Tutorial 8 - Belief Propagation and Sampling

COMP9418 – Advanced Topics in Statistical Machine Learning

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Lecture: Belief Propagation and Sampling

Topic: Questions from lecture topics

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Question 1

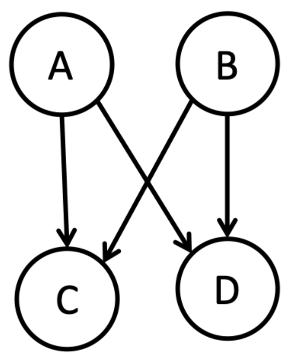
Consider the following Bayesian network:

$P(A)$

+a	.5
-a	.5

$P(B)$

+b	.25
-b	.75



$P(C|A,B)$

+a	+b	+c	.9
+a	+b	-c	.1
+a	-b	+c	.8
+a	-b	-c	.2
-a	+b	+c	.7
-a	+b	-c	.3
-a	-b	+c	.6
-a	-b	-c	.4

$P(D|A,B)$

+a	+b	+d	.6
+a	+b	-d	.4
+a	-b	+d	.4
+a	-b	-d	.6
-a	+b	+d	.3
-a	+b	-d	.7
-a	-b	+d	.3
-a	-b	-d	.7

Suppose we condition on evidence $e : D = true$. Suppose that we have run the Parallel Iterative Belief Propagation (IBP) algorithm on the network, where it converges and yields the following set of messages and family marginals:

A	$\pi_C(A)$	$\pi_D(A)$	$\lambda_C(A)$	$\lambda_D(A)$
+a		.5	.5	.6
-a		.5	.5	.4

B	$\pi_C(B)$	$\pi_D(B)$	$\lambda_C(B)$	$\lambda_D(B)$
+b	.3	.25	.5	
-b	.7	.75	.5	

A	B	C	$\beta(A,B,C)$
+a	+b	+c	.162
+a	+b	-c	.018
+a	-b	+c	
+a	-b	-c	.084
-a	+b	+c	.084
-a	+b	-c	.036
-a	-b	+c	.168
-a	-b	-c	.112

A	B	D	$\beta(A,B,D)$
+a	+b	+d	.2
+a	+b	-d	0
+a	-b	+d	
+a	-b	-d	0
-a	+b	+d	.1
-a	+b	-d	0
-a	-b	+d	.3
-a	-b	-d	0

a. Fill in the missing values for IBP messages.
b. Fill in the missing values for family marginals.
c. Compute marginals $\beta(A)$ and $\beta(B)$ using the IBP messages and those computed in (a).
d. Compute marginals $\beta(A)$ and $\beta(B)$ by summing out the appropriate variables from the family marginals $\beta(ABC)$ as well as $\beta(ABD)$.
e. Compute joint marginal $\beta(AB)$ by summing out the appropriate variables from family marginals $\beta(ABC)$ as well as $\beta(ABD)$.
f. Are the marginals computed in (d) consistent? What about those computed in (e)?

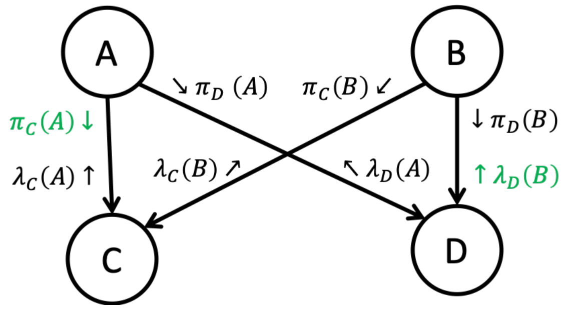
Consider the following IBP algorithm:

Data: N: Bayesian network
Data: $\mathbf{e}$: evidence
Result: Approximate marginals, $\beta(X\mathbf{U})$ of $P(X\mathbf{U}|e)$ for each family $X\mathbf{U}$ in N

begin
    $t \leftarrow 0$;
    initialize all messages;
    while *messages have not converged* **do**
        $t \leftarrow t+1$;
        for *each node X with parents $\boldsymbol{U}$* **do**
            for *each parent U_i* **do**
                $\lambda^t_X(U_i) = \eta \sum_{X\mathbf{U}\setminus\{U_i\}} \lambda_{\mathbf{e}}(X)\phi_X(X,\mathbf{U}) \prod_{k\neq i} \pi^{t-1}_X(U_k) \prod_j \lambda^{t-1}_{Y_j}(X)$;
            end
            for *each child Y_j* **do**
                $\pi^t_{Y_j}(X) = \eta \sum_{\mathbf{U}} \lambda_{\mathbf{e}}(X)\phi_X(X,\mathbf{U}) \prod_i \pi^{t-1}_X(U_i) \prod_{k\neq j} \lambda^{t-1}_{Y_k}(X)$;
            end
        end
    end
end
return $\beta(X\mathbf{U}) = \eta\lambda_{\mathbf{e}}(X)\phi_X(X,\mathbf{U}) \prod_i \pi^t_X(U_i) \prod_j \lambda^t_{Y_j}(X)$

Answer

a. The following figure shows all messages:



Let's start computing $\pi_C(A)$. According to the IBP algorithm:

$\pi_C(A) = \eta \phi_A(A) \lambda_D(A)$

This simplified equation comes from the fact that node A has no parents and no evidence. Also, this message does not consider the message $\lambda_C(A)$ coming from node C because we are computing a message directed to the same node C.

