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Fuzzy cross-entropy, mean, variance, skewness models for portfolio selection

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Abstract In this paper, fuzzy stock portfolio selection models that maximize mean and skewness as well as minimize portfolio variance and cross-entropy are proposed. Because returns are typically asymmetric, in addition to typical mean and variance considerations, third order moment skewness is also considered in generating a larger payoff. Cross-entropy is used to quantify the level of discrimination in a return for a given satisfactory return value. As returns are uncertain, stock returns are considered triangular fuzzy numbers. Stock price data from the Bombay Stock Exchange are used to illustrate the effectiveness of the proposed model. The solutions are done by genetic algorithms.

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1. Introduction

When an investor invests his/her money in a financial market, the aim is to generate maximum profit. As a layman, it seems that if we select only stocks with the highest recent returns, we may generate a substantial amount of profit over a particular interval of time. This idea may sometimes be effective in the short term; but in the long run, it will be difficult to implement because the financial market is generally volatile and uncertain.

Thus, a proper plan for selecting a group of assets is necessary to endure in the market. If a party invests his/her money in a capital market, all we know that the investors undertake the risk. However, it is difficult to measure the risk for a particular portfolio.

Markowitz's (1952) the mean-variance model (MVM), which is based on the assumption that asset returns follow a normal distribution, has been accepted as the pioneer in modeling portfolio selection. The MVM depends only on the first and second order moments of return. However, these moments are typically inadequate for explaining portfolios with non-normal return distributions. Therefore, many studies have discussed whether higher moments may account for this problem. In particular, Chunhachinda et al. (1997), Arditti (1967), as well as Arditti and Levy (1975) assert that higher moments cannot be ignored unless the asset returns are distributed normally. Prakash et al. (2003) and Ibbotson (1975) discuss higher moments in asset allocation where the returns do not follow a symmetric probability distribution. Moreover, they show that

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when skewness is included in the decision-making process, an investor can generate a higher return. Thus, the MVM is extended to a mean–variance–skewness model (MVSM) by including return skewness. As a result, in recent studies (Arditti and Levy, 1975; Prakash et al., 2003; Bhattacharyya et al., 2009, 2011; Bhattacharyya and Kar, 2011a,b) the notion of a mean-variance trade-off has been extended to include skewness in portfolio selection modeling.

Certain studies indicate that the portfolio weights from the MVM and MVSM often focus on a few assets or extreme positions even if an important aim of asset distribution is diversification (Bera and Park, 2008; DeMiguel et al., 2009) because diversification reduces unsystematic risk in the portfolio selection problem. Where the portfolio weights are more diversified, the risk for the portfolio is reduced (Kapur and Kesavan, 1992; Bera and Park, 2005). Diversified portfolios also have lower idiosyncratic volatility than the individual assets (Gilmore et al., 2005). Moreover, the portfolio variance decreases as portfolio diversification increases. To assess diversification, entropy is an established measure of diversity. Greater entropy values in portfolio weights yield higher portfolio diversification. Furthermore, Bera and Park (2005, 2008) have generated asset allocation models based on entropy and cross-entropy to produce a well-diversified portfolio. When entropy is used as an objective function, the weights generated are automatically positive. This means that a model with entropy inherent yields no short-selling, which is preferable in portfolio selection. On the other hand, the liaison between diversification and asymmetry has also been studied in the literature (Simkowitz and Beedles, 1978; Sears and Trennepohl, 1986; Cromwell et al., 2000; Hueng and Yau, 2006).

Given the complexity of the financial system and volatility of the stock market, investors cannot produce accurate expectations for return, risk and additional higher moments. Therefore, the fuzzy set theory, which was proposed by Ammar and Khalifa (2003), is a helpful tool for managing imprecise conditions and attributes in portfolio selection. Instead of the crisp representations used in the related research, in many cases (Bhattacharyya et al., 2009, 2011; Bhattacharyya and Kar, 2011a,b; Zadeh, 1978), the return rates are represented as fuzzy numbers to reflect uncertainty at the evaluation stage. Rather than precisely predicting future return rates, we present the future return rates as fuzzy numbers.

The primary focus of this study is to propose fuzzy cross-entropy-mean–variance–skewness models for portfolio optimization under several constraints. The result facilitates a more reasonable investment decision more suitable for the imprecise financial environment. The results are also compared with models that use the three objectives other than cross-entropy.

This paper is organized as follows. In Section 2, the introductory information required for development of this paper is discussed. In subsection 2.1, the notion of fuzzy cross-entropy is discussed. In subsection 2.2, the credibility theory is used to evaluate mean, variance, skewness and cross-entropy for a fuzzy return. In Section 3, a tetra objective portfolio selection model that maximizes return and skewness as well as minimizes cross-entropy and variance for a portfolio is developed. In addition, several constraints related to the problem are developed to increase the model's effectiveness. In Section 4, a multiple objective genetic algorithm (MOGA) is proposed to resolve the proposed model. In Section 5, numerical results are used to illustrate the method through a case study on stocks from the Bombay Stock Exchange (BSE) in India. Section 6 comprises a comparative study using the proposals herein and other relevant articles. Finally, concluding remarks are in Section 6.

