Math 101 HW 19

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1

> **Q** : Let $\{x_n\}$ be an increasing sequence, and suppose $\{x_{n_k}\}$ is a subsequence which converges to l. Prove that $x_n \to l$.

WTS $x_n \to l$. Let $\varepsilon > 0$. Since $x_{n_k} \to l$ we know $\exists N \in \mathbb{N}$ s.t. if $k > N$, $|x_n - l| < \varepsilon$. Let $k > N$. Then $-\varepsilon < x_{n_k} - l < \varepsilon$. Let $n > n_k$. Then $x_n > x_{n_k}$. So $-\varepsilon < x_{n_k} - l < x_n - l$. We also know that $n_k \to \infty$ and so $\exists j \in \mathbb{N}$ s.t. $n_j > n$. So $x_{n_j} - l > x_n - l$. We know that $j > k > n$ so $x_n - l < \varepsilon$ thus $-\varepsilon < x_{n_k} - l < x_n - l < x_{n_j} - l < \varepsilon$ so $-\varepsilon < x_n - l < \varepsilon$. $\therefore x_n \to l$.

2

> **Q** : Let $a \in \mathbb{R}$. Let $\{x_n\}$ have subsequences $\{x_{n_k}\}$ and $\{x_{m_j}\}$, such that every term of $\{x_n\}$ is either equal to a or is contained in one of these two subsequences. Suppose that both subsequences converge to a. Prove that $x_n \to a$

Let $\varepsilon > 0$. Since $\{x_{n_k}\} \to a$ we know $\exists N_1 \in \mathbb{N}$ s.t. if $k > N_1$ then $|x_{n_k} - l| < \varepsilon$. Since $\{x_{m_j}\} \to a$ we know $\exists N_2 \in \mathbb{N}$ s.t. if $j > N_2$ then $|x_{m_j} - l| < \varepsilon$. Now let $n > \max\{n_{N_1}, m_{N_2}\}$. We know that every term of $\{x_n\}$ is either equal to a or is contained in one of the two subsequences. If x_n is not in either of those two subsequences then $x_n = a$ so $|x_n - a| = 0 < \varepsilon$. If $\exists k$ s.t. $n = n_k > n_{N_1}$, so $k > N_1 \Rightarrow |x_n - a| = |x_{n_k} - a| < \varepsilon$. If $\exists j$ s.t. $m = m_j > m_{N_2}$, so $j > N_2 \Rightarrow |x_m - a| = |x_{m_j} - a| < \varepsilon$. Thus $x_n \to a$.

3

Q : Give examples of sequences and subsequences which satisfy the following. You don't have to prove all claims about your examples.

(a) $\{x_n\}$ is not increasing, but $\{x_n\}$ has an increasing subsequence.

(b) $\{x_n\}$ is unbounded, but $\{x_n\}$ has a bounded subsequence.

(c) A sequence of integers, $\{x_n\}$ which diverges, but which has infinitely many distinct subsequential limits.

(a)
Let $\{x_n\} = \{(-1)^n n\}$. Let $n \in \mathbb{N}$. Either n is even or n is odd.

n is even: Let n be even then $(-1)^n n > 0$, but $n+1$ is odd so $(-1)^{n+1}(n+1) < 0 < (-1)^n n$. Thus $\{x_n\}$ is not increasing.

✓

n is odd: Let n be odd then $(-1)^n n < 0$, but $n-1$ is odd so $(-1)^{n-1}(n-1) > 0 > (-1)^n n$. Thus $\{x_n\}$ is not increasing.

✓

Now let $\{n_k\} = 2k \quad \forall k \in \mathbb{N}$. Note that $\{n_k\} = \{2, 4, 6, 8, ...\}$ is an increasing sequence. Thus, $\{x_{n_k}\}$ is a valid subsequence. We have $\{x_{n_k}\} = \{x_2, x_4, x_6, x_8, ...\} = \{(-1)^2 2, (-1)^4 4, (-1)^6 6, (-1)^8, ...\} = \{2, 4, 6, 8, ...\}$ is an increasing sequence. Thus we have found an example where $\{x_n\}$ is not an increasing sequence but $\{x_{n_k}\}$ is increasing.

(b)
Let $\{x_n\} = \{n + ((-1)^n n)\} = \{0, 4, 0, 8, 0, 12, ...\}$. Clearly this is unbounded. Let $n_k = 2k \quad \forall k \in \mathbb{N}$. Then $\{n_k\} = \{2, 4, 6, 8, ...\}$ is increasing so $\{x_{n_k}\}$ is a valid subsequence. $\{x_{n_k}\} = \{0, 0, 0, ...\}$ which clearly converges to 0. Thus $\{x_n\}$ is unbounded, but has a bounded subsequence.

(c)
Let $\{x_n\} = \{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, ...\}$. By the way this is constructed, there will be a subsequence that converges to every single integer

and there are an infinite number of integers. Thus, this sequence has infinitely many distinct subsequential limits.