The power of modeling lies in the fact that if a situation has the characteristics described in Part III, we can use the solution developed in Section 5.2 without carrying out the full analysis again. This is very similar to having a formula to find the solution to a problem.

Part IV. The parentheses problem in Section 5.2 was originally described by a recurrence relation. This provides another way that we can recognize this model and obtain a solution.

1. Describe how to model the parentheses problem in order to develop the recurrence relation. State the recurrence relation for the parentheses problem. Give the solution of this recurrence relation.
2. Consider the problem of counting the number of nonisomorphic binary trees with n vertices.
   - **(a)** Draw and count the nonisomorphic binary trees with one vertex, two vertices, and three vertices.
   - **(b)** Explain how this recurrence relation is related to that of the parentheses problem and give the explicit solution for the tree problem.
3. Here is another counting problem. Consider a convex polygon with $n + 2$ edges and $n + 2$ vertices like that in Figure 1. The problem is to count the maximum number of triangles created by dividing the polygon by drawing lines between nonadjacent vertices such that the lines do not intersect. For example, Figure 2 shows the cases of $n = 2$ and $n = 3$.

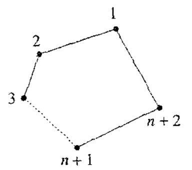


Figure 1

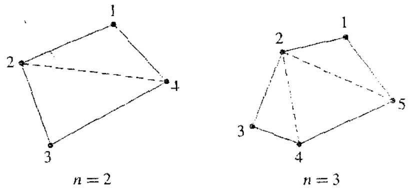


Figure 2

   - **(a)** Give a recurrence relation for the number of triangles that can be created in this way. Clearly identify the variables in the relation.
   - **(b)** Explain how this recurrence relation is related to that of the parentheses problem and give the explicit solution for the triangle problem.
4. Discuss the similarities of the parentheses, tree, and triangle counting problems. Describe how to recognize another instance of this model.

Answers to Odd-Numbered Exercises

Note: We have not included "solutions" for Coding Exercises for several reasons. We believe each department (or instructor) should have very specific programming standards and any code we presented would certainly violate someone's standards and set up unnecessary conflicts. More importantly, different programming languages support different constructions and what is a good choice in one can be a very bad one in another.

CHAPTER 1

Exercise Set 1.1, page 4

1. **(a)** True. **(b)** False. **(c)** False.
(d) False. **(e)** True. **(f)** False.

3. **(a)** {A, R, D, V, K}. **(b)** {B, O, K}.
(c) {M, I, S, P}.

5. **(a)** False. **(b)** True. **(c)** False.
(d) True. **(e)** False. **(f)** False.

7. $\{x \mid x \text{ is a vowel}\}$

9. $\{x \mid x \in Z \text{ and } x^2 < 5\}$

11. (b), (c), (e).

13. { }, {BASIC}, {PASCAL}, {ADA}, {BASIC, PASCAL}, {BASIC, ADA}, {PASCAL, ADA}, {BASIC, PASCAL, ADA}.

15. **(a)** True. **(b)** False. **(c)** False.
(d) True. **(e)** True. **(f)** True.
(g) True. **(h)** True.

17. **(a)** $\subseteq$ **(b)** $\subseteq$ **(c)** $\nsubseteq$
(d) $\subseteq$ **(e)** $\nsubseteq$ **(f)** $\subseteq$

19. $\{1, 2, 3\}$

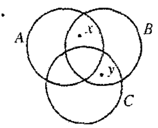
21. Yes, Yes, the complement of a set would not be defined unambiguously.

23. **(a)** False. **(b)** False. **(c)** Insufficient information.
(d) False. **(e)** True. **(f)** True.

25. Eight. There are three parts that represent what is left of each set when common parts are removed, three regions that each represent the part shared by one of the three pairs of sets, a region that represents what all three sets have in common, and a region outside all three sets.

27. $B = \{m, n\}$. **29.** $B = \{a, b, c\}$.

31.



is one solution

33. $\varnothing \subseteq Z^+ \subseteq N \subseteq Z \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

35. B; B. **37.** 4; 8.

Exercise Set 1.2, page 11

1. **(a)** $\{a, b, c, d, e, f, g\}$. **(b)** $\{a, c, d, e, f, g\}$.
(c) $\{a, c\}$. **(d)** $\{f\}$.
(e) $\{b, g, d, e\}$. **(f)** $\{a, b, c\}$.
(g) $\{d, e, f, h, k\}$. **(h)** $\{a, b, c, d, e, f\}$.
(i) $\{b, g, f\}$. **(j)** $\{g\}$.

3. **(a)** $\{a, b, c, d, e, f, g\}$. **(b)** { }.
(c) $\{a, c, g\}$. **(d)** $\{a, c, f\}$.
(e) $\{h, k\}$. **(f)** $\{a, b, c, d, e, f, h, k\}$.

5. **(a)** $\{1, 2, 4, 5, 6, 8, 9\}$. **(b)** $\{1, 2, 3, 4, 6, 8\}$.
(c) $\{1, 2, 4, 6, 7, 8\}$. **(d)** $\{1, 2, 3, 4, 5, 9\}$.
(e) $\{1, 2, 4\}$. **(f)** $\{8\}$.
(g) $\{2, 4\}$. **(h)** { }.

7. **(a)** $\{1, 2, 3, 4, 5, 6, 8, 9\}$. **(b)** $\{2, 4\}$.
(c) $\{1, 2, 4\}$. **(d)** $\{8\}$.
(e) $\{3, 7\}$. **(f)** $\{1, 3, 5, 6, 7, 8, 9\}$.

9. **(a)** $\{b, d, e, h\}$. **(b)** $\{b, c, d, f, g, h\}$.
(c) $\{b, d, h\}$. **(d)** $\{b, c, d, e, f, g, h\}$.
(e) { }. **(f)** $\{c, f, g\}$.

11. **(a)** All real numbers except -1 and 1.
(b) All real numbers except -1 and 4.
(c) All real numbers except -1, 1, and 4.
(d) All real numbers except -1.

13. **(a)** True. **(b)** True. **(c)** False. **(d)** False.

15. 1.

17. **(a)** $|A \cup B| = 10$, $|A| = 6$, $|B| = 7$, $|A \cap B| = 3$. Hence $|A \cup B| = |A| + |B| - |A \cap B|$.
(b) $|A \cup B| = 11$, $|A| = 5$, $|B| = 6$, $|A \cap B| = 0$. Hence $|A \cup B| = |A| + |B| - |A \cap B|$.

19. B must be the empty set.

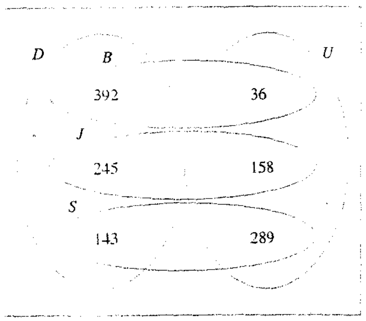
21. $|A| = 6$, $|B| = 5$, $|C| = 6$, $|A \cap B| = 2$, $|A \cap C| = 3$, $|B \cap C| = 3$, $|A \cap B \cap C| = 2$, $|A \cup B \cup C| = 11$. Hence $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

23. **(a)** 106. **(b)** 60.

25. 16; 23

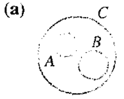
27. **(a)** 162. **(b)** 118. **(c)** 236.
(d) 290. **(e)** 264.

29.



31. A and to B.

33. **(a)**



(b) $A \cup B$.

(c) Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. Since $A \subseteq C$ and $B \subseteq C$, $x \in A$ or $x \in B$ means that $x \in C$. Hence $A \cup B \subseteq C$.

35. Yes. Suppose $x \in B$. Either $x \in A$ or $x \notin A$. If $x \in A$, then $x \notin A \oplus B = A \oplus C$. But then x must be in C. If $x \notin A$, then $x \in A \oplus B = A \oplus C$, and again x must be in C. So $B \subseteq C$. A similar argument shows that $C \subseteq B$, so $B = C$.

37. No. Let $A = \{1, 2, 3\}$, $B = \{4\}$, and $C = \{3, 4\}$. Then $A \cup B = A \cup C$ and $B \neq C$.

39. **(a)** Let $x \in A \cup C$. Then $x \in A$ or $x \in C$, so $x \in B$ or $x \in D$ and $x \in B \cup D$. Hence $A \cup C \subseteq B \cup D$.

(b) Let $x \in A \cap C$. Then $x \in A$ and $x \in C$, so $x \in B$ and $x \in D$. Hence $x \in B \cap D$. Thus $A \cap C \subseteq B \cap D$.

41. We must subtract $|B \cap C|$, because each element in $B \cap C$ has been counted twice in $|A| + |B| + |C|$. But when we subtract both $|B \cap C|$ and $|A \cap C|$, we have "uncounted" all the elements of C that also belong to B and A. These (and the similar elements of A and B) are counted again by adding $|A \cap B \cap C|$.

43. The cardinality of the union of n sets is the sum of the cardinalities of each of the n sets minus the sum of the cardinalities of the ${}_nC_2$ different intersections of two of the sets plus the sum of the ${}_nC_3$ different intersections of three of the sets and so on, alternating plus and minus the sum of the ${}_nC_k$ k-set intersections, $k = 4, \ldots, n$.

Exercise Set 1.3, page 19

1. $\{1, 2\}$.

3. $\{a, b, c, \ldots, z\}$.

5. Possible answers include $xyzxyz\ldots$, $xxyyzzxxyyzz$, and $yzxyzx\ldots$.

7. 5, 25, 125, 625.

9. 1, 2, 6, 24.

11. 2.5, 4, 5.5, 7.

13. 0, −2, −4, −6.

15. $a_n = a_{n-1} + 2$, $a_1 = 1$, recursive; $a_n = 2n - 1$, explicit.

17. $c_n = (-1)^{n+1}$, explicit.

19. $e_n = e_{n-1} + 3$, $e_1 = 1$, recursive.

21. $a_n = 2 + 3(n - 1)$.

23. A, uncountable; B, finite; C, countable; D, finite; E, finite.

25. **(a)** Yes. **(b)** No. **(c)** Yes.
(d) Yes. **(e)** No. **(f)** No.

27. **(a)** 1. **(b)** 0.

(c) f_B: 10000000, f_C: 01010011, f_D: 01000101.

(d) 11010011, 01010111, 01000001.

29.
$$\begin{aligned} f_{(A\oplus B)\oplus C} &= f_{A\oplus B} + f_C - 2f_{A\oplus B}f_C \quad \text{by Theorem 4}\\ &= (f_A + f_B - 2f_Af_B)\\ &\quad + f_C - 2(f_A + f_B - 2f_Af_B)f_C\\ &= f_A + (f_B + f_C - 2f_Bf_C)\\ &\quad - 2f_A(f_B + f_C - 2f_Bf_C)\\ &= f_A + f_{B\oplus C} - 2f_Af_{B\oplus C}\\ &= f_{A\oplus(B\oplus C)} \end{aligned}$$
Since the characteristic functions are the same, the sets must be the same.

31. **(a)** Yes. **(b)** Yes. **(c)** Yes.

33. Possible answers include $\vee ac \quad a\vee \quad ab($.

35. **(a)** 01^*0. **(b)** $0(00)^* \vee (00)^*1$.

37. By (1), 8 is an S-number. By (3), 1 is an S-number. By (2), all multiples of 1, that is, all integers are S-numbers.

39. 1, 2, 3, 7, 16.

Exercise Set 1.4, page 30

1. $20 = 6 \cdot 3 + 2$.

3. $3 = 0 \cdot 22 + 3$.

5. **(a)** $828 = 2^2 \cdot 3^2 \cdot 23$. **(b)** $1666 = 2 \cdot 7^2 \cdot 17$.
(c) $1781 = 13 \cdot 137$. **(d)** $1125 = 3^2 \cdot 5^3$.
(e) 107.

7. $d = 3$; $3 = 3 \cdot 45 - 4 \cdot 33$.

9. $d = 1$; $1 = 5 \cdot 77 - 3 \cdot 128$.

11. 1050. **13.** 864.

15. **(a)** 6. **(b)** 1. **(c)** 0.
(d) 1. **(e)** 20. **(f)** 14.

17. **(a)** 10. **(b)** 22. **(c)** 2. **(d)** 14.

19. $f(a) + f(b)$ may be greater than n.

21. **(a)** $\{2, 7, 12, 17, \ldots\}$. **(b)** $\{1, 6, 11, 16, \ldots\}$.

23. The only divisors of p are $\pm p$ and ± 1, but p does not divide a. (Multiply both sides by b.) $p \mid sab$ and $p \mid tpb$. If p divides the right side of the equation, then it must divide the left side also.

25. Because $a \mid m$, $ac \mid mc$ and because $c \mid m$, $ac \mid am$. If $\text{GCD}(a, c) = 1$, there are integers s, t such that $1 = sa + tc$. Thus $m = sam + tcm$. But ac divides each term on the left so $ac \mid m$.

27. Let $d = \text{GCD}(a, b)$. Then $cd \mid ca$ and $cd \mid cb$; that is, cd is a common divisor of ca and cb. Let $e = \text{GCD}(ca, cd)$. Then $cd \mid e$ and $e = cdk$ is a divisor of ca and cb. But then $dk \mid a$ and $dk \mid b$. Because d is the greatest common divisor of a and b, k must be 1 and $e = cd$.

29. By Theorem 6, $\text{GCD}(a, b) \cdot \text{LCM}(a, b) = ab$. Since $\text{GCD}(a, b) = 1$, we have $\text{LCM}(a, b) = ab$.

31. $a \mid b$ means $b = am$. $b \mid a$ means $a = bn$. Thus $b = am = bnm$. Hence $nm = 1$ and $n = m = 1$, because a and b are positive.

33. No; consider $a = 6$, $b = 4$, $c = 3$.

35. Yes. Using the same reasoning as in Exercise 34, m and n share no prime factors, so for mn to be a perfect tth power, each prime in the factorizations of m and n must appear a multiple of t times. But this means each of m and n are also perfect tth powers.

37. **(a)** 112. **(b)** 10. **(c)** 30.

39. **(a)** **(i)** $(104)_5$. **(ii)** $(243)_5$. **(iii)** $(1330)_5$.**(iv)** $(10412)_5$.

(b) **(i)** 49. **(ii)** 85. **(iii)** 197. **(iv)** 816.

41. **(a)** $(11101)_2$. $(1001001)_2$. $(11010111)_2$. $(1011011100)_2$.

(b) $(131)_4$. $(1021)_4$. $(3113)_4$. $(23130)_4$.

(c) $(1D)_{16}$. $(49)_{16}$. $(D7)_{16}$. $(2DC)_{16}$.

43. **(a)** Answers will vary, but the pattern of italicized and nonitalized letters should match
AAAAAAAAAAAAAAAAAAAAAAAAAA
AAAAAAAAA

(b) STUDY WELL.

45. BIJS.

Exercise Set 1.5, page 39

1. **(a)** −2, 1, 2. **(b)** 3, 4 **(c)** 4, −1, 8.
(d) 2, 6, 8.

3. a is 3, b is 1, c is 8, and d is −2.

5. **(a)** $\begin{bmatrix} 4 & 0 & 2 \\ 9 & 6 & 2 \\ 3 & 2 & 4 \end{bmatrix}$.

(b) $\mathbf{AB} = \begin{bmatrix} 7 & 13 \\ -3 & 0 \end{bmatrix}$.

(c) Not possible. **(d)** $\begin{bmatrix} 21 & 14 \\ -7 & 17 \end{bmatrix}$.

7. **(a)** **EB** is 3 × 2 and **FA** is 2 × 3; the sum is undefined.

(b) $\mathbf{B} + \mathbf{D}$ does not exist.

(c) $\begin{bmatrix} 10 & 0 & -25 \\ 40 & 14 & 12 \end{bmatrix}$. **(d)** **DE** does not exist.

9. **(a)** $\begin{bmatrix} 22 & 34 \\ 3 & 11 \\ -31 & 3 \end{bmatrix}$. **(b)** **BC** is not defined.

(c) $\begin{bmatrix} 25 & 5 & 26 \\ 20 & -3 & 32 \end{bmatrix}$.

(d) $\mathbf{D}^T + \mathbf{E}$ is not defined.

11. Let $\mathbf{B} = \left[\, b_{jk} \,\right] = \mathbf{I}_m\mathbf{A}$. Then $b_{jk} = \sum_{l=1}^{m} i_{jl}a_{lk}$, for $1 \le j \le m$ and $1 \le k \le n$. But $i_{jj} = 1$ and $i_{jl} = 0$ if $j \neq l$. Hence $b_{jk} = i_{jj}a_{jk}$, $1 \le j \le m$, $1 \le k \le n$. This means $\mathbf{B} = \mathbf{I}_m\mathbf{A} = \mathbf{A}$. Similarly, if $\mathbf{C} = \mathbf{A}\mathbf{I}_n = \left[\, c_{jk} \,\right]$, $c_{jk} = \sum_{l=1}^{n} a_{jl}i_{lk} = a_{jk}i_{kk} = a_{jk}$ for $1 \le j \le m$, $1 \le k \le n$.

13. $\mathbf{A}^3 = \begin{bmatrix} 27 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 64 \end{bmatrix}$ or $\begin{bmatrix} 3^3 & 0 & 0 \\ 0 & (-2)^3 & 0 \\ 0 & 0 & 4^3 \end{bmatrix}$,
$\mathbf{A}^k = \begin{bmatrix} 3^k & 0 & 0 \\ 0 & (-2)^k & 0 \\ 0 & 0 & 4^k \end{bmatrix}$.

15. The entries of $\mathbf{I}_n^T$ satisfy $i'_{kj} = i_{jk}$. But $i_{jk} = 1$ if $j = k$ and is 0 otherwise. Thus $i'_{kj} = 1$ if $k = j$ and is 0 if $k \neq j$ for $1 \le j \le n$, $1 \le k \le n$.

17. The jth column of **AB** has entries $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. Let $\mathbf{D} = \left[\, d_{ij} \,\right] = \mathbf{AB}_j$, where $\mathbf{B}_j$ is the jth column of **B**. Then $d_{ij} = \sum_{m=1}^{n} a_{im}b_{mj} = c_{ij}$.

19. $\mathbf{A}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2$.

21. **(a)** $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$ by Theorem 3. Since **A** and **B** are symmetric, $\mathbf{A}^T + \mathbf{B}^T = \mathbf{A} + \mathbf{B}$ and $\mathbf{A} + \mathbf{B}$ is also symmetric.

(b) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T = \mathbf{BA}$, but this may not be **AB**, so **AB** may not be symmetric. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$. Then **AB** is not symmetric.

23. **(a)** The i, jth element of $(\mathbf{A}^T)^T$ is the j, ith element of $\mathbf{A}^T$. But the j, ith element of $\mathbf{A}^T$ is the i, jth element of **A**. Thus $(\mathbf{A}^T)^T = \mathbf{A}$.

(b) The i, jth element of $(\mathbf{A} + \mathbf{B})^T$ is the j, ith element of $\mathbf{A} + \mathbf{B}$, $a_{ji} + b_{ji}$. But this is the sum of the j, ith entry of **A** and the j, ith entry of **B**. It is also the sum of the i, jth entry of $\mathbf{A}^T$ and the i, jth entry of $\mathbf{B}^T$. Thus $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

(c) Let $\mathbf{C} = \left[\, c_{ij} \,\right] = (\mathbf{AB})^T$. Then $c_{ij} = \sum_{k=1}^{n} a_{jk}b_{ki}$, the j, ith entry of **AB**. Let $\mathbf{D} = \left[\, d_{ij} \,\right] = \mathbf{B}^T\mathbf{A}^T$, then

$$d_{ij} = \sum_{k=1}^{n} b'_{ik}a'_{kj} = \sum_{k=1}^{n} b_{ki}a_{jk} = \sum_{k=1}^{n} a_{jk}b_{ki} = c_{ij}.$$

Hence $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

25. **(a)** $\begin{bmatrix} \frac{7}{9} & -\frac{1}{9} \\ -\frac{5}{9} & \frac{2}{9} \end{bmatrix}$. **(b)** $\begin{bmatrix} -\frac{9}{27} & \frac{4}{27} \\ \frac{0}{27} & \frac{3}{27} \end{bmatrix}$.

(c) $\begin{bmatrix} \frac{2}{32} & \frac{5}{32} \\ \frac{4}{32} & -\frac{6}{32} \end{bmatrix}$.

27. **(a)** $\mathbf{C} = \mathbf{A}^{-1}$ since $\mathbf{CA} = \mathbf{I}_3$.

(b) $\mathbf{D} = \mathbf{B}^{-1}$ since $\mathbf{DB} = \mathbf{I}_3$.

29. Since
$(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{B}^{-1}\mathbf{I}_n\mathbf{B} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}_n$,
$(\mathbf{B}^{-1}\mathbf{A}^{-1}) = (\mathbf{AB})^{-1}$.

31. **(a)** $\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$; $\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$;

$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(b) $\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$;

$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(c) $\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$; $\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$;

$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

33. Let $\mathbf{C} = [\, c_{ij} \,] = \mathbf{A} \vee \mathbf{B}$ and $\mathbf{D} = [\, d_{ij} \,] = \mathbf{B} \vee \mathbf{A}$.

$$c_{ij} = \begin{cases} 1 & a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 0 & a_{ij} = 0 = b_{ij} \end{cases} = d_{ij}$$

Hence $\mathbf{C} = \mathbf{D}$.

35. Let $[\, d_{ij} \,] = \mathbf{B} \vee \mathbf{C}$, $[\, e_{ij} \,] = \mathbf{A} \vee (\mathbf{B} \vee \mathbf{C})$,
$[\, f_{ij} \,] = \mathbf{A} \vee \mathbf{B}$, and $[\, g_{ij} \,] = (\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C}$. Then

$$d_{ij} = \begin{cases} 1 & b_{ij} = 1 \text{ or } c_{ij} = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$e_{ij} = \begin{cases} 1 & a_{ij} = 1 \text{ or } d_{ij} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

But this means $d_{ij} = 1$ if $a_{ij} = 1$ or $b_{ij} = 1$ or $c_{ij} = 1$ and $d_{ij} = 0$ otherwise.

$$f_{ij} = \begin{cases} 1 & a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$g_{ij} = \begin{cases} 1 & f_{ij} = 1 \text{ or } c_{ij} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

But this means $g_{ij} = 1$ if $a_{ij} = 1$ or $b_{ij} = 1$ or $c_{ij} = 1$ and $g_{ij} = 0$ otherwise. Hence $\mathbf{A} \vee (\mathbf{B} \vee \mathbf{C}) = (\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C}$.

37. Let $[\, d_{ij} \,] = \mathbf{B} \odot \mathbf{C}$, $[\, e_{ij} \,] = \mathbf{A} \odot (\mathbf{B} \odot \mathbf{C})$,
$[\, f_{ij} \,] = \mathbf{A} \odot \mathbf{B}$, and $[\, g_{ij} \,] = (\mathbf{A} \odot \mathbf{B}) \odot \mathbf{C}$. Then

$$d_{ij} = \begin{cases} 1 & b_{ik} = c_{kj} \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

and

$$e_{ij} = \begin{cases} 1 & a_{il} = 1 = d_{lj} \text{ for some } l \\ 0 & \text{otherwise.} \end{cases}$$

But this means

$$e_{ij} = \begin{cases} 1 & a_{il} = 1 = b_{lk} = c_{kj} \text{ for some } k, l \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{ij} = \begin{cases} 1 & a_{ik} = 1 = b_{kj} \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_{ij} = \begin{cases} 1 & f_{il} = 1 = c_{lj} \text{ for some } l \\ 0 & \text{otherwise.} \end{cases}$$

But then

$$g_{ij} = \begin{cases} 1 & a_{ik} = 1 = b_{kl} = c_{lj} \text{ for some } k, l \\ 0 & \text{otherwise} \end{cases}$$

and $\mathbf{A} \odot (\mathbf{B} \odot \mathbf{C}) = (\mathbf{A} \odot \mathbf{B}) \odot \mathbf{C}$.

39. An argument similar to that in Exercise 38 shows that $\mathbf{A} \vee (\mathbf{B} \wedge \mathbf{C}) = (\mathbf{A} \vee \mathbf{B}) \wedge (\mathbf{A} \vee \mathbf{C})$.

41. Since $c_{ij} = \sum_{t=1}^{n} a_{it}b_{tj}$ and $k \mid a_{it}$ for any i and t, k divides each term in c_{ij}, and thus $k \mid c_{ij}$ for all i and j.

43. **(a)** $\begin{bmatrix} 6 & -9 & -3 \\ 0 & 15 & 6 \\ 12 & -12 & 18 \end{bmatrix}$.

(b) $\begin{bmatrix} 10 & 20 & -30 \\ 20 & 0 & 45 \\ 35 & -5 & 15 \end{bmatrix}$.

(c) $\begin{bmatrix} -4 & 0 \\ -3 & -1 \\ 2 & -5 \end{bmatrix}$.

45. Let $\mathbf{A} = [\, a_{ij} \,]$ and $\mathbf{B} = [\, b_{ij} \,]$ be two $m \times n$ matrices. Then $k(\mathbf{A} + \mathbf{B}) = k[\, a_{ij} + b_{ij} \,] = [\, k(a_{ij} + b_{ij}) \,] = [\, ka_{ij} + kb_{ij} \,] = [\, ka_{ij} \,] + [\, kb_{ij} \,] = k\mathbf{A} + k\mathbf{B}$.

47. Let $\mathbf{K}$ be the $m \times m$ diagonal matrix with each diagonal entry equal to k. Then $\mathbf{KA} = k\mathbf{A}$.

Exercise Set 1.6, page 44

1. **(a)** Yes. **(b)** Yes.

3. **(a)** No. **(b)** Yes.

5. $A \oplus B = \{x \mid (x \in A \cup B) \text{ and } (x \notin A \cap B)\} = \{x \mid (x \in B \cup A) \text{ and } (x \notin B \cap A)\} = B \oplus A$.

7.

$x\ y\ z$	$y \Box z$	$x \nabla (y \Box z)$	$x \nabla y$	$x \nabla z$	$(x \nabla y) \Box (x \nabla z)$
0 0 0	0	0	0	0	0
0 0 1	1	0	0	0	0
0 1 0	1	0	0	0	0
0 1 1	0	0	0	0	0
1 0 0	0	0	0	0	0
1 0 1	1	1	0	1	1
1 1 0	1	1	1	0	1
1 1 1	0	0	1	1	0
		(A)			(B)

Since columns (A) and (B) are identical, the distributive property $x \nabla (y \Box z) = (x \nabla y) \Box (x \nabla z)$ holds.

9. 5×5 zero matrix for $\vee$; 5×5 matrix of 1's for $\wedge$; $\mathbf{I}_5$ for $\odot$.

