Linear Regression Analysis (4): Polynomial Regression & Interaction

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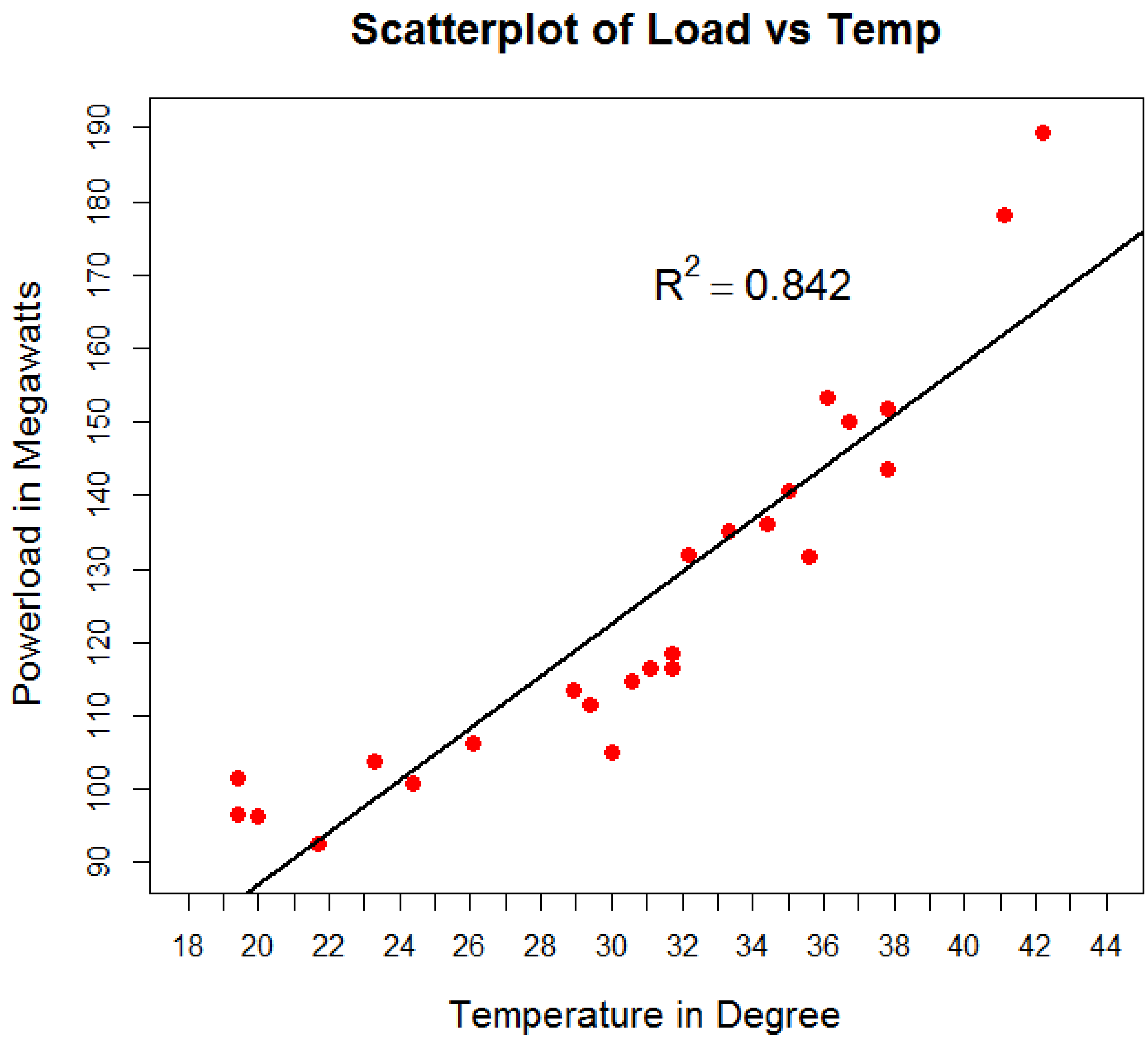
Polynomial Regression

Case Study: Peak Power Load & Temperature

- To operate efficiently, power companies must be able to predict the peak power load at their various stations.
- Peak power load is the maximum amount of power that must be generated each day to meet demand.
- A power company wants to use daily high temperature, x, to model daily peak power load, y, during the summer months when demand is greatest.
- Although the company expects peak load to increase as the temperature increases, the *rate* of increase in $E(y)$ might not remain constant as x increases.

- For example, a 1-unit increase in high temperature from 36°C to 37°C might result in a larger increase in power demand than would a 1-unit increase from 26°C to 27°C.
- Therefore, the company postulates that the relationship between temperature and power load may not be linear.
- A random sample of 25 summer days is selected and both the peak load (measured in megawatts) and high temperature (in Celsius degrees) recorded for each day.

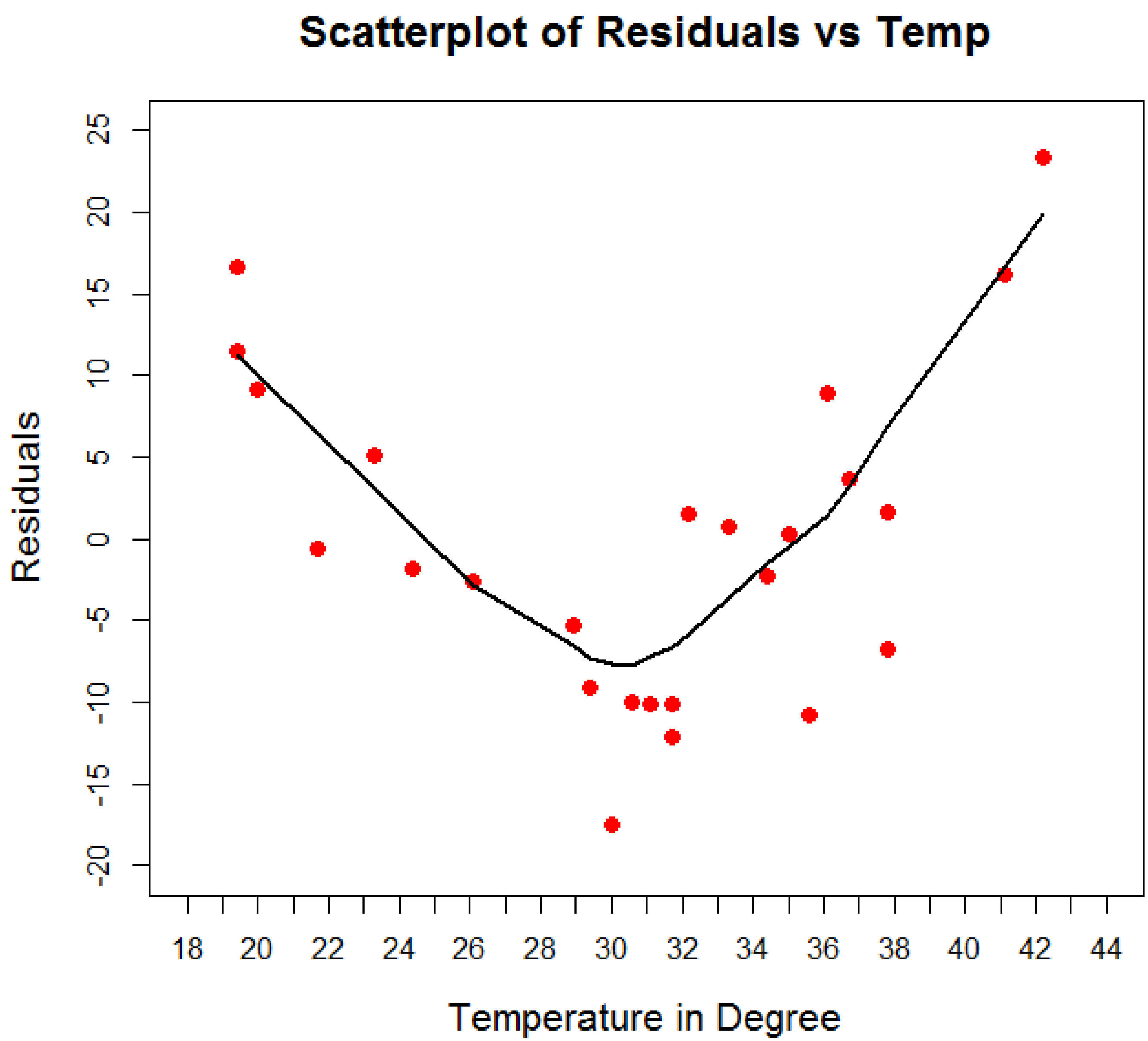
Linear Regression Model



Load	Celsius
136	34.4
131.7	35.6
140.7	35.0
189.3	42.2
96.5	19.4
116.4	31.1
118.5	31.7
113.4	28.9
132	32.2
178.2	41.1
101.6	19.4
92.5	21.7
151.9	37.8
106.2	26.1
153.2	36.1
150.1	36.7
114.7	30.6
100.9	24.4
96.3	20.0
135.1	33.3
143.6	37.8
111.4	29.4
116.5	31.7
103.9	23.3
105.1	30.0

```
> lm1<-lm(Load~Celsius, data=powerload)
> summary(lm1)

Residuals:
     Min       1Q   Median       3Q      Max 
-17.5024  -9.0725  -0.6394   5.0810  23.3906 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  16.1096    10.0946   1.596    0.124    
Celsius       3.5498     0.3208  11.066 1.09e-10 ***

Residual standard error: 10.37 on 23 degrees of freedom
Multiple R-squared:  0.8419,    Adjusted R-squared:  0.835 
F-statistic: 122.4 on 1 and 23 DF,  p-value: 1.095e-10
```

