

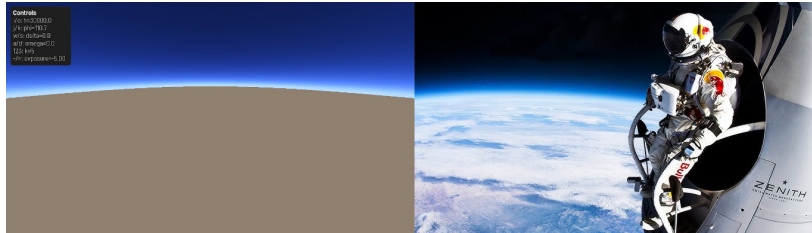
# Atmosphere Plus

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## About

These notes describe the atmospheric rendering theory and implementation for the Atmosphere Plus demo.

The following shows a side-by-side comparison between the rendered atmosphere and a picture of Felix Baumgartner (I Jumped From Space).



## 1 Background

### 1.1 Light and Color

#### 1.1.1 Wavelength of Light

The wavelength of light is a continuous value, however, we typically select 3 discrete wavelengths to represent red, green and blue light.

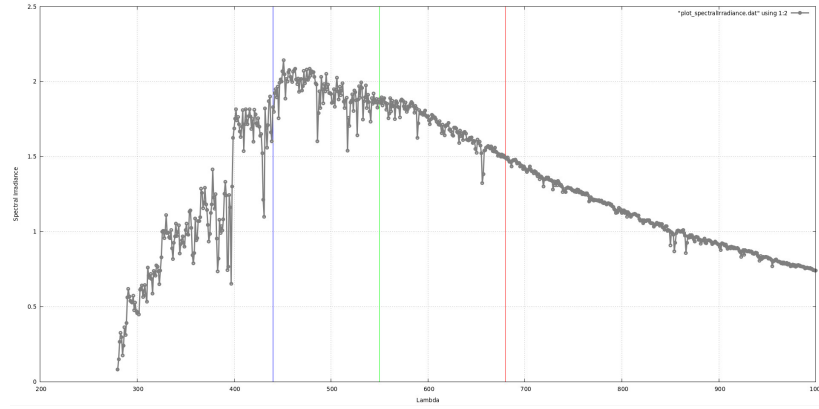
The typical range of wavelengths include:

- $\lambda_r = (620, 750)$  nm
- $\lambda_g = (495, 570)$  nm
- $\lambda_b = (380, 500)$  nm

### 1.1.2 Spectral Irradiance

The spectral irradiance measures the power density of solar radiation at specific wavelengths at the top of atmosphere (TOA) and may be found in tables of measured values.

The following diagram shows the spectral irradiance over the visible spectrum from the ASTM G173 data set.



### 1.1.3 Spectral to RGB Color Conversion

To convert a spectral wavelength  $\lambda$  to an RGB color, we will follow a two step process. First, the wavelengths must be converted to a CIE XYZ (1931) color space by looking up the wavelength in a table. Next, the XYZ coordinates can be transformed into a color space via matrix multiplication. For our purposes we will select the sRGB color space since it has been designed to produce outputs that are optimized for modern displays.

$$C_{rgb} = \text{Normalize} (M^{-1} \cdot XYZ(\lambda))$$

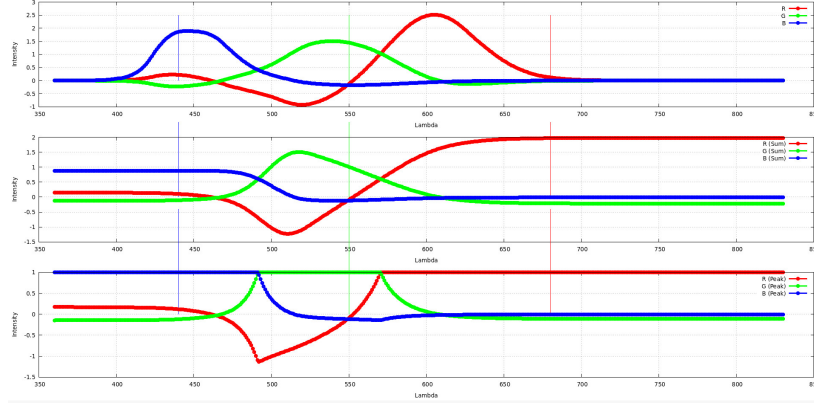
Where the sRGB transformation matrix  $M^{-1}$  is defined.

$$M^{-1} = \begin{bmatrix} 3.2404542 & -1.5371385 & -0.4985314 \\ -0.9692660 & 1.8760108 & 0.0415560 \\ 0.0556434 & -0.2040259 & 1.0572252 \end{bmatrix}$$

The following plot shows the spectral to RGB conversion with various optional normalization techniques.

- None: No normalization
- Sum: Scale XYZ by the sum of XYZ values

- Peak: Scale RGB by the peak RGB value



**Note:** RGB colors should be clamped to (0, 1) for display.

### 1.1.4 Spectral Intensity of Incident Light

The spectral intensity of incident light  $I_I$  is defined by the intensity of light at the top of atmosphere (TOA). The computation of  $I_I$  involves computing the integral of the spectral intensity over all wavelengths while simultaneously converting wavelengths into RGB colors. Do not apply clamping or normalization in the integration process. The resulting output is a high dynamic range (HDR) color.

$$I_I(r, g, b) = M^{-1} \cdot \left( \int_{\lambda} \text{SpectralIrradiance}(\lambda) \cdot XYZ(\lambda) \cdot d\lambda \right)$$

The computed spectral intensity of incident light  $I_I$ .

$$I_I(r, g, b) = (213.865952, 190.346115, 183.806488)$$

## 1.2 Exposure, Tone Mapping and Gamma Correction

The atmospheric simulation inherently outputs high dynamic range intensities that must be transformed for display on standard devices as the final step.