11. Let $\mathbf{A}, \mathbf{B}$ be $n \times n$ diagonal matrices. Let $[\, c_{ij} \,] = \mathbf{AB}$. Then $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, but $a_{ik} = 0$ if $i \neq k$. Hence $c_{ij} = a_{ii}b_{ij}$. But $b_{ij} = 0$ if $i \neq j$. Thus $c_{ij} = 0$ if $i \neq j$ and $\mathbf{AB}$ is an $n \times n$ diagonal matrix.

13. Yes, the $n \times n$ zero matrix, which is a diagonal matrix.

15. $-\mathbf{A}$ is the diagonal matrix with i, ith entry $-a_{ii}$.

17. $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} a+b & 0 \\ 0 & 0 \end{bmatrix}$ belongs to M.

19. $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}^T$ or $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ belongs to M.

21. Yes, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

23. If $a \neq 0$, then $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 0 \end{bmatrix}$.

25. Yes. **27.** No.

29. **(a)** Yes. **(b)** Yes. **(c)** Yes.

(d) Yes, $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$. **(e)** Yes, $\begin{bmatrix} -x \\ -y-2 \end{bmatrix}$.

31. **(a)** Yes. **(b)** Yes. **(c)** No.

33. **(a)** $k\left(\begin{bmatrix} x \\ y \end{bmatrix} \nabla \begin{bmatrix} w \\ z \end{bmatrix}\right) = \left(k\begin{bmatrix} x \\ y \end{bmatrix} \nabla k\begin{bmatrix} w \\ z \end{bmatrix}\right)$.

(b) No.

35. Let $\mathbf{C} = [\, c_{ij} \,] = \text{comp}(\mathbf{A} \vee \mathbf{B})$ and $\mathbf{D} = [\, d_{ij} \,] = \text{comp}(\mathbf{A}) \wedge \text{comp}(\mathbf{B})$. Then

$$c_{ij} = \begin{cases} 0 & a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 1 & a_{ij} = 0 = b_{ij} \end{cases}$$

and

$$d_{ij} = \begin{cases} 0 & a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 1 & a_{ij} = 0 = b_{ij}. \end{cases}$$

Hence, $\mathbf{C} = \mathbf{D}$. Similarly, we can show that $\text{comp}(\mathbf{A} \wedge \mathbf{B}) = \text{comp}(\mathbf{A}) \vee \text{comp}(\mathbf{B})$.

37. $A \cap \overline{B}$. **39.** $\{\,\}$.

Review Questions, page 47

1. $P(A)$ is a set of sets.

2. $|P(A)|$ is a counting number or infinity.

3. LCM(a, b) is a positive integer.

4. $k\mathbf{A}$ is a matrix of the same size as $\mathbf{A}$.

5. A mathematical structure consists of a set of objects, operations on those objects, and the properties of those operations.

Chapter 2

Exercise Set 2.1, page 55

1. (b), (d), and (e) are statements.

3. **(a)** It will not rain tomorrow and it will not snow tomorrow.

(b) It is not the case that if you drive, I will walk.

5. **(a)** I will drive my car and I will be late.
I will drive my car or I will be late.

(b) $10 < NUM \leq 15$. $NUM > 10$ or $NUM \leq 15$.

7. **(a)** True. **(b)** True. **(c)** True. **(d)** False.

9. **(a)** False. **(b)** True. **(c)** True. **(d)** False.

11. (d) is the negation.

13. **(a)** The dish did not run away with the spoon and the grass is wet.

(b) The grass is dry or the dish ran away with the spoon.

(c) It is not true that today is Monday or the grass is wet.

(d) Today is Monday or the dish did not run away with the spoon.

15. **(a)** For all x there exists a y such that $x + y$ is even.

(b) There exists an x such that, for all y, $x + y$ is even.

17. **(a)** It is not true that there is an x such that x is even.

(b) It is not true that, for all x, x is a prime number.

19. 14: **(a)** False. **(b)** True.
15: **(a)** True. **(b)** False.
16: **(a)** False. **(b)** True.
17: **(a)** False. **(b)** True.
18: **(a)** False. **(b)** False. **(c)** False. **(d)** False.

21. $\forall A \forall B (\overline{A \cup B}) = \overline{A} \cap \overline{B}$; $\forall A \forall B (\overline{A \cap B}) = \overline{A} \cup \overline{B}$.

23.

p	q	$(\sim p$	$\wedge\ q)$	$\vee\ p$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	F
		(1)	(2)	↑

25.

p	q	r	$(p \vee q)$	$\wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F
			(1)	↑

27.

p	q	r	$(p \downarrow q)$	$\downarrow r$
T	T	T	F	F
T	T	F	F	T
T	F	T	F	F
T	F	F	F	T
F	T	T	F	F
F	T	F	F	T
F	F	T	T	F
F	F	F	T	F
			(1)	↑

29.

p	q	r	$(p \downarrow q)$	$\downarrow$	$(p \downarrow r)$
T	T	T	F	T	F
T	T	F	F	T	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	F	T
F	F	T	T	F	F
F	F	F	T	F	T
			(1)	↑	(2)

31.

p	q	$(p \wedge q)$	$\triangle\ p$
T	T	T	F
T	F	F	T
F	T	T	T
F	F	F	F
		(1)	↑

33. $(x = max)$ or $(y \leq 4)$

35. WHILE (item = sought or index $\geq$ 101) take action

Exercise Set 2.2, page 60

1. **(a)** $p \Rightarrow q$. **(b)** $r \Rightarrow p$.

(c) $q \Rightarrow p$. **(d)** $\sim r \Rightarrow p$.

3. **(a)** If I am not the Queen of England, then $2 + 2 = 4$.

(b) If I walk to work, then I am not the President of the United States.

(c) If I did not take the train to work, then I am late.

(d) If I go to the store, then I have time and I am not too tired.

(e) If I buy a car and I buy a house, then I have enough money.

5. **(a)** True. **(b)** False.

(c) True. **(d)** True.

7. **(a)** If I do not study discrete structures and I go to a movie, then I am in a good mood.

(b) If I am in a good mood, then I will study discrete structures or I will go to a movie.

(c) If I am not in a good mood, then I will not go to a movie or I will study discrete structures.

(d) I will go to a movie and I will not study discrete structures if and only if I am in a good mood.

9. **(a)** If $4 > 1$ and $2 > 2$, then $4 < 5$.

(b) It is not true that $3 \leq 3$ and $4 < 5$.

(c) If $3 > 3$, then $4 > 1$.

11. **(a)**

p	q	$p \Rightarrow$	$(q \Rightarrow p)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	T	T
		↑	

tautology

(b)

p	q	$q \Rightarrow$	$(q \Rightarrow p)$
T	T	T	T
T	F	T	T
F	T	F	F
F	F	T	T
		↑	

contingency

13. Yes. If $p \Rightarrow q$ is false, then p is true and q is false. Hence $p \wedge q$ is false, $\sim(p \wedge q)$ is true, and $(\sim(p \wedge q)) \Rightarrow q$ is false.

15. No, because if $p \Rightarrow q$ is true, it may be that both p and q are true and then $(p \wedge q) \Rightarrow \sim q$ is false. But it could also be that p and q are both false, and then $(p \wedge q) \Rightarrow \sim q$ is true.

17. **(a)** False. **(b)** True. **(c)** False. **(d)** True.

19.

p	q	$(p \wedge q)$	$(p \downarrow p)$	$\downarrow$	$(q \downarrow q)$
T	T	T	F	T	F
T	F	F	F	F	T
F	T	F	T	F	F
F	F	F	T	F	T
		(A)		(B)	

Since columns (A) and (B) are the same, the statements are equivalent.

21. **(a)** Jack did eat fat or he did not eat broccoli.

(b) Mary did not lose her lamb and the wolf did not eat the lamb.

(c) Tom stole a pie and ran away and the three pigs have some supper.

23.

p	q	r	$p\ \wedge$	$(q \vee r)$	$(p \wedge q)$	$\vee$	$(p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	F
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	F	T	F	F	F
F	T	F	F	T	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F
			(A)			(B)	

Because (A) and (B) are the same, the statements are equivalent.

25.

p	q	$\sim$	$(p \Leftrightarrow q)$	$(p \wedge\ \sim q)$	$\vee$	$(q \wedge \sim p)$
T	T	F	T	F	F	F
T	F	T	F	T	T	F
F	T	T	F	F	T	T
F	F	F	T	F	F	F
		(A)			(B)	

Because columns (A) and (B) are the same, the statements are equivalent.

27. The statement $\forall x(\mathrm{P}(x) \wedge \mathrm{Q}(x))$ is true if and only if $\forall x$ both $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ are true, but this means $\forall x \mathrm{P}(x)$ is true and $\forall x \mathrm{Q}(x)$ is true. This holds if and only if $\forall x \mathrm{P}(x) \wedge \forall x \mathrm{Q}(x)$is true.

29.

p	q	$q \Rightarrow$	$(p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	T	F
		↑	

31.

p	q	r	$((p \Rightarrow q)$	$\wedge$	$(q \Rightarrow r))$	$\Rightarrow$	$(p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T
			(1)	(3)	(2)	↑	(4)

33. Because $(p \wedge q) \equiv (q \wedge p)$, parts (a) and (b) of Theorem 4 say the same thing; that is, either p or q can be considered the first statement.

Exercise Set 2.3, page 66

1. Valid: $((d \Rightarrow t) \wedge \sim t) \Rightarrow \sim d$.

3. Invalid.

5. Valid: $((f \vee \sim w) \wedge w) \Rightarrow f$.

7. Valid: $[(ht \Rightarrow m) \wedge (m \Rightarrow hp)] \Rightarrow [\sim hp \Rightarrow \sim ht]$.

9. Invalid.

11. **(a)** $((p \vee q) \wedge \sim q) \Rightarrow p$.

(b) $((p \Rightarrow q) \wedge \sim p) \Rightarrow \sim q$.

13. **(a)** $(((p \Rightarrow q) \wedge (q \Rightarrow r)) \wedge ((\sim q) \wedge r)) \Rightarrow p$

(b) $(\sim(p \Rightarrow q) \wedge p) \Rightarrow \sim q$

15. Suppose m and n are odd. Then there exist integers j and k such that $m = 2j + 1$ and $n = 2k + 1$. $m + n = (2j + 1) + (2k + 1) = 2j + 2k + 2 = 2(j + k + 1)$. Since $j + k + 1$ is an integer, $m + n$ is even.

17. Suppose that m and n are odd. Then there exist integers j and k such that $m = 2j + 1$ and $n = 2k + 1$. $m \cdot n = 2j \cdot 2k + 2j + 2k + 1 = 2(2jk + j + k) + 1$. Since $2jk + j + k$ is an integer, $m \cdot n$ is odd and the system is closed with respect to multiplication.

19. If $A = B$, then, clearly, $A \subseteq B$ and $B \subseteq A$. If $A \subseteq B$ and $B \subseteq A$, then $A \subseteq B \subseteq A$ and B must be the same as A.

21. **(a)** If $A \subseteq B$, then $A \cup B \subseteq B$. But $B \subseteq A \cup B$. Hence $A \cup B = B$. If $A \cup B = B$, then since $A \subseteq A \cup B$, we have $A \subseteq B$.

(b) If $A \subseteq B$, then $A \subseteq A \cap B$. But $A \cap B \subseteq A$. Hence $A \cap B = A$. If $A \cap B = A$, then since $A \cap B \subseteq B$, we have $A \subseteq B$.

23. For $n = 41$, we have a counterexample. $41^2 + 41 \cdot 41 + 41$ is $41(41 + 41 + 1)$ or $41 \cdot 83$.

25. $n^3 - n = n(n - 1)(n + 1)$, the product of three consecutive integers. One of these must be a multiple of 3, so $3 \mid n^3 - n$.

27. Invalid. Multiplying by $x - 1$ may or may not preserve the order of the inequality.

29. Valid.

31. Let x and y be prime numbers, each larger than 2. Then x and y are odd and their sum is even (Exercise 15). The only even prime is 2, so $x + y$ is not a prime.

33. Suppose $x + y$ is rational. Then there are integers a and b such that $x + y = \frac{a}{b}$. Since x is rational, we can write $x = \frac{c}{d}$ with integers c and d. But now $y = x + y - x = \frac{a}{b} - \frac{c}{d}$. $y = \frac{ad-bc}{bd}$. Both $ad - bc$ and bd are integers. This is a contradiction since y is an irrational number and cannot be expressed as the quotient of two integers.

Exercise Set 2.4, page 72

Note: Only the outlines of the induction proofs are given. These are not complete proofs.

1. Basis step: $n = 1$ P(1): $2(1) = 1(1 + 1)$ is true.
Induction step: P(k): $2 + 4 + \cdots + 2k = k(k + 1)$.
P($k + 1$): $2 + 4 + \cdots + 2(k + 1) = (k + 1)(k + 2)$
LHS of P($k + 1$): $2 + 4 + \cdots + 2k + 2(k + 1) = k(k + 1) + 2(k + 1) = (k + 1)(k + 2)$
RHS of P($k + 1$).

3. Basis step: $n = 0$ P(0): $2^0 = 2^{0+1} - 1$ is true.
Induction step: LHS of P($k + 1$):
$1 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} = (2^{k+1} - 1) + 2^{k+1} = 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1$ RHS of P($k + 1$).

5. Basis step: $n = 1$ P(1): $1^2 = \dfrac{1(1 + 1)(2 + 1)}{6}$ is true.
Induction step: LHS of P($k + 1$):

$$
\begin{aligned}
1^2 + 2^2 + \cdots + k^2 + (k + 1)^2
&= \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2 \\
&= (k + 1)\left(\frac{k(2k + 1)}{6} + (k + 1)\right) \\
&= \frac{k + 1}{6}(2k^2 + k + 6(k + 1)) \\
&= \frac{k + 1}{6}(2k^2 + 7k + 6) \\
&= \frac{(k + 1)(k + 2)(2k + 3)}{6} \\
&= \frac{(k + 1)((k + 1) + 1)(2(k + 1) + 1)}{6}
\end{aligned}
$$

RHS of P($k + 1$).

7. Basis step: $n = 1$ P(1): $a = \dfrac{a(1 - r^1)}{1 - r}$ is true.
Induction step: LHS of P($k + 1$):
$a + ar + \cdots + ar^{k-1} + ar^k = \dfrac{a(1 - r^k)}{1 - r} + ar^k = \dfrac{a - ar^k + ar^k - ar^{k+1}}{1 - r} = \dfrac{a(1 - r^{k+1})}{1 - r}$ RHS of P($k + 1$).

9. **(a)** LHS of P($k + 1$): $1 + 5 + 9 + \cdots + (4(k + 1) - 3) = (2k + 1)(k - 1) + 4(k + 1) - 3 = 2k^2 + 3k = (2k + 3)(k) = (2(k + 1) + 1)((k + 1) - 1)$ RHS of P($k + 1$).
(b) No; P(1): $1 = (2 \cdot 1 + 1)(1 - 1)$ is false.

11. Basis step: $n = 2$ P(2): $2 < 2^2$ is true.
Induction step: LHS of P($k + 1$):
$k + 1 < 2^k + 1 < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ RHS of P($k + 1$).

13. Basis step: $n = 5$ P(5): $1 + 5^2 < 2^5$ is true.
Induction step: LHS of P($k + 1$):
$1 + (k + 1)^2 = k^2 + 1 + 2k + 1 < 2^k + 2k + 1 < 2^k + k^2 + 1 < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ RHS of P($k + 1$).

15. Basis step: $n = 0$ $A = \{\ \}$ and $P(A) = \{\{\ \}\}$, so $|P(A)| = 2^0$ and P(0) is true.
Induction step: Use P(k): If $|A| = k$, then $|P(A)| = 2^k$ to show P($k + 1$): If $|A| = k + 1$, then $|P(A)| = 2^{k+1}$. Suppose that $|A| = k + 1$. Set aside one element x of A. Then $|A - \{x\}| = k$ and $A - \{x\}$ has 2^k subsets. These subsets are also subsets of A. We can form another 2^k subsets of A by forming the union of $\{x\}$ with each subset of $A - \{x\}$. None of these subsets are duplicates. Now A has $2^k + 2^k$, or 2^{k+1}, subsets.

17. Basis step: $n = 1$ P(1): $\overline{A_1} = \overline{A_1}$ is true.
Induction step: LHS of P($k + 1$):

$$
\begin{aligned}
\overline{\bigcap_{i=1}^{k+1} A_i} &= \overline{\left(\bigcap_{i=1}^{k} A_i\right) \cap A_{k+1}} \\
&= \overline{\bigcap_{i=1}^{k} A_i} \cup \overline{A_{k+1}} \quad \text{(De Morgan's laws)} \\
&= \left(\bigcup_{i=1}^{k} \overline{A_i}\right) \cup \overline{A_{k+1}} \\
&= \bigcup_{i=1}^{k+1} \overline{A_i} \quad \text{RHS of P}(k + 1).
\end{aligned}
$$

19. Basis step: $n = 1$ P(1): $A_1 \cup B = A_1 \cup B$ is true.
Induction step: LHS of P($k + 1$):

$$
\begin{aligned}
\left(\bigcap_{i=1}^{k+1} A_i\right) \cup B &= \left(\left(\bigcap_{i=1}^{k} A_i\right) \cap A_{k+1}\right) \cup B \\
&= \left(\left(\bigcap_{i=1}^{k} A_i\right) \cup B\right) \cap (A_{k+1} \cup B) \\
&\quad \text{(distributive property)} \\
&= \left(\bigcap_{i=1}^{k} (A_i \cup B)\right) \cap (A_{k+1} \cup B) \\
&= \bigcap_{i=1}^{k+1} (A_i \cup B) \quad \text{RHS of P}(k + 1).
\end{aligned}
$$

21. **(a)** $(k + 1)^2 + (k + 1) = k^2 + 2k + 1 + k + 1 = k^2 + k + 2(k + 1)$. Using P($k$), $k^2 + k$ is odd; $2(k + 1)$ is clearly even. Hence their sum is odd.
(b) No, P(1) is false.

23. The flaw is that to carry out the procedure in the induction step, you must have at least three trucks, but the basis step was done for one truck.

25. Basis step: $n = 1$ P(1): $\mathbf{A}^2 \cdot \mathbf{A} = \mathbf{A}^{2+1}$ is true.
Induction step: LHS of P($k + 1$):
$\mathbf{A}^2 \cdot \mathbf{A}^{k+1} = \mathbf{A}^2(\mathbf{A}^k \cdot \mathbf{A}) = (\mathbf{A}^2 \cdot \mathbf{A}^k) \cdot \mathbf{A} = \mathbf{A}^{2+k} \cdot \mathbf{A} = \mathbf{A}^{2+k+1}$
RHS of P($k + 1$).

27. Basis step: $n = 5$ P(5): A restaurant bill of \$5 can be paid exactly with a \$5 bill.
Induction step: We use P(j): A restaurant bill of \$$j$ can be paid exactly using \$2 and \$5 bills for $j = 5, 6, \ldots, k$ to show P($k + 1$). Write $k + 1$ as $2m + r$, $0 \leq r < 2$. If $r = 0$, then the bill can be paid with m \$2 bills. If $r = 1$, then $2m + r = 2m + 1 = 2(m - 2) + 5$. Since $k + 1 \geq 7$, $m \geq 2$, and $m - 2 \geq 0$. Thus a bill of $k + 1$ dollars can be paid exactly with $(m - 2)$ \$2 bills and a \$5 bill.

29. Basis step: $n = 1$ P(1): If p is prime and $p \mid a^1$, then $p \mid a$ is true.
Induction step: If $p \mid a^{k+1}$, then $p \mid a^k \cdot a$ and either $p \mid a^k$ or $p \mid a$. If $p \mid a^k$, then (using P(k)), $p \mid a$.

31. **(a)** 5.
(b) Basis step: $n = 5$ P(5): $2^5 > 5^2$ is true.
Induction step: LHS of P($k + 1$): $2^{k+1} = 2 \cdot 2^k = 2^k + 2^k > k^2 + k^2 > k^2 + 3k > k^2 + 2k + 1$ (since $k \geq 5) = (k + 1)^2$ RHS of P($k + 1$).

33. Basis step: $n = 1$ P(1): $x - y$ divides $x - y$ is true.
Induction step: $x^{k+1} - y^{k+1} = x \cdot x^k - y \cdot y^k = x \cdot x^k - x \cdot y^k + x \cdot y^k - y \cdot y^k = x(x^k - y^k) + y^k(x - y)$.
This rewriting gives an expression where each term is divisible by $x - y$ and so the sum is as well.

35. Loop invariant check:
Basis step: $n = 0$ P(0): $X - Z_0 + W_0 = Y$ is true, because $Z_0 = X$ and $W_0 = Y$.
Induction step: LHS of P($k + 1$): $X - Z_{k+1} + W_{k+1} = X - (Z_k - 1) + (W_k - 1) = X - Z_k + W_k = Y$
RHS of P($k + 1$).
Exit condition check: $W = 0$ $X - Z + W = Y$ yields $X - Z = Y$ or $Z = X - Y$.

37. Loop invariant check:
Basis step: $n = 0$ P(0): $Z_0 + (X \times W_0) = X \times Y$ is true, because $Z_0 = 0$ and $W_0 = Y$.
Induction step: LHS of P($k + 1$): $Z_{k+1} + (X \times W_{k+1}) = Z_k + X + (X \times (W_k - 1)) = Z_k + (X \times W_k) = X \times Y$
RHS of P($k + 1$).
Exit condition check: $W = 0$ $Z + (X \times W) = X \times Y$ yields $Z = X \times Y$.
Loop invariant check:
Basis step: $n = 0$ P(0): $Z_0 + (X \times Y \times W_0) = X + Y^2$ is true, because $Z_0 = X \times Y$ and $W_0 = Y - 1$.
Induction step: LHS of P($k + 1$):
$Z_{k+1} + (X \times Y \times W_{k-1}) = Z_k + X \times Y + (X \times Y + (X \times Y \times (W_k - 1)) = Z_k + X \times Y \times W_k = X + Y^2$
RHS of P($k + 1$).
Exit condition check: $W = 0$
$Z + X \times Y \times W = X + Y^2$ yields $Z = X + Y^2$.

39. Loop invariant check:
Basis step: $n = 0$ P(0): $Z_0 + (X \times W_0) = Y + X^2$ is true, because $Z_0 = Y$ and $W_0 = X$.
Induction step: LHS of P($k + 1$): $Z_{k+1} + (X \times W_{k+1}) = (Z_k + X) + (X \times (W_k - 1)) = Z_k + X \times W_k = Y + X^2$
RHS of P($k + 1$).
Exit condition check: $W = 0$ $Z + (X \times W) = Y + X^2$ yields $Z = Y + X^2$.
Loop invariant check:
Basis step: $n = 0$ P(0): $Z_0 + (Y \times W_0) = X^2 + Y^2$ is true, because $Z_0 = Y + X^2$ and $W_0 = Y - 1$.
Induction step: LHS of P($k + 1$): $Z_{k+1} + (Y \times W_{k+1}) = Z_k + Y + Y \times (W_k - 1) = Z_k + Y \times W_k = X^2 + Y^2$
RHS of P($k + 1$).
Exit condition check: $W = 0$ $Z + Y \times W = X^2 + Y^2$ yields $Z = X^2 + Y^2$.

Review Questions, page 75

1. The converse is not equivalent to the original statement, but the contrapositive is. In some cases, the contrapositive may be easier to prove than the original statement.

2. In the strong form of induction all statements $P(n_0), P(n_0 + 1), \ldots, P(k)$ may be used to show $P(k + 1)$, not just $P(k)$.

3. The mathematical structure of sets, union, intersection, and complement has the same properties as (logical statements, $\vee$, $\wedge$, $\sim$).

4. An indirect proof proves the contrapositive of the statement or assumes the statement is false and derives a contradiction.

5. A proof by contradiction proceeds by assuming the negation of the conclusion of the statement to be proved. Then definitions, previous theorems, and commonly known facts are used to derive a contradiction.

Chapter 3

Exercise Set 3.1, page 82

1. 67,600. **3.** 16. **5.** 1296.
7. **(a)** 0. **(b)** 1.
9. **(a)** $n!$ **(b)** $\dfrac{n!}{2}$ **(c)** $\dfrac{(n + 1)!}{2}$
11. 120. **13.** 4! or 24. **15.** 30.
17. **(a)** 479,001,600. **(b)** 1,036,800.
19. 240. **21.** 360.
23. 39,916,800. **25.** $(n - 1)!$. **27.** 67,200.
29. $n \cdot {}_{n-1}P_{n-1} = n \cdot (n - 1)(n - 2) \cdots 2 \cdot 1 = n! = {}_nP_n$.
31. 190.
33. 2; 6; 12.
35. **(a)** 14. **(b)** 11.
37. **(a)** 16. **(b)** 12.

Exercise Set 3.2, page 86

1. **(a)** 1. **(b)** 35. **(c)** 4368.
3. ${}_nC_r = \dfrac{n!}{r!(n - r)!} = \dfrac{n!}{(n - (n - r))!(n - r)!} = {}_nC_{n-r}$.
5. 20,358,520.
7. **(a)** 1. **(b)** 360.
9. **(a)** One of size 0, four of size 1, six of size 2, four of size 3, and one of size 4.
(b) For each r, $0 \leq r \leq n$, there are ${}_nC_r$ subsets of size r.
11. **(a)** 980. **(b)** 1176.
13. 2702.
15. Because three people can be arranged in only one way from youngest to oldest, the problem is to count the number of ways to choose three people from seven.
17. 177,100.

19. $$ {}_nC_{r-1} + {}_nC_r = \frac{n!}{(r-1)!\,(n-(r-1))!} + \frac{n!}{r!\,(n-r)!} $$
$$ = \frac{n!r + n!\,(n-r+1)}{r!\,(n-r+1)!} = \frac{n!\,(n+1)}{r!\,(n+1-r)!} $$
$$ = \frac{(n+1)!}{r!\,(n+1-r)!} = {}_{n+1}C_r. $$

21. **(a)** 32. **(b)** 5. **(c)** 10.

23. **(a)** 2^n. **(b)** ${}_nC_3$. **(c)** ${}_nC_k$.

25. 525.

27. **(a)**

1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

(b) Begin the row with a 1; write the sum of each consecutive pair of numbers in the previous row, moving left to right; end the row with a 1.

29. Exercise 19 shows another way to express the results of Exercise 27(b) and Exercise 28.

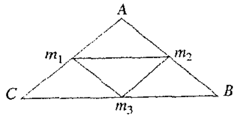
31. **(a)** 2. **(b)** 4. **(c)** 8.

33. 15.

Exercise Set 3.3, page 90

1. Let the birth months play the role of the pigeons and the calendar months, the pigeonholes. Then there are 13 pigeons and 12 pigeonholes. By the pigeonhole principle, at least two people were born in the same month.

