Product Spaces

Definition: Projection

Let X and Y be sets. The *projection* functions $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are defined by:

$$\pi_X(x, y) = x$$
$$\pi_Y(x, y) = y$$

Definition: Product Topology

Let X and Y be topological spaces. The *product topology* on the product $X \times Y$ is the topology with basis $\mathcal{B}$ given by:

$$\mathcal{B} = \{U \times V \,|\, U \in \mathscr{T}_X \text{ and } V \in \mathscr{T}_Y\}$$

Theorem

The basis for a product topology is in fact a basis.

Proof. Assume that X and Y are topological spaces and let:

$$\mathcal{B} = \{U \times V \,|\, U \in \mathscr{T}_X \text{ and } V \in \mathscr{T}_Y\}$$

Assume that $(a, b) \in X \times Y$. Since $a \in X$, there exists $U \in \mathscr{T}_X$ such that $a \in U$. Likewise, since $b \in Y$, there exists $V \in \mathscr{T}_Y$ such that $b \in V$. Therefore, $(a, b) \in U \times V \in \mathcal{B}$.

Now, assume that $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$ and assume that $(a, b) \in U_1 \times V_1 \cap U_2 \times V_2$. This means that $a \in U_1 \cap U_2$ and $b \in V_1 \cap V_2$. But U_1 and U_2 are generated by basic sets in $\mathscr{T}_X$, and V_1 and V_2 are generated by basic sets in $\mathscr{T}_Y$. So there exists basic sets $W_1 \in \mathscr{T}_X$ and $W_2 \in \mathscr{T}_Y$ such that $a \in W_1 \subset U_1 \cap U_2$ and $b \in W_2 \subset V_1 \cap V_2$. Therefore, $W_1 \times W_2 \in \mathcal{B}$ and $(a, b) \in W_1 \times W_2 \subset U_1 \times V_1 \cap U_2 \times V_2$.

Therefore $\mathcal{B}$ is a basis for a topology on $X \times Y$. ∎

Theorem

Let X, Y be topological spaces. If $A \subset X$ and $B \subset Y$ are closed sets then $A \times B$ is closed in $X \times Y$.

Proof. Since A and B are closed, $X - A$ and $Y - B$ are open. And so:

$$(X - A) \times (X - B) = (X \times Y) - (A \times B)$$

is open. Therefore $A \times B$ is closed. ∎

Theorem

Let X and Y be topological spaces. The product topology on $X \times Y$ is the same as the topology generated by the subbasis of inverse images of open sets under the projection functions, that is, the basis is given by:

$$\mathcal{B} = \left\{\pi_X^{-1}(U) \,\middle|\, U \in \mathscr{T}_X\right\} \cup \left\{\pi_Y^{-1}(V) \,\middle|\, V \in \mathscr{T}_Y\right\}$$

Proof. Assume $U \in \mathscr{T}_X$ and $V \in \mathscr{T}_y$:

$$\pi_X^{-1}(U) = \{(x,y) \,|\, x \in U, y \in Y\} = U \times Y$$
$$\pi_Y^{-1}(V) = \{(x,y) \,|\, x \in X, y \in V\} = X \times V$$

$$\pi_X^{-1}(U) \cap \pi_Y^{-1}(V) = (U \times Y) \cap (X \times V) = (U \cap X, V \cap Y) = (U, V)$$

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Example

The standard topology on $\mathbb{R}^2$ is not the same as the product topology on $\mathbb{R} \times \mathbb{R}$. The basic open sets in the standard topology are the open balls $B(p, \epsilon)$. But these open balls can be generated in the product topology by arbitrary unions of basic sets of the form $(p-\epsilon, p+\epsilon) \times (p-\epsilon, p+\epsilon)$, and thus the product topology is finer than the standard topology.

Notation

Let $\{X_i : i \in [n]\}$ be a finite family of topological spaces:

$$\prod_{i=1}^{n} X_i = X_1 \times X_2 \times \cdots \times X_n$$

An element $x = (x_1, x_2, ldots, x_n$ can be views as a function $f : [n] \to \bigcup_{i=1}^{n} X_i$ where $f(i) \in X_i$.

Definition: Infinite Product

Let $\{X_\alpha : \alpha \in \lambda\}$ be an arbitrary collection of topological spaces. The infinite *product* of these spaces is given by:

$$\prod_{\alpha \in \lambda} X_\alpha = \left\{ f : \lambda \to \bigcup_{\alpha \in \lambda} X_\alpha \,\middle|\, \forall \alpha \in \lambda, f(\alpha) \in X_\alpha \right\}$$

Definition

Let $\{X_\alpha : \alpha \in \lambda\}$ be an arbitrary collection of topological spaces and for each $\beta \in \lambda$ define the *projection* function $\pi_\beta : \prod_{\alpha \in \lambda} X_\alpha \to X_\beta$ by $\pi_\beta(f) = f(\beta)$. The *product topology* on $\prod_{\alpha \in \lambda} X_\alpha$ is the one generated by the subbasis of sets of the form $\pi_\beta^{-1}(U_\beta)$ where $U_\beta \in \mathscr{T}_{X_\beta}$.

Example

Let $\{X_n : n \in \mathbb{N}\}$ be a family of topological spaces and $\{U_n : n \in \mathbb{N}\}$ be a family of sets such that $U_n \in \mathscr{T}_{X_n}$:

$$\begin{aligned}\pi_{X_1}^{-1}(U_1) \cap \pi_{X_3}^{-1}(U_3) \cap \pi_{X_5}^{-1}(U_4) &= (U_1 \times X_2 \times X_3 \times X_4 \times X_5 \times \cdots) \cap \\ &\quad (X_1 \times X_2 \times U_3 \times X_4 \times X_5 \times \cdots) \cap \\ &\quad (X_1 \times X_2 \times X_3 \times U_4 \times X_5 \times \cdots) \\ &= (U_1 \times X_2 \times U_3 \times U_4 \times X_5 \times \cdots)\end{aligned}$$

The basis elements are the entire space except for a finite number of components.