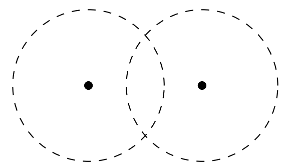
Separation

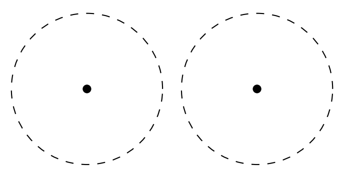
Definition: T_1

Let X be a topological space. To say that X is a T_1-*space* means that for all distinct $x, y \in X$ there exists $U, V \in \mathscr{T}$ such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.



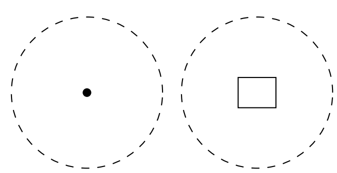
Definition: T_2

Let X be a topological space. To say that X is T_2-*space* or *Hausdorff* means that for all distinct $x, y \in X$ there exists disjoint $U, V \in \mathscr{T}$ such that $x \in U$ and $y \in V$.



Definition: Regular

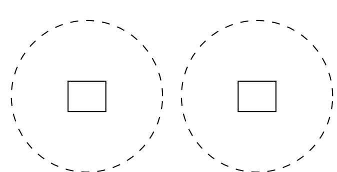
Let X be a topological space. To say that X is *regular* means that for all $x \in X$ and closed sets $A \subset X$ such that $x \notin A$ there exists disjoint $U, V \in \mathscr{T}$ such that $x \in U$ and $A \subset V$.



To say that X is a T_3-*space* means that X is regular and T_1.

Definition: Normal

Let X be a topological space. To say that X is *regular* means that for all disjoint closed sets $A, B \subset X$ there exists disjoint $U, V \in \mathscr{T}$ such that $A \subset U$ and $B \subset V$.



To say that X is a T_4-*space* means that X is normal and T_1.

Theorem

Let X be a topological space. X is T_1 iff every point in X is a closed set.

Proof. Assume $x, y \in X$ such that $x \neq y$.

$\Longrightarrow$ Assume X is T_1.

So there exists $U \in \mathscr{T}$ such that $x \notin U$ and $y \in U$. This means that $U \cap \{x\} = \emptyset$ and so y is not a limit point of $\{x\}$.

Therefore, $\{x\}$ is closed.

$\Longleftarrow$ Assume that every point in X is a closed set.

So x is not a limit point of $\{y\}$ and y is not a limit point of $\{x\}$. This means that there exists $U, V \in \mathscr{T}$ such that $x \in U$ and $U \cap \{y\} = \emptyset$ and likewise $y \in V$ and $V \cap \{x\} = \emptyset$. Hence $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.

Therefore X is T_1.

■

Theorem

Let X be a topological space. If X is cofinite then X is T_1.

Proof. Assume that X is cofinite and assume that $x \in X$. But $X - \{x\}$ is open in the cofinite topology, and so $\{x\}$ is closed. Therefore, by the previous theorem, X is T_1. ■

Theorem

$\mathbb{R}_{\text{std}}$ is T_2.

Proof. Assume that $a, b \in \mathbb{R}$ such that $a \neq b$ and let $\epsilon = \frac{|b-a|}{3}$. Now let $U = (a - \epsilon, a + \epsilon) \in \mathscr{T}$ and let $V = (b - \epsilon, b + \epsilon) \in \mathscr{T}$. So $a \in U$ and $b \in V$ and $U \cap V = \emptyset$.

Therefore $\mathbb{R}_{\text{std}}$ is T_2. ■

Theorem

$\mathbb{R}_{LL}$ is normal.

