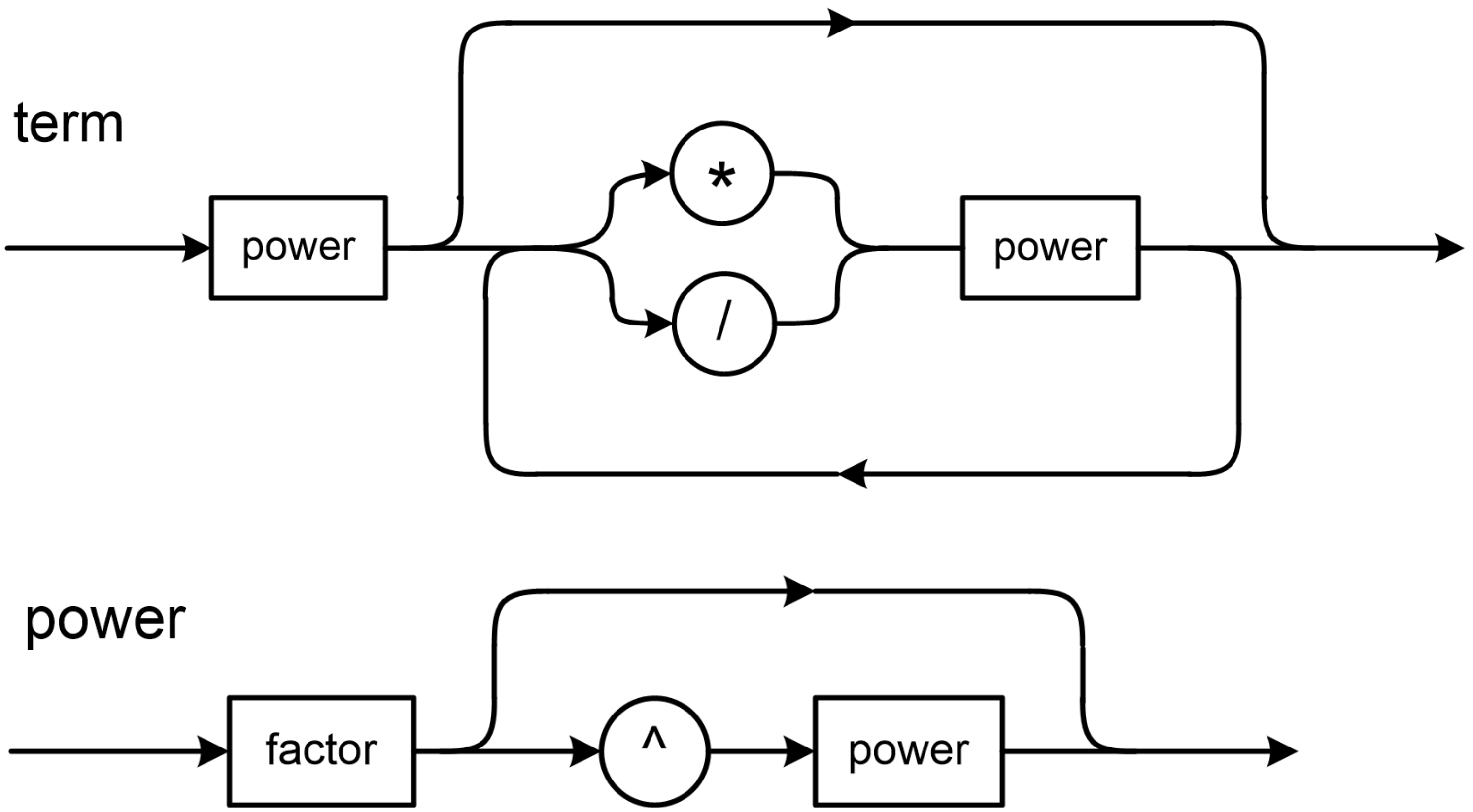
- A leading + or – sign.

expression



Assignment #10 Extra Credit, *cont'd*

- Exponentiation



Introduction to Algorithm Analysis

- To analyze an algorithm, we measure it.
- A convenient measure must be:
  - A resource we care about (elapsed time, memory usage, etc.).
  - Quantitative, to make comparisons possible.
  - Easy to compute.
  - A good predictor of the "goodness" of the algorithm.

In this class, we will be concerned mostly with elapsed time.

Introduction to Algorithm Analysis, *cont'd*

- Our concern generally is <u>not how long</u> a particular run of an algorithm will take, but <u>how well the algorithm scales</u>.

- How does the run time increase as the amount of input increases?
  - Example: How does the reading time of a book increase as the number of pages increases?
  - Example: How does the run time of a particular sort algorithm increase as the number of items to be sorted increases?

Example: Reading Books

- Algorithm: Read a book.
- Measure: Length of time to read a book.

- Given a set of books to read, can we predict how long it will take to read each one, without actually reading it?

