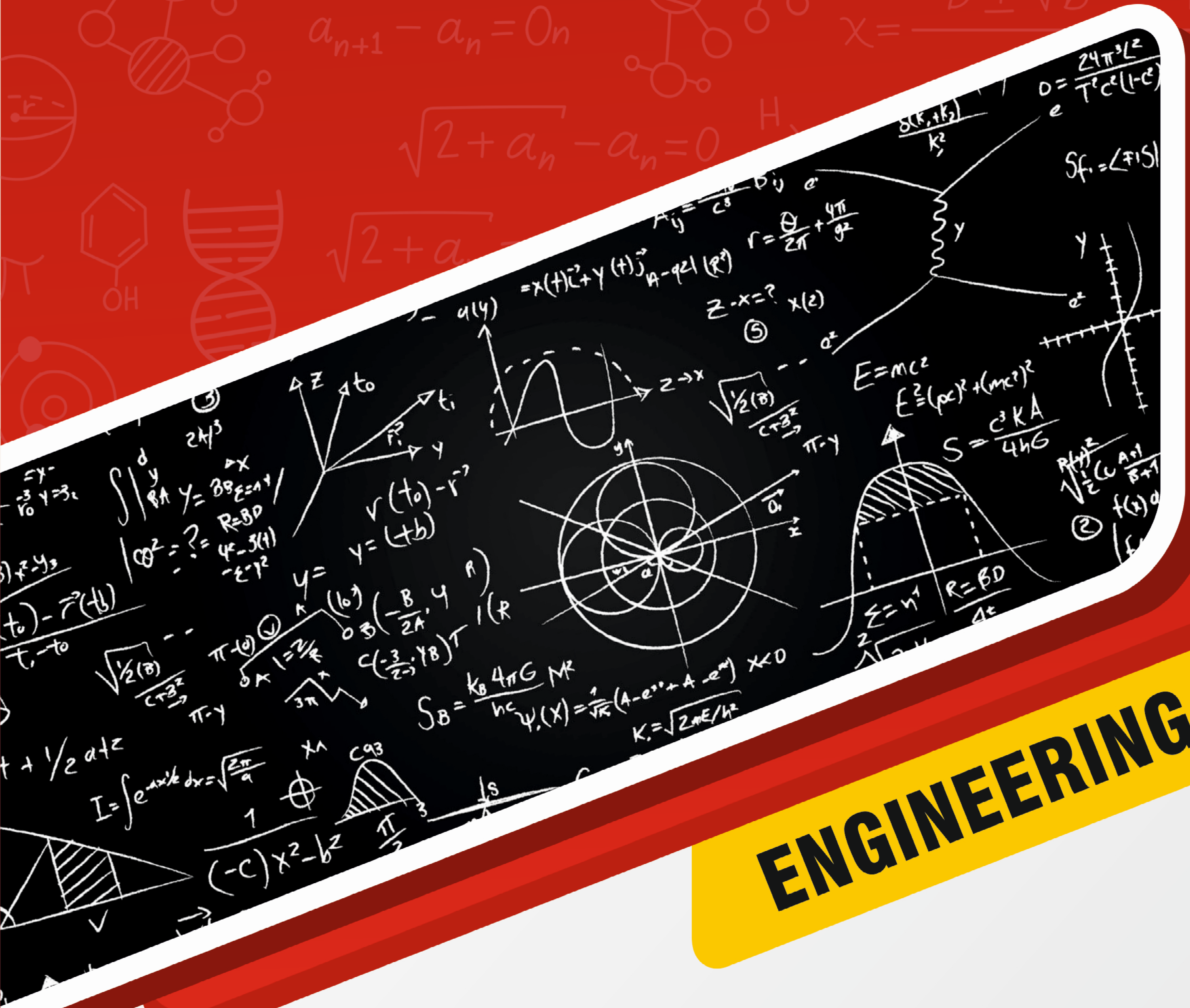
MATHEMATICS



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BINOMIAL

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BINOMIAL

CONTENT

BINOMIAL THEOREM

1.1 BINOMIAL EXPRESSION :

An algebric expression consisting of two different terms is called a binomial expression.
e.g. (1) $x + y$ (2) $x^3 + y^3$
But $(x + 2x)$ is not a binomial, it is called a monomial.

