Math 101 HW 24

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Please grade 17.11, 1, and 2

17.11

Q : Let f be a real-valued function with $\text{dom}(f) \subseteq \mathbb{R}$. Prove f is continuous at x_0 if and only if, for every *monotonic* sequence (x_n) in $\text{dom}(f)$ converging to x_0, we have $\lim f(x_n) = f(x_0)$. *Hint:* Don't forget **Theorem 11.4**.

$(\Rightarrow)$
Assume f is continuous at x_0. Then for every sequence $\{x_n\} \subseteq \text{dom}(f)$ s.t. $x_n \to x_0$ we have $f(x_n) \to f(x_0)$. Note that every sequence in $\text{dom}(f)$ includes every *monotonic* sequence in $\text{dom}(f)$. So for every monotonic sequence $\{x_n\}$ in $\text{dom}(f)$ converging to x_0 we have $\lim f(x_n) = f(x_0)$.

$(\Leftarrow)$
Assume that for every *monotonic* sequence (x_n) in $\text{dom}(f)$ converging to x_0, we have $\lim f(x_n) = f(x_0)$. Let $\{x_n\} \subseteq \text{dom}(f)$ s.t. $x_n \to x_0$. Suppose that $f(x_n) \nrightarrow f(x_0)$. Thus $\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}\ \exists n > N$ s.t. $|f(x_n) - f(x_0)| \geq \varepsilon$. Let $\{x_{n_k}\}$be a subsequence of all those elements of $\{x_n\}$ for which the above is true. That is let $\{x_{n_k}\}\subseteq \{x_n\}$ s.t. $\forall k$, $|f(x_{n_k}) - f(x_0)| \geq \varepsilon$. By theorem 11.4 $\exists \{x_{n_{k_j}}\} \subseteq \{x_{n_k}\}$s.t. $\{x_{n_{k_j}}\}$ is monotonic. Since $x_n \to x_0 \Rightarrow x_{n_k} \to x_0$ we know $x_{n_{k_j}} \to x_0$. But since $\forall k$, $|f(x_{n_k}) - f(x_0)| \geq \varepsilon$ we must have $f(x_{n_k}) \nrightarrow f(x_0) \Rightarrow\Leftarrow$ because we assumed every monotonic sequence $\{y_n\}$ converging to $x_0 \Rightarrow f(y_n) \to f(x_0)$.

17.15

> **Q** : Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Show f is continuous at x_0 in $\text{dom}(f)$ if and only if, for every sequence $\{x_n\}$ in $\text{dom}(f)\backslash\{x_0\}$ converging to x_0, we have $\lim f(x_n) = f(x_0)$.

($\Rightarrow$)
Assume that f is continuous at x_0 in $\text{dom}(f)$.

1

> **Q** : Let $f : [a, b] \to \mathbb{R}$ be a continuous function which has the property that for each $x \in [a, b]$ there is a $y \in [a, b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$.
> (a) Prove that there is a sequence $\{x_n\} \subset [a, b]$ such that $f(x_n) \to 0$.
> (b) Prove that there is a point $c \in [a, b]$ such that $f(c) = 0$.

(a) Let $y \in [a, b]$. Then $\exists x_1 \in [a, b]$ s.t. $|f(x_1)| \leq \frac{1}{2}|f(y)| \Rightarrow -\frac{1}{2}|f(y)| \leq f(x_1) \leq \frac{1}{2}|f(y)|$. And $\exists x_2 \in [a, b]$ s.t. $|f(x_2)| \leq \frac{1}{2}|f(x_1)| \leq \frac{1}{2^2}|f(y)| \Rightarrow -\frac{1}{2^2}|f(y)| \leq f(x_2) \leq \frac{1}{2^2}|f(y)|$. Continuing in this fashion we get the sequence $\{x_n\}$ where $-\frac{1}{2^n}|f(y)| \leq f(x_n) \leq \frac{1}{2^n}|f(y)|$. We know $-\frac{1}{2^n}|f(y)| \to 0$ and $\frac{1}{2^n}|f(y)| \to 0$ and thus, by the squeeze theorem, $f(x_n) \to 0$.

2

> **Q** : Let $a \in \mathbb{R}$ and $f : \mathbb{R} - \{a\} \to \mathbb{R}$. Suppose that for every sequence $\{x_n\} \subseteq \mathbb{R} - \{a\}$ such that $x_n \to a$, we have $f(x_n) \to -\infty$. Prove that for every $M \in \mathbb{R}$ there is a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) < M$.

Assume that $\exists M \in R$ s.t. $\forall \delta > 0$ $\exists x \in \mathbb{R} - \{a\}$ s.t. $0 < |x - a| < \delta$ and $f(x) \geq M$. Let $\{x_n\} \subseteq \mathbb{R} - \{a\}$ s.t. $x_n \to a$. Let $\delta > 0$. Then $\exists N_1 \in \mathbb{N}$ s.t. if $n > N_1$ then $|x_n - a| < \delta$. Since $f(x_n) \to -\infty$ we know that $\forall M < 0$ $\exists N_2 \in \mathbb{N}$ s.t. if $n > N_2$ then $f(x_n) < M$. Let $N = \max\{N_1, N_2\}$ and let $n > N$. Let $M \in \mathbb{R}$. Let $x = x_n$. Since $n > N_1$ we know $|x - a| < \delta$. And since $n > N_2$ we know that $f(x_n) < M$. But since M is arbitrary we know that $f(x_n) < M$ is true for any M. Thus $\Rightarrow\Leftarrow$ because we assumed that there exists $M \in \mathbb{R}$ s.t. $\forall \delta > 0\ \exists x \in \mathbb{R} - \{a\}$ s.t. $0 < |x - a| < \delta$ and $f(x) \geq M$. Thus for every $M \in \mathbb{R}$ there is a $\delta > 0$ s.t. if $0 < |x - a| < \delta$ then $f(x) < M$.