CHAPTER

3 Counting

Prerequisites: Chapter 1

Techniques for counting are important in mathematics and in computer science, especially in the analysis of algorithms. In Section 1.2, the addition principle was introduced. In this chapter, we present other counting techniques, in particular those for permutations and combinations, and we look at two applications of counting, the pigeonhole principle and probability. In addition, recurrence relations, another tool for the analysis of computer programs, are discussed.

LOOKING BACK

An early contributor to the study of combinations was Abraham ben Meir ibn Ezra (1092–1167), who was born and died in Spain. Rabbi ben Ezra also worked in astrology, astronomy, philosophy, and medicine. Another early researcher in the area of permutations and combinations was the French mathematician, astronomer, and philosopher Levi ben Gerson (1288–1344).

In 1654 Blaise Pascal (1623–1662), a French mathematician who had been a child prodigy, exchanged a small series of letters with Pierre de Fermat (1601–1665), a French lawyer for whom mathematics was a hobby. This series of letters laid the foundation for the theory of probability. The correspondence between Pascal and Fermat developed when Pascal's friend, the Chevalier de Méré, asked him to solve several dice problems. In addition to making many other important contributions in mathematics and hydrostatics, Pascal invented a mechanical calculator that was very similar to the mechanical calculators used in the 1940s, just before the development of the digital electronic computer.

Blaise Pascal

Pierre de Fermat

3.1 Permutations

We begin with a simple but general result we will use frequently in this section and elsewhere.

Theorem 1 Suppose that two tasks T_1 and T_2 are to be performed in sequence. If T_1 can be performed in n_1 ways, and for each of these ways T_2 can be performed in n_2 ways, then the sequence T_1T_2 can be performed in n_1n_2 ways.

Proof

Each choice of a method of performing T_1 will result in a different way of performing the task sequence. There are n_1 such methods, and for each of these we may choose n_2 ways of performing T_2. Thus, in all, there will be n_1n_2 ways of performing the sequence T_1T_2. See Figure 3.1 for the case where n_1 is 3 and n_2 is 4. ■

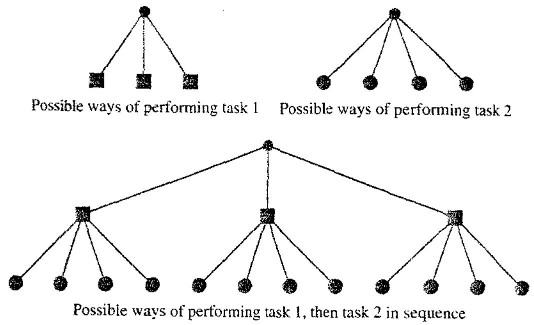


Figure 3.1

Theorem 1 is sometimes called the **multiplication principle of counting**. (You should compare it carefully with the addition principle of counting from Section 1.2.) It is an easy matter to extend the multiplication principle as follows.

Theorem 2 Suppose that tasks $T_1, T_2, \ldots, T_k$ are to be performed in sequence. If T_1 can be performed in n_1 ways, and for each of these ways T_2 can be performed in n_2 ways, and for each of these n_1n_2 ways of performing T_1T_2 in sequence, T_3 can be performed in n_3 ways, and so on, then the sequence $T_1T_2\cdots T_k$ can be performed in exactly $n_1n_2\cdots n_k$ ways.

Proof

This result can be proved by using the principle of mathematical induction on k. ■

EXAMPLE 1 A label identifier, for a computer system, consists of one letter followed by three digits. If repetitions are allowed, how many distinct label identifiers are possible?

Solution

There are 26 possibilities for the beginning letter and there are 10 possibilities for each of the three digits. Thus, by the extended multiplication principle, there are $26 \times 10 \times 10 \times 10$ or 26,000 possible label identifiers. ■

EXAMPLE 2 Let A be a set with n elements. How many subsets does A have?

Solution

We know from Section 1.3 that each subset of A is determined by its characteristic function, and if A has n elements, this function may be described as an array of 0's and 1's having length n. The first element of the array can be filled in two ways (with a 0 or a 1), and this is true for all succeeding elements as well. Thus, by the extended multiplication principle, there are

$$\underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{n \text{ factors}} = 2^n$$

ways of filling the array, and therefore 2^n subsets of A. ■

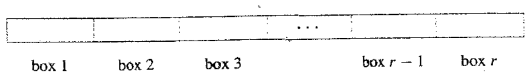
We now turn our attention to the following counting problem. Let A be any set with n elements, and suppose that $1 \le r \le n$.

Problem 1 How many different sequences, each of length r, can be formed using elements from A if

(a) elements in the sequence may be repeated?

(b) all elements in the sequence must be distinct?

First we note that any sequence of length r can be formed by filling r boxes in order from left to right with elements of A. In case (a) we may use copies of elements of A.



Let T_1 be the task "fill box 1," let T_2 be the task "fill box 2," and so on. Then the combined task $T_1 T_2 \cdots T_r$ represents the formation of the sequence.

Case (a). T_1 can be accomplished in n ways, since we may copy any element of A for the first position of the sequence. The same is true for each of the tasks $T_2, T_3, \ldots, T_r$. Then by the extended multiplication principle, the number of sequences that can be formed is

$$\underbrace{n \cdot n \cdot \cdots \cdot n}_{r \text{ factors}} = n^r.$$

We have therefore proved the following result.

Theorem 3 Let A be a set with n elements and $1 \le r \le n$. Then the number of sequences of length r that can be formed from elements of A, allowing repetitions, is n^r. ■

EXAMPLE 3 How many three-letter "words" can be formed from letters in the set $\{a, b, y, z\}$ if repeated letters are allowed?

Solution

Here n is 4 and r is 3, so the number of such words is 4^3 or 64, by Theorem 3. ■

Now we consider case (b) of Problem 1. Here also T_1 can be performed in n ways, since any element of A can be chosen for the first position. Whichever element is chosen, only $(n-1)$ elements remain, so that T_2 can be performed in $(n-1)$ ways, and so on, until finally T_r can be performed in $n-(r-1)$) or

$(n-r+1)$ ways. Thus, by the extended principle of multiplication, a sequence of r distinct elements from A can be formed in $n(n-1)(n-2)\cdots(n-r+1)$ ways.

A sequence of r distinct elements of A is often called a permutation of A taken r at a time. This terminology is standard, and therefore we adopt it, but it is confusing. A better terminology might be a "permutation of r elements chosen from A." Many sequences of interest are permutations of some set of n objects taken r at a time. The preceding discussion shows that the number of such sequences depends only on n and r, not on A. This number is often written ${}_nP_r$ and is called the **number of permutations of n objects taken r at a time**. We have just proved the following result.

Theorem 4 If $1 \le r \le n$, then ${}_nP_r$, the number of permutations of n objects taken r at a time, is $n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-r+1)$. ■

EXAMPLE 4 Let A be $\{1, 2, 3, 4\}$. Then the sequences 124, 421, 341, and 243 are some permutations of A taken 3 at a time. The sequences 12, 43, 31, 24, and 21 are examples of different permutations of A taken two at a time. By Theorem 4, the total number of permutations of A taken three at a time is ${}_4P_3$ or $4 \cdot 3 \cdot 2$ or 24. The total number of permutations of A taken two at a time is ${}_4P_2$ or $4 \cdot 3$ or 12. ■

When $r = n$, we are counting the distinct arrangements of the elements of A, with $|A| = n$, into sequences of length n. Such a sequence is simply called a **permutation** of A. In Chapter 5 we use the term "permutation" in a slightly different way to increase its utility. The number of permutations of A is thus ${}_nP_n$ or $n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$, if $n \ge 1$. This number is also written $n!$ and is read **n factorial**. Both ${}_nP_r$ and $n!$ are built-in functions on many calculators.

EXAMPLE 5 Let A be $\{a, b, c\}$. Then the possible permutations of A are the sequences abc, acb, bac, bca, cab, and cba. ■

For convenience, we define $0!$ to be 1. Then for every $n \ge 0$ the number of permutations of n objects is $n!$. If $n \ge 0$ and $0 \le r \le n$, we can now give a more compact form for ${}_nP_r$ as follows:

$$\begin{aligned}
{}_nP_r &= n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-r+1) \\
&= \frac{n \cdot (n-1) \cdot \cdots \cdot (n-r+1) \cdot (n-r) \cdot (n-r-1) \cdot \cdots \cdot 2 \cdot 1}{(n-r) \cdot (n-r-1) \cdot \cdots \cdot 2 \cdot 1} \\
&= \frac{n!}{(n-r)!}.
\end{aligned}$$

EXAMPLE 6 Let A consist of all 52 cards in an ordinary deck of playing cards. Suppose that this deck is shuffled and a hand of five cards is dealt. A list of cards in this hand, in the order in which they were dealt, is a permutation of A taken five at a time. Examples would include AH, 3D, 5C, 2H, JS; 2H, 3H, 5H, QH, KD; JH, JD, JS, 4H, 4C; and 3D, 2H, AH, JS, 5C. Note that the first and last hands are the same, but they represent different permutations since they were dealt in a different order. The number of permutations of A taken five at a time is ${}_{52}P_5 = \frac{52!}{47!}$ or $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ or 311,875,200. This is the number of five-card hands that can be dealt if we consider the order in which they were dealt. ■

EXAMPLE 7 If A is the set in Example 5, then n is 3 and the number of permutations of A is $3!$ or 6. Thus, all the permutations of A are listed in Example 5, as claimed. ■

EXAMPLE 8 How many "words" of three distinct letters can be formed from the letters of the word MAST?

Solution

The number is ${}_4P_3 = \dfrac{4!}{(4-3)!}$ or $\dfrac{4!}{1!}$ or 24. ∎

In Example 8 if the word had been MASS, ${}_4P_3$ would count as distinct some permutations that cannot be distinguished. For example, if we tag the two S's as S_1 and S_2, then S_1AS_2 and S_2AS_1 are two of the 24 permutations counted, but without the tags, these are the same "word." We have one more case to consider, permutations with limited repeats.

EXAMPLE 9 How many distinguishable permutations of the letters in the word BANANA are there?

Solution

We begin by tagging the A's and N's in order to distinguish between them temporarily. For the letters B, A_1, N_1, A_2, N_2, A_3, there are 6! or 720 permutations. Some of these permutations are identical except for the order in which the N's appear; for example, $A_1A_2A_3BN_1N_2$ and $A_1A_2A_3BN_2N_1$. In fact, the 720 permutations can be listed in pairs whose members differ only in the order of the two N's. This means that if the tags are dropped from the N's only $\frac{720}{2}$ or 360 distinguishable permutations remain. Reasoning in a similar way we see that these can be grouped in groups of 3! or 6 that differ only in the order of the three A's. For example, one group of 6 consists of $BNNA_1A_2A_3$, $BNNA_1A_3A_2$, $BNNA_2A_1A_3$, $BNNA_2A_3A_1$, $BNNA_3A_1A_2$, $BNNA_3A_2A_1$. Dropping the tags would change these 6 into the single permutation BNNAAA. Thus, there are $\frac{360}{6}$ or 60 distinguishable permutations of the letters of BANANA. ∎

The following theorem describes the general situation for permutations with limited repeats.

Theorem 5 The number of distinguishable permutations that can be formed from a collection of n objects where the first object appears k_1 times, the second object k_2 times, and so on, is

$$\frac{n!}{k_1!\,k_2!\cdots k_t!}, \quad \text{where } k_1 + k_2 + \cdots + k_t = n.$$

EXAMPLE 10 The number of distinguishable "words" that can be formed from the letters of MISSISSIPPI is $\frac{11!}{1!4!4!2!}$ or 34,650. ∎

3.1 Exercises

1. A bank password consists of two letters of the English alphabet followed by two digits. How many different passwords are there?
2. In a psychological experiment, a person must arrange a square, a cube, a circle, a triangle, and a pentagon in a row. How many different arrangements are possible?
3. A coin is tossed four times and the result of each toss is recorded. How many different sequences of heads and tails are possible?
4. A catered menu is to include a soup, a main course, a dessert, and a beverage. Suppose a customer can select from four soups, five main courses, three desserts, and two beverages. How many different menus can be selected?
5. A fair six-sided die is tossed four times and the numbers shown are recorded in a sequence. How many different sequences are there?

