Math101 Homework 2

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1

It is given that $a > 0$ and $-a < x < a$. WTS: $x^2 < a^2$

Case 1: $x > 0$
$-a < x < a \Rightarrow -a^2 < ax < a^2$
$-a < x < a \Rightarrow -ax < x^2 < ax$
$\Rightarrow x^2 < ax < a^2$
$\Rightarrow x^2 < a^2$

✓

Case 2: $x < 0$
$-a < x < a \Rightarrow -ax > x^2 > ax$
$-a < x < a \Rightarrow -a^2 < ax < a^2 \Rightarrow a^2 > -ax > -a^2$
$\Rightarrow a^2 > -ax > x^2$
$\Rightarrow a^2 > x^2$

✓

Case 3: $x = 0$
$0^2 = 0, a > 0 \Rightarrow a^2 > 0$
$a^2 > 0^2$
$\therefore a^2 > x^2$

$\square$

2

Let $a > 0$ and suppose that $x^2 < a^2$.
WTS: $-a < x < a$

Suppose $x \leq a$:
$ax \geq a^2$
$x^2 \geq ax$

$\Rightarrow a^2 \geq x^2$
$\Rightarrow\Leftarrow$ because we assumed that $x^2 < a^2$

Suppose $x \leq -a$:
$x^2 \geq -ax$ and $-ax \geq a^2$
$\Rightarrow a^2 \geq x^2$
$\Rightarrow\Leftarrow$ because we assumed that $x^2 < a^2$

$\square$

3

Let $p(n)$ = a natural number n has the form $3k$, $3k+1$, or $3k+2$ for some non-negative integer k.

Base Case: $n_0 = 1$
Let $k = 0$. $1 = 3k + 1$

$\checkmark$

Inductive Step:
Suppose $p(n)$ is true for some $n \in N$.

WTS: the natural number n+1 has the form $3k$, $3k+1$, or $3k+2$ for some non-negative integer k.

Case 1: $n = 3k$
$\Rightarrow n + 1 = 3k + 1$

$\checkmark$

Case 2: $n = 3k + 1$
$\Rightarrow n + 1 = 3k + 2$

$\checkmark$

Case 3: $n = 3k + 2$
$\Rightarrow n + 1 = 3k + 3$
$\Rightarrow n + 1 = 3(k + 1)$ and $(k + 1) \in Z_+$

$\square$

4

Prove that $\sqrt{3}$ is irrational

This can be proved by contradiction. Suppose $\sqrt{3}$ is rational. By definition of $\sqrt{x}$, $\sqrt{3} \geq 0$ and we know $\sqrt{3} > 0$. Thus, $\exists p, q \in N$ s.t. $\sqrt{3} = \frac{p}{q}$ and p and q have no common factors (otherwise we could cancel the common factor). Now $3q^2 = p^2$.

Case 1: q is even
$\Rightarrow q^2$ is even
$\Rightarrow p^2 = 3 \cdot \text{even} \Rightarrow p^2$ is even $\Rightarrow p$ is even

If p, q are both even then they at least have a common factor of 2. So $\frac{p}{q}$ can be reduced further $\Rightarrow\Leftarrow$ because $\frac{p}{q}$ should be fully reduced.

✓

Case 2: q is odd
$\Rightarrow q^2$ is odd $\Rightarrow p^2 = 3 \cdot$ odd $\Rightarrow p^2$ is odd $\Rightarrow p$ is odd.

Let $k, j \in Z$ s.t. $p = 2k + 1$ and $q = 2j + 1$.

$3q^2 = p^2$
$3(2k + 1)^2 = (2j + 1)^2$
$3(4k^2 + 4k + 1) = 4j^2 + 4j + 1$
$12k^2 + 12k + 3 = 4j^2 + 4j + 1$
$12k^2 + 12k + 2 = 4j^2 + 4j$
$2(6k^2 + 6k + 1) = 2(2j^2 + 2j)$
$6k^2 + 6k + 1 = 2j^2 + 2j$
$6k^2 + 6k + 1 = 2(j^2 + j)$ call this equation (*)

Note $6k^2 + 6k + 1 = 2(3k^2) + 2(3k) + 1$ has the form even+even+odd = odd.
$2(j^2 + j)$ is even.

Thus the equation (*) has the form odd = even $\Rightarrow\Leftarrow$.

$\square$