Residual Plot



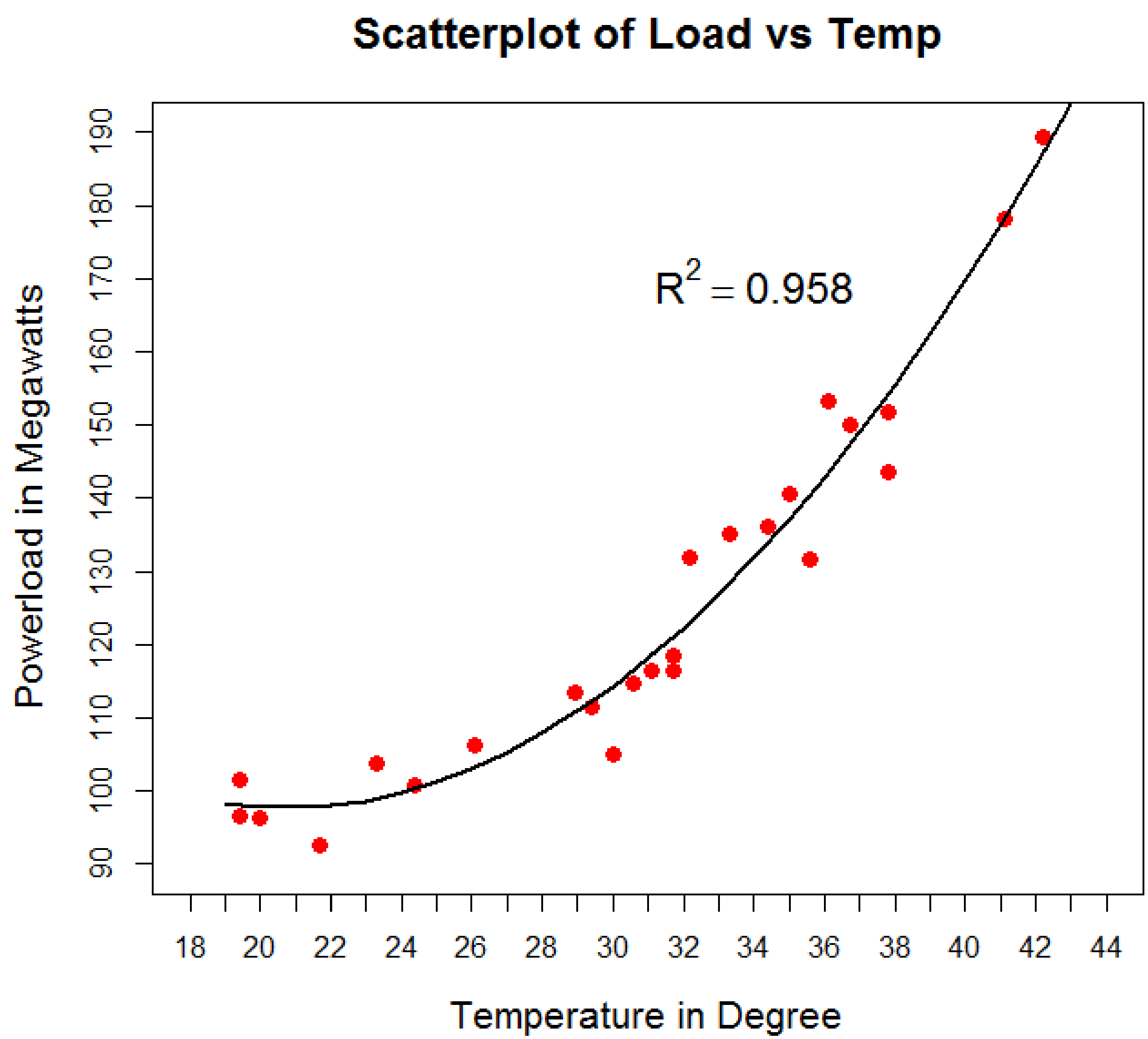
- It is quite obvious the relationship between power load and temperature is not linear
- There are several ways to estimate a curvilinear or non-linear relationship, such as polynomials, spline and nonlinear functions.
- Polynomial regression is the most widely used method to model a non-linear relationship between an explanatory variable and the outcome

Polynomial Regression

- First-order model with one covariate: $\hat{y} = b_0 + b_1x_1$
- Second-order (quadratic) model with one covariate: $\hat{y} = b_0 + b_1x_1 + b_2x_1^2$
- Third-order (cubic) model with one covariate: $\hat{y} = b_0 + b_1x_1 + b_2x_1^2 + b_3x_1^3$
- Pth-order model with one covariate: $\hat{y} = b_0 + b_1x_1 + b_2x_1^2 + \cdots + b_px_1^p$
- Although the fitted line is not a straight line, these models are still "linear" model, usually known as curvilinear models to be distinguished from non-linear models

- To fit a quadratic model for our case study, we first create a new variable Celsius2, which is squared Celsius

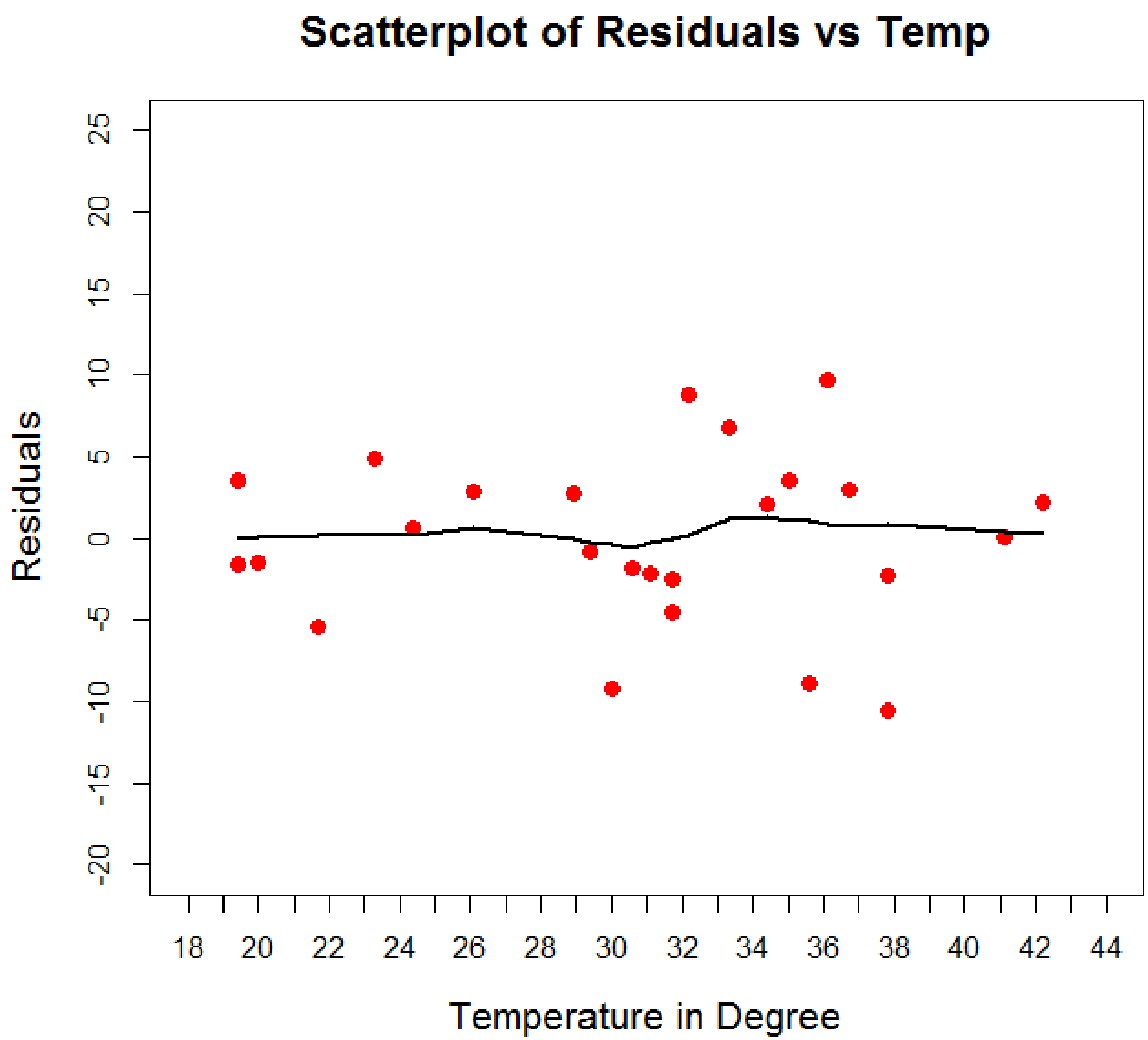
```
> Celsius2 <- powerload$Celsius^2
> lm2<-lm(Load~Celsius+Celsius2, data=powerload)
> summary(lm2)
Residuals:
     Min       1Q    Median       3Q      Max
-10.5597  -2.2597   0.0827   2.9870   9.7328
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 181.17159   21.71016   8.345 2.91e-08 ***
Celsius      -8.04877    1.48894  -5.406 1.98e-05 ***
Celsius2      0.19403    0.02475   7.840 8.24e-08 ***

Residual standard error: 5.445 on 22 degrees of freedom
Multiple R-squared:  0.9583,    Adjusted R-squared:  0.9545
F-statistic: 252.9 on 2 and 22 DF,  p-value: 6.595e-16
```