6. Let $A = \{0, 1\}$.
   (a) How many strings of length three are there in A^*?
   (b) How many strings of length seven are there in A^*?
7. (a) Compute the number of strings of length four in the set corresponding to the regular expression $(01)^*1$.
   (b) Compute the number of strings of length five in the set corresponding to the regular expression $(01)^*1$.
8. Compute each of the following.
   (a) ${}_4P_4$ (b) ${}_6P_5$ (c) ${}_7P_2$
9. Compute each of the following.
   (a) ${}_nP_{n-1}$ (b) ${}_nP_{n-2}$ (c) ${}_{n+1}P_{n-1}$

In Exercises 10 through 13, compute the number of permutations of the given set.

10. $\{r, s, t, u\}$
11. $\{1, 2, 3, 4, 5\}$
12. $\{a, b, 1, 2, 3, c\}$
13. $\{4, 7, 10, 13\}$

In Exercises 14 through 16, find the number of permutations of A taken r at a time.

14. $A = \{1, 2, 3, 4, 5, 6, 7\}$, $r = 3$
15. $A = \{a, b, c, d, e, f\}$, $r = 2$
16. $A = \{x \mid x \text{ is an integer and } x^2 < 16\}$, $r = 4$
17. In how many ways can six men and six women be seated in a row if
    (a) any person may sit next to any other?
    (b) men and women must occupy alternate seats?
18. Find the number of different permutations of the letters in the word GROUP.
19. How many different arrangements of the letters in the word BOUGHT can be formed if the vowels must be kept next to each other?
20. Find the number of distinguishable permutations of the letters in BOOLEAN.
21. Find the number of distinguishable permutations of the letters in PASCAL.
22. Find the number of distinguishable permutations of the letters in ASSOCIATIVE.
23. Find the number of distinguishable permutations of the letters in REQUIREMENTS.
24. In how many ways can seven people be seated in a circle?
25. How many different ways can n people be seated around a circular table?
26. Give a proof of your result for Exercise 25.
27. A bookshelf is to be used to display six new books. Suppose there are eight computer science books and five French books from which to choose. If we decide to show four computer science books and two French books and we are required to keep the books in each subject together, how many different displays are possible?
28. Three fair six-sided dice are tossed and the numbers showing on the top faces are recorded as a triple. How many different records are possible?
29. Prove that $n \cdot {}_{n-1}P_{n-1} = {}_nP_n$.
30. Most versions of Pascal allow variable names to consist of eight letters or digits with the requirement that the first character must be a letter. How many eight-character variable names are possible?
31. Until recently, U.S. telephone area codes were three-digit numbers whose middle digit was 0 or 1. Codes whose last two digits are 1's are used for other purposes (for example, 911). With these conditions how many area codes were available?
32. How many Social Security numbers can be assigned at any one time? Identify any assumptions you have made.
33. How many zeros are there at the end of 12!? at the end of 26!? at the end of 53!?
34. Give a procedure for determining the number of zeros at the end of $n!$. Justify your procedure.

In Exercises 35 through 37, use the following information. There are three routes from Atlanta to Athens, four routes from Athens to Augusta, and two routes from Atlanta to Augusta.

35. (a) How many ways are there to travel from Atlanta to Augusta?
    (b) How many ways are there to travel from Athens to Atlanta?
36. (a) How many different ways can the round trip between Atlanta and Augusta be made?
    (b) How many different ways can the round trip between Atlanta and Augusta be made if each route is used only once?
37. (a) How many different ways can the round trip between Augusta and Athens be made if the trip does not go through Atlanta?
    (b) How many different ways can the round trip between Augusta and Athens be made if the trip does not go through Atlanta and each route is used only once?

3.2 Combinations

The multiplication principle and the counting methods for permutations all apply to situations where order matters. In this section we look at some counting problems where order does not matter.

Problem 2 Let A be any set with n elements and $0 \le r \le n$. How many different subsets of A are there, each with r elements?

The traditional name for an r-element subset of an n-element set A is a **combination of A, taken r at a time.**

EXAMPLE 1 Let $A = \{1, 2, 3, 4\}$. The following are all distinct combinations of A, taken three at a time: $A_1 = \{1, 2, 3\}$, $A_2 = \{1, 2, 4\}$, $A_3 = \{1, 3, 4\}$, $A_4 = \{2, 3, 4\}$. Note that these are subsets, not sequences. Thus $A_1 = \{2, 1, 3\} = \{2, 3, 1\} = \{1, 3, 2\} = \{3, 1, 2\} = \{3, 2, 1\}$. In other words, when it comes to combinations, unlike permutations, the order of the elements is irrelevant. ■

EXAMPLE 2 Let A be the set of all 52 cards in an ordinary deck of playing cards. Then a combination of A, taken five at a time, is just a hand of five cards regardless of how these cards were dealt. ■

We now want to count the number of r-element subsets of an n-element set A. This is most easily accomplished by using what we already know about permutations. Observe that each permutation of the elements of A, taken r at a time, can be produced by performing the following two tasks in sequence.

Task 1: Choose a subset B of A containing r elements.
Task 2: Choose a particular permutation of B.

We are trying to compute the number of ways to choose B. Call this number C. Then task 1 can be performed in C ways, and task 2 can be performed in $r!$ ways. Thus the total number of ways of performing both tasks is, by the multiplication principle, $C \cdot r!$. But it is also ${}_nP_r$. Hence,

$$C \cdot r! = {}_nP_r = \frac{n!}{(n-r)!}.$$

Therefore,

$$C = \frac{n!}{r!\,(n-r)!}.$$

We have proved the following result.

Theorem 1 Let A be a set with $|A| = n$, and let $0 \le r \le n$. Then the number of combinations of the elements of A, taken r at a time, that is, the number of r-element subsets of A, is

$$\frac{n!}{r!\,(n-r)!}.$$ ■

Note again that the number of combinations of A, taken r at a time, does not depend on A, but only on n and r. This number is often written ${}_nC_r$ and is called the **number of combinations of n objects taken r at a time**. We have

$${}_nC_r = \frac{n!}{r!\,(n-r)!}.$$

This computation is a built-in function on many calculators.

EXAMPLE 3 Compute the number of distinct five-card hands that can be dealt from a deck of 52 cards.

Solution

This number is ${}_{52}C_5$ because the order in which the cards were dealt is irrelevant. ${}_{52}C_5 = \frac{52!}{5!\,47!}$ or 2,598,960. Compare this number with the number computed in Example 6, Section 3.1. ■

In the discussion of permutations, we considered cases where repetitions are allowed. We now look at one such case for combinations.

Consider the following situation. A radio station offers a prize of three CDs from the Top Ten list. The choice of CDs is left to the winner, and repeats are allowed. The order in which the choices are made is irrelevant. To determine the number of ways in which prize winners can make their choices, we use a problem-solving technique we have used before; we model the situation with one we already know how to handle.

Suppose choices are recorded by the station's voice mail system. After properly identifying herself, a winner is asked to press 1 if she wants CD number n and to press 2 if she does not. If 1 is pressed, the system asks again about CD number n. When 2 is pressed, the system asks about the next CD on the list. When three 1's have been recorded, the system tells the caller the selected CDs will be shipped. A record must be created for each of these calls. A record will be a sequence of 1's and 2's. Clearly there will be three 1's in the sequence. A sequence may contain as many as nine 2's, for example, if the winner refuses the first nine CDs and chooses three copies of CD number 10. Our model for counting the number of ways a prize winner can choose her three CDs is the following. Each three-CD selection can be represented by an array containing three 1's and nine 2's or blanks, or a total of twelve cells. Some possible records are 222122122221 (selecting numbers 4, 6, 10), 1211*bbbbbbbb* (selecting number 1 and two copies of number 2), and 222222222111 (selecting three copies of number 10). The number of ways to select three cells of the array to hold 1's is ${}_{12}C_3$ since the array has $3 + 9$ or 12 cells and the order in which this selection is made does not matter. The following theorem generalizes this discussion.

Theorem 2 Suppose k selections are to be made from n items without regard to order and repeats are allowed, assuming at least k copies of each of the n items. The number of ways these selections can be made is ${}_{(n+k-1)}C_k$. ■

EXAMPLE 4 In how many ways can a prize winner choose three CDs from the Top Ten list if repeats are allowed?

Solution

Here n is 10 and k is 3. By Theorem 2, there are ${}_{10+3-1}C_3$ or ${}_{12}C_3$ ways to make the selections. The prize winner can make the selection in 220 ways. ■

In general, when order matters, we count the number of sequences or permutations; when order does not matter, we count the number of subsets or combinations.

Some problems require that the counting of permutations and combinations be combined or supplemented by the direct use of the addition or the multiplication principle.

EXAMPLE 5 Suppose that a valid computer password consists of seven characters, the first of which is a letter chosen from the set {A, B, C, D, E, F, G} and the remaining six characters are letters chosen from the English alphabet or a digit. How many different passwords are possible?

Solution

A password can be constructed by performing the tasks T_1 and T_2 in sequence.

Task 1: Choose a starting letter from the set given.

Task 2: Choose a sequence of letter and digits. Repeats are allowed.

Task T_1 can be performed in ${}_7C_1$ or 7 ways. Since there are 26 letters and 10 digits that can be chosen for each of the remaining six characters, and since repeats are allowed, task T_2 can be performed in 36^6 or 2,176,782,336 ways. By the multiplication principle, there are $7 \cdot 2176782336$ or 15,237,476,352 different passwords. ■

EXAMPLE 6 How many different seven-person committees can be formed each containing three women from an available set of 20 women and four men from an available set of 30 men?

Solution

In this case a committee can be formed by performing the following two tasks in succession:

Task 1: Choose three women from the set of 20 women.

Task 2: Choose four men from the set of 30 men.

Here order does not matter in the individual choices, so we are merely counting the number of possible subsets. Thus task 1 can be performed in ${}_{20}C_3$ or 1140 ways and task 2 can be performed in ${}_{30}C_4$ or 27,405 ways. By the multiplication principle, there are (1140)(27405) or 31,241,700 different committees. ■

3.2 Exercises

1. Compute each of the following.
(a) ${}_7C_7$ (b) ${}_7C_4$ (c) ${}_{16}C_5$
2. Compute each of the following.
(a) ${}_nC_{n-1}$ (b) ${}_nC_{n-2}$ (c) ${}_{n+1}C_{n-1}$
3. Show that ${}_nC_r = {}_nC_{n-r}$.
4. In how many ways can a committee of three faculty members and two students be selected from seven faculty members and eight students?
5. In how many ways can a six-card hand be dealt from a deck of 52 cards?
6. At a certain college, the housing office has decided to appoint, for each floor, one male and one female residential advisor. How many different pairs of advisors can be selected for a seven-story building from 12 male candidates and 15 female candidates?
7. A microcomputer manufacturer who is designing an advertising campaign is considering six magazines, three newspapers, two television stations, and four radio stations. In how many ways can six advertisements be run if
(a) all six are to be in magazines?
(b) two are to be in magazines, two are to be in newspapers, one is to be on television, and one is to be on radio?
8. How many different eight-card hands with five red cards and three black cards can be dealt from a deck of 52 cards?
9. (a) Find the number of subsets of each possible size for a set containing four elements.
(b) Find the number of subsets of each possible size for a set containing n elements.

For Exercises 10 through 13, suppose that an urn contains 15 balls, of which eight are red and seven are black.

10. In how many ways can five balls be chosen so that
(a) all five are red?
(b) all five are black?
11. In how many ways can five balls be chosen so that
(a) two are red and three are black?
(b) three are red and two are black?
12. In how many ways can five balls be chosen so that at most three are black?
13. In how many ways can five balls be chosen so that at least two are red?
14. Give a model in terms of combinations to count the number of strings of length 6 in $\{0, 1\}^*$ that have exactly four ones.

