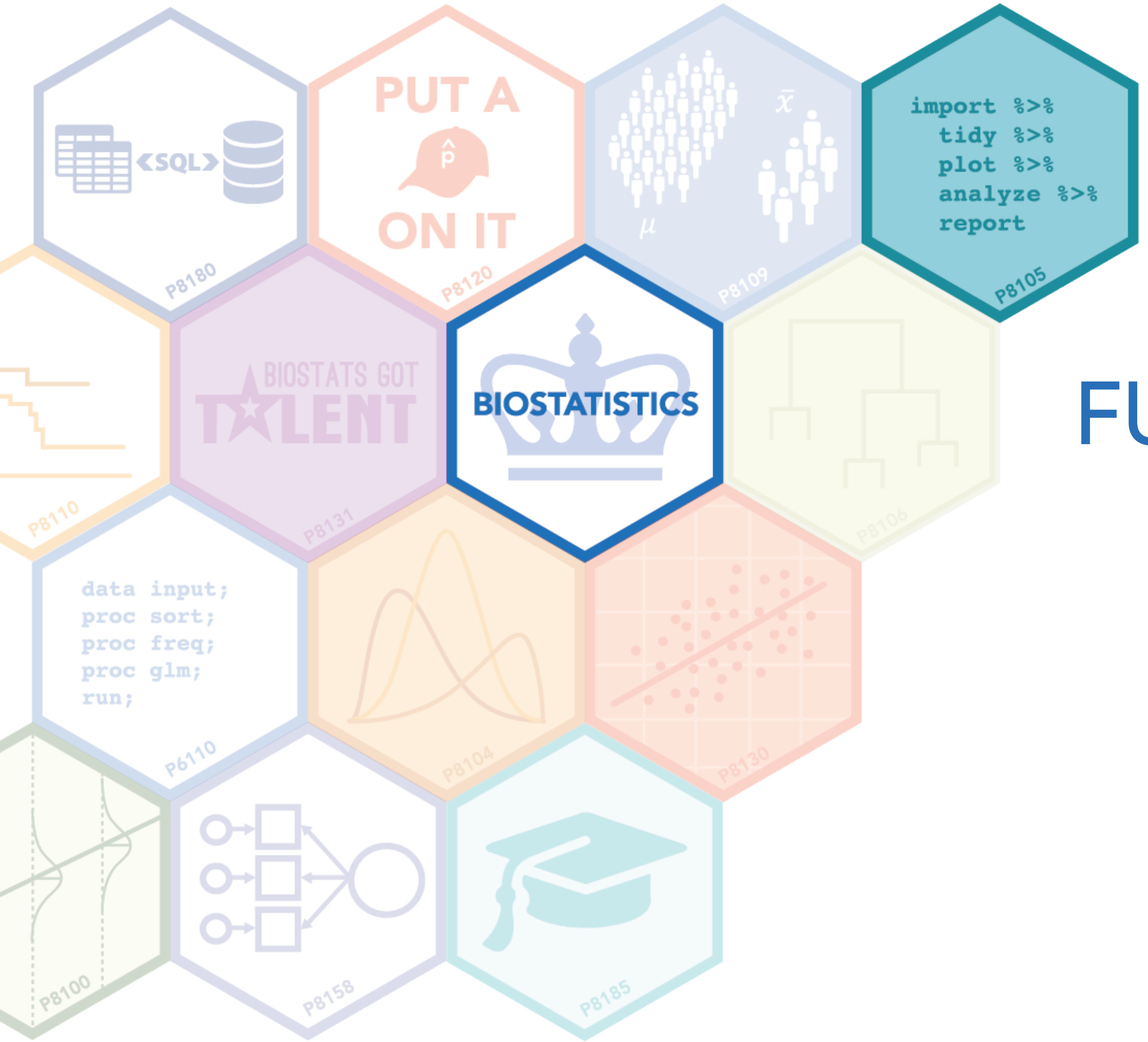

FUNCTION-ON-SCALAR MODELS

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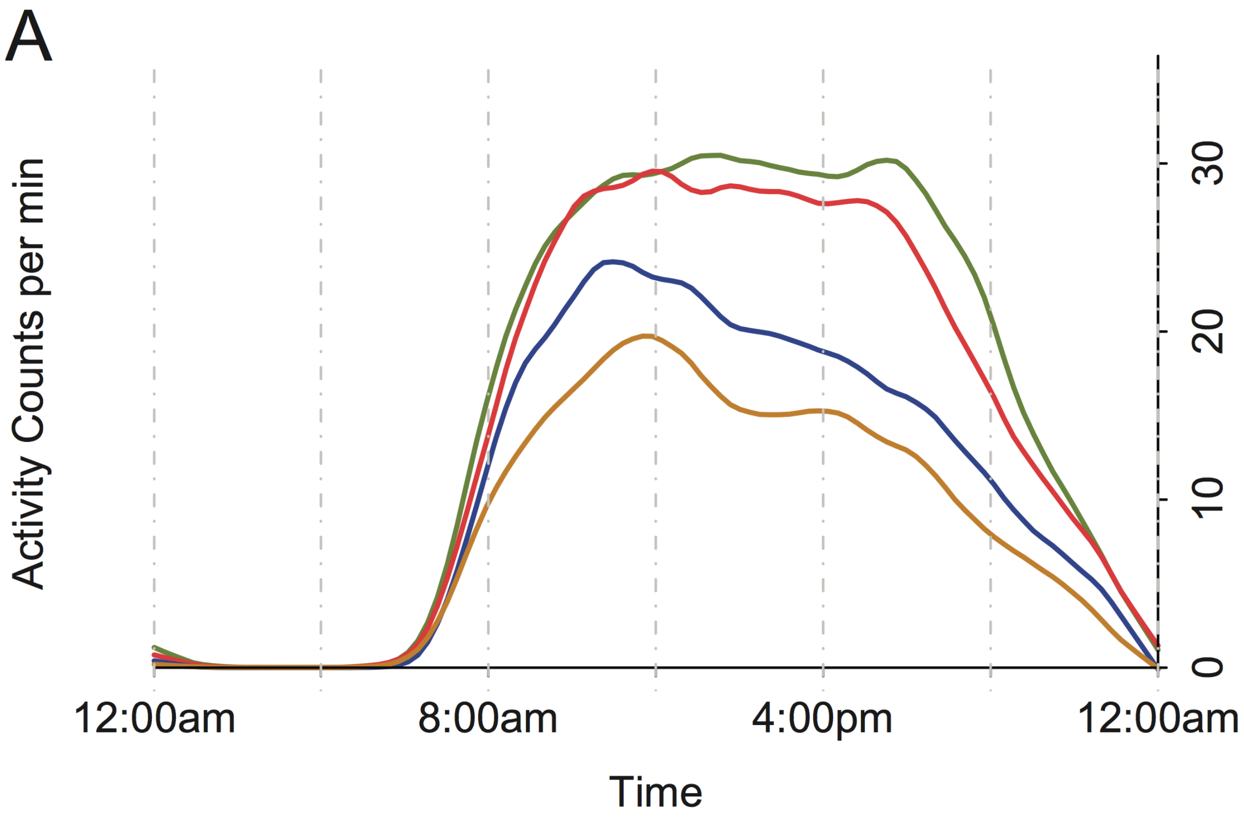
Motivation



J.A. Schrack, V. Zipunnikov, J. Goldsmith, J. Bai, E. M. Simonsick, C. M. Crainiceanu, L. Ferrucci (2014). Assessing the "Physical Cliff": Detailed Quantification of Aging and Physical Activity. Journal of Gerontology: Medical Sciences, 69 973-979.

Questions to address

- Association?
- Confounding?
- Significance?

Functions as outcomes

- Activity may depend on covariates
- Associations may be different at different times of day
- Shift from functions as predictors to functions as outcomes
- “Function-on-scalar” regression

Simple linear regression

- The function-on-scalar model that is analogous to simple linear regression is

$$y_i(t) = \beta_0(t) + \beta_1(t)x_i + \epsilon_i(t)$$

- Functional response y_i
- Scalar predictor x_i
- Functional covariate is of interest
- Linear model
  – Most common approach

Linear FoSR

- The MLR equivalent is

$$y_i(t) = \beta_0(t) + \sum_{l=1}^{p} x_{il}\beta_l(t) + \epsilon_i(t)$$

- Functional response y_i
- Scalar predictor x_i
- Functional covariates are of interest
- Linear model
  - Most common approach

Basis expansion

- The functional coefficients are usually expanded in terms of a basis:

$$\beta_l(t) \approx \sum_{k=1}^{K} \phi_k(t)\beta_{kl}$$

- Several basis options are possible
  - –FPC
  - –Splines (my preference)
  - –Wavelets
  - –Fourier

