Technical Memorandum (TM)

TO: [redacted]

FROM: Jefferson Lee

SUBJECT: Kinematics of crankshaft and connecting rod

DATE: September 25, 2020

EXECUTIVE SUMMARY

A slider crank mechanism can convert between a simple linear motion and a rotary motion. This is applicable in many engineering machineries, such as pistons, engines, turbines, and many more. A crankshaft and its connecting rod is examined as seen in figure 1. The kinematic equations of motion of point B and G can be determined through basic, geometric kinematic constraints, derivative calculations, and vector relationships between two points in a rigid body. The motion of point B can be summarized in the following three equations:

$$x = l_1(\cos\theta) + l_2\left(\sqrt{\frac{-\sin^2\theta + Q^2}{Q^2}}\right)$$

$$\dot{x} = \frac{-l_1\omega\sin(\varphi+\theta)}{cos\varphi}$$

$$\ddot{x} = \frac{-l_1\omega^2\cos(\varphi+\theta) + l_2\dot{\varphi}^2}{cos\varphi}$$

In addition, the motion of point G can be summarized in the following equations:

$$r_{GX} = -l_2\left(\sqrt{\frac{-\sin^2\theta+Q^2}{Q^2}}\right) - \frac{\omega cos\theta}{\sqrt{-\sin^2\theta+Q^2}} * \frac{l_2\frac{\sin\theta}{Q}}{2}$$

$$r_{GY} = \frac{\omega cos\theta}{\sqrt{-\sin^2\theta+Q^2}} * \frac{-l_2\sqrt{\frac{-\sin^2\theta+Q^2}{Q^2}}}{2}$$

$$V_{GX} = \frac{\boldsymbol{-l_1\omega\sin(\varphi+\theta)}}{\boldsymbol{cos\varphi}} + \frac{\boldsymbol{l_1\omega cos\theta}}{\boldsymbol{l_2 cos\varphi}}\left(\frac{l_2\frac{\sin\theta}{Q}}{2}\right)$$

$$V_{GY} = \left(\frac{l_1\omega cos\theta}{l_2 cos\varphi}\right)\frac{-l_2\sqrt{\frac{-\sin^2\theta+Q^2}{Q^2}}}{2}$$

$$|a| = \ddot{x} + \omega^2 sin\theta \left(\frac{cos^2\theta - (Q^2 - sin^2\theta)}{2(Q^2 - sin^2\theta)^{\frac{3}{2}}} \right) l_2 + \frac{l_2}{2} \left(\frac{\omega cos\theta}{\sqrt{Q^2 - sin^2\theta}} \right)^2$$

Plotting these equations with respect to the angle of l_1 with the X axis, or θ, provides visual confirmation of the expected motions.

METHOD

In order to obtain a kinematic equation of motion (displacement, velocity, and acceleration,), a system must be analyzed from a coordinate reference point. As seen in figure 1, the crankshaft is arranged in a manner where θ is the angle of l_1 counterclockwise from the X direction, and φ is the angle of l_2 clockwise from the X direction. The relationship between the two angles, and hence the two similar triangles formed, will create the kinematic constraints used. As the location of point B is of interest, it is crucial to include a displacement value of B from the origin, O. From the two equations, differentiating them will result in a velocity, and acceleration if differentiated once more.

