Math 101 HW 11

Jeff Carney

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8.6

Let (s_n) be a sequence in $\mathbb{R}$.

(a)

Q : Prove lim $s_n = 0$ iff lim $\lvert s_n\rvert = 0$

$(\Rightarrow)$
Assume that lim $s_n = 0$. Thus, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n > N$, then $|s_n| < \varepsilon$. Let $\varepsilon > 0$ and $n > N$. We know that $||s_n|| = |s_n| < \varepsilon$, thus $||s_n|| < \varepsilon$. $\therefore$ lim $|s_n| = 0$.

✓

$(\Leftarrow)$
Assume that lim $|s_n| = 0$. Thus, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n > N$, then $||s_n|| < \varepsilon$. Let $\varepsilon > 0$ and $n > N$. We know that $|s_n| = ||s_n|| < \varepsilon$, thus $|s_n| < \varepsilon$. $\therefore$ lim $s_n = 0$.

□

(b)

Q : Prove that if $s_n = (-1)^n$, then lim $\lvert s_n\rvert$ exists, but lim s_n does not exist.

Proof that lim s_n does not exist:

Let $s_n = (-1)^n$. Suppose that that lim s_n exists. Let l =lim s_n. We know that $s_n = \{-1, 1, -1, 1, ..\}$ is a sequence of integers. By homework 9 problem 2 $\exists N \in \mathbb{N}$ s.t. if $n > N$, then $s_n = l$. Let $n > N$. Either n is odd or n is even.

n is even:

If n is even then $s_n = 1 = l$. Since $n > N$, $n+1 > N \Rightarrow s_{n+1} = l$. But since $n+1$ is odd $s_{n+1} = -1$. So we have $1 = -1 \Rightarrow\Leftarrow$. Thus, lim s_n does not exist.

n is odd:

If n is odd then $s_n = -11 = l$. Since $n > N$, $n+1 > N \Rightarrow s_{n+1} = l$. But since $n+1$ is even $s_{n+1} = 1$. So we have $1 = -1 \Rightarrow\Leftarrow$. Thus, lim s_n does not exist.

$\square$

Proof that lim $|s_n|$ exists:

Let $s_n = (-1)^n = \{-1, 1, -1, 1, ...\}$. Then $|s_n| = |(-1)^n| = \{|-1|, |1|, |-1|, |1|, ...\} = \{1, 1, 1, 1, ...\}$. Let $n \geq 1$. Thus, $\forall \varepsilon > 0$ $||s_n|-1| = |1-1| = 0 < \varepsilon \Rightarrow ||s_n| - 1| < \varepsilon$. Thus, lim $s_n = 1$.

8.9

Let (s_n) be a sequence that converges.

(a)

Q : Show that if $s_n \geq a$ for all but finitely many n, then lim $s_n \geq a$.

The finitely many n for which $s_n < a$ must come from the head because the tail is infinite. Thus, $\exists N_1 \in \mathbb{N}$ s.t. if $n > N_1$, then $s_n \geq a$. Assume that lim $s_n = l$ and $l < a$. Then $\forall \varepsilon > 0$, $\exists N_2 \in \mathbb{N}$ s.t. if $n > N_2$, then $|s_n - l| < \varepsilon$. Let N =max$\{N_1, N_2\}$ and let $n > N$, so $s_n \geq a$. Since $l < a$, $s_n - l > 0$. So we know that $\exists x \in \mathbb{R}$ s.t. $0 < x < s_n - l$. Let $\varepsilon = x$, then $|s_n - l| = s_n - l > \varepsilon \Rightarrow |s_n - l| > \varepsilon \Rightarrow\Leftarrow$. Thus lim $s_n \geq a$.

$\square$

(b)

Q : Show that if $s_n \leq b$ for all but finitely many n, then lim $s_n \leq b$.

The finitely many n for which $s_n > b$ must come from the head because the tail is infinite. Thus, $\exists N_1 \in \mathbb{N}$ s.t. if $n > N_1$, then $s_n \leq b$. Assume that lim $s_n = l$ and $l > b$. Then $\forall \varepsilon > 0$, $\exists N_2 \in \mathbb{N}$ s.t. if $n > N_2$, then $|s_n - l| < \varepsilon$. Let $N =$max$\{N_1, N_2\}$ and let $n > N$, so $s_n \leq b$. Since $l > b$, $s_n - l < 0$. So we know that $\exists x \in \mathbb{R}$ s.t. $s_n - l < x < 0 \Rightarrow l - s_n > x$. Let $\varepsilon = x$, then $|s_n - l| = l - s_n > \varepsilon \Rightarrow |s_n - l| > \varepsilon \Rightarrow\Leftarrow$. Thus lim $s_n \leq b$.

$\square$

(c)

Q : Conclude that if all but finitely many s_n belong to $[a, b]$, then lim s_n belongs to $[a, b]$.

Let all but finitely many $s_n \in [a, b]$, then all but finitely many s_n are such that $a \leq s_n \leq b$. By part (a) we know that lim $s_n \geq a$. By part (b) we know that lim $s_n \leq a$. Thus, $a \leq$ lim $s_n \leq b \Rightarrow$ lim $s_n \in [a, b]$.

$\square$

8.10

Q : Let (s_n) be a convergent sequence, and suppose lim $s_n > a$. Prove there exists a number N such that $n > N$ implies $s_n > a$

Let lim $s_n = l$ s.t. $l > a$. Then $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n > N$ then $|s_n - l| < \varepsilon$. Since $l > a$, $\exists x > 0$ s.t. if $n > N$ then $|s_n - a| \geq x > 0$. Let $n > N$. Thus $s_n - a = |s_n - a| > 0 \Rightarrow s_n - a > 0 \Rightarrow s_n > a$.

$\square$

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(a)

> **Q** : Prove or disprove the following: Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences such that for all $n \in \mathbb{N}$, $x_n < y_n$. Then $\lim x_n < \lim y_n$.

Let $\{x_n\} = \{\frac{1}{n+1}\}$ and $\{y_n\} = \{\frac{1}{n}\}$. Then $\forall n \in \mathbb{N}$, $x_n < y_n$. Let $\varepsilon > 0$ be given. By Archimedes $\exists N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$. Let $n > N > \frac{1}{\varepsilon}$. So, $|\frac{1}{n} - 0| = \frac{1}{n} < \varepsilon$. $\therefore$ $\lim y_n = 0$. Now let $n + 1 > N > \frac{1}{\varepsilon}$. So, $|\frac{1}{n+1} - 0| = \frac{1}{n+1} < \varepsilon$. $\therefore$ $\lim y_n = 0$. Thus, we have disproved the statement.

(b)

> **Q** : Prove or disprove the following: If $\{x_n\}$ diverges to ∞ then $\{x_n\}$ does not converge

WTS $\forall l \in \mathbb{R}$ $\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$ $\exists n > N$ s.t. $|x_n - l| > \varepsilon$. Assume that $\{x_n\}$ diverges to ∞, thus $\forall M > 0$ $\exists N \in \mathbb{N}$ s.t. if $n > N$ then $x_n > M$. Let $M > 0$, $n > N$, $\varepsilon > 0$, and $l \in \mathbb{R}$. Since $x_n > M$, $|x_n - l| > |M - l|$. Thus, $\{x_n\}$ does not converge.

$\square$