Math101 Homework 1

Jeff Carney

January 23, 2017

1

Let $p(n)$ be the statement: if $n \in N$ functions $f_1, f_2, ..., f_n$ from the reals to the reals which are one-to-one and onto, then the composition $f_n \circ f_{n-1} \circ ... \circ f_1$ is one-to-one and onto.

Base Case: $n_0 = 2$

$p(n_0) =$ if 2 functions f_1, f_2 from the reals to the reals which are one-to-one and onto, then the composition $f_2 \circ f_1$ is one-to-one and onto.

Let f_1, f_2 be any two one-to-one, onto functions from the reals to the reals

Onto:

Let $x \in R$
f_2 is onto so $\exists y \in R$ s.t. $f_2(y) = x$
f_1 is onto so $\exists z \in R$ s.t. $f_1(z) = y$

$f_2(f_1(z)) = f_2(y) = x$, $\forall x \in R$
Therefore, $f_2 \circ f_1$ is onto ✓

One-to-One:

Let $x, y \in R$ and assume $f_2(f_1(x)) = f_2(f_1(y))$
f_2 is one-to-one so $f_1(x) = f_1(y)$
f_1 is one-to-one so $x = y$

Thus for any $x, y \in R$ if $f_2(f_1(x)) = f_2(f_1(y))$, then $x = y$
Therefore, $f_2 \circ f_1$ is one-to-one ✓

Inductive Step:

Suppose $p(n)$ is true for some n
WTS: $\forall$ sets of $n + 1 \in N$ functions $f_1, f_2, ..., f_{n+1}$ from reals to reals which are one-to-one and onto, then $f_{n+1} \circ f_n \circ ... \circ f_1$ is one-to-one and onto

Let $f_1, f_2, ..., f_{n+1}$ be any functions from reals to reals which are one-to-one and onto.

$f_{n+1} \circ f_n \circ f_{n-1} \circ ... \circ f_1 = f_{n+1} \circ (f_n \circ f_{n-1} \circ ... \circ f_1)$

$(f_n \circ f_{n-1} \circ ... \circ f_1)$ is one-to-one and onto by our inductive hypothesis. Thus $f_n n + 1 \circ (f_n \circ f_{n-1} \circ ... \circ f_1)$ is the composition of two one-to-one and onto functions. Therefore, by our base case $f_n n + 1 \circ (f_n \circ f_{n-1} \circ ... \circ f_1)$ is one-to-one and onto.

$\square$

2

Let $f_1, f_2, ..., f_n$ be any invertible functions from reals to reals and let $f_1^{-1}, f_2^{-1}, ..., f_n^{-1}$ be there respective inverses. And let $x \in R$

$f_1 \circ ... \circ f_{n-1} \circ f_n \circ f_n^{-1} \circ f_{n-1}^{-1} \circ ... \circ f_1^{-1}(x)$

$= (f_1 \circ ... \circ f_{n-1}) \circ f_n \circ f_n^{-1}(f_{n-1}^{-1} \circ ... \circ f_1^{-1}(x)$

$= (f_1 \circ ... \circ f_{n-2}) \circ f_{n-1} \circ f_{n-1}^{-1}(f_{n-2}^{-1} \circ ... \circ f_1^{-1}(x)$

$\vdots$

$= f_1 \circ f_1^{-1}(x)$

$= x$

Therefore, the inverse of the function $f_1 \circ ... \circ f_{n-1} \circ f_n$ is the function $f_n^{-1} \circ f_{n-1}^{-1} \circ ... \circ f_1^{-1}$

$\square$

3

We are given $A \subseteq R$

Let $p(n) =$ if $B_1, ..., B_n \subseteq R$ then $A \cap (B_1 \cup ... \cup B_n) = (A \cap B_1) \cup ... \cup (A \cap B_n)$

Base Case: $n_0 = 2$

Let $B_1, B_2 \subseteq R$

WTS: $A \cap (B_1 \cup B_2) = (A \cap B_1) \cup (A \cap B_1)$

To show this we will show that $A \cap (B_1 \cup B_2) \subseteq (A \cap B_1) \cup (A \cap B_1)$ and $A \cap (B_1 \cup B_2) \supseteq (A \cap B_1) \cup (A \cap B_1)$