Proof. Assume that $A, B \subset \mathbb{R}$ such that A and B are closed and $A \cap B = \emptyset$. This means that $B \subset \mathbb{R} - A \in \mathscr{T}$ and $A \subset \mathbb{R} - B \in \mathscr{T}$. Now assume $a \in A$ and $b \in B$. Then there exists basic open sets $U_a = [a, \epsilon_a) \subset \mathbb{R} - B$ and $V_b = [b, \epsilon_b) \subset \mathbb{R} - A$ and open sets:

$$U = \bigcup_{a \in A} U_a \supset A$$

$$V = \bigcup_{b \in B} V_b \supset B$$

So ABC that $U \cap V \neq \emptyset$. This means that there exists some $U_a \cap V_b \neq \emptyset$, and hence $\max\{a, b\} \in U_a \cap V_b$.

Case 1: $a \in U_a \cap V_b$

Thus $a \in A$ and $a \in V_b \subset \mathbb{R} - A$, a contradiction.

Case 2: $b \in U_a \cap V_b$

Thus $b \in B$ and $b \in V_a \subset \mathbb{R} - B$, a contradiction.

And so $U \cap V = \emptyset$.

Therefore R_{LL} is normal. ∎

Example

Consider $\mathbb{R}^2$ with the standard topology.

1. Let $p \in \mathbb{R}^2$ and let $A \subset \mathbb{R}^2$ be a closed set such that $p \notin A$. Show that:

   $$\inf \{d(a, p) \mid a \in A\} > 0$$

   Since A is closed and $p \notin A$, p is not a limit point of A. Thus, there exists $\epsilon > 0$ such that $B(p, \epsilon) \cap A = \emptyset$ and so for all $a \in A$ the distance from p to a is at least ϵ.

   Therefore, $\inf \{d(a, p) \mid a \in A\} > \epsilon > 0$.

2. Show that $\mathbb{R}^2$ with the standard topology is regular.

   Assume that $p \in \mathbb{R}^2$ and $A \subset \mathbb{R}^2$ such that $p \notin A$ and A is closed. By (1), there exists some $\epsilon > 0$ such that for all $a \in A$, $d(p, a) > \epsilon$. Let $\delta = \frac{\epsilon}{3}$ and consider $U = B(p, \delta)$ and open set V generated by $\{B(a, \delta_a) \mid a \in A, \delta_a < \delta\}$. Thus, for every point $x \in U$ and $y \in V$, $d(x, y) \geq \delta$ and so $U \cap V = \emptyset$.

   Therefore R^2 is regular.

3. Find two disjoint closed sets $A, B \subset \mathbb{R}^2$ with the standard topology such that:

   $$\inf \{d(a, b) \mid a \in A, b \in B\} = 0$$

   Any two asymptotic functions in R^2 will do. So let:

   $$A = \{(x, 0) \mid x \in [1, \infty)\}$$
   $$B = \left\{ \left(x, \frac{1}{x}\right) \middle| \, x \in [1, \infty) \right\}$$

4. Show that $\mathbb{R}^2$ with the standard topology is normal.

   Assume that $A, B \subset \mathbb{R}^2$ such that A and B are closed and $A \cap B = \emptyset$. By (2), for every $a \in A$ there exists $B(a, \epsilon_a)$ such that $B(a, \epsilon_a) \cap B = \emptyset$. Likewise, for every $b \in B$ there exists $B(b, \epsilon_b)$ such that $B(b, \epsilon_b) \cap A = \emptyset$. So let $\delta_a = \frac{\epsilon_a}{3}$ and let $\delta_b = \frac{\epsilon_b}{3}$ and consider the families of open sets $U_a = B(a, \delta_a)$ and $V_b = B(b, \delta_b)$. Let:

$$U = \bigcup_{a \in A} U_a \supset A$$
$$V = \bigcup_{b \in B} V_b \supset B$$

Now, assume that $a \in A$ and $b \in B$:

$$d(a, b) \geq \max\{\epsilon_a, \epsilon_b\} > \max\{\delta_a, \delta_b\}$$

Thus $U_a \cap V_b = \emptyset$ and hence $U \cap V = \emptyset$.

Therefore R^2 is normal.

Theorem

1. A T_2-space (Hausdorff) is a T_1-space.
2. A T_3-space (regular and T_1) is a T_2-space (Hausdorff).
3. A T_4-space (normal and T_1) is a T_3-space (regular and T_1).

