Workshop Testing and Formal Methods, Week 4

This workshop is about working with sets and relations. Sets and lists are treated in chapter 4 of "The Haskell Road". Ordered sets of pairs (HR, 4.5) are the basic ingredients of relations. Relations are treated in chapter 5 of "The Haskell Road".

If the following exercises are difficult for you, you should study the relevant material (again).

Question 1 Consider the following relations on the natural numbers. Check their properties. The *successor* relation on $\mathbb{N}$ is the relation given by $\{(n, m) \mid n + 1 = m\}$. The divisor relation on $\mathbb{N}$ is $\{(n, m) \mid n \text{ divides } m\}$. The *coprime* relation C on $\mathbb{N}$ is given by $nCm :\equiv \text{GCD}(n, m) = 1$, i.e., the only factor of n that divides m is 1, and vice versa.

	$<$	$\leq$	successor	divisor	coprime
irreflexive					
reflexive					
asymmetric					
antisymmetric					
symmetric					
transitive					
linear					

Question 2 Consider the relation

$$R = \{(0, 2), (0, 3), (1, 0), (1, 3), (2, 0), (2, 3)\}$$

on the set $A = \{0, 1, 2, 3, 4\}$.

1. Determine R^2, R^3 and R^4.
2. Give a relation S on A such that $R \cup (S \circ R) = S$.

Question 3 The transitive closure of a relation R is by definition the smallest transitive relation S such that $R \subseteq S$. Notation: R^+

Consider again the relation

$$R = \{(0, 2), (0, 3), (1, 0), (1, 3), (2, 0), (2, 3)\}$$

on the set $A = \{0, 1, 2, 3, 4\}$. What is the transitive closure of R?

Question 4 A binary relation R is transitive iff $R \circ R \subseteq R$. You should check this!

Next, give an example of a transitive relation R for which $R \circ R = R$ is false.

Question 5 The reflexive transitive closure of a relation R is by definition the smallest transitive and reflexive relation S such that $R \subseteq S$. Notation: R^*.

Give the reflexive transitive closure of the following relation:

$$R = \{(n, n+1) \mid n \in \mathbb{N}\}.$$

Question 6 The inverse of a relation R is the relation $\{(y, x) \mid (x, y) \in R\}$. Notation R^{-1} or $R^{\smallsmile}$.

1. if S is the successor relation on the natural numbers, what is $S^{\smallsmile}$?
2. if S is the successor relation on the natural numbers, what is $S \cup S^{\smallsmile}$?
3. if S is the successor relation on the natural numbers, what is $(S \cup S^{\smallsmile})^*$?

Question 7 Suppose a relation R satisfies $R^{\smallsmile} \subseteq R$.

1. Does it follow from this that R is reflexive?
2. Does it follow from this that R is symmetric?
3. Does it follow from this that R is transitive?

Question 8

1. Is $R \cup R^{\smallsmile}$ symmetric for all relations R? Give a counterexample if your answer is negative.
2. Is $R^* \cup R^{\smallsmile *}$ symmetric for all relations R? Give a counterexample if your answer is negative.
3. Is $R^* \cup R^{\smallsmile *}$ transitive for all relations R? Give a counterexample if your answer is negative.
4. Is $(R \cup R^{\smallsmile})^*$ an equivalence relation (reflexive, transitive and symmetric) for all relations R?