FOURIER TRANSFORMS IN QUANTUM MECHANICS

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THE FOURIER SERIES

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$$F(x) = \sum_{r=-\infty}^{+\infty} a_r |\alpha\rangle$$

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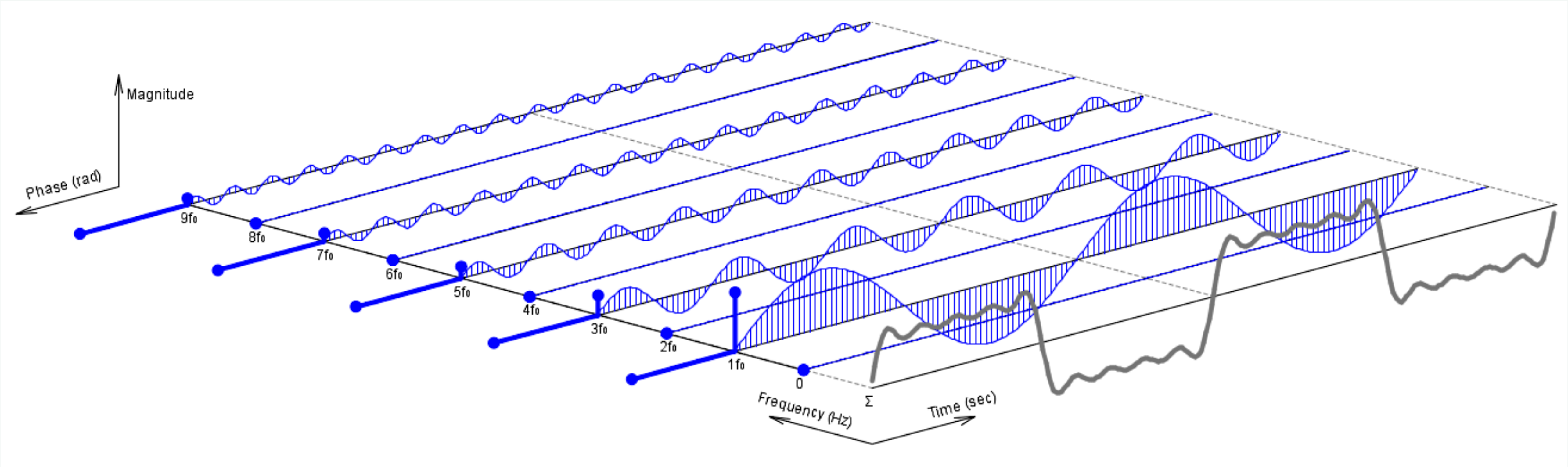
$$F(x) = \sum_{r=-\infty}^{+\infty} a_r e^{ir\omega t}$$

$$f(\chi) = \cdots + \alpha_{-2}\mathrm{e}^{-2\mathrm{i}\omega\chi} + \alpha_{-1}\mathrm{e}^{-\mathrm{i}\omega\chi} + \alpha_0 + \alpha_1\mathrm{e}^{\mathrm{i}\omega\chi} + \alpha_2\mathrm{e}^{2\mathrm{i}\omega\chi} + \alpha_3\mathrm{e}^{3\mathrm{i}\omega\chi} + \cdots,$$

