Math 101 HW 20

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1

Q : Imitate the proof of the MST to prove that every sequence either has a decreasing subsequence or a non-decreasing subsequence.

We can create a new definition for this proof.

Def: Let $\{x_n\}$ be a sequence and $N \in \mathbb{N}$. If $\forall n > N \;\; x_N \leq x_n$ then we say that x_N is **submissive**.

Let $\{x_n\}$ be a sequence. Either $\{x_n\}$ has infinitely many submissive terms or it does not have infinitely many submissive terms.

Case 1: $\{x_n\}$ has infinitely many submissive terms.
Let x_{n_1} be the first submissive term. Let x_{n_2} be the next submissive term.
So $\forall n > n_1 \;\; x_n \geq x_{n_1}$
and $\forall n > n_2 \;\; x_n \geq x_{n_2}$ and $n_2 > n_1 \Rightarrow x_{n_1} \leq x_{n_2}$
Continue this process to get $n_1 < n_2 < ... < n_k$ and $\forall i \leq k \;\; x_{n_i}$ is submissive.
Since $\exists$ infinitely many submissive terms $\exists$ a n_{k+1}^{th} submissive term. So $n_{k+1} > n_k$ and $\forall n > k+1 \;\; x_{n_{k+1}} \leq x_n$. This gives us $n_1 < n_2 < ... < n_k < n_{k+1}$ and $\forall i \leq k+1 \;\; x_{n_i}$ is submissive. This gives us a subsequence $\{x_{n_k}\}$ and since $\forall k \;\; x_{n_{k-1}}$ is submissive $x_{n_{k-1}} \leq x_{n_k}$. So $x_{n_k} \leq x_{n_{k+1}}$. Thus, $\{x_{n_k}\}$ is non-decreasing.

Case 2: $\{x_n\}$ does not have infinitely many submissive terms.
Either $\{x_n\}$ has 0 or finitely many submissive terms. So $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, x_n is not submissive. Let $n_1 = N + 1$. Then x_{n_1} is not submissive. So

$\exists n_2 > n_1$ s.t. $x_{n_2} < x_{n_1}$. Now x_{n_2} is not submissive so $\exists n_3 > n_2$ s.t. $x_{n_3} < x_{n_2} < x_{n_1}$. Continue this process to get $x_{n_1} > x_{n_2} > x_{n_3} > ... > x_{n_k}$ and $n_1 < n_2 < n_3 < ... < n_k$. Since x_{n_k} is not submissive $\exists n_{k+1} > n_k$ s.t. $x_{n_{k+1}} < x_{n_k}$. Now $\{n_k\}$ is increasing so $\{x_{n_k}\}$ is a subsequence and $\{x_{n_k}\}$ is decreasing by construction of x_{n_k}.

□

2

Q : Let a be a limit point of a sequence $\{x_n\}$. Let $\{y_n\}$ be a sequence obtained by rearranging the order of the terms of $\{x_n\}$ and adding a finite or infinite number of additional terms. Prove that a is a limit point of $\{y_n\}$.

Let $\{y_{n_k}\}$be a subsequence of $\{y_n\}$ s.t. the terms of $\{y_{n_k}\}$are a rearrangement of the terms of $\{x_n\}$. Since a is a limit point of $\{x_n\}$ $\exists$ $\{x_{m_j}\}$s.t. $x_{m_j} \to a$. Let $\{y_{n_{m_j}}\}$ be subsequence of $\{y_{n_k}\}$s.t. the terms of $\{y_{n_{m_j}}\}$ are the terms of $\{x_{m_j}\}$as they appear in $\{y_{n_k}\}$. Thus $\{y_{n_{m_j}}\}$ is a rearrangement of the terms of $\{x_{m_j}\}$. We know by HW 14 that this means that since $x_{m_j} \to a$ we can conclude that $y_{n_{m_j}} \to a$. We should in HW 19 that a subsequence of a subsequence of a sequence is indeed a subsequence of the sequence. Thus $\{y_{n_{m_j}}\}$ is a subsequence of $\{y_n\}$. So a is a subsequential limit of $\{y_n\}$. $\therefore$ a is a limit point of $\{y_n\}$.