2. Preliminaries

In this section, we discuss introductory information on portfolio selection model construction.

2.1. Fuzzy cross-entropy

Let $\widetilde{A} = (\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2), \ldots, \mu_{\widetilde{A}}(x_n))$ and $\widetilde{B} = (\mu_{\widetilde{B}}(x_1), \mu_{\widetilde{B}}(x_2), \ldots, \mu_{\widetilde{B}}(x_n))$ be two given possibility distributions. For x_i ($i = 1, 2, \ldots, n$), the cross-entropy of $\widetilde{A}$ from $\widetilde{B}$ can be defined as follows:

$$S(\widetilde{A}, \widetilde{B}) = \sum_{i=1}^{n} \left\{ \mu_{\widetilde{A}}(x_i) \ln \frac{\mu_{\widetilde{A}}(x_i)}{\mu_{\widetilde{B}}(x_i)} + (1 - \mu_{\widetilde{A}}(x_i)) \ln \frac{1 - \mu_{\widetilde{A}}(x_i)}{1 - \mu_{\widetilde{B}}(x_i)} \right\}. \tag{1}$$

This expression is the same as the fuzzy information for discrimination favoring $\widetilde{A}$ over $\widetilde{B}$ proposed by Bhandary and Pal (1993). However, it has been noted that Eq. (1) has a drawback; when $\mu_{\widetilde{B}}(x_i)$ approaches 0 or 1, its value will tend toward infinity (Liu, 2010). Therefore, it should be modified (Lin, 1991) as follows:

$$T(\widetilde{A}, \widetilde{B}) = \sum_{i=1}^{n} \left\{ \mu_{\widetilde{A}}(x_i) \ln \frac{\mu_{\widetilde{A}}(x_i)}{(1/2)\mu_{\widetilde{A}}(x_i) + (1/2)\mu_{\widetilde{B}}(x_i)} \right. \\ \left. + (1 - \mu_{\widetilde{A}}(x_i)) \ln \frac{1 - \mu_{\widetilde{A}}(x_i)}{1 - (1/2)(\mu_{\widetilde{A}}(x_i) + \mu_{\widetilde{B}}(x_i))} \right\}. \tag{2}$$

$T(\widetilde{A}, \widetilde{B})$ is well-defined and independent of $\mu_{\widetilde{A}}(x_i)$ and $\mu_{\widetilde{B}}(x_i)$, which is referred to as fuzzy cross entropy and can be used as the level of discrimination between $\widetilde{A}$ and $\widetilde{B}$. Thus, it can also be referred to as discrimination information. Lin (1991) provides a more detailed description.

For the indefinite possibility distribution $\widetilde{A}'$, where there is a prior estimation for $\widetilde{A}'$ and new information on $\widetilde{A}'$, of all distributions that conform to certain constraints, the posterior $\widetilde{A}$ with the least fuzzy cross-entropy $T(\widetilde{A}, \widetilde{B})$ should be chosen. $\widetilde{B}'$ is a prior estimation of $\widetilde{A}'$ (Shore and Johnson, 1981), which is referred to as the minimum fuzzy cross-entropy principle.

Note that $T(\widetilde{A}, \widetilde{B})$ is not symmetric. A symmetric discrimination information measure can be defined based on E as follows: $CE(\widetilde{A}, \widetilde{B}) = T(\widetilde{A}, \widetilde{B}) + T(\widetilde{B}, \widetilde{A})$. This equation is a natural extension of T. Obviously, $CE(\widetilde{A}, \widetilde{B}) \geqslant 0$ and $CE(\widetilde{A}, \widetilde{B}) = 0$ if and only if $\widetilde{A} = \widetilde{B} \cdot CE(\widetilde{A}, \widetilde{B})$ is also finite with respect to $\mu_{\widetilde{A}}(x_i)$ and $\mu_{\widetilde{B}}(x_i)$, i.e., $CE(\widetilde{A}, \widetilde{B})$ is well-defined for all value ranges of $\mu_{\widetilde{A}}(x_i)$ and $\mu_{\widetilde{B}}(x_i)$.

2.2. Credibility theory and its application

In this section, we will use the credibility theory (c.f., Liu, 2010) to generate mean, skewness and cross-entropy of a fuzzy variable.