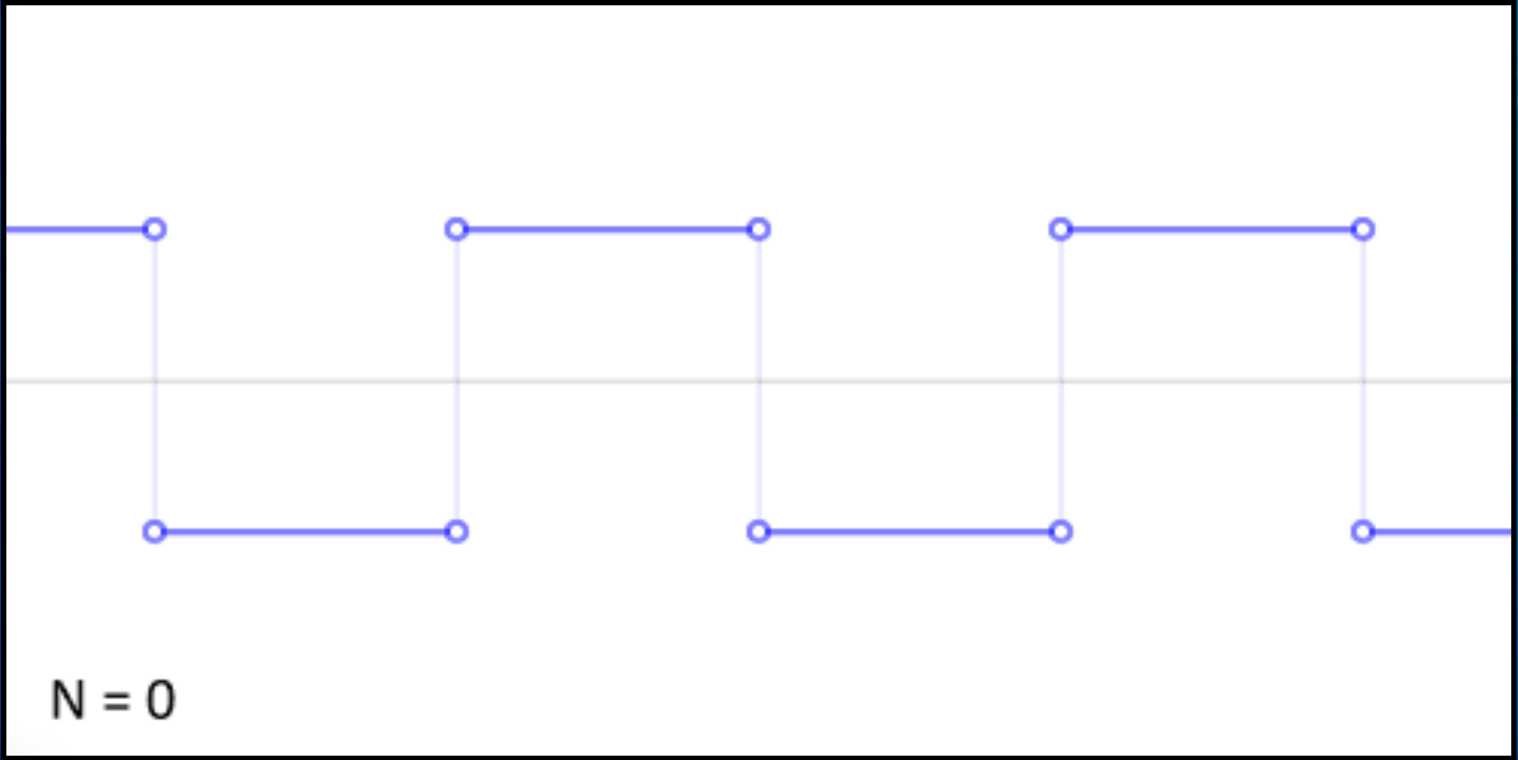
$$\begin{aligned} f(\chi) = {} & c + a_1\cos\omega\chi + b_1\sin\omega\chi + a_2\cos 2\omega\chi + b_2\sin 2\omega\chi + \\ & a_3\cos 3\omega\chi + b_3\sin 3\omega\chi + \cdots, \end{aligned}$$



FOURIER SERIES EXPANSION OF THE SQUARE WAVE FUNCTION

$$f(t) = k \quad \text{if } 0 \leq t < a$$
$$= -k \quad \text{if } a < t < 2a$$

$$f(x) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right).$$



#Any function that is piecewise continuous in [-r,r] can be expressed as a linear combination of the orthogonal set of functions e^ to be typed on the interval [-r,r]