3.



m_1, m_2, m_3 are the midpoints of sides AC, AB, and BC, respectively. Let the four small triangles created be the pigeonholes. For any five points in or on triangle ABC, at least two must be in or on the same small triangle and thus are no more than $\frac{1}{2}$ unit apart.

5. By the extended pigeonhole principle, at least $\lfloor (50-1)/7 \rfloor + 1$ or 8 will be the same color.

7. Let 2161 cents be the pigeons and the six friends, the pigeonholes. Then at least one friend has $\lfloor (2161-1)/6 \rfloor + 1$ or 361 cents.

9. If repetitions are allowed, there are ${}_{16}C_5$ or 4368 choices. At least $\lfloor 4367/175 \rfloor + 1$, or 25, choices have the same cost.

11. You must have at least 49 friends.

13. Consider the first eight rows; one row must have at least 7 ones since there are 51 ones in all. Similarly, there is a column with at least 7 ones. The sum of the entries in this row and this column is at least 14.

15. Label the pigeonholes with 1, 3, 5, …, 25, the odd numbers between 1 and 25 inclusive. Assign each of the selected 14 numbers to the pigeonhole labeled with its odd part. There are only 13 pigeonholes, so two numbers must have the same odd part. One is a multiple of the other.

17. Using an argument similar to that for Exercise 16, the subset must contain at least $\lfloor \frac{n-1}{2} \rfloor + 1$ elements.

19. No. At least one pair of the 12 disks must add up to 21.

21. Consider the six sums $c_1, c_1 + c_2, c_1 + c_2 + c_3, \ldots, c_1 + c_2 + c_3 + c_4 + c_5 + c_6$. If one of these has remainder 0 when divided by 6. then we are done. If none have remainder 0 when divided by 6, then two of them must give the same remainder. The positive difference of these two is a subsequence whose sum is divisible by 6.

23. We consider the cases of 3 or more ones, 2 ones, 1 one, and no ones. If there are at least 3 ones, we are done. If there are 2 ones and no two, then the sum is at least $1 + 1 + 4 \cdot 3$, but this is not possible. So if there are 2 ones. there is at least 1 two and we are done. If there is 1 one and no twos, then the sum is at least $1 + 5 \cdot 3$. This is impossible. If there are no ones and no three, then the sum is $6 \cdot 2$ or at least $5 \cdot 2 + 5$. But again, neither of these is possible.

25. There need to be at least 40 connections. If one printer has seven or fewer connections to PCs. then there can be a set of 5 PCs requesting printer access, but only four printers are available.

Exercise Set 3.4, page 98

1. {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}.

3. {sb, sr, sg, cb, cr, cg}.

5. **(a)** { }, {1}, {2}, {3}, {2, 3}, {1, 2}, {1, 3}, {1, 2, 3}.
   (b) 2^n.

7. **(a)** The card is a red ace.
   (b) The card is black or a diamond or an ace.

9. **(a)** {(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (1, 5), (2, 5), (3, 5), (4, 5), (6, 5)}.
   (b) { }.

11. **(a)** No, 3 satisfies both descriptions.
    (b) No, 2 satisfies both descriptions.
    (c) Yes, $E \cup F = \{3, 4, 5, 1, 2, 3\}$.
    (d) No, $E \cap F = \{3\}$.

13. **(a)** No. **(b)** No. **(c)** No. **(d)** Yes.

15. **(a)** $\{dls, dln, dms, dmn, dus, dun, als, aln\}$.
    (b) $\{als, aln\}$.
    (c) $\{dls, dln, als\}$.

17. $\overline{E} \cup F = \{2, 6, 3\}$, $\overline{F} \cap G = \{4\}$.

19. {club}, {spade}, {diamond}, {heart}.

21. **(a)** $\frac{10}{18}$. **(b)** $\frac{11}{18}$. **(c)** $\frac{12}{18}$. **(d)** $\frac{9}{18}$.

23. **(a)** 0.7. **(b)** 0. **(c)** 0.7. **(d)** 1.

25. $p(A) = \frac{6}{11}$, $p(B) = \frac{3}{11}$, $p(C) = \frac{1}{11}$, $p(D) = \frac{1}{11}$.

27. $\frac{10}{52}$.

29. **(a)** $\frac{1}{6}$. **(b)** 1.

31. **(a)** $\frac{11}{36}$. **(b)** $\frac{27}{36}$. **(c)** $\frac{6}{36}$. **(d)** $\frac{21}{36}$.

33. **(a)** $\frac{35}{220}$. **(b)** $\frac{80}{220}$. **(c)** $\frac{210}{220}$. **(d)** $\frac{210}{220}$.

35. **(a)** $\frac{4}{24}$. **(b)** $\frac{20}{24}$.

37. $\dfrac{n+1}{2}$.

39. $-\frac{11}{9}$ dollars.

41. **(a)** $\frac{8}{52} \cdot \frac{7}{51}$. **(b)** $\frac{8}{52} \cdot \frac{8}{52}$.

Exercise Set 3.5, page 104

1. 4, 10, 25, 62.5. Yes, degree 1.

3. 3, 12, 24, 48. No.

5. 1, 8, 43, 216. No.

7. $s_1 = 2, s_2 = 3, s_n = s_{n-1} + 1$.

9. $A_1 = 100\left(1 + \frac{0.06}{12}\right), A_n = \left(1 + \frac{0.06}{12}\right)(A_{n-1} + 100)$.

11. $c_n = c_{n-2} + c_{n-3}, c_1 = 0, c_2 = 1, c_3 = 1$.

13. $b_n = 3 \cdot 5^{n-1} + \frac{3}{4}(5^{n-1} - 1)$.

15. $d_n = 5(-1.1)^{n-1}$.

17. $g_n = n! \cdot 6$.

19. $b_n = (-2)^n$.

21. $d_n = -\frac{3}{4} \cdot 2^n + \frac{5}{4} \cdot n \cdot 2^n$

23. $g_n = \frac{-1-2i}{2}(1 + i)^n + \frac{-1+2i}{2}(1 - i)^n$.

27. **(a)** $a_n = 2a_{n-1} + 1, a_1 = 0$. **(b)** $a_n = 2^{n-1} - 1$.

29. $c_1 = 2, c_2 = 3, c_n = u\left(\frac{1+\sqrt{5}}{2}\right)^n + v\left(\frac{1-\sqrt{5}}{2}\right)^n, u = \frac{5+3\sqrt{5}}{10}$, $v = \frac{5-3\sqrt{5}}{10}$.

31. For $n \geq 2$, $f_{n+1}^2 - f_n^2 = (f_{n+1} - f_n)(f_{n+1} + f_n) = f_{n-1}f_{n+2}$, by the definition of f_k.

33. $A_1 = 100.5$.
$A_n = (1.005)^{n-1}(100.5) + 20{,}100[(1.005)^{n-1} - 1]$.

35. $a_n = -2(-2)^n + \left(\frac{5}{2} - \sqrt{2}\right)\left(\sqrt{2}\right)^n + \left(\frac{5}{2} + \sqrt{2}\right)\left(-\sqrt{2}\right)^n$

37. Basis step: $n = 0$ P(0): $5 \mid a_1$ is clearly true.
Induction step: We use P(k): $5 \mid a_{3k+1}$ to show P($k + 1$): $5 \mid a_{3(k+1)+1}$. Consider $a_{3(k+1)+1} = 2a_{3(k+1)} + a_{3k+2} = 2(2a_{3k+2} + a_{3k+1}) + a_{3k+2} = 5a_{3k+2} + 2a_{3k+1}$. Clearly $5 \mid 5a_{3k+2}$ and P(k) guarantees $5 \mid a_{3k+1}$.

39. $C_n = C_1C_{n-1} + C_2C_{n-2} + \cdots + C_{n-1}C_1, C_1 = 1$.

Review Questions, page 106

1. If all items to be chosen are not from the same set, then the problem is not a simple combination or permutation problem. If the items to be chosen are from the same set, and if the order in which they are chosen matters. permutations should be counted; otherwise, combinations.

2. Pigeons can often be identified by the phrases "at least k items have the property P(x)" and the corresponding pigeonholes are the possible values for P(x).

3. The recursive form may be easier to find or to justify. but this form may be difficult to evaluate for large n.

4. The solution form may be more efficient to evaluate for large n. but this form may be difficult to find.

5. The counting rules from Sections 3.1 and 3.2 are the primary tools needed to answer the probability questions presented in this chapter.

Chapter 4

Exercise Set 4.1, page 114

1. **(a)** x is 4. **(b)** y is 3.

3. **(a)** x is 4; y is 6. **(b)** x is 4; y is 2.

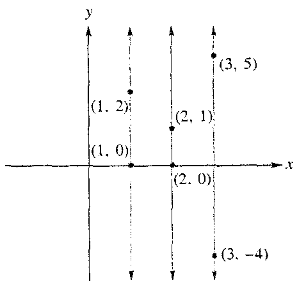
5. **(a)** $\{(a, 4), (a, 5), (a, 6), (b, 4), (b, 5), (b, 6)\}$.
   (b) $\{(4, a), (5, a), (6, a), (4, b), (5, b), (6, b)\}$.

7. **(a)** {(Fine, president), (Fine. vice-president), (Fine, secretary), (Fine, treasurer), (Yang, president), (Yang, vice-president), (Yang, secretary), (Yang, treasurer)}.
   (b) {(president, Fine), (vice-president, Fine), (secretary, Fine), (treasurer, Fine), (president, Yang), (vice-president, Yang), (secretary, Yang), (treasurer, Yang)}.
   (c) {(Fine, Fine), (Fine, Yang), (Yang, Fine), (Yang, Yang)}.

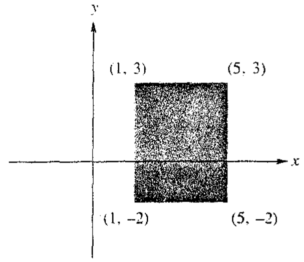
9. gs, ds, gc, dc, gv, dv.

11. (Outline) Basis step: $n = 1$ P(1): If $|A| = 3$ and $|B| = 1$, then $|A \times B| = 3$.
$A \times B = \{(a_1, *), (a_2, *), (a_3, *)\}$. Clearly, $|A \times B| = 3$.
Induction step: Suppose $|B| = k > 1$. Let $x \in B$ and $C = B - \{x\}$. Then $|C| = k - 1 \geq 1$ and using P(k), we have $|A \times C| = 3(k - 1)$. $|A \times \{x\}| = 3$. Since $(A \times C) \cap (A \times \{x\}) = \{\ \}$ and $(A \times C) \cup (A \times \{x\}) = A \times B$, $|A \times B| = 3(k - 1) + 3$ or $3k$.

13.



15.



17. Let $(x, y) \in A \times B$, then $x \in A$ and $y \in B$. Since $A \subseteq C$ and $B \subseteq D$, $x \in C$ and $y \in D$. Hence $(x, y) \in C \times D$.

19. One possible answer is *project*(*select* Employees[Department = Human Resources]) [Last Name].

21. One possible answer is *project*(*select* Employees[Department = Research]) [Years with Company].

23. **(a)** Yes. **(b)** No.

25. Answers will vary.

27. No. $|A| = 26$.

29. {{1}, {2}, {3}}, {{1}, {2, 3}}, {{2}, {1, 3}}, {{3}, {1, 2}}, {{1, 2,.3}}.

31. 3. There are three 2-element partitions listed in the solution to Exercise 29.

33. 6.

35. Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$, $y \in B \cup C$. Hence $(x, y) \in A \times B$ or $(x, y) \in A \times C$. Thus $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$. Let $(x, y) \in (A \times B) \cup (A \times C)$. Then $x \in A$, $y \in B$ or $y \in C$. Hence $y \in B \cup C$ and $(x, y) \in A \times (B \cup C)$. So, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Exercise Set 4.2, page 122

1. **(a)** No. **(b)** No. **(c)** Yes.
(d) Yes. **(e)** Yes. **(f)** Yes.

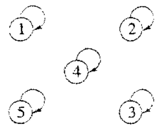
3. **(a)** No. **(b)** No. **(c)** Yes.
(d) Yes. **(e)** No. **(f)** Only if $n = 1$.

5. Domain: {IBM, Dell, COMPAQ, Gateway}, Range: {750C, 466V, 450SV, PS60};

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

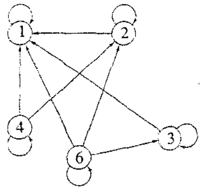
7. Domain: {1, 2, 3, 4, 8}, Range: {1, 2, 3, 4, 8};

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$



9. Domain: {1, 2, 3, 4, 6}, Range: {1, 2, 3, 4, 6};

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$



11. Domain: {3, 5, 7, 9}, Range: {2. 4, 6, 8};

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

13. **(a)** No. **(b)** No. **(c)** Yes.
(d) Yes. **(e)** No. **(f)** No.

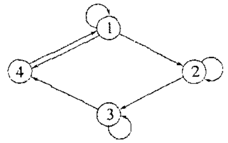
15. $\text{Dom}(R) = [-5, 5]$, $\text{Ran}(R) = [-5, 5]$.

17. **(a)** {1, 3}. **(b)** {1, 2, 3, 6}. **(c)** {1, 2, 4, 3, 6}.

19. **(a)** {3}. **(b)** {2, 4}. **(c)** {3, 2, 4}. **(d)** {2, 4}.

21. $a\ R\ b$ if and only if $0 \le a \le 3$ and $0 \le b \le 2$.

23. $R =$ {(1, 1), (1, 2), (1, 4), (2, 2), (2, 3), (3, 3), (3, 4), (4, 1)}.



25. $R = \{(1, 2), (2, 2), (2, 3), (3, 4), (4, 4), (5, 1), (5, 4)\}$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

27.

Vertex	1	2	3	4	5
In-degree	1	2	1	3	0
Out-degree	1	2	1	1	2

29. The in-degree of a vertex is the number of ones in the column labeled by that vertex. The out-degree of a vertex is the number of ones in the row labeled by that vertex.

31. {(2, 3), (3, 6)}.

33. Delete any vertex labeled by an element of $A - B$. Then delete any edges that do not point to a vertex.

35. **(a)** The elements of $R(a_k)$ are those elements of A that can be reached from a_k in one step.

(b) The elements of $R(\{a_1 . a_j . a_n\})$ are those elements of A that can be reached from a_1, a_j, or a_n in one step.

37. 2^{mn}.

Exercise Set 4.3, page 128

1. 1, 2 1, 6 2, 3 3, 3 3, 4 4, 3 4, 5 4, 1 6, 4.

3. **(a)** 3, 3, 3, 3 3, 3, 4, 3 3, 3, 4, 5 3, 4, 1, 6
3, 4, 1, 2 3, 4, 3, 3 3, 4, 3, 4 3, 3, 4, 1
3, 3, 3, 4.

(b) In addition to those in part (a), 1, 2, 3, 3
1, 2, 3, 4 1, 6, 4, 1 1, 6, 4, 5 2, 3, 3, 3
2, 3, 3, 4 2, 3, 4, 3 2, 3, 4, 5 4, 1, 2, 3
4, 1, 6, 4 6, 4, 3, 3 6, 4, 3, 4 6, 4, 1, 2
6, 4, 1, 6 1, 6, 4, 3 2, 3, 4, 1 4, 3, 3, 3
4, 3, 4, 3 4, 3, 4, 1 4, 3, 4, 5 4, 3, 3, 4.

5. One is 6, 4, 1, 6.

7.
$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

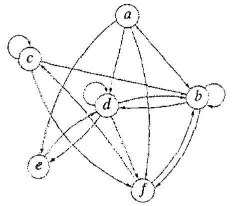
9. a, c a, b b, b b, f c, d c, e d, c d, b
e, f f, d.

11. **(a)** a, c, d, c a, c, d, b a, c, e, f a, b, b, b
a, b, b, f a, b, f, d.

(b) In addition to those in part (a), b, b, b, b
b, b, b, f b, b, f, d b, f, d, b b, f, d, c
c, d, c, d c, d, c, e c, d, b, b c, d, b, f
c, e, f, d d, c, d, c d, c, e, f d, b, b, b
d, b, b, f d, b, f, d d, c, e, f e, f, d, b
e, f, d, c f, d, c, d f, d, c, e f, d, b, b
f, d, b, f.

13. One is d, b, f, d.

15.



17. **(a)**
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(b) $\{(a, c), (a, d), (a, b), (a, e), (a, f), (b, b), (b, c), (b, d), (b, e), (b, f), (c, b), (c, c), (c, d), (c, e), (c, f), (d, b), (d, c), (d, d), (d, e), (d, f), (e, b), (e, c), (e, d), (e, e), (e, f), (f, b), (f, c), (f, d), (f, e), (f, f)\}$.

19. $x_i\ R^*\ x_j$ if and only if $x_i = x_j$ or $x_i\ R^n\ x_j$ for some n. The i, jth entry of $\mathbf{M}_{R^*}$ is 1 if and only if $i = j$ or the i, jth entry of $\mathbf{M}_{R^n}$ is 1 for some n. Since $R^\infty = \bigcup_{k=1}^{\infty} R^k$, the i, jth entry of M_{R^*} is 1 if and only if $i = j$ or the i, jth entry of $\mathbf{M}_{R^\infty}$ is 1. Hence $\mathbf{M}_{R^*} = \mathbf{I}_n \vee \mathbf{M}_{R^\infty}$.

21. 1, 7, 5, 6, 7, 4, 3.

23. 2, 3, 5, 6, 7, 5, 6, 4

25. 7, 4, 3, 5, 6, 7 is a possible answer.

27. The ij-entry of $\mathbf{M}_R \cdot \mathbf{M}_R$ is the number of paths from i to j of length two, because it is also the number of k's such that $a_{ik} = b_{kj} = 1$.

29. Direct; Boolean multiplication.

31. Suppose each vertex has out-degree at least one. Choose a vertex, say v_i. Construct a path R $v_i, v_{i+1}, v_{i+2}, \ldots$. This is possible since each vertex has an edge leaving it. But there are only a finite number of vertices so for some k and j, $v_j = v_k$ and a cycle is created.

33. The essentials of the digraph are the connections made by the arrows. Compare the arrows leaving each vertex in turn to pairs in R with that vertex as first element.

Exercise Set 4.4, page 134

1. Reflexive, symmetric, transitive.

3. None.

5. Irreflexive, symmetric, asymmetric, antisymmetric, transitive.

7. Transitive.

9. Antisymmetric, transitive.

11. Irreflexive, symmetric.

13. Reflexive.

15. Reflexive, antisymmetric, transitive.

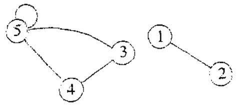
17. Irreflexive, symmetric.

19. Symmetric.

21. Reflexive, symmetric, transitive.

23. Reflexive, symmetric, transitive.

25.



27. $\{(1,5), (5,1), (1,6), (6,1), (5,6), (6,5), (1,2), (2,1), (2,7), (7,2), (2,3), (3,2)\}$.

29. Let $a_1, a_2, \ldots, a_n$ be the elements of the base set. The graph of R is connected if for each a_i, there is a 1 in the ith column of $(\mathbf{M}_R)^k_{\odot}$ for some k.

31. Let R be transitive and irreflexive. Suppose $a\ R\ b$ and $b\ R\ a$. Then $a\ R\ a$ since R is transitive. But this contradicts the fact that R is irreflexive. Hence R is asymmetric.

33. Let $R \neq \{\ \}$ be symmetric and transitive. There exists $(x, y) \in R$ and $(y, x) \in R$. Since R is transitive, we have $(x, x) \in R$, and R is not irreflexive.

35. (Outline) Basis step: $n = 1$ P(1): If R is symmetric, then R^1 is symmetric is true.
Induction step: Use P(k): If R is symmetric, then R^k is symmetric to show P($k + 1$). Suppose that $a\ R^{k+1}\ b$. Then there is a $c \in A$ such that $a\ R^k\ c$ and $c\ R\ b$. We have $b\ R\ c$ and $c\ R^k\ a$. Hence $b\ R^{k+1}\ a$.

37. **(a)** One answer is $\{(a,a), (b,b), (c,c), (d,d)\}$.

(b) One answer is $\{(a,a), (b,b), (c,c), (d,d), (a,b)\}$.

39. **(a)** One answer is $\{(a,a), (b,b), (c,c), (d,d)\}$.

(b) One answer is $\{(a,b), (b,c), (a,c)\}$.

Exercise Set 4.5, page 139

1. Yes. **3.** Yes. **5.** No.

7. No. **9.** Yes. **11.** Yes.

13. $\{(a,a), (a,c), (a,e), (c,a), (c,c), (c,e), (e,a), (e,c), (e,e), (b,b), (b,f), (b,d), (d,b), (d,d), (d,f), (f,b), (f,d), (f,f)\}$.

15. $\{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7), (8,8), (9,9), (10,10), (1,3), (3,1), (1,5), (5,1), (1,7), (7,1), (1,9), (9,1), (3,5), (5,3), (3,7), (7,3), (3,9), (9,3), (5,7), (7,5), (5,9), (9,5), (7,9), (9,7), (2,4), (4,2), (2,6), (6,2), (2,8), (8,2), (2,10), (10,2), (4,6), (6,4), (4,8), (8,4), (4,10), (10,4), (6,8), (8,6), (6,10), (10,6), (8,10), (10,8)\}$.

17. $\{\{\ldots, -3, -1, 1, 3, 5, \ldots\}, \{\ldots, -4, -2, 0, 2, 4, \ldots\}\}$.

19. **(a)** R is reflexive because $a^2 + b^2 = a^2 + b^2$. R is clearly symmetric. R is transitive because if $a^2 + b^2 = c^2 + d^2$ and $c^2 + d^2 = e^2 + f^2$, certainly $a^2 + b^2 = e^2 + f^2$.

(b) The equivalence classes of A/R are circles with center at (0, 0), including the circle with radius 0.

21. **(a)** $(a, b)\ R\ (a, b)$ because $ab = ba$. Hence R is reflexive. If $(a, b)\ R\ (a', b')$, then $ab' = ba'$. Then $a'b = b'a$ and $(a', b')\ R\ (a, b)$. Hence R is symmetric. Now suppose that $(a, b)\ R\ (a', b')$ and $(a', b')\ R\ (a'', b'')$. Then $ab' = ba'$ and $a'b'' = b'a''$.

$$ab'' = a\frac{b'a''}{a'} = ab'\frac{a''}{a'} = ba'\frac{a''}{a'} = ba''.$$

Hence $(a, b)\ R\ (a'', b'')$ and R is transitive.

(b) $\{\{(1,1), (2,2), (3,3), (4,4), (5,5)\}, \{(1,2), (2,4)\}, \{(1,3)\}, \{(1,4)\}, \{(1,5)\}, \{(2,1), (4,2)\}, \{(2,3)\}, \{(2,5)\}, \{(3,1)\}, \{(3,2)\}, \{(3,4)\}, \{(3,5)\}, \{(4,1)\}, \{(4,3)\}, \{(4,5)\}, \{(5,1)\}, \{(5,2)\}, \{(5,3)\}, \{(5,4)\}\}$.

23. Let R be reflexive and circular. If $a\ R\ b$, then $a\ R\ b$ and $b\ R\ b$, so $b\ R\ a$. Hence R is symmetric. If $a\ R\ b$ and $b\ R\ c$, then $c\ R\ a$. But R is symmetric, so $a\ R\ c$, and R is transitive.

Let R be an equivalence relation. Then R is reflexive. If $a\ R\ b$ and $b\ R\ c$, then $a\ R\ c$ (transitivity) and $c\ R\ a$ (symmetry), so R is also circular.

25. $a\ R\ b$ if and only if $ab > 0$.

27. If z is even (or odd), then $R(z)$ is the set of even (or odd) integers. Thus, if a and b are both even (or odd), then $R(a) + R(b) = \{x \mid x = s + t, s \in R(a), t \in R(b)\} = \{x \mid x \text{ is even}\} = R(a + b)$. If a and b have opposite parity, then $R(a) + R(b) = \{x \mid x = s + t, s \in R(a), t \in R(b)\} = \{x \mid x \text{ is odd}\} = R(a + b)$.

29. $(1,2)\ R\ (2,4)$ and $(1,3)\ R\ (1,3)$, but $((1,2) + (1,3))\ \not R\ ((2,4) + (1,3))$ so the set $R((a,b)) + R((a',b'))$ is not an equivalence class.

Exercise Set 4.6, page 146

1. VERT[1] = 9 (1, 6) NEXT[9] = 10 (1, 3)
NEXT[10] = 1 (1, 2) NEXT[1] = 0
VERT[2] = 3 (2, 1) NEXT[3] = 2 (2, 3)
NEXT[2] = 0
VERT[3] = 6 (3, 4) NEXT[6] = 4 (3, 5)
NEXT[4] = 7 (3, 6) NEXT[7] = 0
VERT[4] = 0
VERT[5] = 5 (5, 4) NEXT[5] = 0
VERT[6] = 8 (6, 1) NEXT[8] = 0

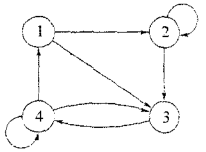
3. On average, EDGE must look at the average number of edges from any vertex. If R has P edges and N vertices, then EDGE examines $\dfrac{\sum P_{ij}}{N} = \dfrac{P}{N}$ edges on average.

5.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

VERT	TAIL	HEAD	NEXT
1	1	1	2
4	1	2	3
6	1	3	0
8	2	3	5
	2	4	0
	3	1	7
	3	4	0
	4	2	0

7. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$



9.

VERT	TAIL	HEAD	NEXT
1	1	1	2
3	1	4	0
5	2	2	4
6	2	3	0
9	3	4	0
	4	1	7
	4	3	8
	4	5	0
	5	2	10
	5	5	0

11.

VERT	TAIL	HEAD	NEXT
1	a	a	2
4	a	b	3
6	a	d	0
9	b	b	5
	b	c	0
	c	b	7
	c	c	8
	c	d	0
	d	a	10
	d	b	11
	d	c	12
	d	d	0

Exercise Set 4.7, page 155

1. **(a)** $\{(1,3), (2,1), (2,2), (3,2), (3,3)\}$.

(b) $\{(3,1)\}$.

(c) $\{(1,1), (1,2), (2,1), (2,3), (3,1), (3,2), (3,3)\}$.

(d) $\{(1,2), (1,\underline{3}), (2,3), (3,3)\}$.

3. $\{(a, b) \mid a, b$ are sisters or a, b are brothers$\}$.

5. $a\ (R \cup S)\ b$ if and only if a is a parent of b.

7. **(a)** $\{(2,1), (3,1), (3,2), (3,3), (4,2), (4,3), (4,4), (1,4)\}$.

(b) $\{(1,1), (1,2), (2,2), (2,3), (2,4), (4,1), (3,4)\}$.

