Eigenvalues and Eigenvectors

Definition

Let $A \in M_n(\mathbb{C})$ and let $\vec{x}$ be a $n \times 1$ column vector. The *eigen equation* for A is given by:

$$A\vec{x} = \lambda\vec{x}$$

To say that λ is an *eigenvalue* of A means that there exists a non-zero $\vec{x}$ such that the eigen equation is true. Such a λ is called an $eigenvalue$ of A and the corresponding non-zero $\vec{x}$ is called an *eigenvector* of A with respect to λ. The ordered pair $(\lambda, \vec{x})$ is called an *eigen pair* of A.

Example

$$A = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \lambda = -3$$

$$A\vec{x} = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -3\vec{x}$$

Definition

Let $A \in M_n(\mathbb{C})$. To say that A is *nilpotent* means $\exists\, k \in \mathbb{Z}^+$ such that $A^k = 0$.

Theorem

Let $A \in M_n(\mathbb{C})$ be nilpotent:

λ is an eigenvalue of $A \implies \lambda = 0$

Proof

Assume λ is an eigenvalue of A
$\exists\, \vec{x} \neq \vec{0}, A\vec{x} = \lambda\vec{x}$
$A^{k-1}(A\vec{x}) = A^{k-1}(\lambda\vec{x})$
$A^k\vec{x} = \lambda(A^{k-1}\vec{x})$
$0\vec{x} = \lambda(\lambda^{k-1}\vec{x})$
$\vec{0} = \lambda^k\vec{x}$
But $\vec{x}$ is an eigenvector and thus $\vec{x} \neq \vec{0}$, so:
$\lambda^k = 0$

$\therefore \lambda = 0$

Definition

Let $A \in M_n(\mathbb{C})$. To say that A is *idempotent* means $A^2 = A$.

Theorem

Let $A \in M_n(\mathbb{C})$ be idempotent:

λ is an eigenvalue of $A \implies \lambda = 0$ or 1

Proof

Assume λ is an eigenvalue of A
$\exists\, \vec{x} \neq \vec{0}, A\vec{x} = \lambda\vec{x}$
$A(A\vec{x}) = A(\lambda\vec{x})$
$A^2\vec{x} = \lambda(A\vec{x})$
$A^2\vec{x} = \lambda(\lambda\vec{x})$
$A\vec{x} = \lambda^2\vec{x}$
$\lambda^2\vec{x} = \lambda\vec{x}$
$(\lambda^2 - \lambda)\vec{x} = \vec{0}$
$\lambda(\lambda - 1)\vec{x} = \vec{0}$
But $\vec{x}$ is an eigenvector and thus $\vec{x} \neq \vec{0}$, so:
$\lambda(\lambda - 1) = 0$

$\therefore \lambda = 0$ or 1

Notation

Let $A \in M_n(\mathbb{C})$. The set of all distinct eigenvalues of A is denoted by $\sigma(A)$.

Theorem: Eigenvalue Criteria

Let $A \in M_n(\mathbb{C})$:

λ is an eigenvalue of $A \iff \det(\lambda I_n - A) = 0$

Thus, $\lambda I_n - A$ is singular.

Proof

λ is an eigenvalue of A
$\iff A\vec{x} = \lambda\vec{x}$
$\iff A\vec{x} - \lambda\vec{x} = \vec{0}$
$\iff (A - \lambda I_n)\vec{x} = \vec{0}$
$\iff (A - \lambda I_n)\vec{x} = \vec{0}$ has non-trivial solutions
$\iff A - \lambda I_n$ is singular
$\iff \det(A - \lambda I_n) = 0$
$\iff \det(\lambda I_n - A) = 0$

Example

$A = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix} \qquad A - \lambda I_2 = \begin{bmatrix} \lambda + 1 & -2 \\ -3 & \lambda \end{bmatrix}$

$\det(A - \lambda I_2) = \lambda(\lambda + 1) - 6 = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2) = 0$

$\sigma(A) = \{-3, 2\}$