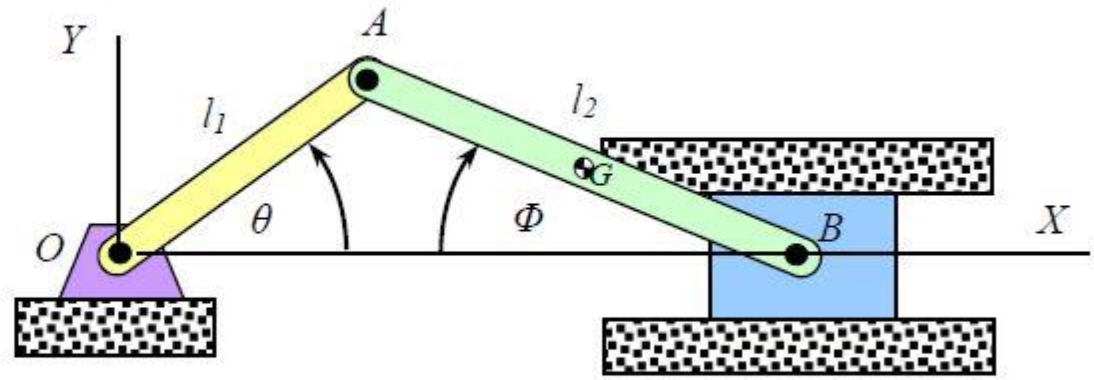


Figure 1: Crankshaft and connecting rod with coordinate references (Texas A&M University)

In order to find a kinematic equation of motion for point G, which is along the solid link AB, the following vector relationship approach is needed.

$$V_G = V_B + \dot{\varphi} \times r_{BG}$$

and

$$a_G = a_B + \ddot{\varphi} \times r_{BG} + \dot{\varphi} \times (\dot{\varphi} \times r_{BG})$$

In creating plots to visualize the motion of the crankshaft, python was used along with libraries NumPy and Matplotlib.

PROCEDURE

From an understanding of similar triangles, it is known that the slider crank mechanism is related by the following:

$$l_1(\sin\theta) = l_2(\sin\varphi)$$

This relationship can be re arranged into the following where $Q = \frac{l_2}{l_1}$:

$$\frac{\sin\theta}{\frac{l_2}{l_1}} = \sin\varphi = \frac{\sin\theta}{Q}$$

In addition, the geometry of the system denotes that the displacement, x, from an initial position of point O is as follows:

$$x = l_1(\cos\theta) + l_2(\cos\varphi)$$

Differentiating both equations twice results in the following:

$$\dot{\varphi}\cos\varphi = \frac{\dot{\theta}\cos\theta}{Q}$$

$$-\dot{\varphi}^2\sin\varphi \ + \ddot{\varphi}cos\varphi \ = \frac{-\dot{\theta}^2\sin\theta + \ddot{\theta}\cos\theta}{Q}$$

and

$$\dot{x} = -\dot{\theta}l_1 sin\theta - \dot{\varphi}l_2 sin\varphi$$

$$\ddot{x} = -\dot{\theta}^2 l_1 cos\theta - \ddot{\theta}l_1 sin\theta - \dot{\varphi}^2 l_2 cos\varphi - \ddot{\varphi}l_2 sin\varphi$$

However, as the values focused on has a crankcase of a constant angular velocity of 180 rpm, the second differentiation equations above can be simplified; These equations can be simplified due to the fact $\ddot{\theta}$ or $\dot{\omega}$ = 0.

$$-\dot{\varphi}^2\sin\varphi \ + \ddot{\varphi}cos\varphi \ = \frac{-\dot{\theta}^2\sin\theta}{Q}$$

$$\ddot{x} = -\dot{\theta}^2 l_1 cos\theta - \dot{\varphi}^2 l_2 cos\varphi - \ddot{\varphi}l_2 sin\varphi$$

Formatting these equations into matrices result in the following:

$$\begin{bmatrix} cos\varphi & 0 \\ l_2 sin\varphi & 1 \end{bmatrix}\begin{bmatrix} \dot{\varphi} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \frac{\dot{\theta}cos\theta}{Q} \\ -\dot{\theta}l_1 sin\theta \end{bmatrix}$$

$$\begin{bmatrix} cos\varphi & 0 \\ l_2 sin\varphi & 1 \end{bmatrix} \begin{bmatrix} \ddot{\varphi} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \frac{-\dot{\theta}^2 sin\theta}{Q} + \dot{\varphi}^2 sin\varphi \\ -\dot{\theta}^2 l_1 cos\theta - \dot{\varphi}^2 l_2 cos\varphi \end{bmatrix}$$

Which can then be easily solved using Cramer's Rule.