Basis expansion

- For response data on a common finite grid, the model can be expressed

$$Y = XB\Phi^T + E$$

- Y is the matrix of row-stacked responses
- X is the usual design matrix
- Φ is the matrix of basis functions evaluated over the common grid
- B is the matrix of basis coefficients
- E is the matrix of row-stacked errors

Recast model

- By vectorizing the response and the linear predictor, we obtain the equivalent model formulation

$$vec(Y^T) = (X \otimes \Phi)vec(B^T) + vec(E^T)$$

- vec() concatenates the columns of the matrix argument
- $\otimes$ is the kronecker product
- This reformulates function-on-scalar regression as a usual least-squares problem
- Goal is to estimate the columns of B or, equivalently, the elements of vec(B)

Correlated errors

- Errors $\epsilon_i(t)$ are correlated within a subject, in a way that's complex

- Three possible approaches:
  - Ignore this issue
  - Use GLS in place of OLS by "pre-whitening" the left and right side of the matrix formulation of the model:
    - I.e. define $Y^* = Y(L^{-1})^T$ where $\Sigma = LL^T$ is the error covariance matrix, and similarly modify the RHS
  - Jointly model the coefficient vector and the residual covariance
    - Easiest in a Bayesian setting

Smoothness constraints

- The preceding does not include smoothness constraints on estimated coefficients
- Such constraints often take the form of a penalty

$$\lambda_l \int [\beta_l(t)'']^2 dt$$

- Can be expressed in terms of a penalty on the basis coefficients
- Alternatively, one can be careful about the dimension of the basis

Testing

- “Global” test: under the null, there is no association between the predictor and outcome at any time

$$H_0 : \beta_k(t) = 0 \text{ for all } t$$

- “Local” test: under the null, there is no association between the predictor and outcome at time t

$$H_0 : \beta_k(t) = 0 \text{ for a specific } t$$

- Both can be implemented by focusing on basis coefficients
- The former avoids some multiple comparisons issues
- Analogous to F and t tests in MLR

Connection to MLR

- Integrating over t for each subject gives the average activity for a subject

$$\int y_i(t)\,dt = \bar{y}_i$$

- Can similarly average over coefficients in FoSR to approximate estimates in MLR

$$\int \beta_0(t)\,dt = \bar{\beta}_0 \text{ and } \int \beta_1(t)\,dt = \bar{\beta}_1$$

- Will differ slightly from an MLR fit to average activity counts
- Illustrates the connection (and difference) between approaches