Scatterplot of Load vs Temp
Powerload in Megawatts
90
100
110
120
130
140
150
160
170
180
190
$R^2 = 0.958$
18
20
22
24
26
28
30
32
34
36
38
40
42
44
Temperature in Degree

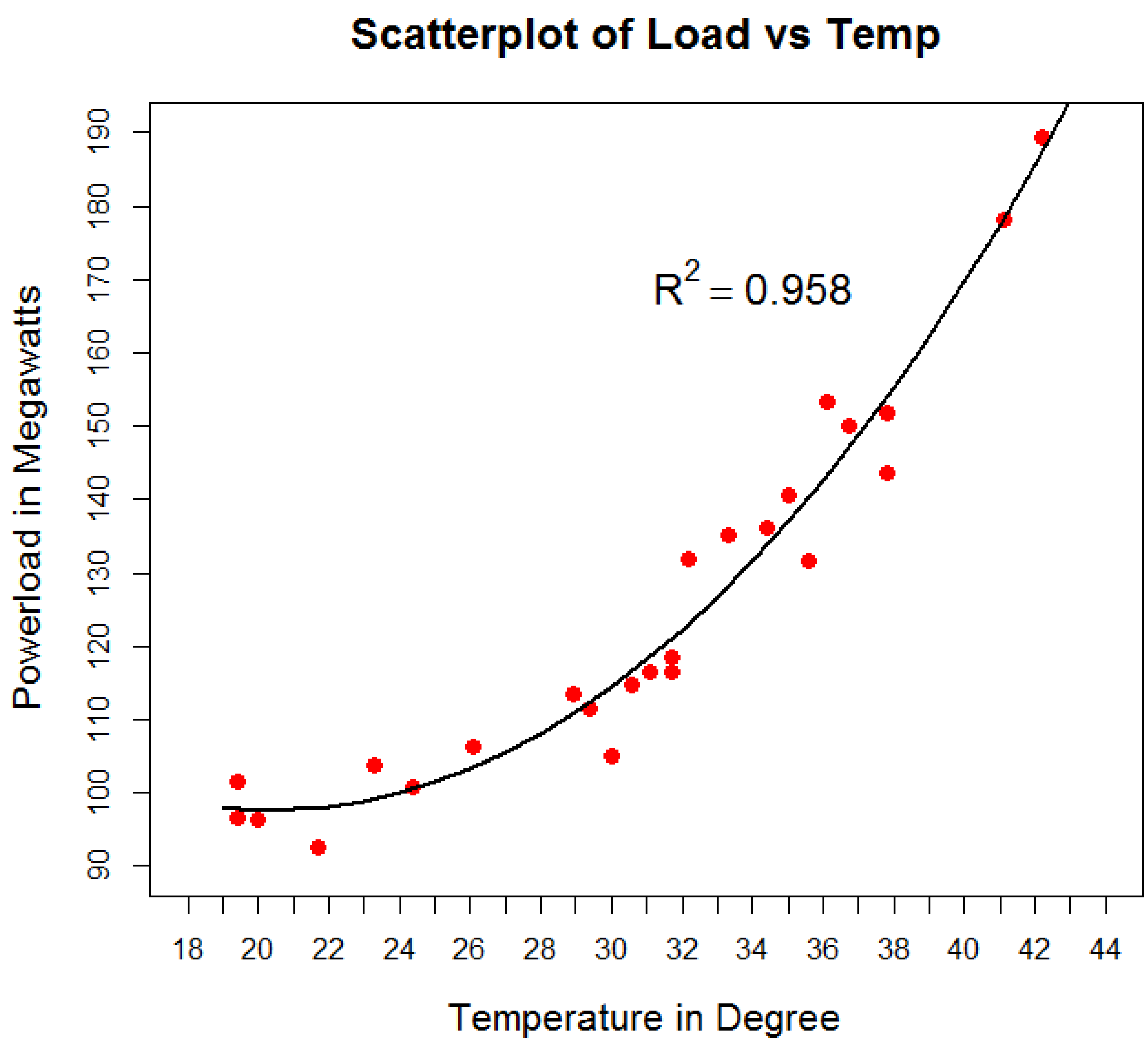
- Both Celsius and Celsius2 are highly significant
- The overall model is also significant. This is reflected by the increased R^2 (from 0.842 to 0.958) and the significant F-test result (F = 252.9, df = (2, 22))

Residual Plot



- Now, let us see if a cubic model will further improve the model fit:

```
> Celsius3 <- powerload$Celsius^3
> lm3<-lm(Load~Celsius+Celsius2+Celsius3, data=powerload)
> summary(lm3)

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.645e+02  1.159e+02   1.419    0.171
Celsius     -6.293e+00  1.205e+01  -0.522    0.607
Celsius2     1.349e-01  4.036e-01   0.334    0.742
Celsius3     6.429e-04  4.378e-03   0.147    0.885

Residual standard error: 5.57 on 21 degrees of freedom
Multiple R-squared:  0.9584,   Adjusted R-squared:  0.9524
F-statistic: 161.1 on 3 and 21 DF,  p-value: 1.186e-14
```


Scatterplot of Load vs Temp
$R^2 = 0.958$
Powerload in Megawatts
90
100
110
120
130
140
150
160
170
180
190
18
20
22
24
26
28
30
32
34
36
38
40
42
44
Temperature in Degree

- The model R^2 only increases marginally from 0.9583 to 0.9584.
- However, none of the explanatory variables is statistically significant! But the *F*-test remains highly significant
- So the model is good but none of the explanatory variables make "important" contribution. How did this happen?
- This is because the linear, quadratic and cubic terms are highly collinear (around 0.97)!
- But why did collinearity not cause any problem for quadratic model?