15. Give a model in terms of combinations to count the number of ways to arrange three of seven people in order from youngest to oldest.
16. A committee of six people with one person designated as chair of the committee is to be chosen. How many different committees of this type can be chosen from a group of 10 people?
17. A gift certificate at a local bookstore allows the recipient to choose six books from the combined list of ten bestselling fiction books and ten bestselling nonfiction books. In how many different ways can the selection of six books be made?
18. The college food plan allows a student to chose three pieces of fruit each day. The fruits available are apples, bananas, peaches, pears, and plums. For how many days can a student make a different selection?
19. Show that ${}_{n+1}C_r = {}_nC_{r-1} + {}_nC_r$.
20. (a) How many ways can a student choose eight out of ten questions to answer on an exam?
(b) How many ways can a student choose eight out of ten questions to answer on an exam if the first three questions must be answered?
21. Five fair coins are tossed and the results are recorded.
(a) How many different sequences of heads and tails are possible?
(b) How many of the sequences in part (a) have exactly one head recorded?
(c) How many of the sequences in part (a) have exactly three heads recorded?
22. Three fair six-sided dice are tossed and the numbers showing on top are recorded.
(a) How many different record sequences are possible?
(b) How many of the records in part (a) contain exactly one six?
(c) How many of the records in part (a) contain exactly two fours?
23. If n fair coins are tossed and the results recorded, how many
(a) record sequences are possible?
(b) sequences contain exactly three tails, assuming $n \geq 3$?
(c) sequences contain exactly k heads, assuming $n \geq k$?
24. If n fair six-sided dice are tossed and the numbers showing on top are recorded, how many
(a) record sequences are possible?
(b) sequences contain exactly one six?
(c) sequences contain exactly four twos, assuming $n \geq 4$?
25. How many ways can you choose three of seven fiction books and two of six nonfiction books to take with you on your vacation?
26. For the driving part of your vacation you will take 6 of the 35 rock cassettes in your collection, 3 of the 22 classical cassettes, and 1 of the 8 comedy cassettes. In how many ways can you make your choice(s)?
27. The array commonly called Pascal's triangle can be defined by giving enough information to establish its pattern.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

(a) Write the next three rows of Pascal's triangle.
(b) Give a rule for building the next row from the previous row(s).
28. Pascal's triangle can also be defined by an explicit pattern. Use the results of Exercises 9 and 27 to give an explicit rule for building the nth row of Pascal's triangle.
29. Explain the connections between Exercises 19, 27, and 28.
30. The list of numbers in any row of Pascal's triangle reads the same from left to right as it does from right to left. Such a sequence is called a *palindrome*. Use the results of Exercise 28 to prove that each row of Pascal's triangle is a palindrome.
31. (a) The sum of the entries in the second row of Pascal's triangle is ______.
(b) The sum of the entries in the third row of Pascal's triangle is ______.
(c) The sum of the entries in the fourth row of Pascal's triangle is ______.
32. Make a conjecture about the sum of the entries in the nth row of Pascal's triangle and prove it.
33. Marcy wants to buy a book of poems. If she wants to read a different set of three poems every day for a year (365 days), what is the minimum number of poems the book should contain?

3.3 Pigeonhole Principle

In this section we introduce another proof technique, one that often makes use of the counting methods we have discussed.

Theorem 1
The Pigeonhole Principle

If n pigeons are assigned to m pigeonholes, and $m < n$, then at least one pigeonhole contains two or more pigeons.

Proof
Suppose each pigeonhole contains at most 1 pigeon. Then at most m pigeons have been assigned. But since $m < n$, not all pigeons have been assigned pigeonholes. This is a contradiction. At least one pigeonhole contains two or more pigeons. ■

This informal and almost trivial sounding theorem is easy to use and has unexpected power in proving interesting consequences.

EXAMPLE 1

If eight people are chosen in any way from some group, at least two of them will have been born on the same day of the week. Here each person (pigeon) is assigned to the day of the week (pigeonhole) on which he or she was born. Since there are eight people and only seven days of the week, the pigeonhole principle tells us that at least two people must be assigned to the same day of the week. ■

Note that the pigeonhole principle provides an **existence proof**; there must be an object or objects with a certain characteristic. In Example 1, this characteristic is having been born on the same day of the week. The pigeonhole principle guarantees that there are at least two people with this characteristic but gives no information on identifying these people. Only their existence is guaranteed. In contrast, a **constructive proof** guarantees the existence of an object or objects with a certain characteristic by actually constructing such an object or objects. For example, we could prove that given two rational numbers p and q there is a rational number between them by showing that $\frac{p+q}{2}$ is between p and q.

In order to use the pigeonhole principle we must identify pigeons (objects) and pigeonholes (categories of the desired characteristic) and be able to count the number of pigeons and the number of pigeonholes.

EXAMPLE 2

Show that if any five numbers from 1 to 8 are chosen, then two of them will add to 9.

Solution

Construct four different sets, each containing two numbers that add up to 9 as follows: $A_1 = \{1, 8\}$, $A_2 = \{2, 7\}$, $A_3 = \{3, 6\}$, $A_4 = \{4, 5\}$. Each of the five numbers chosen must belong to one of these sets. Since there are only four sets, the pigeonhole principle tells us that two of the chosen numbers belong to the same set. These numbers add up to 9. ■

EXAMPLE 3

Show that if any 11 numbers are chosen from the set $\{1, 2, \ldots, 20\}$, then one of them will be a multiple of another.

Solution

The key to solving this problem is to create 10 or fewer pigeonholes in such a way that each number chosen can be assigned to only one pigeonhole, and when x and y are assigned to the same pigeonhole we are guaranteed that either $x \mid y$ or $y \mid x$. Factors are a natural feature to explore. There are eight prime numbers between 1 and 20, but knowing that x and y are multiples of the same prime will not guarantee that either $x \mid y$ or $y \mid x$. We try again. There are ten odd numbers between 1 and 20. Every positive integer n can be written as $n = 2^k m$, where m is odd and $k \geq 0$. This can be seen by simply factoring all powers of 2 (if any) out of n. In this case let us call m the odd part of n. If 11 numbers are chosen from the set $\{1, 2, \ldots, 20\}$, then two of them must have the same odd part. This follows from the pigeonhole principle since there are 11 numbers (pigeons), but only 10 odd numbers between 1 and 20 (pigeonholes) that can be odd parts of these numbers.

Let n_1 and n_2 be two chosen numbers with the same odd part. We must have $n_1 = 2^{k_1} m$ and $n_2 = 2^{k_2} m$, for some k_1 and k_2. If $k_1 \geq k_2$, then n_1 is a multiple of n_2; otherwise, n_2 is a multiple of n_1. ■

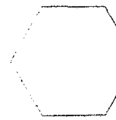


Figure 3.2

EXAMPLE 4

Consider the region shown in Figure 3.2. It is bounded by a regular hexagon whose sides are of length 1 unit. Show that if any seven points are chosen in this region, then two of them must be no farther apart than 1 unit.

Solution

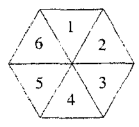


Figure 3.3

Divide the region into six equilateral triangles, as shown in Figure 3.3. If seven points are chosen in the region, we can assign each of them to a triangle that contains it. If the point belongs to several triangles, arbitrarily assign it to one of them. Then the seven points are assigned to six triangular regions, so by the pigeonhole principle, at least two points must belong to the same region. These two cannot be more than one unit apart. (Why?) ■

EXAMPLE 5

Shirts numbered consecutively from 1 to 20 are worn by the 20 members of a bowling league. When any three of these members are chosen to be a team, the league proposes to use the sum of their shirt numbers as a code number for the team. Show that if any eight of the 20 are selected, then from these eight one may form at least two different teams having the same code number.

Solution

From the eight selected bowlers, we can form a total of ${}_8C_3$ or 56 different teams. These will play the role of pigeons. The largest possible team code number is $18 + 19 + 20$ or 57, and the smallest possible is $1 + 2 + 3$ or 6. Thus only the 52 code numbers (pigeonholes) between 6 and 57 inclusive are available for the 56 possible teams. By the pigeonhole principle, at least two teams will have the same code number. The league should use another way to assign team numbers. ■

■ The Extended Pigeonhole Principle

Note that if there are m pigeonholes and more than $2m$ pigeons, three or more pigeons will have to be assigned to at least one of the pigeonholes. (Consider the most even distribution of pigeons you can make.) In general, if the number of pigeons is much larger than the number of pigeonholes, Theorem 1 can be restated to give a stronger conclusion.

First a word about notation. If n and m are positive integers, then $\lfloor n/m \rfloor$ stands for the largest integer less than or equal to the rational number n/m. Thus $\lfloor 3/2 \rfloor$ is 1, $\lfloor 9/4 \rfloor$ is 2, and $\lfloor 6/3 \rfloor$ is 2.

Theorem 2 *The Extended Pigeonhole Principle* If n pigeons are assigned to m pigeonholes, then one of the pigeonholes must contain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons.

Proof (by contradiction)
If each pigeonhole contains no more than $\lfloor (n-1)/m \rfloor$ pigeons, then there are at most $m \cdot \lfloor (n-1)/m \rfloor \leq m \cdot (n-1)/m = n-1$ pigeons in all. This contradicts our hypothesis, so one of the pigeonholes must contain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons. ∎

This proof by contradiction uses the fact that there are two ways to count the total number of pigeons, the original count n and as the product of the number of pigeonholes times the number of pigeons per pigeonhole.

EXAMPLE 6 We give an extension of Example 1. Show that if any 30 people are selected, then one may choose a subset of five so that all five were born on the same day of the week.

Solution
Assign each person to the day of the week on which she or he was born. Then 30 pigeons are being assigned to 7 pigeonholes. By the extended pigeonhole principle with $n = 30$ and $m = 7$, at least $\lfloor (30-1)/7 \rfloor + 1$ or 5 of the people must have been born on the same day of the week. ■

EXAMPLE 7 Show that if 30 dictionaries in a library contain a total of 61,327 pages, then one of the dictionaries must have at least 2045 pages.

Solution
Let the pages be the pigeons and the dictionaries the pigeonholes. Assign each page to the dictionary in which it appears. Then by the extended pigeonhole principle, one dictionary must contain at least $\lfloor 61{,}326/30 \rfloor + 1$ or 2045 pages. ■

3.3 Exercises

1. If thirteen people are assembled in a room, show that at least two of them must have their birthday in the same month.
2. Show that if seven integers from 1 to 12 are chosen, then two of them will add up to 13.
3. Let T be an equilateral triangle whose sides are of length 1 unit. Show that if any five points are chosen lying on or inside the triangle, then two of them must be no more than $\frac{1}{2}$ unit apart.
4. Show that if any eight positive integers are chosen, two of them will have the same remainder when divided by 7.
5. Show that if seven colors are used to paint 50 bicycles, at least eight bicycles will be the same color.
6. Ten people volunteer for a three-person committee. Every possible committee of three that can be formed from these ten names is written on a slip of paper, one slip for each possible committee, and the slips are put in ten hats. Show that at least one hat contains 12 or more slips of paper.
7. Six friends discover that they have a total of \$21.61 with them on a trip to the movies. Show that one or more of them must have at least \$3.61.
8. A store has an introductory sale on 12 types of candy bars. A customer may choose one bar of any five different types and will be charged no more than \$1.75. Show that although different choices may cost different amounts, there must be at least two different ways to choose so that the cost will be the same for both choices.
9. If the store in Exercise 8 allows repetitions in the choices, show that there must be at least ten ways to make different choices that have the same cost.
10. Show that there must be at least 90 ways to choose six integers from 1 to 15 so that all the choices have the same sum.
11. How many friends must you have to guarantee at least five of them will have birthdays in the same month?
12. Show that if five points are selected in a square whose sides have length 1 inch, at least two of the points must be no more than $\sqrt{2}$ inches apart.