Switch to code

Code

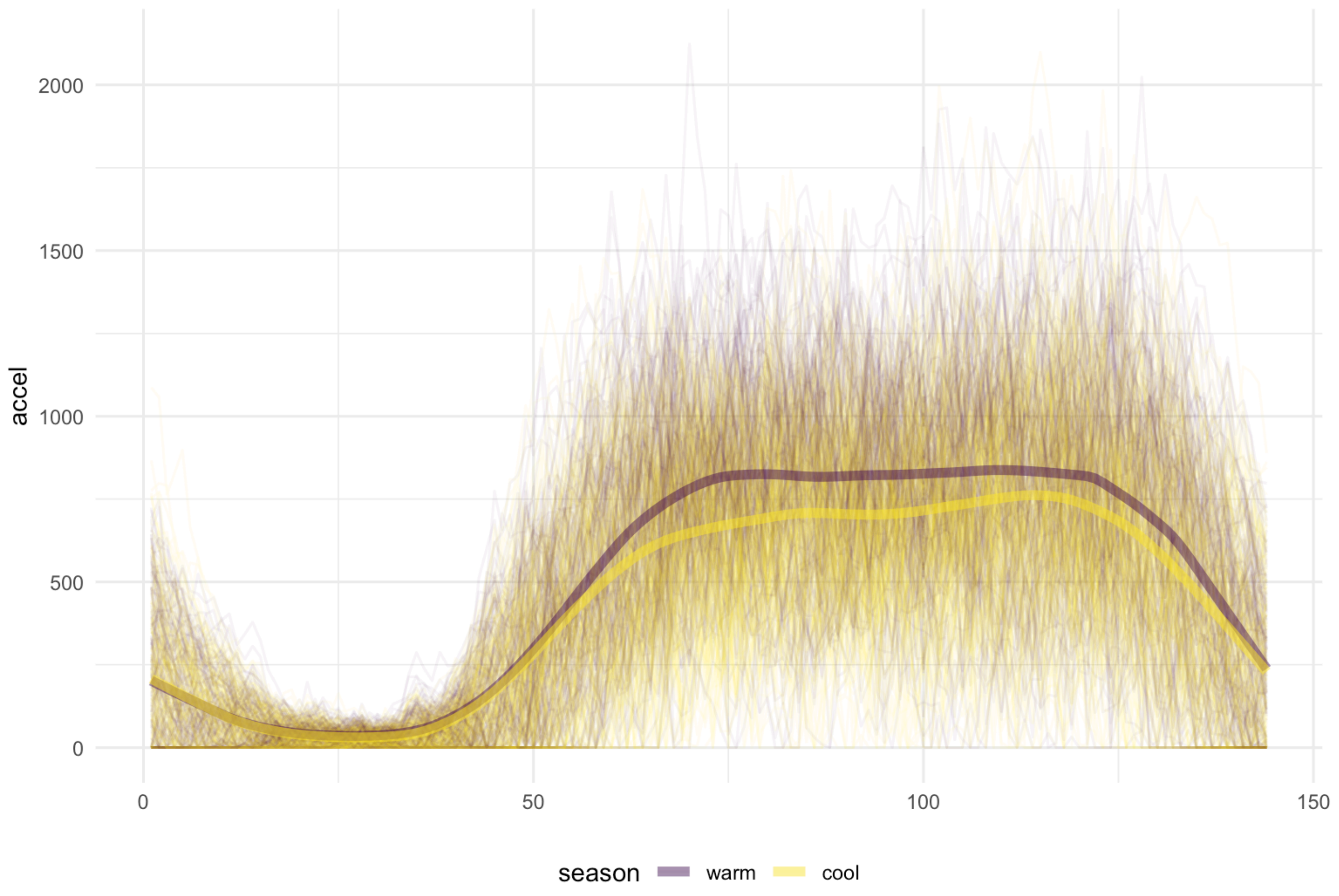
```
load("./DataCode/HeadStart.RDA")

covariate_data =
  covariate_data %>%
  mutate(accel = tfd(accel))

mean_df =
  covariate_data %>%
  group_by(season) %>%
  summarize(mean_act = mean(accel)) %>%
  mutate(smooth_mean = tf_smooth(mean_act))
## using f = 0.15 as smoother span for lowess

covariate_data %>%
  ggplot(aes(y = accel, color = season)) +
  geom_spaghetti(alpha = .05) +
  geom_spaghetti(data = mean_df, aes(y = smooth_mean), size = 2)
```