$$\dot{\varphi} = \frac{l_1 \omega cos\theta}{l_2 cos\varphi}$$

$$\dot{x} = \frac{-l_1 \omega \sin{(\varphi + \theta)}}{cos\varphi}$$

$$\ddot{\varphi} = \frac{-l_1 \omega^2 sin\theta + l_2 \dot{\varphi}^2 sin\varphi}{l_2 cos\varphi}$$

$$\ddot{x} = \frac{-l_1 \omega^2 \cos(\varphi + \theta) + l_2 \dot{\varphi}^2}{cos\varphi}$$

However, it is known from the relationship $\sin\varphi = \frac{\sin\theta}{Q}$, that $\varphi = sin^{-1}(\frac{\sin\theta}{Q})$, and $\cos\varphi = \sqrt{\frac{-\sin^2\theta + Q^2}{Q^2}}$ from basic trigonometric identities. From these, the equations of $\dot{\varphi}$, $\dot{x}$, $\ddot{\varphi}$, and $\ddot{x}$ can be dependent on only the variable θ and ω ($\dot{\theta}$).

$$\dot{\varphi} = \frac{\omega cos\theta}{\sqrt{-\sin^2\theta + Q^2}}$$

$$\dot{x} = \frac{(-2Q sin\theta l_1 \sqrt{-\sin^2\theta + Q^2} - l_2 \dot{\theta} sin2\theta)\sqrt{-\sin^2\theta + Q^2}}{-2Q\sin^2\theta + 2Q^3}$$

$$\ddot{\varphi} = \frac{(\dot{\varphi} sin\theta - \omega^2 sin\theta)\sqrt{-\sin^2\theta + Q^2}}{-\sin^2\theta + Q^2}$$

$$\ddot{x} = \frac{[\left(\omega^2 l_2 sin^2\theta - Q^2 \omega^2 l_2 \sin^{-1}\left(\frac{sin\theta}{Q}\right)\right) - Q\omega^2 l_1 cos\theta \sqrt{-\sin^2\theta + Q^2}]\sqrt{-\sin^2\theta + Q^2}}{-\mathrm{Q}\sin^2\theta + Q^3}$$

For the motion of point G, $r_G = r_B + \dot{\varphi} \times r_{BG}$. This results in the following:

$$r_G = l_2\left(\sqrt{\frac{-\sin^2\theta + Q^2}{Q^2}}\right) + \frac{\omega cos\theta}{\sqrt{-\sin^2\theta + Q^2}}(\hat{k}) \times (\frac{-l_2\sqrt{\frac{-\sin^2\theta + Q^2}{Q^2}}}{2}\ \hat{\imath} + \frac{l_2 \frac{\sin\theta}{Q}}{2}\hat{\jmath}).$$

Which simplifies to the following:

$$r_{GX} = -l_2\left(\sqrt{\frac{-\sin^2\theta+Q^2}{Q^2}}\right) - \frac{\omega cos\theta}{\sqrt{-\sin^2\theta+Q^2}} * \frac{l_2\frac{\sin\theta}{Q}}{2}$$

$$r_{GY} = \frac{\omega cos\theta}{\sqrt{-\sin^2\theta+Q^2}} * \frac{-l_2\sqrt{\frac{-\sin^2\theta+Q^2}{Q^2}}}{2}$$

For the acceleration of point G, $a_G = a_B + \ddot{\varphi} \times r_{BG} + \dot{\varphi} \times (\dot{\varphi} \times r_{BG})$, resulting in a vector in the x direction and in the y direction. $\sqrt{a_x{}^2 + a_y{}^2}$ will result in a magnitude of point G's acceleration.

$$a = \ddot{x} + \omega^2 sin\theta\left(\frac{cos^2\theta - (Q^2 - sin^2\theta)}{2(Q^2 - sin^2\theta)^{\frac{3}{2}}}\right)l_2 + \frac{l_2}{2}\left(\frac{\omega cos\theta}{\sqrt{Q^2 - sin^2\theta}}\right)^2$$

In finding the magnitudes of $l_1$and l_2 is derived from the relationship $l_1 + l_2 = 10ft$ and where $P = \frac{l_1}{l_2}$. From P = 1.5, $l_1$and l_2 is 4 and 6. From P = 3, $l_1$and l_2 is 2.5 and 7.5. From P = 4, $l_1$and l_2 is 2 and 8.