(c) $\{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,4)\}$.

(d) $\{(1,1), (2,1), (2,2), (1,4), (4,1), (2,3), (3,2), (1,3), (4,2), (3,4), (4,4)\}$.

9. **(a)** $\{(1,1), (1,4), (2,2), (2,3), (3,3), (3,4)\}$.

(b) $\{(1,2), (2,4), (3,1), (3,2)\}$.

(c) $\{(1,1), (1,2), (1,3), (1,4), (2,1), (2,4), (3,1), (3,2), (3,3)\}$.

(d) $\{(1,1), (2,1), (4,1), (4,2), (1,3), (2,3), (3,3)\}$.

11. **(a)** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. **(b)** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. **(d)** $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

13. **(a)** $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. **(b)** $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. **(d)** $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$.

15. $R \cap S = \{(a,a), (b,b), (b,c), (c,b), (c,c), (d,d), (e,e)\}$
$\{\{a\}, \{b,c\}, \{d\}, \{e\}\}$.

17. **(a)** $\{(a,a), (a,d), (a,e), (b,b), (b,c), (b,e), (c,a), (c,b), (c,c), (d,b), (d,c), (d,d), (e,c), (e,e)\}$.

(b) $\{(a,a), (a,d), (d,a), (a,e), (e,a), (b,c), (c,b), (b,e), (e,b), (c,a), (a,c), (c,c), (d,b), (b,d), (d,c), (c,d), (e,c), (c,e), (e,e)\}$.

19. The definitions of irreflexive, asymmetric, and antisymmetric each require that a certain pair does not belong to R. We cannot "fix" this by including more pairs in R.

21. **(a)** Yes. **(b)** Yes.

(c) $x\ (S \circ R)\ y$ if and only if $x \leq 6y$.

23. **(a)** Reflexive $a\ R\ a \wedge a\ S\ a \Rightarrow a\ S \circ R\ a$.
Irreflexive No $1\ R\ 2 \wedge 2\ S\ 1 \Rightarrow 1\ S \circ R\ 1$.
Symmetric No $1\ R\ 3, 3\ R\ 1, 3\ S\ 2, 2\ S\ 3 \Rightarrow 1\ S \circ R\ 2$, but $2\ \cancel{S \circ R}\ 1$.
Asymmetric No $R = \{(1,2), (3,4)\}$ and $S = \{(2,3), (4,1)\}$ provide a counterexample.
Antisymmetric No $R = \{(a,b), (c,d)\}$ and $S = \{(b,c), (d,a)\}$ provide a counterexample.
Transitive No $R = \{(a,d), (b,e)\}$ and $S = \{(d,b), (e,c)\}$ provide a counterexample.

(b) No, symmetric and transitive properties are not preserved.

25. **(a)** $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$.

27. $R \cap S$ is antisymmetric. If $a(R \cap S)b$ and $b(R \cap S)a$, then $a\ R\ b$ and $b\ R\ a$. Hence $a = b$ because R is antisymmetric. $R \cup S$ may not be antisymmetric. Let $R = \{(1, 2)\}$, $S = \{(2, 1)\}$.

29. Let $R = \{(x, z), (x, m)\}$, $S = \{(z, y)\}$, and $T = \{(m, y)\}$. Then $(x, y) \in (S \circ R) \cap (T \circ R)$, but $(S \cap T) \circ R = \{\ \}$.

31. **(a)** Let $\mathbf{M}_{R \cap S} = [\, m_{ij} \,]$, $\mathbf{M}_R = [\, r_{ij} \,]$, $\mathbf{M}_S = [\, s_{ij} \,]$. $m_{ij} = 1$ if and only if $(i, j) \in R \cap S$. $(i, j) \in R$ if and only if $r_{ij} = 1$ and $(i, j) \in S$ if and only if $s_{ij} = 1$. But this happens if and only if the i, jth entry of $\mathbf{M}_R \wedge \mathbf{M}_S$ is 1.

(b) Let $\mathbf{M}_{R \cup S} = [\, m_{ij} \,]$, $\mathbf{M}_R = [\, r_{ij} \,]$, $\mathbf{M}_S = [\, s_{ij} \,]$. $m_{ij} = 1$ if and only if $(i, j) \in R \cup S$. $(i, j) \in R$ if and only if $r_{ij} = 1$ or $(i, j) \in S$ if and only if $s_{ij} = 1$. But this happens if and only if the i, jth entry of $\mathbf{M}_R \vee \mathbf{M}_S$ is 1.

(c) The i, jth entry of $\mathbf{M}_{R^{-1}}$ is 1 if and only if $(i, j) \in R^{-1}$ if and only if $(j, i) \in R$ if and only if the j, ith entry of $\mathbf{M}_R$ is 1 if and only if the i, jth entry of $\mathbf{M}_R^T$ is 1.

(d) The i, jth entry of $\mathbf{M}_{\overline{R}}$ is 1 if and only if $(i, j) \in \overline{R}$ if and only if $(i, j) \notin R$ if and only if the i, jth entry of $\mathbf{M}_R$ is 0 if and only if the i, jth entry of $\overline{\mathbf{M}_R}$ is 1.

33. To form the digraph of R^{-1}, reverse the arrows in the digraph of R.

35. The edges of the digraph of $R \cup S$ are the edges that appear on either the digraph of R or the digraph of S.

37. **(a)** R is symmetric if and only if $x\ R\ y \Rightarrow y\ R\ x$ if and only if $R \subseteq R^{-1} \subseteq R$.

(b) Suppose R is antisymmetric. Let $(x, y) \in R \cap R^{-1}$, then $x = y$ and $(x, y) \in \Delta$. Suppose $R \cap R^{-1} \subseteq \Delta$. If $x\ R\ y$ and $y\ R\ x$, then $x\ R\ y$ and $x\ R^{-1}\ y$. Thus $(x, y) \in \Delta$, so $x = y$.

(c) Suppose R is asymmetric. Let $(x, y) \in R \cap R^{-1}$. This contradicts the fact that R is asymmetric. Hence $R \cap R^{-1} = \varnothing$. Let $x\ R\ y$ and $y\ R\ x$. Then $(x, y) \in R \cap R^{-1} = \varnothing$. Hence R is asymmetric.

Exercise Set 4.8, page 162

1. **(a)** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(b) $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$.

3. $W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$, $W_2 = W_1 = W_3$.

5. $a\ R^{\infty}\ b$ if and only if $b > a$.

7. Let R be reflexive and transitive. Suppose that $x\ R^n\ y$. Then $x, a_1, a_2, \ldots, a_{n-1}, y$ is a path of length n from x to y. $x\ R\ a_1 \wedge a_1\ R\ a_2 \Rightarrow x\ R\ a_2$. Similarly, we have $x\ R\ a_k \wedge a_k\ R\ a_{k+1} \Rightarrow x\ R\ a_{k+1}$ and finally $x\ R\ a_{n-1} \wedge a_{n-1}\ R\ y \Rightarrow x\ R\ y$. Hence $R^n \subseteq R$. If $x\ R\ y$, then since R is reflexive we can build a path of length n, $x, x, x, \ldots, x, y$ from x to y and $x\ R^n\ y$.

9. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

11. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

13. The sets are the same. This can be shown by computing each set.

15. $A \times A$.

17. $A/R = \{\{1, 2, 3\}, \{4, 5\}\}$, $A/S = \{\{1\}, \{2, 3, 4\}, \{5\}\}$, $A/(R \cup S)^{\infty} = \{A\}$.

19. The collection of elements in A/R and A/S can be separated into subcollections of nondisjoint sets. Each element of $A/(R \cup S)^{\infty}$ is the union of the sets in one of these subcollections.

21. $\{(1, 1), (1, 2), (1, 4), (2, 2), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4), (1, 3)\}$.

23. We first show R^{∞} is transitive. Then we show it is the smallest relation that contains R. It is a direct proof.

25. $\mathbb{R} \times \mathbb{R}$.

Review Questions, page 165

1. $A \times B$ is a set of ordered pairs.

2. A partition of A is a set of nonempty subsets of A.

3. A relation may be given by a verbal description, a set of ordered pairs, a digraph (in some cases) and as a matrix (in some cases).

4. Unless $|R|$ is very small, using the matrix representation will be the most efficient way to determine if R is transitive.

5. The transitive closure of R is the connectivity relation R^{∞}; it is also the smallest transitive relation containing R.

Chapter 5

Exercise Set 5.1, page 176

1. (a) Yes. $\text{Ran}(R) = \{1, 2\}$. (b) No.

3. Yes.

5. Each integer has a unique square that is also an integer.

7. Each $r \in \mathbb{R}$ is either an integer or it is not.

9. **(a)** 3. **(b)** 1. **(c)** $(x - 1)^2$.
(d) $x^2 - 1$. **(e)** $y - 2$. **(f)** y^4.

11. **(a)** Both. **(b)** Neither.

13. **(a)** Both. **(b)** Onto.

15. **(a)** Both. **(b)** Onto.

17. Neither.

19. **(a)** $(g \circ f)(a) = g\left(\dfrac{a+1}{2}\right) = 2\left(\dfrac{a+1}{2}\right) - 1 = a$.

(b) $(g \circ f)(a) = g(a^2 - 1) = \sqrt{a^2 + 1 - 1} = |a| = a$, since $a \geq 0$.

21. **(a)** $f^{-1}(b) = b^2 - 1$. **(b)** $f^{-1}(b) = \sqrt[3]{b - 1}$.

23. **(a)** Suppose $f(x, y) = f(w, z)$. Then $2x - y = 2w - z$ and $x - 2y = w - 2z$. Solving this system of equations gives $x = w$, $y = z$. Therefore, f is one to one.

(b) $f^{-1}(x, y) = \left(\dfrac{2x - y}{3}, \dfrac{x - 2y}{3}\right)$.

25. Yes. **27.** n^n. **29.** n^m.

31. C; B; because g is onto; A; because f is onto.

33. Suppose $g \circ f$ is onto. Let $c \in C$. Then $\exists a \in A$ such that $(g \circ f)(a) = c$. But $(g \circ f)(a) = g(f(a))$, $f(a) \in B$, so g is onto.

35. **(a)** Let $\text{Dom}(f) = \{a_1, a_2, \ldots, a_n\}$. Then $\text{Ran}(f) = \{f(a_1), f(a_2), \ldots, f(a_n)\}$. Since f is one to one, the $f(a_i)$ are all distinct, so $\text{Ran}(f)$ has n elements. Thus $m = n$.

(b) If f is not one to one, some $f(a_i) = f(a_j)$ and $m < n$.

37. One possible one-to-one correspondence is $f\colon Z^+ \to B$, where $f(z) = 2z$.

39. Yes, the cardinality of a set is determined by creating a one-to-one correspondence between the set and a subset of Z^+. Here we use all of Z^+.

41. **(a)** and **(b)**. Consider the table for $\diamond$.

$\diamond$	0	1
0	0	1
1	1	0

Since $f(0) = \textit{true}$ and $f(1) = \textit{false}$, we see this is not the table for either $\vee$ or $\wedge$.

43. JCCGKFMNUQPJKNFEQKJUM.

Exercise Set 5.2, page 181

1. **(a)** 7. **(b)** 8. **(c)** 3.

3. **(a)** 1. **(b)** 0. **(c)** 1.

5. **(a)** 2. **(b)** -3. **(c)** 14.
(d) -18. **(e)** 21.

7. $n = ak + r, 0 \leq r < n$. Since $k < 2k < 3k < \cdots < ak \leq n$ the number of multiples of k between 1 and n is a. But $\frac{n}{k} = a + \frac{r}{n}$ with $0 \leq \frac{r}{n} < 1$ so $\lfloor \frac{n}{k} \rfloor = a$.

9. **(a)** 26. **(b)** 866. **(c)** 74. **(d)** 431.

11. **(a)** 2. **(b)** 8. **(c)** 32. **(d)** 1024.

13. **(a)** 4. **(b)** 7. **(c)** 9. **(d)** 10.

15. **(a)** 5; 6 **(b)** 6; 7

17. For any 2×2 matrix $\mathbf{M}$, $\mathbf{M}^T$ exists so t is everywhere defined. If $\mathbf{M}$ is a 2×2 matrix, then $t(\mathbf{M}^T) = \mathbf{M}$, so t is onto. Suppose $\mathbf{M}^T = \mathbf{N}^T$. Then $(\mathbf{M}^T)^T = (\mathbf{N}^T)^T$: that is, $\mathbf{M} = \mathbf{N}$ and t is one to one.

19. Every relation R on A defines a unique matrix $\mathbf{M}_R$ so f is everywhere defined and one to one. Any $n \times n$ Boolean matrix $\mathbf{M}$ defines a relation on A so f is onto.

21.

x	y	z	$f(x, y, z)$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

23. **(a)** True. **(b)** False. **(c)** False. **(d)** True.

25. **(a)** 31. **(b)** 0. **(c)** 36.

27. $\dfrac{\lfloor \frac{m}{k} + 2 \rfloor}{2}$.

29. $f^{-1}(1)$ is the set of elements of A.

31. ( ); (( )), ( )( ); ((( ))), (( ))( ), (( )( )), ( )(( )), ( )( )( ); $C_1 = 1$, $C_2 = 2$, $C_3 = 5$.

33. Because there are more right than left parentheses: one less: n left and n right. $s_1 s_2 \ldots s_{2n}$ is not well formed since in $s_1 s_2 \ldots s_k$ there are more right than left parentheses.

35. ${}_{2n}C_n - {}_{2n}C_{n-1}$.

37. **(a)** $0 + 2 + 9 + 0 + 40 + 18 + 7 + 32 + 27 = 135$; $135 + 30 = 165$ or 0 (mod 11).

(b) **(i)** 9. **(ii)** X. **(iii)** 5. **(iv)** 3.

Exercise Set 5.3, page 187

1. **(a)** The number of steps remains 1001.

(b) The number of steps doubles.

(c) The number of steps quadruples.

(d) The number of steps increases eightfold.

3. $|n!| = |n(n-1)(n-2) \cdots 2 \cdot 1| \leq 1 \cdot |n \cdot n \cdots n|, n \geq 1$.

5. $|8n + lg(n)| \leq |8n + n| = 9|n|, n \geq 1$.

7. $|n\, lg(n)| \leq |n \cdot n| = n^2, n \geq 1$. Suppose there exist c and k such that $n^2 \leq c \cdot n\, lg(n), n \geq k$. Choose $N > k$ with $N > c \cdot lg(N)$. Then $N^2 \leq c \cdot N \cdot lg(N) < N^2$, a contradiction.

9. $|5n^2 + 4n + 3| \leq |5n^2 + 500n|, n \geq 1$; $|5n^2 + 500n| \leq 5|n^2 + 100n|$. We have $|n^2 + 100n| = |n^2 + 4 \cdot 25n| \leq |n^2 + 4n^2|, n \geq 25$. But $|5n^2| \leq |5n^2 + 4n + 3|$.

11. $\{f_5\}, \{f_6, f_{10}, f_{11}\}, \{f_7\}, \{f_4\}, \{f_8\}, \{f_1\}, \{f_2\}, \{f_3\}, \{f_9\}, \{f_{12}\}$.

13. $f_1, \Theta(n\, lg\, n)$; $f_2, \Theta(n^2)$; $f_4, \Theta(lg\, n)$; $f_5, \Theta(1)$; $f_6, \Theta(n)$; $f_{10}, \Theta(n)$; $f_{11}, \Theta(n)$.

15. $f(n) = 2 + 4 \cdot 5 + 1$ or $f(n) = 23 \quad \Theta(1)$.

17. $f(n) = 1 + n \cdot 2 \quad \Theta(n)$.

19. $f(n) = 2 + 2n + \frac{n(n+1)}{2} \quad \Theta(n^2)$.

21. $f(n, m, q) = 1 + nq + 3nmq + 1$. Let $N = \max(n, m, q)$, then f is $\Theta(N^3)$.

23. **(a)** $P_n = P_{n-1} + (n-2) + (n-3)$, $P_3 = 1$, $P_4 = 4$.
(b) $\theta(n^2)$.

25. Suppose $h(n) > 0$, $\forall n$, $\Theta(f)$ lower than (or the same as) $\Theta(g)$. $|f(n)| \le c \cdot |g(n)|, n \ge k$ (and $|g(n)| \le d \cdot |f(n)|$, $n \ge l$). $h(n)|f(n)| \le c \cdot h(n) \cdot |g(n)|, n \ge k$ (and $h(n)|g(n)| \le d \cdot h(n) \cdot |f(n)|, n \ge l$). Hence $|f(n) \cdot h(n)| \le c \cdot |g(n) \cdot h(n)|, n \ge k$ (and $|g(n) \cdot h(n)| \le d \cdot |f(n) \cdot h(n)|, n \ge l$). Hence $\Theta(fh)$ is lower than (or the same as) $\Theta(gh)$. Note that if $\Theta(f)$ is strictly lower than $\Theta(g)$, then $\Theta(fh)$ must be strictly lower than $\Theta(gh)$.

27. There exist c_1, k_1 such that $|f(n)| \le c_1|g(n)|, \forall n \ge k_1$, so $|cf(n)| = |c| \cdot |f(n)| \le c_1|c| \cdot |g(n)|, \forall n \ge k_1$. Also, there exist c_2, k_2 such that $|g(n)| \le c_2|f(n)|, \forall n \ge k_2$ and so $|g(n)| \le c_2|f(n)| = \frac{c_2}{|c|} \cdot |cf(n)|, \forall n \ge k_2$.

29. $|rf(n)| \le |r| \cdot |f(n)|, n \ge 1$, so rf is $O(f)$. Choose c such that $c \cdot |r| \ge 1$, then $|f(n)| \le c \cdot |r| \cdot |f(n)| = c \cdot |rf(n)|, n \ge 1$ and f is $O(rf)$.

Exercise Set 5.4, page 195

1. **(a)** Yes. **(b)** No.

3. **(a)** Yes. **(b)** No.

5. **(a)** $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix}$.

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 6 & 3 & 1 & 4 \end{pmatrix}$.

7. **(a)** $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 5 & 4 & 1 \end{pmatrix}$.

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 6 & 2 & 5 \end{pmatrix}$.

9. **(a)** $(1, 5, 7, 8, 3, 2)$. **(b)** $(2, 7, 8, 3, 4, 6)$.

11. **(a)** $(a, f, g) \circ (b, c, d, e)$.
(b) $(a, c) \circ (b, g, f)$.

13. **(a)** $(1, 6, 3, 7, 2, 5, 4, 8)$.
(b) $(5, 6, 7, 8) \circ (1, 2, 3)$.

15. **(a)** $(2, 6) \circ (2, 8) \circ (2, 5) \circ (2, 4) \circ (2, 1)$.
(b) $(3, 6) \circ (3, 1) \circ (4, 5) \circ (4, 2) \circ (4, 8)$.

17. I AM NOT AT HOME.

19. **(a)** EOXMEFKNRAAEMFX.

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 4 & 10 & 7 & 1 & 13 & 5 & 11 & 8 & 2 & 14 & 6 & 12 & 9 & 3 & 15 \end{pmatrix}$.

21. **(a)** Even. **(b)** Odd.

23. Suppose p_1 is the product of $2k_1 + 1$ transpositions and p_2 is the product of $2k_2 + 1$ transpositions. Then $p_2 \circ p_1$ can be written as the product of $2(k_1 + k_2) + 2$ transpositions. By Theorem 3, $p_2 \circ p_1$ is even.

25. **(a)** $(1, 5, 2, 3, 4)$. **(b)** $(1, 4, 2, 5, 3)$.

27. **(a)** $(1, 2, 4)$.

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 2 & 5 & 6 \end{pmatrix}$.

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 2 & 5 & 6 \end{pmatrix}$.

(d) 3.

29. **(a)** Basis step: $n = 1$. If p is a permutation of a finite set A, then p^1 is a permutation of A is true.
Induction step: The argument in Exercise 26 also shows that if p^{n-1} is a permutation of A, then $p^{n-1} \circ p$ is a permutation of A. Hence p^n is a permutation of A.

(b) If $|A| = n$, then there are $n!$ permutations of A. Hence, the sequence $1_A, p, p^2, p^3, \dots$ is finite and $p^i = p^j$ for some $i \ne j$. Suppose $i < j$. Then $p^{-i} \circ p^i = 1_A = p^{-i} \circ p^j$. So $p^{j-i} = 1_A, j - i \in Z$.

31.

	1_A	p_1	p_2	p_3	p_4	p_5
1_A	1_A	p_1	p_2	p_3	p_4	p_5
p_1	p_1	1_A	p_4	p_5	p_2	p_3
p_2	p_2	p_3	1_A	p_1	p_5	p_4
p_3	p_3	p_2	p_5	p_4	1_A	p_1
p_4	p_4	p_5	p_1	1_A	p_3	p_2
p_5	p_5	p_4	p_3	p_2	p_1	1_A

33. $\{1_A\}, \{1_A, p_1\}, \{1_A, p_2\}, \{1_A, p_5\}, \{1_A, p_3, p_4\}, \{1_A, p_1, p_2, p_3, p_4, p_5\}$.

35. There is exactly one of each kind.

37. **(a)** 3. **(b)** 6.

39. For each increasing sequence of length $\lceil \frac{n}{2} \rceil$, there is exactly one associated up-down permutation of A, because there is just one way to arrange the remaining elements of A in decreasing order and insert them to fill the even positions.

Review Questions, page 197

1. Let $f: A \to B$ be a function, then $|f(a)| \le 1, a \in A$.

2. Assume that $f(a_1) = f(a_2)$ and show that $a_1 = a_2$.

3. Let $f: A \to B$. Choose $b \in B$ and find $a \in A$ such that $f(a) = b$.

4. A hashing function is designed to assign items to a limited number of storage places, and any mod-n function has only n outputs.

5. The Θ-class of a function f represents an approximation of how values $f(n)$ grow as n grows.

Chapter 6

Exercise Set 6.1, page 209

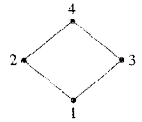
1. **(a)** No. **(b)** No.

3. **(a)** Yes. **(b)** Yes.

5. $\{(a, a), (b, b), (c, c), (a, b)\}$,
$\{(a, a), (b, b), (c, c), (a, b), (a, c)\}$,
$\{(a, a), (b, b), (c, c), (a, b), (c, b)\}$,
$\{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$,
$\{(a, a), (b, b), (c, c), (a, b), (c, b), (c, a)\}$,
$\{(a, a), (b, b), (c, c), (a, c), (c, b), (a, b)\}$.

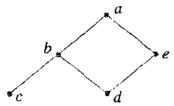
7. The structure of the proof is to check directly each of the three properties required for a partial order.

9.

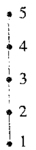


11. $\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.

13.



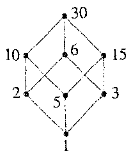
15.



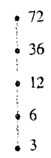
17. $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

19. ACE, BASE, CAP, CAPE, MACE, MAP, MOP, MOPE.

21.



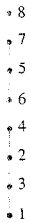
23. Linear



25. If the main diagonal of $\mathbf{M}_R$ is all 1's, then R is reflexive. If $\mathbf{M}_R \odot \mathbf{M}_R = \mathbf{M}_R$, then R is transitive. If $a_{ij} = 1$, $i \ne j$, in $\mathbf{M}_R$, then a_{ji} must be 0 in order for R to be antisymmetric.

27. **(a)** $\{2, 3\}$. **(b)** $\{b, c, d\}$. **(c)** $\{3\}$. **(d)** $\{2, 3\}$. **(e)** $\{2, 3, 7, 8\}$.

29.



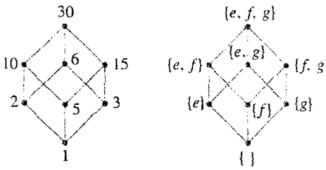
31. $a \le a$ gives $a \le' a$ for all $a \in A'$. $\le'$ is reflexive.
Suppose $a \le' b$ and $b \le' a$. Then $a \le b$, $b \le a$, and $a = b$. Hence $\le'$ is antisymmetric.
Suppose $a \le' b$ and $b \le' c$. Then $a \le b$, $b \le c$, and $a \le c$. Hence $a \le' c$ and $\le'$ is transitive.

33. Suppose $U \subset T$ and $T \subset V$. Then $U \subset V$ and R is transitive. No set is a proper subset of itself so R is irreflexive. Hence R is a quasiorder.

35. Suppose $a\ R^{-1}\ b$ and $b\ R^{-1}\ c$. Then $c\ R\ b$, $b\ R\ a$, and $c\ R\ a$. Hence $a\ R^{-1}\ c$ and R^{-1} is transitive. Suppose that $x\ R^{-1}\ x$. Then $x\ R\ x$, but this is a contradiction. Hence R^{-1} is irreflexive and a quasiorder.

37. $(a, b) \prec (a, b)$ since $a \mid a$ and $b \le b$. Thus $\prec$ is reflexive. Suppose $(a, b) \prec (c, d)$ and $(c, d) \prec (a, b)$. Then $a \mid c$ and $c \mid a$. This means $c = ka = k(mc)$ and for a and c in B, $km = 1$ implies $k = m = 1$. Hence $a = c$. Also, $b \le d$ and $d \le b$ so $b = d$. Thus $\prec$ is antisymmetric.

39.



Define F as follows: $F(1) = \{\ \}$; $F(2) = \{e\}$; $F(5) = \{f\}$; $F(3) = \{g\}$; $F(10) = \{e, f\}$; $F(6) = \{e, g\}$; $F(15) = \{f, g\}$; $F(30) = \{e, f, g\}$.

41. Let $U = \{a, b, c, d\}$, S a subset of U, and f_S be the characteristic function of S (relative to U). Define

$$g(S) = \begin{bmatrix} f_S(a) & f_S(b) \\ f_S(c) & f_S(d) \end{bmatrix}.$$

Then g is a one-to-one correspondence between $P(U)$ and A. If $S \le T$ in $(U, \subseteq)$, then $f_S(x) \le f_T(x)$ for all x in U. Hence $g(S) \le g(T)$.

Exercise Set 6.2, page 215

1. Maximal: 3, 5; minimal: 1, 6.

3. Maximal: e, f; minimal: a.

5. Maximal: none; minimal: none.

7. Maximal: 1; minimal: none.

9. Greatest: f; least: a.

11. No greatest or least.

13. Greatest: none; least: none.

15. Greatest: 72; least: 2.

17. No, a may be maximal and there exists an element of A, b, such that a and b are incomparable.

19. **(a)** True. There cannot be $a_1 < a_2 < \cdots$ since A is finite.
(b) False. Not all elements have to be comparable.
(c) True. There cannot be $\cdots < a_2 < a_1$ since A is finite.
(d) False. Not all elements have to be comparable.

21. Suppose a and b are least elements of $(A, \leq)$. Then $a \leq b$ and $b \leq a$. Since $\leq$ is antisymmetric, $a = b$. Note: This is a restatement of Theorem 2.

