Math 101 HW 30

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> **Q** : Prove Let f be a strictly increasing function on an interval J such that $f(J)$ is an interval. Then f is continuous on J. Same example from class but prove for x_0 is an endpoint of J

Let x_0 be an endpoint J. WLOG x_0 is the lower endpoint. Let $\varepsilon > 0$. Since x_0 is the lower endpoint of J, $\exists x_1 \in J$ s.t. $x_0 < x_1$. Let $y_1 = \min\{f(x_0) + \varepsilon, f(x_1)\}$. We know $y_1 > f(x_0)$ because f is strictly increasing. So $f(x_0) < y_1 \leq f(x_1)$. Since $f(J)$ is an interval it has the interval property, so $y_1 \in f(J)$. So $\exists a_1 \in J$ s.t. $f(a_1) = y_1$. We also know that $x_0 < a_1$ since f is strictly increasing. Now $f(a_1) \leq f(x_0)+\varepsilon$. Let $\delta = a_1 - x_0$. Let $x \in J$ s.t. $|x - x_0| < \delta$. Then $x_0 - \delta < x < x_0 + \delta = a_1$. Since f is strictly increasing and $x < a_1$ we know that $f(x) < f(a_1) \leq f(x_0) + \varepsilon$ and so $f(x) < f(x_0) + \varepsilon$. Now since $x \in J$ and x_0 is an endpoint of J we know that $x_0 \leq x \Rightarrow f(x_0) - \varepsilon < f(x_0) \leq f(x) \Rightarrow f(x_0) - \varepsilon < f(x)$. Thus we have $f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon \Rightarrow |f(x) - f(x_0)| < \varepsilon$. $\therefore$ f is continuous at x_0. Hence f is continuous at the endpoints of J.

$\square$

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Q : Recall from the lecture that a set J is said to be an *interval* if for any $x, y \in J$ and $x < z < y$, then $z \in J$. Let I and J be intervals. Suppose that I is bounded and contains its lub but not its glb. Prove that I has the form $(a, b]$. Suppose that J is bounded above and does not contain its lub and is unbounded below. Prove that J has the form $(-\infty, b)$.

Interval I = (a, b]:
Let $a = \text{glb}(I)$ and $b = \text{lub}(I)$.
$(\subseteq)$
Let $x \in I$. Since $a = \text{glb}(I)$ and I does not contain its glb, we know $a < x$. Since $b = \text{lub}(I)$ and I contains its lub then we know $x \leq b$. Thus $x \in (a, b]$.

✓

$(\supseteq)$
Let $x \in (a, b]$. So $x > a = \text{glb}(I)$. Thus $\exists y \in I$ s.t. $y < x$. Since b is contained in I and $b = \text{glb}(I)$ we know $x \leq b$. So $y < x \leq b$ and $y, b \in I$. By the interval property $x \in I$. $\therefore I = (a, b]$

$\square$

Interval $J = (-\infty, b)$**:**
Let $b = \text{lub}(J)$.

$(\subseteq)$
Let $x \in J$. Since $b = \text{lub}(J)$ and b is not contained in J we know $x < b$. Thus, $x \in (-\infty, b)$.

✓

$(\supseteq)$
Let $x \in (-\infty, b)$. So $x < b = \text{lub}(J)$. Thus $\exists y \in I$ s.t. $x < y$. Since J is unbounded below $\exists z \in J$ s.t. $z < x$. So $z < x < y$ and $z, y \in J$. By the interval property $x \in J$. $\therefore J = (-\infty, b)$.

$\square$

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Q : Let $\{a_n\}$ be a bounded non-decreasing sequence. Define a sequence $\{b_n\}$ by

$$b_n = \frac{a_1 + a_2 + ... + a_n}{n}$$

Prove that $\{b_n\}$ is bounded and non-decreasing, and therefore converges.
Hint: First prove that for every $n \in \mathbb{N}$

$$(n+1)(a_1 + ... + a_n) \leq n(a_1 + ... + a_{n+1})$$

Let $n \in \mathbb{N}$. Then since $\{a_n\}$ is non-decreasing we know that $a_n \leq a_{n+1}$. Since $n > 0$ we have $n(a_n) \leq n(a_{n+1})$. Note that since $\{a_n\}$ is non-decreasing $a_1 + ... + a_n \leq n(a_n) \leq n(a_{n+1}) \Rightarrow a_1 + ... + a_n \leq n(a_{n+1})$. Now if we add $n(a_1 + ... + a_n)$ to both sides of that inequality we get $n(a_1 + ... + a_n) + (a_1 + ... + a_n) \leq n(a_1 + ... + a_{n+1}) \Rightarrow (n+1)(a_1 + ... + a_n) \leq n(a_1 + ... + a_{n+1})$. And so $b_n = \frac{a_1+...+a_n}{n} \leq \frac{a_1+...+a_{n+1}}{n+1} = b_{n+1}$ and so $\{b_n\}$ is non-decreasing. Now let u be an upper bound for $\{a_n\}$. Then $a_1 + ... + a_n \leq (n)(u)$ and so $b_n = \frac{a_1+...+a_n}{n} \leq \frac{(n)(u)}{n} = u$. Thus, u is an upper bound for $\{b_n\}$ and $\{b_n\}$ is bounded above. Now let l be a lower bound for $\{a_n\}$. Then $(n)(l) \leq a_1 + ... + a_n$ and so $l = \frac{(n)(l)}{n} \leq \frac{a_1+...+a_n}{n} = b_n$. Thus, l is lower bound for $\{b_n\}$ and $\{b_n\}$ is bounded below. $\therefore$ $\{b_n\}$ is bounded and non-decreasing and therefore converges.

$\square$