$A \cap (B_1 \cup B_2) \supseteq (A \cap B_1) \cup (A \cap B_1)$:

Let $x \in (A \cap B_1) \cup (A \cap B_1)$

$(A \cap B_1) \subseteq B_1$ and $(A \cap B_2) \subseteq B_2 \Rightarrow x \in (B_1 \cup B_2)$

$(A \cap B_1) \subseteq A$ and $(A \cap B_2) \subseteq A \Rightarrow x \in A$
Thus, $x \in A \cap (B_1 \cup B_2)$

✓

$\underline{A \cap (B_1 \cup B_2) \subseteq (A \cap B_1) \cup (A \cap B_1) :}$

Let $x \in A \cap (B_1 \cup B_2)$, so $x \in A, x \in (B_1 \cup B_2)$.

Either $x \in B_1$ or $x \in B_2$
If $x \in B_1 \Rightarrow x \in (A \cup B_1)$
If $x \in B_2 \Rightarrow x \in (A \cup B_2)$
So, $x \in (A \cup B_1)$ or $x \in (A \cup B_2)$
$x \in (A \cup B_1) \cup (A \cup B_2)$

✓

Inductive Step :
Suppose p(n) is true for some n

Let $B_1, ..., B_{n+1} \subseteq R$

WTS: $A \cap (B_1 \cup ... \cup B_{n+1}) = (A \cap B_1) \cup ... \cup (A \cap B_{n+1})$

By the inductive hypothesis: $(A \cap B_1) \cup ... \cup (A \cap B_n) = A \cap (B_1 \cup ... \cup B_n)$.
Thus, $(A \cap B_1) \cup ... \cup (A \cap B_n) \cup (A \cap B_{n+1}) = ((A \cap (B_1 \cup ... \cup B_n)) \cup (A \cap B_{n+1})$
$= A \cap (B_1 \cup ... \cup B_{n+1})$ by our base case

□

4

Let $p(n)$ = a set of n real numbers has a smallest element

Base Case: $n_0 = 2$
Let $\{a, b\} \subseteq R$. By definition of a set $a \neq b$.

Case 1: $a < b$
$\Rightarrow$ a is the smallest element in the set

✓

Case 2: $a > b$
$\Rightarrow$ b is the smallest element in the set

✓

So there must be a smallest element in a set of two real numbers

✓

Inductive Step:

Suppose $p(n)$ is true for some n.

WTS: a set of n+1 real numbers has a smallest element

Let $A = x_1, ..., x_{n+1} \subseteq R$. Let $x_a \in x_1, ..., x_n \subseteq A$ s.t. $1 \leq a \leq n$ where x_a is the smallest element in the set $x_1, ..., x_n$. Note that x_a exists by the inductive hypothesis.

In the set A, there are two possible cases:
Case 1: $x_a < x_{n+1} \Rightarrow x_a$ is the smallest element in A
Case 1: $x_a > x_{n+1} \Rightarrow x_{n+1}$ is the smallest element in A
In either case there is a smallest element.

□

5

Claim: The product of any finite number of odd numbers is odd.
Let $p(n) =$ product of n odd numbers is odd

Base Case: $n_0 = 2$

$p(n_0) =$ product of 2 odd numbers is odd

Let l, m be any odd numbers $\therefore \exists k, j \in Z$ s.t. $l = 2k + 1$ and $m = 2j + 1$

$l \cdot m = (2k + 1)(2l + 1)$
$= 4kj + 2k + 2j + 1$
$= 2(2kj + k + j) + 1$

(2kj + k + j) is an integer so, by the definition of an odd number, $l \cdot m$ is odd. ✓

Inductive Step:

Suppose $p(n)$ is true for some n

Let $m_1, m_2, ..., m_{n+1}$ be odd numbers.
$m_1 \cdot m_2 \cdot ... \cdot m_n$ is an odd number by the inductive hypothesis.
Thus, $(m_1 \cdot m_2 \cdot ... \cdot m_n) \cdot m_{n+1}$ is a product of two odd numbers, which, by our base case, is odd.

□