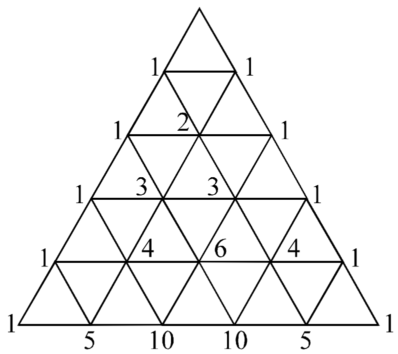
1.2 BINOMIAL THEOREM :

In elementary algebra, the binomial theorem describes the algebric expansion of powers of a binomial expression.According to the theorem it is possible to expand the powers $(x + y)^n$ into a sum involving terms of the form ax^by^c, where exponents b and c are non-negative integers with $b + c = n$ and the coefficient 'a' of each term is a specific positive integer depending on n and b.
This theorem was given by Newton.

Binomial Theorem $\Rightarrow$ $(x + y)^n$
- If $n \in N$
  (Form a finite series)
- Any index $n \notin N$
  (Form an infinite series)

1.3 HISTORICAL DEVELOPMENT :

Earlier people used to multiply the brackets to expand the given binomial of known index.
Then came the Pascal triangle



$(x + y)^2 = (x + y)^2 = x^2 + 2xy + y^2$
$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

Note that
(a) The powers of x go down until it reaches reaches zero, starting value is n.
(b) The power of y goes up from zero untill it reaches n.
(c) The n^{th} row of the Pascals triangle will be the coefficients of the expanded binomial.

1.4 STATEMENT OF THE THEOREM :

$(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2\, x^{n-2}\, y^2 + ... + {}^nC_rx^{n-r} \cdot y^r + ... + {}^nC_n\, x^{n-n}\, y^n.$

We observe $T_1 = {}^nC_0\, x^n$

$T_2 = {}^nC_1\, x^{n-1} \cdot y^1$

$\Rightarrow$ General term in the expansion of $(x + y)^n$ is

$T_{r+1} = {}^nC_r\, x^{n-r} \cdot y^r$

Where ${}^nC_r$ is called as combinatorial or binomial coefficient also denoted by $\binom{n}{r}$.

Also, $(x+y)^n = \sum\limits_{r=0}^{n} {}^nC_r\, x^{n-r} \cdot y^r$

1.5 IMPORTANT POINTS OF EXPANSION :

(1) Number of terms in expansion of $(x + y)^n$ is $n + 1$ i.e., one more than index.

(2) Sum of indices of x and y in each term in the expansion of $(x + y)^n$ is n.

(3) New expansions by $(x + y)^n$

(a) We have $(x + y)^n = {}^nC_0x^n + {}^nC_1\, x^{n-1}y + {}^nC_2\, x^{n-2}\, y^2 + ... + {}^nC_n\, y^n$ ....(1)

i.e., $(x + y)^n = \sum\limits_{r=0}^{r=n} {}^nC_r\, x^{n-r} \cdot y^r$.

(b) Replace y by $-$ y

$(x - y)^n = {}^nC_0\, x^n - {}^nC_1\, x^{n-1}y + {}^nC_2\, x^{n-2}\, y^2 + ... + {}^nC_r\, x^{n-r} \cdot y^r\, (-1)^r + ... + {}^nC_n\, (-1)^n \cdot y^n$

i.e., $(x - y)^n = \sum\limits_{r=0}^{r=n} {}^nC_r\, (-1)^r\, x^{n-r} \cdot y^r$.