Therefore,

A	$\pi_C(A)$
+a	.6
-a	.4

Remember that the multiplication by the constant η renormalizes the message $\pi_C(A)$ to sum to one.

Now, we compute $\lambda_D(B)$ using the equation from the IBP algorithm:

$\lambda_D(B) = \eta \sum_{A,B} \lambda_{\mathbf{e}}(D) \phi_D(D, A, B) \pi_D(A)$

Also, notice that node B has no children, and we should not consider the message $\pi_D(B)$ when computing a message to B.

We first compute the multiplication $\lambda_{\mathbf{e}}(D) \phi_D(D, A, B) \pi_D(A)$.

A	B	D	$\lambda_{\mathbf{e}}(D)\phi_D(D, A, B)\pi_D(A)$
+a	+b	+d	.3
+a	+b	-d	0
+a	-b	+d	.2
+a	-b	-d	0
-a	+b	+d	.15
-a	+b	-d	0
-a	-b	+d	.15
-a	-b	-d	0

After the elimination of variables A and B and renormalization, we get the message $\lambda_D(B)$

B	$\lambda_D(B)$
+b	.5625
-b	.4375

b. We also use the equation from IBP algorithm to compute the family marginals

$\beta(+a, -b, +c) = \eta\phi_C(+c, +a, -b)\pi_C(+a)\pi_C(-b) = \eta 0.8 \times 0.6 \times 0.7 = \eta 0.336$. Since this factor is natually normalized, $\eta = 1$.

$\beta(+a, -b, +d) = \eta\phi_D(+d, +a, -b)\pi_D(+a)\pi_D(-b) = \eta 0.4 \times 0.5 \times 0.75 = \eta 0.15$. This factor does not sum to one naturally since we have evidence on D. In this case $\eta \approx 2.667$ and $\beta(+a, -b, +d) = 0.4$.

c. According to the IBP algorithm:

$\beta(A) = \eta\phi_A(A)\lambda_C(A)\lambda_D(A)$. Therefore:

A	$\beta(A)$
+a	.6
-a	.4

Also, $\beta(B) = \eta\phi_B(B)\lambda_C(B)\lambda_D(B)$.

B	$\beta(B)$
+b	.3
-b	.7

d. We can compute $\beta(A)$ by summing out variables B and C from $\beta(A, B, C)$:

$\beta(A) = \sum_{B,C} \beta(A, B, C)$

A	B	C	$\beta(A, B, C)$
+a	+b	+c	.162
+a	+b	-c	.018
+a	-b	+c	.336
+a	-b	-c	.084
-a	+b	+c	.084
-a	+b	-c	.036
-a	-b	+c	.168
-a	-b	-c	.112

Summing out C, we get:

A	B	$\beta(A, B)$
+a	+b	.18
+a	-b	.42
-a	+b	.12
-a	-b	.28

Summing out B, we get:

A	$\beta(A)$
+a	.6
-a	.4

We can also compute $\beta(B)$ from $\beta(A, B)$ eliminating A

B	$\beta(B)$
+b	.3
-b	.7

We will now sum out variables from the family marginal $\beta(A, B, D)$. We can compute $\beta(A)$ by summing out variables B and D from $\beta(A, B, D)$:

A	B	D	$\beta(A, B, D)$
+a	+b	+d	.2
+a	+b	-d	0
+a	-b	+d	.4
+a	-b	-d	0
-a	+b	+d	.1
-a	+b	-d	0
-a	-b	+d	.3
-a	-b	-d	0

Summing out D, we get:

A	B	$\beta(A, B)$
+a	+b	.2
+a	-b	.4
-a	+b	.1
-a	-b	.3

Summing out B, we get:

A	$\beta(A)$
+a	.6
-a	.4

We can also compute $\beta(B)$ from $\beta(A, B)$ eliminating A

B	$\beta(B)$
+b	.3
-b	.7

e. We did this as an intermediate step of the previous answer. We copy the answer above to facilitate reference.