Centering

- Centering is useful for reducing the collinearity between a variable and its power terms
- Note that centering does not work for other collinearities

```
> ## centering Celsius
> Celsius.c <- powerload$Celsius -
mean(powerload$Celsius)
> Celsius.c2 <- Celsius.c^2
> Celsius.c3 <- Celsius.c^3
> lm4<-lm(Load~Celsius.c+Celsius.c2+Celsius.c3,
data=powerload)
```

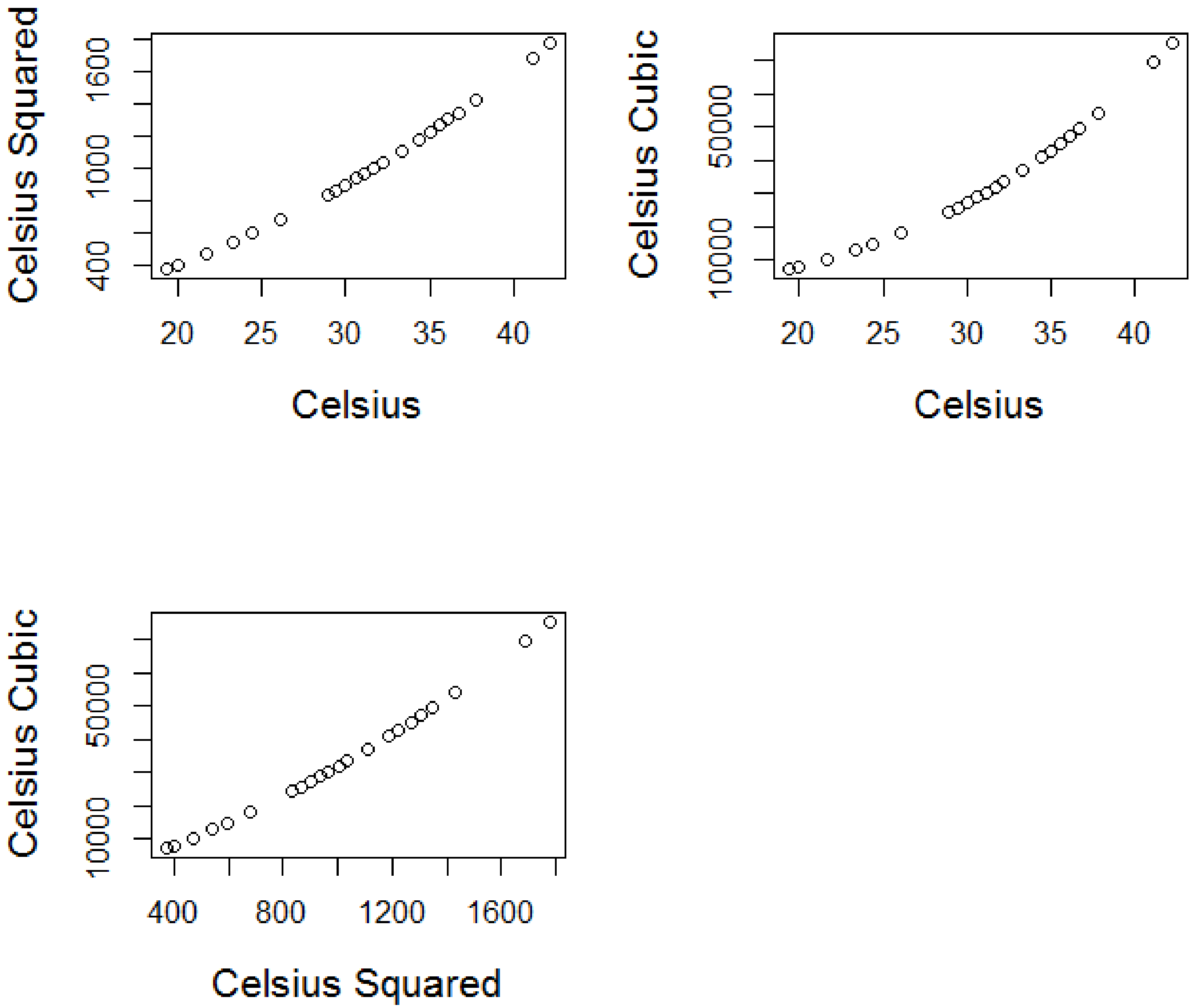
- Note that the mean of Celsius is 30.8

```
> summary(lm4)

Residuals:
     Min       1Q   Median       3Q      Max 
-10.4228  -2.1391  -0.0845   3.1520   9.9008 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) 1.174e+02  1.559e+00  75.280  < 2e-16 ***
Celsius.c   3.843e+00  4.353e-01   8.829 1.64e-08 ***
Celsius.c2  1.943e-01  2.537e-02   7.656 1.65e-07 ***
Celsius.c3  6.429e-04  4.378e-03   0.147    0.885    

Residual standard error: 5.57 on 21 degrees of freedom
Multiple R-squared:  0.9584,	Adjusted R-squared:  0.9524 
F-statistic: 161.1 on 3 and 21 DF,  p-value: 1.186e-14
```

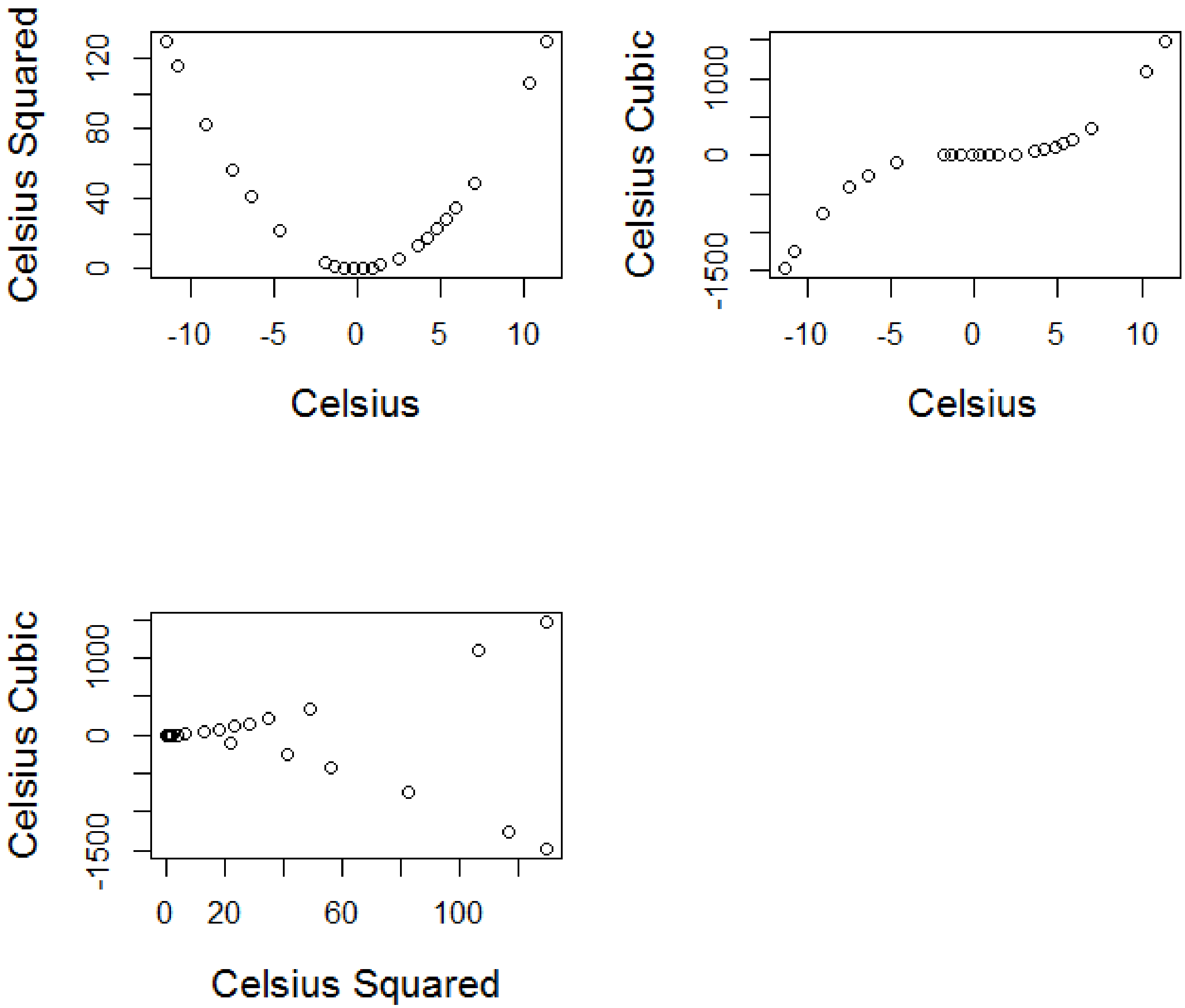
How Does Centering Work?

- After centering, the model correctly shows that both the linear and quadratic terms are statistically significant, while the cubic term is not.
- Note that centered model has the same R^2 as the original model. These two models are identical.

Correlations between Celsius & Its Power Terms



Correlations between Celsius & Its Power Terms After Centering



Impact of Centering

We now use the quadratic model to illustrate the impact of centering on polynomial regression model.

