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On construction of general classes of bivariate distributions

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ABSTRACT

In this study, taking into account the physics of failure (death) and the interrelationship between the items involved, we propose a general methodology for constructing new 'classes of bivariate distributions'. The approach is based on the stochastic modeling of the residual lifetime of an item after the failure of the other item. We derive the joint and the corresponding marginal distributions in a class constructed from the modeling of conditional failure rate. It is shown that the proposed new class includes several well-known bivariate distributions as special bivariate models. As illustrated, a number of new families of bivariate distributions are generated from the new class proposed in this paper. Furthermore, we briefly discuss the relationship to Freund's bivariate exponential distribution.

1. Introduction

실제는 indep경우보다 dep경우가 더 많음

Bivariate distributions are very important in modeling dependent random quantities in many different areas such as reliability, survival analysis, queueing analysis, insurance risk analysis, life insurance, and so forth (see, e.g., Iyer and Manjunath [20]). Especially, in the area of the lifetime modeling and analysis, the lifetimes of organisms or items are most often stochastically dependent. For an example, it is observed that there is a very high positive correlation between the times of deaths of coupled lives (see Carriere [6]). It is also addressed that, after the marital bereavement, the risk of mortality is significantly increased (see Jagger and Sutton [21]). In this regard, there have been numerous bivariate models proposed in the literature.

Traditionally, during the initial period of study on the topic, many researchers tried to extend the exponential distribution to the bivariate case. For instance, the work by Gumbel [16] can be considered as the initial work in this direction. Then, Freund [13] proposed a bivariate extension of the exponential distribution, which is absolutely continuous. Marshall and Olkin [27] also proposed a very important bivariate exponential distribution but it is not absolutely continuous. Thus, as illustrated in Section 9 of Block and Basu [5], there are clearly situations when the model of Marshall and Olkin [27] cannot be applied. Further extensions and discussions were also performed by, e.g., Downton [11], Hawkes [17], Block and Basu [5], Shaked [31], Sarkar [30] and Hayakawa [18]. Motivated by the work of Freund [13], Spurrier and Weier [33] derived a bivariate survival model based on the Weibull distribution. In the literature, extensions of other distributions to the bivariate models have also been proposed (see e.g., Lee [26], Sarhan and Balakrishnan [29]). A review on several theoretical bivariate gamma distributions with gamma marginals can be found in Yue et al. [34]. A nice review on the modeling of multivariate survival models can also be found in Hougaard [19]. An excellent encyclopaedic survey for various bivariate distributions can be found in Balakrishnan and Lai [4].

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Until now, in addition to the aforementioned works, numerous models for modeling general bivariate distributions have been proposed. One of the most convenient ways of constructing a bivariate distribution is that based on a copula function (see Nelson [28]). However, in the approach based on copula functions, the models are frequently lacking in the physics of dependency which lies behind the models. Also in the approaches other than the copula function, very few have studied whether the derivation of the bivariate models actually fits the way how the stochastic dependency is supposed to be generated.

On the contrary, in this paper, taking into account the physics of failure (death) of items (organisms) and the interrelationship between them, we propose and discuss a general methodology for constructing new 'general classes of bivariate distributions'. Our approach is based on the stochastic modeling of the residual lifetime of the remaining item after the failure of the other item. It is worthy of note again that this paper proposes not a specific parametric family of distributions, but general methodology for constructing 'classes of parametric families' of distributions. It is also shown that the proposed classes allow much flexibility in modeling bivariate distributions. Although our study had started independent of the well-known bivariate distributions such as Freund [13] and Block and Basu [5], it is shown that the proposed new class includes these several well-known bivariate distributions as special bivariate distributions belonging to the class. One of the most important contributions of this paper is that it provides a new 'general insight' and 'new perspective' on the modeling of the bivariate distributions.

The construction of this paper is as follows. In Section 2, a general methodology for constructing new classes of bivariate distributions is suggested and a new class is constructed. We start our main discussion with the modeling based on the concept of failure rate process and the failure rate order relationship. It will be briefly explained how other stochastic orderings applied to the modeling of residual lifetimes can generate other classes of distributions. In Section 3, a number of new families of bivariate distributions are generated from the new class proposed in Section 2. Especially, the relationship to Freund's bivariate exponential distribution will be discussed in detail. Finally, in Section 4, the results in this paper are briefly summarized. Furthermore, some topics for the future study are suggested and concluding remarks are given.

2. Construction of general class of bivariate distributions

In this section, we will construct a new class of bivariate distributions considering the physics of failure (death) of items (organisms) and the interrelationship between them. For a convenient description of the approach and procedure, we will discuss the model in terms of reliability context such as 'component' and 'failures'. However, the application of the proposed model is not necessarily limited to the area of reliability, but it can generally be applied to the modeling of dependent lifetimes in survival analysis, life insurance, biomedical areas, and so on.

Consider a system composed of two components. In many practical situations, when the failure of one component occurs in such a system, the remaining component may suffer from more stress or increased load. This could significantly affect the residual lifetime of the remaining component, eventually shortening the residual lifetime of the remaining component. This kind of dependency can frequently be observed in different practical situations, for example, in the survival of the paired organs such as person's eyes, ears, kidneys and lungs, or in the survival of two-engine airplane, or in the failures of adjacent pumps on a cooling system.

We will now start to discuss our model in a more detail. Suppose that the system is composed of two components: component 1 and component 2 and they start to operate at time $t = 0$. The original lifetimes of components 1 and 2, when they start to operate, are described by the corresponding failure rates $\lambda_1(t)$ and $\lambda_2(t)$, respectively. These original lifetimes of components 1 and 2 are denoted by X_1^* and X_2^*, respectively, assuming that X_1^* and X_2^* are stochastically independent.

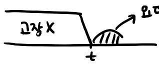
We consider the practical situation when the failure of one component increases the stress of the other component, which results in the shortened residual lifetime of the remaining component. Thus, there is a change point, $\min\{X_1^*, X_2^*\}$, after which the residual lifetime distribution of the remaining component changes. Under this type of dependency, we denote the corresponding eventual lifetimes of components 1 and 2 by X_1 and X_2, respectively. In order to describe the stochastic dependence model more precisely, 'for a moment' in the next paragraph, we need to discuss the concept of 'conditional failure rate' under a general setting, which is crucial for a proper understanding of our model to be discussed.