Code

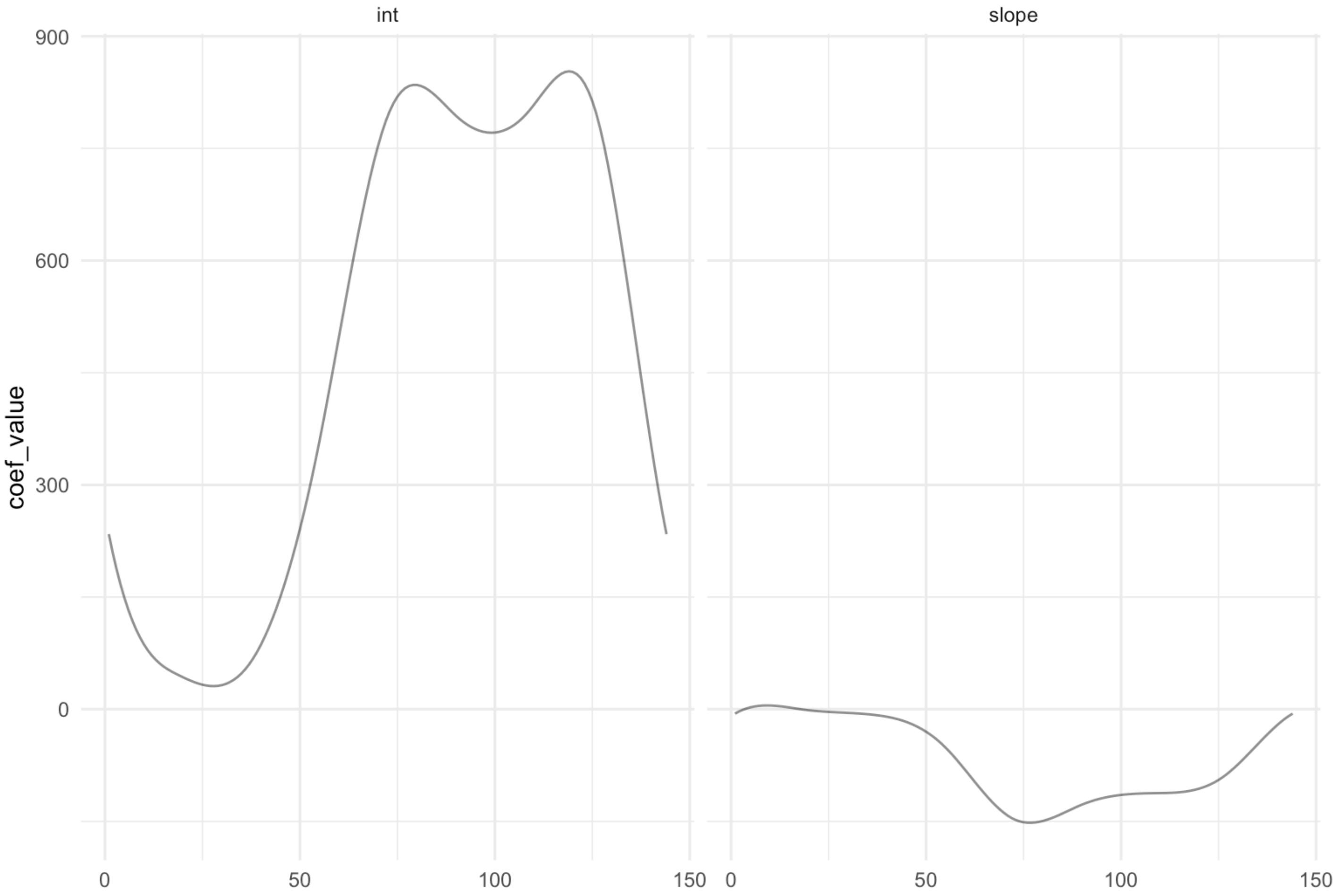


Code

```
regress_df = covariate_data
regress_df$accel = accel

fosr_slr = bayes_fosr(accel ~ season, data = regress_df,
                                            est.method = "GLS", Kt = 8, basis = "pbs")
tibble(
  coef_name = c("int", "slope"),
  coef_value = tfd(fosr_slr$beta.hat)) %>%
    ggplot(aes(y = coef_value)) +
  geom_spaghetti() +
  facet_grid(~coef_name)
```

