Cauchy-Riemann Equations

Theorem

Let $f(z) = u(x,y) + iv(x,y)$ be differentiable on a domain D. The first partial derivatives (u_x, u_y, v_x, v_y) exist in D and satisfy the Cauchy-Rieman equations:

$u_x = v_y$ and $u_y = -v_x$

so that:

$$f'(z) = u_x + iv_x = v_y - iu_y$$

Proof

Assume $z \in D$

$$\begin{aligned}
f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\
&= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y} \\
&= \lim_{(\Delta x, \Delta y) \to (0,0)} \left[\frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i\frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y}\right]
\end{aligned}$$

Consider the path along the x-axis: $(\Delta x, 0) \to (0,0)$:

$$\begin{aligned}
f'(z) &= \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right] \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \\
&= u_x + iv_x
\end{aligned}$$

Now, consider the path along the y-axis: $(0, i\Delta y) \to (0,0)$:

$$\begin{aligned}
f'(z) &= \lim_{i\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}\right] \\
&= \lim_{i\Delta y \to 0} \left[\frac{v(x, y+\Delta y) - v(x,y)}{\Delta y} - i\frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}\right] \\
&= \lim_{i\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y} - i \lim_{i\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} \\
&= v_y - iu_y
\end{aligned}$$

Thus, in order for the limit to exist:

$$f'(z) = u_x + iv_x = v_y - iu_y$$

$\therefore u_x = u_y$ and $v_x = -u_y$

Corollary

Let $f(z) = u(x, y) + iv(x, y)$ be differentiable on a domain D:

$$f'(z) = f_x = -if_y$$

Proof

The CR equations hold in D
$f'(z) = u_x + iv_x = f_x$
$f'(z) = v_y - iu_y = -i(u_y + iv_y) = -if_y$

Example

$f(z) = z^2 = (x^2 - y^2) + i2xy$
$f'(z) = 2z$

$$\begin{aligned} u &= x^2 - y^2 & \qquad v &= 2xy \\ u_x &= 2x & \qquad v_x &= 2y \\ u_y &= -2y & \qquad v_y &= 2x \end{aligned}$$

$u_x = v_y = 2x$ and $v_x = -u_y = 2y$

$f'(z) = f_x = 2x + i2y = 2(x + iy) = 2z$
$f'(z) = -if_y = -i(-2y + i2x) = 2x + i2y = 2(x + iy) = 2z$