Recall the original model:

```
> lm2<-lm(Load~Celsius+Celsius2, data=powerload)
> summary(lm2)

Estimate Std. Error t value Pr(>|t|)
(Intercept) 181.17159   21.71016   8.345 2.91e-08 ***
Celsius      -8.04877    1.48894  -5.406 1.98e-05 ***
Celsius2      0.19403    0.02475   7.840 8.24e-08 ***

Residual standard error: 5.445 on 22 degrees of freedom
Multiple R-squared:  0.9583,   Adjusted R-squared:  0.9545
F-statistic: 252.9 on 2 and 22 DF,  p-value: 6.595e-16
```

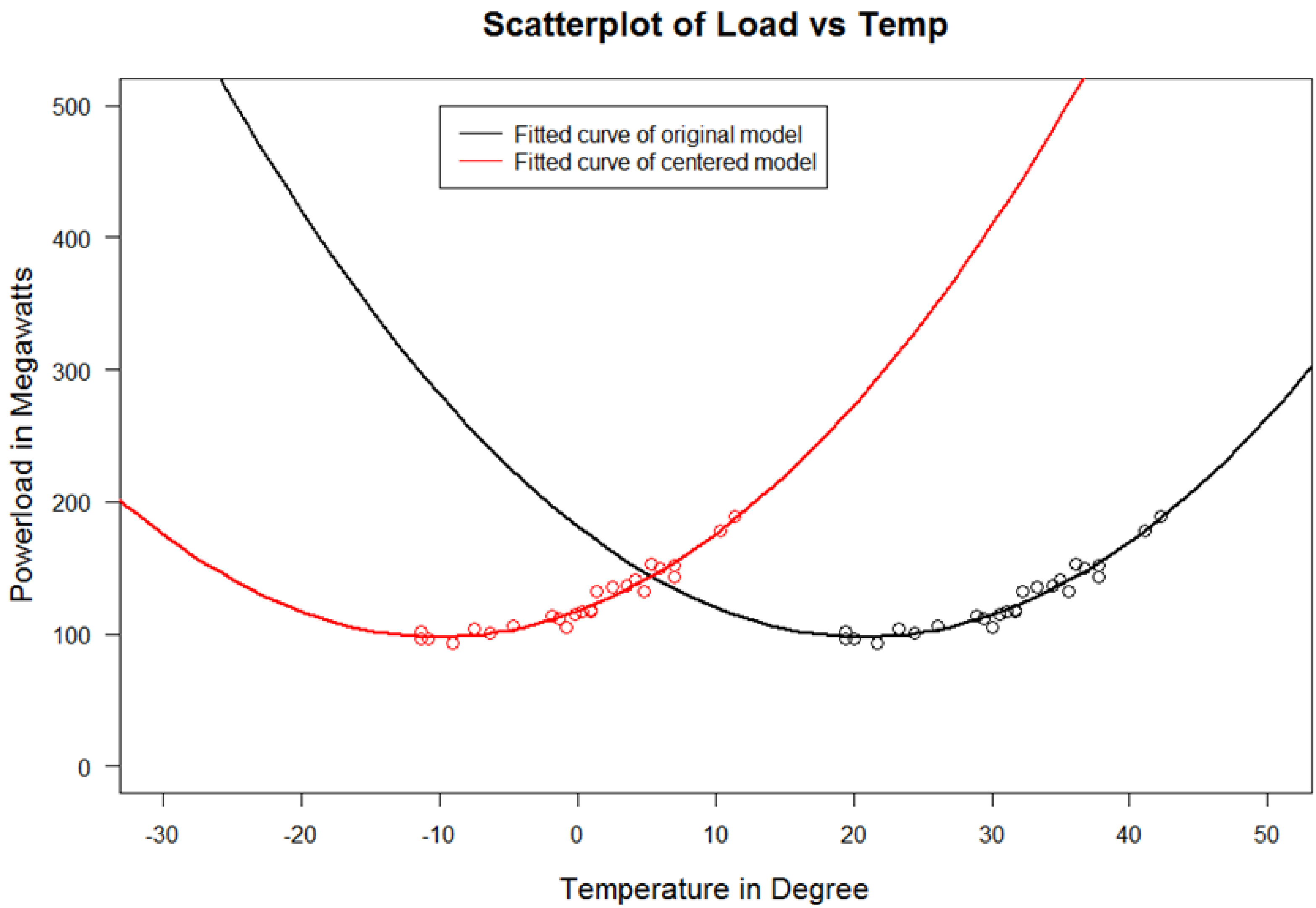
Impact of Centering

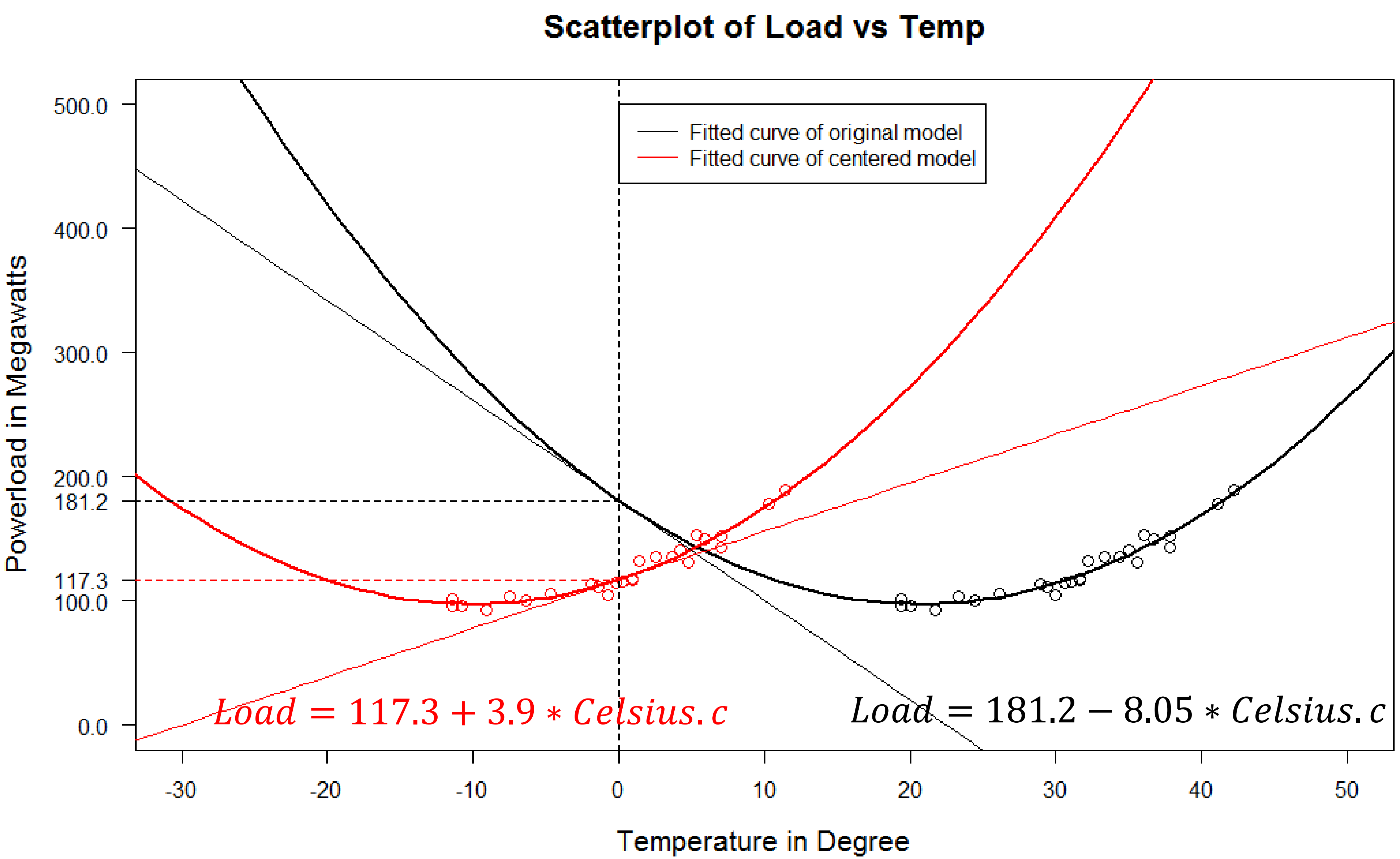
We now re-fit the model using the centered Celsius:

```
> # fit the centered quadratic model
> lm5<-lm(Load~Celsius.c+Celsius.c2, data=powerload)
> summary(lm5)

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 117.31439    1.50227   78.09  < 2e-16 ***
Celsius.c     3.90166    0.17427   22.39  < 2e-16 ***
Celsius.c2    0.19403    0.02475    7.84 8.24e-08 ***
---
Residual standard error: 5.445 on 22 degrees of freedom
Multiple R-squared:  0.9583,    Adjusted R-squared:
0.9545
F-statistic: 252.9 on 2 and 22 DF,  p-value: 6.595e-16
```

