Math 101 HW 21

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1

> **Q** : Let $\{x_n\}$ be a sequence of limit points of $\{y_n\}$. Suppose $x_n \to l$. Prove that l is a limit point of $\{y_n\}$.

Let $\varepsilon > 0$. WTS $\forall N \in \mathbb{N}\ \exists n > N$ s.t. $|y_n - l| < \varepsilon$. Since $x_n \to l$ we know $\exists N_1 \in \mathbb{N}$ s.t. $\forall n > N_1\ |x_n - l| < \varepsilon/2$. Since $\{x_n\}$ is a sequence of limit points of $\{y_n\}$ $\forall m \in \mathbb{N}, \forall N \in \mathbb{N}\ \exists n > N$ s.t. $|y_n - x_m| < \varepsilon/2$. Let $m > N_1$. Then let $N \in \mathbb{N}$. We know $\exists n > N$ s.t. $|y_n - x_m| < \varepsilon/2$. So, $|y_n - l| = |y_n - x_m + x_m - l| \leq |y_n - x_m| + |x_m - l|$ by the triangle inequality. And since $m > N_1$ we have $|y_n - l| \leq |y_n - x_m| + |x_m - l| < \varepsilon/2 + \varepsilon/2 = \varepsilon \Rightarrow |y_n - l| < \varepsilon$. Thus, l is a limit point of $\{y_n\}$.

$\square$

2

> **Q** : Let $\{x_n\}$ be a sequence of real numbers such that $|x_n| \nrightarrow \infty$. Prove that $\{x_n\}$ has a limit point. Give an example of a sequence $\{x_n\}$ such that $x_n \nrightarrow \infty$ and $x_n \nrightarrow -\infty$ and $\{x_n\}$ has no limit points.

Since $|x_n| \nrightarrow \infty$ $\exists M_1 > 0$ s.t. $\forall N \in \mathbb{N}\ \exists n > N$ s.t. $|x_n| \leq M_1$. Thus there is no tail of $\{x_n\}$ that is bigger than M_1 which means that 0 or only a finite amount of terms are bigger than M_1. Let A be the set of terms of $\{x_n\}$ that are bigger than M_1. Let $M = \max\{M_1, A\}$. Then $\forall n \in \mathbb{N}\ |x_n| \leq M$. Thus $\{x_n\}$ is bounded. By Bolzano-Wieierstrass $\{x_n\}$ has a convergent subsequence $\Rightarrow \{x_n\}$ has a limit point.

$\square$

3

Q : Let $\{x_n\}$ be a bounded divergent sequence. Prove that $\{x_n\}$ has at least two limit points.

By Bolzano-Wieierstrass $\{x_n\}$ has a convergent subsequence $\Rightarrow$ $\{x_n\}$ has a limit point. Now assume that $\{x_n\}$ has no more than one limit point. That means that $\{x_n\}$ has only one subsequential limit. If xn has only one subsequential limit then $\{x_n\}$ converges. But by the statement of the problem, $\{x_n\}$ is a divergent sequence $\Rightarrow\Leftarrow$. Thus $\{x_n\}$ has at least two limit points.