13. Let $\mathbf{A}$ be an 8×8 Boolean matrix. If the sum of the entries in $\mathbf{A}$ is 51, prove that there is a row i and a column j in $\mathbf{A}$ such that the entries in row i and in column j add up to more than 13.
14. Write an exercise similar to Exercise 13 for a 12×12 Boolean matrix.
15. Prove that if any 14 integers from 1 to 25 are chosen, then one of them is a multiple of another.
16. How large a subset of the integers from 1 to 50 must be chosen to guarantee that one of the numbers in the subset is a multiple of another number in the subset?
17. How large a subset of the integers from 1 to n must be chosen to guarantee that one of the numbers in the subset is a multiple of another number in the subset?
18. Twenty disks numbered 1 through 20 are placed face down on a table. Disks are selected one at a time and turned over until 10 disks have been chosen. If two of the disks add up to 21, the player loses. Is it possible to win this game?
19. Suppose the game in Exercise 18 has been changed so that 12 disks are chosen. Is it possible to win this game?
20. Complete the following proof. It is not possible to arrange the numbers $1, 2, 3, \ldots, 10$ in a circle so that every triple of consecutively placed numbers has a sum less than 15.

    Proof: In any arrangement of $1, 2, 3, \ldots, 10$ in a circle, there are ____ triples of consecutively placed numbers, because ____. Each number appears in ____ of these triples. If the sum of each triple were less than 15, then the total sum of all triples would be less than ____ times 15 or ____. But $1+2+3+\cdots+10$ is 55 and since each number appears in ____ triples, the total sum should be ____ times 55. This is a contradiction so not all triples can have a sum less than 15.
21. Prove that any sequence of six integers must contain a subsequence whose sum is divisible by six. (*Hint*: Consider the sums $c_1, c_1+c_2, c_1+c_2+c_3, \ldots$ and the possible remainders when dividing by six.)
22. Prove that any sequence of n integers must contain a subsequence whose sum is divisible by n.
23. Show that any set of six positive integers whose sum is 13 must contain a subset whose sum is three.
24. Use the pigeonhole principle to prove that any rational number can be expressed as a number with a finite or repeating decimal part.
25. The computer classroom has 12 PCs and 5 printers. What is the minimum number of connections that must be made to guarantee that any set of 5 or fewer PCs can access printers at the same time?

3.4 Elements of Probability

Another area where counting techniques are important is probability theory. In this section we present a brief introduction to probability.

Many experiments do not yield exactly the same results when performed repeatedly. For example, if we toss a coin, we are not sure if we will get heads or tails, and if we toss a die, we have no way of knowing which of the six possible numbers will turn up. Experiments of this type are called **probabilistic**, in contrast to **deterministic** experiments, whose outcome is always the same.

■ Sample Spaces

A set A consisting of all the outcomes of an experiment is called a **sample space** of the experiment. With a given experiment, we can often associate more than one sample space, depending on what the observer chooses to record as an outcome.

EXAMPLE 1 Suppose that a nickel and a quarter are tossed in the air. We describe three possible sample spaces that can be associated with this experiment.

1. If the observer decides to record as an outcome the number of heads observed, the sample space is $A = \{0, 1, 2\}$.
2. If the observer decides to record the sequence of heads (H) and tails (T) observed, listing the condition of the nickel first and then that of the quarter, then the sample space is $A = \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$.
3. If the observer decides to record the fact that the coins match (M) (both heads or both tails) or do not match (N), then the sample space is $A = \{\text{M}, \text{N}\}$. ■

We thus see that in addition to describing the experiment, we must indicate exactly what the observer wishes to record. Then the set of all outcomes of this type becomes the sample space for the experiment.

A sample space may contain a finite or an infinite number of outcomes, but in this chapter, we need only finite sample spaces.

EXAMPLE 2

Determine the sample space for an experiment consisting of tossing a six-sided die twice and recording the sequence of numbers showing on the top face of the die after each toss.

Solution

An outcome of the experiment can be represented by an ordered pair of numbers (n, m), where n and m can be 1, 2, 3, 4, 5, or 6. Thus the sample space A contains 6×6 or 36 elements (by the multiplication principle). ■

EXAMPLE 3

An experiment consists of drawing three coins in succession from a box containing four pennies and five dimes, and recording the sequence of results. Determine the sample space of this experiment.

Solution

An outcome can be recorded as a sequence of length 3 constructed from the letters P (penny) and D (dime). Thus the sample space A is {PPP, PPD, PDP, PDD, DPP, DPD, DDP, DDD}. ■

■ Events

A statement about the outcome of an experiment, which for a particular outcome will be either true or false, is said to describe an **event**. Thus for Example 2, the statements, "Each of the numbers recorded is less than 3" and "The sum of the numbers recorded is 4" would describe events. The event described by a statement is taken to be the set of all outcomes for which the statement is true. With this interpretation, any event can be considered a subset of the sample space. Thus the event E described by the first statement is $E = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Similarly, the event F described by the second statement is $F = \{(1, 3), (2, 2), (3, 1)\}$.

EXAMPLE 4

Consider the experiment in Example 2. Determine the events described by each of the following statements.

(a) The sum of the numbers showing on the top faces is 8.
(b) The sum of the numbers showing on the top faces is at least 10.

Solution

(a) The event consists of all ordered pairs whose sum is 8. Thus the event is $\{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$.
(b) The event consists of all ordered pairs whose sum is 10, 11, or 12. Thus the event is $\{(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\}$. ■

If A is a sample space of an experiment, then A itself is an event called the **certain event** and the empty subset of A is called the **impossible event**.

Since events are sets, we can combine them by applying the operations of union, intersection, and complementation to form new events. The sample space A is the universal set for these events. Thus if E and F are events, we can form the new events $E \cup F$, $E \cap F$, and $\overline{E}$. What do these new events mean in terms of the experiment? An outcome of the experiment belongs to $E \cup F$ when it belongs to E or F (or both). In other words, the event $E \cup F$ occurs exactly when E or F occurs. Similarly, the event $E \cap F$ occurs if and only if both E and F occur. Finally, $\overline{E}$ occurs if and only if E does not occur.

EXAMPLE 5

Consider the experiment of tossing a die and recording the number on the top face. Let E be the event that the number is even and let F be the event that the number is prime. Then $E = \{2, 4, 6\}$ and $F = \{2, 3, 5\}$. The event that the number showing is either even or prime is $E \cup F = \{2, 3, 4, 5, 6\}$. The event that the number showing is an even prime is $E \cap F = \{2\}$. Finally, the event that the number showing is not even is $\overline{E} = \{1, 3, 5\}$ and the event that the number showing is not prime is $\overline{F} = \{1, 4, 6\}$. ■

Events E and F are said to be **mutually exclusive** or **disjoint** if $E \cap F = \{\ \}$. If E and F are mutually exclusive events, then E and F cannot both occur at the same time; if E occurs, then F does not occur, and if F occurs, then E does not. If $E_1, E_2, \ldots, E_n$ are all events, then we say that these sets are **mutually exclusive**, or **disjoint**, if each pair of them is mutually exclusive. Again, this means that at most one of the events can occur on any given outcome of the experiment.

■ Assigning Probabilities to Events

In probability theory, we assume that each event E has been assigned a number $p(E)$ called the **probability of the event E**. We now look at probabilities. We will investigate ways in which they can be assigned, properties they must satisfy, and the meaning that can be given to them.

The number $p(E)$ reflects our assessment of the likelihood that the event E will occur. More precisely, suppose the underlying experiment is performed repeatedly, and that after n such performances, the event E has occurred n_E times. Then the fraction $f_E = n_E/n$, called the **frequency of occurrence of E in n trials**, is a measure of the likelihood that E will occur. When we assign the probability $p(E)$ to the event E, it means that in our judgment or experience, we believe that the fraction f_E will tend ever closer to a certain number as n becomes larger, and that $p(E)$ is this number. Thus probabilities can be thought of as idealized frequencies of occurrence of events, to which actual frequencies of occurrence will tend when the experiment is performed repeatedly.

EXAMPLE 6

Suppose an experiment is performed 2000 times, and the frequency of occurrence f_E of an event E is recorded after 100, 500, 1000, and 2000 trials. Table 3.1 summarizes the results.

TABLE 3.1

Number of Repetitions of the Experiment	n_E	$f_E = n_E/n$
100	48	0.48
500	259	0.518
1000	496	0.496
2000	1002	0.501

Based on this table, it appears that the frequency f_E approaches $\frac{1}{2}$ as n becomes larger. It could therefore be argued that $p(E)$ should be set equal to $\frac{1}{2}$. On the other hand, one might require more extensive evidence before assigning $\frac{1}{2}$ as the value of $p(E)$. In any case, this sort of evidence can never "prove" that $p(E)$ is $\frac{1}{2}$. It only serves to make this a plausible assumption. ■

If probabilities assigned to various events are to represent frequencies of occurrence of the events meaningfully, as explained previously, then they cannot be assigned in a totally arbitrary way. They must satisfy certain conditions. In the first place, since every frequency f_E must satisfy the inequalities $0 \le f_E \le 1$, it is only reasonable to assume that

$$\text{P1:} \quad 0 \le p(E) \le 1 \text{ for every event } E \text{ in } A.$$

Also, since the event A must occur every time (every outcome belongs to A), and the event $\varnothing$ cannot occur, we assume that

$$\text{P2:} \quad p(A) = 1 \text{ and } p(\varnothing) = 0.$$

Finally, if $E_1, E_2, \ldots, E_k$ are mutually exclusive events, then

$$n_{(E_1 \cup E_2 \cup \cdots \cup E_k)} = n_{E_1} + n_{E_2} + \cdots + n_{E_k},$$

since only one of these events can occur at a time. If we divide both sides of this equation by n, we see that the frequencies of occurrence must satisfy a similar equation. We therefore assume

$$\text{P3:} \quad p(E_1 \cup E_2 \cup \cdots \cup E_k) = p(E_1) + p(E_2) + \cdots + p(E_k)$$

whenever the events are mutually exclusive. If the probabilities are assigned to all events in such a way that P1, P2, and P3 are always satisfied, then we have a **probability space**. We call P1, P2, and P3 the **axioms for a probability space**.

It is important to realize that mathematically, no demands are made on a probability space except those given by the probability axioms P1, P2, and P3. Probability theory begins with all probabilities assigned, and then investigates consequences of any relations between these probabilities. No mention is made of how the probabilities were assigned. However, the mathematical conclusions will be useful in an actual situation only if the probabilities assigned reflect what actually occurs in that situation.

Experimentation is not the only way to determine reasonable probabilities for events. The probability axioms can sometimes provide logical arguments for choosing certain probabilities.

EXAMPLE 7

Consider the experiment of tossing a coin and recording whether heads or tails results. Consider the events E: heads turns up and F: tails turns up. The mechanics of the toss are not controllable in detail. Thus in the absence of any defect in the coin that might unbalance it, one may argue that E and F are equally likely to occur. There is a symmetry in the situation that makes it impossible to prefer one outcome over the other. This argument lets us compute what the probabilities of E and F must be.

We have assumed that $p(E) = p(F)$, and it is clear that E and F are mutually exclusive events and $A = E \cup F$. Thus, using the properties P2 and P3, we see that $1 = p(A) = p(E) + p(F) = 2p(E)$ since $p(E) = p(F)$. This shows that $p(E) = \frac{1}{2} = p(F)$. One may often assign appropriate probabilities to events by combining the symmetry of situations with the axioms of probability. ■

Finally, we will show that the problem of assigning probabilities to events can be reduced to the consideration of the simplest cases. Let A be a probability space. We assume that A is finite, that is, $A = \{x_1, x_2, \ldots, x_n\}$. Then each event $\{x_k\}$, consisting of just one outcome, is called an **elementary event**. For simplicity, let us write $p_k = p(\{x_k\})$. Then p_k is called the **elementary probability corresponding to the outcome** x_k. Since the elementary events are mutually exclusive and their union is A, the axioms of probability tell us that

$$\text{EP1:} \quad 0 \le p_k \le 1 \text{ for all } k$$

$$\text{EP2:} \quad p_1 + p_2 + \cdots + p_n = 1.$$

If E is any event in A, say $E = \{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$, then we can write $E = \{x_{i_1}\} \cup \{x_{i_2}\} \cup \cdots \cup \{x_{i_m}\}$. This means, by axiom P2, that $p(E) = p_{i_1} + p_{i_2} + \cdots + p_{i_m}$. Thus if we know the elementary probabilities, then we can compute the probability of any event E.

