Computation and Verification of A174511(14)

Number of Isomorphism Types of Subgroups of S_14

Result: A174511(14) = 7,766

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1. Introduction

OEIS sequence A174511 counts the number of isomorphism types of subgroups of the symmetric group S_n. Two subgroups are considered isomorphic if they are isomorphic as abstract groups, not as permutation groups. This report documents the computation of a(14) = 7,766, extending the sequence beyond the previously known a(13) = 3,845.

The computation was carried out using GAP (Groups, Algorithms, Programming) version 4.15.1 on a Windows system with Cygwin and WSL environments. The result was verified through four independent rounds of checking (original computation, double check, triple check, and quadruple check).

2. Known Values

The complete sequence A174511 through n = 14:

n	a(n)	Ratio
1	1	-
2	2	2.000
3	4	2.000
4	9	2.250
5	16	1.778
6	29	1.812
7	55	1.897
8	137	2.491
9	241	1.759
10	453	1.880
11	894	1.974
12	2,065	2.310
13	3,845	1.862
14	7,766	2.020

The ratio a(14)/a(13) = 2.020, consistent with the observed growth pattern of approximately 1.5x to 2.2x per increment.

3. Methodology

3.1 Overview

The computation proceeds in three stages:
(1) Enumerate all conjugacy classes of subgroups of S_14.
(2) Classify each subgroup by abstract isomorphism type.
(3) Deduplicate to count distinct isomorphism types.

3.2 Stage 1: Conjugacy Class Enumeration

We first computed A000638(14) = 75,154, the number of conjugacy classes of subgroups of S_14. This was done via maximal subgroup decomposition: every subgroup of S_14 is contained in at least one maximal subgroup. The maximal subgroups of S_14 are:

- 7 intransitive subgroups: S_k x S_(14-k) for k = 1..7
- 2 wreath products: S_2 wr S_7 and S_7 wr S_2
- Primitive groups of degree 14 (from GAP's primitive groups library)
- The alternating group A_14

Subgroup lattices of each maximal subgroup were computed in parallel (11 workers), producing 600,634 candidate subgroups. These were deduplicated using invariant-based bucketing and pairwise S_14-conjugacy testing, yielding 75,154 conjugacy class representatives. This matches the known value of A000638(14).

3.3 Stage 2: Isomorphism Type Classification

Each of the 75,154 conjugacy class representatives was classified into one of two categories:

- IdGroup-compatible (order < 2,000 and not in {512, 768, 1024, 1536}): 64,467 groups, yielding 4,602 unique IdGroup types.
- Large groups (order >= 2,000 or in {512, 768, 1024, 1536}): 10,687 groups requiring isomorphism deduplication.

GAP's IdGroup function assigns a canonical identifier [order, id] to groups of order less than 2,000 (excluding orders 512, 768, 1024, and 1536 where the small groups library is incomplete). Groups with the same IdGroup are isomorphic.

3.4 Stage 3: Large Group Deduplication

The 10,687 large groups were deduplicated using a multi-level approach:

- Invariant bucketing: Groups were sorted into buckets by a signature key [order, derived_size, conjugacy_classes, derived_length, abelian_invariants] combined with an element-order/fixed-point histogram. Groups in different buckets are guaranteed non-isomorphic.
- Direct product decomposition: 7,431 of the 10,687 groups decompose as direct products of smaller groups. Factor-level isomorphism testing (comparing sorted factor IdGroups via bipartite matching) replaces expensive full-group tests.
- 2-group testing: 336 groups of order 512 were tested using the ANUPQ package's IsIsomorphicPGroup function, which is optimized for p-groups.
- Full isomorphism testing: The remaining non-DP, non-2-group buckets (2,095 groups across 508 buckets) used GAP's IsomorphismGroups for pairwise comparison.

Result: 10,687 large groups deduplicated to 3,164 unique isomorphism types.

4. Result Breakdown

The final count of A174511(14) = 7,766 is composed of:

Category	Groups	Types
IdGroup-compatible	64,467	4,602
Large: Direct products	7,431	2,269
Large: 2-groups (order 512)	336	10
Large: Regular (non-DP, non-2-group)	2,908	884
Large: Difficult bucket (order 2,592)	4	1
TOTAL	**75,154 ***	**7,766**

** 75,154 = A000638(14), the number of conjugacy classes of subgroups of S_14.*

5. Verification History

5.1 Original Computation (January 2026)

The initial computation used the partition-based method from a174511.g, processing all 34 integer partitions of 14. Groups were classified by IdGroup where possible and deduplicated using invariant-based bucketing with CompareByFactorsV3 for direct products and IsomorphismGroups for non-decomposable groups.

5.2 Double Check (January 2026)

An independent 6-way parallel computation re-covered all 34 partitions (3,878 group combinations). Cross-deduplication against S_13 large groups identified 758 duplicates. A bug in the original CompareByFactorsV3 algorithm was discovered: when groups had two semidirect factors in the old ambiguousFactorGens format, only one factor was compared. This caused an undercount of 1 group.