3

Q : 3) Suppose that $x_n \nrightarrow \infty$. Prove that either $\{x_n\}$ has a subsequence which converges or $\{x_n\}$ has a subsequence which diverges to $-\infty$.

Either $\{x_n\}$ is bounded or it is not bounded below.

<u>Case 1:</u> $\{x_n\}$ is bounded
Since $\{x_n\}$ is bounded then, by Bolzano-Wieierstrass, it has a convergent subsequence.

Case 2: $\{x_n\}$ is not bounded below
By 4(c), $\{x_n\}$ has a subsequence which diverges to $-\infty$.

4

Q : Prove or disprove each of the following statements. You can disprove something with a counterexample, you don't have to prove your claims about the counterexample.

a) Every sequence has an increasing subsequence.
b) Every sequence has a bounded subsequence.
c) Every unbounded sequence has a subsequence which either diverges to ∞ or to $-\infty$.
d) If a sequence has a greatest term, then every subsequence has a greatest term.

(a)
Let $\{x_n\} = \{1^n\} = 1, 1, 1, 1, ...$ Every subsequence of $\{x_n\}$ will be a sequence of 1's and therefore will not be increasing.

(b)
Let $\{x_n\} = \{n\}$. Then $\{x_n\}$ does not have a bounded subsequence.

(c)
Let $\{x_n\}$ be an unbounded sequence. Then either $\{x_n\}$ is not bounded above or $\{x_n\}$ is not bounded below (or is not bounded above or below).

Case 1: $\{x_n\}$ is not bounded above
Then $\forall M > 0\ \exists n \in \mathbb{N}$ s.t. $x_n > M$.
$\exists n_1 \in \mathbb{N}$ s.t. $x_{n_1} > 1$.
$\exists n_2 \in \mathbb{N}$ s.t. $x_{n_2} > \max\{2, x_1, x_2, ..., x_{n_1}\} \Rightarrow n_2 > n_1$
$\exists n_3 \in \mathbb{N}$ s.t. $x_{n_3} > \max\{3, x_1, x_2, ..., x_{n_2}\} \Rightarrow n_3 > n_2$
Continuing this we get $n_1 < n_2 < n_3 < ... < n_k$ s.t. $\forall i \leq k\ \ x_{n_i} > i$.
$\exists n_{k+1} \in \mathbb{N}$ s.t. $x_{n_{k+1}} > \max\{k+1, x_1, x_2, ..., x_{n_{k+1}}\} \Rightarrow n_{k+1} > n_{k+1}$
Thus $\{n_k\}$ is an increasing sequence and $\forall k \in \mathbb{N}\ \ x_{n_k} > k$. We know $k \to \infty$ and so $\{x_{n_k}\} \to \infty$.

Case 2: $\{x_n\}$ is not bounded below

Then $\forall M < 0 \ \exists n \in \mathbb{N}$ s.t. $x_n < M$.
$\exists n_1 \in \mathbb{N}$ s.t. $x_{n_1} < -1$.
$\exists n_2 \in \mathbb{N}$ s.t. $x_{n_2} < \min\{-2, x_1, x_2, ..., x_{n_1}\} \Rightarrow n_2 > n_1$
$\exists n_3 \in \mathbb{N}$ s.t. $x_{n_3} < \min\{-3, x_1, x_2, ..., x_{n_2}\} \Rightarrow n_3 > n_2$
Continuing this we get $n_1 < n_2 < n_3 < ... < n_k$ s.t. $\forall i \leq k \ \ x_{n_i} < -i$.
$\exists n_{k+1} \in \mathbb{N}$ s.t. $x_{n_{k+1}} < \min\{-(k+1), x_1, x_2, ..., x_{n_{k+1}}\} \Rightarrow n_{k+1} > n_{k+1}$
Thus $\{n_k\}$ is an increasing sequence and $\forall k \in \mathbb{N} \ x_{n_k} < -k$. We know $-k \to -\infty$ and so $\{x_{n_k}\} \to -\infty$.

Thus every unbounded sequence has a subsequence which either diverges to ∞ or to $-\infty$.

$\square$

(d)
Let $\{x_n\} = \{1, -1, -1/2, -1/3, ...\}$ then taking every other term starting with -1 will give a subsequence that does not have a greatest element.