Math 101 Homework 4

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1

Let a, b be rational numbers s.t. $\sqrt{ab}$ is irrational. WTS: this implies that $\sqrt{a}+\sqrt{b}$ is irrational. We can prove this by contradiction. Assume that $\sqrt{a}+\sqrt{b}$ is rational. $\sqrt{ab} = \sqrt{a}\sqrt{b}$, thus either $\sqrt{a}$ is irrational, $\sqrt{b}$ is irrational, or both are irrational. By our assumption $\sqrt{a} + \sqrt{b} = \frac{j}{k} \in \mathbb{Q}$, where $j, k \in \mathbb{Z}$. $\Rightarrow k \sqrt{a} + k\sqrt{b} = j$. Given that one or both of $\sqrt{a}$, $\sqrt{b}$ is irrational, by homework 0 problem 4, one or both of $k\sqrt{a}$, $k\sqrt{b}$ is irrational. Again by homework 0 problem 4, we know that the sum of an irrational and a rational number is an irrational number. Thus, the sum $k\sqrt{a} + k\sqrt{b}$ is an irrational number. But $k\sqrt{a} + k\sqrt{b} = j$ which is a rational number $\Rightarrow\Leftarrow$.

$\square$

2

We are given that A is a non-empty set of reals and $B = \{-a | a \in A\}$. Suppose A is bounded below. By the Greatest Lower Bound Axiom $\exists x \in A$ s.t. $glb(A) = x$. That is $\forall y \in A$, $x \leq y$. Now let $b \in B, \Rightarrow -b \in A \Rightarrow x \leq -b \Rightarrow -x \geq b$ where b is any element in B. Thus B is bounded above by -x. So, B is bounded above.

$\square$

3

We are given that $A \subseteq \mathbb{R}$ s.t. $A \neq \emptyset$ where $p = lub(A)$ and $B = \{-a|a \in A\}$. WTS: $-p = glb(B)$. In order to show this we must show:

$$1)\ \forall x \in B, x \geq -p$$
$$2)\ \forall - z > -p, \exists - y \in B \text{ s.t. } -y < -z$$

By the previous problem, we know that part 1) is true. Now we must prove that part 2) is true. Given that $p = lub(A)$, we know that if we let $z \in A$ s.t. $z < p$, $\exists y \in A$ s.t. $y > z$. From this we know that $-z \in B$ and that $-z > -p$. But we also know that $-y \in B$ and because $y > z$ we know that $-y < -z$. Thus, $-p = glb(B)$.

$\square$

4

We are given that $X \subseteq \mathbb{R}$ and $X = \emptyset$ s.t. $\forall x \in X$, $a \leq x \leq b$. WTS: there exists a positive real number c s.t. $\forall x \in X$, $|x| \leq c$. Let $c = max(|a|, |b|)$. From this we know that $c \geq |a| \Rightarrow -c \leq -|a| \leq a \Rightarrow -c \leq a$. We also know that $c \geq |b| \geq b \Rightarrow c \geq b$. Combining what we know we have $-c \leq a \leq x \leq b \leq c \Rightarrow -c \leq x \leq c$. This implies that $|x| \leq c$.

$\square$