23. **(a)** f, g, h. **(b)** a, b, c. **(c)** f. **(d)** c.

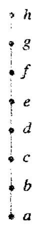
25. **(a)** d, e, f. **(b)** b, a **(c)** d. **(d)** b.

27. **(a)** None. **(b)** b. **(c)** None. **(d)** b.

29. **(a)** $x \in [2, \infty)$. **(b)** $x \in (-\infty, 1]$.
(c) 2. **(d)** 1.

31. **(a)** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. **(b)** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
(c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. **(d)** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

33.



35. The least element of A is the label on the row that is all ones. The greatest element of A is the label on the column that is all ones.

37. **(a)** 49. **(b)** $\{2, 4, 8, 16, 32, 64\}$.

Exercise Set 6.3, page 224

1. Yes, all the properties are satisfied.

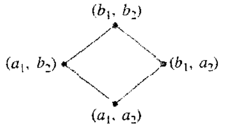
3. No. GLB($\{e, b\}$) does not exist.

5. Yes, all the properties are satisfied.

7. No.

9. Yes, LUB(**M**, **N**) $= \left[\, a_{ij} = \max\{m_{ij}, n_{ij}\} \,\right]$ and GLB(**M**, **N**) $= \left[\, b_{ij} = \min\{m_{ij}, n_{ij}\} \,\right]$

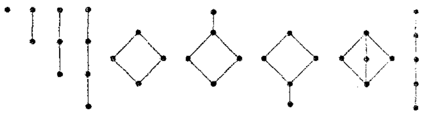
11.



13. For each $T_1, T_2 \subseteq T$, $T_1 \cap T_2$ and $T_1 \cup T_2$ are subsets of T so $P(T)$ is a sublattice of $P(S)$.

15. For any elements x, y of a linearly ordered poset, $x \leq y$ or $y \leq x$. Say $x \leq y$. Then $x = x \wedge y$ and $y = x \vee y$. Hence any subset of a linearly ordered poset is a sublattice.

17.



19. Suppose $a \wedge b = a$.
$a \leq a \vee b = (a \wedge b) \vee b = (a \vee b) \wedge b \leq b$. Thus $a \leq b$.
Suppose $a \leq b$. $a \wedge b \leq a$ and $a \leq a$, $a \leq b$ gives $a \leq a \wedge b$. Hence $a \wedge b = a$.

21. **(a)** 12. **(b)** Figure 6.44(a): 3 Figure 6.44(b): 3.

23.
$$\begin{aligned}(a_1, a_2) &\wedge ((b_1, b_2) \vee (c_1, c_2)) \\ &= (a_1, a_2) \wedge (b_1 \vee c_1, b_2 \vee c_2 \\ &= (a_1 \wedge (b_1 \vee c_1), a_2 \wedge (b_2 \vee c_2)) \\ &= ((a_1 \wedge b_1) \vee (a_1 \wedge c_1), (a_2 \wedge b_2) \vee (a_2 \wedge c_2)) \\ &= ((a_1, a_2) \wedge (b_1, b_2)) \vee ((a_1, a_2) \wedge (c_1, c_2)).\end{aligned}$$
A similar argument establishes the other distributive property.

25. Suppose $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$. Then

$$\begin{aligned} y \leq y \vee (y \wedge a) &= (y \wedge y) \vee (y \wedge a) \\ &= y \wedge (y \vee a) \\ &= y \wedge (a \vee x) \\ &= (y \wedge a) \vee (y \wedge x) \\ &= (a \wedge x) \vee (y \wedge x) \\ &= x \wedge (a \vee y) \leq x. \end{aligned}$$

Hence $y \leq x$. A similar argument shows $x \leq y$. Thus $x = y$.

27. $1' = 42$, $42' = 1$, $2' = 21$, $21' = 2$, $3' = 14$, $14' = 3$, $7' = 6$, $6' = 7$.

29. Neither.

31. Distributive, but not complemented.

33. If $x = x'$, then $x = x \vee x = I$ and $x = x \wedge x = 0$. But by Exercise 18, $0 \neq I$. Hence, $x \neq x'$.

35. Suppose $\mathcal{P}_1 \leq \mathcal{P}_2$. Then $R_1 \subseteq R_2$. Let $x \in A_i$. Then $A_i = \{y \mid a \ R_1 \ y\}$ and $A_i \subseteq \{y \mid x \ R_2 \ y\} = B_j$, where $x \in B_j$. Suppose each $A_i \subseteq B_j$. Then $x \ R_1 \ y$ implies $x \ R_2 \ y$ and $R_1 \subseteq R_2$. Thus $\mathcal{P}_1 \leq \mathcal{P}_2$.

37. The sublattice $\{a, b, d\}$ of Figure 6.57 is not complemented.

39. For any a, b, c in the sublattice with $a \leq c$, $a \vee (b \wedge c) = (a \vee b) \wedge c$, because this is true in the full lattice.

Exercise Set 6.4, page 231

1. No, it has 6 elements, not 2^n elements.

3. No, it has 6 elements, not 2^n elements.

5. Yes, it is B_3.

7. Yes, it is B_1.

9. Yes; $385 = 5 \cdot 7 \cdot 11$.

11. No, each Boolean algebra must have 2^n elements.

13. Suppose $a = b$.
$(a \wedge b') \vee (a' \wedge b) = (b \wedge b') \vee (a' \wedge a) = 0 \vee 0 = 0$.
Suppose $(a \wedge b') \vee (a' \wedge b) = 0$. Then $a \wedge b' = 0$ and $a' \wedge b = 0$. We have $I = 0' = (a \wedge b')' = a' \vee b$. So a' is the complement of b; $b' = a'$.

15. Suppose $a \leq b$. Then $a \wedge c \leq a \leq b$ and $a \wedge c \leq c$ so $a \wedge c \leq b \wedge c$.

17. $(a \wedge b) \vee (a \wedge b') = a \wedge (b \vee b') = a \wedge I = a$.

19. $(a \wedge b \wedge c) \vee (b \wedge c) = (a \vee I) \wedge (b \wedge c) = I \wedge (b \wedge c) = b \wedge c$.

21. Suppose $a \leq b$. Then
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = b \wedge (a \vee c)$.

23. R is reflexive because $m_{ii} = 1$, $i = 1, 2, \ldots, 8$. R is antisymmetric since if $m_{ij} = 1$ and $i \neq j$, then $m_{ji} = 0$. R is transitive, because $\mathbf{M}_R \odot \mathbf{M}_R$ shows that $R^2 \subseteq R$.

25. Complement pairs are a, h; b, g; c, f; d, e. Since each element has a unique complement, (A, R) is complemented.

27. (A, R) is not a Boolean algebra; complements are not unique.

29. **(a)** $\{a\}, \{b\}, \{c\}$. **(b)** 2, 3, 5.

31. Matrices with exactly one 1.

Exercise Set 6.5, page 236

1.

x	y	z	x	$\wedge$	$(y \vee z')$
0	0	0		0	1
0	0	1		0	0
0	1	0		0	1
0	1	1		0	1
1	0	0		1	1
1	0	1		0	0
1	1	0		1	1
1	1	1		1	1
				↑	

3.

x	y	z	$(x \vee y')$	$\vee$	$(y \wedge$	$(x' \vee y)$
0	0	0	0	0	0	1
0	0	1	0	0	0	1
0	1	0	0	1	1	1
0	1	1	0	1	1	1
1	0	0	1	1	0	0
1	0	1	1	1	0	0
1	1	0	0	1	1	1
1	1	1	0	1	1	1
			(1)	(4)	(3)	(2)

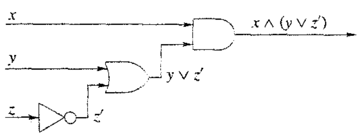
5. $(x \vee y) \wedge (x' \vee y) = (x \wedge x') \vee y = 0 \vee y = y$.

7. $(z' \vee x) \wedge ((x \wedge y) \vee z) \wedge (z' \vee y)) =$
$(z' \vee (x \wedge y)) \wedge ((x \wedge y) \vee z) = (x \wedge y) \vee (z' \wedge z) =$
$(x \wedge y) \vee 0 = x \wedge y$.

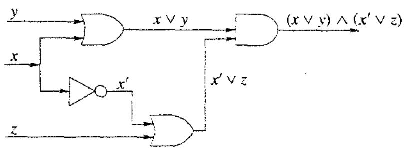
9. $(x' \vee y)' \vee z \vee x \wedge ((y \wedge z) \vee (y' \wedge z')) =$
$(x \wedge y') \vee z \vee (x \wedge y \wedge z) \vee (x \wedge y' \wedge z') =$
$((x \wedge y') \wedge (1 \vee z')) \vee z \wedge (1 \vee (x \wedge y)) = (x \wedge y') \vee z$.

11. $x \wedge z$. **13.** $y \vee x'$.

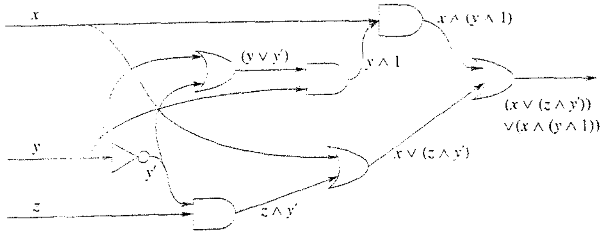
15. **(a)**



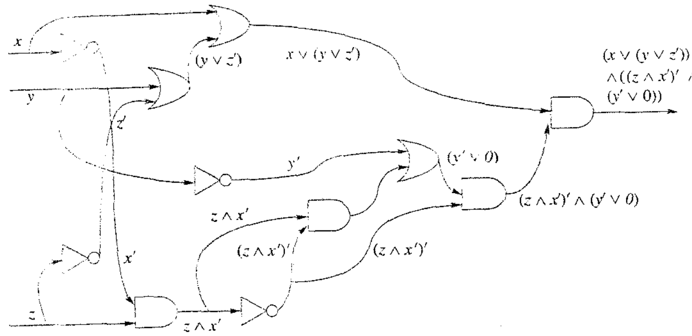
(b)



17. **(a)**

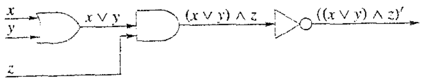


(b)

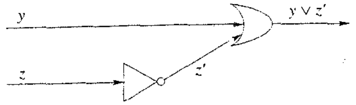


19. $((x \wedge y) \vee (y \wedge z))'$.

21. $((x \vee y) \wedge z)'$

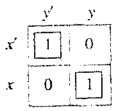


23. $y \vee z'$

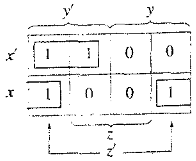


Exercise Set 6.6, page 246

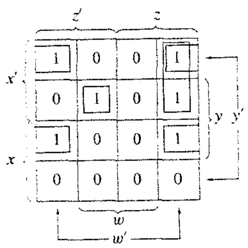
1.



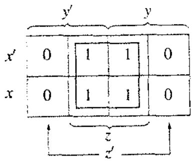
3.



5.



7.



9. $(x' \wedge y') \vee (x \wedge y)$.

11. $z' \vee (x' \wedge z)$.

13. $(z' \wedge y) \vee (x \wedge y') \vee (y' \wedge z)$.

15. $(z \wedge x') \vee (w' \wedge x \wedge y) \vee (w \wedge x \wedge y')$.

17. $(x' \wedge y') \vee (x \wedge y)$.

19. $(x' \wedge y') \vee (x \wedge z')$.

---

21. z.

23. $(x' \wedge y' \wedge w') \vee (y \wedge z \wedge w') \vee (x' \wedge z' \wedge y \wedge w)$.

25. **(a)** $x' \wedge y', x' \wedge y, x \wedge y'$

(b) Since $\wedge$ is commutative and associative, we need only consider the case $(w_1 \wedge w_2 \wedge \cdots \wedge w_n \wedge y) \vee (w_1 \wedge w_2 \wedge \cdots \wedge w_n \wedge y')$. But this is equivalent to $w_1 \wedge w_2 \wedge \cdots \wedge w_n$.

27. $x' \wedge z', y' \wedge z', y \wedge z', x \wedge z', x \wedge y$.

29. **(a)** $(x \wedge y) \vee z'$.

(b) A simple check of the values of $f(x, y, z)$ will verify this.

Review Questions, page 249

1. Partial order is a generalization of less than or equal for the real numbers.
2. In a partial order not every pair of elements must have a least upper bound and a greatest lower bound.
3. A model for any Boolean algebra is the power set of a set and the subset relation.
4. The Karnaugh map of a function is an $n \times n$ array of 0's and 1's used to create a Boolean expression that produces the function.
5. Every Boolean function can be produced by a Boolean expression; every Boolean expression produces a Boolean function.

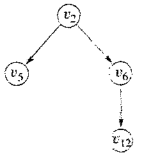
Chapter 7

Exercise Set 7.1, page 258

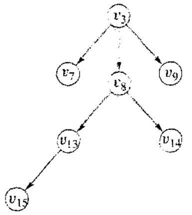
1. Yes, the root is b.
3. Yes, the root is f.
5. No.
7. Yes, the root is t.
9. **(a)** $v_{12}, v_{10}, v_{11}, v_{13}, v_{14}$.

   (b) $v_{10}, v_{11}, v_5, v_{12}, v_7, v_{15}, v_{14}, v_9$.

11. **(a)**



(b)



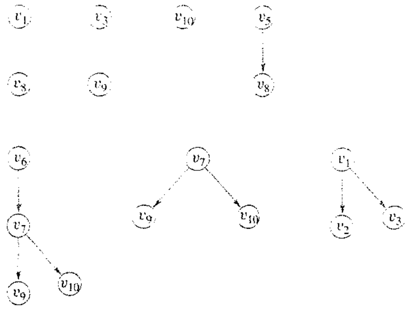
13. (T, v_0) may be an n-tree for $n \geq 3$. It is not a complete 3-tree.

15. **(a)** v_1, v_3.

(b) $v_6, v_7, v_8, v_{13}, v_{14}, v_{16}, v_{10}$.

17. **(a)** 4. **(b)** 2.

19.



21. Basis step: $n = 1$. An n-tree of height 1 can have at most n leaves by definition.
Induction step: Use P(i): An n-tree of height i has at most n^i leaves to show P($i+1$): An n-tree of height $i+1$ has at most n^{i+1} leaves. The leaves of a tree T of height $i+1$ belong to the subtrees of T whose roots are at level 1. Each of these subtrees has height at most i, and there are at most n of them. Hence the maximum number of leaves of T is $n \cdot n^i$ or n^{i+1}.

23. The total number of vertices is $1 + kn$, where k is the number of nonleaves and 1 counts the root, because every vertex except the root is an offspring. Since $l = m - k$, $l = 1 + kn - k$ or $1 + k(n - 1)$.

25. If both $v\ T\ u$ and $u\ T\ v$, then v, u, v is a cycle in T. Thus $v\ T\ u$ implies $u\ T\ v$. T is asymmetric.

27. Each vertex except the root has in-degree 1. Thus $s = r - 1$.

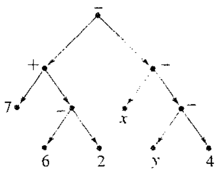
29. 4. The tree of maximum height has one vertex on each level.

31. Assume that the in-degree of $v_0 \neq 0$. Then there is a cycle that begins and ends at v_0. This is impossible. Hence the in-degree of v_0 must be 0.

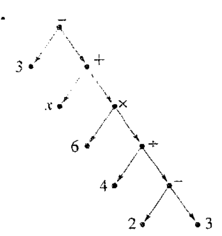
33. **(a)** $2 \leq n$. **(b)** $1 \leq k \leq 7$.

Exercise Set 7.2, page 263

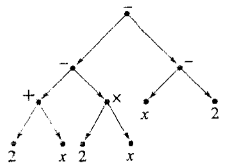
1.



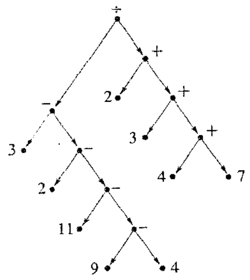
3.



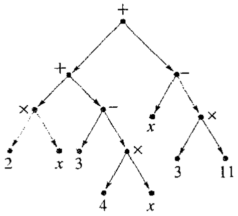
5.



7.

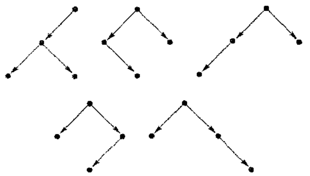


9.



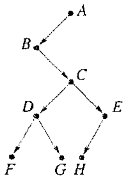
11. 6.

13.

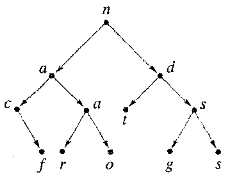


15. 721.

17.



19.



21.

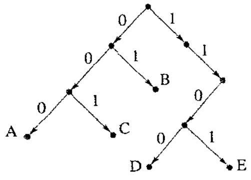
LEFT	DATA	RIGHT
2		0
3	−	4
5	+	6
9	−	10
0	7	0
7	−	8
0	6	0
0	2	0
0	x	0
11	−	12
0	y	0
0	4	0

23.

LEFT	DATA	RIGHT
2		0
3	−	4
5	−	6
9	−	10
7	+	8
11	×	12
0	2	0
0	x	0
0	x	0
0	2	0
0	2	0
0	x	0

25. **(a)** CAR. **(b)** SEAR. **(c)** RACE. **(d)** SCAR.

27.



Exercise Set 7.3, page 271

1. $x\ y\ s\ z\ t\ u\ v$.

3. $a\ b\ c\ g\ h\ i\ d\ k\ e\ j\ f$.

5. TSAMZWEDQMLCKFNTRGJ.

7. $2 + 3 - 1 \times 2$.

9. 6 4 2 1 3 5 7.

11. $s\ y\ v\ u\ t\ z\ x$.

13. $g\ h\ c\ i\ b\ k\ j\ f\ e\ d\ a$.

15. ZWMADQESCNTFKLJGRMT.

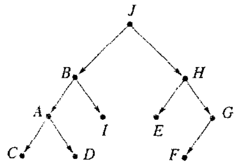
17. NEVER I COW A SAW PURPLE ONE SEE I NEVER HOPE I.

19. 4.

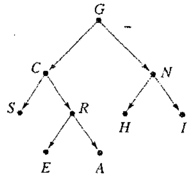
21. $\frac{15}{16}$.

23. $\frac{8}{6}$.

25.



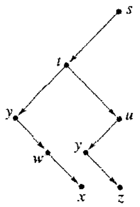
27.



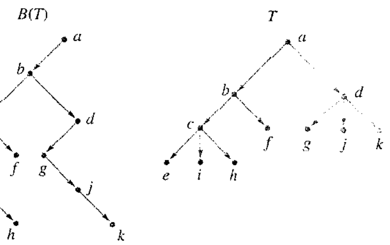
29. **(a)** The root must be labeled J; if J has a left offspring, it must be labeled B, otherwise the right offspring is labeled B.

(b) The root must be labeled G; if G has a left offspring, it must be labeled N, otherwise the right offspring is labeled N.

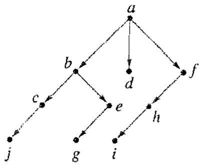
31.



33.



35.

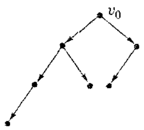


37. **(a)**

(b)

(c)

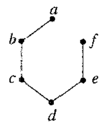
(d)



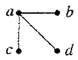
39. $AVL_n = AVL_{n-1} + AVL_{n-2} + 1$. Let v be a new root. Let $T(v_L)$ be an AVL tree of height $n - 1$ using a minimal number of vertices. Let $T(v_R)$ be an AVL tree of height $n - 2$ using a minimal number of vertices. Then $T(v)$, where the left offspring of v is v_L and its right offspring is v_R, is an AVL tree of height n using a minimal number of vertices.

Exercise Set 7.4, page 278

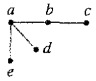
1.



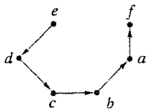
3.



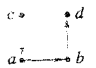
5.



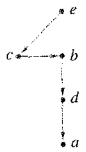
7.



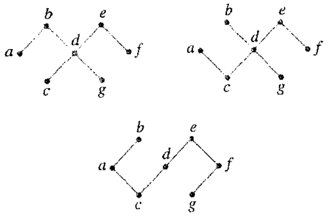
9.



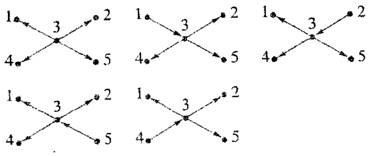
11.



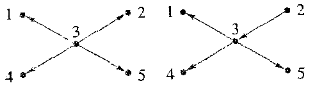
13.



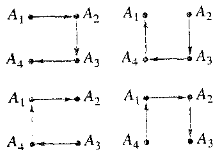
15. There are 5 spanning trees.



17.



19.

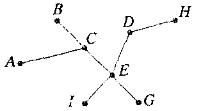


21. Five. 23. n.

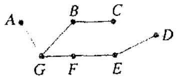
25. There is only one. Any spanning tree is formed by omitting one edge from the graph. A clockwise shift of the labels gives an isomorphism between any spanning tree and the tree formed by omitting the edge (v_n, v_1).

Exercise Set 7.5, page 285

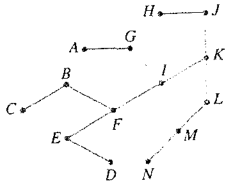
1.



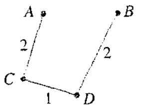
3.



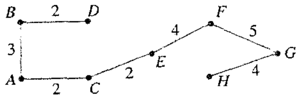
5.



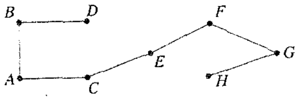
7.



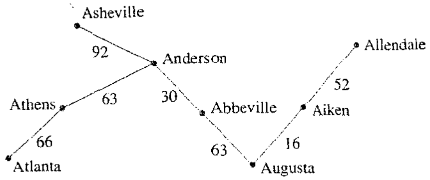
9.



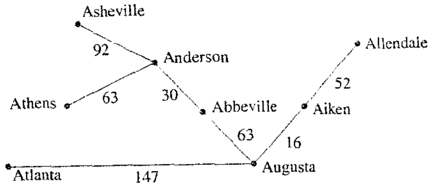
11.



13.

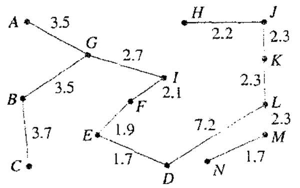


15.

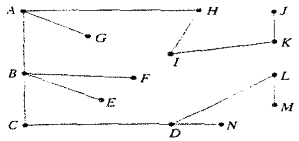


It would be 81 miles longer.

17.



19.



21. The maximal spanning tree is the same as that in Exercise 19.

23. If each edge has a distinct weight, there will be a unique maximal spanning tree since only one choice can be made at each step.

25. One example is the sum 50. The greedy algorithm would select four 11-xebec coins and then six 1-xebec coins for a total of 10 coins. But the amount can be made from seven 7-xebec coins and one 1-xebec coin, a total of eight coins.

Review Questions, page 287

1. A tree is a relation with certain conditions.

2. The only change in performing a preorder, inorder, or postorder search is when the root is visited.

3. The other three sequences are (i) right, root, left, (ii) right, left, root, and (iii) root, right, left.

4. A complete n-tree usually has smaller height than a general n-tree with the same number of vertices. This would shorten a search of the tree that begins at the root.

5. An estimate of how many vertices and edges the graph has would help in the decision process.

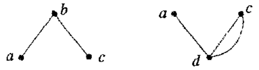
Chapter 8

Exercise Set 8.1, page 295

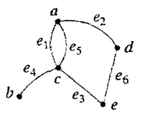
1. $V = \{a, b, c, d\}$, $E = \{\{a, b\}, \{b, c\}, \{b, d\}, \{c, c\}\}$.

3. $V = \{a, b, c, d\}$, $E = \{\{a, b\}, \{b, c\}, \{d, a\}, \{d, c\}\}$. All edges are double edges.

5. Possible answers are



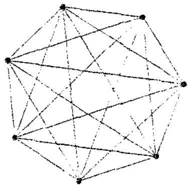
7.



9. Degree of a is 2; degree of b is 3; degree of c is 3; degree of d is 1.

11. a, c; a, b, c; a, c, d; a, c, e.

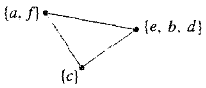
13.



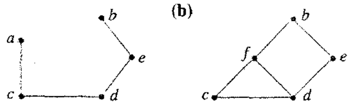
15. Only the graph given in Exercise 3 is regular.

17. One possible solution is 

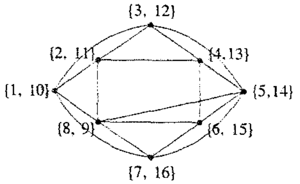
19.



21. (a) (b)



23.



25. $n - 1$. The two "endpoints" have degree 1; the other $n - 2$ vertices each have degree 2. Hence the number of edges is

$$\frac{2(1) + 2(n-2)}{2}$$

or $n - 1$, since each edge is counted twice in the sum of the degrees.

27. The graphs in Figures 8.24(a) and 8.24(b) are not isomorphic, because the one in Figure 8.24(a) has two vertices of degree 1 and the one in Figure 8.24(b) has only one vertex of degree 1.

29. Define a function f as follows: $f(a) = 1$, $f(b) = 3$, $f(c) = 5$, $f(d) = 2$, $f(e) = 4$. This function is a one-to-one correspondence between the vertices of the two graphs with the property that (u, v) is an edge in Figure 8.24(a) if and only if $(f(u), f(v))$ is an edge in Figure 8.24(d). Thus, the graphs are isomorphic.

31. If G has no loops or multiple edges, then each edge contributes 1 to the degree of each of its end points. Thus, the sum of the degrees of all vertices is twice the number of edges.

Exercise Set 8.2, page 302

1. Neither. There are 4 vertices of odd degree.

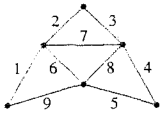
3. Euler circuit. All vertices have even degree.

5. Euler path only, since exactly two vertices have odd degree.

7. Neither. The graph is disconnected.

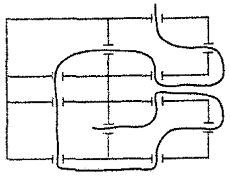
9. Yes, all vertices have even degree.

11.

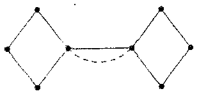


is one possible answer.

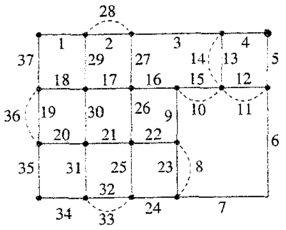
13. Yes. Note that if a circuit is required, it is not possible.