Assume that a device or an organism is operating (living) in a random environment described by a certain (covariate) stochastic process $\{Z(t), t \geq 0\}$. The lifetime of this device or organism is denoted by T. For example, the stochastic process $\{Z(t), t \geq 0\}$ can represent the randomly changing time-dependent external temperature, electric or mechanical load, or some other randomly changing external stress, etc. Then, the *conditional failure rate* can formally be defined (see Kalbfleisch and Prentice [22]):

조건부 고장률 함수
=사력

시작

$$r(t|z(s), 0 \leq s \leq t) \equiv \lim_{\Delta t \to 0} \frac{P(t < T \leq t + \Delta t | Z(s) = z(s), 0 \leq s \leq t, T > t)}{\Delta t}.$$



요때 바로 고장날 확률
: failure rate ftn
즉, hazard rate

Note that this conditional failure rate can be specified for a realization of the covariate process. With the covariate process not fixed yet, it is also obviously a stochastic process, which is usually referred to as the 'failure (hazard) rate process' (or random failure rate). For details, see, e.g., Kebir [23], Aven and Jensen [2] and Finkelstein and Cha [12]. Then, based on it, the conditional survival function can now be specified as

condition

$$P(T > t|\boxed{Z(s) = z(s), 0 \leq s \leq t}) = \exp\left\{-\int_0^t r(w|z(s), 0 \leq s \leq w)dw\right\}. \quad (1)$$

(*)

(*)

(1)식 내용

이제까지는 $\begin{cases} r(t) = \dfrac{f(t)}{S(t)} \quad (= P(X=x \mid X>x)) \\ S(t) = e^{-\int_0^t r(u)du} \end{cases}$ → 이 공식은 condition이 fix되어 있을 때를 의미함

(*) (1)식에서 알 수 있듯이 Stochastic Process 개념을 도입해서, 시간에 따라 변화하는 환경 $Z(S)$를 정의하며 시간(t)에 따라 변화하며 살아온 환경이 given 되었다는 가정하에 ex) 빈도, 온도 등의 condition이 변해왔던 것을 가정

(*) t시점이후의 생존할 확률을 구하는 것

condition	failure rate (baseline ftn)	survival ftn
ex) $z=1$	$r(t \mid z=1) = 1 \times \underline{r_0(t)}$	$P(T>t \mid z=1) = e^{-\int_0^t r(u)du}$
$z=2$	$r(t \mid z=2) = 2 \times \underline{r_0(t)}$	$P(T>t \mid z=2) = e^{-\int_0^t 2r(u)du}$

Q. Stochastic Order?

$$\left[\begin{array}{l} \dfrac{\sum^n X_i}{n} \xrightarrow{n\uparrow} E(X_i) = \mu \\ \dfrac{2n+1}{n} \xrightarrow{n\uparrow} 2 \end{array} \right.$$

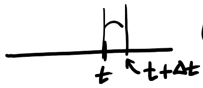
← n이 확률적, 여전히 확률변수

← n이 fix되면 상수값 (fixed constant)

→ 확률변수의 크기 비교를 위해 도입한 개념

→ 크기 비교를 어떤 tool을 이용할 것인가? 에 따라서 fr, st, lr 방법으로 나뉘는 것

→ if, $\underline{r_X(t)} \le r_Y(t)$, $\forall t \ge 0$

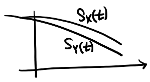


($r(t)$의 의미가 t시점에 바로 고장/사망 할 확률이니까)

고장날 확률 더 적은거니까, 더 수명이 길 것이다.
but, 무조건 오래사는 것은 아님

if, $X \ge Y$

$S_X(t) \ge S_Y(t)$, $\forall t \ge 0$



↳ 생존함수가 크다고해서, 무조건 오래사는것은아님

ex) X_i: $0 <$ 1 ($\frac{1}{2}$), 0 ($\frac{1}{2}$) | Y_i: $0 <$ 1 ($\frac{3}{4}$), 0 ($\frac{1}{4}$)

위와같은 경우에는 Y_i가 이기는 경우(1을 갖는경우)가 더많을것,
하지만 그렇다고 해서 Y_i가 항상 이기는 것은 아님.
(즉, 단지 확률이 클 뿐임

See also Cha and Mi [9,10], Cha and Finkelstein [7,8] for its applications to various shock models and other applications in reliability.

Let us now return to our issue and continue the discussion on the construction of new class of bivariate distributions. As mentioned before, in our setting, the failure of one component increases the stress of the other component and, thus, the covariate process which corresponds to one component is the process of the state of the other component. It thus naturally allows us to define the indicator variables for representing the states of the components at an arbitrary time. Define $\Psi_1(t) = 1$ $(\Psi_2(t) = 1)$ if component 1 (component 2) is functioning at time t, whereas $\Psi_1(t) = 0$ $(\Psi_2(t) = 0)$ if component 1 (component 2) is at a failed state at time t. For notational convenience, let $\tilde{i} = 2$ when $i = 1$, whereas $\tilde{i} = 1$ when $i = 2$. Then, for component i $(i = 1, 2)$ the corresponding covariate process is $\{\Psi_{\tilde{i}}(t), t \geq 0\}$ and we assume

부품 ĩ 상태 부품 i

$$r_i(t|\Psi_{\tilde{i}}(s) = 1, 0 \leq s \leq t) \equiv \lim_{\Delta t \to 0} P(t < X_i \leq t + \Delta t|\Psi_{\tilde{i}}(s) = 1, 0 \leq s \leq t, X_i > t) = \boxed{\lambda_i(t)}, \quad i = 1, 2, \tag{2}$$