- Possible ways to compute reading time:
  - weight of the book
  - physical size (width, height, thickness) of the book
  - total number of words
  - total number of pages

Introduction to Algorithm Analysis, *cont'd*

- When we compare two algorithms, we want to compare how well they scale.
  - How do their elapsed run times grow as the size of the input grows?
  - How do their growth rates compare?
- Can we do this comparison without actually running the algorithms?
  - Some algorithms may be too expensive to run.

How Well Does an Algorithm Scale?

Function	Name
c	Constant
$\log N$	Logarithmic
$\log^2 N$	Log-squared
N	Linear
$N \log N$	
N^2	Quadratic
N^3	Cubic
2^N	Exponential

Figure 2.1 Typical growth rates

Data Structures and Algorithms in Java, 3rd ed.
by Mark Allen Weiss
Pearson Education, Inc., 2012
ISBN 978-0-13-257627-7

How Well Does an Algorithm Scale? *cont'd*

	Algorithm Time			
Input Size	1 $O(N^3)$	2 $O(N^2)$	3 $O(N \log N)$	4 $O(N)$
$N = 100$	0.000159	0.000006	0.000005	0.000002
$N = 1{,}000$	0.095857	0.000371	0.000060	0.000022
$N = 10{,}000$	86.67	0.033322	0.000619	0.000222
$N = 100{,}000$	NA	3.33	0.006700	0.002205
$N = 1{,}000{,}000$	NA	NA	0.074870	0.022711

Figure 2.2 Running times of several algorithms for maximum subsequence sum (in seconds)

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How Well Does an Algorithm Scale? *cont'd*

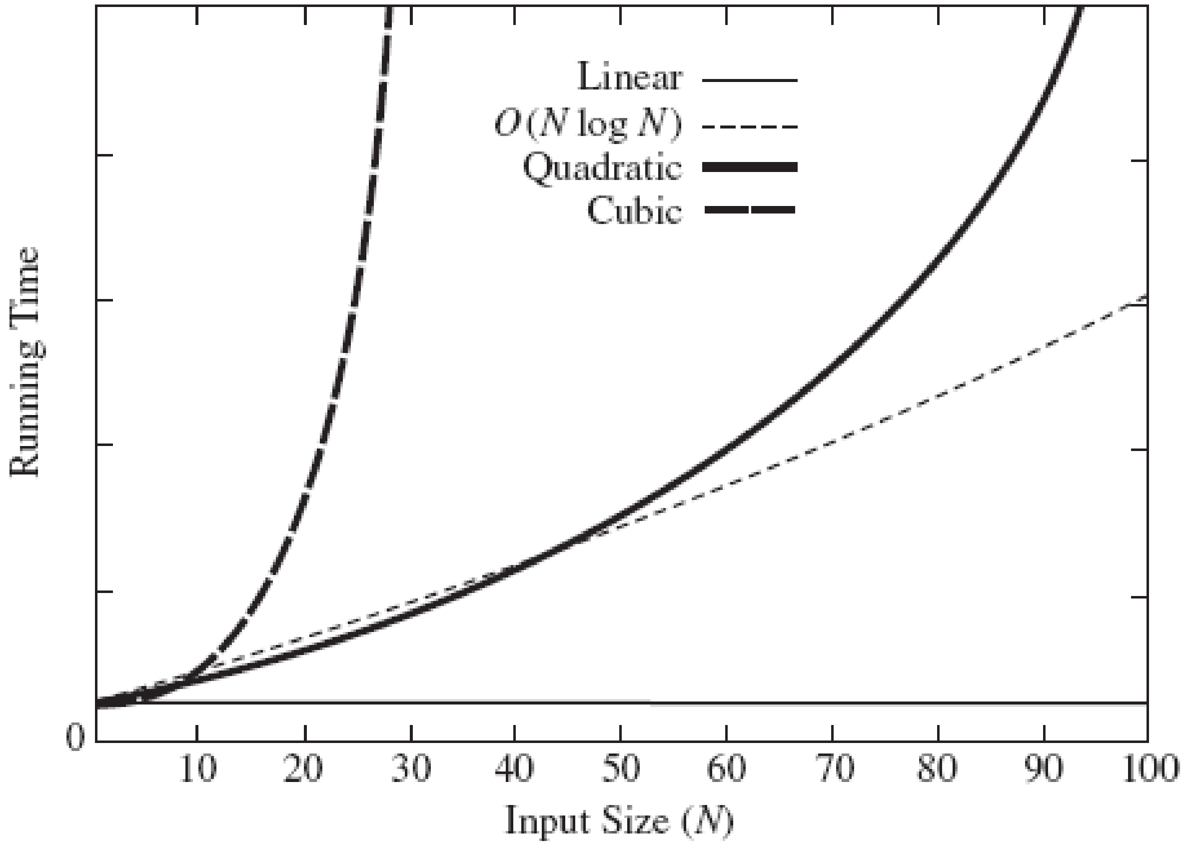

Figure 2.3 Plot (N vs. time) of various algorithms

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How Well Does an Algorithm Scale? *cont'd*