Proof. Let X be a topological space.

1. Assume that X is T_2.

   Assume $x, y \in X$ such that $x \neq y$. Since X is T_2, there exists $U, V \in \mathscr{T}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Thus, $x \in U$, $y \notin U$, $x \notin V$, and $y \in V$.

   Therefore X is T_1.

2. Assume that X is T_3.

   Assume $x, y \in X$ such that $x \neq y$. Since X is T_1, $\{y\}$ is closed, and since X is T_3, there exists $U, V \in \mathscr{T}$ such that $x \in U$, $\{y\} \subset V$ ($y \in V$), and $U \cap V = \emptyset$.

   Therefore X is T_2.

3. Assume that X is T_4.

   Assume $x \in X$ and $A \subset X$ such that A is closed and $x \notin A$. Since X is T_1, $\{x\}$ is closed, and since X is T_4, there exists $U, V \in \mathscr{T}$ such that $\{x\} \subset U$ and $A \subset V$ and $U \cap V = \emptyset$.

   Therefore X is regular and T_1 and hence T_3.

■

Theorem

Let X be a topological space. X is regular iff for all $p \in X$ and $U \in \mathcal{U}_p$, there exists $V \in \mathcal{U}_p$ such that $\bar{V} \subset U$.

Proof.

$\Longrightarrow$ Assume that X is regular.

Assume $p \in X$ and assume $U \in \mathcal{U}_p$. Since U is open, $X - U$ is closed. So, since X is regular, there exists $V, W \in \mathscr{T}$ such that $p \in V$, $X - U \subset W$, and $V \cap W = \emptyset$. Now, since $X - U \subset W$:

$$X - (X - U) \supset X - W$$

and so $X - W \subset U$. Next, since $V \cap W = \emptyset$, it must be the case that $V \subset X - W$. But since W is open, $X - W$ is closed. Therefore:

$$\bar{V} \subset \overline{X - W} = X - W \subset U$$

$\Longleftarrow$ Assume that $\forall\, p \in X, \forall\, U \in \mathcal{U}_p, \exists\, V \in \mathcal{U}_p, \bar{V} \subset U$.

Assume $p \in X$ and $A \subset X$ such that A is closed and $p \notin A$. This means that p is not a limit point of A and so there exists $U \in \mathcal{U}_p$ such that $U \cap A = \emptyset$. Furthermore, there exists $V \in \mathcal{U}_p$ such that $V \subset \bar{V} \subset U$, and so $\bar{V} \cap A = \emptyset$. This means that $A \subset X - \bar{V}$, with $X - \bar{V}$ open. But $V \cap X - \bar{V} = \emptyset$.

Therefore X is regular.

■

<u>Theorem</u>

Let X be a topological space. X is normal iff for all closed sets $A \subset X$ and for all $U \in \mathcal{U}_A$ there exists $V \in \mathcal{U}_A$ such that $\bar{V} \subset U$.

Proof.

$\Longrightarrow$ Assume that X is normal.

Assume $A \subset X$ and assume $U \in \mathcal{U}_A$. Since U is open, $X - U$ is closed. So, since X is normal, there exists $V, W \in \mathscr{T}$ such that $A \subset V$, $X - U \subset W$, and $V \cap W = \emptyset$. Now, since $X - U \subset W$:

$$X - (X - U) \supset X - W$$

and so $X - W \subset U$. Next, since $V \cap W = \emptyset$, it must be the case that $V \subset X - W$. But since W is open, $X - W$ is closed. Therefore:

$$\bar{V} \subset \overline{X - W} = X - W \subset U$$

$\Longleftarrow$ Assume that for all closed sets $A \subset X$ and for all $U \in \mathcal{U}_A$ there exists $V \in \mathcal{U}_A$ such that $\bar{V} \subset U$.