### 1.2.1 Exposure

Exposure scales the overall intensity of color values and mimics the behavior of camera f-stops (e.g. one stop doubles or halves the intensity).

$$Exposure(C) = 2^{exposure} \cdot C$$

Where the default exposure is defined.

$$exposure = -2.5$$

### 1.2.2 Tone Mapping Operation (TMO)

Tone mapping operations transform the high dynamic range intensities such that the colors are mapped to the displayable range (0.0, 1.0) while preserving contrast in low-intensity areas and compressing high-intensity values. The transformation from HDR-to-LDR (High Dynamic Range to Low Dynamic Range) is a lossy operation which produces subjective results. Experimental results suggest that the Hable (Uncharted 2) TMO is suitable for atmospheric simulations.

The  $TMO_{HP}$  defines a partial operation that is used to compute  $TMO_H$ .

$$A, B, C, D, E, F = (0.15, 0.50, 0.10, 0.20, 0.02, 0.30)$$

$$TMO_{HP}(X) = \left( \frac{X \cdot (A \cdot X + C \cdot B) + D \cdot E}{X \cdot (A \cdot X + B) + D \cdot F} \right) - \frac{E}{F}$$

The  $TMO_H$  operator is referred to as a filmic operation since it is designed to emulate real film.

$$ExposureBias = 2.0$$

$$W = vec3(11.2)$$

$$TMO_H(X) = \frac{TMO_{HP}(X \cdot ExposureBias)}{TMO_{HP}(W)}$$

### 1.2.3 Gamma Correction

Gamma correction helps to distribute the compressed values more evenly across the perceptual range of human vision.

$$Gamma(C) = C^{1.0/2.2}$$

### 1.2.4 HDR-to-LDR Transformation

The HDR-to-LDR conversion may include exposure, tone mapping and gamma correction.

$$C_{LDR} = \text{Gamma}(\text{TMO}(\text{Exposure}(C_{HDR})))$$

## 1.3 Atmospheric Boundary

The atmospheric simulation models the atmosphere as the region where the effects of the atmosphere are non-negligible, extending from the Earth's surface to an atmospheric boundary.

The Scalable and Production Ready Sky and Atmosphere Rendering Technique paper uses the following constants for these radii.

- Radius of the planet:  $R_p = 6360000$  m
- Radius of the atmospheric boundary:  $R_a = 6460000$  m

## 1.4 Atmospheric Particles

The atmospheric simulation models three types of particles.

- Rayleigh Scattering: Smaller particles consisting of air molecules such as oxygen, nitrogen and carbon dioxide.
- Mie Scattering and Absorption: Larger particles consisting of ice crystals, water droplets, dust and air pollution.
- Ozone Absorption: Ozone particles consist of  $O_3$ .

## 1.5 Density Function

### 1.5.1 Definition of the Density Function

The density function expresses the decrease in atmospheric density in dependence on  $h$ , the altitude of a point  $P$  over the ground.

Rayleigh/Mie density function,  $p_{R,M}(h)$ :

$$p_{R,M}(h) = \exp\left(-\frac{h}{H_{R,M}}\right)$$

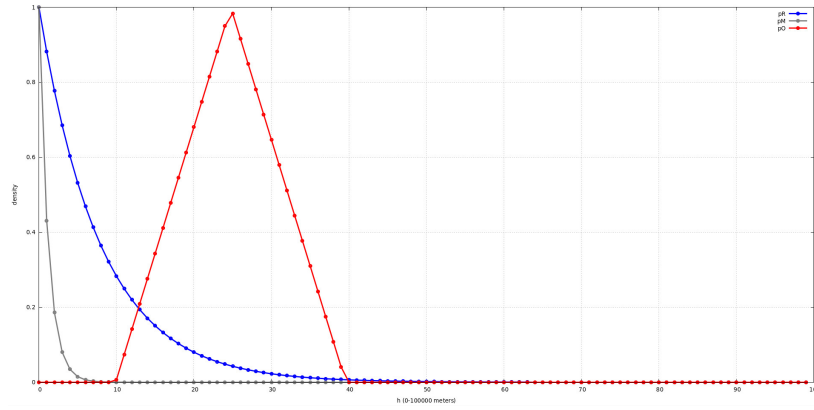
Where  $H_R \approx 8000m$  and  $H_M \approx 1200m$  are the Rayleigh/Mie scale heights (e.g. the altitude where the density of particles scales down by a  $\frac{1}{e}$  term).

Ozone density function,  $p_O(h)$ :

$$p_O(h) = \max\left(0, 1 - \frac{|h - 25000|}{15000}\right)$$

### 1.5.2 Visualization of the Density Function

The following diagram shows the exponential decay of the density (Rayleigh, Mie and Ozone) as a function of height.



### 1.6 Phase Function

The phase function expresses the relative amount of light that is scattered in a particular direction due to interactions with a particle.