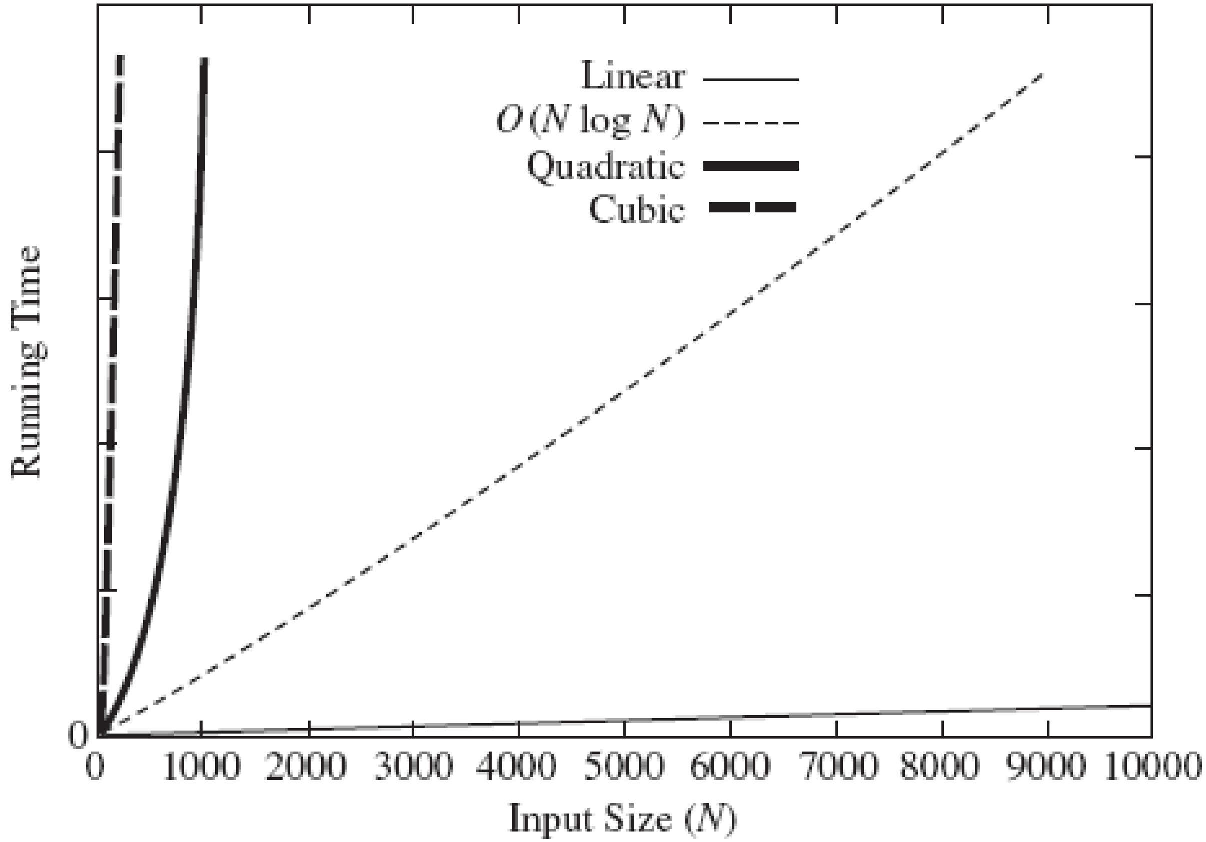

Figure 2.4 Plot (N vs. time) of various algorithms

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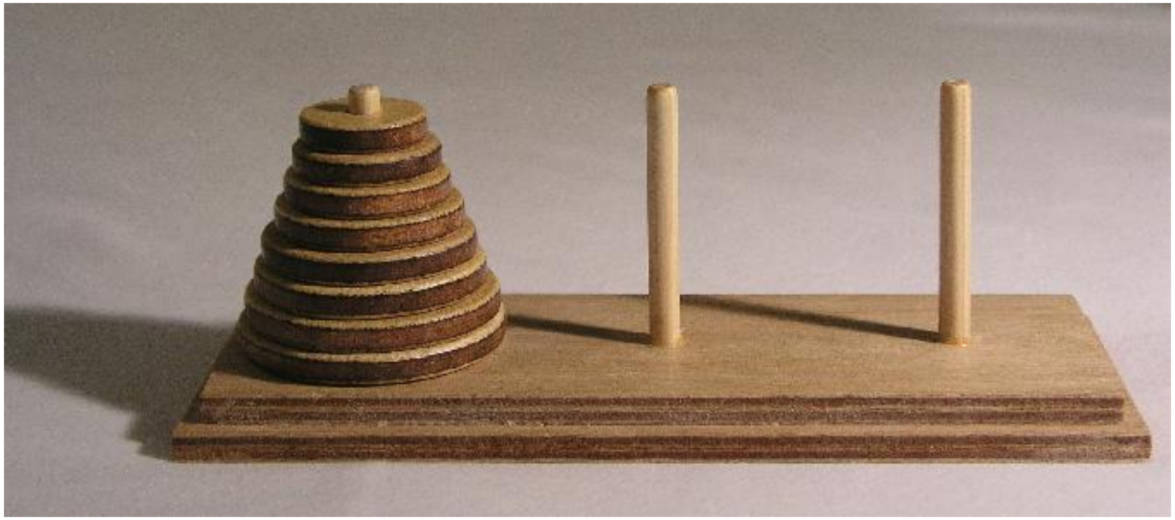
Towers of Hanoi



- **Goal:** Move the stack of disks from the source pin to the destination pin.
  - You can move only one disk at a time.
  - You cannot put a larger disk on top of a smaller disk.
  - Use the third pin for temporary disk storage.