(c) Now replace x by 1 and y by x in Ist

$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + ... + {}^nC_rx^r + ... + {}^nC_n\, y^n.$

Also, i.e., $(1 + x)^n = \sum\limits_{r=0}^{r=n} {}^nC_r\, x^r$.

(d) $(1 - x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2x^2 + ...+ {}^nC_r\, (-1)^r\, x^r + ... + {}^nC_n\, (-1)^n\, x^n.$

i.e., $(1 - x)^n = \sum\limits_{r=0}^{r=n} (-1)^r\, {}^nC_r\, x^r$.

(e) Also remember

$(1 + x)^n + (1 - x)^n = 2\, [{}^nC_0 + {}^nC_2x^2 + ...]$

and $(1 + x)^n - (1 - x)^n = 2\, [{}^nC_1 + {}^nC_3\, x^3 + {}^nC_5\, x^5 + ...]$

$\therefore$ Coefficient of x^r in the expansion of $(1 + x)^n$ is ${}^nC_r$.

$T_{r+1} = {}^nC_r\, x^r$

$\therefore$ coefficient of $(r + 1)^{th}$ term = coefficient of $x^r = {}^nC_r$ in the expansion of $(1 + x)^n$.

For e.g., Find the coefficient of x^6 in $(1 + 3x + 3x^2 + x^3)^{15} = [(1 + x)^3]^{15} = (1 + x)^{45}$

$\therefore$ $T_{r+1} = {}^{45}C_rx^r$

$\therefore$ $r = 6$.

$\therefore$ coefficient is ${}^{45}C_6$.

(4) ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots\ldots\ldots, {}^nC_n$ are called binomial coefficient or combinatorial coefficients and may be simply written as $C_0, C_1, C_2, \ldots\ldots\ldots\ldots, C_n$.

$$(x + y)^n = {}^nC_0\, x^n\, y^0 + {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}\, y^2 + \ldots\ldots\ldots + {}^nC_0\, x^{n-n}\, y^n.$$

Also, the sum of all the combinatorial coefficient.

i.e., ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots + {}^nC_n = 2^n$.

(5) Binomial coefficients of the term equidistant from beginning and end are equal.

$$(2x + 3y)^2 = {}^2C_0\,(2x)^2 + {}^2C_1\,(2x)^1\,(3y)^1 + {}^2C_2\,(2x)^0\,(3y)^2.$$

Now coefficient of 1^{st} and last term is same ${}^2C_0 = {}^2C_2$.

2.1 IMPORTANT TERMS IN BINOMIAL :

(A) General term **(B)** Term independent of x. **(C)** Middle term

(A) GENERAL TERM :

$(T_{r+1})^{th}$ term is called as general term in $(x + y)^n$ and general term is given by

$$\boxed{T_{r+1} = {}^nC_r x^{n-r}.y^r}$$

(B) TERM INDEPENDENT OF x :

It means term containing x^0.

Illustration :

Find term independent of x in $\left(x^2 + \frac{1}{x^2} - 2\right)^{10}$.

Sol. $\left(x^2 + \frac{1}{x^2} - 2\right)^{10} = \left(x - \frac{1}{x}\right)^{20}$

$\Rightarrow \quad T_{r+1} = {}^{20}C_r\, x^{20-r}\,(-1)^r\, \frac{1}{x^r} = {}^{20}C_r\, x^{20-2r}\,(-1)^r$

$\Rightarrow \quad 20 - 2r = 0\ ;\ r = 10$

$\Rightarrow \quad 11^{th}$ *term is independent of x.*

(C) MIDDLE TERM :

Let T_m is middle term in expansion of $(x + y)^n$ then

Case I : If n is odd, then number of terms will be even so there is two middle terms

$$\left(\frac{n+1}{2}\right)^{th} \textbf{ and } \left(\frac{n+3}{2}\right)^{th}.$$

Case II : If n is even, then number of terms will be odd so only one term is middle term $\left(\frac{n}{2}+1\right)^{th}$.

Note : Binomial coefficient of middle term is greatest.