Workshop Testing and Formal Methods, Week 4: Answers

This workshop is about working with sets and relations. Sets and lists are treated in chapter 4 of “The Haskell Road”. Ordered sets of pairs (HR, 4.5) are the basic ingredients of relations. Relations are treated in chapter 5 of “The Haskell Road”.

If the following exercises are difficult for you, you should study the relevant material (again).

Question 1 Consider the following relations on the natural numbers. Check their properties. The *successor* relation on $\mathbb{N}$ is the relation given by $\{(n,m) \mid n+1=m\}$. The divisor relation on $\mathbb{N}$ is $\{(n,m) \mid n \text{ divides } m\}$. The *coprime* relation C on $\mathbb{N}$ is given by $nCm :\equiv \text{GCD}(n,m)=1$, i.e., the only factor of n that divides m is 1, and vice versa.

	$<$	$\leq$	successor	divisor	coprime
irreflexive					
reflexive					
asymmetric					
antisymmetric					
symmetric					
transitive					
linear					

Answer:

	$<$	$\leq$	successor	divisor	coprime
irreflexive	$\surd$		$\surd$		
reflexive		$\surd$		$\surd$	
asymmetric	$\surd$		$\surd$		
antisymmetric	$\surd$	$\surd$	$\surd$	$\surd$	
symmetric					$\surd$
transitive	$\surd$	$\surd$		$\surd$	
linear	$\surd$	$\surd$			

Note that the *coprime* relation is not irreflexive, for 1 and 1 are coprime.

Question 2 Consider the relation

$$R = \{(0,2),(0,3),(1,0),(1,3),(2,0),(2,3)\}$$

on the set $A = \{0,1,2,3,4\}$.

1. Determine R^2, R^3 and R^4.

2. Give a relation S on A such that $R \cup (S \circ R) = S$.

Answer: $R^2 = \{(0,0),(0,3),(1,2),(1,3),(2,2),(2,3)\}$,
$R^3 = \{(0,2),(0,3),(1,0),(1,3),(2,0),(2,3)\}$
and $R^4 = \{(0,0),(0,3),(1,2),(1,3),(2,2),(2,3)\}$.
From these results we see that $R \cup R^2$ is a good candidate for S. And indeed, if we put

$$S = \{(0,0),(0,2),(0,3),(1,0),(1,2),(1,3),(2,0),(2,2),(2,3)\},$$

we get that $R \cup (S \circ R) = S$.

Question 3 The transitive closure of a relation R is by definition the smallest transitive relation S such that $R \subseteq S$. Notation: R^+

Consider again the relation

$$R = \{(0,2),(0,3),(1,0),(1,3),(2,0),(2,3)\}$$

on the set $A = \{0,1,2,3,4\}$. What is the transitive closure of R?

Answer:

$$\{(0,0),(0,2),(0,3),(1,0),(1,2),(1,3),(2,0),(2,2),(2,3)\}.$$

Same as the S we found in the previous exercise.

Question 4 A binary relation R is transitive iff $R \circ R \subseteq R$. You should check this!

Give an example of a transitive relation R for which $R \circ R = R$ is false.

Answer: What the requirement $R \circ R \subseteq R$ expresses is that if you can get from x to y in two R-steps, then also in one. This is precisely what the transitivity requirement says.

An example of a transitive relation R for which $R \circ R \neq R$ is $<$ on $\mathbb{N}$. We have that $0 < 1$, but $\neg 0 (< \circ <) 1$.

Question 5 The reflexive transitive closure of a relation R is by definition the smallest transitive and reflexive relation S such that $R \subseteq S$. Notation: R^*.

Give the reflexive transitive closure of the following relation:

$$R = \{(n, n+1) \mid n \in \mathbb{N}\}.$$

Answer:

$$\{(n, n+1) \mid n \in \mathbb{N}\}^* = \leq .$$

Question 6 The inverse of a relation R is the relation $\{(y,x) \mid (x,y) \in R\}$. Notation R^{-1} or $R^{\smallsmile}$.

1. if S is the successor relation on the natural numbers, what is $S^{\smile}$?
2. if S is the successor relation on the natural numbers, what is $S \cup S^{\smile}$?
3. if S is the successor relation on the natural numbers, what is $(S \cup S^{\smile})^*$?

Answer:

1. If S is the successor relation on $\mathbb{N}$, then $S^{\smile}$ is the predecessor relation on $\mathbb{N}$.
2. If S is the successor relation on $\mathbb{N}$, then $S \cup S^{\smile}$? is the relation that links 0 to 1, and every number $n+1$ to both n and $n+2$. In a picture:

$$0 - 1 - 2 - 3 - 4 - 5 \cdots$$

3. If S is the successor relation on $\mathbb{N}$, $(S \cup S^{\smile})^*$ is the total relation on $\mathbb{N}$.

Question 7 Suppose a relation R satisfies $R^{\smile} \subseteq R$.

1. Does it follow from this that R is reflexive?
2. Does it follow from this that R is symmetric?
3. Does it follow from this that R is transitive?

Answer: If R satisfies $R^{\smile} \subseteq R$, then

1. it does not follow that R is reflexive (counterexample: $R = \{(0,1),(1,0)\}$);
2. it does follow that R is symmetric;
3. it does not follow that R is transitive (counterexample: $\{R = \{(0,1),(1,0)\}$).

Question 8

1. Is $R \cup R^{\smile}$ symmetric for all relations R? Give a counterexample if your answer is negative.
2. Is $R^* \cup R^{\smile *}$ symmetric for all relations R? Give a counterexample if your answer is negative.
3. Is $R^* \cup R^{\smile *}$ transitive for all relations R? Give a counterexample if your answer is negative.
4. Is $(R \cup R^{\smile})^*$ an equivalence relation (reflexive, transitive and symmetric) for all relations R?

Answer:

1. Is $R \cup R^{\smile}$ symmetric for all relations R? Yes.
2. Is $R^* \cup R^{\smile *}$ symmetric for all relations R? Yes.
3. Is $R^* \cup R^{\smile *}$ transitive for all relations R? No. Counterexample: $R = \{(0,1),(0,2)\}$. This gives

$$\begin{aligned} R^* &= \{(0,1),(0,2),(0,0),(1,1),(2,2)\}, \\ R^{\smile} &= \{(1,0),(2,0)\}, \\ R^{\smile *} &= \{(1,0),(2,0),(0,0),(1,1),(2,2)\}, \end{aligned}$$

   and finally,

$$R^* \cup R^{\smile *} = \{(0,1),(0,2),(1,0),(2,0),(0,0),(1,1),(2,2)\},$$

   which is not a transitive relation, because $(1,2)$ and $(2,1)$ are lacking.
4. Is $(R \cup R^{\smile})^*$ an equivalence relation (reflexive, transitive and symmetric) for all relations R? Yes.