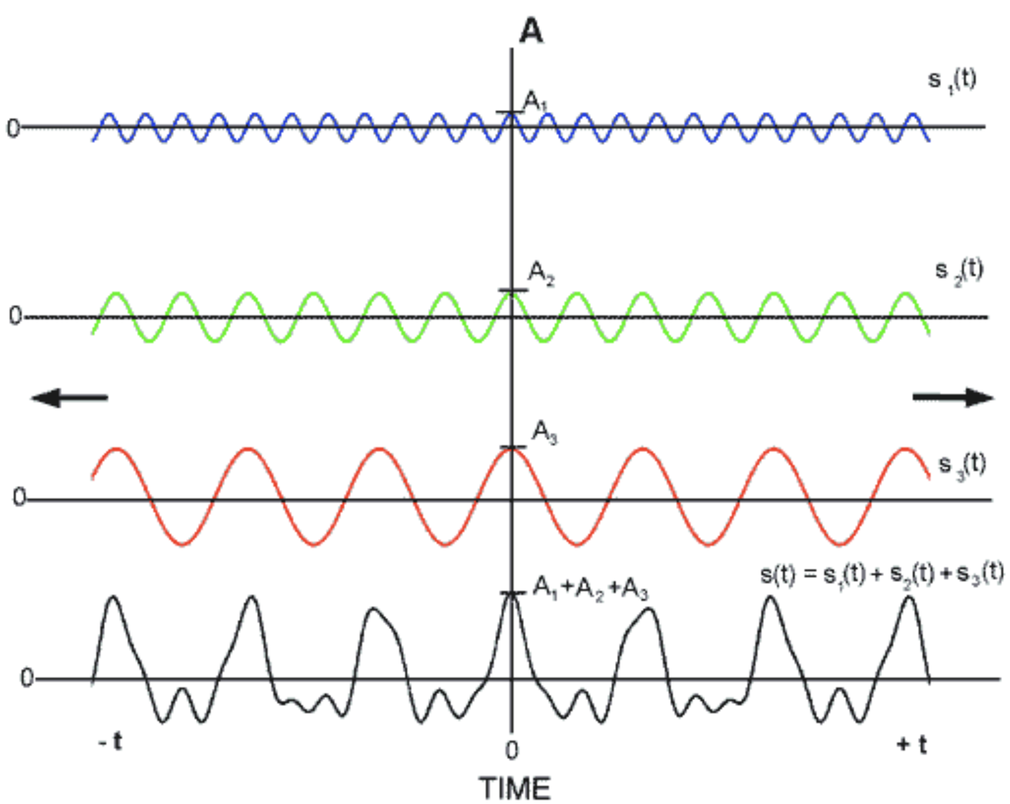
#This set of mutually orthogonal functions span an infinite dimensional Hilbert space (the L^2 Hilbert space)

