Math 101 HW 25

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1

Q : Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function whose range is contained in $\mathbb{Q}$. Prove that f is a constant function.

Assume f is not a constant function. Then there are at least two distinct rationals, p and q, s.t. $\exists x, y \in \mathbb{R}$ s.t. $f(x) = p$ and $f(y) = q$. WLOG p ¡ q. By the density of the irrationals, $\exists w \in (\mathbb{R} - \mathbb{Q})$ s.t. $p < w < q$. By the Intermediate Value Theorem $\exists z \in \mathbb{R}$ s.t. $f(z) = w$. But $w \notin \mathbb{Q} \Rightarrow\Leftarrow$. Thus, f is a constant function.

2

Q : Let $f : A \to \mathbb{R}$ and let $c \in A$. Suppose that for every $\{x_n\} \subset \mathbb{Q} \cap A$ such that $x_n \to c$, $f(x_n) \to f(c)$; and for every $\{x_n\} \subset (\mathbb{R} - \mathbb{Q}) \cap A$ such that $x_n \to c$, $f(x_n) \to f(c)$. Prove that f is continuous at c.

Let $c \in A$. Let $\{x_n\} \subset A$ s.t. $x_n \to c$. There are three possibilities for $\{x_n\}$: $\{x_n\}$ has infinitely many rationals and infinitely many rationals, $\{x_n\}$ has infinitely many rationals and finitely many irrationals, or $\{x_n\}$ has finitely many rationals and infinitely many irrationals.

<u>Case 1:</u> $\{x_n\}$ has infinitely many rationals and infinitely many rationals
Let $\{x_{n_k}\}$be a subsequence of $\{x_n\}$ that contains all the rationals of $\{x_n\}$. Let $\{x_{m_j}\}$be a subsequence of $\{x_n\}$ that contains all the irrationals of $\{x_n\}$. Since $\{x_{n_k}\}$and $\{x_{m_j}\}$are subsequences of $\{x_n\}$ we know $\{x_{n_k}\}\to c$ and $\{x_{m_j}\}\to c$.

Since $\{x_{n_k}\}$is contained in $\mathbb{Q} \cap A$ we know $f(x_{n_k}) \to f(c)$ and since $\{x_{m_j}\}$is contained in $(\mathbb{R}-\mathbb{Q}) \cap A$ we know $f(x_{m_j}) \to f(c)$. Since $\{x_{n_k}\}$and $\{x_{m_j}\}$divide $\{x_n\}$ into two, we know $f(x_n) \to f(c)$.

✓

<u>Case 2:</u> $\{x_n\}$ has infinitely many rationals and finitely many irrationals
Then $\exists N \in \mathbb{N}$ s.t. $\{x_{n+N}\} \in \mathbb{Q}$. Since $x_{n+N} \to c$ and $\{x_{n+N}\} \in \mathbb{Q}$ we know $f(x_{n+N}) \to f(c)$. We know $lim f(x_n) = lim f(x_{n+N}) = f(c)$. And so $f(x_n) \to f(c)$.

✓

<u>Case 3:</u> $\{x_n\}$ has finitely many rationals and infinitely many irrationals.
Then $\exists N \in \mathbb{N}$ s.t. $\{x_{n+N}\} \in \mathbb{R} - \mathbb{Q}$. Since $x_{n+N} \to c$ and $\{x_{n+N}\} \in \mathbb{R} - \mathbb{Q}$ we know $f(x_{n+N}) \to f(c)$. We know $lim f(x_n) = lim f(x_{n+N}) = f(c)$. And so $f(x_n) \to f(c)$.

□

3

Q : Let $f : [a, b] \to \mathbb{R}$ and $g : [b, c] \to \mathbb{R}$ be continuous. Let

$$h(x) = \begin{cases} f(x) & \text{if } x \in [a, b] \\ g(x) & \text{if } x \in (b, c] \end{cases}$$

Prove that $h(x)$ is continuous iff $f(b) = g(b)$.

($\Rightarrow$)
Assume that $h(x)$ is continuous. WTS $f(b) = g(b)$. Suppose that $f(b) \neq g(b)$. Since g is continuous $\exists$ $\{x_n\} \subset [b, c]$ s.t. $x_n \to b$ and $g(x_n) \to g(b)$. Note that since $\{x_n\} \subset [b, c] \subset [a, c]$, $\{x_n\} \subset [a, c]$. And we have $x_n \to b$ and $g(x_n) \to g(b)$. But $\Rightarrow\Leftarrow$ b/c since h is continuous, $\forall$ $\{y_n\} \subset [a, c]$ s.t. $y_n \to b$ we have $h(y_n) \to h(b) = f(b) \neq g(b)$.

✓

($\Leftarrow$)
Assume $f(b) = g(b)$. We can see that if $d \in [a, b)$ then $\forall \{x_n\} \subseteq [a, c]$ s.t. $x_n \to d$ we have $h(x_n) = f(x_n) \to f(d) = h(d)$ since f is continuous. It is also obvious that if $d \in (b, c]$ then $\forall \{x_n\} \subseteq [a, c]$ s.t. $x_n \to d$ we have $h(x_n) = g(x_n) \to g(d) = h(d)$ since g is continuous. Now let $\{x_n\} \subset [a, c]$ s.t. $x_n \to b$. Either $\{x_n\}$ has infinitely many elements in $[a, b]$ and infinitely many elements in $(b, c]$, $\{x_n\}$ has infinitely many elements in $[a, b]$ and finitely many elements in $(b, c]$, or $\{x_n\}$ has finitely many elements in $[a, b]$ and infinitely many elements in $(b, c]$. By problem 2, we know $h(x_n) \to h(b)$. And so h is continuous.

$\square$

4

Q : Prove that the following function is continuous at just one point.

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 1 - x & \text{if } x \notin \mathbb{Q} \end{cases}$$

WTS that f is only continuous at $\frac{1}{2}$. Let $\varepsilon > 0$. Let $\delta = \varepsilon$. Let $x \in \mathbb{R}$ s.t. $|x - \frac{1}{2}| < \delta$. Either $x \in \mathbb{Q}$ or $x \notin \mathbb{Q}$.

<u>Case 1:</u> $x \in \mathbb{Q}$
Then we have $|f(x) - f(\frac{1}{2})| = |x - \frac{1}{2}| < \delta = \varepsilon \Rightarrow |f(x) - f(\frac{1}{2})| < \varepsilon$. Thus, f is continuous at $\frac{1}{2}$.

<u>Case 2:</u> $x \notin \mathbb{Q}$
Then we have $|f(x) - f(\frac{1}{2})| = |1 - x - \frac{1}{2}| = |\frac{1}{2} - x| = |x - \frac{1}{2}| < \delta = \varepsilon \Rightarrow |f(x) - f(\frac{1}{2})| < \varepsilon$. Thus f is continuous at $\frac{1}{2}$.

Now Let $p \in \mathbb{R}$ s.t. $p \neq \frac{1}{2}$. Let $\{x_n\}$ be a sequence of rationals s.t. $x_n \to p$. Then $f(x_n) = x_n \to p$ so $f(x_n) \to p$. Let $\{y_n\}$ be a sequence of irrationals s.t. $y_n \to p$. Then $f(y_n) = 1 - y_n \to 1 - p$ so $f(y_n) \to 1 - p$. In order for f to be continuous we require that $1 - p = p \Rightarrow 1 = 2p \Rightarrow p = \frac{1}{2}$ but $p \neq \frac{1}{2}$. Thus, f is not continuous at p.

$\square$