↳ 아무것도 고장나지 않았을 때 (각각의 것으로 존재하고, 각각 아무런 영향을 받지 않음)

and

부품 ĩ 상태 부품 i

$$\begin{aligned} &r_i(t|\Psi_{\tilde{i}}(s) = 1, 0 \leq s < u; \Psi_{\tilde{i}}(s) = 0, u \leq s \leq t) \\ &\equiv \lim_{\Delta t \to 0} P(t < X_i \leq t + \Delta t|\Psi_{\tilde{i}}(s) = 1, 0 \leq s < u; \Psi_{\tilde{i}}(s) = 0, u \leq s \leq t, X_i > t) \\ &= \boxed{\alpha_i(u, t - u)\lambda_i(t),} \quad t \geq u, i = 1, 2, \end{aligned} \tag{3}$$

where $\alpha_i(s, w) \geq 1$, for all $s, w \geq 0$, $i = 1, 2$. The assumptions stated in Eqs. (2) and (3) imply: (i) Case I: when component 2 (component 1) is at a functioning state, the failure rate of X_1 (X_2) is the same as that of the original lifetime X_1^* (X_2^*); (ii) Case II: when component 2 (component 1) is at a failed state, the failure rate of component 1 (component 2) is increased compared with Case I. Therefore, the dependence model defined by (2) and (3) can represent the physical situation under consideration in a proper way. Note that the effect of the change point is modeled by the argument "s" and the effect of the elapsed time from the change point to the point of interest can be modeled by the argument "w". That is, for example, if the effect of the increased stress becomes more and more significant as the operating time increases then $\alpha_i(s, w)$ should be increasing in w, and vice versa. Similarly, if the effect of the increased stress becomes severer when the change point occurs later, then $\alpha_i(s, w)$ should be increasing in s, and vice versa. Obviously, the cases of $\alpha_i(s, w) \equiv 1$, for all $s, w \geq 0$, $i = 1, 2$, in Eq. (3) correspond to the independence of X_1 and X_2.

s: 고장시점 = change point
w: s~t 경과시간

Let $\tilde{X}_i$, $i = 1, 2$, denote the lifetimes of component 1 and component 2 when the two lifetimes are completely independent, i.e., when $\alpha_i(s, w) \equiv 1$, $i = 1, 2$. Then, according to (1),

indep ⇒

$$\begin{aligned} &P(\tilde{X}_i > t|\Psi_{\tilde{i}}(s) = 1, 0 \leq s < u, \Psi_{\tilde{i}}(s) = 0, s \geq u, \tilde{X}_i > u) \\ &= P(\tilde{X}_i > t|\tilde{X}_i > u) = \exp\left(-\int_0^{t-u} \boxed{\lambda_i(u + w)}dw\right), \quad i = 1, 2, \end{aligned}$$

$$\frac{P(\tilde{X}_i > t)}{P(\tilde{X}_i > u)} = \frac{e^{-\int_0^t \lambda_i(s)ds}}{e^{-\int_0^u \lambda_i(s)ds}} = e^{-\int_u^t \lambda_i(s)ds}$$

??

whereas when the two lifetimes X_1 and X_2 are dependent in accordance with the above described model,

dep ⇒

$$P(X_i > t|\Psi_{\tilde{i}}(s) = 1, 0 \leq s < u, \Psi_{\tilde{i}}(s) = 0, s \geq u, X_i > u) = \exp\left(-\int_0^{t-u} \boxed{\alpha_i(u, w)\lambda_i(u + w)}dw\right), \quad i = 1, 2.$$

In the following theorem, we obtain the joint distribution of X_1 and X_2 under the assumed model. In the following discussions, the notations $S(x_1, x_2)$, $f(x_1, x_2)$ will be used to denote the joint survival function and the joint pdf of X_1 and X_2, respectively. Furthermore, we will use $S_{X_1}(x_1)$, $f_{X_1}(x_1)$ and $S_{X_2}(x_2)$, $f_{X_2}(x_2)$ to denote the marginal survival function and the corresponding pdf of X_1 and X_2, respectively.

Theorem 1. *Under the conditional failure rate model stated in (2)–(3), the joint survival function $S(x_1, x_2)$, for $0 < x_1 < x_2$, is given by*

$$\begin{aligned} S(x_1, x_2) = &\int_{x_1}^{x_2} \lambda_1(u) \exp\left(-\int_0^{x_2-u} \alpha_2(u, w)\lambda_2(u + w)dw\right) \exp\left(-\int_0^u \lambda_1(w) + \lambda_2(w)dw\right) du \\ &+ \exp\left(-\int_0^{x_2} \lambda_1(w) + \lambda_2(w)dw\right), \quad \text{for } 0 < x_1 < x_2, \end{aligned} \tag{4}$$

and $S(x_1, x_2)$, for $0 < x_2 \leq x_1$, can be obtained symmetrically by replacing $x_1, x_2, \lambda_1(\cdot), \lambda_2(\cdot), \alpha_1(\cdot, \cdot), \alpha_2(\cdot, \cdot)$ on the right-hand side of Eq. (4) with respective opposite components. The corresponding joint pdf, for $0 < x_1 < x_2$, is given by

첫번째 고장 두번째 고장
0 x_1 x_2

$$\begin{aligned} f(x_1, x_2) = &\lambda_1(x_1)\lambda_2(x_2)\alpha_2(x_1, x_2 - x_1) \exp\left(-\int_0^{x_1} \lambda_1(w) + \lambda_2(w)dw\right) \\ &\times \exp\left(-\int_0^{x_2-x_1} \alpha_2(x_1, w)\lambda_2(x_1 + w)dw\right), \quad \text{for } 0 < x_1 < x_2, \end{aligned}$$

×Δx₁Δx₂로 해석해!

: x₁의 pdf
: x₂의 pdf

and $f(x_1, x_2)$, for $0 < x_2 \leq x_1$, can also be obtained symmetrically.

$= P(x_1 < X_1 \leq x_1 + \Delta x, x_2 < X_2 \leq x_2 + \Delta x_2)$
↑ x₁ 근방에서 고장 ↑ x₂ 근방에서 고장

첫번째가 x₁에 고장 후 두번째는 x₁까지 살고, x₂에서 고장나야함!