Rayleigh scattering phase function,  $F_R$ :

$$F_R(\cos(\theta)) = \frac{3}{4} (1 + \cos^2(\theta))$$

Where the scattering angle  $\theta$  represents the angle between the incoming light ray and the scattered light ray.

$$\cos(\theta) = \text{dot}(L_{\text{Incoming}}, L_{\text{Scattered}})$$

The Rayleigh scattering phase function  $F_R$  may be modified, in practice, to produce more natural results given simplifications introduced for the scattering intensity parameterization.

$$F_R(\cos(\theta)) = \frac{8}{10} \left( \frac{7}{5} + \frac{1}{2} \cos(\theta) \right)$$

Mie scattering phase function,  $F_M$ :

$$F_M(\cos(\theta)) = \frac{3(1-g^2)}{2(2+g^2)} \frac{(1+\cos^2(\theta))}{(1+g^2-2g\cos(\theta))^{\frac{3}{2}}}$$

The parameter  $g$  is an asymmetry factor denoting the width of the forward lobe and is in the range  $(-1, 1)$ .

The Scalable and Production Ready Sky and Atmosphere Rendering Technique paper uses  $g = 0.8$ .

## 1.7 Scattering, Absorption and Extinction Coefficients

The scattering/absorption/extinction coefficients represents the probability of light being scattered/absorbed/scattered+absorbed as it travels through a medium.

The Scalable and Production Ready Sky and Atmosphere Rendering Technique paper uses the following coefficients.

$$\beta_R(r, g, b) = (5.802e-6, 13.558e-6, 33.1e-6)$$

$$\beta_M^s = 3.996e-6$$

$$\beta_M^a = 4.40e-6$$

$$\beta_O^a = (0.65e-6, 1.881e-6, 0.085e-6)$$

$$\beta_{R,M}^e = \beta_{R,M}^s + \beta_{M,O}^a$$

The Rayleigh coefficients were determined assuming the following wavelengths.

- $\lambda_r = 680$  nm
- $\lambda_g = 550$  nm
- $\lambda_b = 440$  nm

## 1.8 Transmittance

### 1.8.1 Definition of Transmittance

The transmittance function  $T(P_1, P_2, \lambda)$  expresses the amount of attenuated light after it passes between two points through a medium.

$$T(P_1, P_2, \lambda) = \exp(-t(P_1, P_2, \lambda))$$

$$t(P_1, P_2, \lambda) = \sum_{i \in R, M, O} \beta_i^e(\lambda) \int_{P_1}^{P_2} p_i(h(P)) ds$$

Where the parameters are defined:

- $T$ : Transmittance,  $T \in (0, 1)$
- $t$ : Optical depth,  $t \in (0, \infty)$
- $P$ : Sample point parameterized by  $s$ ,  $P = P_1 + s(P_2 - P_1)$
- $h$ : Height is parameterized by  $P$ ,  $h(P) = |P| - R_p$

The Rendering Parametrizable Planetary Atmospheres with Multiple Scattering in Real-Time paper uses numerical integration with  $n = 30$  steps to compute the transmittance.

The light ray may change direction as it interacts with particles while passing through a medium. Consider a light ray which passes between the points  $P_1, P_2, P_3$  where the point  $P_2$  represents an interaction which changes the direction of the light ray. The transmittance between the points  $P_1, P_2, P_3$  may be computed as the product of the transmittance between each segment.

$$T(P_1, P_2, \lambda) \cdot T(P_2, P_3, \lambda) = \exp(-t(P_1, P_2, \lambda)) \cdot \exp(-t(P_2, P_3, \lambda))$$

The implementation may utilize the laws of exponents rule for product of powers to optimize the computation.

$$a^m \cdot a^n = a^{m+n}$$

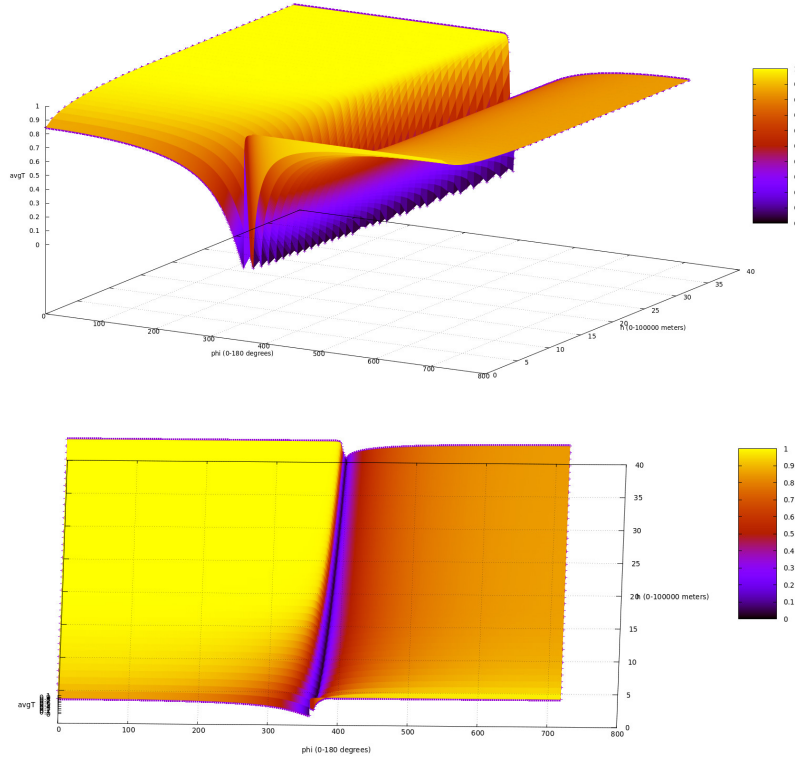
Where the functions  $t(\dots)$  may be rearranged in order to group similar terms.

$$T(P_1, P_2, \lambda) \cdot T(P_2, P_3, \lambda) = \exp(-t(P_1, P_2, \lambda) - t(P_2, P_3, \lambda))$$



### 1.8.2 Visualization of Transmittance

The following diagrams show the average transmittance as a function of altitude  $h$  and the View-Zenith Angle  $\phi$ . Notice how the valley follows the horizon as the height increases. The transmittance is much lower at the horizon since the atmosphere is much thinner in the vertical direction than the horizontal direction. See the Scattering Intensity Parameterization section for more details.