Definition 2.2.1. The expected value of the fuzzy variable $\breve{\xi}$ is defined as follows:

$$E[\breve{\xi}] = \int_0^{\infty} Cr\{\breve{\xi} \geqslant r\} dr - \int_{-\infty}^{0} Cr\{\breve{\xi} \leqslant r\} dr. \tag{3}$$

The expected value of a triangular fuzzy variable $\breve{\xi} = (a, b, c)$ is as follows:

$$E[\breve{\xi}] = \frac{a + 2b + c}{4}. \tag{4}$$

Definition 2.2.2. Let us suppose that $\breve{\xi}$ is a fuzzy variable with a finite expected value. The variance of $\breve{\xi}$ is defined as follows:

$$V[\breve{\xi}] = E[(\breve{\xi} - E[\breve{\xi}])^2]. \tag{5}$$

The variance for the triangular fuzzy variable $\breve{\xi} = (a, b, c)$ is as follows:

$$V[\breve{\xi}] = \frac{33\alpha^3 + 21\alpha^2\beta + 11\alpha\beta^2 - \beta^3}{384\alpha}, \tag{6}$$

where $\alpha = \max\{b - a, c - b\}$ and $\beta = \min\{b - a, c - b\}$.
Especially, if $b - a = c - b$, then

$$V[\breve{\xi}] = \frac{1}{6}(b - a)^2. \tag{7}$$

Definition 2.2.3. Let us suppose that $\breve{\xi}$ is a fuzzy variable with a finite expected value. The skewness of $\breve{\xi}$ is defined as follows:

$$S[\breve{\xi}] = E[(\breve{\xi} - E[\breve{\xi}])^3]/(V[\xi])^{3/2}. \tag{8}$$

The skewness of a triangular fuzzy variable $\breve{\xi} = (a, b, c)$ is as follows:

$$S[\breve{\xi}] = \frac{(c - a)^2(c - 2b + a)}{32(V[\xi])^{3/2}}, \tag{9}$$

where $V[\xi]$ is generated using Eq. (6).

Definition 2.2.4. $\breve{\xi}$ and $\tilde{\eta}$ are continuous fuzzy variables and $T(s, t) = s \ln\left(\frac{s}{t}\right) + (1 - s) \ln\left(\frac{1-s}{1-t}\right)$. The cross-entropy of $\breve{\xi}$ from $\tilde{\eta}$ is then defined in the following equation.

$$CE[\breve{\xi}; \tilde{\eta}] = \int_{-\infty}^{\infty} T(Cr\{\breve{\xi} = x\}, Cr\{\tilde{\eta} = x\})dx. \tag{10}$$

μ and v are membership functions of $\breve{\xi}$ and $\tilde{\eta}$, respectively. Thus, $Cr\{\breve{\xi} = x\} = \mu(x)/2$ and $Cr\{\tilde{\eta} = x\} = v(x)/2$.
Thus, the cross-entropy for $\breve{\xi}$ and $\tilde{\eta}$ can be written as follows.

$$CE[\breve{\xi}; \tilde{\eta}] = \int_{-\infty}^{\infty} \left(\frac{\mu(x)}{2} \ln\left(\frac{\mu(x)}{v(x)}\right) \right)$$
$$+ \left(1 - \frac{\mu(x)}{2}\right) \ln\left(\frac{2 - \mu(x)}{2 - v(x)}\right) dx. \tag{11}$$

Let $\breve{\xi} = (a, b, c)$ be a triangular fuzzy variable, and $\tilde{\eta}$ is an equipossible fuzzy variable for $[a, c]$. Thus, the cross-entropy for $\breve{\xi}$ and $\tilde{\eta}$ is as follows:

$$CE[\breve{\xi}; \tilde{\eta}] = \left(\ln 2 - \frac{1}{2}\right)(c - a). \tag{12}$$

Theorem 2.2.5. Let $\tilde{r}_i = (a_i, b_i, c_i)[i = 1, 2, \ldots, n]$ be independent triangular fuzzy numbers. Thus,

$$E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n] = \frac{1}{4}\sum_{i=1}^{n}(a_i + b_i + c_i)x_i; \tag{13}$$

$$V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n] = \frac{33\alpha^3 + 21\alpha^2\beta + 11\alpha\beta^2 - \beta^3}{384\alpha}; \tag{14}$$

where

$$\alpha = \max\left\{\sum_{i=1}^{n}b_i x_i - \sum_{i=1}^{n}a_i x_i, \sum_{i=1}^{n}c_i x_i - \sum_{i=1}^{n}b_i x_i\right\}; \tag{15}$$

$$\beta = \min\left\{\sum_{i=1}^{n}b_i x_i - \sum_{i=1}^{n}a_i x_i, \sum_{i=1}^{n}c_i x_i - \sum_{i=1}^{n}b_i x_i\right\}; \tag{16}$$

$$S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n] = \frac{1}{32V^{3/2}}\left(\sum_{i=1}^{n}(c_i - a_i)x_i\right)^2$$
$$\sum_{i=1}^{n}(c_i - 2b_i + a_i)x_i; \tag{17}$$

where V is generated using Eq. (14).

$$CE[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n; \eta]$$
$$= \left(\ln 2 - \frac{1}{2}\right)\sum_{i=1}^{n}(c_i - a_i)x_i. \tag{18}$$

3. Fuzzy portfolio selection model formulation

In this section, we construct the proposed stock portfolio selection models.