A	B	$\beta(A, B)$ from $\beta(A, B, C)$
+a	+b	.18
+a	-b	.42
-a	+b	.12

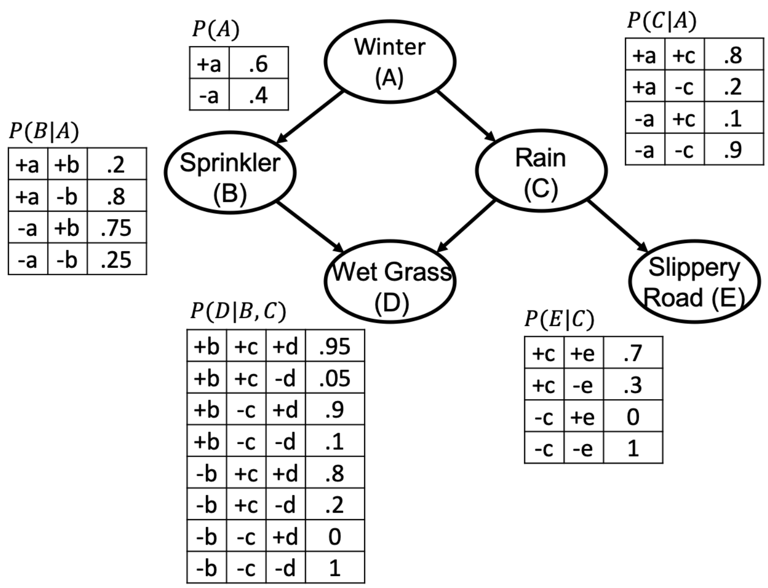
A	B	$\beta(A,B)$ from $\beta(A,B,C)$
-a	-b	.28

A	B	$\beta(A,B)$ from $\beta(A,B,D)$
+a	+b	.2
+a	-b	.4
-a	+b	.1
-a	-b	.3

f. The marginals computed in (d) are consistent, while the ones in computed in (e) are not consistent.

Question 2

Consider the following Bayesian network:



and the parameter $\theta_{\bar{a}}$ representing the probability of $A = false$ (i.e., it is not winter). For each of the following values of this parameter, 0.01, 0.4 and 0.99 do the following:

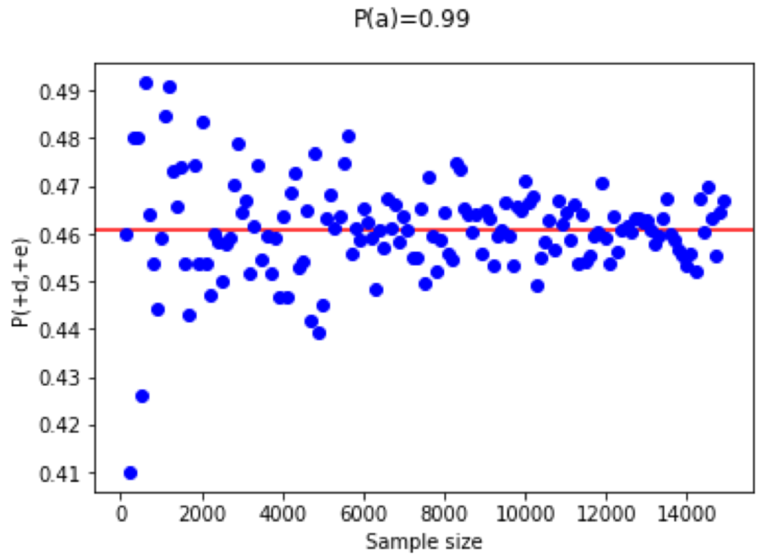
a. Compute the probability of $P(+d, +e)$: wet grass and slippery road. You can use the code from previous tutorials.
b. Estimate $P(+d, +e)$ using forward sampling with sample sizes ranging from $n = 100$ to $n = 15,000$.
c. Generate a plot with n on the x-axis and the exact value of $P(+d, +e)$ and the estimate for $P(+d, +e)$ on the y-axis.
d. Generate a plot with n on the x-axis and the exact variance of the estimate for $P(+d, +e)$ and the sample variance on the y-axis.