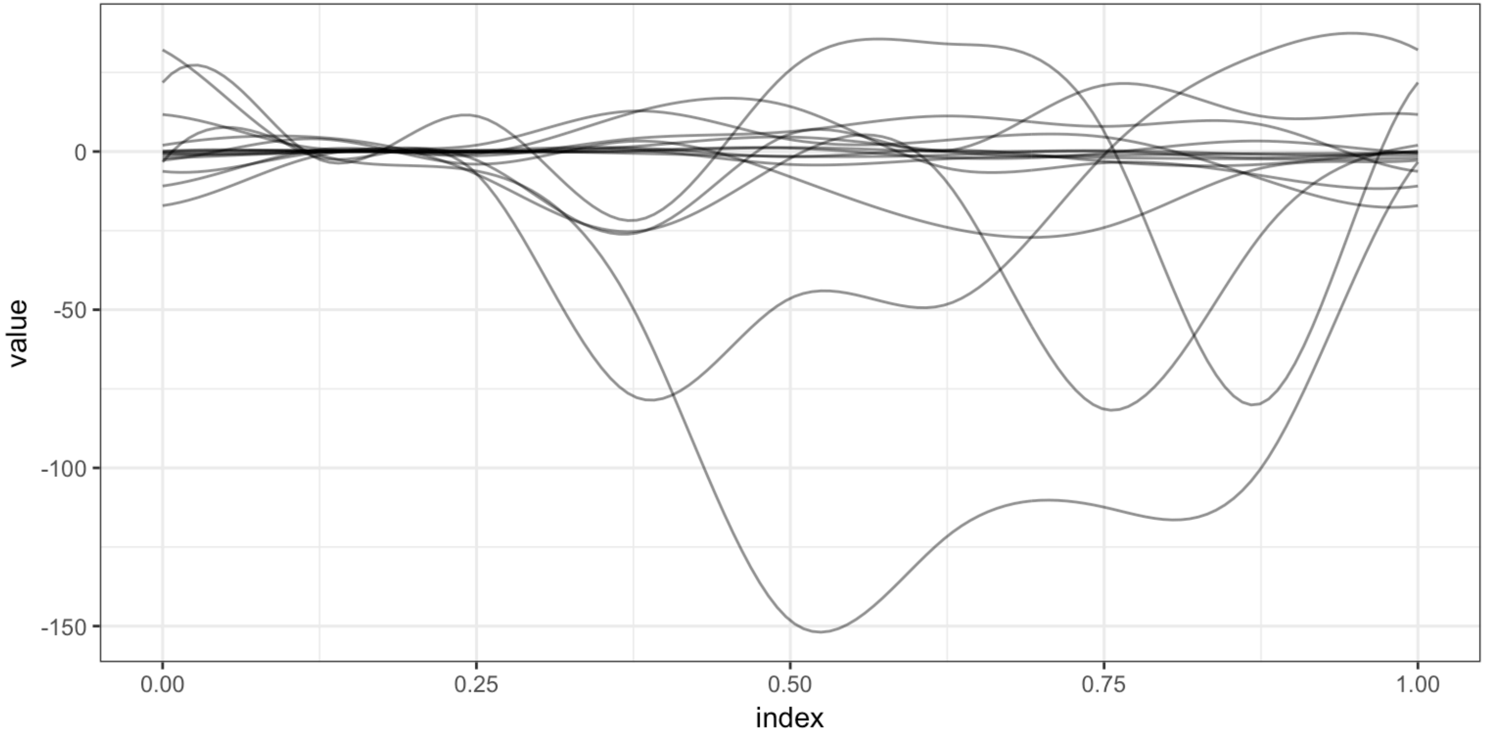
Code



Code

```
fosr_mlr = bayes_fosr(accel ~ season + sex + BMIZ + TV +
                     videogames + mom_work + asthma + child_age +
                     mom_age + educ_mom + num_rooms + mom_born_US + tricep +
                     subscap + skinfold, data = regress_df,
                                          est.method = "GLS", Kt = 8, basis = "pbs")
```

Code



Code

```
regress_df$mean_accel = apply(accel, 1, mean)

mlr = lm(mean_accel ~ season + sex + BMIZ + TV +
                      videogames + mom_work + asthma + child_age +
                      mom_age + educ_mom + num_rooms + mom_born_US + tricep +
                      subscap + skinfold, data = regress_df)

mlr %>%
    broom::tidy() %>%
    knitr::kable(digits = 2)
```

Code

term	estimate	std.error	statistic	p.value
(Intercept)	585.54	53.19	11.01	0.00
seasoncool	-60.17	10.53	-5.72	0.00
sexmale	-9.38	10.14	-0.92	0.36
BMIZ	-0.10	3.56	-0.03	0.98
TV>=2h	-16.71	10.18	-1.64	0.10
videogames>=1h	-7.47	11.47	-0.65	0.52
mom_workyes	-4.44	10.15	-0.44	0.66
asthmayes	-2.99	10.73	-0.28	0.78
child_age	-0.91	0.65	-1.40	0.16
mom_age	-0.33	0.87	-0.38	0.71
educ_mom	-0.84	1.68	-0.50	0.62
num_rooms	5.17	4.79	1.08	0.28
mom_born_USyes	3.14	12.76	0.25	0.81
tricep	0.83	1.48	0.56	0.58
subscap	0.01	1.36	0.01	0.99
skinfold	-0.66	0.79	-0.83	0.41

Code

	beta.mlr	beta.fosr
(Intercept)	585.54	541.60
seasoncool	-60.17	-67.39
sexmale	-9.38	-14.54
BMIZ	-0.10	-0.17
TV>=2h	-16.71	-18.08
videogames>=1h	-7.47	-0.44
mom_workyes	-4.44	-7.57
asthmayes	-2.99	2.81
child_age	-0.91	-0.59
mom_age	-0.33	0.19
educ_mom	-0.84	-0.97
num_rooms	5.17	3.70
mom_born_USyes	3.14	-0.13
tricep	0.83	-0.10
subscap	0.01	-0.82
skinfold	-0.66	-0.07