## 2 Sky Rendering

### 2.1 Single-scattering

The single-scattering equation  $I_{S_{R,M}}^{(1)}$  describes the intensity of light that reaches an observer  $P_0$  looking in the direction  $V$  after exactly one scattering event.

$$I_{S_{R,M}}^{(1)}(P_0, V, L, \lambda) = I_I(\lambda) \cdot F_{R,M}(\text{dot}(L, -V)) \cdot \frac{\beta_{R,M}^s(\lambda)}{4\pi} \cdot \int_{P_a}^{P_b} p_{R,M}(h(P)) \cdot T(P_c, P, \lambda) \cdot T(P, P_a, \lambda) ds$$

Where the following parameters are defined:

- $P$ : Sample point parameterized by  $s$ ,  $P = P_a + s(P_b - P_a)$
- $h$ : Height is parameterized by  $P$ ,  $h(P) = |P| - R_p$
- $V$ : Viewing direction,  $V = \text{normalize}(P_b - P_a)$
- $L$ : Direction of light from the Sun
- $\theta$ : Scattering angle,  $\cos(\theta) = \text{dot}(L, -V)$
- $P_a$ : First point along  $V$  where the atmosphere is nonzero (e.g.  $P_0$  or the atmospheric boundary).
- $P_b$ : Last point along  $V$  where the atmosphere is nonzero (e.g. the atmospheric boundary or the Earth's surface).
- $P_c$ : The nearest intersection point along the ray from  $P$  in the direction of the Sun (e.g. the atmosphere boundary or the Earth's surface). If the ray intersects the Earth then the point  $P$  is shadowed by the Earth and does not contribute to the single-scattering intensity. However, the point  $P$  may still contribute to the multiple scattering intensity as described in the next section.

The phase function  $F_{R,M}$  can be excluded from integration if we assume all light rays coming from the Sun are parallel.

The total intensity of the single-scattering light:

$$I_S^{(1)} = I_{S_R}^{(1)} + I_{S_M}^{(1)}$$

## 2.2 Multiple-scattering

The multiple-scattering equation  $I_{S_{R,M}}^{(k)}$  describes the intensity of light that reaches an observer  $P_0$  looking in the direction  $V$  for the  $k$ th scattering event.

$$I_{S_{R,M}}^{(k)}(P_0, V, L, \lambda) = \frac{\beta_{R,M}^s(\lambda)}{4\pi} \cdot \int_{P_a}^{P_b} G_{R,M}^{(k-1)}(P, V, L, \lambda) \cdot p_{R,M}(h(P)) \cdot T(P, P_a, \lambda) ds$$

Where the gather-scattering equation  $G_{R,M}^{(k)}$  describes the intensity of light that reaches the point  $P$  that is scattered from all directions  $\omega$  towards the observer for the  $k$ th order scattering event.

$$G_{R,M}^{(k)}(P, V, L, \lambda) = \int_{4\pi} F_{R,M}(\text{dot}(-\omega, -V)) \cdot I_{S_{R,M}}^{(k)}(P, \omega, L, \lambda) d\omega$$

Where the following parameters are defined:

- $\omega$ : Gathering direction for the kth order scattered light source
- $\theta$ : Scattering angle,  $\cos(\theta) = \text{dot}(-\omega, -V)$

The first order of  $I_{S_{R,M}}^{(k)}$  is initialized with the single-scattering output  $I_{S_{R,M}}^{(1)}$  while subsequent orders of  $I_{S_{R,M}}^{(k)}$  are computed iteratively by sampling the previous output  $I_{S_{R,M}}^{(k-1)}$ .

Observe that the light vector  $L$  is used in computation of the phase function  $F_{R,M}$  for the single-scattering equation  $I_{S_{R,M}}^{(1)}$ , but not for the gather-scattering equation  $G_{R,M}^{(k)}$ . As a result, the phase function  $F_{R,M}$  in the gather-scattering equation  $G_{R,M}^{(k)}$  may not be excluded from integration because  $\theta$  depends on the integration variable  $\omega$ . The light vector  $L$  is also required to parameterize the Sun-Zenith angle for all scattering orders as shown in subsequent sections.

The total intensity of the multiple-scattering light:

$$I_S = \sum_{k=1}^K I_{S_R}^{(k)} + I_{S_M}^{(k)} = I_{S_R} + I_{S_M}$$

### 2.3 Scattering Equation Factorization

The following derivation factors the constant phase function  $F_{R,M}(\text{dot}(L, -V))$  and the spectral intensity of incident light  $I_I(\lambda)$  from the scattering equations.

