Interlacing Inequalities

Theorem: Rank-One Interlace

Let $A, B \in M_n$ such that $B = \vec{x}\vec{x}^*$ for some $\vec{x} \in \mathbb{C}^n$:

$$\lambda_1(A) \leq \lambda_1(A+B) \leq \lambda_2(A) \leq \lambda_2(A+B) \leq \cdots \leq \lambda_n(A) \leq \lambda_n(A+B)$$

Proof

Recall that $\mathrm{Sp}(B) = \mathrm{Sp}(\vec{x}\vec{x}^*) = \{0^{(n-1)}, \vec{x}^*\vec{x}\}$.

From Weyl with $j = i$ and $k = 1$:

$$\lambda_i(A) + \lambda_1(B) \leq \lambda_i(A+B)$$

But $\lambda_1(B) = 0$

$\therefore \lambda_i(A) \leq \lambda_i(A+B)$

Now, from Weyl with $j = i+1$ and $k = n-1$:

$$\lambda_i(A+B) \leq \lambda_{i+1}(A) + \lambda_{n-1}(B)$$

But $\lambda_{n-1}(B) = 0$

$\therefore \lambda_i(A+B) \leq \lambda_{i+1}(A)$

Note that the converse is also true: Given interlacing sets of n numbers $\{\lambda_k\}$ and $\{\mu_k\}$, there exists Hermitian matrices A and $B = \vec{x}\vec{x}^*$ with these interlacing sets as their respective eigenvalues (proof omitted). To find such a $\vec{x}$, solve the following SOLE:

$$\mathrm{Sp}\left\{\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix} + \vec{x}\vec{x}^*\right\} = \{\mu_k \mid 1 \leq k \leq n\}$$

Theorem

Let $A \in M_n$ be Hermitian and let B be the leading principal submatrix of A (also Hermitian):

$$\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \lambda_2(B) \leq \cdots \leq \lambda_{n-1}(A) \leq \lambda_{n-1}(B) \leq \lambda_n(A)$$

Example

$$A = \begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix} \qquad B = \begin{bmatrix}1\end{bmatrix}$$

$$\begin{aligned}(1-\lambda)(4-\lambda) &= 2(2)\\ 4 - 5\lambda + \lambda^2 &= 4\\ \lambda^2 - 5\lambda &= 0\\ \lambda(\lambda - 5) &= 0\\ \lambda &= 0, 5\end{aligned}$$

$0 \leq 1 \leq 5$

<u>Proof</u>

Let $A = \begin{bmatrix} B & \vec{y} \\ \vec{y}^* & a \end{bmatrix}$ where B is the leading principal submatrix of A, $\vec{y} \in \mathbb{C}^{n-1}$, and $a \in \mathbb{C}$.

Let $\{\vec{u}_1, \dots, \vec{u}_n\}$ be eigenvectors of A corresponding to $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$.
Let $\{\vec{v}_1, \dots, \vec{v}_{n-1}\}$ be eigenvectors of B corresponding to $\lambda_1(B) \leq \cdots \leq \lambda_{n-1}(B)$.
Let $\vec{w}_k = \begin{bmatrix} \vec{v}_k \\ 0 \end{bmatrix}$ for $1 \leq k \leq n-1$.

Let $S_1 = \operatorname{span}\{\vec{u}_i, \dots \vec{u}_n\}$
Let $S_2 = \operatorname{span}\{\vec{w}_1, \dots \vec{w}_i\}$
$\dim(S_1 \cap S_2) \geq \dim(S_1) + \dim(S_2) - n = (n - i + 1) + i - n = 1$
Thus, there exists $\vec{x} \in S_1 \cap S_2$ such that $\vec{x} \neq \vec{0}$

