Math 101 HW 8

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Please grade 4.16, 2, and 3.

4.16

Q : Show lub$\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$

Let $a \in \mathbb{R}$ and A $= \{r \in \mathbb{Q} : r < a\}$. By the definition of A, $\forall r \in$ A, $r < a$. Thus a is an upper bound for A. Now let $z \in \mathbb{R}$ s.t. $z < a$. By the Density of the Rationals we know $\exists q \in \mathbb{Q}$ s.t. $z < q < a$. But if $q \in \mathbb{Q}$ is less than a then it is in the set A. Thus, z cannot be an upper bound for A. Therefore, a = lub(A).

2

Q : Prove if $\{x_n\}$ converges to 0 then $\forall c \in \mathbb{R}$, $\{cx_n\}$ converges to 0.

$\{x_n\}$converges to 0 $\Rightarrow \forall \epsilon > 0 \; \exists N \in \mathbb{N}$ s.t. if $n > N$ then $|x_n - 0| < \epsilon$. If c = 0, $\{cx_N\} = \{0, 0, 0, ...\}$ in which case $\{cx_n\}$ converges to 0. Assume $c \neq 0$, $\Rightarrow |c| > 0$. Let $\epsilon > 0$ be given $\Rightarrow \frac{\epsilon}{|c|} > 0$. $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|x_n| < \frac{\epsilon}{|c|} \Rightarrow \forall n > N$, $|cx_n| < \epsilon$. Thus, $\{cx_n\}$ converges to 0.

3

Q : Prove that $\{x_n\}$ converges to 0 iff for every $\epsilon > 0$, $(-\epsilon, \epsilon)$ contains either all the terms of $\{x_n\}$ or all but finitely many terms of $\{x_n\}$

$(\Rightarrow)$
Assume that $\{x_n\}$ converges to 0. $\Rightarrow \forall\epsilon > 0\ \exists N \in \mathbb{N}$ s.t. if $n > N$ then $|x_n - 0| < \epsilon$. Thus, $\forall\epsilon > 0\ \exists N \in \mathbb{N}$ s.t. if $n > N$ then $|x_n| < \epsilon \Rightarrow -\epsilon < x_n < \epsilon$. If $\{x_n\} = \{0, 0, 0, 0, ...\}$ then $\forall\epsilon > 0$ all of the terms of $\{x_n\}$are contained in $(-\epsilon, \epsilon)$. If $\{x_n\} \neq \{0, 0, 0, 0, ...\}$ then we still know that $\forall\epsilon > 0\ \exists N \in \mathbb{N}$ s.t. if $n > N$ then $-\epsilon < x_n < \epsilon$, which tells us that for all $n > N$ the terms of $\{x_n\}$are contained in $(-\epsilon, \epsilon)$ and the first N (note that N is a finite number) terms of $\{x_n\}$are not in $(-\epsilon, \epsilon)$.

$(\Leftarrow)$
Assume that for every $\epsilon > 0$, $(-\epsilon, \epsilon)$ contains either all the terms of $\{x_n\}$ or all but finitely many terms of $\{x_n\}$. If every term of $\{x_n\}$is contained in the interval $(-\epsilon, \epsilon)$, then $\{x_n\}= \{0, 0, 0, ...\}$. If there are finitely many terms of $\{x_n\}$not in $(-\epsilon, \epsilon)$ then the terms not in the interval must come from the head because the tail is infinite. Thus, $\exists N \in \mathbb{N}$ s.t. $\{x_1, x_2, ..., x_N\}$ are not in the interval $(-\epsilon, \epsilon)$ and so $\forall n > N$, x_n is contained in the interval $(-\epsilon, \epsilon)$, which means that $|x_n| < \epsilon \Rightarrow |x_n - 0| < \epsilon$. Thus, $\{x_n\}$converges to 0.

4

$\boxed{\textbf{Q} : \text{Prove that } \{x_n\} \to l \text{ iff } \{x_n - l\} \text{ is null.}}$

$(\Rightarrow)$
Assume that $\{x_n\}\to l$. $\Rightarrow \forall\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n > N$, then $|x_n - l| < \epsilon \Rightarrow |(x_n - l) - 0| < \epsilon$. Thus, $\{x_n - l\}$ is null.

$(\Leftarrow)$
Assume that $\{x_n - l\}$ is null. $\Rightarrow \forall\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n > N$, then $|(x_n - l) - 0| < \epsilon \Rightarrow |x_n - l| < \epsilon$. Thus, $\{x_n\}\to l$.