Data Mining HW 4

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1

Q : Given $k_1(x, y)$ is a valid kernel show the scalar product $k(x, y) = ck_1(x, y)$ is a valid kernel.

Since $k_1(x, y)$ is a valid kernel $\exists \phi_1$ s.t. $k_1(x, y) =< \phi_1(x), \phi_1(y) >$. Note that $ck_1(x, y) = c < \phi_1(x), \phi_1(y) >=< \sqrt{c}\phi_1(x), \sqrt{c}\phi_1(y) >$. Let $\phi = \sqrt{c}\phi_1$. Then $k(x, y) = ck_1(x, y) =< \phi(x), \phi(y) >$. Thus $k(x, y)$ is a valid kernel.

2

Q : Given $k_1(x, y)$ and $k_2(x, y)$ are valid kernels show the sum $k(x, y) = k_1(x, y) + k_2(x, y)$ is a valid kernel.

Since $k_1(x, y)$ is a valid kernel $\exists \phi_1$ s.t. $k_1(x, y) =< \phi_1(x), \phi_1(y) >$. Since $k_2(x, y)$ is a valid kernel $\exists \phi_2$ s.t. $k_2(x, y) =< \phi_2(x), \phi_2(y) >$. Note that $k_1(x, y)+k_2(x, y) =< \phi_1(x), \phi_1(y) > + < \phi_2(x), \phi_2(y) >=< \phi_1(x)+\phi_2(x), \phi_1(y)+ \phi_2(y) >$. Let $\phi(x) = \phi_1(x) + \phi_2(x)$. Then $k(x, y) = k_1(x, y) + k_2(x, y) =< \phi(x), \phi(y) >$. Thus, $k(x, y)$ is a valid kernel.

5

Q : For support vectors $< \beta, x_* > \leq < \beta, x >$ for $x \in C_1$ and $< \beta, x_\# > \geq < \beta, x >$ for $x \in C_0$, show that the support vectors must satisfy:

$$\alpha_{opt} + < \beta_{opt}, x_* > = 1, \quad \alpha_{opt} + < \beta_{opt}, x_\# > = -1$$

Where x_* is the support vector for class 1 and $x_\#$ is the support vector for class 0.

We know that if $\alpha + < \beta, x > \geq 1$ then $x \in C_1$ and if $\alpha + < \beta, x > \leq -1$ then $x \in C_0$. Let $x_1 \in C_1$. Then $\alpha + < \beta, x_1 > \geq 1$ and $< \beta, x_* > \leq < \beta, x_1 >$. In order to satisfy these two inequalities without a contradiction, we must have that $\alpha_{opt} + < \beta_{opt}, x_* > = 1$. Now let $x_0 \in C_0$. Then $\alpha + < \beta, x_0 > \leq -1$ and $< \beta, x_\# > \geq < \beta, x_0 >$. In order to satisfy these two inequalities without a contradiction, we must have that $\alpha_{opt} + < \beta_{opt}, x_\# > = -1$.

7b

Q : Show the symmetric normalized Graph Laplacian and symmetric normalized Adjacency matrix share the same eigenvalues, i.e. λ is an eigenvalue of $D^{-1/2}(D-A)D^{-1/2}$ if and only if $1-\lambda$ is an eigenvalue of $D^{-1/2}AD^{-1/2}$.

($\Rightarrow$)
Assume that $D^{-1/2}(D-A)D^{-1/2}x = \lambda x$. Then

$$(D^{-1/2}DD^{-1/2} - D^{-1/2}AD^{-1/2})x = \lambda x$$
$$\Rightarrow (1 - D^{-1/2}AD^{-1/2})x = \lambda x$$
$$\Rightarrow x - D^{-1/2}AD^{-1/2}x = \lambda x$$
$$\Rightarrow D^{-1/2}AD^{-1/2}x = x - \lambda x$$
$$\Rightarrow D^{-1/2}AD^{-1/2}x = (1-\lambda)x$$

Thus, $1-\lambda$ is an eigenvalue of $D^{-1/2}AD^{-1/2}$.

✓

($\Leftarrow$)
Assume $D^{-1/2}AD^{-1/2}x = (1-\lambda)x$. Then:

$$\Rightarrow D^{-1/2}AD^{-1/2}x = x - \lambda x$$
$$\Rightarrow x - D^{-1/2}AD^{-1/2}x = \lambda x$$
$$\Rightarrow (1 - D^{-1/2}AD^{-1/2})x = \lambda x$$
$$\Rightarrow (D^{-1/2}DD^{-1/2} - D^{-1/2}AD^{-1/2})x = \lambda x$$
$$\Rightarrow D^{-1/2}(D-A)D^{-1/2}x = \lambda x$$

Thus, λ is an eigenvalue of $D^{-1/2}(D-A)D^{-1/2}$.

✓

Now we have shown that both directions of the if and only if statement are true. Therefore we can conclude that λ is an eigenvalue of $D^{-1/2}(D-A)D^{-1/2}$ if and only if $1-\lambda$ is an eigenvalue of $D^{-1/2}AD^{-1/2}$.