Towers of Hanoi, *cont'd*

- Label the pins A, B, and C. Initial roles:
  - A: source
  - B: destination
  - C: temporary

During recursive calls, the pins will assume different roles.

- Base case: n = 1 disk
  - Move disk from A to B *(source → destination)*

- Simpler but similar case: n-1 disks
  - Solve for n-1 disks: A to C *(source → temp)*
  - Move 1 disk from A to B *(source → destination)*
  - Solve for n-1 disks: C to B *(temp → destination)*

Towers of Hanoi: Analysis

- How can we measure how long it will take to solve the puzzle for n disks?
- What's a good predictor?
  - The number times we move a disk from one pin to another.
  - Therefore, let's count the number of moves.

Towers of Hanoi: Analysis, *cont'd*

- Solve n disks
  - Solve for n-1 disks *(source → temp)*
  - Move 1 disk *(source → destination)*
  - Solve for n-1 disks *(temp → destination)*

- What is the pattern in the number of moves as n increases?
  - Let $f(n)$ be the number of moves for n disks.

$$f(n) = \begin{cases} 1 & n = 1 \\ 2f(n\text{-}1) + 1 & n > 1 \end{cases}$$

Towers of Hanoi: Analysis

$$f(n) = \begin{cases} 1 & n = 1 \\ 2f(n\text{-}1) + 1 & n > 1 \end{cases}$$

- This is a recurrence relation.
  - f shows up in its own definition: $f(n) = 2f(n\text{-}1) + 1$
  - The mathematical analogy of recursion.

- Can we find the definition of function f ?
  - Observation:
    Since $f(n) = 2f(n\text{-}1) + 1$, we know that $f(n) > 2f(n\text{-}1)$.
  - Therefore, if we increase the number of disks from n to n+1, the number of moves will at least double.

Towers of Hanoi: Count Moves

Hanoi2.cpp

```
int main()
{
    cout << "Disks Moves" << endl;

    for (int n = 1; n <= 10; n++) {
        int count = 0;
        solve(n, Pin::A, Pin::B, Pin::C, count);
        cout << setw(5) << n << setw(5) << count << endl;
    }
}

void move(Pin from, Pin to, int& count)
{
    count++;
}
```

Don’t print. Just count moves.

Towers of Hanoi: Analysis

Disks	Moves
1	1
2	3
3	7
4	15
5	31
6	63
7	127
8	255
9	511
10	1023

- What's the pattern?

$$f(n) = 2^n - 1$$

- Can we prove this?
  - Just because this formula holds for the first 10 values of n, does it hold for all values of $n \geq 1$?

Proof by Induction: Base Case

Prove that <u>if</u>:

$$f(n) = \begin{cases} 1 & n = 1 \\ 2f(n-1) + 1 & n > 1 \end{cases}$$

<u>then</u>:

$f(n) = 2^n - 1$
for all $n \geq 1$

- Let $n = 1$.
- Then $f(1) = 2^1 - 1 = 1$ is true.

Proof by Induction: Inductive Step

Prove that <u>if</u>:

$$f(n) = \begin{cases} 1 & n = 1 \\ 2f(n\text{-}1) + 1 & n > 1 \end{cases}$$

<u>then</u>:

$f(n) = 2^n - 1$
for all $n \geq 1$

- Let $n > 1$.
- **Inductive hypothesis:**
  Assume that $f(k) = 2^k - 1$ is true for all $k < n$, where $n > 1$
- Since $n\text{-}1 < n$, then by our hypothesis: $\boxed{f(n\text{-}1)} = \boxed{2^{n-1} - 1}$.
- From the recurrence relation:
  $f(n) = 2\boxed{f(n\text{-}1)} + 1 = 2\boxed{(2^{n-1} - 1)} + 1 = 2^n - 1$.
- So $f(n) = 2^n - 1$ for all $n > 1$.
- Therefore, if $f(k) = 2^k - 1$ is true for all $k < n$,
  it must also be true for n as well.

Proof by Induction: What Happened?