3.1. Formulation of the objective functions

Let us consider n risky stocks with the returns $\tilde{r}_i = (a_i, b_i, c_i)$. Let $x = (x_1, x_2, \ldots, x_n)$ be a portfolio. We have considered four objective functions. They are return, skewness, cross-entropy and variance. The portfolio is assumed to strike a balance between maximizing the return and minimizing the risk in investment decisions. Return is quantified as the mean, and risk is the variance for the security portfolio. People interested in considering skewness prefer a portfolio with a higher chance of large payoffs when the mean and variance are constant. Cross-entropy measures the level of return discrimination for a given satisfactory return. Cross-entropy should be minimized. Thus, we maximize both the expected return as well as the skewness and minimize both the variance as well as the cross-entropy for the portfolio x. Thus, we consider the following objective functions.

Maximize $E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n]$

$$= \frac{1}{4}\sum_{i=1}^{n}(a_i + 2b_i + c_i)x_i. \tag{19}$$

Maximize $S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n]$

$$= \frac{1}{32}\left(\sum_{i=1}^{n}(c_i - a_i)x_i\right)^2 \sum_{i=1}^{n}(c_i + a_i - 2b_i)x_i. \tag{20}$$

$$Minimize\ V[\tilde{r}_1x_1 + \tilde{r}_2x_2 + \cdots + \tilde{r}_nx_n]$$
$$= \frac{33\alpha^3 + 21\alpha^2\beta + 11\alpha\beta^2 - \beta^3}{384\alpha}. \tag{21}$$

$$Minimize\ CE[\tilde{r}_1x_1 + \tilde{r}_2x_2 + \cdots + \tilde{r}_nx_n;\eta]$$
$$= \left(\ln 2 - \frac{1}{2}\right)\sum_{i=1}^{n}(c_i - a_i)x_i. \tag{22}$$

3.2. Construction of constraints

In this subsection, we describe the constraints used in the proposed model.

3.2.1. Constraints on short-term and long-term returns

Let, $R_i^{(12)}$ = the average 12 months' return and $R_i^{(36)}$ = the average 36 months' return for an ith security.

For the portfolio $x = (x_1, x_2, \ldots, x_n)$, the expected short-term return is expressed as follows:

$$R_{st}(x) = \sum_{i=0}^{n} R_i^{(12)} x_i. \tag{23}$$

For the portfolio $x = (x_1, x_2, \ldots, x_n)$, the expected long-term return is expressed as follows:

$$R_{lt}(x) = \sum_{i=0}^{n} R_i^{(36)} x_i. \tag{24}$$

Certain investors may plan asset allocation for the short term, long term or both. Such investors would prefer minimum short term, long term or both returns. Thus, investors may consider the following two types of constraints:

$$\left.\begin{array}{l} R_{st} \geqslant \varsigma \\ R_{lt} \geqslant \tau \end{array}\right\}, \tag{25}$$

where ζ and τ are allocated by the investors.

3.2.2. Constraint on the dividend

Dividends are payments made by a company to its shareholders. It is the portion of corporate profits paid to the investors. Let d_i = the estimated annual dividend for the ith security in the next year. For the portfolio $x = (x_1, x_2, \ldots, x_n)$, the annual dividend is expressed as follows:

$$D(x) = \sum_{i=1}^{n} d_i x_i. \tag{26}$$

Certain investors may prefer a portfolio that yields a high dividend. Considering this preference, we propose the following constraint.

$$D(x) \geqslant d. \tag{27}$$

3.2.3. Constraint on the number of allowable assets in the portfolio

Let y_i = the binary variable indicating whether the ith asset is in the portfolio or not. $y_i = 1$, if the ith asset is in the portfolio and 0, otherwise. Let the number of assets an investor can effectively manage in his portfolio be $k(1 \leqslant k \leqslant n)$. Thus,

$$\sum_{i=1}^{n} y_i = k. \tag{28}$$

3.2.4. Constraint on the maximum and minimum investment proportion for a single asset

Let the maximum fraction of the capital that can be invested in a single selected asset i be M_i. Thus,

$$x_i \leqslant M_i y_i. \tag{29}$$

Let the minimum fraction of the capital that can be invested in a single selected asset i be m_i. Thus,

$$x_i \geqslant m_i y_i. \tag{30}$$

The above two constraints ensure that neither a large nor a small portion of the assets is assigned to a single stock in the portfolio. A large investment of the assets in a single stock opposes the standard in selecting a portfolio (i.e., investment diversification). On the other hand, a negligible investment in a portfolio is impractical. For example, investing neither 80% nor 0.0005% of the assets in a single stock in the portfolio is preferred. Note that for $(n - k)$ number of stocks, $x_i = 0$. For the selected stocks, $m_i \leqslant x_i \leqslant M_i$.

