Math 101 HW 22

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Please grade 1, 2, and 3

1

Q : Let $\{a_n\}$ be a sequence.
Let $A = \{x \in \mathbb{R} | a_n \geq x \text{ for infinitely many } n \in \mathbb{N}\}$ and
$B = \{x \in \mathbb{R} | a_n \leq x \text{ for infinitely many } n \in \mathbb{N}\}$. Suppose that lub($A$) = glb($B$). Prove that $\{a_n\}$ converges.

Let $a \in A$. Then by definition of A $\exists N_1 \in \mathbb{N}$ s.t. $\forall n > N_1$ $a_n \geq a$. Let $b \in B$. Then by definition of B $\exists N_2 \in \mathbb{N}$ s.t. $\forall n > N_2$ $a_n \leq b$. Let $N = max\{N_1, N_2\}$. Let $n > N$. By definition of lub, we have $a \leq \text{lub}(A) \leq a_n$. And by definition of glb, we have $a_n \leq \text{glb}(B) \leq b$. Combining what we know, we have $\text{lub}(A) \leq a_n \leq \text{glb}(B)$. We know $\text{lub}(A) \rightarrow \text{lub}(A) = \text{glb}(B)$ and $\text{glb}(B) \rightarrow \text{glb}(B) = \text{lub}(A)$. Thus, by squeeze theorem, $a_n \rightarrow \text{glb}(B) = \text{lub}(A)$. So $\{a_n\}$ converges.

2

Q : Let $f : [0, \infty) \rightarrow \mathbb{R}$ be given by $f(x) = \sqrt{x}$. Prove that f is continuous.

Let $c \in [0, \infty)$. And let $\{x_n\} \subseteq [0, \infty)$ s.t. $x_n \rightarrow c$. Let $\varepsilon > 0 \Rightarrow \sqrt{c}\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t. if $n > N$ then $|x_n - c| < \sqrt{c}\varepsilon$. Let $n > N$. So $|x_n - c| < \sqrt{c}\varepsilon \Rightarrow \frac{|x_n - c|}{\sqrt{c}} < \varepsilon$. So, $|\sqrt{x_n} - \sqrt{c}| = |\sqrt{x_n} - \sqrt{c}|\left(\frac{\sqrt{x_n}+\sqrt{c}}{\sqrt{x_n}+\sqrt{c}}\right) = \frac{|x_n - c|}{\sqrt{x_n}+\sqrt{c}} \leq \frac{|x_n - c|}{\sqrt{c}} < \varepsilon$. Thus, $|\sqrt{x_n} - \sqrt{c}| < \varepsilon \Rightarrow \sqrt{x_n} \rightarrow \sqrt{c}$. $\therefore f$ is continuous.

3

Q : Suppose $f(x)$ is a continuous function on $\mathbb{R}$ such that $f(q) = 0$ for all $q \in \mathbb{Q}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

Assume $\exists x \in \mathbb{R}$ s.t. $f(x) \neq 0$. By homework 15 question 1, $\exists$ a sequence of rationals $\{x_n\}$ s.t $x_n \to x$. Since f is continuous $f(x_n) \to f(x) \neq 0$. But since $\{x_n\}$ is a sequence of rationals, $\forall n \in \mathbb{N}$ $f(x_n) = 0$, thus $f(x_n) \to 0$. Thus $\{f(x_n)\}$ has two limits but by uniqueness of limits this is not possible $\Rightarrow\Leftarrow$.

4

Q : Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $\{x_n\}$ be a sequence such that $f(x_n) \to f(l)$. Does it follow that $x_n \to l$? Give a proof or counterexample. Also, give an example of a continuous function $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence $\{x_n\}$ such that $\{f(x_n)\}$ is not Cauchy. You don't have to prove all claims about your example(s).

Let $f : \mathbb{R} \to [0, \infty)$ be given by $f(x) = |x|$. Let $\{x_n\} = \{-1^n\}$. Then $f(x_n) = |x_n| \to 1$ but $x_n \nrightarrow 1$.