Math 101 Homework 5

Jeff Carney

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4.6

(a) WTS glb(S) $\leq$ lub(S). Let $g =$ glb(S) and $l =$ lub(S). Let $s \in$ S. By the definition of a greatest lower bound we know that $g \leq s$. By the definition of a least upper bound we know that $s \leq l$. Combining what we know we have $g \leq s \leq l \Rightarrow g \leq l$. Thus, glb(S) $\leq$ lub(S).

$\square$

(b) Assume glb(S) = lub(S). Let x = glb(S) = lub(S). From this we can say that S contains only one element, x. This can be proved. We are given that S is nonempty so it must contain at least one element. Let $a \in$ S. glb(S) $\leq a \leq$ lub(S). If glb(S) = lub(S) then glb(S) = a = lub(S). Thus a must be the only element in S.

1

We are given $x > 0$. WTS $\exists n \in \mathbb{N}$ s.t. $n - 1 \leq x < n$. Let A be the set of integers s.t. $\forall a \in A$, $x < a$. By the definition of A we know that it is bounded below by x. We also know by Archimedes Property that $\forall x \in \mathbb{R} \exists n \in \mathbb{N}$ s.t. $n > x$. Thus, we know that A is non-empty. Given that A is a non-empty set of integers that is bounded below, by the Well Ordering Principle, we know that A has a smallest element. Call this smallest element n. By the way A is defined, we know that $x < n$. We also know that $n - 1 \leq x$ because $n - 1 < n$ and n is the smallest integer in the set of integers greater than x. Combining what we know $n - 1 \leq x < n$.

$\square$

2

We are given that a is an upper bound for a set X. WTS $a = lub(X)$ iff for every $\varepsilon > 0$ there is an element of X in $[a - \varepsilon, a]$.
($\Rightarrow$)
Assume $a = lub(X)$. From this we know that $\forall z < a$, $\exists y \in X$ s.t. $y > z$. Let $z = a - \varepsilon$ where $\varepsilon > 0$, so $z < a$. But we know that $\forall z < a$, $\exists y \in X$ s.t. $y > z$, thus $y > a - \varepsilon$. Note that $y < a$ because a is an upper bound for X, therefore y must be in $[a - \varepsilon, a]$.

✓

($\Leftarrow$)
Assume that for every $\varepsilon > 0$ there is an element of X in $[a - \varepsilon, a]$. WTS this implies that any number smaller than a is not an upper bound for X since part 1) of being a least upper bound is satisfied by a being given as an upper bound of X. Let z be any number s.t. $z < a$. z can be expressed as $z = a - \varepsilon$ for some $\varepsilon > 0$. By our assumption, for every $\varepsilon > 0$ there is an element of X in $[a - \varepsilon, a]$. Thus, $\exists y \in X$ s.t. y is in $[a - \varepsilon, a]$. Any such y will be greater than z, therefore an number smaller than a is not an upper bound for X.

□