3.2.5. Constraint on short selling

No short selling is considered in the portfolio here. Therefore,

$$x_i \geqslant 0\ \forall \quad i = 1, 2, \ldots, n. \tag{31}$$

3.2.6. Capital budget constraint

The well-known capital budget constraint on the assets is as follows:

$$\sum_{i=1}^{n} x_i = 1. \tag{32}$$

Notably, the above constraints are one approach to the problem. It entirely depends upon the investors' perspective.

3.3. Portfolio selection models

Depending on the investors' preference, we propose the following portfolio selection models.

Model 3.3.1: When minimal expected return (ω), maximum variance (ξ) and maximum cross-entropy (θ) are known, the investor will prefer a portfolio with large skewness, which can be modeled as follows:

$$\left.\begin{array}{l} Maximize\ S[\tilde{r}_1x_1 + \tilde{r}_2x_2 + \cdots + \tilde{r}_nx_n] \\ Subject\ to\ the\ constraints \\ E[\tilde{r}_1x_1 + \tilde{r}_2x_2 + \cdots + \tilde{r}_nx_n] \geqslant \omega \\ V[\tilde{r}_1x_1 + \tilde{r}_2x_2 + \cdots + \tilde{r}_nx_n] \leqslant \xi \\ CE[\tilde{r}_1x_1 + \tilde{r}_2x_2 + \cdots + \tilde{r}_nx_n;\eta] \leqslant \theta \\ R_{st}(x) \geqslant \varsigma, R_{lt}(x) \geqslant \tau, D(x) \geqslant d, \sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} y_i = k, \\ x_i \leqslant M_iy_i, x_i \geqslant m_iy_i, x_i \geqslant 0, y_i \in \{0, 1\}, \\ i = 1, 2, 3, \ldots, n. \end{array}\right\} \tag{33}$$

Model 3.3.2: When minimal expected return (ω), maximum variance (ξ) and minimal skewness (ψ) are known, the investor will prefer a portfolio with a small cross-entropy, which can be modeled as follows:

$$
\left.
\begin{aligned}
&\textit{Minimize } CE[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n; \eta] \\
&\textit{Subject to the constraints} \\
&E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n] \geqslant \omega \\
&V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n] \leqslant \xi \\
&S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n] \geqslant \psi \\
&R_{st}(x) \geqslant \varsigma, R_{lt}(x) \geqslant \tau, D(x) \geqslant d, \sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} y_i = k, \\
&x_i \leqslant M_i y_i, x_i \geqslant m_i y_i, x_i \geqslant 0, y_i \in \{0,1\}, \\
&i = 1, 2, 3, \ldots, n.
\end{aligned}
\right\}
$$

(34)

Model 3.3.3: When minimal expected return (ω), minimal skewness (ψ) and maximum cross-entropy (θ) are known, the investor will prefer a portfolio with a small variance, which can be modeled as follows:

$$
\left.
\begin{aligned}
&\textit{Minimize } V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n] \\
&\textit{Subject to the constraints} \\
&E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n] \geqslant \omega \\
&S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n] \geqslant \psi \\
&CE[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n; \eta] \leqslant \theta \\
&R_{st}(x) \geqslant \varsigma, R_{lt}(x) \geqslant \tau, D(x) \geqslant d, \sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} y_i = k, \\
&x_i \leqslant M_i y_i, x_i \geqslant m_i y_i, x_i \geqslant 0, y_i \in \{0,1\}, \\
&i = 1, 2, 3, \ldots, n.
\end{aligned}
\right\}
$$

(35)

Model 3.3.4: When maximum variance (ξ), minimal skewness (ψ) and maximum cross-entropy (θ) are known, the investor will prefer a portfolio with a large expected return, which can be modeled as follows:

$$
\left.
\begin{aligned}
&\textit{Maximize } E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n] \\
&\textit{Subject to the constraints} \\
&V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n] \leqslant \xi \\
&S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n] \geqslant \psi \\
&CE[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n; \eta] \leqslant \theta \\
&R_{st}(x) \geqslant \varsigma, R_{lt}(x) \geqslant \tau, D(x) \geqslant d, \sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} y_i = k, \\
&x_i \leqslant M_i y_i, x_i \geqslant m_i y_i, x_i \geqslant 0, y_i \in \{0,1\}, \\
&i = 1, 2, 3, \ldots, n.
\end{aligned}
\right\}
$$

(36)