EXAMPLE 8

Suppose that an experiment has a sample space $A = \{1, 2, 3, 4, 5, 6\}$ and that the elementary probabilities have been determined as follows:

$$p_1 = \frac{1}{12}, \quad p_2 = \frac{1}{12}, \quad p_3 = \frac{1}{3}, \quad p_4 = \frac{1}{6}, \quad p_5 = \frac{1}{4}, \quad p_6 = \frac{1}{12}.$$

Let E be the event "The outcome is an even number." Compute $p(E)$.

Solution

Since $E = \{2, 4, 6\}$, we see that

$$p(E) = p_2 + p_4 + p_6 = \frac{1}{12} + \frac{1}{6} + \frac{1}{12} \quad \text{or} \quad \frac{1}{3}.$$

In a similar way we can determine the probability of any event in A. ■

Thus we see that the problem of assigning probabilities to all events in a consistent way can be reduced to the problem of finding numbers $p_1, p_2, \ldots, p_n$ that satisfy EP1 and EP2. Again, mathematically speaking, there are no other restrictions on the p_k's. However, if the mathematical structure that results is to be useful in a particular situation, then the p_k's must reflect the actual behavior occurring in that situation.

■ Equally Likely Outcomes

Let us assume that all outcomes in a sample space A are equally likely to occur. This is, of course, an assumption, and so cannot be proved. We would make such an assumption if experimental evidence or symmetry indicated that it was appropriate in a particular situation (see Example 7). Actually these situations arise commonly. One additional piece of terminology is customary. Sometimes experiments involve choosing an object, in a nondeterministic way, from some collection. If the selection is made in such a way that all objects have an equal probability of being chosen, we say that we have made a **random selection** or **chosen an object at random** from the collection. We will often use this terminology to specify examples of experiments with equally likely outcomes.

Suppose that $|A| = n$ and these n outcomes are equally likely. Then the elementary probabilities are all equal, and since they must add up to 1, this means that each elementary probability is $1/n$. Now let E be an event that contains k outcomes, say $E = \{x_1, x_2, \ldots, x_k\}$. Since all elementary probabilities are $1/n$, we must have

$$p(E) = \underbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{k \text{ summands}} = \frac{k}{n}.$$

Since $k = |E|$, we have the following principle: If all outcomes are equally likely, then for every event E

$$p(E) = \frac{|E|}{|A|} = \frac{\text{total number of outcomes in } E}{\text{total number of outcomes}}.$$

In this case, the computation of probabilities reduces to counting numbers of elements in sets. For this reason, the methods of counting discussed in the earlier sections of this chapter are quite useful.

EXAMPLE 9 Choose four cards at random from a standard 52-card deck. What is the probability that four kings will be chosen?

Solution

The outcomes of this experiment are four-card hands; each is equally likely to be chosen. The number of four-card hands is ${}_{52}C_4$ or 270,725. Let E be the event that all four cards are kings. The event E contains only one outcome. Thus $p(E) = \frac{1}{270,725}$ or approximately 0.000003694. This is an extremely unlikely event. ■

EXAMPLE 10 A box contains six red balls and four green balls. Four balls are selected at random from the box. What is the probability that two of the selected balls will be red and two will be green?

Solution

The total number of outcomes is the number of ways to select four objects out of ten, without regard to order. This is ${}_{10}C_4$ or 210. Now the event E, that two of the balls are red and two of them are green, can be thought of as the result of performing two tasks in succession.

Task 1: Choose two red balls from the six red balls in the box.

Task 2: Choose two green balls from the four green balls in the box.

Task 1 can be done in ${}_6C_2$ or 15 ways and task 2 can be done in ${}_4C_2$ or 6 ways. Thus, event E can occur in $15 \cdot 6$ or 90 ways, and therefore $p(E) = \frac{90}{210}$ or $\frac{3}{7}$. ■

EXAMPLE 11 A fair six-sided die is tossed three times and the resulting sequence of numbers is recorded. What is the probability of the event E that either all three numbers are equal or none of them is a 4?

Solution

Since the die is assumed to be fair, all outcomes are equally likely. First, we compute the total number of outcomes of the experiment. This is the number of sequences of length 3, allowing repetitions, that can be constructed from the set $\{1, 2, 3, 4, 5, 6\}$. This number is 6^3 or 216.

Event E cannot be described as the result of performing two successive tasks as in Example 10. We can, however, write E as the union of two simpler events. Let F be the event that all three numbers recorded are equal, and let G be the event that none of the numbers recorded is a 4. Then $E = F \cup G$. By the addition principle (Theorem 2, Section 1.2), $|F \cup G| = |F| + |G| - |F \cap G|$.

There are only six outcomes in which the numbers are equal, so $|F|$ is 6. The event G consists of all sequences of length 3 that can be formed from the set $\{1, 2, 3, 5, 6\}$. Thus $|G|$ is 5^3 or 125. Finally, the event $F \cap G$ consists of all sequences for which the three numbers are equal and none is a 4. Clearly, there are five ways for this to happen, so $|F \cap G|$ is 5. Using the addition principle, $|E| = |F \cup G| = 6 + 125 - 5$ or 126. Thus, we have $p(E) = \frac{126}{216}$ or $\frac{7}{12}$. ■

EXAMPLE 12 Consider again the experiment in Example 10, in which four balls are selected at random from a box containing six red balls and four green balls.

(a) If E is the event that no more than two of the balls are red, compute the probability of E.

(b) If F is the event that no more than three of the balls are red, compute the probability of F.

Solution

(a) Here E can be decomposed as the union of mutually exclusive events. Let E_0 be the event that none of the chosen balls are red, let E_1 be the event that exactly one of the chosen balls is red, and let E_2 be the event that exactly two of the chosen balls are red. Then E_0, E_1, and E_2 are mutually exclusive and $E = E_0 \cup E_1 \cup E_2$. Using the addition principle twice, $|E| = |E_0| + |E_1| + |E_2|$. If none of the balls is red, then all four must be green. Since there are only four green balls in the box, there is only one way for event E_0 to occur. Thus $|E_0| = 1$. If one ball is red, then the other three must be green. To make such a choice, we must choose one red ball from a set of six, and then three green balls from a set of four. Thus, the number of outcomes in E_1 is $({}_6C_1)({}_4C_3)$ or 24.

In exactly the same way, we can show that the number of outcomes in E_2 is $({}_6C_2)({}_4C_2)$ or 90. Thus, $|E| = 1 + 24 + 90$ or 115. On the other hand, the total number of ways of choosing four balls from the box is ${}_{10}C_4$ or 210, so $p(E) = \frac{115}{210}$ or $\frac{23}{42}$.

(b) We could compute $|F|$ in the same way we computed $|E|$ in part (a), by decomposing F into four mutually exclusive events. The analysis would, however, be even longer than that of part (a). We choose instead to illustrate another approach that is frequently useful.

Let $\overline{F}$ be the complementary event to F. Since F and $\overline{F}$ are mutually exclusive and their union is the sample space, we must have $p(F) + p(\overline{F}) = 1$. This formula holds for any event F and is used when the complementary event is easier to analyze. This is the case here, since $\overline{F}$ is the event that all four balls chosen are red. These four red balls can be chosen from the six red balls in ${}_6C_4$ or 15 ways, so $p(\overline{F}) = \frac{15}{210}$ or $\frac{1}{14}$. This means that $p(F) = 1 - \frac{1}{14}$ or $\frac{13}{14}$. ■

A common use of probability in computer science is in analyzing the efficiency of algorithms. For example, this may be done by considering the number of steps we "expect" the algorithm to execute on an "average" run. Here is a simple case to consider. If a fair coin is tossed 500 times, we expect 250 $\left(\frac{1}{2} \cdot 500\right)$ heads to occur. Of course, we would not be surprised if the number of heads were not exactly 250. This idea leads to the following definition. The **expected value** of an experiment is the sum of the value of each outcome times its probability. Roughly speaking, the expected value describes the "average" value for a large number of trials.

EXAMPLE 13 An array of length 10 is searched for a key word. The number of steps needed to find it is recorded. Assuming that the key is equally likely to be in any position of the array, the expected value of this experiment is $1 \cdot \frac{1}{10} + 2 \cdot \frac{1}{10} + \cdots + 10 \cdot \frac{1}{10}$ or $\frac{55}{10}$. On the average, we can expect to find a key word in 5.5 steps. ■

3.4 Exercises

In Exercises 1 through 4, describe the associated sample space.

1. A coin is tossed three times and the sequence of heads and tails is recorded.
2. Two letters are selected simultaneously at random from the letters a, b, c, d.
3. A silver urn and a copper urn contain blue, red, and green balls. An urn is chosen at random and then a ball is selected at random from this urn.
4. A box contains 12 items, four of which are defective. An item is chosen at random and not replaced. This is continued until all four defective items have been selected. The total number of items selected is recorded.
5. (a) Suppose that the sample space of an experiment is $\{1, 2, 3\}$. Determine all possible events.
   (b) Let S be a sample space containing n elements. How many events are there for the associated experiment?

In Exercises 6 through 8, use the following assumptions. A card is selected at random from a standard deck. Let E, F, and G be the following events.

E: The card is black.
F: The card is a diamond.
G: The card is an ace.

Describe the following events in complete sentences.

6. (a) $E \cup G$ (b) $E \cap G$
7. (a) $\overline{E} \cap G$ (b) $E \cup F \cup G$
8. (a) $E \cup \overline{F} \cup G$ (b) $(F \cap \overline{G}) \cup E$

In Exercises 9 and 10, assume that a die is tossed twice and the numbers showing on the top faces are recorded in sequence. Determine the elements in each of the given events.

9. (a) At least one of the numbers is a 5.
   (b) At least one of the numbers is an 8.
10. (a) The sum of the numbers is less than 7.
    (b) The sum of the numbers is greater than 8.
11. A die is tossed and the number showing on the top face is recorded. Let E, F, and G be the following events.

    E: The number is at least 3.
    F: The number is at most 3.
    G: The number is divisible by 2.

    (a) Are E and F mutually exclusive? Justify your answer.
    (b) Are F and G mutually exclusive? Justify your answer.
    (c) Is $E \cup F$ the certain event? Justify your answer.
    (d) Is $E \cap F$ the impossible event? Justify your answer.
12. A card is chosen from a standard deck of 52 cards. Consider the following events.

    E_1: The card drawn is a face card.
    E_2: The card drawn is a heart.
    E_3: The card drawn has an even number on it.
    E_4: The card drawn is a red card.

    Compute each of the following.
    (a) $p(E_1)$ (b) $p(E_2 \cap E_3)$ (c) $p(\overline{E_3} \cup E_2)$
13. For the events defined in Exercise 12, which of the following pairs is a pair of mutually exclusive events?
    (a) E_2, E_3 (b) E_1, E_2
    (c) E_3, E_4 (d) E_1, E_3
14. Let E be an event for an experiment with sample space A. Show that
    (a) $E \cup \overline{E}$ is the certain event.
    (b) $E \cap \overline{E}$ is the impossible event.
15. A medical team classifies people according to the following characteristics.

    Drinking habits: drinks (d), abstains (a)
    Income level: low (l), middle (m), upper (u)
    Smoking habits: smoker (s), nonsmoker (n)

    Let E, F, and G be the following events.