The factored single-scattering equation  $\bar{I}_{S_{R,M}}^{(1)}$ :

$$\bar{I}_{S_{R,M}}^{(1)}(P_0, V, L, \lambda) = \frac{\beta_{R,M}^s(\lambda)}{4\pi}.$$

$$\cdot \int_{P_a}^{P_b} p_{R,M}(h(P)) \cdot T(P_c, P, \lambda) \cdot T(P, P_a, \lambda) ds$$

Where  $I_{S_{R,M}}^{(1)}$ :

$$I_{S_{R,M}}^{(1)}(P_0, V, L, \lambda) = I_I(\lambda) \cdot F_{R,M}(\text{dot}(L, -V)) \cdot \bar{I}_{S_{R,M}}^{(1)}(P_0, V, L, \lambda)$$

The factored multiple-scattering equation  $\bar{I}_{S_{R,M}}^{(k)}$ :

$$\bar{I}_{S_{R,M}}^{(k)}(P_0, V, L, \lambda) = \frac{\beta_{R,M}^s(\lambda)}{4\pi} \cdot \int_{P_a}^{P_b} \bar{G}_{R,M}^{(k-1)}(P, V, L, \lambda) \cdot p_{R,M}(h(P)) \cdot T(P, P_a, \lambda) ds$$

Where  $I_{S_{R,M}}^{(k)}$ :

$$I_{S_{R,M}}^{(k)}(P_0, V, L, \lambda) = I_I(\lambda) \cdot F_{R,M}(\text{dot}(L, -V)) \cdot \bar{I}_{S_{R,M}}^{(k)}(P_0, V, L, \lambda)$$

The factored gather-scattering equation  $\bar{G}_{R,M}^{(k)}$ :

$$\bar{G}_{R,M}^{(k)}(P, V, L, \lambda) = \int_{4\pi} F_{R,M}(\text{dot}(-\omega, -V)) \cdot \bar{I}_{S_{R,M}}^{(k)}(P, \omega, L, \lambda) d\omega$$

Where  $G_{R,M}^{(k)}$ :

$$G_{R,M}^{(k)}(P, V, L, \lambda) = I_I(\lambda) \cdot F_{R,M}(\text{dot}(L, -V)) \cdot \bar{G}_{R,M}^{(k)}(P, V, L, \lambda)$$

The total scattering intensity using the factored equations:

$$I_S = I_I(\lambda) \cdot (F_R(\text{dot}(L, -V)) \cdot \bar{I}_{S_R} + F_M(\text{dot}(L, -V)) \cdot \bar{I}_{S_M})$$

This factorization includes two important properties that facilitate an efficient rendering implementation. First, the wavelength dependent components,  $I_I(\lambda)$  and  $\bar{I}_{S_R}$ , may be separated from the wavelength independent component  $\bar{I}_{S_M}$ . The spectral intensity of incident light  $I_I(\lambda)$  may be applied directly in the fragment shader and the factored scattering intensity  $\bar{I}_S$  may be determined by performing a single 3D texture fetch. The Rayleigh factored scattering intensity  $\bar{I}_{S_R}$  is stored in the RGB channels while the Mie factored scattering intensity  $\bar{I}_{S_M}$  is stored in the alpha channel. Second, the constant phase function  $F_{R,M}(\text{dot}(L, -V))$  may be applied directly in the fragment shader which partially accounts for the omitted Sun-View Azimuth parameter. These properties are fundamental to the scattering intensity parameterization that is described in the next section.

## 2.4 Scattering Intensity Parameterization

### 2.4.1 Canonical Form

To precompute every scattering intensity from every position  $P_0(x, y, z)$ , in every viewing direction  $V(x, y, z)$  and every light direction  $L(x, y, z)$  would require 9 parameters. However, by taking advantage of symmetries and making a few assumptions the parameter count may be reduced to 4 scalar parameters.

- Altitude:  $h \in [0, H_a]$  where  $H_a = R_a - R_p$
- View-Zenith Angle:  $\phi \in [0, \pi]$  where  $\cos(\phi) = \text{dot}(V, \text{Zenith})$
- Sun-Zenith Angle  $\delta \in [0, \pi]$  where  $\cos(\delta) = \text{dot}(-L, \text{Zenith})$
- Sun-View Azimuth  $\omega \in [0, \pi]$  where  $\cos(\omega) = \text{dot}(\text{proj}(V), \text{proj}(-L))$

**Note:** The the Sun-View Azimuth  $\omega$  is a separate variable from the gathering direction  $\omega$  described in other sections.

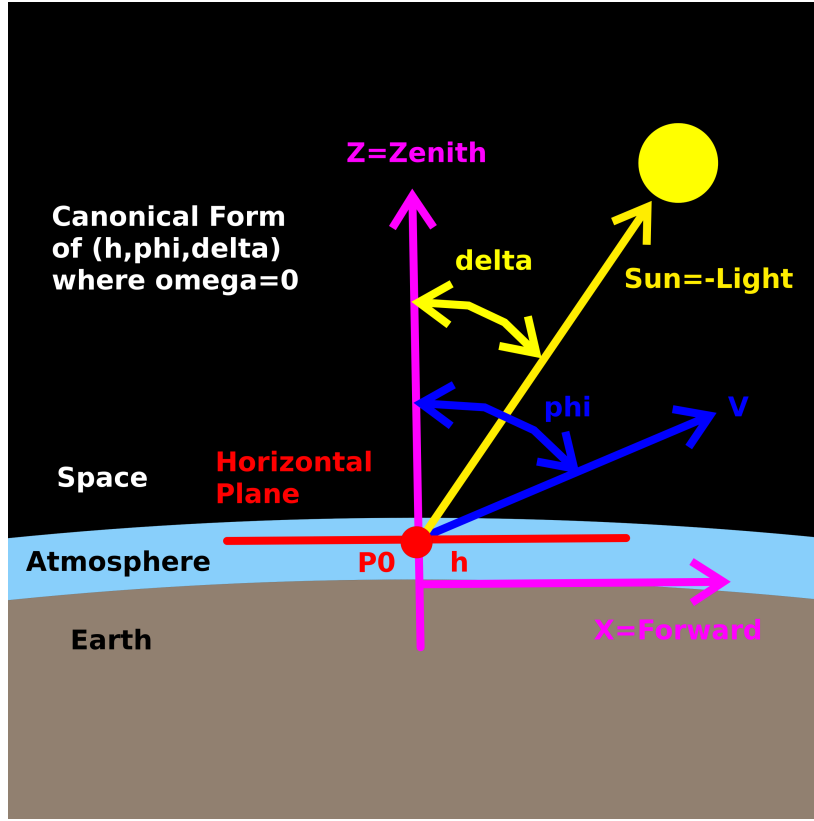
The Zenith is the up vector for the point  $P_0$ .