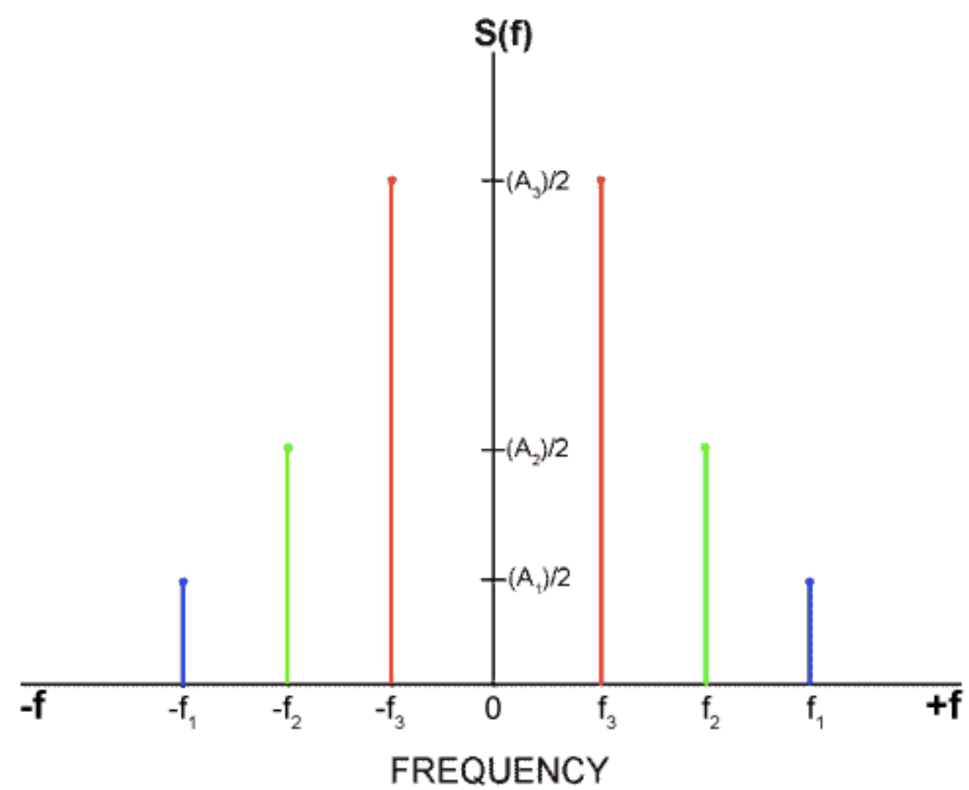
THE FOURIER TRANSFORM

- The Fourier transform of a function f(t) is given by:

$$g(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt$$

- The transformation maps a function in the time (t) domain to a function in the frequency (ω) domain.





THE FOURIER TRANSFORM

The $f(x)$

LIM

The Fourier series expansion

$A + a\cos(\,) + b\sin(\,) + a\cos 2(\,) + b\sin 2(\,) + \ldots\ldots\ldots\ldots\ldots$

The Fourier transform

A function of frequency g(k)

FOURIER TRANSFORMS IN QUANTUM MECHANICS

- SOME EXAMPLES:
- Transformation of a wave function from momentum space to ordinary space (and vice versa).
- Transformation of the Yukawa potential from momentum space to ordinary space (and vice versa)
- Transformation of the Dirac delta function.
- Transformation of the Exponential function.

	The position operator x	The infinitesimal translation operator	The momentum operator p
The operation	$x\|x'\rangle = x'\|x'\rangle,$	$\mathscr{T}(d\mathbf{x}')\|\mathbf{x}'\rangle = \|\mathbf{x}' + d\mathbf{x}'\rangle,$	$p\|p'\rangle = p'\|p'\rangle$
Orthogonality	$\langle x''\|x'\rangle = \delta(x'' - x').$		$\langle p'\|p''\rangle = \delta(p' - p'').$
Expansion of an arbitrary vector	$\|\alpha\rangle = \int dx'\|x'\rangle\langle x'\|\alpha\rangle,$		$\|\alpha\rangle = \int dp'\|p'\rangle\langle p'\|\alpha\rangle.$
The wave function	$\langle x'\|\alpha\rangle = \psi_\alpha(x').$		$\langle p'\|\alpha\rangle = \phi_\alpha(p').$
Probability	$\langle x'\|\alpha\rangle = \psi_\alpha(x').$		$\int dp'\langle\alpha\|p'\rangle\langle p'\|\alpha\rangle = \int dp'\|\phi_\alpha(p')\|^2 = 1.$

The commutation relations between the operators

$$[x_i, x_j] = 0, \qquad [p_i, p_j] = 0, \qquad [x_i, p_j] = i\hbar\delta_{ij}.$$

A DERIVATION

- Consider the infinitesimal translation operator with the action

$$\mathscr{T}(d\mathbf{x}')|\mathbf{x}'\rangle = |\mathbf{x}' + d\mathbf{x}'\rangle$$

- This operator is given in terms of the operator ‘p’ as

$$\mathscr{T}(d\mathbf{x}') = 1 - i\mathbf{p}\cdot d\mathbf{x}'/\hbar,$$

- Let us operate on an arbitrary state which is expanded in terms of the eigenkets of the operator ‘x’

$$\begin{aligned}\left(1 - \frac{ip\Delta x'}{\hbar}\right)|\alpha\rangle &= \int dx' \mathscr{T}(\Delta x')|x'\rangle\langle x'|\alpha\rangle \\ &= \int dx' |x' + \Delta x'\rangle\langle x'|\alpha\rangle \\ &= \int dx' |x'\rangle\langle x' - \Delta x'|\alpha\rangle \\ &= \int dx' |x'\rangle\left(\langle x'|\alpha\rangle - \Delta x' \frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right).\end{aligned}$$

- Comparing the LHS and RHS, we get,

A DERIVATION

- we get the relation,

$$p|\alpha\rangle = \int dx'|x'\rangle\left(-i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right)$$

- And, multiplying with the specified bra,

$$\langle x'|p|\alpha\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle,$$

- Now, let us replace the arbitrary ket with an eigenket of the momentum operator 'p'

$$\langle x'|p|p'\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle$$

- but, WKT:

$$p|p'\rangle = p'|p'\rangle$$

- Therefore we can write:

$$p'\langle x'|p'\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle$$

- But the above equation represents a differential equation, whose solution is given by ....

