Math 101 HW 27

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1

Q : Let $f : (a, b) \to \mathbb{R}$ be uniformly continuous.

(a) Prove that there exists a $p \in \mathbb{R}$ such that for any $\{x_n\} \subseteq (a, b)$ with $x_n \to a$, then $f(x_n) \to p$.

(b) Let $g : [a, b) \to \mathbb{R}$ be defined by $g(x) = \begin{cases} p & \text{if } x = a \\ f(x) & \text{if } x \in (a, b) \end{cases}$

Prove that g is continuous.

(a) Let $\{x_n\} \subseteq (a, b)$ s.t. $x_n \to a$. Since $\{x_n\}$ converges we know that $\{x_n\}$ is Cauchy. Since f is uniformly continuous and $\{x_n\}$ is Cauchy, we know that $\{f(x_n)\}$ is Cauchy. Since $\{f(x_n)\}$ is Cauchy, we know $\exists p \in \mathbb{R}$ s.t. $f(x_n) \to p$. Let $\{y_n\} \subseteq (a, b)$ s.t. $y_n \to a$. Since $\{y_n\}$ converges, then $\{y_n\}$ is Cauchy. Since f is uniformly continuous we know that $\{f(y_n)\}$ is Cauchy. Since $\{f(y_n)\}$ is Cauchy we know that $\{f(y_n)\}$ converges. Now let $\{z_n\} \subseteq (a, b)$ s.t. $z_{2n} = x_n$ and $z_{2n-1} = x_n$ $\forall n$. Since $\{x_n\}$ and $\{y_n\}$ are two subsequences of $\{z_n\}$ that span all of $\{z_n\}$ and they both converge to a then we know $\{z_n\}$ converges to a. Since $\{z_n\}$ converges we know $\{z_n\}$ is Cauchy and thus $\{f(z_n)\}$ is Cauchy. Since $\{f(z_n)\}$ is Cauchy it converges. We know that $\lim f(z_n) = \lim f(x_n) = p$. We also know that $\lim f(y_n) = \lim f(z_n) = p$. Since $\{y_n\}$ was an arbitrary sequence we know that $\exists p \in \mathbb{R}$ s.t. for any $\{x_n\} \subseteq (a, b)$ with $x_n \to a$ then $f(x_n) \to p$.

$\square$

(b) Let $\{x_n\} \subseteq [a, b)$ s.t. $x_n \to a$. By homework 19 Q2, we know that $g(x_n) \to p$. Thus, g is continuous at a. Let $\varepsilon > 0$. Then $\exists \delta_f > 0$ s.t. if $x, y \in (a, b)$

and $|x-y|<\delta_f$, then $|g(x)-g(y)|=|f(x)-f(y)|<\varepsilon$. Since g is continuous at a, $\exists\delta_a>0$ s.t. if $x\in[a,b)$ and $|x-a|<\delta_a$, then $|g(x)-g(a)|<\varepsilon$. Let $\delta=\min\{d_f,d_a\}$. Let $x,y\in[a,b)$ s.t. $|x-y|<\delta$. If $x,y>a$, then $|x-y|<\delta_f$, so $|g(x)-g(y)|=|f(x)-f(y)|<\varepsilon$. If WLOG $y=a$, then $|x-y|=|x-a|<\delta_a$, so $|g(x)-g(y)|<\varepsilon$. $\therefore$ g is uniformly continuous.

2

> **Q** : Prove that every bounded infinite subset of $\mathbb{R}$ has an accumulation point.

Let $A\subseteq\mathbb{R}$ be a bounded infinite set. Then $\exists$ $\{x_n\}\subseteq A$ of distinct points. Since A is bounded we know $\{x_n\}$ is bounded and so $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Let $x\in\mathbb{R}$ s.t. $x_{n_k}\to x$. Since all terms of $\{x_{n_k}\}$are distinct, there is at most one k s.t. $x_{n_k}=x$. Let $\{y_n\}=\{x_{n_k}\}$if x is not one of the terms of $\{x_{n_k}\}$or $\{y_n\}=\{x_{n_1},x_{n_2},...,x_{n_{k-1}},x_{n_{k+1}},...\}$ where $x_{n_k}=x$. Then $\{y_n\}\subseteq A-\{x\}$ of distinct points s.t. $y_n\to x$. By problem 3, x is an accumulation point of A.

$\square$

3

> **Q** : Let $A\subseteq\mathbb{R}$ and $x\in\mathbb{R}$. Prove that x is an accumulation point of A iff there exists a sequence $\{x_n\}$ of distinct points in $A-\{x\}$ that converges to x.

$(\Rightarrow)$
Assume that x is an accumulation point of A. We know $1>0$, so $\exists x_1\in A$ s.t. $0<|x_1-x|<1$. Now $\forall n>1$, $\exists x_n\in A$ s.t. $0<|x_n-x|<\min\{\frac{1}{n},x_{n-1}\}$. Let $\{x_n\}$ be as above. Then $\forall n\in\mathbb{N}$ $|x_n-x|<frac1n\Rightarrow x-\frac{1}{n}<x_n<x+\frac{1}{n}$ and so, by squeeze theorem, $x_n\to x$. Let $n,m\in\mathbb{N}$ and assume WLOG $n>M$. Then $|x_n-x|<|x_{n-1}-x|<...<|x_{m+1}-x|<|x_m-x|\Rightarrow x_n\neq x_m$. Thus $\{x_n\}$ is a sequence of distinct points, $\{x_n\}\subseteq A-\{x\}$ s.t. $x_n\to x$.

$(\Leftarrow)$
Assume that there exists a sequence $\{x_n\}$ of distinct points in $A-\{x\}$ that

converges to x. Let $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t. if $n > N$ then $|x_n - x| < \varepsilon$. Let $n > N$. We know $0 < |x_n - x|$ because $\{x_n\}$ is a sequence of distinct points in $A - \{x\}$ and we know $|x_n - x| < \varepsilon$ because $n > N$. Thus x is an accumulation point of A.