

Composite Field Mapping Example

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This document will provide an example of mapping from Galois Field $GF(2^8)$ to $GF(((2^2)^2)^2)$, focusing on how the mapping matrix is generated.

Here is a list of the finite fields and the polynomials they are based on used for this example:

$GF(2^8) : x^8 + x^4 + x^3 + x^2 + x + 1$, with primitive element: $\alpha(x)$ to be determined.

$GF(((2^2)^2)^2) : x^2 + x + 1100_2$, with primitive element: $\beta(x) = x + 0$

$GF((2^2)^2) : x^2 + x + 10_2$, with primitive element: $x + 0$

$GF(2^2) : x^2 + x + 1$, with primitive element: $x + 0$

$GF(2^8)$ is to be mapped to $GF(((2^2)^2)^2)$ with the constraints that the mapping will be isomorphic in addition (xor) and multiplication. Let $\text{map}()$ represent the mapping function. While operating in $GF(((2^2)^2)^2)$ the constraints can be stated as (using \cdot to represent multiplication):

$$\text{map}(a + b) = \text{map}(a) + \text{map}(b)$$

$$\text{map}(a \cdot b) = \text{map}(a) \cdot \text{map}(b)$$

The mapping converts like powers of the primitive elements and there is an inverse mapping:

$$\beta^k = \text{map}(\alpha^k)$$

$$\alpha^k = \text{map}^{-1}(\beta^k)$$

To meet the constraints, a search is done for any primitive element of $GF(2^8)$ that will satisfy the mapping constraints, and $\alpha(x) = x^4 + x^3 + x^2 + x + 1$ is one of those elements.

The mapping is done with an 8 row by 8 bit matrix and the inverse mapping is done by the inverse of that 8 row by 8 bit matrix, as shown on the next two pages. The mapping is done via a matrix multiply in $GF(2)$, treating the element to be mapped as 8 row by 1 bit matrix, resulting in an 8 row by 1 bit mapped element. It's easier to understand the mapping matrix by noting that the columns of the matrix correspond to the bits 7 through 0 of elements in $GF(2^8)$, and represent powers of α that result in the values 10000000_2 , 01000000_2 , 00100000_2 , 00010000_2 , 00001000_2 , 00000100_2 , 00000010_2 , 00000001_2 .

The indexes to the columns correspond to powers of α shown in hex. This is easier to understand if the identity matrix is used:

α^{64}	α^{c3}	α^{23}	α^{82}	α^{e1}	α^{41}	α^{a0}	α^{00}
1	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0
0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	1

For the mapping matrix, the values in the columns correspond to powers of β , which are the same powers as in the matrix above, since $\beta^k = \text{map}(\alpha^k)$:

β^{64}	β^{c3}	β^{23}	β^{82}	β^{e1}	β^{41}	β^{a0}	β^{00}
1	0	1	0	0	0	0	0
1	1	0	1	1	1	1	0
1	0	1	0	1	1	0	0
1	0	1	0	1	1	1	0
1	1	0	0	0	1	1	0
1	0	0	1	1	1	1	0
0	1	0	1	0	0	1	0
0	1	0	0	0	0	1	1
fc	4b	b0	46	74	7c	5f	01

For the inverse mapping matrix, the column indexes correspond to powers of β . Again starting with the identity matrix:

β^{67}	β^{bc}	β^{ab}	β^{01}	β^{66}	β^{bb}	β^{aa}	β^{00}
1	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0
0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	1

The values in the inverse matrix columns correspond to powers of α , which are the same powers as in the matrix above, since $\alpha^k = \text{map}^{-1}(\beta^k)$:

α^{67}	α^{bc}	α^{ab}	α^{01}	α^{66}	α^{bb}	α^{aa}	α^{00}
1	1	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	0	0	0	1	0
0	1	1	1	0	1	1	0
0	0	1	1	1	1	1	0
1	0	0	1	1	1	1	0
0	0	1	1	0	0	0	0
0	1	1	1	0	1	0	1
84	f1	bb	1f	0c	5d	bc	01