$$\text{Zenith} = \text{normalize}(P_0)$$

The  $\text{proj}(X)$  function projects the unit vector  $X$  onto the horizontal plane perpendicular to the Zenith.

$$\text{proj}(X) = \text{normalize}(X - \text{dot}(X, \text{Zenith}) \cdot \text{Zenith})$$

The following diagram shows canonical form of the scattering intensity parameterization. Observe that the atmosphere is much thinner in the vertical direction than the horizontal direction.



The parameter  $h$  may be converted to the canonical form  $P_0$ .

$$P_0 = (0, 0, h + R_p)$$

The parameters  $(h, \phi, \delta)$  may be converted to the canonical form vectors  $V$  and  $L$  by using the spherical coordinate system.

$$x = r \sin(\Theta) \cos(\Phi)$$

$$y = r \sin(\Theta) \sin(\Phi)$$

$$z = r \cos(\Theta)$$

Therefore  $V$ ,  $Sun$  and  $L$  are defined where  $r = 1$ ,  $\Theta = \phi$  for  $V$ ,  $\Theta = \delta$  for  $Sun$  and  $\Phi = \omega = 0$ .

$$V = (\sin(\phi), 0, \cos(\phi))$$

$$Sun = (\sin(\delta), 0, \cos(\delta))$$

$$L = -Sun$$

### 2.4.2 Linear Mapping

The parameters  $(h, \cos(\phi), \cos(\delta))$  may be converted to 3D texture coordinates  $(u, v, w)$  for shader lookups.

The most straightforward mapping described by the Efficient and Dynamic Atmospheric Scattering paper is a linear mapping.

$$u = \frac{h}{H_a}$$

$$v = \frac{\cos(\phi) + 1}{2}$$

$$w = \frac{\cos(\delta) + 1}{2}$$

The linear mappings may be converted from 3D texture coordinates as follows.

$$h = u \cdot H_a$$

$$\cos(\phi) = 2v - 1$$

$$\cos(\delta) = 2w - 1$$

### 2.4.3 Power Mapping

Linear mappings are typically replaced with nonlinear mappings in order to optimize the parameterization space. This is achieved by eliminating unused values and improving precision for critical values by compressing less critical values. The linear and nonlinear mappings may be generalized for an arbitrary power as follows.

$$u(h, p_u) = \left( \frac{h}{H_a} \right)^{\frac{1}{p_u}}$$

$$v(\phi, p_v) = \frac{\text{sign}(\cos(\phi)) \cdot |\cos(\phi)|^{\frac{1}{p_v}} + 1}{2}$$

$$w(\delta, p_w) = \frac{\text{sign}(\cos(\delta)) \cdot |\cos(\delta)|^{\frac{1}{p_w}} + 1}{2}$$

The power mappings may be converted from 3D texture coordinates as follows.

$$h = u^{p_u} \cdot H_a$$

$$\cos(\phi) = \text{sign}(2v - 1) \cdot |2v - 1|^{p_v}$$

$$\cos(\delta) = \text{sign}(2w - 1) \cdot |2w - 1|^{p_w}$$

#### 2.4.4 Weighted Power Mapping

The View-Zenith Angle is especially difficult to generalize due to the following conditions which lead to inefficient parameterization.

- The horizon  $\phi_H$  increases from 90 degrees at ground level towards 180 degrees as the height increases. However, the published approaches typically fix the horizon  $\phi_H$  at 90 degrees for all heights which causes incorrect angles to be compressed as the height increases.
- The horizon also vanishes quickly as the height increases leading to ineffective compression at different altitudes.
- At ground level there is no atmosphere when looking downwards, whereas at the TOA there is no atmosphere when looking upwards. A large percentage of the texture space is dedicated to parameters which represent little or no atmosphere.

The weighted power mapping addresses these problems by incorporating the height  $h$  into the parameterization for the  $v$  coordinate.

The horizon  $\phi_H$  is defined by a triangle consisting of a point at the center of the planet, a point tangent to the planet and a point at the observer. The function  $\cos(\phi_H)$  may be derived given the trigonometric identities  $\cos(\theta) = \frac{adj}{hyp}$ ,  $a^2 + b^2 = c^2$  and the fact that the horizon  $\phi_H \geq 90$  (e.g.  $\cos(\phi_H) \leq 0$ ).

$$\cos(\phi_H) = -\frac{\sqrt{(R_p + h)^2 - R_p^2}}{R_p + h}$$



The cosine of the horizon angle  $\cos(\phi_H)$  and the horizontal angle  $\cos(\phi_{90}) = 0$  are used to define several regions ( $U$ : upper atmosphere,  $L$ : lower atmosphere,  $S$ : surface) which may be parameterized independently.