    E: A person drinks.
    F: A person's income level is low.
    G: A person smokes.

    List the elements in each of the following events.
    (a) $E \cup F$ (b) $\overline{E} \cap F$ (c) $(E \cup G) \cap F$

In Exercises 16 and 17, let $S = \{1, 2, 3, 4, 5, 6\}$ be the sample space of an experiment and let

$$E = \{1, 3, 4, 5\}, \quad F = \{2, 3\}, \quad G = \{4\}.$$

16. Compute the events $E \cup F$, $E \cap F$, and $\overline{F}$.
17. Compute the following events: $\overline{E} \cup F$ and $\overline{F} \cap G$.

In Exercises 18 and 19, list the elementary events for the given experiments.

18. A vowel is selected at random from the set of all vowels a, e, i, o, u.
19. A card is selected at random from a standard deck and it is recorded whether the card is a club, spade, diamond, or heart.
20. (a) What is the probability of correctly guessing a person's four-digit PIN?
    (b) People often use the four digits of their birthday (MM-DD) to create a PIN. What is the probability of correctly guessing a PIN created this way, if the birthday is known?
21. When a certain defective die is tossed, the numbers from 1 to 6 will be on the top face with the following probabilities.

$$p_1 = \frac{2}{18}, \quad p_2 = \frac{3}{18}, \quad p_3 = \frac{4}{18}, \quad p_4 = \frac{3}{18}$$
$$p_5 = \frac{4}{18}, \quad p_6 = \frac{2}{18}$$

    Find the probability that
    (a) an odd number is on top.
    (b) a prime number is on top.
    (c) a number less than 5 is on top.
    (d) a number greater than 3 is on top.
22. Repeat Exercise 21, assuming that the die is not defective.
23. Suppose that E and F are mutually exclusive events such that $p(E) = 0.3$ and $p(F) = 0.4$. Find the probability that
    (a) E does not occur. (b) E and F occur.
    (c) E or F occurs.
    (d) E does not occur or F does not occur.
24. Consider an experiment with sample space $A = \{x_1, x_2, x_3, x_4\}$ for which

$$p_1 = \frac{2}{7}, \quad p_2 = \frac{3}{7}, \quad p_3 = \frac{1}{7}, \quad p_4 = \frac{1}{7}.$$

    Find the probability of the given event.
    (a) $E = \{x_1, x_2\}$ (b) $F = \{x_1, x_3, x_4\}$
25. There are four candidates for president, A, B, C, and D. Suppose A is twice as likely to be elected as B, B is three times as likely as C, and C and D are equally likely to be elected. What is the probability of being elected for each candidate?
26. The outcome of a particular game of chance is an integer from 1 to 5. Integers 1, 2, and 3 are equally likely to occur, and integers 4 and 5 are equally likely to occur. The probability that the outcome is greater than 2 is $\frac{1}{2}$. Find the probability of each possible outcome.
27. A fair coin is tossed five times. What is the probability of obtaining three heads and two tails?

In Exercises 28 through 30, suppose a fair die is tossed and the number showing on the top face is recorded. Let E, F, and G be the following events.

$$E: \{1, 2, 3, 5\}, \quad F: \{2, 4\}, \quad G: \{1, 4, 6\}$$

Compute the probability of the event indicated.

28. (a) $E \cup F$ (b) $E \cap F$
29. (a) $\overline{E} \cap F$ (b) $E \cup G$

30. (a) $\overline{E} \cup \overline{G}$ (b) $\overline{E} \cap \overline{G}$
31. Suppose two dice are tossed and the numbers on the top faces recorded. What is the probability that
    (a) a 4 was tossed?
    (b) a prime number was tossed?
    (c) the sum of the numbers is less than 5?
    (d) the sum of the numbers is at least 7?
32. Suppose that two cards are selected at random from a standard 52-card deck. What is the probability that both cards are less than 10 and neither of them is red?
33. Suppose that three balls are selected at random from an urn containing seven red balls and five black balls. Compute the probability that
    (a) all three balls are red.
    (b) at least two balls are black.
    (c) at most two balls are black.
    (d) at least one ball is red.
34. A fair die is tossed three times in succession. Find the probability that the three resulting numbers
    (a) include exactly two 3's.
    (b) form an increasing sequence.
    (c) include at least one 3.
    (d) include at most one 3.
    (e) include no 3's.
35. There are four cards numbered 1, 2, 3, 4. Choose three cards at random and lay them face up side by side.
    (a) What is the probability that the cards chosen show numbers in increasing order from left to right?
    (b) What is the probability that the cards chosen show numbers that are not in decreasing order from left to right?
36. Each day five secretaries draw numbers to determine the order in which they will take their breaks.
    (a) What is the probability that today's order is exactly the same as yesterday's order?
    (b) What is the probability that in today's order four secretaries have the same position as they had yesterday?
    (c) What is the probability that at least one secretary has the same position as yesterday?
37. An array of length n is searched for a key word. On the average, how many steps will it take to find the key?
38. How should the analysis in Exercise 37 be changed if we do not assume that the key word is in the array?
39. A game is played by rolling two dice and paying the player an amount (in dollars) equal to the sum of the numbers on top if this is 10 or greater. The player must pay \$3 for each game. What is the expected value of this game?
40. For the game described in Exercise 39, what would be a "fair" cost to play the game? Justify your answer.

41. Suppose two cards are selected at random from a standard 52-card deck.
(a) If both cards are drawn at the same time, what is the probability that both cards have an odd number on them and neither is black?
(b) If one card is drawn and replaced before the second card is drawn, what is the probability that both cards have an odd number on them and neither is black?

3.5 Recurrence Relations

The recursive definitions of sequences in Section 1.3 are examples of recurrence relations. When the problem is to find an explicit formula for a recursively defined sequence, the recursive formula is called a **recurrence relation**. Remember that to define a sequence recursively, a recursive formula must be accompanied by information about the beginning of the sequence. This information is called the **initial condition** or **conditions** for the sequence.

EXAMPLE 1

(a) The recurrence relation $a_n = a_{n-1} + 3$ with $a_1 = 4$ recursively defines the sequence $4, 7, 10, 13, \ldots$.
(b) The recurrence relation $f_n = f_{n-1} + f_{n-2}$, $f_1 = f_2 = 1$, defines the **Fibonacci sequence** $1, 1, 2, 3, 5, 8, 13, 21, \ldots$. The initial conditions are $f_1 = 1$ and $f_2 = 1$. ■

Recurrence relations arise naturally in many counting problems and in analyzing programming problems.

EXAMPLE 2

Let $A = \{0, 1\}$. Give a recurrence relation for c_n, the number of strings of length n in A^* that do not contain adjacent 0's.

Solution

Since 0 and 1 are the only strings of length 1, $c_1 = 2$. Also, $c_2 = 3$: the only such strings are 01, 10, 11. In general, any string w of length $n-1$ that does not contain 00 can be catenated with 1 to form a string $1 \cdot w$, a string of length n that does not contain 00. The only other possible beginning for a "good" string of length n is 01. But any of these strings must be of the form $01 \cdot v$, where v is a "good" string of length $n-2$. Hence, $c_n = c_{n-1} + c_{n-2}$ with the initial conditions $c_1 = 2$ and $c_2 = 3$. ■

EXAMPLE 3

Suppose we wish to print out all n-element sequences without repeats that can be made from the set $\{1, 2, 3, \ldots, n\}$. One approach to this problem is to proceed recursively as follows.

Step 1: Produce a list of all sequences without repeats that can be made from $\{1, 2, 3, \ldots, n-1\}$.

Step 2: For each sequence in step 1, insert n in turn in each of the n available places (at the front, at the end, and between every pair of numbers in the sequence), print the result, and remove n.

The number of insert-print-remove actions is the number of n-element sequences. It is also clearly n times the number of sequences produced in step 1. Thus we have

number of n-element sequences $= n \times$ (number of $(n-1)$-sequences).

This gives a recursive formula for the number of n-element sequences. What is the initial condition? ■

One technique for finding an explicit formula for the sequence defined by a recurrence relation is **backtracking**, as illustrated in the following example.

EXAMPLE 4

The recurrence relation $a_n = a_{n-1} + 3$ with $a_1 = 2$ defines the sequence $2, 5, 8, \ldots$. We backtrack the value of a_n by substituting the definition of a_{n-1}, a_{n-2}, and so on until a pattern is clear.

$$\begin{aligned} a_n &= a_{n-1} + 3 \\ &= (a_{n-2} + 3) + 3 \\ &= ((a_{n-3} + 3) + 3) + 3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} a_n &= a_{n-1} + 3 \\ &= a_{n-2} + 2 \cdot 3 \\ &= a_{n-3} + 3 \cdot 3 \end{aligned}$$

Eventually this process will produce

$$\begin{aligned} a_n &= a_{n-(n-1)} + (n-1) \cdot 3 \\ &= a_1 + (n-1) \cdot 3 \\ &= 2 + (n-1) \cdot 3. \end{aligned}$$

An explicit formula for the sequence is $a_n = 2 + (n-1)3$. ■

EXAMPLE 5

Backtrack to find an explicit formula for the sequence defined by the recurrence relation $b_n = 2b_{n-1} + 1$ with initial condition $b_1 = 7$.

Solution

We begin by substituting the definition of the previous term in the defining formula.

$$\begin{aligned} b_n &= 2b_{n-1} + 1 \\ &= 2(2b_{n-2} + 1) + 1 \\ &= 2[2(2b_{n-3} + 1) + 1] + 1 \\ &= 2^3 b_{n-3} + 4 + 2 + 1 \\ &= 2^3 b_{n-3} + 2^2 + 2^1 + 1. \end{aligned}$$

A pattern is emerging with these rewritings of b_n. (*Note*: There are no set rules for how to rewrite these expressions and a certain amount of experimentation may be necessary.) The backtracking will end at

$$\begin{aligned} b_n &= 2^{n-1} b_{n-(n-1)} + 2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2^1 + 1 \\ &= 2^{n-1} b_1 + 2^{n-1} - 1 && \text{using Exercise 3, Section 2.4} \\ &= 7 \cdot 2^{n-1} + 2^{n-1} - 1 && \text{using } b_1 = 7 \\ &= 8 \cdot 2^{n-1} - 1 \quad \text{or} \quad 2^{n+2} - 1. \end{aligned}$$ ■

Two useful summing rules were proved in Section 2.4. We record them again for use in this section.

S1. $1 + a + a^2 + a^3 + \cdots + a^{n-1} = \dfrac{a^n - 1}{a - 1}$

S2. $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

Backtracking may not reveal an explicit pattern for the sequence defined by a recurrence relation. We now introduce a more general technique for solving a recurrence relation. First we give a definition. A recurrence relation is a **linear homogeneous relation of degree k** if it is of the form

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k} \qquad \text{with the } r_i\text{'s constants.}$$

Note that on the right-hand side, the summands are each built the same (homogeneous) way, as a multiple of one of the k (degree k) previous terms (linear).

EXAMPLE 6

(a) The relation $c_n = (-2)c_{n-1}$ is a linear homogeneous recurrence relation of degree 1.

(b) The relation $a_n = a_{n-1} + 3$ is not a linear homogeneous recurrence relation.

(c) The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous relation of degree 2.

(d) The recurrence relation $g_n = g_{n-1}^2 + g_{n-2}$ is not a linear homogeneous relation. ■

For a linear homogeneous recurrence relation of degree k, $a_n = r_1a_{n-1} + r_2a_{n-2} + \cdots + r_ka_{n-k}$, we call the associated polynomial of degree k, $x^k = r_1x^{k-1} + r_2x^{k-2} + \cdots + r_k$, its **characteristic equation.** The roots of the characteristic equation play a key role in the explicit formula for the sequence defined by the recurrence relation and the initial conditions. While the problem can be solved in general, we give a theorem for degree 2 only. Here it is common to write the characteristic equation as $x^2 - r_1x - r_2 = 0$.