변화한 고장률

X_2입장에서 $[0, x_2-u]$에서 $[0, u]$에 잘 작동하고 $[u, x_2-u]$ 사이에도 잘 작동할 확률

Theorem 1. *Under the conditional failure rate model stated in (2)–(3), the joint survival function $S(x_1, x_2)$, for $0 < x_1 < x_2$, is given by*

$$S(x_1, x_2) = \int_{x_1}^{x_2} \lambda_1(u) \boxed{\exp\left(-\int_0^{x_2-u} \alpha_2(u, w)\lambda_2(u+w)dw\right)} \exp\left(-\int_0^{u} \lambda_1(w) \boxed{+\lambda_2(w)dw}\right) du$$
$$+ \boxed{\exp\left(-\int_0^{x_2} \lambda_1(w) + \lambda_2(w)dw\right)}, \quad \text{for } 0 < x_1 < x_2, \tag{4}$$

↳ original lifetime (둘다 살아있는 경우) ②

→ X_1이 u에서 고장날 확률

x_1 → ①

x_2 → ②

★ 의미를 살펴보자

joint survival ftn $= S(x_1, x_2)$, $x_2 > x_1$

$= P(X_1 > x_1, X_2 > x_2) = P(\underbrace{\{X_1 > x_1\}}_{①} \cap \underbrace{\{X_2 > x_2\}}_{②})$

$= P(\underbrace{\{x_1 < X_1 < x_2\} \cup \{X_1 > x_2\}}_{①} \cap \underbrace{\{X_2 > x_2\}}_{②})$ (배반)

$(A \cup B) \cap C = (A \cap C) + (B \cap C)$ 배반 ($A \cap C$는 A의 일부, $B \cap C$는 B의 일부)

$= P(x_1 < X_1 < x_2, X_2 > x_2) + P(X_1 > x_2, X_2 > x_2)$

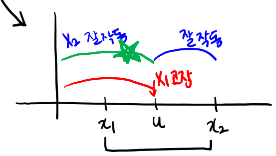
$= P(x_1 < X_1^* < x_2, X_2 > x_2) + \boxed{P(X_1^* > x_2, X_2^* > x_2)}$ ②

다른 component에 영향을 안받은 경우를 X^*로 표시

$= \int_0^{\infty} P(x_1 < X_1^* < x_2, X_2 > x_2 \mid X_1^* = u) \cdot f_{X_1^*}(u)\, du$ → $u \in [0, \infty]$, $X_1^* \in [x_1, x_2]$ 조건이 $X_1^* = u$일때 니까 u는 $[x_1, x_2]$ 안에 존재해야 함

$= \int_{x_1}^{x_2} P(x_1 < X_1^* < x_2, X_2 > x_2 \mid X_1^* = u)\, f_{X_1^*}(u)\, du$ 구간안에 반영됨

$= \int_{x_1}^{x_2} P(X_2 > x_2 \mid X_1^* = u) \cdot f_{X_1^*}(u)\, du$ → 의미 : 첫번째 component가 $[x_1, x_2]$에서 고장났을 때 두번째 component가 x_2 이후에 고장



$\lambda(t) = \frac{f(t)}{S(t)}$

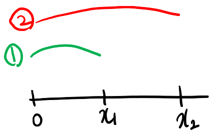
$\Rightarrow f(t) = S(t) \cdot \lambda(t)$

t에서 고장 / t까지 살고 / t이후 바로 고장

$u+w = x$
$w = x-u$

$P(X > u+t \mid X > u) = \frac{S(u+t)}{S(u)} = \frac{e^{-\int_0^{u+t} \lambda(x)dx}}{e^{-\int_0^{u} \lambda(x)dx}} = e^{-\int_u^{u+t} \lambda(x)dx} = e^{-\int_0^{t} \lambda(u+w)dw}$

↳ u까지 살아있는데, w까지 고장 X



Proof. Observe that

$$\begin{aligned}P(X_1 > x_1, X_2 > x_2) &= P(X_1 > x_1, X_2 > x_2 | X_1^* > X_2^*) P(X_1^* > X_2^*) \\ &\quad + P(X_1 > x_1, X_2 > x_2 | X_1^* < X_2^*) P(X_1^* < X_2^*).\end{aligned} \tag{5}$$

Here,

$$P(X_1 > x_1, X_2 > x_2 | X_1^* < X_2^*) = \int_0^\infty P(X_1 > x_1, X_2 > x_2 | X_1^* < X_2^*, X_1^* = u) f_{X_1^*|X_1^*<X_2^*}(u) du, \tag{6}$$

and

$$P(X_1 > x_1, X_2 > x_2 | X_1^* > X_2^*) = \int_0^\infty P(X_1 > x_1, X_2 > x_2 | X_1^* > X_2^*, X_2^* = u) f_{X_2^*|X_1^*>X_2^*}(u) du. \tag{7}$$

(*Case* I). Let $0 < x_1 < x_2$. The conditional joint survival function given that component 1 has failed first at u is, for $u < x_1$ or $x_2 \le u$, obviously given by

$$P(X_1 > x_1, X_2 > x_2 | X_1^* < X_2^*, X_1^* = u) = \begin{cases} 0, & u < x_1, \\ 1, & x_2 \le u, \end{cases}$$

whereas, for $x_1 \le u < x_2$, from the assumptions on the conditional failure rates stated in (2)–(3), we have

$$\begin{aligned}P(X_1 > x_1, X_2 > x_2 | X_1^* < X_2^*, X_1^* = u) &= P(X_2 > x_2 | X_1^* < X_2^*, X_1^* = u) \\ &= P(X_2 > x_2 | \Psi_1(s) = 1, 0 \le s < u, \Psi_1(s) = 0, s \ge u, X_2 > u) \\ &= \exp\left(-\int_0^{x_2-u} \alpha_2(u, w)\lambda_2(u+w) dw\right), \quad x_1 \le u < x_2.\end{aligned}$$