A DERIVATION

- The solution is given by:

$$\langle x'|p'\rangle = N\exp\left(\frac{ip'x'}{\hbar}\right)$$

- Where N is the normalisation constant to be determined. From the calculations mentioned in slide [appendix 1], the normalisation constant is found to be:

$$\mathbf{N} = \frac{1}{\sqrt{2\pi\hbar}}$$

- And we have the final expression:

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip'x'}{\hbar}\right)$$

- This equation provides us with a relation between the eigenvectors of the position operator ‘x’ and the eigenvectors of the momentum operator ‘p’

- Using this relation, we can obtain a very interesting and insightful relation between the position space wave function $\psi_\alpha(x')$ and the momentum space wave function…. $\phi_\alpha(p')$.

A DERIVATION

- Rewriting the position space wave function:

$$\langle x'|\alpha\rangle = \int dp' \langle x'|p'\rangle\langle p'|\alpha\rangle$$

- Rewriting the momentum space wave function:

$$\langle p'|\alpha\rangle = \int dx' \langle p'|x'\rangle\langle x'|\alpha\rangle$$

- Giving us the relations

$$\psi_\alpha(x') = \left[\frac{1}{\sqrt{2\pi\hbar}}\right]\int dp' \exp\left(\frac{ip'x'}{\hbar}\right)\phi_\alpha(p')$$

$$\phi_\alpha(p') = \left[\frac{1}{\sqrt{2\pi\hbar}}\right]\int dx' \exp\left(\frac{-ip'x'}{\hbar}\right)\psi_\alpha(x').$$

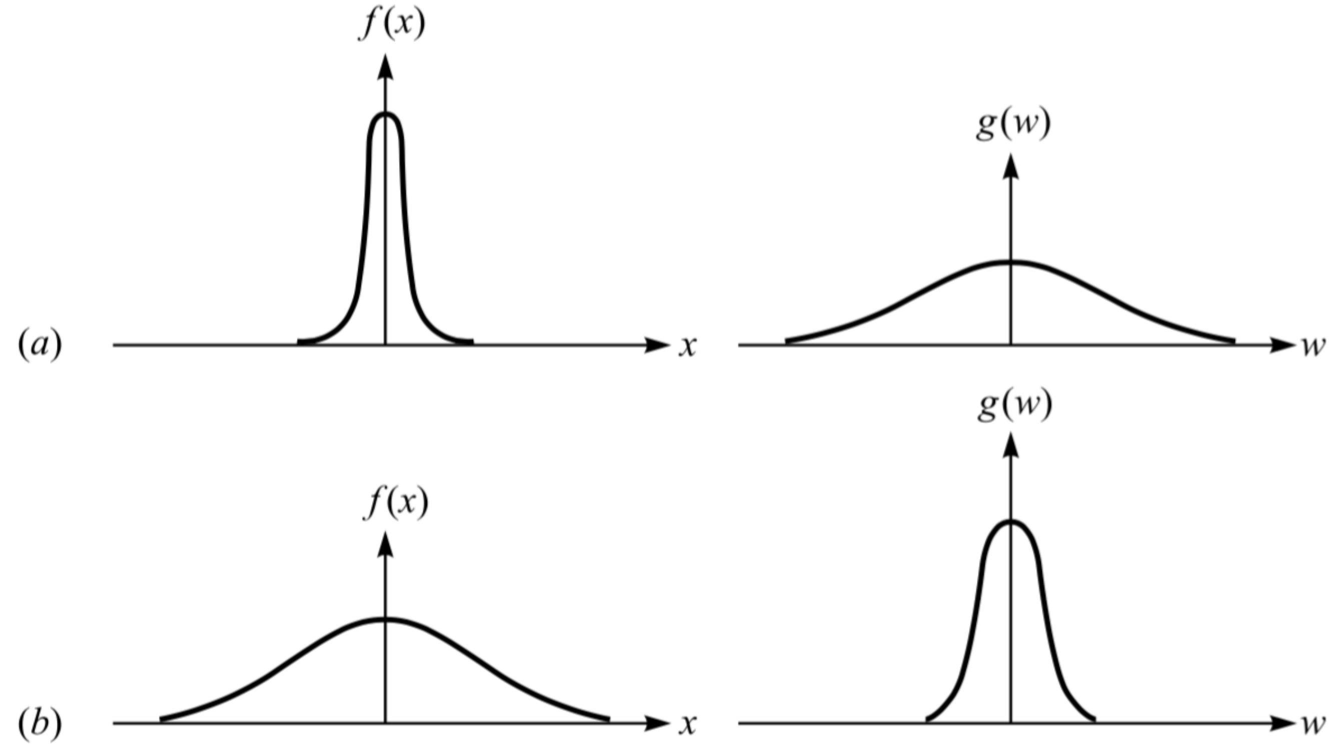
- These relations show that position space wave function and the momentum space wave function of our arbitrary state (alpha) are the fourier transforms of each other.