Theorem 1

(a) If the characteristic equation $x^2 - r_1x - r_2 = 0$ of the recurrence relation $a_n = r_1a_{n-1} + r_2a_{n-2}$ has two distinct roots, s_1 and s_2, then $a_n = us_1^n + vs_2^n$, where u and v depend on the initial conditions, is the explicit formula for the sequence.

(b) If the characteristic equation $x^2 - r_1x - r_2 = 0$ has a single root s, the explicit formula is $a_n = us^n + vns^n$, where u and v depend on the initial conditions.

Proof

(a) Suppose that s_1 and s_2 are roots of $x^2 - r_1x - r_2 = 0$, so $s_1^2 - r_1s_1 - r_2 = 0$, $s_2^2 - r_1s_2 - r_2 = 0$, and $a_n = us_1^n + vs_2^n$, for $n \geq 1$. We show that this definition of a_n defines the same sequence as $a_n = r_1a_{n-1} + r_2a_{n-2}$. First we note that u and v are chosen so that $a_1 = us_1 + vs_2$ and $a_2 = us_1^2 + vs_2^2$ and so the initial conditions are satisfied. Then

$$
\begin{aligned}
a_n &= us_1^n + vs_2^n && \text{split out } s_1^2 \text{ and } s_2^2.\\
&= us_1^{n-2}s_1^2 + vs_2^{n-2}s_2^2 && \text{substitute for } s_1^2 \text{ and } s_2^2.\\
&= us_1^{n-2}(r_1s_1 + r_2) + vs_2^{n-2}(r_1s_2 + r_2)\\
&= r_1us_1^{n-1} + r_2us_1^{n-2} + r_1vs_2^{n-1} + r_2vs_2^{n-2}\\
&= r_1(us_1^{n-1} + vs_2^{n-1}) + r_2(us_1^{n-2} + vs_2^{n-2})\\
&= r_1a_{n-1} + r_2a_{n-2} && \text{use definitions of } a_{n-1} \text{ and } a_{n-2}.
\end{aligned}
$$

(b) This part may be proved in a similar way. □

This direct proof requires that we find a way to use what is known about s_1 and s_2. We know something about s_1^2 and s_2^2, and this suggests the first step of the algebraic rewriting. Finding a useful first step in a proof may involve some false starts. Be persistent.

EXAMPLE 7

Find an explicit formula for the sequence defined by $c_n = 3c_{n-1} - 2c_{n-2}$ with initial conditions $c_1 = 5$ and $c_2 = 3$.

Solution

The recurrence relation $c_n = 3c_{n-1} - 2c_{n-2}$ is a linear homogeneous relation of degree 2. Its associated equation is $x^2 = 3x - 2$. Rewriting this as $x^2 - 3x + 2 = 0$, we see there are two roots, 1 and 2. Theorem 1(a) says we can find u and v so that $c_1 = u(1) + v(2)$ and $c_2 = u(1)^2 + v(2)^2$. Solving this 2×2 system yields u is 7 and v is -1.

By Theorem 1, we have $c_n = 7 \cdot 1^n + (-1) \cdot 2^n$ or $c_n = 7 - 2^n$. Note that using $c_n = 3c_{n-1} - 2c_{n-2}$ with initial conditions $c_1 = 5$ and $c_2 = 3$ gives $5, 3, -1, -9$ as the first four terms of the sequence. The formula $c_n = 7 - 2^n$ also produces $5, 3, -1, -9$ as the first four terms. ■

EXAMPLE 8

Solve the recurrence relation $d_n = 2d_{n-1} - d_{n-2}$ with initial conditions $d_1 = 1.5$ and $d_2 = 3$.

Solution

The associated equation for this linear homogeneous relation is $x^2 - 2x + 1 = 0$. This equation has one (multiple) root, 1. Thus, by Theorem 1(b), $d_n = u(1)^n + vn(1)^n$. Using this formula and the initial conditions, $d_1 = 1.5 = u + v(1)$ and $d_2 = 3 = u + v(2)$, we find that u is 0 and v is 1.5. Then $d_n = 1.5n$. ■

The Fibonacci sequence in Example 1(b) is a well-known sequence whose explicit formula took over two hundred years to find.

EXAMPLE 9

The Fibonacci sequence is defined by a linear homogeneous recurrence relation of degree 2, so by Theorem 1, the roots of the associated equation are needed to describe the explicit formula for the sequence. From $f_n = f_{n-1} + f_{n-2}$ and $f_1 = f_2 = 1$, we have $x^2 - x - 1 = 0$. Using the quadratic formula to obtain the roots, we find

$$s_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad s_2 = \frac{1-\sqrt{5}}{2}.$$

It remains to determine the u and v of Theorem 1. We solve

$$1 = u\left(\frac{1+\sqrt{5}}{2}\right) + v\left(\frac{1-\sqrt{5}}{2}\right) \quad \text{and} \quad 1 = u\left(\frac{1+\sqrt{5}}{2}\right)^2 + v\left(\frac{1-\sqrt{5}}{2}\right)^2.$$

For the given initial conditions, u is $\frac{1}{\sqrt{5}}$ and v is $-\frac{1}{\sqrt{5}}$. The explicit formula for the Fibonacci sequence is

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

■

Sometimes properties of a recurrence relation are useful to know. Because of the close connection between recurrence (recursion) and mathematical induction, proofs of these properties by induction are common.

EXAMPLE 10

For the Fibonacci numbers in Example 1(b), $f_n \leq \left(\frac{5}{3}\right)^n$. This gives a bound on how fast the Fibonacci numbers grow.

Proof (by strong induction)

Basis Step

Here n_0 is 1. $P(1)$ is $1 \leq \frac{5}{3}$ and this is clearly true.

Induction Step

We use $P(j)$, $j \le k$ to show $P(k+1)$: $f_{k+1} \le \left(\frac{5}{3}\right)^{k+1}$. Consider the left-hand side of $P(k+1)$:

$$\begin{aligned} f_{k+1} = f_k + f_{k-1} &\le \left(\frac{5}{3}\right)^{k} + \left(\frac{5}{3}\right)^{k-1} \\ &= \left(\frac{5}{3}\right)^{k-1}\left(\frac{5}{3}+1\right) \\ &= \left(\frac{5}{3}\right)^{k-1}\left(\frac{8}{3}\right) \\ &< \left(\frac{5}{3}\right)^{k-1}\left(\frac{5}{3}\right)^{2} \\ &= \left(\frac{5}{3}\right)^{k+1}, \quad \text{the right-hand side of } P(k+1). \quad \blacksquare \end{aligned}$$

3.5 Exercises

In Exercises 1 through 6, give the first four terms and identify the given recurrence relation as linear homogeneous or not. If the relation is a linear homogeneous relation, give its degree.

1. $a_n = 2.5a_{n-1}$, $a_1 = 4$
2. $b_n = -3b_{n-1} - 2b_{n-2}$, $b_1 = -2$, $b_2 = 4$
3. $c_n = 2^n c_{n-1}$, $c_1 = 3$
4. $d_n = n d_{n-1}$, $d_1 = 2$
5. $e_n = 5e_{n-1} + 3$, $e_1 = 1$
6. $g_n = \sqrt{g_{n-1} + g_{n-2}}$, $g_1 = 1$, $g_2 = 3$
7. Let $A = \{0, 1\}$. Give a recurrence relation for the number of strings of length n in A^* that do not contain 01.
8. Let $A = \{0, 1\}$. Give a recurrence relation for the number of strings of length n in A^* that do not contain 111.
9. On the first of each month Mr. Martinez deposits \$100 in a savings account that pays 6% compounded monthly. Assuming that no withdrawals are made, give a recurrence relation for the total amount of money in the account at the end of n months.
10. An annuity of \$10,000 earns 8% compounded monthly. Each month \$250 is withdrawn from the annuity. Write a recurrence relation for the monthly balance at the end of n months.
11. A game is played by moving a marker ahead either 2 or 3 steps on a linear path. Let c_n be the number of different ways a path of length n can be covered. Give a recurrence relation for c_n.

In Exercises 12 through 17, use the technique of backtracking to find an explicit formula for the sequence defined by the recurrence relation and initial condition(s).

12. $a_n = 2.5a_{n-1}$, $a_1 = 4$
13. $b_n = 5b_{n-1} + 3$, $b_1 = 3$
14. $c_n = c_{n-1} + n$, $c_1 = 4$
15. $d_n = -1.1d_{n-1}$, $d_1 = 5$
16. $e_n = e_{n-1} - 2$, $e_1 = 0$
17. $g_n = n g_{n-1}$, $g_1 = 6$

In Exercises 18 through 23, solve each of the recurrence relations.

18. $a_n = 4a_{n-1} + 5a_{n-2}$, $a_1 = 2$, $a_2 = 6$
19. $b_n = -3b_{n-1} - 2b_{n-2}$, $b_1 = -2$, $b_2 = 4$
20. $c_n = -6c_{n-1} - 9c_{n-2}$, $c_1 = 2.5$, $c_2 = 4.7$
21. $d_n = 4d_{n-1} - 4d_{n-2}$, $d_1 = 1$, $d_2 = 7$
22. $e_n = 2e_{n-2}$, $e_1 = \sqrt{2}$, $e_2 = 6$
23. $g_n = 2g_{n-1} - 2g_{n-2}$, $g_1 = 1$, $g_2 = 4$
24. Develop a general explicit formula for a nonhomogeneous recurrence relation of the form $a_n = ra_{n-1} + s$, where r and s are constants.
25. Test the results of Exercise 24 on Exercises 13 and 16.
26. Let r_n be the number of regions created by n lines in the plane, where each pair of lines has exactly one point of intersection.
    (a) Give a recurrence relation for r_n.
    (b) Solve the recurrence relation of part (a).
27. Let a_n be the number of 2-set partitions of a set with n elements.
    (a) Give a recurrence relation for a_n.
    (b) Solve the recurrence relation of part (a).
28. Prove Theorem 1(b). (*Hint*: Find the condition on r_1 and r_2 that guarantees that there is one solution s.)

29. Solve the recurrence relation of Example 2.
30. Using the argument in Example 3 for ${}_nP_r$ would produce ${}_nP_r = r \cdot {}_nP_{r-1}$. But this is easily shown to be false for nearly all choices of n and r. Explain why the argument is not valid.
31. For the Fibonacci sequence, prove that for $n \ge 2$, $f_{n+1}^2 - f_n^2 = f_{n-1} f_{n+2}$.
32. Solve the recurrence relation of Exercise 7.
33. Solve the recurrence relation of Exercise 9.

Theorem 1 can be extended to a linear homogeneous relation of degree k, $a_n = r_1a_{n-1} + r_2a_{n-2} + \cdots + r_ka_{n-k}$. If the characteristic equation has k distinct roots $s_1, s_2, \ldots, s_k$, then $a_n = u_1s_1^n + u_2s_2^n + \cdots + u_ks_k^n$, where $u_1, u_2, \ldots, u_k$ depend on the initial conditions.

34. Let $a_n = 7a_{n-2} + 6a_{n-3}$, $a_1 = 3$, $a_2 = 6$, $a_3 = 10$.
    (a) What is the degree of this linear homogeneous relation?
    (b) Solve the recurrence relation.
35. Solve the recurrence relation $a_n = -2a_{n-1} + 2a_{n-2} + 4a_{n-3}$, $a_1 = 0$, $a_2 = 2$, $a_3 = 8$.
36. Use mathematical induction to prove that for the recurrence relation $b_n = b_{n-1} + 2b_{n-2}$, $b_1 = 1$, $b_2 = 3$, $b_n < \left(\frac{5}{2}\right)^n$.
37. Use mathematical induction to prove that for the recurrence relation $a_n = 2a_{n-1} + a_{n-2}$, $a_1 = 10$, $a_2 = 12$, $5 \mid a_{3n-1}$, $n \ge 0$.
38. Let $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_{n+1}$ each be a $k \times k$ matrix. Let C_n be the number of ways to evaluate the product $\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3 \times \cdots \times \mathbf{A}_{n+1}$ by choosing different orders in which to do the n multiplications. Compute C_1, C_2, C_3, C_4, C_5.
39. Give a recurrence relation for C_n (defined in Exercise 38).
40. Verify that $C_n = \dfrac{{}_{2n}C_n}{n+1}$ is a possible solution to the recurrence relation of Exercise 39 by showing that this formula produces the first five values as found in Exercise 38. (The terms of this sequence are called the **Catalan numbers**.)

