Math 101 HW 16

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March 1, 2017

1

> **Q** : Suppose that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Prove that $\{x_n + y_n\}$ is Cauchy.

Let $\varepsilon > 0$. Since $\{x_n\}$ is a Cauchy sequence we know that $\exists N_1 \in \mathbb{N}$ s.t. if $n, m > N_1$ then $|x_n - x_m| < \varepsilon/2$. Since $\{y_n\}$ is a Cauchy sequence we know that $\exists N_2 \in \mathbb{N}$ s.t. if $n, m > N_2$ then $|y_n - y_m| < \varepsilon/2$. Let $m, n >$max$\{N_1, N_2\}$. Then $|(x_n+y_n)-(x_m+y_m)| = |(x_n-x_m)+(y_n-y_m)| \leq |x_n-x_m|+|y_n-y_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence, $|(x_n - x_m) + (y_n - y_m)| < \varepsilon$. $\therefore$ $\{x_n + y_n\}$ is Cauchy.

$\square$

2

> **Q** : Suppose that $\{x_n\}$ is a sequence of integers which is Cauchy. Prove that there exists an $N \in \mathbb{N}$ s.t. for every $n > N$ $x_n = x_N$.

Let $\varepsilon = 1/47$ be given. Since $\{x_n\}$ is Cauchy $\exists N_1 \in \mathbb{N}$ s.t. if $n, m > N_1$ then $|x_n - x_m| < \varepsilon$. Let $N > N_1$ and $n > N$. Then $|x_N - x_n| < 1/47 \Rightarrow -1/47 < x_N - x_n < 1/47 \Rightarrow x_n - 1/47 < x_N < x_n + 1/47$. Thus $x_N \in (x_n - 1/47, x_n + 1/47)$. Note that the length of the interval $(x_n - 1/47, x_n + 1/47)$ is $2/47$ and so it contains at most one integer. We know that x_N is equal to the one integer in the interval $(x_n - 1/47, x_n + 1/47)$ because it is a term in a sequence of integers and it is contained in that interval. We also know that $x_n - 1/47 < x_n < x_n + 1/47$ and so $x_n \in (x_n - 1/47, x_n + 1/47)$. And since x_n comes from a sequence of integers and it is contained in the interval $(x_n - 1/47, x_n + 1/47)$ we know that it is also equal to the one integer contained in the interval. Thus $x_n = x_N$. $\therefore$ there exists an $N \in \mathbb{N}$ s.t. for every $n > N$ $x_n = x_N$.

$\square$

4

Q : Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences. Prove that $\{x_n y_n\}$ is Cauchy.

Let $\varepsilon > 0$. Since $\{x_n\}$ and $\{y_n\}$ are Cauchy and because Cauchy sequences are bounded $y_n \in \mathbb{R}$ and $\exists N_1 \in \mathbb{N}$ s.t. if $n, m > N_1$ then $|x_n - x_m| < \varepsilon/2|y_n|$. Since $\{y_n\}$ and $\{x_n\}$ are Cauchy and because Cauchy sequences are bounded $x_n \in \mathbb{R}$ and $\exists N_2 \in \mathbb{N}$ s.t. if $n, m > N_2$ then $|y_n - y_m| < \varepsilon/2|x_m|$. Let $n, m > \max\{N_1, N_2\}$. WTS $|x_n y_n - x_m y_m| < \varepsilon$. Note that $|x_n y_n - x_m y_m| = |y_n(x_n - x_m) + x_m(y_n - y_m)| \leq |y_n(x_n - x_m)| + |x_m(y_n - y_m)|$ by the triangle inequality. Hence, $|x_n y_n - x_m y_m| \leq |y_n(x_n - x_m)| + |x_m(y_n - y_m)| \leq |y_n||(x_n - x_m)| + |x_m||(y_n - y_m)| < |y_n|(\varepsilon/2|y_n|) + |x_n|(\varepsilon/2|x_n|) = \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus, $|x_n y_n - x_m y_m| < \varepsilon$. $\therefore$ $\{x_n y_n\}$ is Cauchy.

$\square$