Model 3.3.5: A fuzzy tetra-objective optimization model that maximizes both the expected return and the skewness as well as minimizes both the variance and the cross-entropy of the portfolio x is proposed in this model.

$$
\left.
\begin{aligned}
&\textit{Maximize } E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n] \\
&\textit{Minimize } V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n] \\
&\textit{Maximize } S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n] \\
&\textit{Minimize } CE[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \cdots + \tilde{r}_n x_n; \eta] \\
&\textit{Subject to the constraints} \\
&R_{st}(x) \geqslant \varsigma, R_{lt}(x) \geqslant \tau, D(x) \geqslant d, \sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} y_i = k, \\
&x_i \leqslant M_i y_i, x_i \geqslant m_i y_i, x_i \geqslant 0, y_i \in \{0,1\}, \\
&i = 1, 2, 3, \ldots, n.
\end{aligned}
\right\}
$$

(37)

4. Case study: Bombay Stock Exchange stocks

In this section, we apply our portfolio selection model to a historic data set extracted from the BSE, which is the oldest stock exchange in Asia with a rich heritage comprising over 133 years.

We have considered stock returns from the State Bank of India (SBI), Tata Iron and Steel Company (TISCO), Infosys (INFY), Larsen & Tubro (LT), and Reliance Industries Limited (RIL) from April 2005 to March 2010.

Table 1 shows the stocks names and the returns as triangular fuzzy numbers and dividends.

For the above data and models (33)–(37), we consider Examples 1–5, where x_1, x_2, x_3, x_4, and x_5, respectively, represent the proportion of investments for SBI, TISCO, INFY, LT and RIL.

Example 1.

$$
\left.
\begin{aligned}
&\textit{Maximize } S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \\
&\textit{Subject to the constraints} \\
&E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \geqslant 0.38 \\
&V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \leqslant 0.00009 \\
&CE[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5; \eta] \leqslant 0.023 \\
&R_{st}(x) \geqslant 0.034, R_{lt}(x) \geqslant 0.034, D(x) \geqslant 0.2, \sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} y_i = 3, \\
&x_i \leqslant 0.6 y_i, x_i \geqslant 0.05 y_i, x_i \geqslant 0, y_i \in \{0,1\}, \\
&i = 1, 2, 3, 4, 5.
\end{aligned}
\right\}
$$

(38)

Example 2.

$$
\left.
\begin{aligned}
&\textit{Minimize } CE[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5; \eta] \\
&\textit{Subject to the constraints} \\
&E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \geqslant 0.038 \\
&V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \leqslant 0.00009 \\
&S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \geqslant 0.5 \\
&R_{st}(x) \geqslant 0.034, R_{lt}(x) \geqslant 0.034, D(x) \geqslant 0.2, \sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} y_i = 2, \\
&x_i \leqslant 0.6 y_i, x_i \geqslant 0.05 y_i, x_i \geqslant 0, y_i \in \{0,1\}, \\
&i = 1, 2, 3, 4, 5.
\end{aligned}
\right\}
$$

(39)

Example 3.

$$Minimize \; V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5]$$
$$Subject \; to \; the \; constraints$$
$$E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \geqslant 0.38$$
$$S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \geqslant 0.5$$
$$CE[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5; \eta] \leqslant 0.023$$
$$R_{st}(x) \geqslant 0.034, R_{lt}(x) \geqslant 0.034, D(x) \geqslant 0.2, \sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} y_i = 3,$$
$$x_i \leqslant 0.6 y_i, x_i \geqslant 0.05 y_i, x_i \geqslant 0, y_i \in \{0,1\},$$
$$i = 1, 2, 3, 4, 5. \tag{40}$$

Example 4.

$$Maximize \; E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5]$$
$$Subject \; to \; the \; constraints$$
$$V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \leqslant 0.00009$$
$$S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \geqslant 0.5$$
$$CE[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5; \eta] \leqslant 0.023 \tag{41}$$
$$R_{st}(x) \geqslant 0.034, R_{lt}(x) \geqslant 0.034, D(x) \geqslant 0.2,$$
$$\sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} y_i = 2, x_i \leqslant 0.60 y_i, x_i \geqslant 0.05 y_i,$$
$$x_i \geqslant 0, y_i \in \{0,1\}, i = 1, 2, 3, 4, 5.$$

Example 5.

$$Maximize \; E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5],$$
$$Maximize \; S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5]$$
$$Minimize \; V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5],$$
$$Minimize \; CE[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5]$$
$$subject \; to \tag{42}$$
$$R_{st}(x) \geqslant 0.034, R_{lt}(x) \geqslant 0.034, D(x) \geqslant 0.2,$$
$$\sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} y_i = 3, x_i \leqslant 0.60 y_i, x_i \geqslant 0.05 y_i,$$
$$x_i \geqslant 0, y_i \in \{0,1\}, i = 1, 2, 3, 4, 5.$$

We have used a genetic algorithm (GA) to solve problems (38)–(42).