Let $\vec{x} \in S_1 \cap S_2$ such that $\vec{x} = \begin{bmatrix} \vec{v} \\ 0 \end{bmatrix}$ where $\vec{v} \in \operatorname{Sp}\{\vec{v}_1, \dots, \vec{v}_i\}$.
Since $\vec{x} \in S_1$, by the key lemma:

$$\lambda_i(A) \leq \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}}$$

Since $\vec{v} \in \operatorname{Sp}\{\vec{v}_1, \dots, \vec{v}_i\}$, by the key lemma:

$$\frac{\vec{v}^* B \vec{v}}{\vec{v}^* \vec{v}} \leq \lambda_i(B)$$

But:

$$\frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} = \frac{\begin{bmatrix} \vec{v}^* & 0 \end{bmatrix} \begin{bmatrix} B & \vec{y} \\ \vec{y}^* & a \end{bmatrix} \begin{bmatrix} \vec{v} \\ 0 \end{bmatrix}}{\begin{bmatrix} \vec{v}^* & 0 \end{bmatrix} \begin{bmatrix} \vec{v} \\ 0 \end{bmatrix}} = \frac{\begin{bmatrix} \vec{v}^* B & \vec{v}^* \vec{y} \end{bmatrix} \begin{bmatrix} \vec{v} \\ 0 \end{bmatrix}}{\vec{v}^* \vec{x}} = \frac{\vec{v}^* B \vec{v}}{\vec{v}^* \vec{x}}$$

Therefore:

$$\lambda_i(A) \leq \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} = \frac{\vec{v}^* B \vec{v}}{\vec{v}^* \vec{x}} \leq \lambda_i(B)$$

Let $S_3 = \operatorname{span}\{\vec{u}_1, \dots, \vec{u}_{i+1}\}$
Let $S_4 = \operatorname{span}\{\vec{w}_i, \dots, \vec{w}_{n-1}\}$
$\dim(S_3 \cap S_4) \geq \dim(S_3) + \dim(S_4) - n = (i + 1) + [(n - 1) - i + 1] - n = 1$
Thus, there exists $\vec{x} \in S_3 \cap S_4$ such that $\vec{x} \neq \vec{0}$

Let $\vec{x} \in S_3 \cap S_4$ such that $\vec{x} = \begin{bmatrix} \vec{v} \\ 0 \end{bmatrix}$ where $\vec{v} \in \operatorname{Sp}\{\vec{v}_i, \dots, \vec{v}_{n-1}\}$.
Since $\vec{x} \in S_3$, by the key lemma:

$$\frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} \leq \lambda_{i+1}(A)$$

Since $\vec{v} \in \mathrm{Sp}\{\vec{v}_i, \ldots, \vec{v}_{n-1}\}$, by the key lemma:

$$\lambda_i(B) \leq \frac{\vec{v}^* B \vec{v}}{\vec{v}^* \vec{v}}$$

Therefore:

$$\lambda_i(B) \leq \frac{\vec{v}^* B \vec{v}}{\vec{v}^* \vec{x}} = \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} \leq \lambda_{i+1}(A)$$

Note that the converse is also true: Given interlacing sets of n numbers $\{\lambda_k \mid 1 \leq k \leq n\}$ and $\{\mu_k \mid 1 \leq k \leq n-1\}$, there exists Hermitian matrix A with leading princple submatrix B with these interlacing sets as their respective eigenvalues (proof omitted). To find such an A, solve the following SOLE for a suitable $\vec{y} \in \mathbb{C}^{n-1}$:

$$\mathrm{Sp}\left\{\left[\begin{array}{ccc|c} \mu_1 & & 0 & y_1 \\ & \ddots & & \vdots \\ 0 & & \mu_{n-1} & y_{n-1} \\ \hline \bar{y}_1 & \cdots & \bar{y}_{n-1} & \mu' \end{array}\right]\right\} = \{\lambda_1, \ldots, \lambda_n$$

where:

$$\mu' = \sum_{k=1}^{n} \lambda_k - \sum_{k=1}^{n-1} \mu_k$$