Counting

Since simple events are by definition mutually exclusive, the following corollary follows from the previous axioms and theorems:

Corollary

Let E be an event composed of a countable number of outcomes ω:

$$P(E) = \sum_{\omega \in E} P(\omega)$$

Theorem

Let $\mathcal{S}$ be a sample space consisting of a finite number of equally likely outcomes. The probability of each outcome ω is given by:

$$P(\omega) = \frac{1}{|\mathcal{S}|}$$

Proof. Let $p =$ the equally likely probability of each ω:

$$1 = \sum_{i=1}^{|\mathcal{S}|} P(\omega) = \sum_{i=1}^{|\mathcal{S}|} p = p|\mathcal{S}|$$

Therefore:

$$p = \frac{1}{|\mathcal{S}|}$$

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Corollary

Let $\mathcal{S}$ be a sample space consisting of a finite number of equally likely outcomes and let $E \subseteq S$:

$$P(E) = \frac{|E|}{|\mathcal{S}|}$$

Examples

- Flip a fair coin: $P(H) = P(T) = \frac{1}{2}$
- Toss a fair die:
  - Simple events: $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$
  - $P\left(\{\text{an even number}\}\right) = \frac{3}{6} = \frac{1}{2}$
  - $P\left(\{\text{at least 5}\}\right) = \frac{2}{6} = \frac{1}{3}$
  - $P\left(\{\text{not 3}\}\right) = \frac{5}{6}$

- Toss a fair coin 5 times:
  - $P(\{\text{at least one head}\}) = 1 - P(\{\text{no heads}\}) = 1 - \frac{1}{2^5} = \frac{31}{32}$

When events are large and complicated, combinatorics can help.

Theorem: Fundamental Counting Principle

Suppose an experiment can be performed in a sequence of k steps such that there are n_i ways to perform the i^{th} step. The total number of outcomes for the experiment is given by:

$$N = \prod_{i=1}^{k} n_i$$

Examples

1. A subway restautant provides 5 kinds of bread, 4 kinds of cheese, 4 kinds of meat, and 6 kinds of sauces. In how many ways can you order a sandwich:

   $N = 5 \cdot 4 \cdot 4 \cdot 6 = 480$ ways

2. How many different CA driver's licenses are there (1 capital letter followed by 7 numbers)? How many driver's licenses have all repeated digits? All distinct digits?

   total: $26 \cdot 10^7 = 260,000,000$

   repeated: $26 \cdot 10 = 260$

   distinct: $26 \cdot P(10, 7) = 26 \cdot \frac{10!}{(10-3)!} = 15,724,800$

3. How many ordered lists of size 3 can be made from a set $S = \{a, b, c, d\}$?
   (a) with repetition allowed: 4^3
   (b) with repetition not allowed: $4 \cdot 3 \cdot 2 = 24$

Definition: Permutation

A *permutation* of k elements from a set with n elements is a set of ordered lists with:

$$P(n, k) = \frac{n!}{(n-k)!}$$

elements.

Example

List all permutations of size $r = 3$ chosen from the set $S = \{a, b, c, d\}$. How many are there? What if $r = 4$?

abc	bac	cab	dab
abd	bad	cad	dac
acb	bca	cba	dba
acd	bcd	cbd	dbc
adb	bda	cda	dca
adc	bdc	cdb	dcb

$$P(4,3) = \frac{4!}{(4-3)!} = 24$$

$$P(4,4) = \frac{4!}{(4-4)!} = 24$$

Example

In how many different ways can 5 people be arranged in a row? Along a circle?

$$5! = 120$$

First, seat one person, then the other four. Each pattern is simply repeated depending on where the first person sits:

$$1 \cdot 4! = 24$$

Example

How many 3-digit numbers are divisible by 5?

The number cannot start with a 0 and must end with a 0 or 5:

$$9 \cdot 10 \cdot 2 = 180$$

Example: The Birthday Problem

Find the probability p that no two people in a class of 35 have a common birthday (i.e., all have different birthdays). Assume that people's birthdays are equally likely to occur among the 365 days of the year and ignore leap years.

$$\frac{P(365,35)}{365^{35}} = 0.1856$$

Definition: Combination

A *combination* of k elements from a set with n elements is a set of subsets (unordered lists) with:

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

elements.

Example

List all combinations of size 3 chosen from the set $S = \{a, b, c, d\}$:

$$\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}$$

$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = 4$$

Example

Consider the problem of choosing 4 members from a group of 10 to work on a special project.

a) Suppose two people A and B really like each other, so that they must be simultaneously chosen or skipped. How many distinct four-person teams can be chosen?

$$\binom{8}{2} + \binom{8}{2}$$

b) Suppose two people A and B really hate each other, so that they cannot both be selected for the project. How many distinct four-person teams can be chosen?

$$\binom{8}{4} + \binom{8}{3} + \binom{8}{3}$$

Example

An urn has 5 red balls and 7 blue balls. Suppose you randomly select 5 balls from the urn. What is the probability that your hand has exactly 3 red balls?

$$\frac{\binom{5}{3}\binom{7}{2}}{\binom{12}{5}}$$

Example

An ordinary deck of 52 cards is divided into 4 suits (hearts, diamonds, spades, and clubs) and 13 ranks (2,3,4,5,6,7,8,9,10,J,Q,K,A). Suppose you randomly draw 5 cards from a deck of 52. What is the probability that you have a:

a) four of a kind (4 cards of the same rank, and one side card)?

$$\frac{\binom{13}{1}\binom{48}{1}}{\binom{52}{5}}$$

b) flush (5 cards of the same suit)?

$$\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}$$