

## Workshop Testing and Formal Methods, Week 4

This workshop is about working with sets and relations. Sets and lists are treated in chapter 4 of “The Haskell Road”. Ordered sets of pairs (HR, 4.5) are the basic ingredients of relations. Relations are treated in chapter 5 of “The Haskell Road”.

If the following exercises are difficult for you, you should study the relevant material (again).

**Question 1** Consider the following relations on the natural numbers. Check their properties. The *successor* relation on  $\mathbb{N}$  is the relation given by  $\{(n, m) \mid n + 1 = m\}$ . The *divisor* relation on  $\mathbb{N}$  is  $\{(n, m) \mid n \text{ divides } m\}$ . The *coprime* relation  $C$  on  $\mathbb{N}$  is given by  $nCm \equiv \text{GCD}(n, m) = 1$ , i.e., the only factor of  $n$  that divides  $m$  is 1, and vice versa.

	$<$	$\leq$	successor	divisor	coprime
irreflexive					
reflexive					
asymmetric					
antisymmetric					
symmetric					
transitive					
linear					

**Question 2** Consider the relation

$$R = \{(0, 2), (0, 3), (1, 0), (1, 3), (2, 0), (2, 3)\}$$

on the set  $A = \{0, 1, 2, 3, 4\}$ .

1. Determine  $R^2$ ,  $R^3$  and  $R^4$ .
2. Give a relation  $S$  on  $A$  such that  $R \cup (S \circ R) = S$ .

**Question 3** The transitive closure of a relation  $R$  is by definition the smallest transitive relation  $S$  such that  $R \subseteq S$ . Notation:  $R^+$

Consider again the relation

$$R = \{(0, 2), (0, 3), (1, 0), (1, 3), (2, 0), (2, 3)\}$$

on the set  $A = \{0, 1, 2, 3, 4\}$ . What is the transitive closure of  $R$ ?

**Question 4** A binary relation  $R$  is transitive iff  $R \circ R \subseteq R$ . You should check this!

Next, give an example of a transitive relation  $R$  for which  $R \circ R = R$  is false.

**Question 5** The reflexive transitive closure of a relation  $R$  is by definition the smallest transitive and reflexive relation  $S$  such that  $R \subseteq S$ . Notation:  $R^*$ .

Give the reflexive transitive closure of the following relation:

$$R = \{(n, n+1) \mid n \in \mathbb{N}\}.$$

**Question 6** The inverse of a relation  $R$  is the relation  $\{(y, x) \mid (x, y) \in R\}$ . Notation  $R^{-1}$  or  $R^\sim$ .

1. if  $S$  is the successor relation on the natural numbers, what is  $S^\sim$ ?
2. if  $S$  is the successor relation on the natural numbers, what is  $S \cup S^\sim$ ?
3. if  $S$  is the successor relation on the natural numbers, what is  $(S \cup S^\sim)^*$ ?

**Question 7** Suppose a relation  $R$  satisfies  $R^\sim \subseteq R$ .

1. Does it follow from this that  $R$  is reflexive?
2. Does it follow from this that  $R$  is symmetric?
3. Does it follow from this that  $R$  is transitive?

**Question 8**

1. Is  $R \cup R^\sim$  symmetric for all relations  $R$ ? Give a counterexample if your answer is negative.
2. Is  $R^* \cup R^{\sim*}$  symmetric for all relations  $R$ ? Give a counterexample if your answer is negative.
3. Is  $R^* \cup R^{\sim*}$  transitive for all relations  $R$ ? Give a counterexample if your answer is negative.
4. Is  $(R \cup R^\sim)^*$  an equivalence relation (reflexive, transitive and symmetric) for all relations  $R$ ?