RESULTS AND DISCUSSION

The displacement (x), velocity ($\dot{x}$), and acceleration ($\ddot{x}$) for point B is equal to the following:

$$x = l_1(\cos\theta) + l_2\left(\sqrt{\frac{-\sin^2\theta + Q^2}{Q^2}}\right)$$

$$\dot{x} = \frac{-l_1\omega\sin(\varphi + \theta)}{cos\varphi}$$

$$\ddot{x} = \frac{-l_1\omega^2\cos(\varphi + \theta) + l_2\dot{\varphi}^2}{cos\varphi}$$

In addition, the angular displacement (φ), angular velocity ($\dot{\varphi}$), and angular acceleration ($\ddot{\varphi}$), for point G is equal to the following:

$$\varphi = sin^{-1}\left(\frac{\sin\theta}{Q}\right)$$

$$\dot{\varphi} = \frac{l_1\omega cos\theta}{l_2 cos\varphi}$$

$$\ddot{\varphi} = \frac{-l_1\omega^2 sin\theta + l_2\dot{\varphi}^2 sin\varphi}{l_2 cos\varphi}$$

Which leads to the equations of motion for point G to be the following:

$$r_G = r_B + \dot{\varphi} \times r_{BG}$$

$$r_{GX} = -l_2\left(\sqrt{\frac{-\sin^2\theta + Q^2}{Q^2}}\right) - \frac{\omega cos\theta}{\sqrt{-\sin^2\theta + Q^2}} * \frac{l_2\frac{\sin\theta}{Q}}{2}$$

$$r_{GY} = \frac{\omega cos\theta}{\sqrt{-\sin^2\theta + Q^2}} * \frac{-l_2\sqrt{\frac{-\sin^2\theta + Q^2}{Q^2}}}{2}$$

and

$$V_G = V_B + \dot{\varphi} \times r_{BG}$$

$$V_{GX} = \frac{-l_1\omega \sin(\varphi+\theta)}{cos\varphi} + \frac{l_1\omega cos\theta}{l_2 cos\varphi}\left(\frac{l_2\frac{\sin\theta}{Q}}{2}\right)$$

$$V_{GY} = \left(\frac{l_1\omega cos\theta}{l_2 cos\varphi}\right)\frac{-l_2\sqrt{\frac{-\sin^2\theta + Q^2}{Q^2}}}{2}$$

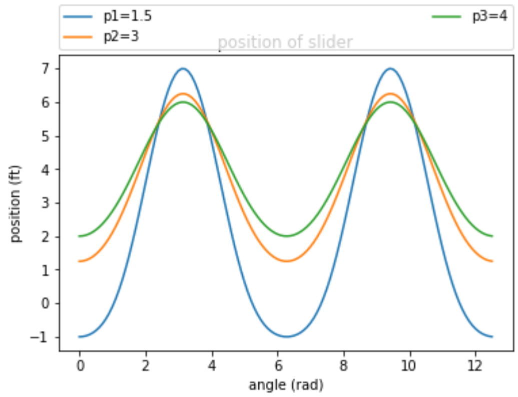
and

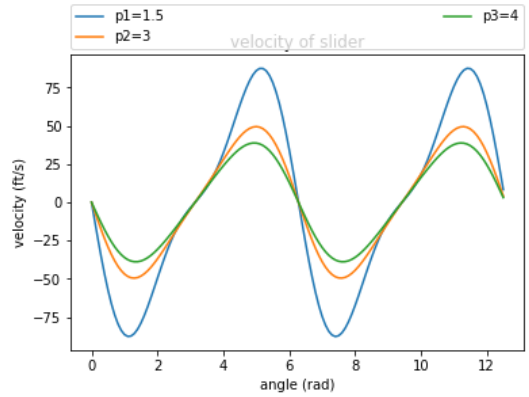
$$a_G = a_B + \ddot{\varphi} \times r_{BG} + \dot{\varphi} \times (\dot{\varphi} \times r_{BG})$$