Furthermore, due to the stochastic independence of X_1^* and X_2^*,

$$f_{X_1^*|X_1^*<X_2^*}(u) = \frac{\lambda_1(u) \exp\left(-\int_0^u \lambda_1(w) + \lambda_2(w) dw\right)}{P(X_1^* < X_2^*)}.$$

On the other hand, the conditional joint survival function given that component 2 has failed first at u is

$$P(X_1 > x_1, X_2 > x_2 | X_1^* > X_2^*, X_2^* = u) = \begin{cases} 0, & u < x_2, \\ 1, & u \ge x_2. \end{cases}$$

Furthermore,

$$f_{X_2^*|X_1^*>X_2^*}(u) = \frac{\lambda_2(u) \exp\left(-\int_0^u \lambda_1(w) + \lambda_2(w) dw\right)}{P(X_1^* > X_2^*)}.$$

Now combining (5)–(7) based on the above derivations, we now have

$$\begin{aligned}S(x_1, x_2) &= \int_{x_1}^{x_2} \lambda_1(u) \exp\left(-\int_0^{x_2-u} \alpha_2(u, w)\lambda_2(u+w) dw\right) \exp\left(-\int_0^u \lambda_1(w) + \lambda_2(w) dw\right) du \\ &\quad + \exp\left(-\int_0^{x_2} \lambda_1(w) + \lambda_2(w) dw\right), \quad \text{for } 0 < x_1 < x_2.\end{aligned}$$

(*Case* II). Let $0 < x_2 \le x_1$. As the roles of (X_1^*, X_1) and (X_2^*, X_2) are symmetric, the result can be obtained symmetrically.

The joint pdf $f(x_1, x_2)$ can be obtained by

$$f(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} S(x_1, x_2). \quad \blacksquare$$

By letting $x_1 \equiv 0$ or $x_2 \equiv 0$ in $S(x_1, x_2)$, the marginal distributions can be obtained. The survival functions and the pdfs of the marginal distributions are given in the following corollary.

Corollary 1. *The marginal distribution of X_1 is given by*

$$\begin{aligned}S_{X_1}(x_1) &= \int_0^{x_1} \lambda_2(u) \exp\left(-\int_0^{x_1-u} \alpha_1(u, w)\lambda_1(u+w) dw\right) \exp\left(-\int_0^u \lambda_1(w) + \lambda_2(w) dw\right) du \\ &\quad + \exp\left(-\int_0^{x_1} \lambda_1(w) + \lambda_2(w) dw\right), \quad x_1 \ge 0,\end{aligned}$$

and

$$f_{X_1}(x_1) = \lambda_1(x_1)\int_0^{x_1} \lambda_2(u)\alpha_1(u, x_1-u)\exp\left(-\int_0^{x_1-u}\alpha_1(u,w)\lambda_1(u+w)dw\right)\exp\left(-\int_0^u \lambda_1(w)+\lambda_2(w)dw\right)du$$
$$+\lambda_1(x_1)\exp\left(-\int_0^{x_1}\lambda_1(w)+\lambda_2(w)dw\right), \quad x_1 \geq 0.$$

The marginal distribution of X_2 can be obtained symmetrically by replacing $x_1, \lambda_1(\cdot), \lambda_2(\cdot), \alpha_1(\cdot,\cdot)$ on the right-hand sides of the above equations with respective opposite components. ■ 하나고장 후 남아있는 것의 잔여수명이 감소한다는 것을 보이기 위하여 독립인 경우와 비교해야 함

Remark 1. In the above model, the crucial point for constructing the class of bivariate distributions exists at the modeling of the residual lifetime distribution of the remaining component. Note that the 'effect of increased stress', compared with the independent case, was mathematically modeled by the following 'stochastic order' between lifetimes:

dep (영향받음)

$$(X_1|\Psi_2(s)=1, 0\leq s<u, \Psi_2(s)=0, s\geq u, X_1>u)$$

고장률 더 높아

indep. original

$$\leq_{fr} (\tilde{X}_1|\Psi_2(s)=1, 0\leq s<u, \Psi_2(s)=0, s\geq u, \tilde{X}_1>u), \quad \text{for all } u>0, \tag{8}$$

where "$\leq_{fr}$" stands for the failure rate order (hazard rate order) between two random variables (see Shaked and Shanthikumar [32]). That is, the failure rate of the random variable

$$(\tilde{X}_1|\Psi_2(s)=1, 0\leq s<u, \Psi_2(s)=0, s\geq u, \tilde{X}_1>u)$$

is given by $\lambda_1(u+t), t\geq 0$, whereas the failure rate of the random variable

$$(X_1|\Psi_2(s)=1, 0\leq s<u, \Psi_2(s)=0, s\geq u, X_1>u)$$

is given by $\alpha_1(u,t)\lambda_1(u+t), t\geq 0$. Therefore,

original 분포 VS stress effect 후 변화한 분포

Stress effect parameter ftn

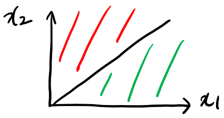
$$\alpha_1(u,t)\lambda_1(u+t)\geq\lambda_1(u+t), \quad \text{for all } t\geq 0, u>0,$$

which implies the relationship in (8). It is now clear that if the 'effect of increased stress' can be modeled by the other 'stochastic orderings', then other classes can be constructed in a similar way. In accordance with this motivation, the 'usual stochastic order' or the 'likelihood ratio order' can alternatively be employed in (8) instead of the failure rate order. For the corresponding definitions and general discussions on these stochastic orders, see also Shaked and Shanthikumar [32].

