Math 101 HW 26

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19.4

Q : (a) Prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S. *Hint*: Assume not. Use Theorems 11.5 (BW) and 19.4 (If f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence).

(b) Use (a) to give yet another proof that $\frac{1}{x^2}$ is not uniformly continuous on (0, 1).

(a) Let f be uniformly continuous on a bounded set S. Suppose f is not a bounded function on S. Then $\exists$ $\{x_n\}$ in S s.t. $f(x_n) \to \infty$. S is bounded and so $\{x_n\}$ is bounded. Since $\{x_n\}$ is bounded, by Bolzano-Wieierstrass, there is a convergent subsequence of $\{x_n\}$, $\{x_{n_k}\}$. Since $\{x_{n_k}\}$converges, it is Cauchy. Since $\{x_{n_k}\}$is Cauchy and f is uniformly continuous, $f(x_{n_k})$ is Cauchy $\Rightarrow$ $f(x_{n_k})$ is bounded. But $\lim f(x_{n_k}) = \lim f(x_n) = \infty \Rightarrow\Leftarrow$ because we assumed $\{x_{n_k}\}$converges and since f is continuous $\lim f(x_{n_k}) = \lim x_{n_k} \neq \infty$. $\therefore$ if f is uniformly continuous on a bounded set S, then f is a bounded function on S.

$\square$

(b) Let $\{x_n\} = \frac{1}{\sqrt{n}}$. Then $f(x_n) = \{n\}$ but $\{n\}$ is not bounded on $(0, 1) \Rightarrow\Leftarrow$. And so $\frac{1}{x^2}$ is not uniformly continuous on (0, 1).

$\square$

1

Q : Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that for every $\alpha > 0$ there is an $M > 0$ such that if $|x| \geq M$, then $|f(x)| < \alpha$. Prove that f is uniformly continuous.

Let $\varepsilon > 0$ and $\alpha = \varepsilon/2$. WTS $\exists \delta > 0$ s.t. if $x, y \in \mathbb{R}$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. We know $\exists M > 0$ s.t. if $|x| \geq M$, then $|f(x)| < \varepsilon/2$. Let f_m be the function f with its domain restricted to the interval to $[-M, M]$. We know f_m is uniformly continuous. Therefore $\exists \delta_1 > 0$ s.t. if $x, y \in [-M, M]$ and $|x - y| < \delta_1$, $|f(x) - f(y)| < \varepsilon/2$. We have that f is continuous at $M \Rightarrow \exists \delta_2 > 0$ s.t. if $|x - M| < \delta_2$, then $|f(x) - f(M)| < \varepsilon/2$. Since f is continuous at $-M \exists \delta_3 > 0$ s.t. if $|x + M| < \delta_3$, then $|f(x) - f(-M)| < \varepsilon/2$. Let $x, y \in \mathbb{R}$ s.t. $|x - y| < \delta$ There are 3 cases:

$\underline{x, y \in [-M, M]}$:
Since $|x| \geq M$ and $|y| \geq M$ we have $|f(x)| < \varepsilon/2$ and $|f(y)| < \varepsilon/2$. By the triangle inequality $|f(x) - f(y)| \geq |f(x)| + |f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

✓

$\underline{x, y \in [-M, M]}$:
Since $|x - y| < \delta < \delta_1$ we have $|f(x) - f(y)| < \varepsilon/2 < \varepsilon$.

✓

<u>WLOG $x \in [-M, M]$ and $y \notin [-M, M]$:</u>

Either $-M \in [y, x]$ or $M \in [x, y]$.

$\underline{-M \in [y, x]}$:
Then $|x + M|, |y + M| \leq |x - y| < \delta \leq \delta_3$. Thus $|f(x) - f(y)| \leq |f(x) - f(-M)| + |f(-M) - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

✓

$\underline{M \in [x, y]}$:
Then $|x - M|, |y - M| \leq |x - y| < \delta \leq \delta_2$. Thus $|f(x) - f(y)| \leq |f(x) - f(M)| + |f(M) - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

✓

$\therefore f$ is uniformly continuous.

$\square$

2

Q : Let f and g be continuous functions on the interval $[a, b]$. Suppose that for all $x \in [a, b]$, $f(x) < g(x)$. Prove that there is an $\alpha < 1$ such that for all $x \in [a, b]$, $f(x) < \alpha g(x)$

Assume $\forall \alpha < 1$, $\exists x \in [a, b]$ s.t. $f(x) \geq g(x)$. We know $1 - \frac{1}{n}$ $\forall n$, so $\forall n$ $\exists x_n \in [a, b]$ s.t. $f(x_n) \geq (1 - \frac{1}{n})g(x)$. By Bolzano-Wieierstrass, $\exists l \in [a, b]$ and $\{x_{n_k}\}$s.t. $x_{n_k} \to l$. Since g and f are continuous $g(x_{n_k}) \to g(l)$ and $f(x_{n_k}) \to f(l)$. Since $\forall n$, $f(x_n) \geq (1 - \frac{1}{n})g(x_n)$, $\lim f(x_{n_k}) \geq \lim ((1 - \frac{1}{n})g(x_{n_k}))$. By the multiplication theorem $\lim ((1 - \frac{1}{n})g(x_{n_k})) = f(l)$. So $f(l) \geq g(l)$ but $\Rightarrow\Leftarrow$ because $l \in [a, b]$ and $\forall x \in [a, b], f(x) < g(x)$.

$\square$