Tips for Proofs

Proofs based on the pigeonhole principle are introduced in this chapter. Two situations are possible; the pigeons and pigeonholes are implicitly defined in the statement of the problem (Section 3.3, Exercise 5) or you must create pigeons and pigeonholes by defining categories into which the objects must fall (Section 3.3, Exercises 12 and 13). In the first case, the phrases "at least k objects" "have the same property" identify the pigeons (objects) and the labels on the pigeonholes (possible properties).

Proofs of statements about ${}_nC_r$ and ${}_nP_r$ are usually direct proofs based on the definitions and elementary algebra. Remember that a direct proof is generally the first approach to try.

Key Ideas for Review

- Theorem (The Multiplication Principle): Suppose two tasks T_1 and T_2 are to be performed in sequence. If T_1 can be performed in n_1 ways and for each of these ways T_2 can be performed in n_2 ways, then the sequence T_1T_2 can be performed in n_1n_2 ways.
- Theorem (The Extended Multiplication Principle): see page 79
- Theorem: Let A be a set with n elements and $1 \le r \le n$. Then the number of sequences of length r that can be formed from elements of A, allowing repetitions, is n^r.
- Permutation of n objects taken r at a time ($1 \le r \le n$): a sequence of length r formed from distinct elements
- Theorem: If $1 \le r \le n$, then ${}_nP_r$, the number of permutations of n objects taken r at a time, is $n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-r+1)$ or $\frac{n!}{(n-r)!}$.
- Permutation: an arrangement of n elements of a set A into a sequence of length n
- Theorem: The number of distinguishable permutations that can be formed from a collection of n objects where the first object appears k_1 times, the second object k_2 times, and so on, is $\frac{n!}{k_1!k_2!\cdots k_t!}$, where $k_1 + k_2 + \cdots + k_t = n$.
- Combination of n objects taken r at a time: a subset of r elements taken from a set with n elements
- Theorem: Let A be a set with $|A| = n$ and let $0 \le r \le n$. Then ${}_nC_r$, the number of combinations of the elements of A, taken r at a time, is $\frac{n!}{r!(n-r)!}$.
- Theorem: Suppose k selections are to be made from n items without regard to order and that repeats are allowed, assuming at least k copies of each of the n items. The number of ways these selections can be made is ${}_{(n+k-1)}C_k$.
- The pigeonhole principle: see page 88
- The extended pigeonhole principle: see page 90

- Sample space: the set of all outcomes of an experiment
- Event: a subset of the sample space
- Certain event: an event certain to occur
- Impossible event: the empty subset of the sample space
- Mutually exclusive events: any two events E and F with $E \cap F = \{\ \}$
- f_E: the frequency of occurrence of the event E in n trials
- $p(E)$: the probability of event E
- Probability space: see page 94
- Elementary event: an event consisting of just one outcome
- Random selection: see page 95
- Expected value: the sum of the products (value of a_i) $\cdot$ $(p(a_i))$ for all outcomes a_i of an experiment
- Recurrence relation: a recursive formula for a sequence
- Initial conditions: information about the beginning of a recursively defined sequence
- Linear homogeneous relation of degree k: a recurrence relation of the form $a_n = r_1a_{n-1} + r_2a_{n-2} + \cdots + r_ka_{n-k}$ with the r_i's constants
- Characteristic equation: see page 102
- Catalan numbers: see page 105

Review Questions

1. How can you decide whether combinations or permutations or neither should be counted in a particular counting problem?
2. What are some clues for deciding how to define pigeons and pigeonholes?
3. What are some of the advantages and disadvantages of the recursive form of a recurrence relation?
4. What are some of the advantages and disadvantages of the solution form of a recurrence relation?
5. What is our primary tool (from previous work) for answering probability questions?

Chapter 3 Self-Test

1. Compute the number of
   (a) five-digit binary numbers.
   (b) five-card hands from a deck of 52 cards.
   (c) distinct arrangements of the letters of DISCRETE.
2. A computer program is used to generate all possible six-letter names for a new medication. Suppose that all the letters of the English alphabet may be used. How many possible names can be formed
   (a) if the letters are to be distinct?
   (b) if exactly two letters are repeated?
3. A fair six-sided die is rolled five times. If the results of each roll are recorded, how many
   (a) record sequences are possible?
   (b) record sequences begin 1, 2?
4. The IC Shoppe has 14 flavors of ice cream today. If you allow repeats, how many different triple-scoop ice cream cones can be chosen? (The order in which the scooping is done does not matter.)
5. The Spring Dance Committee must have 3 freshman and 5 sophomore members. If there are 23 eligible freshman and 18 eligible sophomores, how many different committees are possible?
6. Show that ${}_{2n}C_2 = 2 \cdot {}_nC_2 + n^2$.
7. Pizza Quik always puts 50 pepperoni slices on a pepperoni pizza. If you cut a pepperoni pizza into eight equal size pieces, at least one piece must have ______ pepperoni slices. Justify your answer.
8. Complete and prove the following statement. At least ______ months of the year must begin on the same day of the week.
9. What is the probability that exactly two coins will land heads up when five fair coins are tossed?
10. Let $p(A) = 0.29$, $p(B) = 0.41$, and $p(A \cup B) = 0.65$. Are A and B mutually exclusive events? Justify your answer.
11. Solve the recurrence relation $b_n = 7b_{n-1} - 12b_{n-2}$, $b_1 = 1$, $b_2 = 7$.
12. Develop a formula for the solution of a recurrence relation of the form $a_n = ma_{n-1} - 1$, $a_1 = m$.

Coding Exercises

For each of the following, write the requested program or subroutine in pseudocode (as described in Appendix A) or in a programming language that you know. Test your code either with a paper-and-pencil trace or with a computer run.

1. Write a subroutine that accepts two positive integers n and r, and if $r \leq n$, returns the number of permutations of n objects taken r at a time.
2. Write a program that has as input positive integers n and r and, if $r \leq n$, prints the permutations of $1, 2, 3, \ldots, n$ taken r at a time.

3. Write a subroutine that accepts two positive integers n and r and, if $r \leq n$, returns the number of combinations of n objects taken r at a time.
4. Write a program that has as input positive integers n and r and, if $r \leq n$, prints the combinations of $1, 2, 3, \ldots, n$ taken r at a time.
5. (a) Write a recursive subroutine that with input k prints the first k Fibonacci numbers.
   (b) Write a nonrecursive subroutine that with input k prints the kth Fibonacci number.

Experiment 3

The purpose of this experiment is to introduce the concept of a Markov chain. The investigations will use your knowledge of probability and matrices.

Suppose that the weather in Acia is either rainy or dry. We say that the weather has two possible **states**. As a result of extensive record keeping, it has been determined that the probability of a rainy day following a dry day is $\frac{1}{3}$, and the probability of a rainy day following a rainy day is $\frac{1}{2}$. If we know the weather today, then we can predict the probability that it will be rainy tomorrow. In fact, if we know the state in which the weather is today, then we can predict the probability for each possible state tomorrow. A **Markov chain** is a process in which the probability of a system's being in a particular state at a given observation period depends only on its state at the immediately preceding observation period. Let t_{ij} be the probability that if the system is in state j at a certain observation period, it will be in state i at the next period; t_{ij} is called a **transition probability**. It is convenient to arrange the transition probabilities for a system with n possible states as an $n \times n$ **transition matrix**. A transition matrix for Acia's weather is

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{D} & \mathrm{R} \end{array} \\ \mathbf{T} = & \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{array}{c} \mathrm{D} \\ \mathrm{R} \end{array} \end{array}$$

1. What is the sum of the entries in each column of $\mathbf{T}$? Explain why this must be the same for each column of any transition matrix.

The transition matrix of a Markov chain can be used to determine the probability of the system being in any of its n possible states at future times. Let

$$\mathbf{P}^{(k)} = \begin{bmatrix} p_1^{(k)} \\ p_2^{(k)} \\ \vdots \\ p_n^{(k)} \end{bmatrix}$$

denote the **state vector** of the Markov chain at the observation period k, where $p_j^{(k)}$ is the probability that the system is in state j at the observation period k. The state vector $\mathbf{P}^{(0)}$ is called the initial state vector.

2. Suppose today, a Wednesday, is dry in Acia and this is observation period 0.
   (a) Give the initial state vector for the system.
   (b) What is the probability that it will be dry tomorrow? What is the probability that it will be rainy tomorrow? Give $\mathbf{P}^{(1)}$.
   (c) Compute $\mathbf{TP}^{(0)}$. What is the relationship between $\mathbf{TP}^{(0)}$ and $\mathbf{P}^{(1)}$?

It can be shown that, in general, $\mathbf{P}^{(k)} = \mathbf{T}^k\mathbf{P}^{(0)}$. Thus the transition matrix and the initial state vector completely determine every other state vector.

3. Using the initial state vector from part 2, what is the state vector for next
   (a) Friday?
   (b) Sunday?
   (c) Monday?
   (d) What appears to be the long-term behavior of this system?

In some cases the Markov chain reaches an equilibrium state, because the state vectors converge to a fixed vector. This vector is called the **steady-state vector**. The most common use of Markov chains is to determine long-term behavior, so it is important to know if a particular Markov chain has a steady-state vector.

4. Let

$$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{P}^{(0)} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.$$

Compute enough state vectors to determine the long-term behavior of this Markov chain.

A transition matrix $\mathbf{T}$ of a Markov chain is called **regular** if all the entries in some power of $\mathbf{T}$ are positive. If a Markov chain has a regular transition matrix, then the process has a steady-state vector. One way to find the steady-state vector, if it exists, is to proceed as in question 3; that is, calculate enough successive state vectors to identify the vector to which they are converging. Another method requires the solution of a system of linear equations. The steady-state vector $\mathbf{U}$ must be a solution of the matrix equation $\mathbf{TU} = \mathbf{U}$, and the entries of $\mathbf{U}$ have a sum equal to 1.

5. Verify that the transition matrix for the weather in Acia is regular and that the transition matrix in part 4 is not regular.

6. Solve

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

with the condition that $x + y = 1$. Compare your solution with the results of part 3.

7. Consider a plant that can have red (R), pink (P), or white (W) flowers depending on the genotypes RR, RW, and WW. When we cross each of these genotypes with genotype RW, we have the following transition matrix.

$$\begin{array}{ccc} & & \text{Flowers of parent plant} \\ & & \begin{array}{ccc} \text{R} & \text{P} & \text{W} \end{array} \\ \text{Flowers of offspring plant} & \begin{array}{c} \text{R} \\ \text{P} \\ \text{W} \end{array} & \begin{bmatrix} 0.5 & 0.25 & 0.0 \\ 0.5 & 0.50 & 0.5 \\ 0.0 & 0.25 & 0.5 \end{bmatrix} \end{array}$$

Suppose that each successive generation is produced by crossing only with plants of RW genotype.
   (a) Will the process reach an equilibrium state? Why or why not?
   (b) If there is a steady-state vector for this Markov chain, what are the long-term percentages of plants with red, pink, and white flowers?

8. In Acia there are two companies that produce widgets, Widgets, Inc., and Acia Widgets. Each year Widgets, Inc., keeps one-fourth of its customers while three-fourths switch to Acia Widgets. Each year Acia Widgets keeps

two-thirds of its customers and one-third switch to Widgets, Inc. Both companies began business the same year and in that first year Widgets, Inc., had three-fifths of the market and Acia Widgets had the other two-fifths of the market. Under these conditions, will Acia Widgets ever run Widgets, Inc., out of business? Justify your answer.