5.3 Triple Check (February 2026)

A completely independent approach using conjugacy class representatives from A000638(14) = 75,154. The factorGens refactor replaced the buggy ambiguousFactorGens with positionally-aligned factorGens for ALL direct product factors. This found 11 additional IdGroup types (1 of order 128 + 10 of order 256) that were lost during the original merge. Result: 4,602 + 3,164 = 7,766.

5.4 Quadruple Check (February 2026)

The quadruple check independently verified the triple check's 3,164 large group representatives using two completely different approaches:

Phase 1: Fresh DP Deduplication (Factor IdGroup Canonicalization)
Instead of pairwise bipartite factor matching (CompareByFactorsV3), a canonical key was computed for each direct product group by taking the sorted list of per-factor IdGroups. Two DP groups are isomorphic iff their sorted factor IdGroup lists match. For factors without IdGroup (order 512, 1024, etc.), extended invariants were used with fallback to pairwise IsomorphismGroups on individual factors. Result: 2,271 DP reps (2 of which overlap with regular buckets in the triple check's partitioning). Cross-check: 2,271 + 10 + 884 + 1 - 2 = 3,164.

Phase 2: Non-DP Bucket Verification
For every non-DP bucket from the triple check (508 multi-group buckets + 6 2-group buckets), the quadruple check verified: (a) all representatives are mutually non-isomorphic, and (b) every non-representative is isomorphic to at least one representative. Six parallel Cygwin workers handled regular buckets; one WSL worker with ANUPQ handled 2-group buckets. Two hard buckets with expensive isomorphism tests (order 2,592 and order 10,368) were re-verified using saved explicit homomorphism proofs.

Quadruple Check Result: ALL PHASES PASSED

Phase	Detail	Result
Phase 1 (DP)	2,271 fresh reps	PASS
Phase 2B Regular (6 workers)	508 buckets, 0 errors	PASS
Phase 2B 2-groups (1 worker)	6 buckets, 0 errors	PASS
Phase 2C Difficult proof	4 groups -> 1 rep	PASS
Phase 2C Hard proof	8 groups -> 1 rep	PASS

6. Correction History

The value of a(14) underwent several corrections during computation:

- 7,095: Initial partition-based computation
- 7,739: After fixing missing partition [8,2,2,2]
- 7,740: After finding 1 additional group from DC verification
- 7,754: After cross-deduplication corrections
- 7,756: After additional bucket analysis
- 7,755: After fixing CompareByFactorsV3 bug (two-semidirect-factor case)
- 7,766: Triple check: 11 missing IdGroup types found (final, verified by quadruple check)

The final value of 7,766 has been independently confirmed by the quadruple check using a completely different DP deduplication algorithm and exhaustive verification of all non-DP isomorphism results.

7. Computational Resources

The computation was performed on a single Windows machine with the following setup:

- GAP 4.15.1 (via Cygwin bash) for group-theoretic computations
- WSL (Windows Subsystem for Linux) with ANUPQ package for 2-group testing
- Python 3.11 for orchestration, parallel worker management, and data processing
- Up to 11 parallel GAP workers, each allocated 8-50 GB memory

Approximate wall-clock times for major computation phases:

- A000638(14) = 75,154 conjugacy classes: ~12 hours (11 parallel workers)
- IdGroup classification (64,467 groups): ~2 hours (8 parallel workers)
- Large group deduplication (10,687 groups): ~6 hours (12 parallel workers)
- Quadruple check verification: ~6 hours (8 parallel workers)

8. Software and Reproducibility

All computation code is available at: https://github.com/jeff87654/symmetric-group-subgroups

Key files:

- triple_check/process_s14_subgroups.g - Main conjugacy class processing
- triple_check/dedupe/ - Isomorphism deduplication pipeline
- triple_check/quad_check/ - Quadruple check verification
- Partition/a174511.g - Original partition-based algorithm
- Partition/tests/test_groups_static.g - 41-group validation test suite

The computation can be independently verified by: (1) computing A000638(14) = 75,154 conjugacy class representatives using ConjugacyClassesSubgroups(SymmetricGroup(14)), then (2) classifying each representative by IdGroup where applicable and deduplicating the remainder by isomorphism testing.

9. Related Sequences

This computation also verified/produced values for related OEIS sequences:

- A000638(14) = 75,154 (conjugacy classes of subgroups of S_14) - matches known value
- A174511(14) = 7,766 (isomorphism types of subgroups of S_14) - NEW

Conclusion: A174511(14) = 7,766

The value a(14) = 7,766 has been computed via conjugacy class enumeration and isomorphism deduplication, and independently verified through four rounds of checking using multiple algorithms. The result is 4,602 IdGroup types plus 3,164 large group representatives = 7,766 total isomorphism types.