A DERIVATION

- We don’t get any physical/mathematical inference out of the above equations unless we specify our wave functions. In that direction lets us set our position space wave function as the x-space wave function of a gaussian wave packet. The gaussian wave packet describes semi-localised particles such as electron/protons.

$$\langle x'|\alpha\rangle = \left[\frac{1}{\pi^{1/4}\sqrt{d}}\right]\exp\left[ikx' - \frac{x'^2}{2d^2}\right]$$

- Without going into the mathematics of the Fourier transformation of the above wave function into a function in momentum space, let us look at the below plots which explain a whole lot more:



1. Heisenberg’s uncertainty principle occurs as a natural progression of these ideas. (ref. Modern quantum mechanics [2nd ed by JJ Sakurai]) p.57-58

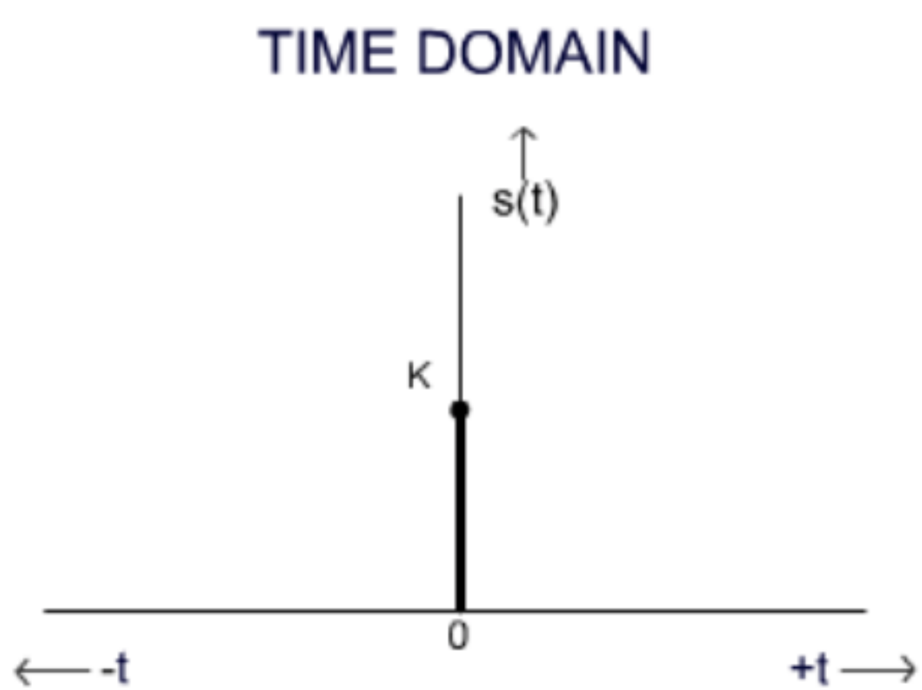
FOURIER TRANSFORM OF THE **DIRAC DELTA FUNCTION**