15. One more edge.



17. Seven edges. One possible solution.



19. See the solution for Exercise 17. The consecutively numbered edges are one possible circuit.

21. If n is odd, each vertex in K_n has degree $n - 1$, an even number. In this case, K_n has an Euler circuit. If n is even, then each vertex of K_n has odd degree; K_n does not have an Euler circuit.

23. Suppose the strings $a_1a_2 \dots a_n$ and $b_1b_2 \dots b_n$ differ only in the ith position. Then a_i or b_i is 1 (and the other is 0); say $a_i = 1$. Let A be the subset represented by $a_1a_2 \dots a_n$ and B, the one represented by $b_1b_2 \dots b_n$. Then B is a subset of A, and there is no proper subset of A that contains B. Hence there is an edge in B_n between these vertices.

Suppose there is an edge between the vertices labeled $a_1a_2 \dots a_n$ and $b_1b_2 \dots b_n$. Then $a_i \leq b_i$, $1 \leq i \leq n$ (or vice versa). Hence there are at least as many 1's in $b_1b_2 \dots b_n$ as in $a_1a_2 \dots a_n$. If the strings differ in two or more positions, say $a_j = a_k = 0$ and $b_j = b_k = 1$, consider the label $c_1c_2 \dots c_n$ with $c_i = b_i$, $i \neq j$, and $c_j = 0$. Then $c_1c_2 \dots c_n$ represents an subset between those represented by $a_1a_2 \dots a_n$ and $b_1b_2 \dots b_n$. But this is not possible if there is an edge between the vertices labeled $a_1a_2 \dots a_n$ and $b_1b_2 \dots b_n$.

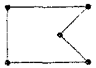
25. If n is even, there is an Euler circuit. Each vertex is labeled with a string of even length. Hence it must have even degree as the value of each position could be changed in turn to create the label of a vertex connected to the original one. If n is odd, by the same reasoning every vertex has odd degree and there is no Euler circuit.

Exercise Set 8.3, page 306

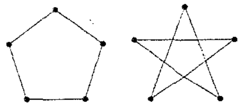
1. Neither.

3. Hamiltonian path, but no Hamiltonian circuit.

5. Hamiltonian circuit.



7.



9. $A, B, D, H, G, E, F, C, A$.

11. A, G, B, C, F, E, D, A.

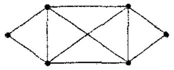
13. $A, B, D, H, G, E, F, C, A$.

15. A, G, B, C, F, E, D, A.

17. $F, E, G, H, D, B, A, C, F$.

19. Choose any vertex, v_1, in K_n, $n \geq 3$. Choose any one of the $n-1$ edges with v_1 as an endpoint. Follow this edge to v_2. Here we have $n-2$ edges from which to choose. Continuing in this way we see there are $(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$ Hamiltonian circuits we can choose.

21. One example is

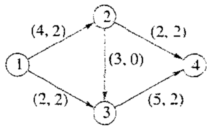


23. One solution is 000, 100, 110, 111, 101, 001, 011, 010, 000.

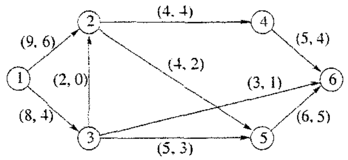
25. Construct a sequence of $2^n + 1$ strings of 0's and 1's of length n such that the first and the last terms are the same, and consecutive terms differ in exactly one position.

Exercise Set 8.4, page 314

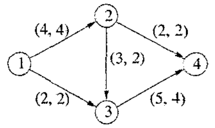
1. One possible solution is



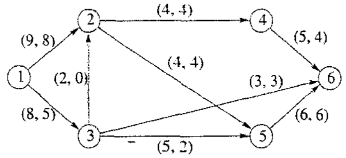
3. One possible solution is



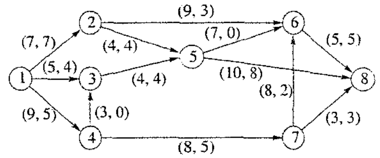
5. $value(F) = 6$.



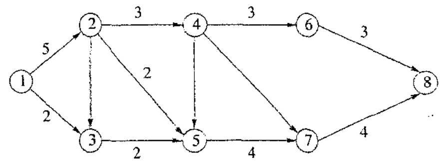
7. $value(F) = 13$.



9. $value(F) = 16$.



11. $value(F) = 7$.



13. Possible cuts include $C_1 = \{(2, 6), (5, 6), (5, 8), (4, 7)\}$ with capacity 34 and $C_2 = \{(6, 8), (7, 8)\}$ with capacity 8.

15. $\{(1, 2), (1, 3)\}$.

17. $\{(2, 4), (3, 6), (5, 6)\}$.

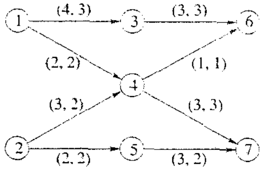
19. $\{(2, 5), (3, 5), (6, 8), (7, 8)\}$.

21. $\{(2, 4), (5, 7)\}$ with capacity 7.

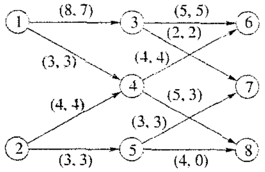
Exercise Set 8.5, page 319

1. Yes. $M(s_2) = b_1$, $M(s_3) = b_3$, $M(s_4) = b_4$, $M(s_5) = b_2$.

3. $value(F) = 9$.



5. $value(F) = 17$.



7. $\{(a, 1), (b, 2), (c, 4), (d, 3)\}$ is a maximal matching.

9. $\{(a, 5), (b, 2), (c, 3), (d, 1), (e, 4)\}$ is a maximal matching.

11. The matchings in Exercises 7, 9, and 10 are complete.

13. **(a)** One set of pairings is Sam-Jane, Kip-Gail, Homer-Rufus, and Kirk-Stacey.

(b) Yes, one such pairing is Sam-Stacey, Kip-Rufus, Homer-Jane, and Kirk-Gail.

15. Let S be any subset of A and E the set of edges that begin in S. Then $k|S| \leq |E|$. Each edge in E must terminate in a node of $R(S)$. There are at most $j|R(S)|$ such nodes. Since $j \leq k$, $j|S| \leq k|S| \leq j|R(S)|$ and $|S| \leq |R(S)|$. By Hall's Marriage theorem, there is a complete matching for A, B, and R.

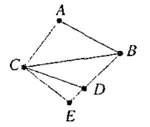
17. **(a)** No, vertices 1, 2, and 6 must be in different subsets, but there are only two sets in the partition.

(b) Yes, $\{\{1, 3, 5, 7\}, \{2, 4, 6, 8\}\}$.

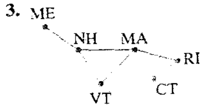
19. A triangle is formed by edges (u, v), (v, w), and (w, u). If u and v are placed in different subsets by a two-set partition, neither subset can contain w.

Exercise Set 8.6, page 324

1.



3.



5. 2.

7. 2.

9. $P_G(x) = x(x-1)(x^2-3x+3)$; $P_G(0) = P_G(1) = 0$, $P_G(2) = 2$.

11. $P_G(x) = x(x-1)(x^4-5x^3+10x^2-10x+5)$; $P_G(0) = P_G(1) = 0$, $P_G(2) = 2$.

13. $P_G(x) = x(x-1)(x-2)^2$; $\chi(G) = 3$.

15. $P_G(x) = x(x-1)(x-2)(x-3)$; $\chi(G) = 4$.

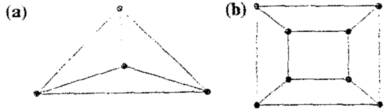
17. $P_G(x) = x(x-1)(x-2)^3$; $\chi(G) = 3$.

19. $P_G(x) = x(x-1)^2(x-2)(x^2-3x+3)$; $\chi(G) = 3$.

21. $\chi(G) = 2$. Let A and B be the subsets that partition the vertices. Color vertices in A one color and those in B the other. It is easy to see that this will give a proper coloring of G.

23. (Outline) Basis Step: $n = 1$ P(1): $P_{L_1}(x) = x$ is true, because L_1 consists of a single vertex.
Induction Step: We use P(k) to show P($k+1$). Let $G = L_{k+1}$ and e be an edge $\{u, v\}$ with $\deg(v) = 1$. Then G_e has two components, L_k and v. Using Theorem 1 and P(k), we have $P_{G_e}(x) = x \cdot x(x-1)^{k-1}$. Merging v with u gives $G^e = L_k$. Thus $P_{G^e}(x) = x(x-1)^{k-1}$. By Theorem 2, $P_{L_{k+1}}(x) = x^2(x-1)^{k-1} - x(x-1)^{k-1} = x(x-1)^{k-1}(x-1)$ or $x(x-1)^k$.

25. These are possible answers.


(a) (b)

27. Label the vertices with the fish species. An edge connects two vertices if one species eats the other. The colors represent the tanks, so $\chi(G)$ will be the minimum number of tanks needed.

Review Questions, page 326

1. A quotient graph is a quotient set of the vertices with a graph structure of inherited edges.

2. An Euler circuit must use all edges exactly once; a Hamiltonian circuit must use all vertices exactly once except for the starting vertex which is also the ending vertex.

3. The subset of edges cannot contain edges whose end points do not belong to the subsets of vertices.

4. Any matching problem can be converted to a maximum flow problem by adding to the graph a supersource and a supersink, and giving each edge capacity 1.

5. Since the chromatic polynomial counts the number of ways the graph can be colored with x colors, the smallest value of x for which $P(x) \neq 0$ is the minimum number of colors needed; that is, the chromatic number.

Chapter 9

Exercise Set 9.1, page 333

1. Yes. 3. No. 5. No. 7. No.

9. Commutative, associative.

11. Not commutative, associative.

13. Commutative, associative.

15. Commutative, associative.

17. Commutative, associative.

19. Commutative; not associative

21. The operation has the idempotent property, because $a*a = \frac{a+a}{2} = a$.

23. One solution is

$*$	a	b	c
a	a	c	c
b	c	b	a
c	c	a	c

25. **(a)** a, a. **(b)** c, b. **(c)** c, a. **(d)** Neither.

27.

$*$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	d	c	d
d	d	c	c	d

29. $n^{\frac{n(n+1)}{2}}$ commutative operations.

31. **(a)** Associative: (1), (5), (8), (9), (10), (11), (15), (16).
(b) Idempotent: (5), (10), (11), (15).

33. A binary operation on a set S must be defined for every a, b in S. According to the earlier definition, $a * b$ may be undefined for some a, b in S. Any binary operation on a set S is a binary operation in the sense of Section 1.6.

Exercise Set 9.2, page 340

1. Semigroup: (b) monoid: (b).

3. Semigroup: (a) monoid: neither.

5. Monoid: identity is 1; commutative.

7. Semigroup.

9. Monoid: identity is S; commutative.

11. Monoid: identity is 12; commutative.

13. Monoid: identity is 0; commutative.

15. monoid; identity $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$; commutative.

17. Neither.

19.

$*$	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

21. Let $f_1(a) = a$, $f_1(b) = a$; $f_2(a) = a$, $f_2(b) = b$; $f_3(a) = b$, $f_3(b) = a$; $f_4(a) = b$, $f_4(b) = b$. These are the only functions on S. It is not commutative.

$\circ$	f_1	f_2	f_3	f_4
f_1	f_1	f_1	f_4	f_4
f_2	f_1	f_2	f_3	f_4
f_3	f_1	f_3	f_2	f_1
f_4	f_1	f_4	f_4	f_4

23. **(a)** *abaccbababc.*
(b) *babcabacabac.*
(c) *babccbaabac.*

25. The subset must form a subsemigroup and the identity element must belong to the subset.

27. By Exercise 26, we need only check that $e \in S_1 \cap S_2$. But $e \in S_1$ and $e \in S_2$, because each is a submonoid of $(S, *)$.

29. Yes. Refer to Exercise 1.

31. Let $x, y \in S_1$.

$$\begin{aligned}(g \circ f)(x *_1 y) &= g(f(x *_1 y)) \\ &= g(f(x) *_2 f(y)) \\ &= g(f(x)) *_3 g(f(y)) \\ &= (g \circ f)(x) *_3 (g \circ f)(y).\end{aligned}$$

Hence $g \circ f$ is a homomorphism from $(S_1, *_1)$ to $(S_3, *_3)$.

33. Onto; homomorphism.

35. Let $x, y \in \mathbb{R}^+$. $\ln(x * y) = \ln(x) + \ln(y)$ so ln is a homomorphism. Suppose $x \in \mathbb{R}$. Then $e^x \in \mathbb{R}^+$ and $\ln(e^x) = x$ so ln is onto $\mathbb{R}^+$. Suppose $\ln(x) = \ln(y)$; then $e^{\ln(x)} = e^{\ln(y)}$ and $x = y$. Hence ln is one to one and an isomorphism between $(\mathbb{R}^+, \times)$ and $(\mathbb{R}, +)$.

Exercise Set 9.3, page 346

1. Let $(s_1, t_1), (s_2, t_2) \in S \times T$. $(s_1, t_1) *'' (s_2, t_2) = (s_1 * s_2, t_1 *' t_2)$, so $*''$ is a binary operation. Consider $(s_1, t_1) *'' ((s_2, t_2) *'' (s_3, t_3)) = (s_1, t_1) *'' (s_2 * s_3, t_2 *' t_3) = (s_1 * (s_2 * s_3), t_1 *' (t_2 *' t_3)) = ((s_1 * s_2) * s_3, (t_1 *' t_2) *' t_3) = ((s_1, t_1) *'' (s_2, t_2)) *'' (s_3, t_3)$. Thus $(S \times T, *'')$ is a semigroup. $(s_1, t_1) *'' (s_2, t_2) = (s_1 * s_2, t_1 *' t_2) = (s_2 * s_1, t_2 *' t_1) = (s_2, t_2) *'' (s_1, t_1)$. Hence $*''$ is commutative.

3. Let $(s_1, t_1), (s_2, t_2) \in S \times T$. Then $f((s_1, t_1) *'' (s_2, t_2)) = f(s_1 * s_2, t_1 *' t_2) = s_1 * s_2 = f(s_1, t_1) * f(s_2, t_2)$. f is a homomorphism.

5. $*''$ is a binary operation, because both $*$ and $*'$ are. Consider $(s_1, t_1) *'' ((s_2, t_2) *'' (s_3, t_3))$.

$$\begin{aligned}(s_1, t_1) *'' ((s_2, t_2) *'' (s_3, t_3)) &= (s_1, t_1) *'' (s_2 * s_3, t_2 *' t_3) \\ &= (s_1 * (s_2 * s_3), t_1 *' (t_2 *' t_3)) \\ &= ((s_1 * s_2) * s_3, (t_1 *' t_2) *' t_3) \\ &= ((s_1, t_1) *'' (s_2, t_2)) *'' (s_3, t_3)\end{aligned}$$

Thus, $*''$ is associative.

7. Yes. 9. Yes. 11. Yes. 13. No. 15. Yes.

17. $S/R = \{[4], [7], [10], [13], [16]\}$.

19. By Exercise 23, Section 4.7, we have that the composition of two equivalence relations need not be an equivalence relation.

21. $S/R = \{[0], [1], [2]\}$. $[0] = \{0, \pm3, \pm6, \ldots\}$, $[1] = \{\pm1, \pm4, \pm7, \ldots\}$, $[2] = \{\pm2, \pm5, \pm8, \ldots\}$.

$\oplus$	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

23. $S/R = \{[0], [1], [2], [3], [4]\}$,
$[a] = \{z \mid z = 5k + a, k \in Z\}$, $a = 0, 1, 2, 3, 4$.

$\oplus$	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

25. **(a)**

$\circledast$	$[a]$	$[b]$
$[a]$	$[a]$	$[b]$
$[b]$	$[b]$	$[b]$

(b) $f_R(e) = [a] = f_R(a)$, $f_R(b) = [b] = f_R(c)$.

27. An examination of the two multiplication tables shows that they are identical and so Z_2 is isomorphic to S/R.

29. This is a direct proof. For part (a), we check the three properties for an equivalence relation and then the property for a congruence relation. In part (b), we first check that $\overline{f}$ is a function and then the properties of an isomorphism are confirmed.

Exercise Set 9.4, page 357

1. No.

3. Yes; Abelian; identity is 0; a^{-1} is $-a$.

5. No. 7. No. 9. No.

11. Yes; Abelian; identity is $\{\ \}$; a^{-1} is a.

13. Since g_1, g_2, g_3 in S_3 each have order 2, they must be paired somehow with f_2, f_3, f_4 of Example 12 if the groups are isomorphic. But no rearrangement of the columns and rows labeled f_2, f_3, f_4 in Example 12 will give the "block" pattern shown by g_1, g_2, g_3 in the table for S_3. Hence the groups are not isomorphic.

15. **(a)** $\frac{8}{3}$. **(b)** $-\frac{4}{5}$.

17. $H_1 = \{1\}$, $H_2 = \{1, -1\}$, $H_3 = \{1, -1, i, -i\}$.

19.

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
f_1	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
f_2	f_2	f_3	f_4	f_1	f_8	f_7	f_5	f_6
f_3	f_3	f_4	f_1	f_2	f_6	f_5	f_8	f_7
f_4	f_4	f_1	f_2	f_3	f_7	f_8	f_6	f_5
f_5	f_5	f_7	f_6	f_8	f_1	f_3	f_2	f_4
f_6	f_6	f_8	f_5	f_7	f_3	f_1	f_4	f_2
f_7	f_7	f_6	f_8	f_5	f_4	f_2	f_1	f_3
f_8	f_8	f_5	f_7	f_6	f_2	f_4	f_3	f_1

21. Consider the sequence $e, a, a^2, a^3, \ldots$. Since G is finite, not all terms of this sequence can be distinct; that is, for some $i \leq j$, $a^i = a^j$. Then $(a^{-1})^i a^i = (a^{-1})^i a^j$ and $e = a^{j-i}$. Note that $j - i \geq 0$.

23. Yes.

25. Clearly, $e \in H$. Let $a, b \in H$. Consider $(ab)y = a(by) = a(yb) = (ay)b = (ya)b = y(ab)\ \forall y \in G$. Hence H is closed under multiplication and is a subgroup of G.

27. The identity permutation is an even permutation. If p_1 and p_2 are even permutations, then each can be written as the product of an even number of transpositions. Then $p_1 \circ p_2$ can be written as the product of these representations of p_1 and p_2. But this gives $p_1 \circ p_2$ as the product of an even number of transpositions. Thus $p_1 \circ p_2 \in A_n$ and A_n is a subgroup of S_n.

29. $\{f_1\}$, $\{f_1, f_2, f_3, f_4\}$, $\{f_1, f_3, f_5, f_6\}$, $\{f_1, f_3, f_7, f_8\}$, $\{f_1, f_5\}$, $\{f_1, f_6\}$, $\{f_1, f_3\}$, $\{f_1, f_7\}$, $\{f_1, f_8\}$, D_4.

31. $|xy| = |x| \cdot |y|$. Thus $f(xy) = f(x)f(y)$.

33. Suppose $f : G \to G$ defined by $f(a) = a^2$ is a homomorphism. Then $f(ab) = f(a)f(b)$ or $(ab)^2 = a^2b^2$. Hence $a^{-1}(abab)b^{-1} = a^{-1}(a^2b^2)b^{-1}$ and $ba = ab$. Suppose G is Abelian. By Exercise 37, $f(ab) = f(a)f(b)$.

35. Let $x, y \in G$. $f_a(xy) = axya^{-1} = axa^{-1}aya^{-1} = f_a(x)f_a(y)$. f_a is a homomorphism. Suppose $x \in G$. Then $f_a(a^{-1}xa) = aa^{-1}xaa^{-1} = x$ so f_a is onto. Suppose $f_a(x) = f_a(y)$, then $axa^{-1} = aya^{-1}$. Now $a^{-1}(axa^{-1})a = a^{-1}(aya^{-1})a$ and $x = y$. Thus f_a is one to one and an isomorphism.

37. (Outline) Basis step: $n = 1$ P(1): $(ab)^1 = a^1b^1$ is true.
Induction step: LHS of P($k + 1$): $(ab)^{k+1} = (ab)^k ab = a^k b^k ab = a^k a b^k b = a^{k+1}b^{k+1}$
RHS of P($k + 1$).

39. One possible table is

$*$	a	b	c
a	b	a	c
b	c	b	a
c	a	c	b

$*$ has no identity element.

Exercise Set 9.5, page 362

1.

	$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{1})$	$(\bar{0}, \bar{2})$	$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{1})$	$(\bar{1}, \bar{2})$
$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{1})$	$(\bar{0}, \bar{2})$	$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{1})$	$(\bar{1}, \bar{2})$
$(\bar{0}, \bar{1})$	$(\bar{0}, \bar{1})$	$(\bar{0}, \bar{2})$	$(\bar{0}, \bar{0})$	$(\bar{1}, \bar{1})$	$(\bar{1}, \bar{2})$	$(\bar{1}, \bar{0})$
$(\bar{0}, \bar{2})$	$(\bar{0}, \bar{2})$	$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{1})$	$(\bar{1}, \bar{2})$	$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{1})$
$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{1})$	$(\bar{1}, \bar{2})$	$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{1})$	$(\bar{0}, \bar{2})$
$(\bar{1}, \bar{1})$	$(\bar{1}, \bar{1})$	$(\bar{1}, \bar{2})$	$(\bar{1}, \bar{0})$	$(\bar{0}, \bar{1})$	$(\bar{0}, \bar{2})$	$(\bar{0}, \bar{0})$
$(\bar{1}, \bar{2})$	$(\bar{1}, \bar{2})$	$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{1})$	$(\bar{0}, \bar{2})$	$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{1})$

3. Define $f : G_1 \to G_2$ by $f((g_1, g_2)) = (g_2, g_1)$. By Exercise 4, Section 9.3, f is an isomorphism.

5.

	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

7. $\ker(f) = \{(e_1, g_2), e_1$, identity of $G_1, g_2 \in G_2\}$.

9. $\{[0]\}, \{[1]\}, \{[2]\}, \{[3]\}$.

11. $\{[0], [1], [2], [3]\}$.

13. The groups are isomorphic. Define $f : G \to Z_4$ by $f(1) = [0]$, $f(i) = [1]$, $f(-1) = [2]$, $f(-i) = [3]$. A comparison of the multiplication tables shows that f preserves the operation.

15. $\{f_1, g_3\}, \{f_2, g_2\}, \{f_3, g_1\}$.

17. $\{f_1\}, \{f_2\}, \{f_3\}, \{g_1\}, \{g_2\}, \{g_3\}$.

19. $\{[0], [4]\}, \{[1], [5]\}, \{[2], [6]\}, \{[3], [7]\}$.

21. $\{(m + x, n + x) \mid x \in Z\}$ for $(m, n) \in Z \times Z$.

23. If N is a normal subgroup of G, Exercise 22 shows that $a^{-1}Na \subseteq N$ for all $a \in G$.

Suppose $a^{-1}Na \subseteq N$ for all $a \in G$. Again the proof in Exercise 22 shows that N is a normal subgroup of G.

25. $\{f_1\}$, $\{f_1, f_3\}$, $\{f_1, f_3, f_5, f_6\}$, $\{f_1, f_2, f_3, f_4\}$, $\{f_1, f_3, f_7, f_8\}$, D_4.

27. Suppose $f_a(h_1) = f_a(h_2)$. Then $ah_1 = ah_2$ and $a^{-1}(ah_1) = a^{-1}(ah_2)$. Hence $h_1 = h_2$ and f_a is one to one. Let $x \in aH$. Then $x = ah$, $h \in H$ and $f_a(h) = x$. Thus f_a is onto and since it is everywhere defined as well, f_a is a one-to-one correspondence between H and aH. Hence $|H| = |aH|$.

29. Suppose $f(aH) = f(bH)$. Then $Ha^{-1} = Hb^{-1}$ and $a^{-1} = hb^{-1}$, $h \in H$. Hence $a = bh^{-1} \in bH$ so $aH \subseteq bH$. Similarly, $bH \subseteq aH$ so $aH = bH$. This means f is one to one. If Hc is a right coset of H, then $f(c^{-1}H) = Hc$ so f is also onto.

31. Consider $f(aba^{-1}b^{-1}) = f(a)f(b)f(a^{-1})f(b^{-1}) = f(a)f(a^{-1})f(b)f(b^{-1}) = f(a)(f(a))^{-1}f(b)(f(b))^{-1}$ (by Theorem 5, Section 9.4) $= ee = e$. Hence $\{aba^{-1}b^{-1} \mid a, b \text{ in } G_1\} \subseteq \ker(f)$.

33. Let $a \notin H$. The left cosets of H are H and aH. The right cosets are H and Ha. $H \cap aH = H \cap Ha = \{\ \}$ and $H \cup aH = H \cup Ha$. Thus $aH = Ha$. Since $a \in H \Rightarrow aH = H$, we have $xH = Hx\ \forall x \in G$. H is a normal subgroup of G.

35. Suppose $f : G \to G'$ is one to one. Let $x \in \ker(f)$. Then $f(x) = e' = f(e)$. Thus $x = e$ and $\ker(f) = \{e\}$. Conversely, suppose $\ker(f) = \{e\}$. If $f(g_1) = f(g_2)$, then $f(g_1g_2^{-1}) = f(g_1)f(g_2^{-1}) = f(g_1)(f(g_2))^{-1} = f(g_1)(f(g_1))^{-1} = e$. Hence $g_1g_2^{-1} \in \ker(f)$. Thus $g_1g_2^{-1} = e$ and $g_1 = g_2$. Hence f is one to one.

37. Since H is a subgroup of G, the identity element e belongs to H. For any $g \in G$, $g = g * e \in gH$, so every element of G belongs to some left coset of H. If aH and bH are distinct left cosets of H, this means that $aH \cap bH = \{\ \}$. Hence the set of distinct left cosets of H forms a partition of G.

Exercise Set 9.6, page 367

1. Noncommutative ring with identity.

3. Not a ring.

5. Commutative ring with identity.

7. This is a ring from Exercise 1. An example of zero divisors are the matrices

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}.$$

9. $\bar{1}, \bar{3}$

11. $\bar{1}, \bar{3}, \bar{7}, \bar{9}$

13. This subset is a subgroup with respect to $+$ since it contains the zero element and $\mathbf{A} - \mathbf{B}$ belongs to the subset if $\mathbf{A}$ and $\mathbf{B}$ do. This subset is a subsemigroup with respect to $*$ since if $\mathbf{A}$ and $\mathbf{B}$ are in the subset so is $\mathbf{A} * \mathbf{B}$.

15. The structures in Exercises 1 and 2 are not fields, because they lack multiplicative inverses. The structure in Exercise 3 is not a ring so it is not a field. The structures in Exercises 4, 5, and 6 are fields.