Impact of Centering

- Note that the regression coefficients for intercept and linear terms are different in the two models, while the coefficient for the quadratic term remain unchanged.
- Centering at mean value of temperature is equivalent to moving the fitted parabolic curve horizontally to the left by 30.8
- Because the shape of the curve remains unaffected, the coefficient for quadratic term remain unchanged


Scatterplot of Load vs Temp
Fitted curve of original model
Fitted curve of centered model
Powerload in Megawatts
Temperature in Degree
0
100
200
300
400
500
-30
-20
-10
0
10
20
30
40
50



Scatterplot of Load vs Temp
Fitted curve of original model
Fitted curve of centered model
500.0
400.0
300.0
200.0
181.2
117.3
100.0
0.0
Powerload in Megawatts
-30
-20
-10
0
10
20
30
40
50
Temperature in Degree
$Load = 117.3 + 3.9 * Celsius.c$
$Load = 181.2 - 8.05 * Celsius.c$

Interpretation of Polynomial Regression Coefficients

For the original polynomial model,

$$Load = 181.2 - 8.05Celsius + 0.194Celsius^2$$

- The coefficient for intercept (181.2) is the estimated power load in megawatts when temperature is at zero Celsius
- The coefficient for slope (−8.05) for Celsius is the slope of the tangent line for the fitted parabolic curve when temperature is at zero Celsius

Interpretation of Polynomial Regression Coefficients

For the centered polynomial model,

$$Load = 117.3 + 3.90Celsius + 0.194Celsius^2$$

- The coefficient for intercept (117.3) is the estimated power load in megawatts when temperature is at 30.8 Celsius
- The coefficient slope (3.90) for Celsius is the slope of the tangent line for the fitted parabolic curve when temperature is at 30.8 Celsius

Interaction

Interaction in Regression Analysis

- Interaction between two binary variables
  - Estimate means for four groups
- Interaction between one binary and one continuous variable
  - Estimate two regression lines with different slopes for the two groups
- Interaction between two continuous variables
  - Estimate a curved plane

Interaction between Two Binary Variables

- We use the FEV example of 654 children to illustrate:

Id	fev	age	gender	smoking	height
1	1.404	3	1	0	131
2	1.072	3	0	0	117
3	0.839	4	0	0	122
4	1.569	4	0	0	127
5	1.577	4	0	0	124
6	0.796	4	1	0	119
7	1.789	4	1	0	132
8	1.102	4	0	0	122
….	….	….	….	….	….
650	4.404	18	1	1	179
651	2.853	18	0	0	152
652	5.102	19	1	0	183
653	3.519	19	0	1	168
654	3.345	19	0	1	166

Interaction between Two Binary Variables

- We first create a new variable agecat which is coded "younger" for children <= 11 y/o and coded "older" for those > 11 y/o
- We then calculate the means for those children stratified by gender and agecat:

```
FEV$agecat <- ifelse(FEV$age > 11, c("older"),
c("younger"))

FEV$sex <- ifelse(FEV$gender == 0, c("girls"),
c("boys"))

aggregate(x = FEV$fev, by = list(FEV$sex,
FEV$agecat), FUN = "mean")
```

Interaction between Two Binary Variables

```
  Group.1 Group.2        x
1    boys   older 3.933056
2   girls   older 2.986833
3    boys younger 2.402467
4   girls younger 2.258880
```

We now run a regression model with sex and agecat as covariates

```
lm1 <- lm(fev ~ sex + agecat, data = FEV)
summary(lm1)
```

Interaction between Two Binary Variables

```
Coefficients:
               Estimate Std. Error t value Pr(>|t|)    
(Intercept)     3.64862    0.05779  63.141  < 2e-16 ***
sex.girls      -0.35704    0.05338  -6.689 4.84e-11 ***
agecat.younger -1.14210    0.06037 -18.917  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '
' 1

Residual standard error: 0.6823 on 651 degrees of freedom
Multiple R-squared:  0.3827,    Adjusted R-squared:  0.3809 
F-statistic: 201.8 on 2 and 651 DF,  p-value: < 2.2e-16
```

The intercept 3.65 is the estimated mean fev for older boys, which is smaller than the real mean fev 3.93

Interaction between Two Binary Variables

- In this model, we assume the difference in mean fev between boys and girls is the same for younger and older children and vice versa,
- i.e. we also assume the difference in mean fev between younger and older children is the same in boys and girls
- However, it is very likely the difference in mean fev between boys and girls is greater for older children
- This is equivalent to an interaction between gender and age groups

Interaction between Two Binary Variables

- To test the interaction, we create a new variable sex.age.i, which is product of agecat and sex

Id	fev	age	sex	smoking	height	agecat	sex.age.i
1	1.404	3	0 (boys)	0	131	1 (younger)	0
2	1.072	3	1 (girls)	0	117	1 (younger)	1
3	0.839	4	1 (girls)	0	122	1 (younger)	1
4	1.569	4	1 (girls)	0	127	1 (younger)	1
5	1.577	4	1 (girls)	0	124	1 (younger)	1
6	0.796	4	0 (boys)	0	119	1 (younger)	0
....	....	....	....	....	....	....	....
650	4.404	18	0 (boys)	1	179	0 (older)	0
651	2.853	18	1 (girls)	0	152	0 (older)	0
652	5.102	19	0 (boys)	0	183	0 (older)	0
653	3.519	19	1 (girls)	1	168	0 (older)	0
654	3.345	19	1 (girls)	1	166	0 (older)	0

Interaction between Two Binary Variables

```
> lm3 <- lm(fev ~ sex + agecat + sex.age.i, data = FEV)
> summary(lm3)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.01706 -0.50003 -0.00888  0.40919  2.23453 

Coefficients:
                Estimate Std. Error t value Pr(>|t|)    
(Intercept)      3.93306    0.06949  56.600  < 2e-16 ***
sex.girls       -0.94622    0.10001  -9.461  < 2e-16 ***
agecat.younger -1.53059    0.08121 -18.847  < 2e-16 ***
sex.age.i        0.80264    0.11673   6.876 1.45e-11 ***
---
Residual standard error: 0.6592 on 650 degrees of freedom
Multiple R-squared:  0.4246,	Adjusted R-squared:  0.4219 
F-statistic: 159.9 on 3 and 650 DF,  p-value: < 2.2e-16
```

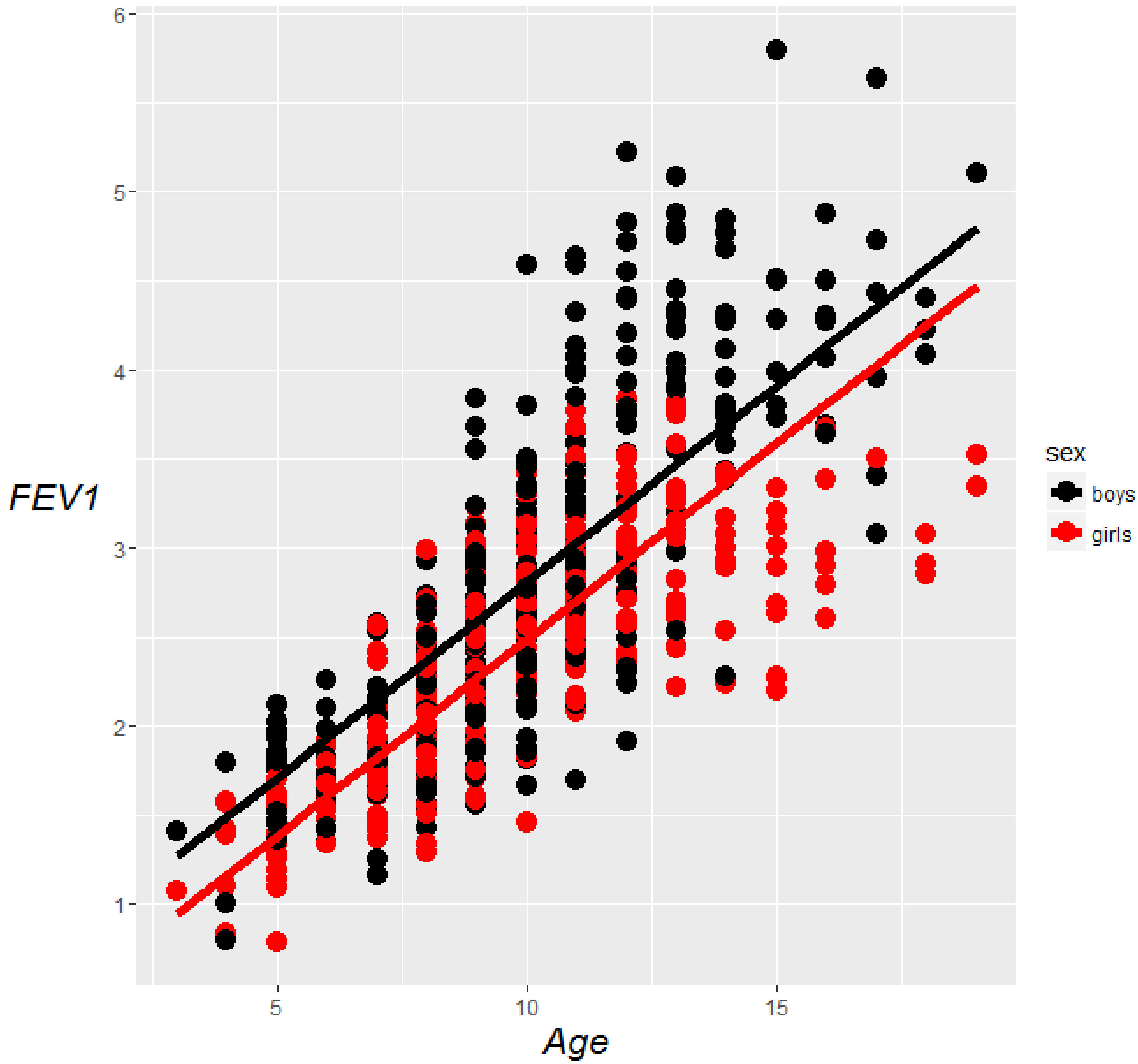
Interaction between Two Binary Variables