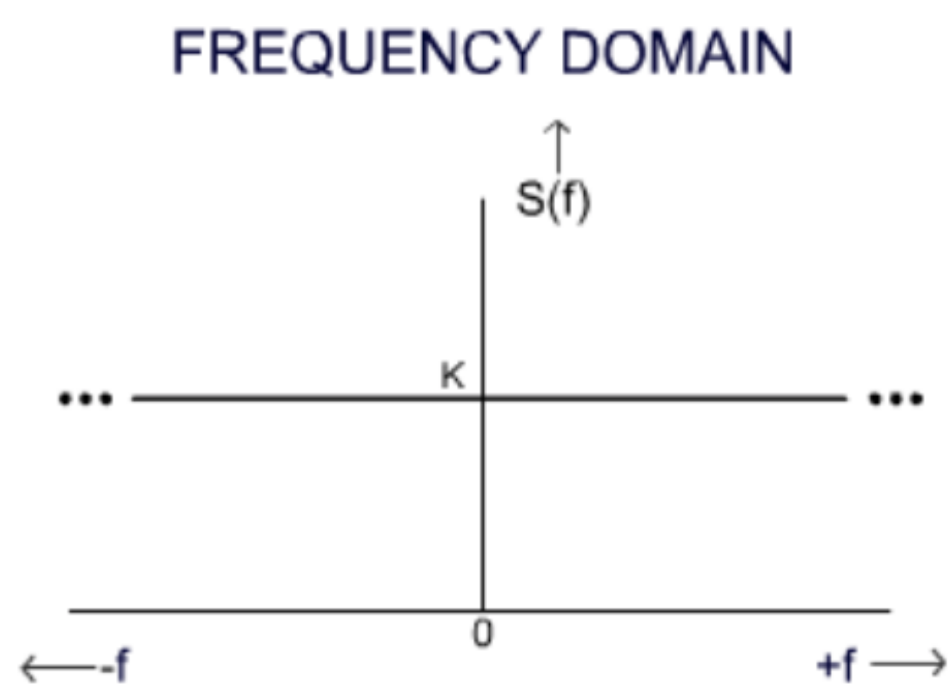
The delta function

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

The Fourier transform

$$\tilde{f}(s) \equiv \int_{-\infty}^{\infty} dt \, \frac{e^{its}}{\sqrt{2\pi}} f(t).$$

