# Math 101 Homework 6

Jeff Carney

February 3, 2017

## 4.8

(a)

Proof that S is bounded above: Let $s \in S$. By the definition of S and T, $\exists t \in T$ s.t. $s \leq t$. Thus, $t$ is an upper bound for S. Therefore S is bounded above.

Proof that T is bounded below: Let $t \in T$. By the definition of S and T, $\exists s \in S$ s.t. $s \leq t$. Thus, $s$ is a lower bound for T. Therefore T is bounded below.

(b)

WTS lub(S) $\leq$ glb(T). We can prove that this by contradiction. Assume that lub(S) $>$ glb(T). Then we know, by the Density of Rationals, that $\exists m, n \in \mathbb{Z}$ s.t. $\frac{m}{n}$ is between lub(S) and glb(T). Let $m$ be the midpoint between lub(S) and glb(T). Note that by the definition of a greatest lower bound, $\exists t \in$ T s.t. $t < m$. And by the definition of a least upper bound, $\exists s \in$ S s.t. $m < s$. So $t < m < s$, but this violates the definition of S and T $\Rightarrow\Leftarrow$. Thus lub(S) $\leq$ glb(T).

(c)

If S = [1, 5] and T = [5, 10], then the definitions of S and T in our problem are satisfied and S $\cap$ T = 5.

(d)

If S = (1, 5) and T = (5, 10), then the definitions of S and T in our problem are satisfied, lub(S) = glb(T) = 5, and S $\cap$ T = $\emptyset$.

## 4.10

We are given that $a > 0$. WTS $\exists n \in \mathbb{N}$ s.t $\frac{1}{n} < a < n$. Let $n = \frac{a^2+1}{a}$. I claim that $\frac{1}{n} < a < n$, that is $\frac{a}{a^2+1} < a < \frac{a^2+1}{a}$. First we will show that $\frac{a}{a^2+1} < a$:

$$\frac{a}{a^2+1} < a$$

$$a < a^3 + a$$

$$0 < a^3$$

This is true because $a > 0$. Thus, $\frac{a}{a^2+1} < a$. Now we will show the second half of the inequality, $a < \frac{a^2+1}{a}$:

$$a < \frac{a^2+1}{a}$$

$$a^2 < a^2 + 1$$

$$0 < 1$$

This is true. Thus, $a < \frac{a^2+1}{a}$. Combining what we know we have that $\frac{a}{a^2+1} < a < \frac{a^2+1}{a}$ as we desired.

$\square$

## 4.14a

WTS lub(A + B) = lub(A) + lub(B).

First we can show that for each $b \in$ B, lub(A + B) - b is an upper bound for A. This can be proved by contradiction. Assume that $\exists b \in$ B s.t. lub(A+B)-b is not an upper bound for A. Then we know $\exists a \in$ A s.t. $a >$ lub(A+B)-b. Thus, $a + b >$ lub(A + B) $\Rightarrow\Leftarrow$ because $a + b \in A + B$, so it cannot be greater than lub(A + B). Therefore, for each $b \in$ B, lub(A + B) - b is an upper bound for A. Thus for each $b \in$ B, lub(A) $\leq$ lub(A + B) - $b \Rightarrow n \leq$ lub(A + B) - lub(A). Thus, lub(A + B) - lub(A) is an upper bound for B. Therefore, lub(B) $\leq$ lub(A + B) - lub(A) $\Rightarrow$ lub(A) + lub(B) $\leq$ lub(A + B).

We know that $\forall a \in$ A, lub(A) $\geq a$. And $\forall b \in$ B, lub(B) $\geq b$. Thus, $\forall a \in$ A and $b \in$ B, lub(A) + lub(B) $\leq a + b \leq$ lub(A + B). So, lub(A) + lub(B) $\leq$ lub(A + B).

Given that lub(A) + lub(B) $\leq$ lub(A + B) and lub(A) + lub(B) $\leq$ lub(A + B), lub(A) + lub(B) = lub(A + B).

$\square$

## 4.15

We are given $a, b \in \mathbb{R}$. WTS $\forall n \in \mathbb{N}$ if $a \leq b + \frac{1}{n}$, then $a \leq b$. Let $A = \{a\}$, $B = \{b\}$, and C $= \{\frac{1}{n} | n \in \mathbb{N}\}$. We proved in the last homework that glb($C$)=0. There is only one element, $b$, in $B$, thus the glb($B$) = b. By 4.14a, we know that glb($B+C$) = glb($B$) + glb($C$) where $B+C = \{b + \frac{1}{n} | \forall n \in \mathbb{N}\}$. Thus glb($B + C$) = b. There is only one element in $A$, thus lub($A$) = a. Given that $\forall n \in \mathbb{N}$ if $a \leq b + \frac{1}{n}$, we know, by 4.8b, that lub($A$) $\leq$ glb($B$) $\Rightarrow a \leq b$.

$\square$