3. Specific families of distributions

② 고장 → ① 모수 α에서 α'로 변화
① 고장 → ② 모수 β에서 β'로 변화

We will now see that the well-known families of bivariate distributions proposed by Freund [13] and Block and Basu [5] belong to the class defined in Section 2 and, especially, we will discuss in detail the relationship to Freund's bivariate distribution. In Freund [13], assuming that the initial lifetime distributions of two components 1 and 2 respectively follow independent exponential distributions with parameters (failure rates) α and β, the dependency of X_1 and X_2 was introduced as follows: the failure of component 2 changes the parameter of the exponential distribution of component 1 from α to α', while the failure of component 1 changes the parameter of the exponential life distribution of component 2 from β to β'. Then the joint pdf of X_1 and X_2 was eventually obtained by



$$f(x_1,x_2)=\begin{cases}\alpha\beta'\exp\{-\beta' x_2-(\alpha+\beta-\beta')x_1\}, & \boxed{0<x_1<x_2;}\\ \beta\alpha'\exp\{-\alpha' x_1-(\beta+\alpha-\alpha')x_2\}, & \boxed{0<x_2\leq x_1.}\end{cases} \tag{9}$$

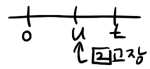
p.20 모형1

We will now show that the above Freund's bivariate exponential distribution can be more meaningfully derived using the modeling process described in this paper. Let $\lambda_1(t)=\lambda_1, \lambda_2(t)=\lambda_2$, and define the parameter functions $\alpha_i(u,t)=\alpha_i$, for all $u, t\geq 0, i=1,2$. From Theorem 1, we have

$$f(x_1,x_2)=\lambda_1\alpha_2\lambda_2\exp\{-\alpha_2\lambda_2 x_2-(\lambda_1+\lambda_2-\alpha_2\lambda_2)x_1\}, \quad 0<x_1<x_2,$$
$$f(x_1,x_2)=\lambda_2\,\alpha_1\lambda_1\exp\{-\alpha_1\lambda_1 x_1-(\lambda_1+\lambda_2-\alpha_1\lambda_1)x_2\}, \quad 0<x_2\leq x_1.$$

Reparameterizating $\lambda_1\equiv\alpha, \lambda_2\equiv\beta, \alpha_1\lambda_1\equiv\alpha'$ and $\alpha_2\lambda_2\equiv\beta'$, this family of distributions becomes the bivariate distribution in (9). Therefore, e.g., for component 1, the parameter changes in Freund's model can be meaningfully explained as the following conditional failure rate modeling:

$$r_1(t|\Psi_2(s)=1, 0\leq s\leq t)=\alpha; \qquad r_1(t|\Psi_2(s)=1, 0\leq s<u; \Psi_2(s)=0, u\leq s\leq t)=\alpha',$$



and the parameter function $\alpha_1(u,t)$, which represents the stress effect, in this case corresponds to $\alpha_1(u,t)=\alpha'/\alpha$.

$$\alpha'=\alpha_1(u,t)\cdot\alpha \Leftrightarrow \alpha_1(u,t)=\frac{\alpha'}{\alpha}$$

On the other hand, by setting

$$\lambda_1\equiv\lambda_1'+\lambda_{12}'\left(\frac{\lambda_1'}{\lambda_1'+\lambda_2'}\right), \qquad \lambda_2\equiv\lambda_2'+\lambda_{12}'\left(\frac{\lambda_2'}{\lambda_1'+\lambda_2'}\right), \qquad \alpha_1\lambda_1\equiv\lambda_1'+\lambda_{12}', \qquad \alpha_2\lambda_2\equiv\lambda_2'+\lambda_{12}',$$

we have $\lambda_1\alpha_2\lambda_2 = \lambda_1(\lambda'_2+\lambda'_{12}) = \left[\lambda'_1 + \lambda'_{12}\left(\frac{\lambda'_1}{\lambda'_1+\lambda'_2}\right)\right](\lambda'_2+\lambda'_{12})$

대입

$$f(x_1, x_2) = \frac{\lambda'_1\lambda'(\lambda'_2+\lambda'_{12})}{\lambda'_1+\lambda'_2}\exp\left\{-\lambda'_1x_1 - (\lambda'_2+\lambda'_{12})x_2\right\}, \quad 0 < x_1 < x_2,$$

$$f(x_1, x_2) = \frac{\lambda'_2\lambda'(\lambda'_1+\lambda'_{12})}{\lambda'_1+\lambda'_2}\exp\left\{-\lambda'_2x_2 - (\lambda'_1+\lambda'_{12})x_1\right\}, \quad 0 < x_2 \le x_1,$$

where $\lambda' \equiv \lambda'_1 + \lambda'_2 + \lambda'_{12}$, which is the well known Block and Basu [5]'s *ACBVE*$(\lambda'_1, \lambda'_2, \lambda'_{12})$ bivariate model.

정리1, bivariate exponential 비슷!

It should also be noted that even the resulting general bivariate distributions given in Theorem 1 share very similar properties with Freund's bivariate exponential distribution: (i) the marginals are mixtures of the baseline distributions; (ii) joint distributions (cdf or pdf) have different expressions depending on $0 < x_1 < x_2$ or $0 < x_2 \le x_1$. Furthermore, a well-defined subclass of the general class defined in Section 2 shares the 'bivariate lack of memory property (BLMP)' with Freund's bivariate exponential distribution. It has been known that Freund [13]'s bivariate exponential distribution is 'one of the few absolutely continuous bivariate distributions' which possess the BLMP. The definition of BLMP is as follows.

└ Freund's bivariate exponential dist^n → 절대연속이변량분포

p.32 정의4

Definition 1 (*Bivariate Lack of Memory Property*). *A bivariate random variable (X_1, X_2) is said to have the bivariate lack of memory property (BLMP) iff*

└고장시점

$$P(X_1 > t + s_1, X_2 > t + s_2 | X_1 > t, X_2 > t) = P(X_1 > s_1, X_2 > s_2), \quad \text{for all } t, s_1, s_2 \ge 0. \tag{10}$$

t $t+s_1$

Interpreting (10), if both of the two items are alive at t, then the joint distribution of their remaining lifetimes is the original joint distribution. The following theorem states that all the members belonging to a well-defined subclass possess BLMP.

p.33 정리5

지수분포(t에무관)

Theorem 2. *Suppose that $\lambda_i(t) = \lambda_i, t \ge 0, i = 1, 2$. If the parameter functions, $\alpha_i(u, t), i = 1, 2$, do not depend on the argument u, then (X_1, X_2) possesses the BLMP.*