Prove that <u>if</u>:

$$f(n) = \begin{cases} 1 & n = 1 \\ 2f(n-1) + 1 & n > 1 \end{cases}$$

<u>then</u>:

$f(n) = 2^n - 1$
for all $n \geq 1$

- First we proved it for $n = 1$ (the base case).
- Then we proved that <u>if</u> it's true for all $k < n$, where $n > 1$ (the induction hypothesis) then it must also be true for n.
- Suppose $n = 2$. Since we know it's true for $n = 1$ (the base case), it must be true for $n = 2$ (from above).
- Suppose $n = 3$. Since we know it's true for $n = 2$ (from above), it must be true for $n = 3$.
- Etc.!

Another Proof By Induction Example

Observe that:

$1 = 1^2$
$1 + 3 = 4 = 2^2$
$1 + 3 + 5 = 9 = 3^2$
$1 + 3 + 5 + 7 = 16 = 4^2$
… ?

Prove:

$1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$ for all $n > 0$

Base case:
Let $n = 1$. Then $1 = 1^2$ is obviously true.

- **Induction hypothesis:**
  Assume that $1 + 3 + 5 + 7 + \ldots + (2k - 1) = k^2$ for some $k > 0$
- **Then show that:** $1 + 3 + 5 + 7 + \ldots + (2(k + 1) - 1) = (k + 1)^2$
- $1 + 3 + 5 + 7 + \ldots\ldots\ldots\ldots\ldots + (2(k + 1) - 1) =$
- $1 + 3 + 5 + 7 + \ldots + (2k - 1) + (2(k + 1) - 1) =$
- $k^2 + (2(k + 1) - 1) = k^2 + 2k + 1$
- $= (k + 1)^2$

True for $n = 1$.
If true for $n = k$
then true for $n = k+1$.

Algorithm Analysis

- An algorithm is a set of operations to perform in order to solve a problem.
- We want to know how an algorithm scales as its input size grows.
- If $T(N)$ is the running time of an algorithm with N input values, then how does $T(N)$ change as N increases?

Big-Oh and its Cousins

- Let $T(N)$ be the running time of an algorithm with N input values.

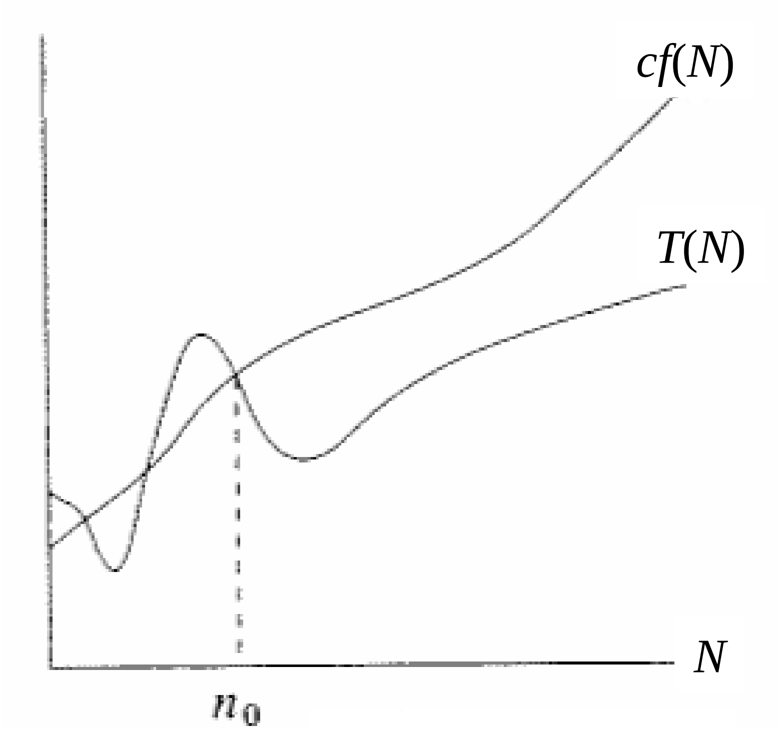
- Big-Oh
  - $T(N) = O(f(N))$ if there are positive constants c and n_0 such that $T(N) \leq cf(N)$ when $N \geq n_0$.
  - In other words, when N is sufficiently large, function $f(N)$ is an upper bound for time function $T(N)$.
    - We don't care about small values of N.
  - $T(N)$ will grow no faster than $f(N)$ as N increases.

Big-Oh and its Cousins, *cont'd*

- Let $T(N)$ be the running time of an algorithm with N input values.