17. $\overline{55} * \overline{57} = \bar{1}$ in Z_{196}.

19. $(\bar{4}, \bar{0})$.

21. **(a)** $\bar{2}, \bar{3}$. **(b)** There are no solutions.

23. 9

25. **(a)** $(x \vee y) \wedge {\sim}(x \wedge y)$. **(b)** $x \wedge y$.

27. The set of units must contain all nonzero elements of R.

29. The statement Z_n is a field implies n is prime is proven in Exercise 28. The converse is shown by using Theorem 4(a), Section 1.4. Any $a \in Z_n$ is relatively prime to n so we have $1 = sa + tn$ for some integers s, t and $\bar{s}$ is the multiplicative inverse of a.

Review Questions, page 369

1. A set is closed with respect to a binary operation if using the operation with any two elements of the set yields an element that belongs to the set.

2. An isomorphism between two semigroups must preserve the operations, but an isomorphism between posets must preserve the orders. In each case the mapping must be a one-to-one correspondence and preserve the defining structure, multiplications for groups and orders for posets.

3. The relation must be reflexive, symmetric, and transitive and preserve the binary operations.

4. Groups are said to have more structure than semigroups, because they must satisfy more conditions.

5. A field must contain a multiplicative inverse for each nonzero element; this is not required in a ring.

Chapter 10

Exercise Set 10.1, page 379

1. $\{x^m y^n z, m \geq 0, n \geq 1\}$.

3. $\{a^{2^{n+1}}, n \geq 0\} \cup \{a^{2^n} b, n \geq 0\}$.

5. $\{\underbrace{((\ldots(}_{k}\underbrace{a + a + \cdots + a}_{n\ a\text{'s}}\underbrace{)\ldots)}_{k}, k \geq 0, n \geq 3\}$.

7. $\{x^m yz^n, m \geq 1, n \geq 0\}$.

9. (a), (c), (e), (h), (i).

11. $L(G)$ is the set of strings from $\{a, b, c, 1, 2, \ldots, 9, 0\}^*$ that begin with a, b, or c.

13. $L(G) = \{(aa)^n bc^k (bb)^j b^k, n \geq 0, k \geq 1, j \geq 0\}$.

15.

v_0	v_0
v_0v_1	v_0v_1
$v_0v_1v_1$	$v_2v_0v_1$
v_2v_0z	xyv_1
xyz	xyz

17.

I	I
LW	LW
aW	LDW
aDW	$LDDW$
$a1W$	$LDDD$
$a1DW$	$aDDD$
$a1DD$	$a1DD$
$a10D$	$a10D$
$a100$	$a100$

19. The languages are not the same; here $L(G)$ contains $aabba$.

21. $G = (V, S, v_0, \mapsto)$, $V = \{v_0, v_1, 0, 1\}$, $S = \{0, 1\}$
$\mapsto$: $v_0 \mapsto 0v_11$, $v_0 \mapsto 1v_10$, $v_1 \mapsto 0v_11$, $v_1 \mapsto 1v_10$, $v_1 \mapsto 01$, $v_1 \mapsto 10$.

23. $G = (V, S, v_0, \mapsto)$, $V = \{v_0, v_1, a, b\}$, $S = \{a, b\}$
$\mapsto$: $v_0 \mapsto aav_1bb$, $v_1 \mapsto av_1b$, $v_1 \mapsto ab$.

25. $G = (V, S, v_0, \mapsto)$, $V = \{v_0, x, y\}$, $S = \{x, y\}$
$\mapsto$: $v_0 \mapsto v_0yy$, $v_0 \mapsto xv_0$, $v_0 \mapsto xx$.

27. $v_0 \mapsto av_0a \quad v_0 \mapsto bv_0b \quad v_0 \mapsto a$
$v_0 \mapsto aav_0 \mapsto b \quad v_0 \mapsto bb$.

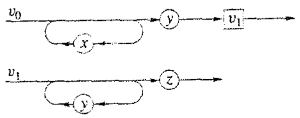
29. By using production rules 1, 2, and 4; production rule 4 to $a^{3k-1}v_2$; 3; $a^{3k}av_3 \Rightarrow a^{3k+1}v_1 \Rightarrow a^{3k+2}v_2 \Rightarrow a^{3(k+1)}$.

31. Let $G_1 = (V_1, S_1, v_0, \mapsto_1)$ and $G_2 = (V_2, S_2, v_0', \mapsto_2)$. Define $G = (V_1 \cup V_2, S_1 \cup S_2, v_0, \mapsto)$ as follows. If $v_i \mapsto_1 wv_k$, then $v_i \mapsto wv_k$. If $v_i \mapsto_1 w$ where w consists of terminal symbols, then $v_i \mapsto wv_0'$. All productions in $\mapsto_2$ become productions in $\mapsto$.

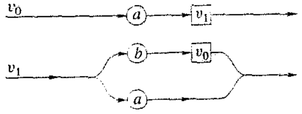
33. $v_0 \mapsto av_0 \quad v_1 \mapsto av_1$
$v_0 \mapsto bv_1 \quad v_1 \mapsto a$
The language is in fact regular.

Exercise Set 10.2, page 389

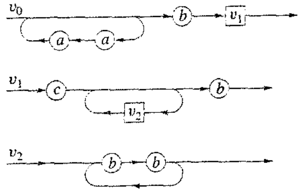
1. $\langle v_0\rangle ::= x\langle v_0\rangle \mid y\langle v_1\rangle$
$\langle v_1\rangle ::= y\langle v_1\rangle \mid z$



3. $\langle v_0\rangle ::= a\langle v_1\rangle$
$\langle v_1\rangle ::= b\langle v_0\rangle \mid a$



5. $\langle v_0\rangle ::= aa\langle v_0\rangle \mid b\langle v_1\rangle$
$\langle v_1\rangle ::= c\langle v_2\rangle b \mid cb$
$\langle v_2\rangle ::= bb\langle v_2\rangle \mid bb$



7. $\langle v_0\rangle ::= x\langle v_0\rangle \mid y\langle v_0\rangle \mid z$

9. $\langle v_0\rangle ::= a\langle v_1\rangle$
$\langle v_1\rangle ::= b\langle v_0\rangle \mid a$

11. $\langle v_0\rangle ::= ab\langle v_1\rangle$
$\langle v_1\rangle ::= c\langle v_1\rangle \mid \langle v_2\rangle$
$\langle v_2\rangle ::= dd\langle v_2\rangle \mid d$

13. $\langle v_0\rangle ::= a\langle v_1\rangle$
$\langle v_1\rangle ::= a\langle v_2\rangle$
$\langle v_2\rangle ::= a\langle v_1\rangle \mid a$

15. $\langle v_0\rangle ::= b\langle v_0\rangle \mid a\langle v_1\rangle \mid b$
$\langle v_1\rangle ::= a\langle v_0\rangle \mid b\langle v_1\rangle \mid a$

17. $(aa)^*aa$.

19. $(()^*(a+a+(a+)^*a())^*$. Note: Right and left parentheses must be matched.

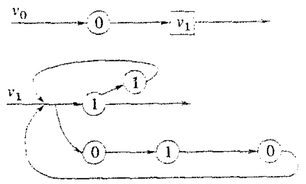
21. $(a \vee b \vee c)(a \vee b \vee c \vee 0 \vee 1 \vee \cdots \vee 9)^*$.

23. $ab(d \vee (d(c \vee d)d))^*$.

25. $ab(abc)^n b, n \geq 1$.

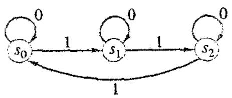
27. $(aab \vee ab)^*$.

29.

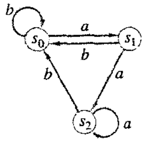


Exercise Set 10.3, page 394

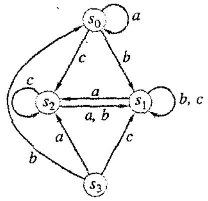
1.



3.



5.



7.

	a	b
s_0	s_1	s_1
s_1	s_1	s_2
s_2	s_0	s_2

9.

	T	F
s_0	s_1	s_0
s_1	s_1	s_1
s_2	s_1	s_2

11.

	a	b	c
s_0	s_0	s_1	s_2
s_1	s_2	s_1	s_3
s_2	s_3	s_3	s_1
s_3	s_3	s_3	s_2

13. Let $x \in I$. Certainly $f_x(s) = f_x(s)$ for all $s \in S$. Thus $x\ R\ x$ and R is reflexive.
Suppose $x\ R\ y$. Then $f_x(s) = f_y(s)\ \forall s \in S$. But then $y\ R\ x$ and R is symmetric.
Suppose $x\ R\ y$, $y\ R\ z$. Then $f_x(s) = f_y(s) = f_z(s)$, $\forall s \in S$. Hence $x\ R\ z$ and R is transitive.

15. Using Exercise 14, we need only show that R is reflexive and symmetric. Let $s \in S$. $s = e * s$ so $f_e(s) = s$ and $s\ R\ s$. Suppose $x\ R\ y$. Then $f_z(x) = y$ for some $z \in S$. $y = z * x \Rightarrow z^{-1} * y = x$ and thus $f_{z^{-1}}(y) = x$. Hence $y\ R\ x$ and R is symmetric.

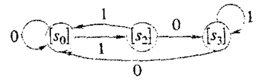
17. **(a)** Inspection of $\mathbf{M}_R$ shows that R is reflexive and symmetric. Since $\mathbf{M}_R \odot \mathbf{M}_R = \mathbf{M}_R$, R is transitive. Thus R is an equivalence relation. The table below shows that it is a machine congruence.

(b)

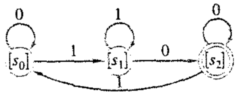
	0	1
[1]	[1]	[1]
[2]	[2]	[2]

19. **(a)** R is clearly an equivalence relation. $s_0\ R\ s_1$ and $f_0(s_0) = s_1$, $f_0(s_1) = s_0$, $f_1(s_0) = s_2 = f(s_1)$ so $f_x(s_0)\ R\ f_x(s_1), \forall x \in I$.

(b)



21. Inspection of $\mathbf{M}_R$ shows that R is reflexive and symmetric. Since $\mathbf{M}_R \odot \mathbf{M}_R = \mathbf{M}_R$, R is transitive. Thus R is an equivalence relation. The digraph below shows that it is a machine congruence.



23. f_x, $x \in I$, is a function on S so f_x is a state transition function. $T \subseteq S$ so the conditions for a Moore machine are met.

25.

	0	1
(s_0, s_0)	(s_0, s_0)	(s_1, s_1)
(s_0, s_1)	(s_0, s_2)	(s_1, s_1)
(s_0, s_2)	(s_0, s_2)	(s_1, s_3)
(s_0, s_3)	(s_0, s_3)	(s_1, s_3)
(s_1, s_0)	(s_1, s_0)	(s_2, s_1)
(s_1, s_1)	(s_1, s_2)	(s_2, s_1)
(s_1, s_2)	(s_1, s_2)	(s_2, s_3)
(s_1, s_3)	(s_1, s_3)	(s_2, s_3)
(s_2, s_0)	(s_2, s_0)	(s_0, s_1)
(s_2, s_1)	(s_2, s_2)	(s_0, s_1)
(s_2, s_2)	(s_2, s_2)	(s_0, s_3)
(s_2, s_3)	(s_2, s_3)	(s_0, s_3)

Exercise Set 10.4, page 399

1. $f_w(s_0) = s_2$, $f_w(s_1) = s_3$, $f_w(s_2) = s_0$, $f_w(s_3) = s_1$.

3. The number of 1's in w is divisible by 4.

5. The number of 1's in w is $2 + 4k$, $k \geq 0$.

7. $f_w(s_0) = s_0$, $f_w(s_1) = s_0$, $f_w(s_2) = s_0$.

9. All words ending in b.

11. Strings of 0's and 1's with $3 + 5k$ 1's. $k \geq 0$.

13. Strings of 0's and 1's that end in 0.

15. Strings of a's and b's that do not contain bb.

17. Strings of 0's and 1's that end in 01.

19. Strings xy and yz.

21. Strings that begin 1 or 00.

23. Let $w, u \in L(M)$. Then $f_w(s_0) = s \in T$, $f_u(s) \in T$. Hence $f_{w \cdot u}(s_0) \in T$, so $w \cdot u \in L(M)$.

25. $w \in L(M') \cup L(M'')$ if and only if $w \in L(M')$ or $w \in L(M'')$. Say $w \in L(M'')$. But $w \in L(M'')$ if and only if $f_w''(s_0'') \in T''$ if and only if $f_w(s_0', s_0'') = (f_w'(s_0'), f_w''(s_0'')) \in T$.

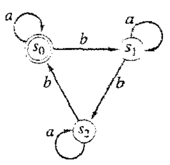
Exercise Set 10.5, page 406

1. $G = (V, I, s_0, \mapsto)$, $V = \{s_0, s_1, s_2, s_3, 0, 1\}$, $I = \{0, 1\}$
$\mapsto : s_0 \mapsto 0s_0, s_0 \mapsto 1s_1, s_1 \mapsto 0s_1, s_1 \mapsto 1s_2, s_2 \mapsto 0s_2, s_2 \mapsto 1s_3, s_2 \mapsto 1, s_3 \mapsto 0s_3, s_3 \mapsto 0, s_3 \mapsto 1s_0$.

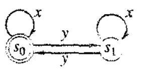
3. $(0 \vee 1)^*1$.

5. $G = (V, I, s_0, \mapsto)$, $V = \{s_0, s_1, s_2, a, b\}$, $I = \{a, b\}$
$\langle s_0\rangle ::= a\langle s_0\rangle \mid b\langle s_1\rangle \mid a \mid b$
$\langle s_1\rangle ::= a\langle s_0\rangle \mid b\langle s_2\rangle \mid a$
$\langle s_2\rangle ::= a\langle s_2\rangle \mid b\langle s_2\rangle$

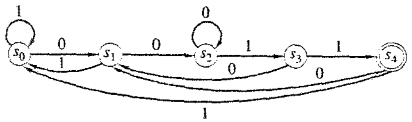
7.



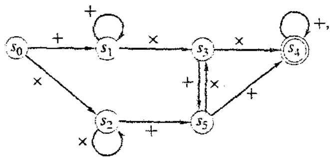
9.



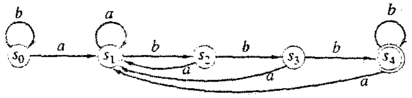
11.



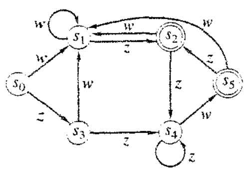
13.



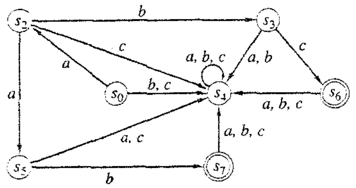
15.



17.



19.



21.

	0	1
s_0	s_1	s_0
s_1	s_1	s_2
s_2	s_3	s_0
s_3	s_1	s_4
s_4	s_3	s_0

$T = \{s_4\}$

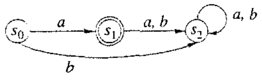
23.

	x	y
s_0	s_1	s_0
s_1	s_2	s_1
s_2	s_3	s_2
s_3	s_3	s_3

$T = \{s_2\}$

25. R is reflexive because $f_w(x) = f_w(x)$. R is symmetric because if $f_w(s_i)$, $f_w(s_j)$ are both (not) in T, then $f_w(s_j)$, $f_w(s_i)$ are both (not) in T. R is transitive because $s_i\ R\ s_j$, $s_j\ R\ s_k$ if and only if $f_w(s_i)$, $f_w(s_j)$, $f_w(s_k)$ are all in (or not in) T.

27.



29. The construction of Exercise 24 of Section 10.3 gives a machine that accepts $\alpha \vee \beta$.

Exercise Set 10.6, page 411

1. $R_0 = \{(s_0, s_0), (s_0, s_1), (s_1, s_0), (s_1, s_1), (s_2, s_2)\}$.

3. $R_1 = \{(s_0, s_0), (s_1, s_1), (s_2, s_2), (s_3, s_3),$
$(s_4, s_4), (s_0, s_3), (s_3, s_0), (s_1, s_2), (s_2, s_1)\}$.

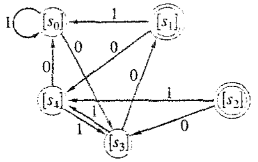
5. $R_{127} = R_1$.

7. $R_2 = R_1$.

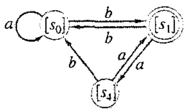
9. $R = \{(s_0, s_0), (s_1, s_1), (s_2, s_2)\}$.

11. $R = R_1$ as given in Exercise 6.

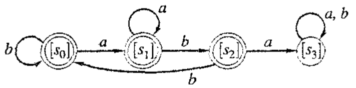
13. $R = \{(s_0, s_0), (s_1, s_1), (s_2, s_2), (s_3, s_3), (s_4, s_4), (s_4, s_5), (s_5, s_4), (s_5, s_5), (s_6, s_6), (s_3, s_6), (s_6, s_3)\}$.



15.



17.

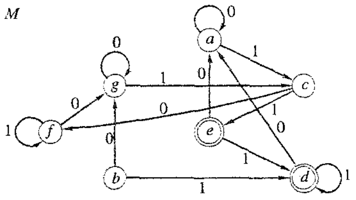


19. $P = \{\{a, g\}, \{f\}, \{b\}, \{c\}, \{d, e\}\}$.

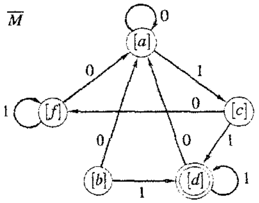
	0	1
$[a]$	$[a]$	$[c]$
$[b]$	$[a]$	$[d]$
$[c]$	$[f]$	$[d]$
$[d]$	$[a]$	$[d]$
$[f]$	$[a]$	$[f]$

21. The equivalence classes each contain a single state, so the table is essentially that given in the statement of the exercise.

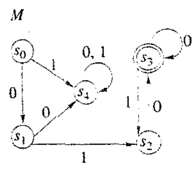
23. M



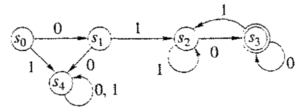
$\overline{M}$



25. M



$\overline{M}$



Review Questions, page 413

1. No, type 0 and type 1 languages cannot be given in BNF.

2. Some type 2 and type 3 grammars can be represented by a syntax diagram.

3. This language is type 2, context-sensitive, but importantly it is not regular.

4. A machine accepts or recognizes a language if and only if every string in the language leads from the starting state to an acceptance state: that is, $f_w(s_0)$ is an acceptance state.

5. A language L is regular if and only if $L = L(M)$ for some Moore machine M.

Chapter 11

Exercise Set 11.1, page 426

1. **(a)** 3. **(b)** 2. **(c)** 3.

3. **(a)** Yes. **(b)** No.

5. **(a)** No. **(b)** No. **(c)** Yes. **(d)** No.

7. **(a)** Yes. **(b)** Yes.

9. Let $a_1a_2\ldots a_{10}$ be an ISBN. Suppose a_i and a_{i+1} are distinct and are transposed in transmission. Consider $a_1+2a_2+3a_3+\cdots+ia_i+(i+1)a_{i+1}+\cdots+10a_{10}-(a_1+2a_2+3a_3+\cdots+ia_{i+1}+(i+1)a_i+\cdots+10a_{10}) = a_{i+1}-a_i$. Because $0 \le a_i, a_{i+1} \le 9$, $a_{i+1} - a_i$ cannot be congruent to 0 (mod 11) and neither can $a_1+2a_2+3a_3+\cdots+ia_{i+1}+(i+1)a_i+\cdots+10a_{10} = a_1+2a_2+3a_3+\cdots+ia_i+(i+1)a_{i+1}+\cdots+10a_{10}-(a_{i+1}-a_i)$. Hence an error would be detected.

11. By definition, $x \oplus y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ and $x_i + y_i = 1$ if and only if $x_i \neq y_i$.

13. **(a)** 2. **(b)** 6.

15. If $x = y$, they differ in 0 positions and $\delta(x, y) = 0$. Conversely, if $\delta(x, y) = 0$, then x and y cannot differ in any position and $x = y$.

17. 1.

19. **(a)** 3. **(b)** 2 or fewer.

21. Let $a = 0000000$, $b = 0010110$, $c = 0101000$, $d = 0111110$, $e = 1000101$, $f = 1010011$, $g = 1101101$, $h = 1111011$.

$\oplus$	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	b	a	d	c	f	e	h	g
c	c	d	a	b	g	h	e	f
d	d	c	b	a	h	g	f	e
e	e	f	g	h	a	b	c	d
f	f	e	h	g	b	a	d	c
g	g	h	e	f	c	d	a	b
h	h	g	f	e	d	c	b	a

23. 2.

25. $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

27. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

29. **(a)** Yes. **(b)** No. **(c)** Yes. **(d)** No.

31. $e_H(000) = 000000$ $e_H(100) = 100100$
$e_H(001) = 001111$ $e_H(101) = 101011$
$e_H(010) = 010011$ $e_H(110) = 110111$
$e_H(011) = 011100$ $e_H(111) = 111000$

33. The statement of Theorem 2 is of the form $p \Leftrightarrow q$. Part one of the proof shows that $q \Rightarrow p$. The second part establishes $p \Rightarrow q$ by showing $\sim q \Rightarrow \sim p$ (the contrapositive).

Exercise Set 11.2, page 435

1. **(a)** 011. **(b)** 101.

3. **(a)** 11. **(b)** 01.

5.

00	01	10	11
0000	1011	0110	1101
1000	0011	1110	0101
0100	1111	0010	1001
0001	1010	0111	1100

7. 0. **9.** 0. **11.** 1.

13. **(a)** 01. **(b)** 11. **(c)** 10.

15. Possible answers: **(a)** 010. **(b)** 110. **(c)** 001.

17. **(a)** Suppose $x \in g_1H \cap g_2H$. Then $x = g_1h_1 = g_2h_2$, for some $h_1, h_2 \in H$. We have $g_1 = g_2h_2h_1^{-1} \in g_2H$ since H is a subgroup. Hence $g_1H \subseteq g_2H$. Similarly, $g_2 = g_1h_1h_2^{-1}$ and $g_2H \subseteq g_1H$. Thus, $g_1H = g_2H$.
    (b) Each element g of G belongs to gH. Part (a) guarantees that there is a set of disjoint left cosets whose union is G.

19. Since we want errors to be detected, no code word should have weight 1. If $1 + n \ge 2^{n-m}$, coset leaders of weight ≤ 1 can be chosen.

21. 00000, 00001, 00010, 00100, 01000, 10000, 01010 (or 10100), 00110 (or 11000).

23. 00, 01, 10, 11. Same order as in Exercise 20.

25. 000, 001, 010, 100, 011, 110, 011, 111. Same order as in Exercise 22.

27. Possible answers: **(a)** 00. **(b)** 01. **(c)** 10.

29. $e(01) = 01110$.

- **(a)** Suppose 01010 is received when 01 is sent. This string is in the column headed 01 so it will be decoded correctly.
- **(b)** Suppose 01011 is received when 01 is sent. This string is in the column headed 11 so it will be decoded incorrectly.

Exercise Set 11.3, page 440

1. Since $391 = 17 \cdot 23$ and 12 is relatively prime to 391, $12^{702} = 12^{2(17-1)(23-1)} \equiv 1 \pmod{391}$.

3. 87. **5.** 211.

7. **(a)** $m = 943$, $n = 880$. **(b)** 601.

9. ACED. **11.** 507. **13.** 11463.

15. 6095. **17.** 151, 131. **19.** CAN.

Review Questions, page 441

1. An (m, n) encoding function can detect at most (minimum distance minus 1) errors.
2. Every possible string that could be received must belong to exactly one coset.
3. A maximum likelihood decoding function will choose the most likely original word by choosing one that produces a string closest to that received.
4. Messages are encoded for efficiency, error-detection, and security purposes.

Appendix A, page 452

1.
```
FUNCTION TAX (INCOME)
1. IF (INCOME ≥ 30,000) THEN
     a. TAXDUE ← 6000
2. ELSE
     a. IF (INCOME ≥ 20,000) THEN
          1. TAXDUE ← 2500
     b. ELSE
          1. TAXDUE ← INCOME × 0.1
3. RETURN (TAXDUE)
END OF FUNCTION TAX
```

3.
```
1. SUM ← 0
2. FOR I = 1 THRU N
     a. SUM ← SUM + X[I]
3. AVERAGE ← SUM/N
```

5.
```
1. DOTPROD ← 0
2. FOR I = 1 THRU 3
     a. DOTPROD ← DOTPROD +
(X[I])(Y[I])
```

7.
```
1. RAD ← $(A[2])^2$ − 4(A[1])(A[3])
2. IF (RAD < 0) THEN
     a. PRINT ('ROOTS ARE IMAGINARY')
3. ELSE
     a. IF (RAD = 0) THEN
          1. R1 ← −A[2]/(2A[1])
          2. PRINT ('ROOTS ARE REAL
                 AND EQUAL')
     b. ELSE
          1. R1 ← (−A[2] +
SQ(RAD))/(2A[1])
          2. R2 ← (−A[2] −
SQ(RAD))/(2A[1])
```

9.
```
1. FOR I = 1 THRU N
     a. IF (A[I] ≠ B[I]) THEN
          1. C[I] ← 1
     b. ELSE
          1. C[I] ← 0
```

11.
```
1. FOR I = 1 THRU N
     a. IF (A[I] = 0 AND B[I] = 0)
THEN
          1. C[I] ← 1
     b. ELSE
          1. C[I] ← 0
```

13.
```
1. SUM ← 0
2. FOR I = 0 THRU 2(N − 1) BY 2
     a. SUM ← SUM + I
```

15.
```
1. PROD ← 1
2. FOR I = 2 THRU 2N BY 2
     a. PROD ← (PROD) × I
```

17.
```
1. SUM ← 0
2. FOR I = 1 THRU 77
     a. SUM ← SUM + $I^2$
```

19.
```
1. SUM ← 0
2. FOR I = 1 THRU 10
     a. SUM ← SUM + (1/(3I + 1))
```

21. MAX returns the larger of X and Y.

23. F returns $|X|$.

25. Assigns 1 to R if $N \mid M$ and assigns 0 otherwise.

27. $X = \sum_{I=1}^{N} I$; I is $N + 1$.

29. $X = 25$; $I = 49$.

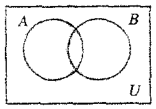
Answers to Chapter Self-Tests

CHAPTER 1 SELF-TEST, page 47

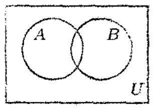
1. **(a)** (i) False. (ii) True. (iii) False.
(iv) True. (v) True.
(b) (i) False. (ii) True. (iii) False.
(iv) False. (v) True.