└이게 u에 의존하지 않는 조건만족하게 설정 → 무기억성 만족
→ 간결해짐. Good 성질임

Proof. According to the assumption, $\alpha_i(u, t), i = 1, 2$, do not depend on the argument u and thus set $\alpha_i(u, t) = \psi_i(t)$, $i = 1, 2$. For $s_2 > s_1$, from Theorem 1,

u에무관한 함수.

$$P(X_1 > t + s_1, X_2 > t + s_2) = \int_{t+s_1}^{t+s_2} \lambda_1 \exp\left(-\lambda_2 \int_0^{t+s_2-u} \psi_2(w)dw\right) \times \exp\{-(\lambda_1+\lambda_2)u\}du + \exp\{-(\lambda_1+\lambda_2)(t+s_2)\}.$$

By letting $u - t = v$ in the integral,

$$\begin{aligned} P(X_1 > t + s_1, X_2 > t + s_2) &= \int_{s_1}^{s_2} \lambda_1 \exp\left(-\lambda_2 \int_0^{s_2-v} \psi_2(w)dw\right) \\ &\quad \times \exp\{-(\lambda_1+\lambda_2)(v+t)\}dv + \exp\{-(\lambda_1+\lambda_2)(t+s_2)\} \\ &= \left(\int_{s_1}^{s_2} \lambda_1 \exp\left(-\lambda_2 \int_0^{s_2-v} \psi_2(w)dw\right)\right. \\ &\quad \left.\times \exp\{-(\lambda_1+\lambda_2)v\}dv + \exp\{-(\lambda_1+\lambda_2)s_2\}\right) \exp\{-(\lambda_1+\lambda_2)t\} \\ &= P(X_1 > s_1, X_2 > s_2)P(X_1 > t, X_2 > t). \end{aligned}$$

For the case $s_1 \ge s_2$, the result can be shown symmetrically. ■

p.21

We will now illustrate how other new families of bivariate distributions can be generated from the proposed class. In Freund [13], as explained before, the approaches are basically based on the change of parameters in the original lifetimes contained in $\lambda_1(t)$ and $\lambda_2(t)$. Therefore, the residual lifetime distributions in Freund [13] are limited only to the same type of distributions, e.g., exponential (original distribution)–exponential (residual lifetime distribution). However, in our modeling approach, there is no such restriction and more general families of distributions can be generated as illustrated in the following model.

초기에 지수

여기서는, 모수만 다른분포로 변하지 않고, 완전 새로운 분포로 변화하기도 한다

p.22 모형3

Model 1. Let $\lambda_1(t) = \lambda_1, \lambda_2(t) = \lambda_2$. We set the parameter functions $\alpha_i(u, t) = \alpha_i t + 1$, where $\alpha_i > 0, i = 1, 2$. Then, from Theorem 1, we have

$$f(x_1, x_2) = \lambda_1\lambda_2(\alpha_2(x_2 - x_1) + 1)\exp\left(-\frac{1}{2}\alpha_2\lambda_2(x_2 - x_1)^2 - \lambda_1x_1 - \lambda_2x_2\right), \quad 0 < x_1 < x_2,$$

$$f(x_1, x_2) = \lambda_1\lambda_2(\alpha_1(x_1 - x_2) + 1)\exp\left(-\frac{1}{2}\alpha_1\lambda_1(x_1 - x_2)^2 - \lambda_1x_1 - \lambda_2x_2\right), \quad 0 < x_2 \le x_1.$$

Observe that the original lifetimes are exponential distributions but, for instance, the residual lifetime of component 1 is now given by

$$\begin{aligned} &P(X_1 > x | \Psi_2(s) = 1, 0 \le s < u, \Psi_2(s) = 0, s \ge u, X_1 > u) \\ &\quad = \exp\left(-\int_0^{x-u} \alpha_1(u, w)\lambda_1(u + w)dw\right) = \exp\left(-\frac{\lambda_1}{2}\alpha_1(x - u)^2 - \lambda_1(x - u)\right), \quad x > u, \end{aligned}$$

Table 1
The values of $Cov(X_1, X_2)$.

α	0	1	3	5	7	9
$Cov(X_1, X_2)$	0	0.142521	0.191950	0.208879	0.217793	0.223379

증가

which is non-exponential distribution. Therefore, in this model, we have exponential (original distribution)—non-exponential (residual lifetime distribution) combination. In this case, for the residual lifetime distribution, the linear (polynomial) failure rate is used. Applications of polynomial failure rates to modeling lifetimes in the reliability and biological contexts can be found, e.g., in Bain [3] and Lawless [25]. ■

We will now see how the parameters in the parameter functions $\alpha_i(u, t)$, $i = 1, 2$, in Model 1 affect the degree of the dependency of the random variables X_1 and X_2.

αλ로 표현되니까, α값 더 커지면 고장률 함수 ↑ → 수명 ↓

p.23 예제1

Example 1. In the above Model 1, we specify the parameters as follows: $\lambda_1(t) = 1$, $\lambda_2(t) = 1$, $\alpha_i(u, t) = \alpha t + 1, i = 1, 2$, where $\alpha \geq 0$. Now we will investigate the effect of the parameter α on $Cov(X_1, X_2)$. The values of $Cov(X_1, X_2)$ corresponding to each value of α are shown in Table 1.

From Table 1, it is observed that as α increases, the degree of the dependency also increases, which is clear from the stochastic modeling of dependency described in Section 2.