- The intercept 3.93 is the estimated mean fev for older boys, which is identical to the real mean fev
- We can work out the remaining means:
  - Older girls = 3.93 – 0.95 = 2.99
  - Younger boys = 3.93 – 1.53 = 2.40
  - Young girls = 3.93 – 0.95 – 1.53 + 0.80 = 2.26
- Those values are identical to their real means

Interaction between One Binary and One Continuous Variables

- Recall that in linear regression with one binary and one continuous covariates, the results are two fitted lines:

```
> lm4 <- lm(fev ~ sex + age, data = FEV)
> summary(lm4)

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.604713   0.078124   7.740 3.79e-14 ***
sex.girls    -0.323335   0.042609  -7.588 1.13e-13 ***
age           0.220445   0.007215  30.553  < 2e-16 ***
```



FEV1
Age
sex
boys
girls
6
5
4
3
2
1
5
10
15

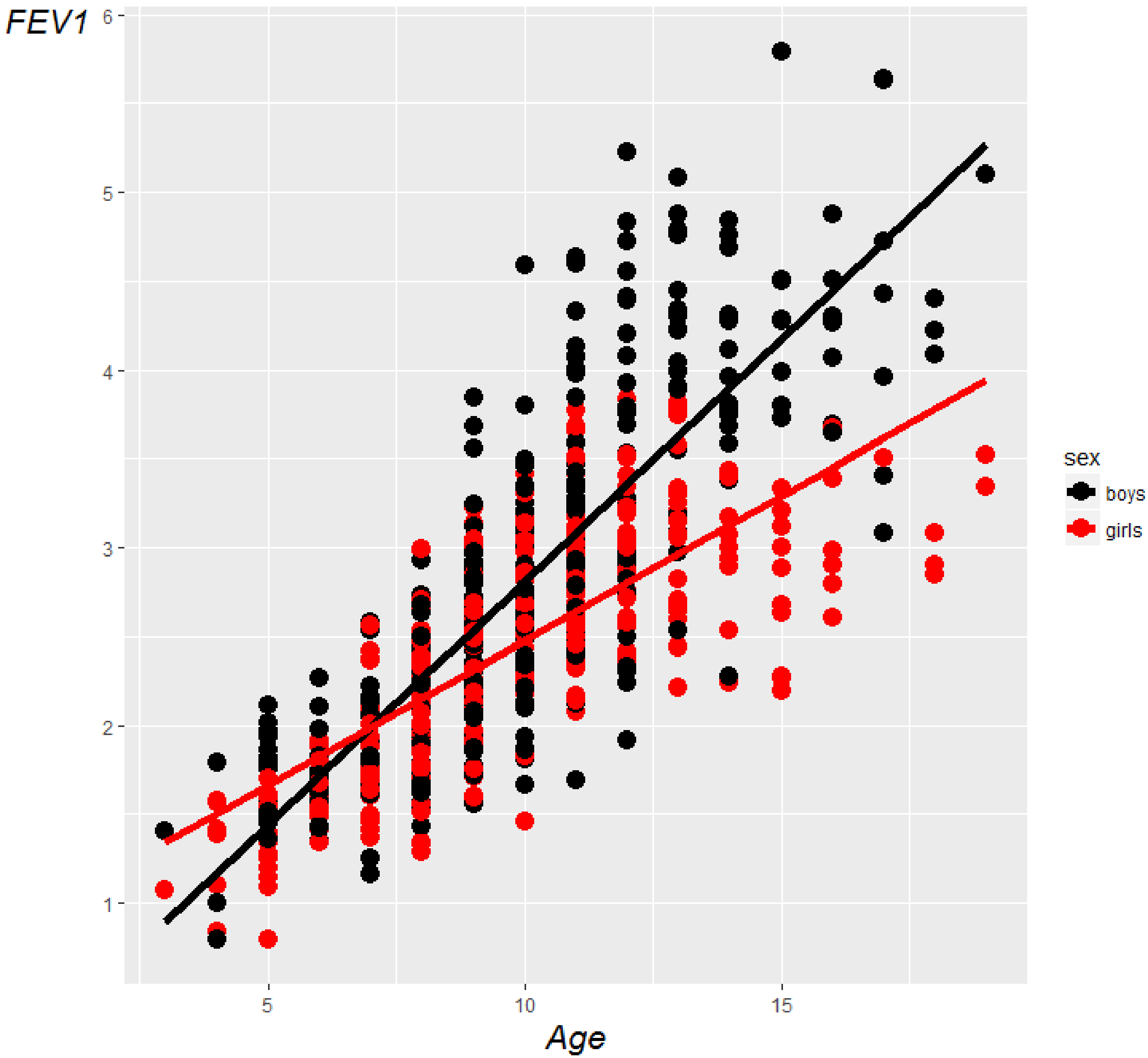
Interaction between One Binary and One Continuous Variables

- The interaction model is to fit two straight lines with different slopes for girls and boys

Id	fev	age	sex	smoking	height	agecat	sex.age
1	1.404	3	0 (boys)	0	131	1	0
2	1.072	3	1 (girls)	0	117	1	3
3	0.839	4	1 (girls)	0	122	1	4
4	1.569	4	1 (girls)	0	127	1	4
5	1.577	4	1 (girls)	0	124	1	4
6	0.796	4	0 (boys)	0	119	1	0
....	....	....	....	....	....	....	....
650	4.404	18	0 (boys)	1	179	0	0
651	2.853	18	1 (girls)	0	152	0	18
652	5.102	19	0 (boys)	0	183	0	0
653	3.519	19	1 (girls)	1	168	0	19
654	3.345	19	1 (girls)	1	166	0	19

Interaction between One Binary and One Continuous Variables

```
> lm5 <- lm(fev ~ sex*age, data = FEV)
> summary(lm5)

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)    0.07360    0.09966   0.739     0.46
sex.girls      0.77587    0.14275   5.435 7.74e-08 ***
age            0.27348    0.00954  28.667  < 2e-16 ***
sex.girls:age -0.11075    0.01379  -8.033 4.47e-15 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '
' 1

Residual standard error: 0.5196 on 650 degrees of freedom
Multiple R-squared:  0.6425,   Adjusted R-squared:  0.6408
F-statistic: 389.4 on 3 and 650 DF,  p-value: < 2.2e-16
```


FEV1
6
5
4
3
2
1
5
10
15
Age
sex
boys
girls

Interaction Between Two Continuous Variables

- We now regress fev on both age and height:

