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CHOOSING A POINT FROM THE SURFACE OF A SPHERE

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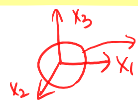
A frequent problem in Monte Carlo is that of sampling uniformly from the surface of the unit 3-sphere $\{(z_1, z_2, z_3) : z_1^2 + z_2^2 + z_3^2 = 1\}$. Such problems arise in connection with random rotations, orientations, directions. Refer to *random directions* in the index of Feller **2** [2] for a number of applications, and to Stephens [4], or Watson and Williams [5], for references on statistical problems associated with uniform distributions on a sphere. It is an interesting problem in probability theory to find functions of uniform variates which produce a point with uniform distribution on the surface of the sphere, and which are practical for use in computers. There are three well-known methods, two of them obvious and the other, by J. M. Cook [1], based on elegant theory but unfortunately too slow to be practical. I will summarize the three methods here and then offer a new method that is simpler and about twice as fast as any of the existing methods.

Method 1. Generate V_1, V_2, V_3, independent uniform on $(-1, 1)$ until $S = V_1^2 + V_2^2 + V_3^2 < 1$ then form

$$(V_1/S^{\frac{1}{2}}, V_2/S^{\frac{1}{2}}, V_3/S^{\frac{1}{2}}) . \tag{1}$$

The idea here is to choose a point in a cube, reject it unless it is in the inscribed sphere, then project the point to the surface of the sphere. Efficiency is $\pi/6$, so the method requires an average of $18/\pi \cong 5.73$ uniform variates.

Method 2. Generate X_1, X_2, X_3, independent standard normal variates, put $S = X_1^2 + X_2^2 + X_3^2$ and form



표면으로부터 sampling !

$$(X_1/S^{\frac{1}{2}}, X_2/S^{\frac{1}{2}}, X_3/S^{\frac{1}{2}}) . \tag{2}$$

This method is obvious to probabilists, but perhaps because of the importance of the problem of sampling from the surface of a sphere, was the subject of a paper by Muller [3]. (Not so obvious is the converse: if independent X_1, X_2, X_3 lead to (2) with a uniform distribution on the sphere, what can be said about the distributions of the X's?) Method 2 is slower than Method 1, and even in higher dimensions there are better methods for getting a point on the n-sphere. (Method 1 is hopeless in higher dimensions.)

Method 3 (Cook, [1]). Generate V_1, V_2, V_3, V_4 independent uniform on $(-1, 1)$ until $S = V_1^2 + V_2^2 + V_3^2 + V_4^2 < 1$, then form

$$(2(V_2V_4 + V_1V_3)/S, 2(V_3V_4 - V_1V_2)/S, (V_1^2 + V_4^2 - V_2^2 - V_3^2)/S) . \tag{3}$$

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The attraction of this method is the proof that it works. It avoids the square root of Methods 1 and 2, but its requirements of an average of $128/\pi^2 \cong 13.0$ uniform variates and the resulting arithmetic make it too slow for general use.

The New Method. Generate V_1, V_2, independent uniform on $(-1, 1)$ until $S = V_1^2 + V_2^2 < 1$, then form

$$(2V_1(1 - S)^{\frac{1}{2}}, 2V_2(1 - S)^{\frac{1}{2}}, 1 - 2S) . \tag{4}$$

This method requires an average of $8/\pi \cong 2.55$ uniform variates and a square root. It is roughly twice as fast as the best of existing methods, depending on the procedure for getting uniform variates, relative speed of arithmetic, square root, etc. To prove that the method works, combine the following two facts:

A. If (Z_1, Z_2, Z_3) is uniform on the surface of the unit 3-sphere, then each Z is uniform on $(-1, 1)$, (the area of a spherical cap is a multiple of its height), and (Z_1, Z_2), for given Z_3, is uniform on the circumference of the circle of radius $(1 - Z_3^2)^{\frac{1}{2}}$.

B. If (V_1, V_2) is uniform over the interior of the unit circle, then $S = V_1^2 + V_2^2$ is uniform on $(0, 1)$ and independent of the point $(V_1/S^{\frac{1}{2}}, V_2/S^{\frac{1}{2}})$.

Combining A and B we conclude that if Z_3 is uniform on $(-1, 1)$ and independent of $(V_1/S^{\frac{1}{2}}, V_2/S^{\frac{1}{2}})$ then

$$\left(\frac{V_1}{S^{\frac{1}{2}}}(1 - Z_3^2)^{\frac{1}{2}}, \frac{V_2}{S^{\frac{1}{2}}}(1 - Z_3^2)^{\frac{1}{2}}, Z_3\right)$$

is uniform on the surface of the 3-sphere. But $1 - 2S$ is uniform on $(-1, 1)$ and independent of $(V_1/S^{\frac{1}{2}}, V_2/S^{\frac{1}{2}})$. Substituting $1 - 2S$ for Z_3 then yields (4).

Similar ideas can be used to improve methods for choosing points on the n-sphere. For example:

Method for choosing a point on the 4-*sphere.* Choose V_1, V_2, independent uniform on $(-1, 1)$ until $S_1 = V_1^2 + V_2^2 < 1$. Choose V_3, V_4, independent uniform on $(-1, 1)$ until $S_2 = V_3^2 + V_4^2 < 1$. Then this point is uniform on the surface of the unit 4-sphere:

$$(V_1, V_2, V_3[(1 - S_1)/S_2]^{\frac{1}{2}}, V_4[(1 - S_1)/S_2]^{\frac{1}{2}}) .$$

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