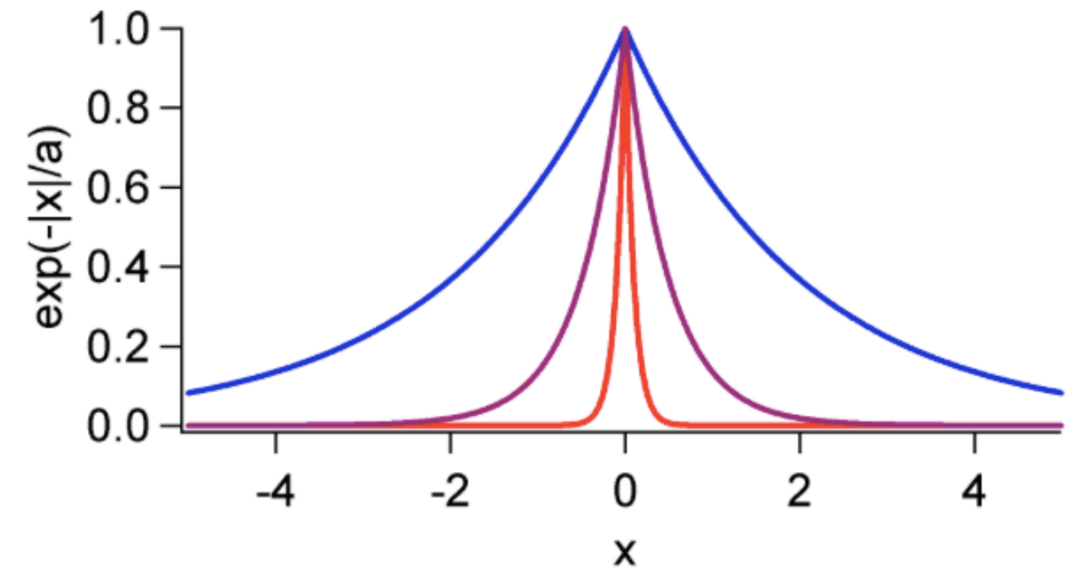




FOURIER TRANSFORM OF THE EXPONENTIAL FUNCTION

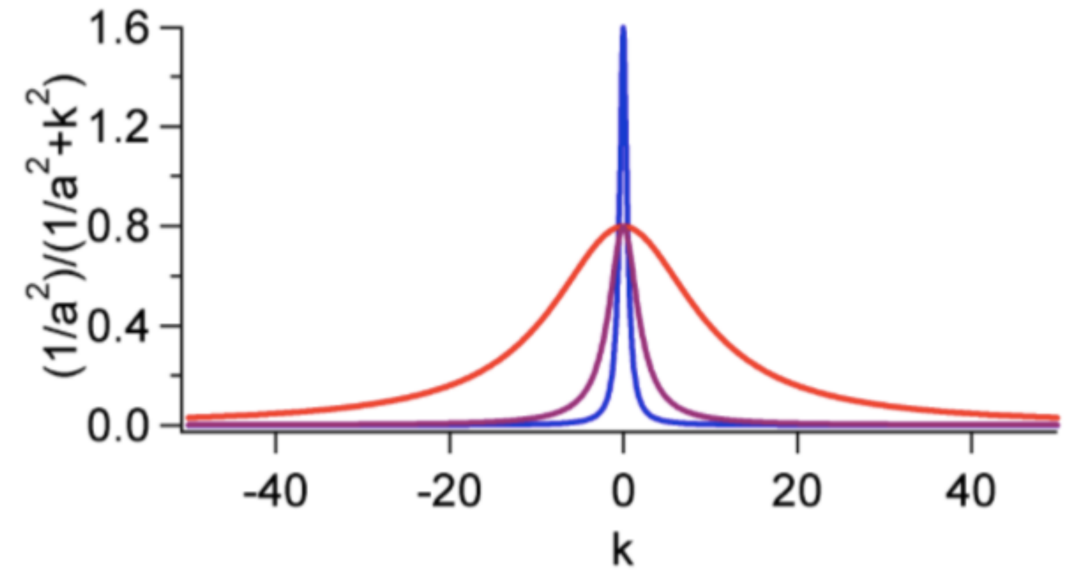
The Exponential function

$$G(x) = \exp\{-|x|/a\}$$



The Fourier transform

$$H(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|/a} e^{-ikx} dx$$



FOURIER TRANSFORM OF THE YUKAWA POTENTIAL FUNCTION

The Yukawa potential function

$$V_{\text{Yukawa}}(r) = -g^2 \frac{e^{-kmr}}{r},$$

The Fourier transform

$$V(\mathbf{r}) = \frac{-g^2}{(2\pi)^3} \int e^{i\mathbf{k}\cdot\mathbf{r}} \frac{4\pi}{k^2 + m^2}\, d^3k$$

The Yukawa potential is the potential between two interacting particles, where the interaction is mediated by an exchange particle.

eg: The electromagnetic coulomb potential between two interacting electrons, the interaction being mediated by a photon of mass m=0

REFERENCES:

1. Modern quantum mechanics (2nd ed.) by J.J. Sakurai and Jim Napolitano
2. Quantum Mechanics (2nd ed.) by Bransden and Joachain
3. Mathematical methods in physics by Tai L Chow

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| *thank you* |

APPENDIX 1

To get the normalization constant N let us first consider

$$\langle x'|x''\rangle = \int dp' \langle x'|p'\rangle\langle p'|x''\rangle. \tag{1.7.30}$$

The left-hand side is just $\delta(x'-x'')$; the right-hand side can be evaluated using the explicit form of $\langle x'|p'\rangle$:

$$\begin{aligned}\delta(x'-x'') &= |N|^2 \int dp' \exp\left[\frac{ip'(x'-x'')}{\hbar}\right] \\ &= 2\pi\hbar|N|^2\delta(x'-x''). \end{aligned} \tag{1.7.31}$$

Choosing N to be purely real and positive by convention, we finally have

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ip'x'}{\hbar}\right). \tag{1.7.32}$$