Math 101 HW 29

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Definition (for 1 and 2): Let A be a set which is not bounded above, let $f : A \to \mathbb{R}$ and let $l \in \mathbb{R}$. We write $\lim_{x\to\infty} f(x) = l$, if for every $\{x_n\} \subseteq A$ such that $x_n \to \infty$, $f(x_n) \to l$.

1

Q : Let A be a set which is not bounded above, let $f : A \to \mathbb{R}$ and let $l \in \mathbb{R}$. Prove that if $\lim_{x\to\infty} f(x) = l$, then for every $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that if $x \in A$ and $x > M$, then $|f(x) - l| < \varepsilon$.

Let $\lim_{x\to\infty} f(x) = l$. Now suppose that $\exists \varepsilon > 0$ s.t. $\forall M \in \mathbb{R}$, $\exists x \in A$ s.t. $x > M$ but $|f(x) - l| \geq \varepsilon$. Let $M_1 \in \mathbb{R}$, then $\exists x_1 \in A$ s.t. $x_1 > M_1$ and $|f(x_1) - l| \geq \varepsilon$. Continuing this we get a sequence s.t. $\forall n \in \mathbb{N}$ where $M_n \in \mathbb{R}$ we have $x_n > M_n$ and $|f(x_n) - l| \geq \varepsilon$. We have $x_n \to \infty$ but $f(x_n) \nrightarrow l$. But this is a contradiction to our assumption $\lim_{x\to\infty} f(x) = l \Rightarrow\Leftarrow$.

$\square$

2

Q : Let A be a set which is not bounded above, let $f : A \to \mathbb{R}$ and let $l \in \mathbb{R}$. Suppose that for every $\varepsilon > 0$ there is an $M \in \mathbb{R}$ such that if $x \in A$ and $x > M$, then $|f(x) - l| < \varepsilon$. Prove that $\lim_{x\to\infty} f(x) = l$.

Let $\varepsilon > 0$. By our hypothesis $\exists M \in \mathbb{R}$ s.t. if $x \in A$ and $x > M$, then $|f(x) - l| < \varepsilon$. Let $\{x_n\} \subseteq A$ s.t. $x_n \to \infty$. Thus $\forall M > 0$ $\exists N \in N$ s.t. if

$n > N$ then $x_n > M$. Let $M > 0$. Then $\exists N \in \mathbb{N}$ s.t. if $n > N$ then $x_n > M$. Let $n > N$. Since $x_n \in A$ and $x_n > M$ we know that $|f(x_n) - l| < \varepsilon$. Thus $f(x_n) \to l$ and by the above definition we know $\lim_{x\to\infty} f(x) \neq l$.

$\square$

3

> **Q** : Let $f : [a, b] \to \mathbb{R}$ be a function and $c \in (a, b)$. Suppose that f is continuous at c. Prove that there exists $\delta > 0$ such that f is bounded on the interval $[c - \delta, c + \delta]$.

Let $\varepsilon = 1$. Since f is continuous at c we know that $\exists \delta > 0$ s.t. if $x \in [a, b]$ and $|x - c| < \delta$ then $|f(x) - f(c)| < 1$. Let $x \in (c - \delta, c + \delta)$. Then $f(c) - 1 < f(x) < f(c) + 1$. Now let $\alpha = \min\{f(c - \delta), f(c + \delta), f(c) - 1\}$ and let $\beta = \max\{f(c - \delta), f(c + \delta), f(c) + 1\}$. Then $\forall x \in [c - \delta, c + \delta]$ we have $\alpha \leq f(x) \leq \beta$. Thus f is bounded on the interval $[c - \delta, c + \delta]$.

$\square$

5

> **Q** : Let $\{b_n\}$ be a null sequence. Suppose that $\{a_n\}$ is a sequence such that for any $m, n \in \mathbb{N}$, if $m \geq n$ then $|a_m - a_n| \leq |b_n|$. Prove that $\{a_n\}$ is Cauchy.

Let $\varepsilon > 0$. Since $\{b_n\}$ is null we know $\exists N \in \mathbb{N}$ s.t. if $n > N$ then $|b_n| < \varepsilon$. Let $n > N$ and $m > n$. Then we have $|a_m - a_n| \leq |b_n| < \varepsilon$. And so $|a_m - a_n| < \varepsilon$. Thus, $\{a_n\}$ is Cauchy.

$\square$