Math 101 HW 23

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Please grade 1, 2, and 4

1

Q : Give a proof or a counterexample to each of the following statements. You don't have to prove all claims about your counterexample(s).
(a) Let $f : (a, b) \to \mathbb{R}$. If $|f|$ is continuous, then f is continuous.
(b) Let $f, g : (a, b) \to \mathbb{R}$. If f and g are both discontinuous at a point $p \in (a, b)$, then $f + g$ is discontinuous at p.
(c)Let $f : [a, b] \to \mathbb{R}$ be continuous at on (a, b). Then f has a maximum and a minimum on $[a, b]$.

(a) Let $f : (a, b) \to \mathbb{R}$ be defined by the following:

$$f(x) = \begin{cases} 2 & \text{if } a < x < (a+b)/2 \text{ or } (a+b)/2 < x < b \\ -2 & \text{if } x = (a+b)/2 \end{cases}$$

Then $|f|$ is continuous but f is not.

(b) Let $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ be defined by the following:

$$f(x) = \begin{cases} 0 & \text{if } a < x < (a+b)/2 \text{ or } (a+b)/2 < x < b \\ -2 & \text{if } x = (a+b)/2 \end{cases}$$
$$g(x) = \begin{cases} 0 & \text{if } a < x < (a+b)/2 \text{ or } (a+b)/2 < x < b \\ 2 & \text{if } x = (a+b)/2 \end{cases}$$

Then f and g are both discontinuous at $(a+b)/2 \in (a,b)$ but $f+g$ is continuous at $(a+b)/2$.

(c) Let $f : [-\pi/2, \pi/2] \to \mathbb{R}$ be defined by the following:

$$f(x) = \begin{cases} \tan(x) & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } |x| = \pi/2 \end{cases}$$

Then f is continuous on $(-\pi/2, \pi/2)$. But f does not have a maximum or a minimum on $[-\pi/2, \pi/2]$.

2

> **Q** : Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at c, and suppose that $f(c) > 0$. Prove that there is a $\delta > 0$ s.t. for all x s.t. $|x - c| < \delta$, $f(x) > 0$.

Let $\varepsilon = f(c)$. Since f is continuous at c, $\exists \delta > 0$ s.t. $\forall x \in \mathbb{R}$ s.t. $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon = f(c)$. Let $x \in \mathbb{R}$ s.t. $|x - c| < \delta$. Assume that $f(x) \leq 0$. Then $|f(x) - f(c)| = |f(c) - f(x)| = f(c) - f(x)$ because $f(c) > 0$ and $f(x) \leq 0$ so $f(c) - f(c) > 0$. But since $f(x) \leq 0$, we know that $|f(x) - f(c)| = f(c) - f(x) \geq f(c) \Rightarrow |f(x) - f(c)| \geq f(c)$. But we stated earlier that $|f(x) - f(c)| \geq f(c)$ so $\Rightarrow\Leftarrow$. Thus there is a $\delta > 0$ s.t. for all x s.t. $|x - c| < \delta$, $f(x) > 0$.

3

Q : A function $f : A \to \mathbb{R}$ is **increasing** if $\forall x, y \in A$ with $x \leq y$, then $f(x) \leq f(y)$. A function $f : A \to \mathbb{R}$ is **decreasing** if $\forall x, y \in A$ with $x \leq y$, then $f(x) \geq f(y)$. If f is either increasing or decreasing, then we say f is **monotonic**. Give a proof or a counterexample to each of the following statements. You don't have to prove all claims about your counterexample(s).
(a) Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are both continuous and monotonic, then $f + g$ is monotonic.
(b) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous. If neither f nor g is monotonic then $f + g$ is not monotonic.
(c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is not monotonic. Then f does not have an inverse.

4

Q : Let $a \in \mathbb{R}$, and let $f(x) = \begin{cases} a & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x \neq 0 \end{cases}$

Prove that $f(x)$ is not continuous at 0 and that $f(x)$ is continuous for all $x \neq 0$.

Let $\{x_n\} = \{\frac{1}{n}\}$. Then $x_n \to 0$. Note that $\{f(x_n)\} = n$ which diverges to ∞ and thus $f(x_n) \nrightarrow f(0)$ and so $f(x)$ is not continuous at 0. Now let $c \in \mathbb{R}$ s.t. $c \neq 0$. Let $\{x_n\}$ be a sequence s.t. $x_n \to c$. WTS $f(x_n) = \frac{1}{x_n} \to f(c) = \frac{1}{c}$. By the homework we know that $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $x_n > \frac{|c|}{2} > 0$. Thus $f(x_{n+N}) = \frac{1}{x_{n+N}}$. Since all the terms of $\{f(x_n)\}$ are non-zero, $f(x_{n+N}) = \frac{1}{x_{n+N}} \to \frac{1}{c}$. We also know that $\lim f(x_n) = \lim f(x_{n+N}) = \frac{1}{c} = f(c)$. Thus, $f(x_n) \to f(c)$. And since c was an arbitrary element s.t. $c \neq 0$, we know that $f(x)$ is continuous for all $x \neq 0$.

$\square$