# Math 101 HW 12

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## 1

**Q** : Suppose that $x_n \to l$. Prove that $\forall m \in \mathbb{N}$, $x_n^m \to l^m$.

Suppose $x_n \to l$.
Let $p(m)$: $x_n^m \to l^m$.

Base Case: $m = 2$

Since $x_n \to l$, by the multiplication theorem, $x_n x_n \to (l)(l)$ i.e. $x_n^2 \to l^2$.

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Inductive Step:

Suppose $p(m)$ is true for some $m$.

WTS $x_n^{m+1} \to l^{m+1}$. We know $x_n^{m+1} = x_n^m x_n$. By the inductive hypothesis $x_n^m \to l^m$ and by our supposition in the problem $x_n \to l$. Thus, by the multiplication theorem $x_n^{m+1} = x_n^m x_n \to (l^m)(l) = l^{m+1}$. $\therefore x_n^{m+1} \to l^{m+1}$.

✓

$\therefore \forall m \in \mathbb{N}, x_n^m \to l^m$.

$\square$

## 2

> **Q** : Let $S$ be a set of real numbers, and let $\{x_n\}$ be a sequence which converges to $l$. Suppose that $\forall n \in \mathbb{N}$, $x_n$ is an upper bound for $S$. Prove that $l$ is an upper bound for $S$.

Assume that $l$ is not an upper bound for $S$. That is, $\exists s \in S$ s.t. $s > l$. Let $n \in \mathbb{N}$. Since $x_n$ is an upper bound for S, $l < s \leq x_n \Rightarrow l < x_n \Rightarrow 0 < x_n - l$. We know that $x_n - l$ and 0 are distinct real numbers, thus, by the density of the rationals, $\exists q \in \mathbb{Q}$ s.t. $0 < q < x_n - l \leq |x_n - l|$. Thus $|x_n - l| > q$. But $q > 0$ and by the definition of convergence $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ s.t. if $n > N$ then $|x_n - l| < \varepsilon \Rightarrow\Leftarrow$ because $|x_n - l| > q$.

## 4

> **Q** : Prove that for all $a, b \in \mathbb{R}$, $||a| - |b|| \leq |a - b|$, and use this inequality to prove that if $x_n \to x$ then $|x_n| \to |x|$.

Let $a, b \in \mathbb{R}$. Either $|a| - |b| \geq 0$ or $|a| - |b| < 0$.

<u>Case 1:</u> $|a| - |b| \geq 0$

Since $|a| - |b| \geq 0$, $||a| - |b|| = |a| - |b|$. Note that $|a| = |(a - b) + b| \leq |a - b| + |b|$ by the triangle inequality. Thus, $|a| \leq |a - b| + |b| \Rightarrow |a| - |b| \leq |a - b|$. Hence, $||a| - |b|| = |a| - |b| \leq |a - b| \Rightarrow ||a| - |b|| \leq |a - b|$.

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<u>Case 2:</u> $|a| - |b| < 0$

Since $|a| - |b| < 0$, $||a| - |b|| = -(|a| - |b|) = |b| - |a|$. Note that $|b| = |(b - a) + a| \leq |b - a| + |a|$ by the triangle inequality. Thus, $|b| \leq |b - a| + |a| \Rightarrow |b| - |a| \leq |b - a| = |a - b| \Rightarrow |b| - |a| \leq |a - b|$. Hence, $||a| - |b|| = |b| - |a| \leq |a - b| \Rightarrow ||a| - |b|| \leq |a - b|$.

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$\therefore \forall a, b \in \mathbb{R} ||a| - |b|| \leq |a - b|$

$\square$

Now assume that $x_n \to x$. Thus, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n > N$ then $|x_n - x| < \varepsilon$. Let $\varepsilon > 0$ and $n > N$. By the above proof $||x_n| - |x|| \leq |x_n - x| < \varepsilon \Rightarrow ||x_n| - |x|| < \varepsilon$. Thus, $|x_n| \to |x|$.