Answer

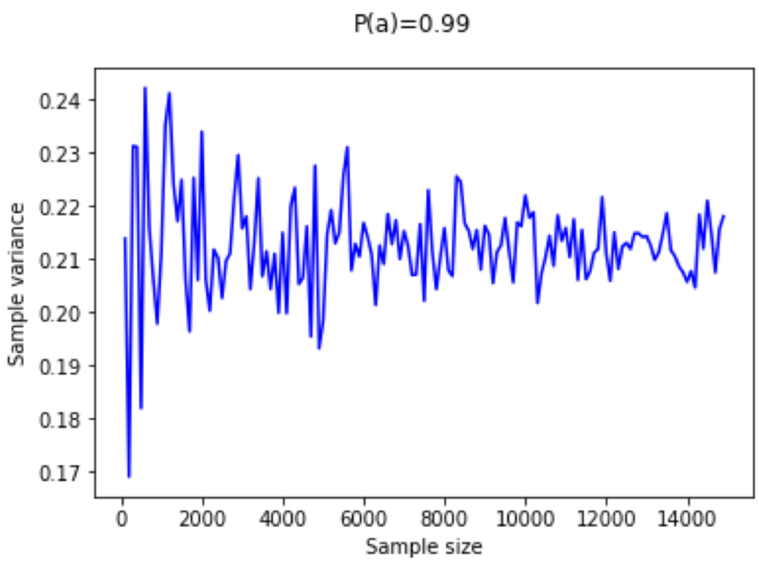
a. We computed $P(d, e)$ using our code from previous tutorials. We obtained the following results: $P(+d, +e) = 0.4608$ for $\theta_{\bar{a}} = 0.01$ $P(+d, +e) = 0.3044$ for $\theta_{\bar{a}} = 0.4$ $P(+d, +e) = 0.0679$ for $\theta_{\bar{a}} = 0.99$
b. The following code will estimate $P(+d, +e)$ using forward sampling:

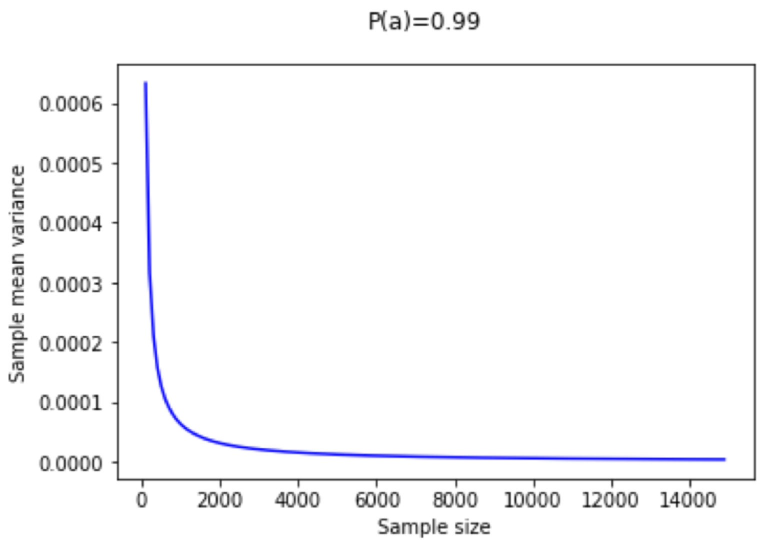
```python
# Declare `factors`: a factor dictionary for the Bayesian network parameters

theta_na_list = [.01, .4, .99]
true_p_d_e_list = [0.4608, 0.3044, 0.0679]
sample_sizes = range(100,15000,100)
for j, theta_na in enumerate(theta_na_list):
    p_d_e_list = []
    sample_var_list = []
    sample_mean_var_list = []
    factors['A']['table'][1] = 1 - theta_na
    for s in sample_sizes:
        d_e_count = 0
        for i in range(s):
            a = random() < factors['A']['table'][1]
            b = random() < factors['B']['table'][(a,1)]
            c = random() < factors['C']['table'][(a,1)]
            d = random() < factors['D']['table'][(b,c,1)]
            e = random() < factors['E']['table'][(c,1)]
            if d and e:
                d_e_count = d_e_count + 1

        p_d_e = d_e_count / s
        p_d_e_list.append(p_d_e)
        sample_var_list.append((sum([(1-p_d_e)**2 for i in range(d_e_count)]) + \
                                sum((0-p_d_e)**2 for i in range(s-d_e_count))) / (s-1))
        sample_mean_var_list.append(true_p_d_e_list[j] * (1-true_p_d_e_list[j])/s)

    plt.figure()
    plt.axhline(y=true_p_d_e_list[j], color='r', linestyle='-')
    plt.suptitle('P(a)='+str(1-theta_na))
    plt.plot(range(100,15000,100), p_d_e_list, 'bo')
    plt.xlabel("Sample size")
    plt.ylabel("P(+d,+e)")
    plt.show()

    plt.figure()
    plt.suptitle('P(a)='+str(1-theta_na))
    plt.plot(range(100,15000,100), sample_var_list, 'b')
    plt.xlabel("Sample size")
    plt.ylabel("Sample variance")
    plt.show()

    plt.figure()
    plt.suptitle('P(a)='+str(1-theta_na))
    plt.plot(range(100,15000,100), sample_mean_var_list, 'b')
    plt.xlabel("Sample size")
    plt.ylabel("Sample mean variance")
    plt.show()
```

c. You should get a plot like this, with the above code



d. Due to the difference in scale, the code creates separated plots for sample variance and the exact variance of the estimate.




P(a)=0.99
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
0.0000
Sample mean variance
0
2000
4000
6000
8000
10000
12000
14000
Sample size