Assume $A, B \subset X$ such that A and B are closed and $A \cap B = \emptyset$. Then $A \subset X - B \in \mathscr{T}$. And so, by assumption, there exists $U \in \mathcal{U}_A$ such that $A \subset U \subset \bar{U} \subset X - B$. This means that $B \subset X - \bar{U} \in \mathscr{T}$. Finally:

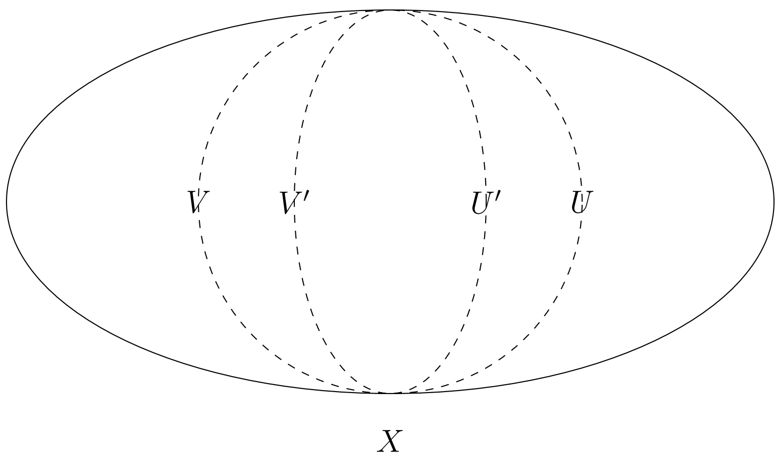
$$U \cap (X - \bar{U}) = (U \cap X) - (U \cap \bar{U}) = U - U = \emptyset$$

Therefore X is normal.

■

Theorem: The Incredible Shrinking Theorem

Let X be a topological space. X is normal iff for all open sets U, V such that $U \cup V = X$, there exist open sets U', V' such that $\overline{U'} \subset U$ and $\overline{V} \subset V$ and $U' \cap V' = X$.



This theorem can be extended to any family of open sets $\{U_\alpha : \alpha \in \lambda\}$ such that:

$$X = \bigcup_{\alpha \in \lambda} U_\alpha$$

Example

$SPACE$	T_1	T_2	REGULAR	NORMAL
R_{std}	✓	✓	✓	✓
R^n_{std}	✓	✓	✓	✓
indiscrete	×	×	×	×
discrete	✓	✓	✓	✓
cofinite	✓	X finite: ✓ X infinite: ×	X finite: ✓ X infinite: ×	X finite: ✓ X infinite: ×
cocountable	✓	X countable: ✓ X uncountable: ×	X countable: ✓ X uncountable: ×	X countable: ✓ X uncountable: ×
R_{LL}	✓	✓	✓	✓
R_{+00}	✓	×	×	×
LOS	✓	✓	✓	✓

R and $\mathbb{R}^n$

Since there is a finite distance between points and closed sets (not containing those points), there is always room for enclosing disjoint balls.

indiscrete

Since the only non-empty set is the entire space, there is no separation.

discrete

Since all disjoint subsets are both open and closed, they are self-enclosed.

cofinite/cocountable

First note that all finite sets are closed. Thus, single points can be viewed as closed sets. So assume p and q are distinct points in X. This means that $X - \{p\}$ and $X - \{q\}$ are open. Furthermore, $p \in X - \{q\}$ but $p \notin X - \{p\}$ and $q \in X - \{p\}$ but $q \notin X - \{q\}$. Thus, cofinite/cocountable is T_1.

Now assume that there exists disjoint $U, V \in \mathscr{T}$. This means that $X - U$ and $X - V$ are finite/countable and since $U \cap V = \emptyset$ it is the case that $X - (U \cap V) = (X - U) \cup (X - V) = X$ and hence X is finite/countable. When X is finite/countable, all subsets are both open and closed, equivalent to the discrete topology, and so cofinite and cocountable are T_2, regular, and normal. However, if X is infinite/uncountable then open sets will always intersect and so cofinite and countable are neither T_2, regular, nor normal.

$\mathbb{R}_{LL}$

Since R_{LL} is finer than $\mathbb{R}$, it has the same separation properties.