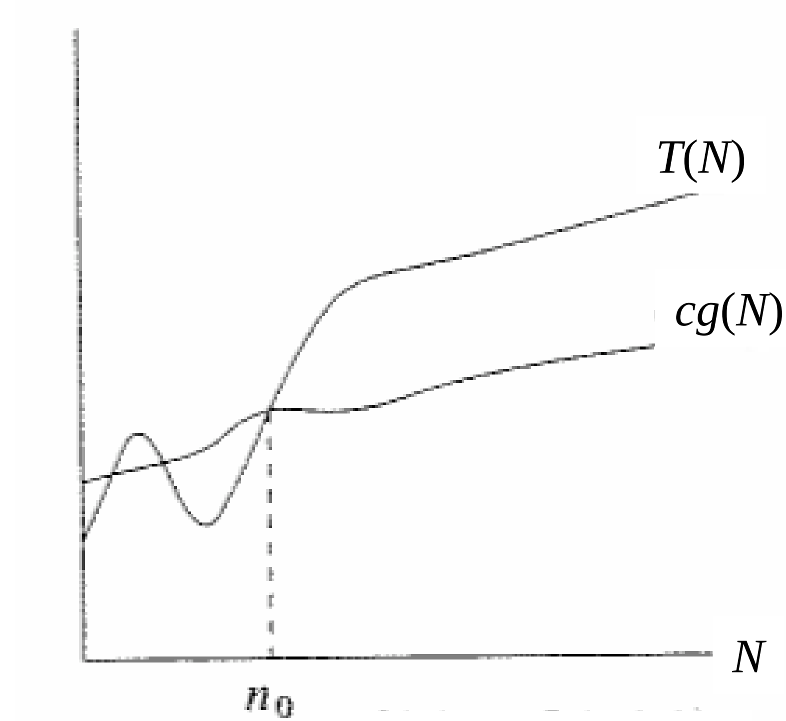
- Omega
  - $T(N) = \Omega(g(N))$ if there are positive constants c and n_0 such that $T(N) \geq cg(N)$ when $N \geq n_0$.
  - In other words, when N is sufficiently large, function $g(N)$ is <u>lower bound</u> for time function $T(N)$.
    - We don't care about small values of N.
  - $T(N)$ will grow <u>at least as fast</u> as $g(N)$ as N increases.

Big-Oh and its Cousins, *cont'd*



$$T(N) = O(f(N))$$

Upper bound



$$T(N) = \Omega(g(N))$$

Lower bound

Big-Oh and its Cousins, *cont'd*

- Let $T(N)$ be the running time of an algorithm with N input values.

- Theta
  - $T(N) = \Theta(h(N))$ if and only if:
    - $T(N) = O(h(N))$ and
    - $T(N) = \Omega(h(N))$
  - In other words, the rate of growth of $T(N)$ equals the rate of growth of $h(N)$.

Big-Oh and its Cousins, *cont'd*

- Let $T(N)$ be the running time of an algorithm with N input values.

- Little-Oh
  - $T(N) = o(p(N))$ if there are positive constants c and n_0 such that $T(N) < cp(N)$ when $N \geq n_0$.
  - $p(N)$ is similar to the upper bound function $f(N)$ but instead of $T(N) \leq cf(N)$ we have $T(N) < cp(N)$.

Big-Oh and its Cousins, *cont'd*

- If $T_1(N) = O(f_1(N))$ and $T_2(N) = O(f_2(N))$ then
  - $T_1(N) + T_2(N) = O(f_1(N) + f_2(N))$ or $O(\max(f_1(N), f_2(N)))$
  - $T_1(N) \text{ x } T_2(N) = O(f_1(N) \text{ x } f_2(N))$
- If $T(N)$ is a polynomial of degree k, then
  - $T(N) = \Theta(N^k)$
- If $T(N) = \log^k N$ for any constant k, then
  - $T(N) = O(N)$
  - Logarithms grow slowly!

Quiz

- Canvas: Quiz 7 – 2017 Apr 13
  - 20 minutes

Break

Towers of Hanoi: Rate of Growth

- We decided that a good predictor of $T(n)$ for solving the Towers of Hanoi problem was $f(n)$.
  - n is the number of disks
  - $f(n) = 2^n-1$ is the number of disk moves
- Therefore,

$$T(n) = \Theta(2^n)$$

Compare Growth Rates

- If we want to compare the growth rates of two functions $f(N)$ and $g(N)$, compute

$$\lim_{N \to \infty} f(N) / g(N)$$

  - The limit is 0: $f(N) = o(g(N))$
    $g(N)$ is an upper bound for $f(N)$.
  - The limit is a constant $c \neq 0$: $f(N) = \Theta(g(N))$
    $f(N)$ *and* $g(N)$ have the same growth rate.
  - The limit is ∞: $g(N) = o(f(N))$
    $f(N)$ is an upper bound for $g(N)$.

General Rules for Computing Running Time

- Consecutive statements
  - Add the running times of the statements.
  - Generally, only consider the statement with the maximum running time.
- Branching statement
  - The running time of the entire statement is at most the maximum running time of its branches.

Computing Running Time, *cont'd*

- Loop
  - The running time of a loop is at most
    the number of iterations times
    the running time of the statements in the loop.

- Nested loops
  - Compute the running time of the statements in the innermost loop, then multiply by the product of the numbers of iterations of all the loops.

Scalability of Different Algorithms

- Problem: Compute the n^{th} Fibonacci number.
- Two algorithms to solve this problem:
  - Start with 1, 1, and repeatedly add the previous two values.
    - LinearGrowthRate: $T(N) = O(N)$
  - Use recursion: $fib(n) = fib(n\text{-}2) + fib(n\text{-}1)$
    - ExponentialGrowthRate: $T(N) = \Omega(1.5^N)$

Why is the growth rate exponential?