A real vector $X = \{x_1, x_2, x_3, x_4, x_5\}$ is used to represent a solution where each $x_i \in [0,1]$. Ten such vectors are generated to construct the initial population.

We applied an arithmetic cross-over with a 0.6 cross-over probability. A typical unary mutation is applied with a 0.2 mutation probability.

The genetic algorithm proposed by Bhattacharyya et al. (2011) is used to solve problems (38)–(41). The multiple objec-

Table 2 Optimum portfolio.

Example	SBI	TISCO	INFY	LT	RIL
Example 1	0.4549779	0	0.3337914	0.2112307	0
Example 2	0.3798270	0	0.3637823	0.2563907	0
Example 3	0.3504521	0	0.2811333	0.3684146	0
Example 4	0.4140770	0	0.2519898	0.3339332	0
Example 5	0.3492028	0	0.2957778	0.3550194	0

Table 3 Optimal values for return, variance, skewness, and cross-entropy.

Example	Return	Variance	Skewness	Cross-entropy
Example 1	0.38	0.0000857	0.635639	0.023
Example 2	0.38	0.0000753	0.5	0.021840
Example 3	0.40857	0.0000733	0.5	0.021994
Example 4	0.40970	0.0000818	0.619204	0.023
Example 5	0.40428	0.0000729	0.486222	0.021884

tive genetic algorithm (MOGA) proposed by Roy et al. (2008) is followed to solve problem (42).

There are many genetic algorithm studies. Additional articles, such as Srinivas and Patnaik (1994), Aiello et al. (2012) as well as Goswami and Mandal (2012), report attempts to solve the problems underlying proposed models.

In each case, the number of iterations is 100.

The solution is shown in Table 2. The optimal values for the objective functions are shown in Table 3.

The result shown in Table 2 implies that, in Example 5, to generate the desired output (as shown in Table 3), the investor must invest 36%, 59% and 5% of the total capital in *SBI*, *INFY* and *LT* respectively. The optimal portfolio produced is shown in Fig. 1. A similar explanation can support Examples 1–4.

5. Comparative study

In Table 4, we compare the proposed models with other established models from the literature on the portfolio selection problem.

We also compared the results in Tables 2 and 3 with other relevant literature to demonstrate how the results from the proposed technique compare with the literature. Thus, the models in (Markowitz, 1952; Bhattacharyya et al., 2009, 2011; Bhattacharyya and Kar, 2011a) are considered with the same dataset as in Table 1.

Table 1 Stocks and their returns.

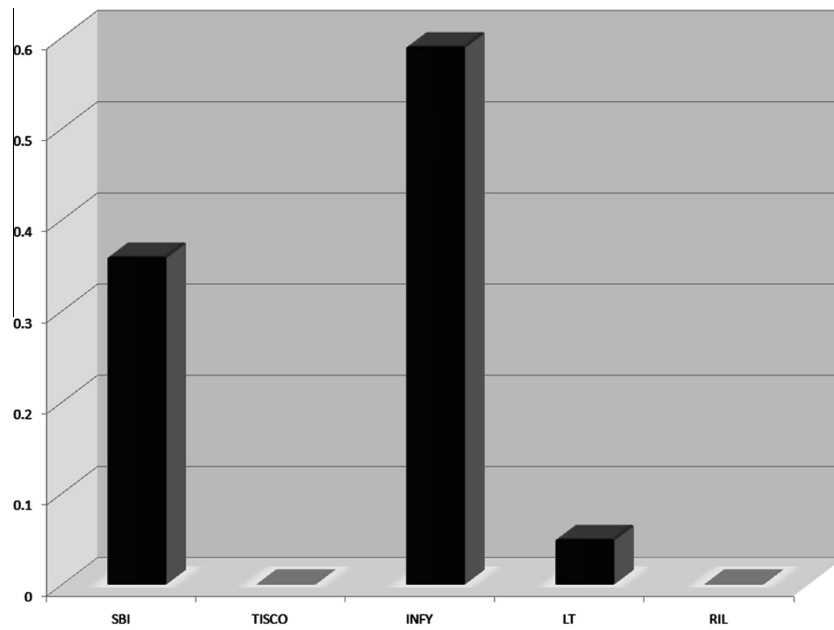| Stock	Return ($\tilde{r}_i$)	Dividend (d_i), %	Short term return (R_{st})	Long term return (R_{lt})
SBI	(0.4000, 0.4054, 0.4500)	20.17	0.4201	0.4180
TISCO	(0.4500, 0.4754, 0.4900)	13.50	0.4701	0.4716
INFY	(0.2200, 0.2366, 0.2400)	25.79	0.2327	0.2320
LT	(0.5200, 0.5370, 0.5500)	15.42	0.5318	0.5369
RIL	(0.2600, 0.2829, 0.3000)	12.50	0.2810	0.2809

Figure 1 Bar diagram of optimum portfolio.

