Math 101 Homework 3

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3.4

Prove (v) and (vii) of Theorem 3.2

(v):

WTS: $0 < 1$

Suppose $0 \geq 1$. Let $a \in \mathbb{R}$ s.t. $a > 0$.
$a \cdot 0 \geq a \cdot 1 \Rightarrow 0 \geq a$
$\Rightarrow\Leftarrow$ because we assumed $a > 0$

$\square$

(vii):

WTS: If $0 < a < b$, then $0 < b^{-1} < a^{-1}$ for some $a, b \in \mathbb{R}$.
Let $a, b \in \mathbb{R}$ s.t. $0 < a < b$.
Note that $0 < a \Rightarrow 0 < a^{-1}$ and $0 < b \Rightarrow 0 < b^{-1}$
$\Rightarrow 0 \cdot a^{-1} < a \cdot a^{-1} < b \cdot a^{-1}$
$\Rightarrow 0 < 1 < b \cdot a^{-1}$
$\Rightarrow b^{-1} \cdot 0 < b^{-1} \cdot 1 < b^{-1} \cdot b \cdot a^{-1}$
$\Rightarrow 0 < b^{-1} < a^{-1}$

$\square$

3.6

(a)

WTS: $|a + b + c| \leq |a| + |b| + |c|$ for all $a, b, c \in \mathbb{R}$.

Let $a, b, c \in \mathbb{R}$.
$|a + b| \leq |a| + |b|$ by the triangle inequality.
$|c| \geq 0 \Rightarrow |a + b| + |c| \leq |a| + |b| + |c|$
$(a + b) \in \mathbb{R} \Rightarrow |(a + b) + c| \leq |a + b| + |c|$
$\Rightarrow |a + b + c| \leq |a| + |b| + |c|$

$\square$

(b)

Let $p(n)$ be the inequality $|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$

<u>Base Case:</u> $n_0 = 2$

Let $a_1, a_2 \in \mathbb{R}$.

By the triangle inequality: $|a_1 + a_2| \leq |a_1| + |a_2|$

$\checkmark$

<u>Inductive Step:</u>

Suppose $p(n)$ is true for some n

Let $a_1, ..., a_{n+1} \in \mathbb{R}$.

WTS: $|a_1 + a_2 + ... + a_n + a_{n+1}| \leq |a_1| + |a_2| + ... + |a_n| + |a_{n+1}|$

$|a_1+a_2+...+a_n| \leq |a_1|+|a_2|+...+|a_n|$ by the inductive hypothesis.
$|a_{n+1}| \geq 0 \Rightarrow |a_1 + a_2 + ... + a_n| + |a_{n+1}| \leq |a_1| + |a_2| + ... + |a_n| + |a_{n+1}|$

$(a_1 + ... + a_n) \in \mathbb{R}$
$\Rightarrow |(a_1 + a_2 + ... + a_n) + a_{n+1}| \leq |(a_1 + a_2 + ... + a_n)| + |a_{n+1}|$ by the triangle inequality.

$\Rightarrow |a_1 + a_2 + ... + a_n + a_{n+1}| \leq |a_1| + |a_2| + ... + |a_n| + |a_{n+1}|$

$\square$

3.7

(a)

WTS: $|b| < a$ if and only if $-a < b < a$

$(\Rightarrow)$

Let $a, b \in \mathbb{R}$, s.t. $|b| < a$.
Either $b < 0$ or $b \geq 0$.

<u>Case 1:</u> Assume $b < 0$. Then $|b| = -b$.

$\Rightarrow -b < a$
$\Rightarrow b > -a$

$-a < b < 0 \Rightarrow -a < 0 \Rightarrow a > 0$
$b < 0 \Rightarrow b < a$

$\Rightarrow -a < b < a$

Case 2: Assume $b \geq 0$. Then $|b| = b$.

$\Rightarrow b < a$

$0 \leq b < a \Rightarrow 0 < a \Rightarrow 0 > -a$
$\Rightarrow -a < 0 \leq b \Rightarrow -a < b$

$-a < b < a$

✓

($\Leftarrow$)

Let $a, b \in \mathbb{R}$ s.t. $-a < b < a$.
Either $b < 0$ or $b \geq 0$.

Case 1: Assume that $b < 0$. Then $|b| = -b$.

We are given that $-a < b < a$
$\Rightarrow a > -b > -a$
$\Rightarrow a > |b| > -a$
$\Rightarrow a > |b|$

✓

Case 2: Assume that $b \geq 0$. Then $|b| = b$.

We are given that $-a < b < a$.
$\Rightarrow -a < |b| < a$
$\Rightarrow |b| < a$

✓

Thus for both cases $|b| < a$.

$\square$

(b)

WTS: $|a - b| < c$ if and only if $b - c < a < b + c$

($\Rightarrow$)

Let $a, b, c \in \mathbb{R}$ s.t. $|a-b| < c$. Either $(a-b) < 0$ or $(a-b) \geq 0$.

Case 1: Assume that $(a - b) < 0$. Then $|a - b| = -(a - b)$.