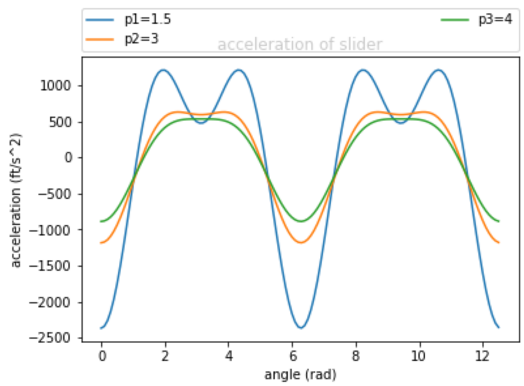
$$|a| = \ddot{x} + \omega^2 sin\theta\left(\frac{cos^2\theta - (Q^2 - sin^2\theta)}{2(Q^2 - sin^2\theta)^{\frac{3}{2}}}\right)l_2 + \frac{l_2}{2}\left(\frac{\omega cos\theta}{\sqrt{Q^2 - sin^2\theta}}\right)^2$$

An important ratio of the system is the relationship between l_2 and l_1, noticeably ‘P’, being $\frac{l_2}{l_1}$. In order to visualize the impact on the motion of the crankshaft from P, the three configurations are displayed: P_1 = 1.5, P_2 = 3, P_3 = 4. In order to find the individual values for l_2 and l_1, the total length of l_2 and l_1 is 10 feet, moreover, $l_2 + l_1 = 10\ ft$. Lastly, the value of ω is 180 rpm for the specific calculations performed.

Plots displaying the horizontal motion, the displacement (x), velocity ($\dot{x}$), and acceleration ($\ddot{x}$), of point B vs. θ (crank angle) for P ratios ($\frac{l_2}{l_1}$) of are as shown:



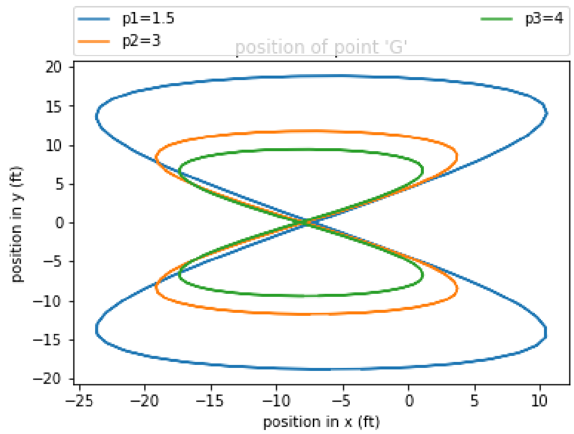


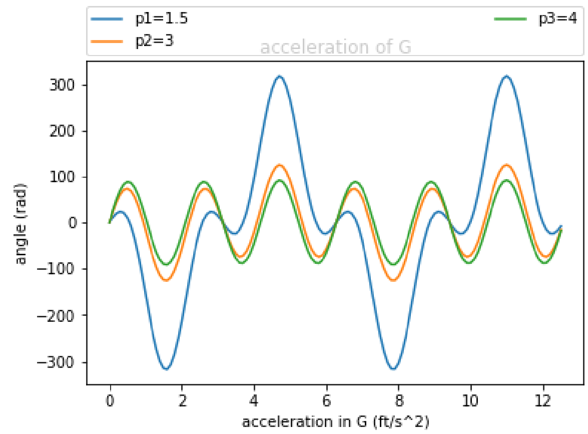


From the graphs, point B is shown to be behave in a function of sinusoidal motion. At an angle of π radians, the position of B achieves its maximum value, with the largest displacement of seven feet in the configuration with a P ratio of 1.5. Similarly, the most negative position achieved is by the P ratio of 1.5 at -1 feet from the starting position. At an angle of approximately one radian, and five, the velocity reaches its peak of an absolute value of 75 $\frac{ft}{s}$. The velocity reaches zero at the angle π radians. The acceleration of point B reaches the highest magnitude at an angle of 2π radians, close to 2500 $\frac{ft}{s^2}$, and an acceleration of zero $\frac{ft}{s^2}$

At approximately five radians. However, these values are not the only points where minimum and maximum acceleration is achieved. The graph shows an interesting shape, almost like the letter 'M' between each period of oscillation. This creates two angles where the acceleration reaches zero.

