Math 101 HW 15

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Please grade 10.2, 10.5, and 1

10.2

Q : Prove all bounded decreasing sequences converge (see Ross Theorem 10.2).

Let $\{x_n\}$ be a bounded decreasing sequence. Let X denote the set $\{x_n | n \in \mathbb{N}\}$, and let $l = \text{glb}(X)$. Since X is bounded, l represents a real number. We show that $x_n \to l$. Let $\varepsilon > 0$. Since $l + \varepsilon$ is not a lower bound for X, $\exists N \in \mathbb{N}$ s.t. $x_N < l + \varepsilon$. Since $\{x_n\}$ is decreasing, we have $x_N \geq x_n$ for all $n \geq N$. Since l is the glb(X), $x_n \geq l$ for all n, so $n > N$ implies $l - \varepsilon < l \leq x_n < l + \varepsilon \Rightarrow l - \varepsilon < x_n < l + \varepsilon \Rightarrow |x_n - l| < \varepsilon$. Thus all bounded decreasing sequences converge.

$\square$

10.5

Q : Prove Theorem 10.4(ii): If $\{s_n\}$ is an unbounded decreasing sequence, then $\lim s_n = -\infty$.

Let $\{s_n\}$ be an unbounded decreasing sequence. Let $M < 0$. Since $\{s_n\}$ is unbounded and decreasing $\exists N \in \mathbb{N}$ s.t. $s_N < M$. Let $n > N$. Since $\{s_n\}$ is decreasing $s_n < s_N < M$. Thus, $\forall n > N$ $s_n < M \Rightarrow \lim s_n = -\infty$.

1

Q : Let $l \in \mathbb{R}$. Prove that there exists a sequence $\{x_n\}$ of rationals and a sequence $\{y_n\}$ of irrationals, such that $x_n \to l$ and $y_n \to l$.

(Possibly use $\{x_n\} = \frac{1}{n}$ and $\{y_n\} = \frac{1}{n+\sqrt{2}}$)

Let $\{x_n\} = \frac{1}{n}$ and $\{y_n\} = \frac{1}{n+\sqrt{2}}$. We know that $1 \in \mathbb{Z}$ and $\forall n \in \mathbb{N}$ $n \in \mathbb{Z} \Rightarrow \frac{1}{n} \in \mathbb{Q}$. We can now prove that all terms in $\{y_n\}$ are irrational by contradiction:

<u>Proof that $\{y_n\}$ is a sequence of irrationals:</u>

> Let $n \in \mathbb{N}$ Assume that $\frac{1}{n+\sqrt{2}} \in \mathbb{Q}$. Thus $\exists p, q \in \mathbb{Z}$ where $q \neq 0$ s.t. $\frac{1}{n+\sqrt{2}} = \frac{p}{q}$. Note that $\frac{1}{n+\sqrt{2}} > 0$ thus $p \neq 0$. So $\frac{q}{p} = n + \sqrt{2}$. But we showed in homework 0 that the sum of a rational and irrational is irrational, thus $n + \sqrt{2}$ is irrational $\Rightarrow\Leftarrow$. So $\{y_n\}$ is a sequence of irrationals.

Let $\varepsilon > 0$. By Archimedes $\exists N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$. Let $n > N > \frac{1}{\varepsilon}$. So, $|\frac{1}{n} - 0| = \frac{1}{n} < \varepsilon$. Thus, $\{x_n\} \to 0$. Also since $|\frac{1}{n}| < \varepsilon \Rightarrow -\varepsilon < 0 < \frac{1}{n+\sqrt{2}} < \frac{1}{n} < \varepsilon \Rightarrow -\varepsilon < \frac{1}{n+\sqrt{2}} < \varepsilon$. Hence, $|\frac{1}{n+\sqrt{2}}| < \varepsilon$. $\therefore$ $\{y_n\} \to 0$.

$\square$

2

Q : Let S be a bounded nonempty subset of real numbers and suppose that $\text{lub}(S) \notin S$. Prove that there is an increasing sequence $\{s_n\}$ of points in S such that $\lim s_n = \text{lub}(S)$.

(WTS $|s_n - \text{lub}(S)| < \varepsilon$. Can use $\text{lub}(S) - \varepsilon < s_n \leq \text{lub}(S) < \text{lub}(S) + \varepsilon$, hence $|s_n - \text{lub}(S)| < \varepsilon$.)

Let $\varepsilon > 0$ be given. Since S is a bounded nonempty subset of real numbers and $\text{lub}(S)$ exists we know that $\exists s \in S$ s.t. $\text{lub}(S) - \varepsilon < s \leq \text{lub}(S)$. We proved in class that between any pair of distinct reals there are infinitely many rationals. Since $\text{lub}(S) - \varepsilon < s$ we know that $\text{lub}(S) - \varepsilon$ and s are distinct real numbers, thus there are an infinite number of rationals between $\text{lub}(S) - \varepsilon$

and s. Let $\{s_n\}$ be a sequence made up of the infinite number of rationals between $\text{lub}(S) - \varepsilon$ and s s.t. $\forall n, m \in \mathbb{N}$ where WLOG $n > m$ then $s_n > s_m$. Thus, $\{s_n\}$ is an increasing sequence of points in S. Since $\text{lub}(S) - \varepsilon < s_n \leq \text{lub}(S) <\text{lub}(S) + \varepsilon$ thus $\text{lub}(S) - \varepsilon < s_n <\text{lub}(S) + \varepsilon$. Hence, $|s_n - \text{lub}(S)| < \varepsilon$. $\therefore \lim s_n = \text{lub}(S)$.