2. **(a)** $\{1, 2, 3, 4, \ldots\}$. **(b)** $\{\ldots, -3, -2, -1, 0\} \cup B$.
(c) $\{2, 4\}$. **(d)** A.
(e) $\{2, 6, 10, 14, \ldots\}$.

3. **(a)**



(b)



4. $A \cap B$ is always a subset of $A \cup B$. $A \cup B \subseteq A \cap B$ if and only if $A = B$.

5. **(a)** 6. **(b)** 18. **(c)** 41.

6. 0, 0, 1, −2, 9, −30.

7.

(a)	1	1	1	1	0	1	0	1	0	1
(b)	1	1	0	0	0	0	0	0	0	0
(c)	1	1	0	0	0	1	0	0	0	0
(d)	0	0	1	0	0	0	1	1	0	1

8. **(a)** Yes. **(b)** Yes. **(c)** No. **(d)** No.

9. $33 = 65(7293) - 108(4389)$.

10. **(a)** **AB** does not exist.

(b) $\mathbf{BA} = \begin{bmatrix} 4 & 12 & 8 \\ -7 & -15 & -10 \end{bmatrix}$.

(c) $\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$. **(d)** $\mathbf{A} + \mathbf{B}$ does not exist.

(e) $\begin{bmatrix} 7 & -1 \\ 3 & 3 \\ 2 & 2 \end{bmatrix}$. **(f)** $\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{3}{2} & 1 \end{bmatrix}$.

(g) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 12 & 8 \end{bmatrix}$.

11. **(a)** $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. **(b)** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

12. **A** has a $\wedge$-inverse if and only if $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, the $\wedge$-identity.

CHAPTER 2 SELF-TEST, page 75

1. **(a)** False. **(b)** True.

2. **(a)** False. **(b)** True.

3.

p	q	r	$(p$	$\wedge$	$\sim p)$	$\vee$	$(\sim$	$(q$	$\wedge$	$r))$
T	T	T		F		F	F		T	
T	T	F		F		T	T		F	
T	F	T		F		T	T		F	
T	F	F		F		T	T		F	
F	T	T		F		F	F		T	
F	T	F		F		T	T		F	
F	F	T		F		T	T		F	
F	F	F		F		T	T		F	
				(1)		↑	(3)		(2)	

4. **(a)** $q \Rightarrow p \quad \sim q \Rightarrow \sim p$.
(b) $q \Rightarrow (\sim r) \vee (\sim s) \quad \sim q \Rightarrow (r \wedge s)$.
(c) $(p \vee s) \Rightarrow q \quad (\sim p \wedge \sim s) \Rightarrow \sim q$.

5. **(a)** If $|2| = |-2|$, then $1 < -1$.
If $|2| \neq |-2|$, then $1 \geq -1$.
(b) If $|2| = |-2|$, then either $-3 \geq -1$ or $1 \geq 3$.
If $|2| \neq |-2|$, then $-3 < -1$ and $1 < 3$.
(c) If $1 < -1$ or $1 < 3$, then $|2| = |-2|$.
If $1 \geq -1$ and $1 \geq 3$, then $|2| \neq |-2|$.

6.

(a)	False	True
(b)	False	True
(c)	True	True

7.

p	q	p *xor* q
T	T	F
T	F	T
F	T	T
F	F	F

8. **(a)** If an Internet business makes less money, then if I start an Internet business, then an Internet business is cheaper to start. An Internet business is not cheaper to start. Therefore, either an Internet business does not make less money or I do not start an Internet business. Valid.

(b) If an Internet business is cheaper to start, then I will start an Internet business. If I start an Internet business, then an Internet business will make less money. An Internet business is cheaper to start. Therefore, an Internet business will make less money. Valid.

9. No, consider $6 \mid -6$ and $-6 \mid 6$. If both m and n are positive (or negative), then $n \mid m$ and $m \mid n$ guarantees that $n = m$.

10. Consider 7, 9, and 11. These are three consecutive odd integers whose sum is not divisible by 6.

11. Basis Step: $n = 0$. P(0): $4^0 - 1$ is divisible by 3 is true since $3 \mid 0$.
Induction Step: We use P(k): 3 divides $4^k - 1$ to show P($k+1$): 3 divides $4^{k+1} - 1$. Consider $4^{k+1} - 1 = 4(4^k - 1) + 3$. By P($k$), $3 \mid (4^k - 1)$ and we have $4^{k+1} - 1 = 3(a + 1)$ where $a = 4^k - 1$. So $3 \mid (4^{k+1} - 1)$.

12. Basis Step: $n = 1$. P(1): $1 < \dfrac{(1+1)^2}{2}$ is true.
Induction Step: We use P(k):

$$1 + 2 + 3 + \cdots + k < \frac{(k+1)^2}{2}$$

to show P($k+1$):

$$1 + 2 + \cdots + (k+1) < \frac{(k+2)^2}{2}.$$

LHS of P($k+1$):

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &< \frac{(k+1)^2}{2} + (k+1) \\ &= \frac{k^2 + 4k + 3}{2} \\ &< \frac{k^2 + 4k + 4}{2} \\ &= \frac{(k+2)^2}{2} \end{aligned}$$

the RHS of P($k+1$).

CHAPTER 3 SELF-TEST, page 106

1. **(a)** 32. **(b)** 2,598,960. **(c)** 20,160.

2. **(a)** 165,765,600. **(b)** 118,404,000.

3. **(a)** 7776. **(b)** 216.

4. 560.

5. 15,173,928.

6.
$$\begin{aligned} 2 \cdot \frac{n!}{(n-2)!\,2!} + n^2 &= \frac{2 \cdot n! + 2!\,(n-2)!\,n^2}{(n-2)!\,2!} \\ &= \frac{(n-2)!\,n(n-1+n)}{(n-2)!} \\ &= n(2n-1) \cdot \frac{(2n-2)!}{(2n-2)!} \cdot \frac{2}{2} \\ &= \frac{(2n)!}{2!\,(2n-2)!} \\ &= {}_{2n}C_2. \end{aligned}$$

7. $\lfloor \frac{50}{8} \rfloor + 1 = 7$ pieces, assuming the pepperoni slices are not cut.

8. At least two months must begin on the same day of the week. Let the seven days of the week be the pigeonholes and the twelve months of the year, the pigeons. Then by the pigeonhole principle, at least $\lfloor \frac{11}{7} \rfloor + 1$, or 2, months begin on the same day of the week.

9. $\frac{10}{32}$.

10. No. $p(A \cap B) = p(A) + p(B) - p(A \cup B) = 0.29 + 0.41 - 0.65 = 0.05$.

11. $b_n = -3^n + 4^n$.

12. $a_n = m^{n-1}a_1 - m^{n-2} - m^{n-3} - \cdots - m^2 - 1 = m^n - \dfrac{m^{n-1} - 1}{m - 1}$.

CHAPTER 4 SELF-TEST, page 165

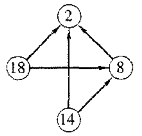
1. **(a)** 12.

(b) $\{(2,1), (2,2), (2,3), (2,4), (5,1), (5,2), (5,3), (5,4), (7,1), (7,2), (7,3), (7,4)\}$.

2. Let $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$, $B = \{2, 3\}$. Then $(2, 5) \in \overline{A \times B}$, but $(2, 5) \notin \bar{A} \times \bar{B}$.

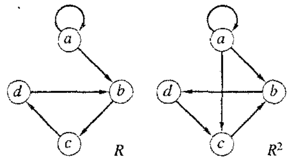
3. $\{\{a,b,c\},\{d,e\}\}, \{\{a,b,d\},\{c,e\}\}, \{\{a,b,e\},\{c,d\}\}, \{\{b,c,d\},\{a,e\}\}, \{\{b,c,e\},\{a,d\}\}, \{\{b,d,e\},\{a,c\}\}, \{\{c,d,e\},\{a,b\}\}, \{\{a,c,d\},\{b,e\}\}, \{\{a,c,e\},\{b,d\}\}, \{\{a,d,e\},\{b,c\}\}, \{\{a,b,c,d\},\{e\}\}, \{\{a,b,d,e\},\{c\}\}, \{\{b,c,d,e\},\{a\}\}, \{\{a,b,c,e\},\{d\}\}, \{\{a,c,d,e\},\{b\}\}$.

4. **(a)**



(b) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$.

5. **(a)**



(b) $\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

$\mathbf{M}_{R^2} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

(c) $\mathbf{M}_{R^\infty} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

6. Reflexive, not irreflexive, not symmetric, not asymmetric, not antisymmetric, not transitive.

7. Not reflexive, not irreflexive, not symmetric, not asymmetric, antisymmetric, not transitive.

8. Since $a\ R\ b$ implies $b \not R\ a$, $a\ R\ b \wedge b\ R\ a$ is always false. Hence $a\ R\ b \wedge b\ R\ a \Rightarrow a = b$ is always true. R must be antisymmetric.

9. **(a)** $(u, v)\ R\ (u, v)$ since $u - v = u - v$ is true. Thus R is reflexive. If $(u, v)\ R\ (x, y)$, then $u - v = x - y$ and $(x, y)\ R\ (u, v)$. Thus R is symmetric.
If $((u, v)\ R\ (x, y)) \wedge ((x, y)\ R\ (w, z))$, then $u - v = x - y = w - z$ and $(u, v)\ R\ (w, z)$. Hence, R is transitive.

(b) $[(2,3)] = \{(2,3), (1,2), (3,4), (4,5)\}$.

(c) $A/R = \{[(2,3)], [(2,4)], [(2,5)], [(2,2)], [(2,1)], [(3,1)], [(4,1)], [(5,1)], [(1,5)]\}$.

10. $\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

11. **(a)** $R^{-1} = \{(a,a), (a,e), (b,a), (c,a), (c,b), (c,d), (e,c)\}$.

(b) $R \circ S = \{(a,a), (a,b), (a,c), (b,b), (b,a), (b,c), (c,e), (c,c), (d,a), (e,c)\}$.

12. $\mathbf{M}_{R^\infty} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

CHAPTER 5 SELF-TEST, page 197

1. **(a)** Yes, $|R(x)| = 1$, $x \in A$.

(b) No, $(1, b), (1, d) \in R^{-1}$.

2. Suppose $f(a) = f(b)$, then $-5a + 8 = -5b + 8$. But then $a = b$ so f is one to one. Let $r \in \mathbb{R}$. Then

$$f\left(\frac{r-8}{-5}\right) = -5\left(\frac{r-8}{-5}\right) + 8 = r - 8 + 8 = r.$$

So f is onto.

3. **(a)** 16. **(b)** −2.

4. **(a)** 17. **(b)** −1.

5. **(a)** 0. **(b)** 6.

6. True; false.

7. **(a)** 89. **(b)** 107. **(c)** 30; 4; 88.

8. $2n^2 + 9n + 5 \leq 2n^2 + n^2 + n^2$, $n \geq 9$. Choose 4 for c and 9 for k. Then $|2n^2 + 9n + 5| \leq 4|n^2|$, $n \geq 9$.

9. $\Theta(2^n)$.

10. $f(N) = 2 + 5\left\lfloor \frac{N}{2} \right\rfloor + 1$; $\Theta(n)$.

11. **(a)** $(1, 4, 5) \circ (2, 3, 6)$ $(1, 5) \circ (1, 4) \circ (2, 6) \circ (2, 3)$.

(b) $(2, 6, 3) \circ (1, 5, 4)$.

12. **(a)** $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 2 & 3 & 7 & 4 & 1 & 6 \end{pmatrix}$.

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 2 & 5 & 6 & 7 & 1 \end{pmatrix}$.

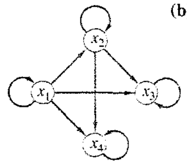
(c) $p_1 = (1, 7, 6, 5, 4) \circ (2, 3) = (1, 4) \circ (1, 5) \circ (1, 6) \circ (1, 7) \circ (2, 3)$ odd.

CHAPTER 6 SELF-TEST, page 249

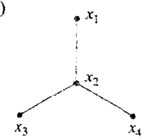
1. **(a)** R is reflexive, antisymmetric, and transitive. Hence, R is a partial order on A.

(b) R is reflexive and transitive, but not antisymmetric. R is not a partial order.

2. **(a)**



(b)



3. **(a)** Minimal: d, e maximal: a.

(b) Least: none greatest: a.

4. **(a)** Upper bounds: 12, 24, 48.

(b) Lower bounds: 2.

(c) LUB(B) = 12. **(d)** GLB(B) = 2.

5. **(a)** $a \wedge (b \vee c) = \begin{cases} a & \text{if } a \leq b \vee c \text{ or } a \leq b, c \quad (1) \\ b \vee c & \text{if } b \vee c \leq a \text{ or } b, c \leq a \quad (2) \end{cases}$

Thus, $(a \wedge b) \vee (a \wedge c) = \begin{cases} a \vee a \text{ or } a & (1) \\ b \vee c & (2) \end{cases}$

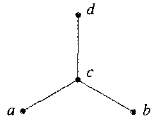
(b) $a \vee (b \wedge c) = \begin{cases} a & \text{if } (b \wedge c) \leq a \text{ or } b, c \leq a \quad (3) \\ b \wedge c & \text{if } a \leq (b \wedge c) \text{ or } a \leq b, c \quad (4) \end{cases}$

Thus, $(a \vee b) \wedge (a \vee c) = \begin{cases} a \wedge a \text{ or } a & (3) \\ b \wedge c & (4) \end{cases}$

6. $1' = 105$; $3' = 35$; $5' = 21$; $7' = 15$; $15' = 7$; $21' = 5$; $35' = 3$; $105' = 1$.

7. **(a)** R is reflexive, because the main diagonal of $\mathbf{M}_R$ is all ones. R is antisymmetric, because if $m_{ij} = 1$, then $m_{ji} = 0$. $\mathbf{M}_{R^2} = \mathbf{M}_R$, so R is transitive.

(b)

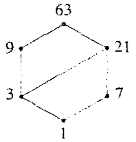


8. Since $a \leq b \leq b \vee d$ and $c \leq d \leq b \vee d$, we have $a \vee c \leq b \vee d$. ($a \vee c$ is the LUB of a and c.) Also, $a \wedge c \leq a \leq b$ and $a \wedge c \leq c \leq d$ since $a \wedge c$ is the GLB of a and c. Thus, $a \wedge c \leq b \wedge d$.

9. **(a)** (1) is not a lattice; $b \vee c$ does not exist.
(2), (3), and (4) are lattices.

(b) (1), (2), and (3) are not Boolean algebras; the number of vertices is not a power of 2. (4) is B_3.

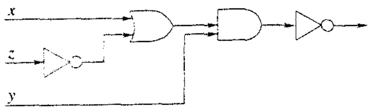
10. **(a)**



(b) D_{63} is not a Boolean algebra; there are 6 elements.

11. **(a)** $((x \wedge y) \vee (y \wedge z'))'$. **(b)** $(y \wedge (x \vee z'))'$.

(c)



12. $(x \wedge y') \vee (y \wedge z')$.

CHAPTER 7 SELF-TEST, page 288

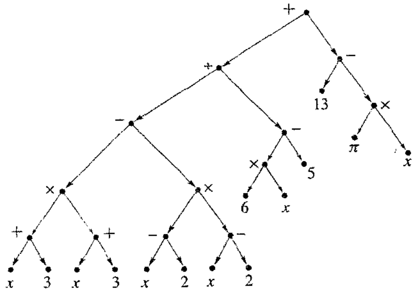
1. It is not a tree; deleting either (2, 3) or (5, 3) will give a tree with root 4.

2. **(a)** 4. **(b)** $v_4, v_{10}, v_6, v_9, v_8$.

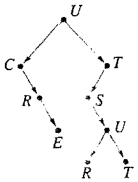
(c) 4. **(d)** v_6, v_8.

3. Every edge (v_i, v_j) in (T, v_0) belongs to a unique path from v_0 to v_j. Hence removing (v_i, v_j) would mean there is no path from v_0 to v_j.

4.

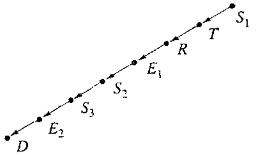


5.

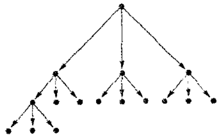


6. **(a)** UCRETSURT. **(b)** ERCRTUSTU.

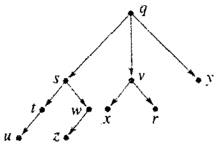
7.



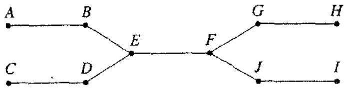
8.



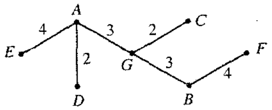
9.



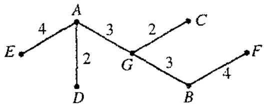
10. One solution is



11. EA, AD, AG, GC, GB, BF.

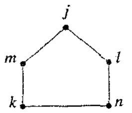


12. GC, AD, GB, GA, BF, AE.

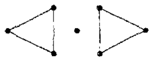


CHAPTER 8 SELF-TEST, page 326

1.



2. One solution is



3.

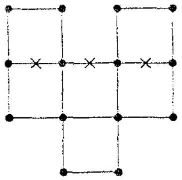
4. **(a)** Neither an Euler circuit nor path. There are more than 2 vertices of odd degree.

(b) An Euler circuit. All vertices have even degree.

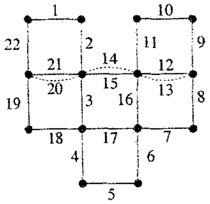
5. **(a)** A Hamiltonian circuit.

(b) A Hamiltonian path, but not a circuit.

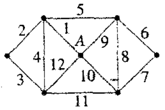
6. **(a)**



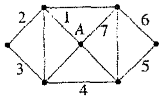
(b) One solution is



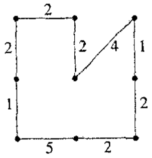
7. One solution is



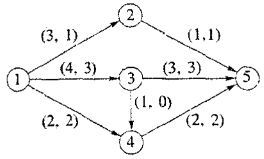
8. One solution is



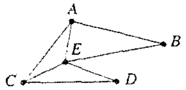
9.



10. $value(F) = 6$.



11. **(a)**



(b) 3.

12. $P_G(x) = x(x-1)(x-2)^3$
$P_G(0) = P_G(1) = P_G(2) = 0$, $P_G(3) = 6$.

CHAPTER 9 SELF-TEST, page 369

1. **(a)** Yes, $\mathbf{A} * \mathbf{B}$ is well defined for all 2×2 Boolean matrices.

(b) Yes, this is ordinary addition for even numbers.

(c) Yes, 2^{ab} is defined uniquely for all a, b in Z^+.

2.

$*$	a	b	c
a	a	c	□
b	c	b	b
c	□	b	c

where □ represents a, b, or c.

3. **(a)** If $a, b \in \mathbb{Q}$, then $a * b$ is also a rational number.

$$\begin{aligned} a * (b * c) &= a * (b + c - bc) \\ &= a + (b + c - bc) - a(b + c - bc) \\ &= a + b + c - bc - ab - ac + abc. \\ (a * b) * c &= (a + b - ab) * c \\ &= a + b - ab + c - (a + b - ab)c \\ &= a + b + c - ab - ac - bc + abc. \end{aligned}$$

Hence, $*$ is associative. Zero is the identity for $(\mathbb{Q}, *)$, which is a monoid.

(b) If $a \neq 1$, then

$$\begin{aligned} a * \frac{a}{a-1} &= a + \frac{a}{a-1} - a\left(\frac{a}{a-1}\right) \\ &= \frac{a^2 - a + a - a^2}{a-1} \\ &= 0. \end{aligned}$$

Thus all rational numbers except 1 have a $*$-inverse.

4. (a) and (b) are monoids. (c) is neither.

5. R has previously been shown to be an equivalence relation, because it is equality for the string lengths. Suppose $a \; R \; b$ and $\alpha \; R \; \beta$, then $\text{length}(a \cdot \alpha) = \text{length}(a) + \text{length}(\alpha) = \text{length}(b) + \text{length}(\beta) = \text{length}(b \cdot \beta)$. Thus $a \cdot \alpha \; R \; b \cdot \beta$ and R is a congruence relation.

6. No, $f(ab) = (ab)^{-1} = b^{-1}a^{-1} \neq f(a) \cdot f(b)$.

7. $\{c, d, e\} = H = He = Hc = Hd$;
$Ha = \{a, b, f\} = Hb = Hf$.

8. Let $g \in G_2$ and $n \in f(N)$. Since f is onto, there is a $g' \in G_1$ such that $f(g') = g$. Since $n \in f(N)$, there is an $n' \in N$ such that $f(n') = n$. Then $gn = f(g')f(n') = f(g'n')$. N is normal in G_1 so

$g'n' = n''g'$ for some $n'' \in N$. Then $f(g'n') = f(n''g') = f(n'')f(g') = f(n'')g \in f(N)g$. Thus $g \cdot f(N) \subseteq f(N) \cdot g$. Similarly, we can show $f(N) \cdot g \subseteq g \cdot f(N)$ and hence $g \cdot f(N) = f(N) \cdot g$ for all $g \in G_2$.

9. Suppose $x^2 = x$. Then $x^{-1}(xx) = x^{-1}x$ and $(x^{-1}x)x = e$. So $x = e$.

10. $f(a+b) = 2(a+b) = 2a+2b = f(a)+f(b)$ so f is a homomorphism. For any even integer n, $n = 2k$, $k \in Z$, and $f(k) = n$ so f is onto. Suppose $f(a) = f(b)$. Then $2a = 2b$ and $a = b$. Hence f is one to one.

11. Since the identity e belongs to every subgroup, $e \in \bigcap_{i=1}^{k} H_i$. Suppose h and h' belong to $\bigcap_{i=1}^{k} H_i$. Then h, h', and hh' belong to each H_i and $hh' \in \bigcap_{i=1}^{k} H_i$. Let $h \in \bigcap_{i=1}^{k} H_i$. Then h and h^{-1} belong to each H_i, since each H_i is a subgroup of G and so $h^{-1} \in \bigcap_{i=1}^{k} H_i$.

12. The inverse of $a + b\sqrt{n}$ must be of the form

$$\frac{a - b\sqrt{n}}{a^2 - nb^2}.$$

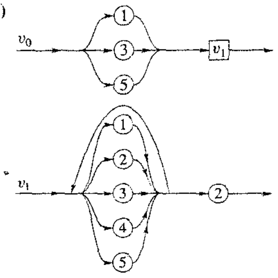
But this exists in the set if and only if $a^2 - nb^2 \neq 0$ if and only if $n \neq \left(\frac{a}{b}\right)$ if and only if $\sqrt{n} \neq \frac{a}{b}$.

CHAPTER 10 SELF-TEST, page 413

1. **(a)** False. **(b)** True. **(c)** True. **(d)** False.

2. $L(G) = \{a^n c^m d b^n, n \geq 0, m \geq 0\}$.

3. **(a)**

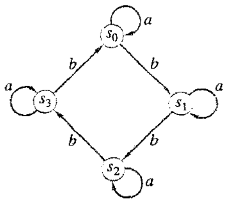


(b) $\langle v_0 \rangle ::= 1\langle v_1 \rangle \mid 3\langle v_1 \rangle \mid 5\langle v_1 \rangle$
$\langle v_1 \rangle ::= 1\langle v_1 \rangle \mid 2\langle v_1 \rangle \mid 3\langle v_1 \rangle \mid 4\langle v_1 \rangle \mid 5\langle v_1 \rangle \mid 2$

4. $L(G) = \{(1 \vee 3 \vee 5)(1 \vee 2 \vee 3 \vee 4 \vee 5)^n 2, n \geq 0\}$.

5. $G = (V, S, v_0, \mapsto)$ with $V = \{v_0, v_1, 0, 1\}$, $S = \{0, 1\}$, and $\mapsto$: $v_0 \mapsto 0v_0$, $v_0 \mapsto 1v_1$, $v_1 \mapsto 0v_1$, $v_1 \mapsto 0$.

6.

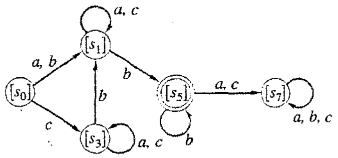


7. Strings of 0's and 1's with $3k$ zeros, $k \geq 0$.

8. **(a)** R is easily seen to be an equivalence relation with equivalence classes $\{s_0\}$, $\{s_3\}$, $\{s_1, s_2, s_4\}$, $\{s_5, s_6\}$, and $\{s_7\}$.

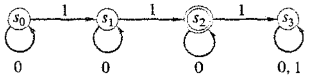
	a	b	c
$[s_0]$	$[s_1]$	$[s_1]$	$[s_3]$
$[s_1]$	$[s_1]$	$[s_5]$	$[s_1]$
$[s_3]$	$[s_3]$	$[s_1]$	$[s_3]$
$[s_5]$	$[s_7]$	$[s_5]$	$[s_7]$
$[s_7]$	$[s_7]$	$[s_7]$	$[s_7]$

(b)



9. Strings with exactly $4k$ b's, $k \geq 0$.

10.

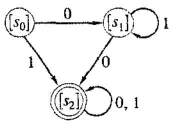


11. $\mathbf{M}_{R_0} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

$R_1 = \{(s_0, s_0), (s_1, s_1), (s_2, s_2), (s_3, s_3), (s_2, s_3), (s_3, s_2)\}$.

12. **(a)** $k = 1$.

(b) $R = \{(s_0, s_0), (s_1, s_1), (s_2, s_2), (s_3, s_3), (s_2, s_3), (s_3, s_2)\}$.



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1. **(a)** Yes. **(b)** No. **(c)** No. **(d)** No.

2. **(a)** Yes. **(b)** Yes. **(c)** Yes.

3. One.

4. Let $c_1 = 00000$, $c_2 = 11110$, $c_3 = 01101$, $c_4 = 10011$, $c_5 = 01010$, $c_6 = 10100$, $c_7 = 00111$, $c_8 = 11001$.

$\oplus$	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
c_1	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
c_2	c_2	c_1	c_4	c_3	c_6	c_5	c_8	c_7
c_3	c_3	c_4	c_1	c_2	c_7	c_8	c_5	c_6
c_4	c_4	c_3	c_2	c_1	c_8	c_7	c_6	c_5
c_5	c_5	c_6	c_7	c_8	c_1	c_2	c_3	c_4
c_6	c_6	c_5	c_8	c_7	c_2	c_1	c_4	c_3
c_7	c_7	c_8	c_5	c_6	c_3	c_4	c_1	c_2
c_8	c_8	c_7	c_6	c_5	c_4	c_3	c_2	c_1

The table shows this subset is closed for $\oplus$, contains the identity for B^5, and contains the inverse of each element.

5. 0.

6. 11.

7. 88.

8. 13, 32.