Plots displaying motion of point G (YG vs XG) as well as the magnitude of the acceleration of point G is as shown:





As seen above, the motion of the link AB follows a figure 8 pattern. The tracing of this makes logical sense as the point G on link AB is not only moving in the axial X direction, but also has an angular component. The acceleration graph of point G also reveals the complex motion of the link that comprises the elements in the vector equation $a_G = a_B + \ddot{\varphi} \times r_{BG} + \dot{\varphi} \times (\dot{\varphi} \times r_{BG})$.

From the graphs, Newton's 2nd law can be applied when analyzing the magnitude of the forces when using a simple mass acceleration calculation. Given the mass of the block/piston at point B, it would be simple in finding the maximum force exerted by the system. This could be useful when analyzing systems such as a piston engine.

From an engineering point of view, the results of the kinematic motions of the crankshaft makes logical sense. This can be deduced from the amplitudes of the displacement figure of point B. The peak amplitude of seven feet is a reasonable distance, while the motion of a crankshaft is not expected to exceed extreme distances. While investigating the peak velocity and acceleration, the peak values seem to reach extremely high values. For example, the peak acceleration reaches an absolute value of 2500 $\frac{ft}{s^2}$. This value is extremely high, being over 760 $\frac{m}{s^2}$. These values could result in the oversimplification done in assuming a constant angular velocity and the frictionless surfaces. However, it is also noticed that in the velocity of point B, a phase angle exists between the different P values. This makes logical sense as the velocity term, $\sin(\varphi + \theta)$, has two angles acting in unison to create the sinusoidal function. From an engineering perspective, the velocity acts with influence from the relationship of the size of l_1 and l_2. This similar property is found in the acceleration motion of the crankshaft as well.

CONCLUSION

The equations of motion of a crankshaft and connecting rod can be easily interpreted and visual through a graph. As from figure 1, the motion of the crankshaft is a function of angle of l_1 with respect to the X axis, or θ. In this study of a crankshaft as in figure 1, points B and G are studied and found to have kinematic equations of motions as follows:

$$x = l_1(\cos\theta) + l_2\left(\sqrt{\frac{-\sin^2\theta + Q^2}{Q^2}}\right)$$

$$\dot{x} = \frac{-l_1\omega\sin(\varphi+\theta)}{cos\varphi}$$

$$\ddot{x} = \frac{-l_1\omega^2\cos(\varphi+\theta) + l_2\dot{\varphi}^2}{cos\varphi}$$

In addition, the motion of point G can be summarized in the following equations:

$$r_{GX} = -l_2\left(\sqrt{\frac{-\sin^2\theta+Q^2}{Q^2}}\right) - \frac{\omega cos\theta}{\sqrt{-\sin^2\theta+Q^2}} * \frac{l_2\frac{\sin\theta}{Q}}{2}$$

$$r_{GY} = \frac{\omega cos\theta}{\sqrt{-\sin^2\theta + Q^2}} * \frac{-l_2\sqrt{\frac{-\sin^2\theta + Q^2}{Q^2}}}{2}$$

$$V_{GX} = \frac{-l_1\omega\sin(\varphi+\theta)}{cos\varphi} + \frac{l_1\omega cos\theta}{l_2 cos\varphi}\left(\frac{l_2\frac{\sin\theta}{Q}}{2}\right)$$

$$V_{GY} = \left(\frac{l_1\omega cos\theta}{l_2 cos\varphi}\right)\frac{-l_2\sqrt{\frac{-\sin^2\theta + Q^2}{Q^2}}}{2}$$

$$|a| = \ddot{x} + \omega^2 sin\theta\left(\frac{cos^2\theta - (Q^2 - sin^2\theta)}{2(Q^2 - sin^2\theta)^{\frac{3}{2}}}\right)l_2 + \frac{l_2}{2}\left(\frac{\omega cos\theta}{\sqrt{Q^2 - sin^2\theta}}\right)^2$$

REFERENCES

"MEEN 363 FALL 2020: Computational Project #1: Crankshaft and connecting rod" Texas A&M University, 2020.