Math 101 HW 18

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Please grade 1, 2, and 4.

1

Q : Let $\{a_n\}$ be a sequence s.t. for every $n \in \mathbb{N}$ $|a_{n+1} - a_n| \leq \frac{1}{2^n}$. Prove that $\{a_n\}$ is Cauchy.

(We can use the fact that $\frac{1}{2^n}$ is Cauchy and therefore converges).

We proved in class that if $b \in (0,1)$ then $\{b^n\}$ converges to 0. Since $\frac{1}{2} \in (0,1)$ then $\frac{1}{2^n}$ converges to 0 and therefore is Cauchy. Thus, $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. if $m, n > N$ then $|\frac{1}{2^m} - \frac{1}{2^n}| < \varepsilon$. Let $m, n > N + 1$. WLOG $m > n$. Then $|a_m - a_n| \leq |a_m - a_{m-1}| + |a_{m-1} + a_{m-2}| + .. + |a_{n+1} - a_n|$ by the triangle inequality. Since for every $j \in \mathbb{N}$ $|a_{j+1} - a_j| \leq \frac{1}{2^n}$, $|a_m - a_{m-1}| + |a_{m-1} + a_{m-2}| + .. + |a_{n+1} - a_n| < \sum_{k=n}^{m-1} \left(\frac{1}{2}\right)^k = \sum_{k=0}^{m-1} \left(\frac{1}{2}\right)^k - \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k$. By the definition of a finite geometric series $\sum_{k=0}^{m-1} \left(\frac{1}{2}\right)^k - \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k = \frac{1-(\frac{1}{2})^m}{1-\frac{1}{2}} = \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = \left(\frac{1}{2^{n-1}} - \frac{1}{2^{m-1}}\right)$. Let $\varepsilon > 0$. Since $\frac{1}{2^n}$ is Cauchy and $m-1, n-1 > N$ then $|a_m - a_n| < \left(\frac{1}{2^{n-1}} - \frac{1}{2^{m-1}}\right) < \varepsilon \Rightarrow |a_m - a_n| < \varepsilon$. $\therefore$ $\{a_n\}$ is Cauchy.

2

Q : Let $\{a_n\}$ and $\{b_n\}$ be sequences such that for each $n \in \mathbb{N}$, $a_n \leq b_n$, and $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3]....$ Prove that there is a point $p \in \cap_{n=1}^{\infty}[a_n, b_n]$. Note that a point p is in $\cap_{n=1}^{\infty}[a_n, b_n]$ if and only if $\forall n \in \mathbb{N}$, $p \in [a_n, b_n]$.

We know by the definition of $\{a_n\}$ and $\{b_n\}$ that $\forall n \in \mathbb{N}$, $a_n \leq b_n \leq b_1$. Thus $\{a_n\}$ is bounded above by b_1 so $\{a_n\}$ has a lub. Let $p = \text{lub}(\{a_n\})$. Let $n \in \mathbb{N}$. Then by definition of a lub, $p \geq a_n$. WTS $b_n \geq p \geq a_n$. Assume that $b_n < p$. Since p is the lub of $\{a_n\}$, then $\exists m \in \mathbb{N}$ s.t. $a_m > b_n$. Either $m = n$, $m < n$, or $m > n$.

<u>$m = n$:</u>

So, $a_m = a_n \leq b_n \Rightarrow a_n \leq b_n \Rightarrow\Leftarrow$

✓

<u>$m < n$:</u>

Then $a_m \leq a_n \leq b_n \Rightarrow a_m \leq b_n \Rightarrow\Leftarrow$.

✓

<u>$m > n$:</u>

Then $b_n < a_m \leq b_m \Rightarrow b_n < b_m$ but $\{b_n\}$ is non-increasing so this is not possible $\Rightarrow\Leftarrow$

□

3

Q : Prove that $\{x_n\}$ diverges iff for every $a \in \mathbb{R}$, there exists an $\varepsilon > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ s.t. for all $k \in \mathbb{N}$, $|x_{n_k} - a| \geq \varepsilon$.

($\Rightarrow$)
Assume that $\{x_n\}$ diverges.
Let $a \in \mathbb{R}$. $\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists n > N$ s.t. $|x_n - a| \geq \varepsilon$ and $\forall k \in \mathbb{N}$, $|x_{n_k} - a| \geq \varepsilon$. We know the following:
$\exists n_1 > 1$ s.t. $|x_{n_1} - a| \geq \varepsilon$,
$\exists n_2 > n_1$ s.t. $|x_{n_2} - a| \geq \varepsilon$,
$\exists n_3 > n_2$ s.t. $|x_{n_3} - a| \geq \varepsilon$,
.
.
.
Continuing in this way $\{n_k\}$ is increasing. Thus $\{x_{n_k}\}$ is a subsequence that satisfies the required conditions.

($\Leftarrow$)
Assume that for every $a \in \mathbb{R}$, there exists an $\varepsilon > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ s.t. for all $k \in \mathbb{N}$, $|x_{n_k} - a| \geq \varepsilon$. A sequence $\{x_n\}$ converges to $l \in \mathbb{R}$ iff all its subsequences converge to l. But by our assumption there is a subsequence that does not converge to a. Thus $\{x_n\}$ diverges.

$\square$

4

Q : Suppose $\{x_n\}$ is Cauchy. Prove that for every $k \in \mathbb{N}$, the sequence $\{x_{n+k} - x_n\}$ is null. Prove that the sequence $\{\sqrt{n}\}$ is a counterexample to the converse.

Let $k \in \mathbb{N}$ and $\varepsilon > 0$. Since $\{x_n\}$ is Cauchy, $\exists N \in \mathbb{N}$ s.t. if $n, m > N$ then $|x_n - x_m| < \varepsilon$. Let $n > N$. Then $n + k > n > N$. Thus $|x_{n+k} - x_n| < \varepsilon$. Thus $\{x_{n+k} - x_n\}$ is null.

✓

Now we want to show that $\forall k$ $\{\sqrt{n+k} - \sqrt{n}\}$ is null and $\sqrt{n}$ is not Cauchy. We showed in HW 16 that $\sqrt{n}$ is not Cauchy. Now we must prove that $\forall k$ $\{\sqrt{n+k} - \sqrt{n}\}$ is null. Let $k \in \mathbb{N}$. Thus, $|\sqrt{n+k} - \sqrt{n}| = |\sqrt{n+k} - \sqrt{n}|\left|\frac{\sqrt{n+k}+\sqrt{n}}{\sqrt{n+k}+\sqrt{n}}\right| = \left|\frac{k}{\sqrt{n+k}+\sqrt{n}}\right| < \left|\frac{k}{\sqrt{n}}\right|$. We know that $\frac{1}{\sqrt{n}} \to 0$ thus $\frac{k}{\sqrt{n}} \to 0$. Let $\varepsilon > 0$. So $\exists N \in \mathbb{N}$ s.t. if $n > N$ then $\left|\frac{k}{\sqrt{n}}\right| < \varepsilon$. Let $n > N$ then $|\sqrt{n+k} - \sqrt{n}| < \left|\frac{k}{\sqrt{n}}\right| < \varepsilon \Rightarrow |\sqrt{n+k} - \sqrt{n}| < \varepsilon$. $\therefore$ $\{\sqrt{n+k} - \sqrt{n}\}$ is null.

□