$\underline{\mathbb{R}_{+00}}$

Any two points can be T_1 separated using the basis elements; however, if one point or closed set contains $0'$ and the other point or closed set contains $0''$ then there is always overlap between the two containing basis elements.

Lexigraphically Ordered Square

Use the alternate definitions. For any point $p \in X$, there exists some containing open set (strip), and it is always possible to use a smaller strip whose closure is contained in the original strip. For any closed set $A \in X$, $X - A$ is an enclosing open set, and likewise, a smaller open set with contained closure is possible.

Theorem

X, Y are $T_2 \implies X \times Y$ is T_2.

Proof. Assume that X and Y are T_2 and assume $p_1, p_2 \in X \times Y$ where $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$. Since X is T_2, there exists $U_1, U_2 \in \mathscr{T}_X$ such that $x_1 \in U_1$ and $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. Likewise, since Y is T_2, there exists $V_1, V_2 \in \mathscr{T}_Y$ such that $y_1 \in V_1$ and $y_2 \in V_2$ and $V_1 \cap V_2 = \emptyset$. So $p_1 \in U_1 \times V_1$ and $p_2 \in U_2 \times V_2$. Furthermore, $U_1 \times V_1, U_2 \times V_2 \in \mathscr{T}_{X \times Y}$ and

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) = \emptyset$$

Therefore $X \times Y$ is T_2. ∎

Lemma

Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$:

$$\overline{A \times B} = \bar{A} \times \bar{B}$$

Proof. Assume that $p \in \overline{A \times B}$. This means that for all $U \in \mathscr{T}_{X \times B}$ such that $p \in U$:

$$U \cap (A \times B) \neq \emptyset$$

Now assume $U_1 \in \mathscr{T}_X$ and $U_2 \in T_Y$ such that $p \in U_1 \times U_2 \in \mathscr{T}_{A \times B}$. Then it must be the case that $(U_1 \times U_2) \cap (A \times B) = (U_1 \cap A) \times (U_2 \cap B) \neq \emptyset$. This is only possible if $U_1 \cap A \neq \emptyset$ and $U_2 \cap B \neq \emptyset$.

Therefore $p \in \bar{A} \times \bar{B}$.

Assume that $p \in \bar{A} \times \bar{B}$. This means that for all $U_1 \in \mathscr{T}_X$ and $U_2 \in \mathscr{T}_Y$ such that $p \in U_1 \times U_2$:

$$(U_1 \cap A) \times (U_2 \cap B) \neq \emptyset$$

Now assume $U \in \mathscr{T}_{A \times B}$ such that $p \in U \in \mathscr{T}_{A \times B}$. Then there exists $U_1 \in \mathscr{T}_X$ and $U_2 \in T_Y$ such that $p \in U_1 \times U_2 = U$. So it must be the case that:

$$U \cap (A \times B) = (U_1 \times U_2) \cap (A \times B) = (U_1 \cap A) \times (U_2 \cap B) \neq \emptyset$$

Therefore $p \in \overline{A \times B}$. ■

Theorem

X, Y are regular $\implies$ $X \times Y$ is regular.

Proof. Assume that X and Y are regular and assume $p \in X \times Y$ and $U \in \mathcal{U}_p$. Then there exists $U_1 \in \mathscr{T}_X$ and $U_2 \in \mathscr{T}_Y$ such that $p \in U_1 \times U_2 \subset U$. Now, since X and Y are regular, there exists $V_1 \in \mathscr{T}_X$ and $V_2 \in \mathscr{T}_y$ such that $p \in V_1 \times V_2$, $V_1 \subset \overline{V_1} \subset U_1$, and $V_2 \subset \overline{V_2} \subset U_2$. Furthermore, since $\overline{V_1}$ is closed in X and $\overline{V_2}$ is closed in Y, $\overline{V_1} \times \overline{V_2}$ (and hence $\overline{V_1 \times V_2}$) is closed in $X \times Y$. And so:

$$p \in V_1 \times V_2 \subset \overline{V_1 \times V_2} = \overline{V_1} \times \overline{V_2} \subset U_1 \times U_2$$

Therefore $X \times Y$ is regular. ■