In general, each region is defined by a weight  $w_i \in (0, 1)$ , a power mapping  $f(\cos(\phi), p_i) \in (0, 1)$  and an offset  $o_i \in (0, 1)$ .

$$v_i = w_i \cdot \left( \frac{\cos(\phi) - \cos(\phi_{min})}{\cos(\phi_{max}) - \cos(\phi_{min})} \right)^{\frac{1}{p_i}} + o_i$$

The upper atmosphere region is defined by the angles,  $0 \leq \cos(\phi) \leq 1$ .

$$v_U(\phi) = w_U \cdot (\cos(\phi))^{\frac{1}{p_U}} + (w_L + w_S)$$

The lower atmosphere region is defined by the angles,  $\cos(\phi_H) \leq \cos(\phi) < 0$ . Since  $\cos(\phi_H) = 0$  when  $h = 0$ , include  $\epsilon = 1e-5$  for numerical stability. Also, observe that  $w_L = 0$  when  $h = 0$  since  $u(h, p_u) = 0$ .

$$v_L(\phi) = w_L \cdot \left( \frac{\cos(\phi) - \cos(\phi_H)}{\max(-\cos(\phi_H), \epsilon)} \right)^{\frac{1}{p_L}} + w_S$$

The surface region is defined by the angles,  $-1 \leq \cos(\phi) < \cos(\phi_H)$ . The one-minus term in  $v_S$  ensures continuity between  $v_L$  and  $v_S$  after crossing the horizon  $\phi_H$ .

$$v_S(\phi) = w_S \cdot \left( 1 - \left( \frac{\cos(\phi) - \cos(\phi_H)}{-1 - \cos(\phi_H)} \right)^{\frac{1}{p_S}} \right)$$

The weights define the amount of texture space allocated for each region.

$$w_U = 1 - w_L - w_S$$

$$w_L = w_L^1 \cdot u$$

$$w_S = w_S^1 \cdot u + w_S^0$$

The sum of the weights is  $w_U + w_L + w_S = 1$ . In order to optimize the texture space near ground level, the weights  $w_L^1$  and  $w_S^1$  are scaled by  $u$ . This causes the weights  $w_L$  and  $w_S$  to grow from a minimum at  $u = 0$  to their maximum at  $u = 1$ . The weight  $w_U$  is defined as the remainder of the other weights, causing it to shrink from the maximum at  $u = 0$  to its minimum at  $u = 1$ . The weight  $w_S^0$  defines a minimal texture parameter space for the surface region at  $u = 0$ .

The weighted power mappings may be converted from 3D texture coordinates as follows.

When  $v \geq w_L + w_S$ :

$$\cos(\phi) = \left( \frac{v - (w_L + w_S)}{w_U} \right)^{p_U}$$

When  $v \geq w_S$ :

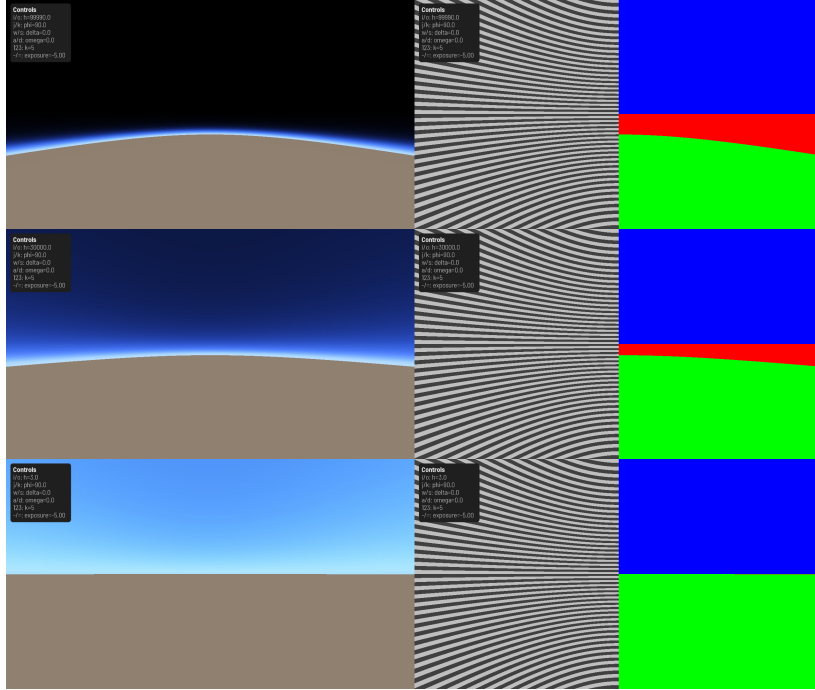
$$\cos(\phi) = -\cos(\phi_H) \left( \frac{v - w_S}{\max(w_L, \epsilon)} \right)^{p_L} + \cos(\phi_H)$$

Otherwise:

$$\cos(\phi) = (-1 - \cos(\phi_H)) \left( 1 - \frac{v}{w_S} \right)^{p_S} + \cos(\phi_H)$$