Scalability of Different Algorithms, *cont'd*

- One set of results for the Fibonacci problem.
  - Times in milliseconds.

n	Linear	Exponential
5	0	0
10	0	1
15	0	0
20	0	2
25	0	3
30	0	4
35	0	45
40	0	504
45	0	5358
50	0	59267

Hash Tables

- Consider an array or a vector.
  - To access a value, you use an integer index.
- The array "maps" the index to a data value stored in the array.
  - The mapping function is very efficient.
  - As long as the index value is within range, there is a strict one-to-one correspondence between an index value and a stored data value.
- We can consider the index value to be the "key" to the corresponding data value.

Hash Tables, *cont'd*

- A hash table also stores data values.
  - Use a key to obtain the corresponding data value.
- The key does not have to be an integer value.
  - For example, the key could be a string.
- There might not be a one-to-one correspondence between keys and data values.
- The mapping function might not be trivial.

Hash Tables, *cont'd*

- We can implement a hash table as an <u>array of cells</u>.
  - Refer to its size as *TableSize*.

- If the hash table's mapping function maps a key value into an integer value in the range 0 to *TableSize* – 1, then we can use this integer value as the index into the underlying array.

Hash Tables, *cont'd*

- Suppose we're storing employee data records into a hash table.
- We use an employee's name as the key.

Hash Tables, *cont'd*

- Suppose that the name
  - *john* hashes (maps) to 3
  - *phil* hashes to 4
  - *dave* hashes to 6
  - *mary* hashes to 7
- This is an ideal situation because each employee record ended up in a different table cell.

0	
1	
2	
3	john 25000
4	phil 31250
5	
6	dave 27500
7	mary 28200
8	
9	

Figure 5.1 An ideal hash table

Data Structures and Algorithms in Java, 3rd ed.
by Mark Allen Weiss
Pearson Education, Inc., 2012
ISBN 0-13-257627-9

Hash Function

- We need an ideal hash function to map each data record into a distinct table cell.
  - It can be very difficult to find such a hash function.

A Simple Hash Function

- Suppose our keys are words and the table size is 10,007 (a prime number).

```
int hash(const string& word, int table_size)
{
    int hashVal = 0;
    for (char ch : word) hashVal += ch;
    return hashVal%table_size;
}
```

- Problem: This hash function does not distribute the keys well if the table is large.
  - The maximum ASCII character value is 127.
  - If a typical word is 8 characters long, the hash function generally has values from 0 to 1,016.

Another Simple Hash Function

```
int hash(const string& word, int table_size)
{
    return (key[0] + 27*key[1] + 729*key[2])%table_size;
}
```

- We use only the first three letters of each word.
  - 27 letters in the alphabet + space
  - $729 = 27^2$
- Good distribution into a table of 10,007 if the first three letters are random.
  - But the English language is not random and many words will start with the same three letters.

A Better Hash Function

```
int hash(const string& word, int table_size)
{
    unsigned int hashVal = 0;
    for (char ch : word) hashVal = 37*hashVal + ch;
    return hashVal%table_size;
}
```

- Calculates a polynomial function
  by nested multiplication (Horner's rule).
- Easy and fast to calculate.
- Distributes the keys well into a large table.

Collisions

- The more data we put into a hash table, the more "collisions" occur.
- A collision is when two or more data records are mapped to the same table cell.
- How can a hash table handle collisions?

Keys for Successful Hashing

- Good hash function
- Good collision resolution
- Size of the underlying array a <u>prime number</u>

Collision Resolution

- Separate chaining
- Open addressing
  - Linear probing
  - Quadratic probing

Collision Resolution: Separate Chaining

- Each cell in a hash table is a pointer to a linked list of all the data records that hash to that entry.
- To retrieve a data record, we first hash to the cell.

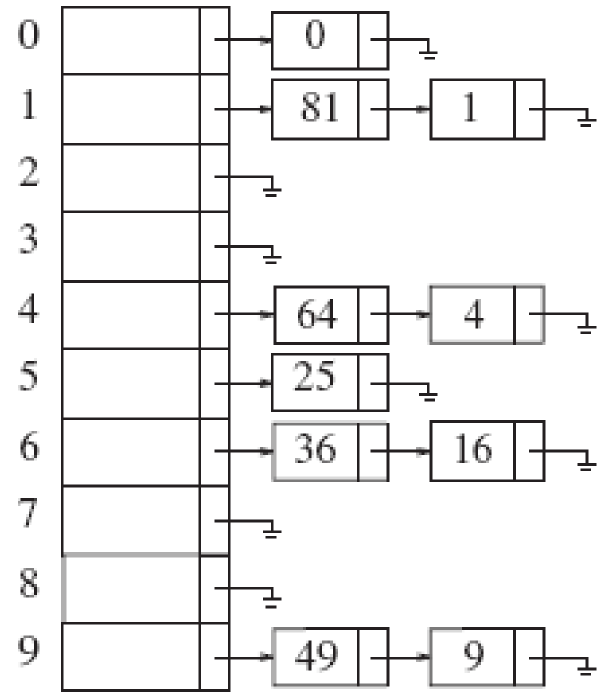


Figure 5.5 A separate chaining hash table

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Collision Resolution: Separate Chaining, *cont'd*

- Then we search the associated linked list for the data record.
- We can sort the linked lists to improve search performance.