Table 4 Performance matrix.

Article	Mean	Variance	Skewness	Cross-entropy	Genetic algorithm	Case study	Additional constraints	Fuzzy
Markowitz (1952)	✓	✓	✗	✗	✗	✗	✗	✗
Bhattacharyya et al. (2009)	✓	✗	✓	✗	✓	✗	✗	✓
Bhattacharyya et al. (2011)	✓	✓	✓	✗	✓	✓	✓	✓
Bhattacharyya and Kar (2011a)	✓	✓	✓	✗	✓	✓	✗	✓
Bhattacharyya and Kar (2011b)	✓	✓	✓	✗	✗	✓	✓	✓
Proposed Article	✓	✓	✓	✓	✓	✓	✓	✓

We also used the following set of constraints (S) for each case:

$$S = \left\{ \begin{array}{l} R_{st}(x) \geqslant 0.034, R_{lt}(x) \geqslant 0.034, D(x) \geqslant 0.2, \sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} y_i = 2, \\ x_i \leqslant 0.60 y_i, x_i \geqslant 0.05 y_i, x_i \geqslant 0, y_i \in \{0,1\}, i = 1,2,3,4,5. \end{array} \right\} \tag{43}$$

5.1. Markowitz (1952) model

We considered the following model:

$$\left. \begin{array}{l} Minimize \ V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \\ Subject \ to \ the \ constraints \\ E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \geqslant 0.38 \\ x \in S \end{array} \right\} \tag{44}$$

The solution is shown in Table 5.

5.2. Bhattacharyya et al. (2009) model

We considered the following model:

$$\left. \begin{array}{l} Minimize \ Entropy[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \\ Subject \ to \ the \ constraints \\ E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \geqslant 0.38 \\ S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \geqslant 0.5 \\ x \in S \end{array} \right\}. \tag{45}$$

The solution is shown in Table 6.

5.3. Bhattacharyya et al. (2011) model

We considered the following model:

Table 5 Solution for model (44).

SBI	TISCO	INFY	LT	RIL	Return	Variance
0.17006	0	0.36376	0.46618	0	0.40535	0.00005145

Table 6 Solution for model (45).

SBI	TISCO	INFY	LT	RIL	Return	Entropy	Skewness
0.37991	0	0.36375	0.25634	0	0.38	0.016980	0.50

Table 7 Solution for model (46).

SBI	TISCO	INFY	LT	RIL	Return	Variance	Skewness
0.60000	0	0.16683	0.23317	0	0.41302	0.00011024	0.89520

Table 8 Solution for model (47).

SBI	TISCO	INFY	LT	RIL	Return	Variance	Skewness
0.35157	0	0.28062	0.36781	0	0.40859	0.0000735	0.5

$$
\left.\begin{array}{l}
Minimize\{\alpha.V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \\
-\beta.E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \\
-\gamma.S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5]\} \\
Subject\ to\ the\ constraints \\
x \in S
\end{array}\right\}. \qquad (46)
$$

The solution using $\alpha = 1/3$ is shown in Table 7.

5.4. *Bhattacharyya and Kar (2011a)* model

We considered the following model:

$$
\left.\begin{array}{l}
Minimize\ V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \\
Subject\ to\ the\ constraints \\
E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \geqslant 0.38 \\
S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \tilde{r}_3 x_3 + \tilde{r}_4 x_4 + \tilde{r}_5 x_5] \geqslant 0.5 \\
x \in S
\end{array}\right\}. \qquad (47)
$$

The solution using $\alpha = 1/3$ is shown in Table 8.

If we compare Tables 5–8 with Tables 2 and 3, the performance of the proposed model is clearly equivalent or better than the established models.

For example, the solution results in Table 8 and Example 3 in Table 3 are approximately identical. However, comparing the solutions in Table 7 with Example 4 in Table 3, both the cases produce approximately the same returns, but the risk (variance) for model 47 is approximately 1.34 times greater than the risk (variance) in the proposed model (Example 4), which implies that the proposed model is useful.

6. Conclusion

In this paper, we introduced a new fuzzy portfolio selection model that maximizes return and skewness as well as minimizes variance and cross-entropy. Cross-entropy was used to measure the degree of dispersion for the fuzzy return from the desired return.

To solve problems using the models herein, genetic algorithms were used for numerical examples extracted from BSE, India. Our experimental results show that our method performed better than the other models.

For future research, three options are suggested:

1. The proposed method can be used comfortably for larger datasets.
2. Other metaheuristic methods, such as Particle Swarm Optimization (PSO), Ant Colony Optimization (ACO) and Differential Evaluation (DE), could be used to solve problems and compare results from the models herein.
3. The algorithm can also be applied to other stock markets.

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