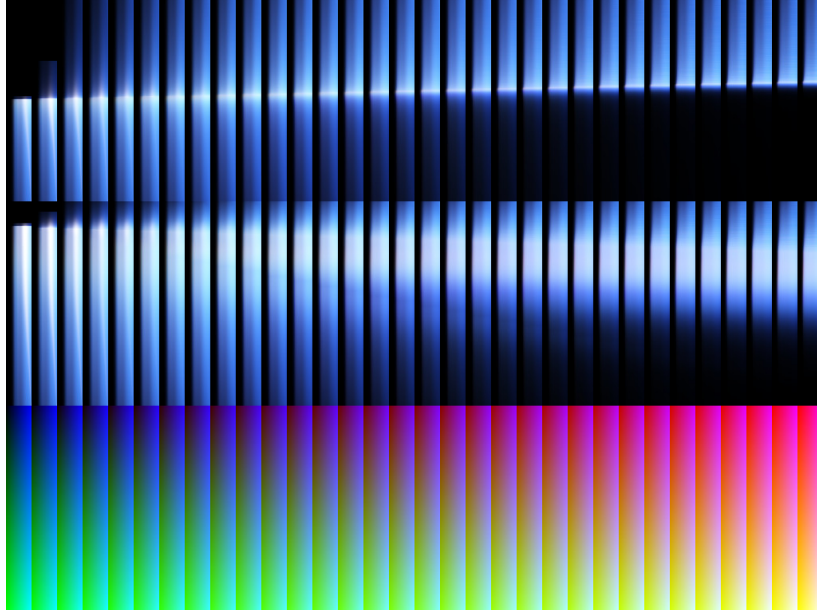
#### 2.4.5 Weighted Power Mapping Visualization

The following diagram shows a visualization of the weighted power mapping regions ( $U$ : blue,  $L$ : red,  $S$ : green) for  $h = H_a$  and  $h = 0$ .



#### 2.4.6 Linear vs Weighted Power Mapping Comparision

The following diagram shows a comparison between the linear mapping and weighted power mapping of the scattering intensity  $I_S$ .



The following parameters were used for the weighted power mapping.

- $p_U, p_L, p_S = 2, 2, 2$
- $w_L^1, w_S^1, w_S^0 = \frac{20}{32}, \frac{4}{32}, \frac{4}{32}$

#### 2.4.7 Sun-View Azimuth Mapping

To reduce the parameter count, the Rendering Parametrizable Planetary Atmospheres with Multiple Scattering in Real-Time paper proposes to omit the parameter  $\omega$  from precomputation. This omission causes uniformity of the atmospheric color with respect to  $\omega$  and is primarily visible during sunsets in parts of the sky where there is no direct illumination. Two techniques are proposed to address the omission of  $\omega$ . First, the modified Rayleigh scattering phase function  $F_R$  ensures that the darkest area of the sky during sunset is on the opposite side of the sky from the Sun. Second, the parameter  $\omega$  is dependent on the scattering angle  $\theta$ . By evaluating the constant phase function  $F_{R,M}(\text{dot}(L, -V))$  during rendering, we are able to reduce the uniformity of the atmospheric color with respect to  $\omega$ .

### 2.4.8 Texture Coordinates and Array Indices

The 3D array indices  $(x, y, z)$  with dimensions  $(width, height, depth)$  may be converted to 3D texture coordinates  $(u, v, w)$  as follows.

$$(u, v, w) = \left( \frac{x}{width - 1}, \frac{y}{height - 1}, \frac{z}{depth - 1} \right)$$

The 3D array indices  $(x, y, z)$  may be interpreted as a 1D array index  $(i)$  as follows.

$$i = x + y \cdot width + z \cdot width \cdot height$$

The Efficient and Dynamic Atmospheric Scattering paper uses the following texture dimensions where optional optimizations discussed for  $v$  component.

$$(width, height, depth) = (32, 256 | 128 | 64, 32)$$

## 3 Numerical Integration

The atmospheric simulation requires the computation of integrals that must be approximated using numerical integration due to complexity of the equations.

### 3.1 1D Trapezoidal Rule

The trapezoidal rule may be used to approximate 1D integrals of the function  $f(x)$ .

$$\int_a^b f(x) dx \approx \sum_{i=1}^n \frac{f(x_{i-1}) + f(x_i)}{2} \cdot dx$$

Where the parameters are defined:

$$dx = \frac{x_n - x_0}{n}$$

$$x_i = x_0 + i \cdot dx$$

The endpoints of the 1D integral are defined as  $x_0 = a$  and  $x_n = b$ .

### 3.2 2D Trapezoidal Rule

The trapezoidal rule may be extended to approximate 2D integrals of the function  $f(x, y)$ .

$$\begin{aligned} \int_a^b \int_c^d f(x, y) dy \cdot dx \approx \\ \frac{1}{4} dx \cdot dy \cdot (f(x_0, y_0) + f(x_n, y_0) + f(x_0, y_m) + f(x_n, y_m) + \\ 2 \sum_{j=1}^{n-1} f(x_j, y_0) + 2 \sum_{j=1}^{n-1} f(x_j, y_m) + 2 \sum_{i=1}^{m-1} f(x_0, y_i) + 2 \sum_{i=1}^{m-1} f(x_n, y_i) + \\ 4 \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} f(x_j, y_i)) \end{aligned}$$

Where the parameters are defined:

$$dx = \frac{x_n - x_0}{n}$$

$$dy = \frac{y_m - y_0}{m}$$

$$x_j = x_0 + j \cdot dx$$

$$y_i = y_0 + i \cdot dy$$

The edges of the 2D integral are defined as  $x_0 = a$ ,  $x_n = b$ ,  $y_0 = c$  and  $y_m = d$ .

### 3.3 Spherical Integration

Spherical integration is the process of evaluating integrals over three-dimensional regions by converting to spherical coordinates.

$$\int_{4\pi} f(\omega) d\omega = \int_{\Phi=0}^{2\pi} \int_{\Theta=0}^{\pi} f(\Theta, \Phi) \sin(\Theta) d\Theta \cdot d\Phi$$

Where the direction vector  $\omega$  and element area  $d\omega$  are defined:

$$\omega = (\sin(\Theta) \cos(\Phi), \sin(\Theta) \sin(\Phi), \cos(\Theta))$$

$$d\omega = \sin(\Theta) d\Theta \cdot d\Phi$$

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