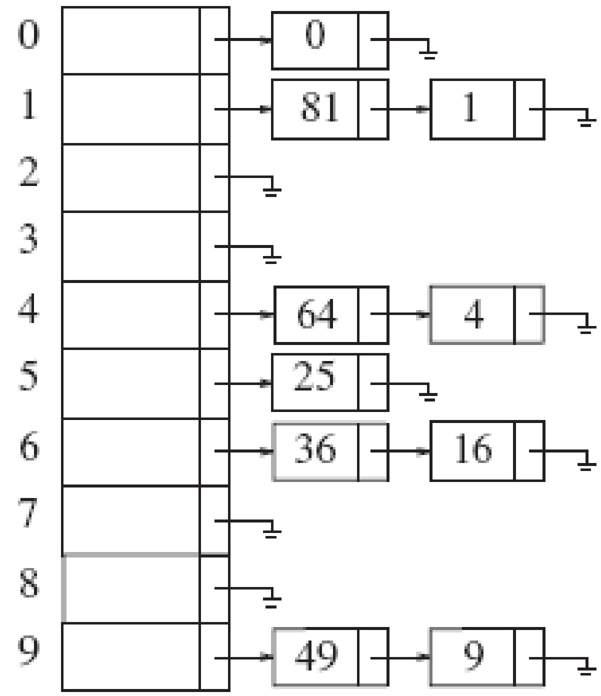

Figure 5.5 A separate chaining hash table

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Collision Resolution: Open Addressing

- Does not use linked lists.
- All the data resides in the table.
- When a collision occurs,
  try a different table cell.
- We will consider two types of open addressing:
  - linear probing
  - quadratic probing

Collision Resolution: Linear Probing

- Try in succession $h_0(x), h_1(x), h_2(x), \ldots$
- $h_i(x) = (hash(x) + f(i)) \bmod TableSize$, with $f(0) = 0$
  - $hash(x)$ produces the home cell.
- Function f is the collision resolution strategy.
- With linear probing, f is a linear function of i, typically, $f(i) = i$

Collision Resolution: Linear Probing, *cont'd*

- Insertion
  - If a cell is filled, look for the next empty cell.
- Search
  - Start searching at the home cell, keep looking at the next cell until you find the matching key is found.
  - If you encounter an <u>empty cell</u>, there is no key match.
- Deletion
  - Empty cells will prematurely terminate a search.
  - Leave deleted items in the hash table but <u>mark them as deleted</u>.

Collision Resolution: Linear Probing, *cont'd*

- Suppose *TableSize* is 10, the keys are integer values, and the hash function is the key value modulo 10.
  - We want to insert keys 89, 18, 49, 58, and 69.

	Empty Table	After 89	After 18	After 49	After 58	After 69
0				49	49	49
1					58	58
2						69
3						
4						
5						
6						
7						
8			18	18	18	18
9		89	89	89	89	89

Figure 5.11 Hash table with linear probing, after each insertion

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Collision Resolution: Quadratic Probing

- Linear probing causes primary clustering.
- Try quadratic probing instead: $f(i) = i^2$.

49 collides with 89: the next empty cell is 1 away.

58 collides with 18: the next cell is filled. Try $2^2 = 4$ cells away from the home cell.

Same for 69.

	Empty Table	After 89	After 18	After 49	After 58	After 69
0				49	49	49
1						
2					58	58
3						69
4						
5						
6						
7						
8			18	18	18	18
9		89	89	89	89	89

Figure 5.13 Hash table with quadratic probing, after each insertion

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Collision Resolution: Quadratic Probing, *cont'd*

- Try quadratic probing instead: $f(i) = i^2$.

- i^2 is easy to compute, for i = 0, 1, 2, ...
  - Remember that we proved that

$1 = 1^2$
$1 + 3 = 4 = 2^2$
$1 + 3 + 5 = 9 = 3^2$
$1 + 3 + 5 + 7 = 16 = 4^2$
…

Load Factor

- The load factor λ of a hash table is the ratio of the number of elements in the table to the table size.
  - λ is much more important than table size.
- For probing collision resolution strategies, it is important to keep λ under 0.5.
  - Don't let the table become more than half full.
- If quadratic probing is used and the table size is a prime number, then a new element can always be inserted if the table is at most half full.

Assignment #11

- Read a copy of the U.S. Constitution and its amendments, and build a concordance table.
  - Concordance: An alphabetical list of words in a text, each word with the number of times it appears.
- Maintain the concordance in both a <u>sorted STL vector</u> and in an <u>STL map</u> (hash table).
- Compare the timings of the vector and the map:
  - Insertion of words
  - Searching for words