Until now, the families of bivariate distributions have been constructed based on the simple original baseline distribution such as exponential. The Weibull distributions can also be used as the baseline distribution to generate a family of bivariate distributions. In the following, a family of bivariate distributions constructed from a more general original baseline distribution will be illustrated.

p.28 모형9

Model 2. Let us take the two-parameter Pareto (Lomax) distribution as the original baseline distributions (survival functions):

$$S_i(t) = \left(\frac{\lambda_i}{t + \lambda_i}\right)^{\theta_i}, \quad \lambda_i > 0, \theta_i > 0, i = 1, 2,$$

which has the corresponding failure rate functions

$$\lambda_i(t) = \frac{\theta_i}{(t + \lambda_i)}, \quad i = 1, 2. \quad = \frac{-S_i'(t)}{S_i(t)} = \frac{d}{dt}\left[-\ln S_i(t)\right] = \frac{d}{dt}\left\{-\theta_i \ln\left(\frac{\lambda_i}{t+\lambda_i}\right)\right\}$$

Define the parameter functions $\alpha_i(u, t) = \alpha_i$, for all $u, t \geq 0$, $i = 1, 2$. Then, from Theorem 1, we have

$$f(x_1, x_2) = \alpha_2 \left(\frac{\theta_1}{x_1 + \lambda_1}\right)\left(\frac{\theta_2}{x_2 + \lambda_2}\right)\left(\frac{\lambda_1}{x_1 + \lambda_1}\right)^{\theta_1}\left(\frac{\lambda_2}{x_1 + \lambda_2}\right)^{\theta_2}\left(\frac{x_1 + \lambda_2}{x_2 + \lambda_2}\right)^{\alpha_2\theta_2}, \quad 0 < x_1 < x_2,$$

$$f(x_1, x_2) = \alpha_1 \left(\frac{\theta_1}{x_1 + \lambda_1}\right)\left(\frac{\theta_2}{x_2 + \lambda_2}\right)\left(\frac{\lambda_1}{x_2 + \lambda_1}\right)^{\theta_1}\left(\frac{\lambda_2}{x_2 + \lambda_2}\right)^{\theta_2}\left(\frac{x_2 + \lambda_1}{x_1 + \lambda_1}\right)^{\alpha_1\theta_1}, \quad 0 < x_2 \leq x_1. \quad ■$$

Note that Models 1 and 2 are just simple illustrations for the application of the general method of constructing bivariate distributions and numerous families of distributions can be generated based on the parametric model suggested in Theorem 1.

stress ↓ 잔여수명↓

In Section 2, based on the physics of failure (death) of items (organisms) and the interrelationship between them, we proposed and discussed a new general class of bivariate distributions. In our modeling approach, the failure of one component increases the load or stress of the other component and this significantly affects the residual lifetime of the remaining component, eventually shortening the residual lifetime of the remaining component. This kind of dependence structure can frequently be observed in many practical situations. The most typical practical example is as follows.

Example 2 (*Load-Sharing Components*). In a load-sharing system, when any one of the components fails, its load is automatically transmitted to the remaining component(s). This results in a higher load on the surviving component(s), thereby inducing a higher failure rate for it(them). This introduces failure dependency among the load sharing components. Examples of load-sharing components include electric generators sharing an electrical load in a power plant, CPUs in a multiprocessor computer system, cables in a suspension bridge, and valves or pumps in a hydraulic system (see Amari and Bergman [1]).

유압 시스템

As mentioned earlier, this kind of load-sharing properties can also be frequently observed in survival analysis. For instance, in the survival of the paired organs such as person's eyes, ears, kidneys and lungs, and so forth (Gross et al. [15]). Various applications of load-sharing dependence properties to other areas such as software reliability, civil engineering, material testing, population sampling and combat modeling can be found in Kvam and Peña [24].

토목공학

4. Concluding remarks

In this paper, a general methodology for constructing new general classes of bivariate distributions has been suggested and a few specific families of distributions have been generated for illustrations. The relationship to Freund's bivariate

다수의

distribution has been discussed in detail. Based on the proposed class, numerous bivariate distributions can further be generated and new issues on the estimation and testing of the model parameters should be discussed in the future studies. In this regard, the impact of this paper to the area of lifetime modeling and analysis would be substantial. Furthermore, when specific parametric families are generated from the class, the dependence structures of the corresponding bivariate models can be studied. Furthermore, the corresponding copula representations which correspond to the specific parametric families could be investigated, which would be useful for inference purpose. However, even for the simplest model of Freund's bivariate distribution, it is well known that there does not exist an analytic expression of the copula and it should be computed numerically (see, e.g., Georges et al. [14]). For the parametric distributions belonging to the suggested class, we observe similar results and the corresponding copula representations cannot be obtained in analytic forms.

식으로 존재X
수학적으로
계산되어야함

In this paper, our discussions are mainly focused on generating bivariate models. However, similar approach could be applied to generate a new class of multivariate distributions. For example, suppose that we have three components. When we had two components, depending on the states of the other component, two kinds of conditional failure rates for each component were needed for the construction (Eqs. (2) and (3)). When we have three components, depending on the states of the other two components, five kinds of conditional failure rates for each component are needed for the construction of a class of trivariate distributions. For example, for component 1, the following five kinds of conditional failure rates should be defined:

$$
\begin{aligned}
& r_1(t|\Psi_2(s) = 1, 0 \le s \le t;\ \Psi_3(s) = 1, 0 \le s \le t); \\
& r_1(t|\Psi_2(s) = 1, 0 \le s < u;\ \Psi_2(s) = 0, u \le s \le t;\ \Psi_3(s) = 1, 0 \le s \le t); \\
& r_1(t|\Psi_3(s) = 1, 0 \le s < u;\ \Psi_3(s) = 0, u \le s \le t;\ \Psi_2(s) = 1, 0 \le s \le t); \\
& r_1(t|\Psi_2(s) = 1, 0 \le s < u_1;\ \Psi_2(s) = 0, u_1 \le s \le t;\ \Psi_3(s) = 1, 0 \le s < u_2;\ \Psi_3(s) = 0, u_2 \le s \le t); \\
& r_1(t|\Psi_3(s) = 1, 0 \le s < u_1;\ \Psi_3(s) = 0, u_1 \le s \le t;\ \Psi_2(s) = 1, 0 \le s < u_2;\ \Psi_2(s) = 0, u_2 \le s \le t),
\end{aligned}
$$

where $u_1 \le u_2$ and $\Psi_i(t)$ represents the state of component i, $i = 2, 3$. Then, applying approach and mechanism similar to those given in the proof of Theorem 1, a new class of trivariate distributions could be constructed.

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