$|a - b| < c$.
$\Rightarrow -(a - b) < c$
$\Rightarrow -a + b < c$
$\Rightarrow -a < -b + c$
$\Rightarrow a > b - c$

$|a - b| \leq 0$ and $|a - b| < c$
$\Rightarrow 0 < c$
We assumed that $a - b < 0$ in this case.
$a - b < 0 < c$

$\Rightarrow a - b < c$
$\Rightarrow a < b + c$

Thus, $b - c < a < b + c$

Case 2: Assume that $(a - b) \geq 0$. Then $|a - b| = a - b$.
$|a - b| < c$
$\Rightarrow a - b < c$
$\Rightarrow a < b + c$

From part (a) we know that if $|a-b| < c$ then
$-c < a - b < c$.
$\Rightarrow -c < a - b$
$\Rightarrow b - c < a$

$\Rightarrow b - c < a < b + c$

✓

($\Leftarrow$)

Let $a, b, c \in \mathbb{R}$ s.t. $b - c < a < b + c$.
$\Rightarrow -c < a - b < c$
Either $(a - b) < 0$ or $(a - b) \geq 0$.

Case 1: Assume that $(a - b) < 0$. Then $|a - b| = -(a - b)$
$-c < a - b < c$
$\Rightarrow c > -(a - b) > -c$
$\Rightarrow c > |a - b| > -c$
$\Rightarrow c > |a - b|$

Case 2: Assume that $(a - b) \geq 0$. Then $|a - b| = a - b$.
$-c < a - b < c$
$\Rightarrow -c < |a - b| < c$
$\Rightarrow |a - b| < c$
Thus for either case $|a - b| < c$

$\square$

(c)

WTS: $|a - b| \leq c$ if and only if $b - c \leq a \leq b + c$

($\Rightarrow$)

Let $a, b, c \in \mathbb{R}$ s.t. $|a-b| \leq c$. Either $(a-b) < 0$ or $(a-b) \geq 0$.

Case 1: Assume that $(a - b) < 0$. Then $|a - b| = -(a - b)$.

$|a-b| \leq c.$
$\Rightarrow -(a-b) \leq c$
$\Rightarrow a-b \geq -c$
$\Rightarrow a \geq b-c$

$|a-b| \leq 0$ and $|a-b| \leq c$
$\Rightarrow 0 \leq c$
We assumed that $a-b<0$ in this case.
$a-b<0 \leq c$
$\Rightarrow a-b \leq c$
$\Rightarrow a \leq b+c$

Thus, $b-c \leq a \leq b+c$

Case 2: Assume that $(a-b) \geq 0$. Then $|a-b| = a-b$.
$|a-b| \leq c$
$\Rightarrow a-b \leq c$
$\Rightarrow a \leq b+c$

From part (a) we know that if $|a-b| \leq c$ then
$-c \leq a-b \leq c.$
$\Rightarrow -c \leq a-b$
$\Rightarrow b-c \leq a$

$\Rightarrow b-c \leq a \leq b+c$

✓

$(\Leftarrow)$

Let $a, b, c \in \mathbb{R}$ s.t. $b-c \leq a \leq b+c.$
$\Rightarrow -c \leq a-b \leq c$
Either $(a-b) < 0$ or $(a-b) \geq 0$.

Case 1: Assume that $(a-b) < 0$. Then $|a-b| = -(a-b)$
$-c \leq a-b \leq c$
$\Rightarrow c \geq -(a-b) \geq -c$
$\Rightarrow c \geq |a-b| \geq -c$
$\Rightarrow c \geq |a-b|$

Case 2: Assume that $(a-b) \geq 0$. Then $|a-b| = a-b$.
$-c \leq a-b \leq c$
$\Rightarrow -c \leq |a-b| \leq c$
$\Rightarrow |a-b| \leq c$
Thus for either case $|a-b| \leq c$

$\square$

3.8

$a, b \in \mathbb{R}$ are given.
WTS: If $a \leq b_1 \forall b_1 > b$, then $a \leq b$.

Original Statement: If $\forall b_1 > b$, $a \leq b_1$, then $a \leq b$

Contrapositive: If $b < a$, then $\exists b_1 > b$ s.t. $a > b_1$

Proof by Contrapositve:

Assume $b < a$ and let $b_1 = \frac{a+b}{2}$.
WTS: $b < b_1 < a$

By assumption $b < a \Rightarrow a + b < a + a$
$\Rightarrow a + b < 2a$
$\Rightarrow \frac{a+b}{2} < \frac{2a}{s}$
$\Rightarrow \frac{a+b}{2} < a$
$\Rightarrow b_1 < a$

Again by assumption $b < a \Rightarrow b + b < a + b$
$\Rightarrow 2b < a + b$
$\Rightarrow \frac{2b}{2} < \frac{a+b}{2}$
$\Rightarrow b < \frac{a+b}{2}$
$\Rightarrow b < b_1$

$\Rightarrow b < b_1 < a$

$\square$