Math 101 HW 17

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Please grade 1, 2, 3

1

Q : Let $a \in (-1, 0)$. Prove that $a^n \to 0$.

Since $a \in (-1, 0)$ then $(-a) \in (0, 1)$. We showed in class that since $(-a) \in (0, 1)$ then $(-a)^n \to 0$. Let $\varepsilon > 0$. Thus, $\exists N \in \mathbb{N}$ s.t. if $n > N$ then $|(-a)^n| < \varepsilon$. Let $n > N$. Note that $|a^n| = |(-a)^n| < \varepsilon$. $\therefore$ $|a^n| < \varepsilon$ so $a^n \to 0$.

$\square$

2

Q : Let $c > 1$. Prove that $c^n \to \infty$.

Note that since $c > 1$ $\forall n \in \mathbb{N} c^n > 0$. Consider the sequence $\{x_n\} = \{1/c^n\}$. WTS that $\{x_n\}$ is decreasing and bounded. We can do this by induction.

Let $p(n) : 1/c^n \in (0, 1)$ and $1/c^{n+1} < 1/c^n$.

<u>Base Case:</u> $n = 1$
$c > 1 \Rightarrow 1/c \in (0, 1)$ and since $1/c < 1$ and $1/c > 0$ then $1/c^2 < 1/c$.

✓

Inductive Step:
Suppose for some n that $1/c^n \in (0,1)$ and $1/c^{n+1} < 1/c^n$.

We know that $0 < 1/c^n < 1$ and since $1/c > 0$ we have $0 < 1/c^{n+1} < 1/c < 1$. Thus $1/c^{n+1} \in (0,1)$. And since $1/c^{n+1} < 1/c^n \Rightarrow 1/c^{n+2} < 1/c^{n+1}$.

Thus $\{1/c^n\}$ is bounded and decreasing.

Since bounded, monotonic sequence converges we know that $\exists \in \mathbb{R}$ s.t. $1/c^n \to l$. Also $1/c \to 1/c$. By the multiplication theorem $1/c^{n+1} \to l/c$. Note $\{1/c^{n+1}\} = \{1/c^2, 1/c^3, ...\} = \{x_{n+1}\}$ where $\{x_n\} = \{1/c^n\}$. In the homework we proved that if $x_n \to l$ then $x_{n+1} \to l$. Thus, $1/c^{n+1} \to l$ since $1/c^n \to l$. By uniqueness of limits $l = l/c \Rightarrow 0 = l/c - l = l((1/c) - 1)$. But$((1/c) - 1) \neq 0$ because $1/c \neq 1$. Therefore $l = 0$ so $1/c^n \to 0$.

Since $\forall n \in \mathbb{N}$ $c^n > 0$ and $1/c^n \to l$ then the lemma of the reciprocal theorem $c^n \to \infty$.

$\square$

3

Q : Let $d < -1$. Prove that $\{d^n\}$ diverges but does not diverge to either ∞ or $-\infty$.

In order to show that $\{d^n\}$ diverges we can show that it is not bounded. Since $|d| > 1$ we know, by problem 2, that $|d|^n = |d^n|$ diverges to infinity. Thus, $\forall M > 0$ $\exists N \in \mathbb{N}$ s.t. if $n > N$ $|d^n| > M$. But note that this is the negation of boundedness, thus $\{d^n\}$ is unbounded. So $\{d^n\}$ diverges. Now we want to show that it does not diverge to either ∞ or $-\infty$.

Proof that $\{d^n\}$ does not diverge to ∞:
Assume that $\{d^n\}$ does diverge to ∞. Then $\forall M > 0$ $\exists N \in \mathbb{N}$ s.t. if $n > N$ $d^n > M$. Let $M > 0$. Let $n > N$. Either n is even or n is odd. If n is odd then d^n is negative $\Rightarrow\Leftarrow$ because d^n cannot be greater than a positive number. If n is even then $n+1$ is odd and d^{n+1} is negative and similarly we have a contradiction.

Proof that $\{d^n\}$ does not diverge to $-\infty$:
Assume that $\{d^n\}$ does diverge to $-\infty$. Then $\forall M < 0\ \exists N \in \mathbb{N}$ s.t. if $n > N$ $d^n < M$. Let $M < 0$. Let $n > N$. Either n is even or n is odd. If n is even then d^n is positive $\Rightarrow\Leftarrow$ because d^n cannot be greater than a negative number. If n is odd then $n+1$ is even and d^{n+1} is positive and similarly we have a contradiction.

$\therefore$ $\{d^n\}$ diverges but does not diverge to either ∞ or $-\infty$.

$\square$

4

Q : Let $N \in \mathbb{N}$. Prove that $x_n \to l$ if and only if $x_{n+N} \to l$.

($\Rightarrow$)
(follows from HW 9 Q4???)

($\Leftarrow$)
Assume that $x_{n+N} \to l$. B (has to be finite) terms s.t. such that $|x_{n+N} - l| < \varepsilon$ is not true. So, at most B + N terms s.t. $|x_n - l| < \varepsilon$ is not true.