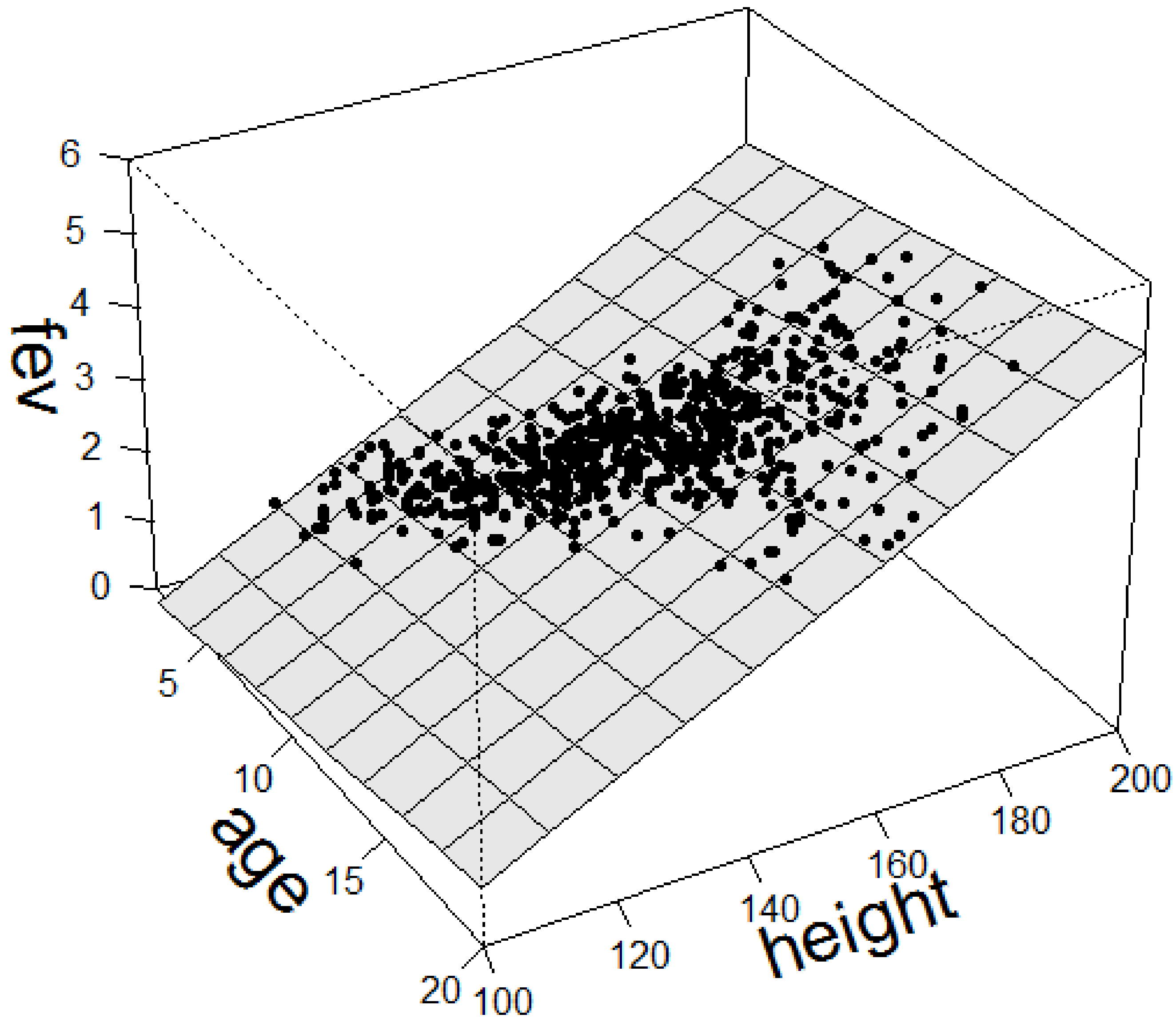
$$\widehat{fev} = b_0 + b_1 age + b_2 height$$

- The fitted values form a plane in a 3-dimensional space
- If we include an interaction between age and height, i.e. a product term into the model:

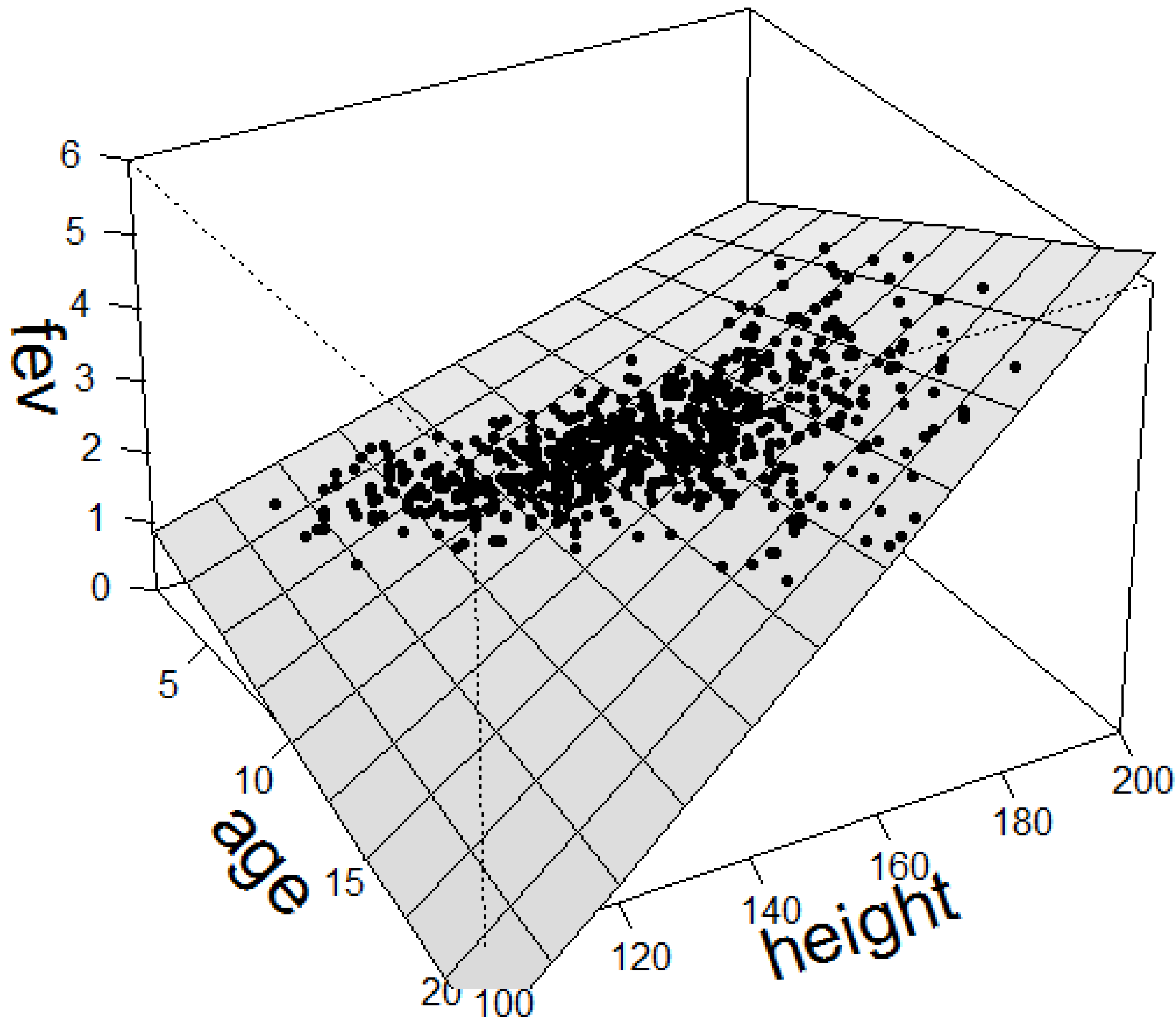
$$\widehat{fev} = b_0 + b_1 age + b_2 height + b_3 age * height$$

- The fitted valued form a curved plane

fev ~ age + height



fev ~ age + height + age*height



Interaction Between Two Continuous Variables

```
> lm7<-lm(fev ~ age*height, data = FEV)
> summary(lm7)

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.7084499  0.5116033  -1.385    0.167
age         -0.4108097  0.0562053  -7.309 7.92e-13 ***
height       0.0182820  0.0034521   5.296 1.62e-07 ***
age:height   0.0029097  0.0003471   8.383 3.19e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '
' 1

Residual standard error: 0.3997 on 650 degrees of freedom
Multiple R-squared:  0.7884,    Adjusted R-squared:  0.7875
F-statistic: 807.4 on 3 and 650 DF,  p-value: < 2.2e-16
```

$$\widehat{fev} = -0.708 - 0.411age + 0.018height + 0.003age * height$$

- We usually only interpret the coefficient for the interaction term, as coefficients for age and height is a little tricky to interpret
- The equation can be re-arranged as:

$$\widehat{fev} = -0.708 + (-0.411 + 0.003height) * age + 0.018height$$

- This means that the effect of age on fev depends on height, i.e. for people with different body heights, the changes in their fev when they become 1 year older are *different*