Math 101 HW 28

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1

> **Q** : Use the $\varepsilon - \delta$ definition of continuity to prove that $f(x) = |x|$ is continuous on $\mathbb{R}$.

Let $c \in \mathbb{R}$. Let $\varepsilon > 0$ be given. WTS $\exists \delta > 0$ s.t. if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$ Let $\delta = \varepsilon$. Let $x \in \mathbb{R}$ s.t. $|x - c| < \delta$. Then by homework 12 Q4 $|f(x) - f(c)| = ||x| - |c|| \leq |x - c| < \delta = \varepsilon \Rightarrow |f(x) - f(c)| < \varepsilon$. Thus f is continuous at c. Since c was arbitrary we know that f is continuous on $\mathbb{R}$.

$\square$

2

> **Q** : Let $f(x) = x^2$. Use the $\varepsilon - \delta$ definition of continuity to prove that $f(x)$ is continuous at 1.

Let $\varepsilon > 0$. Let $\delta = \min\{\varepsilon/3, 1\}$. Let $x \in \mathbb{R}$ s.t. $|x - 1| < \delta \leq 1$. Then $-1 < x - 1 < 1 \Rightarrow -3 < 1 < x + 1 < 3 \Rightarrow |x + 1| < 3$. Now $|f(x) - f(1)| = |x^2 - 1| = |(x + 1)(x - 1)| = |x + 1||x - 1| < 3|x - 1| < 3\frac{\varepsilon}{3} = \varepsilon$ and so $|f(x) - f(1)| < \varepsilon$. Thus f is continuous at 1.

3

Q : Let $a \in \mathbb{R}$, and let $f(x) = \begin{cases} a & \text{if } x = 0 \\ \frac{|x|}{x} & \text{if } x \neq 0 \end{cases}$
Use sequences to prove that $f(x)$ is discontinuous at 0.

Let $\{x_n\} \subseteq \mathbb{R}_-$ s.t. $x_n \to 0$. Then $\{f(x_n)\} = \{-1, -1, -1, ...\}$, so $f(x_n) \to -1$. Now let $\{y_n\} \subseteq \mathbb{R}_+$ s.t. $y_n \to 0$. Then $\{f(y_n)\} = \{1, 1, 1, ...\}$, so $f(y_n) \to 1$. Since we have two sequences that converge to 0 and when the function f is applied to them the sequences converge to different limits, we know that f is discontinuous at 0.

$\square$

4

Q : Let $f : \mathbb{Z} \to \mathbb{R}$. Prove that f is continuous.

Let $a \in \mathbb{Z}$. Let $\{x_n\} \subseteq \mathbb{Z}$ s.t. $x_n \to a$. Then by HW 9 Q2 we know that $\exists N \in \mathbb{N}$ s.t. if $n > N$ then $x_n = a$. Let $\varepsilon > 0$. Let $n > N$. Then $|x_n - a| = |a - a| = 0 < \varepsilon$. Now $|f(x_n) - f(a)| = |f(a) - f(a)| = 0 < \varepsilon$ and so $f(x_n) \to f(a)$. Thus f is continuous at a. Since a was arbitrary we know that f is continuous.