Chi-square Distributions

Assuming $X_i \overset{\text{iid}}{\sim} \mathrm{N}(\mu, \sigma^2)$ where neither μ nor σ is known, it is known that an unbiased estimator for σ^2 is given by:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})$$

Now, S^2 can be used to construct a $1 - \alpha$ confidence interval for σ^2.

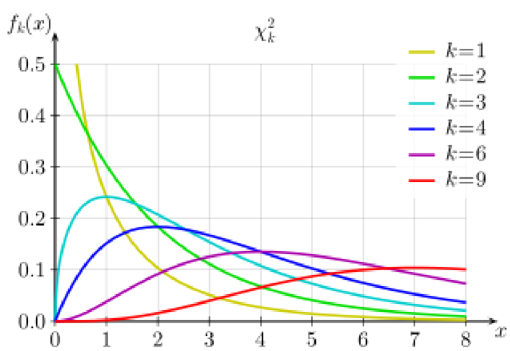
Definition: Chi-square Distribution

The *chi-square distribution* with k degrees of freedom is a continuous distribution whose pdf has the form:

$$f(x) = C \left(\frac{x}{2}\right)^{\frac{k}{2}-1} e^{-\frac{x}{2}}$$

for all $x > 0$.

Properties: Chi-square Distributions



1. If $Z_i \overset{\text{iid}}{\sim} \mathrm{N}(0, 1)$ then $X = \sum Z_i^2 \sim \chi^2(k)$
2. $E(X) = k$
3. $V(X) = 2k$

Theorem

Let $X_i \overset{\text{iid}}{\sim} \mathrm{N}(\mu, \sigma^2)$ such that μ and σ are unknown:

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

Theorem

Let $X_i \overset{\text{iid}}{\sim} \mathrm{N}(\mu, \sigma^2)$ such that μ and σ are unknown. A $1 - \alpha$ confidence interval for σ^2 is given by:

$$\left(\frac{(n-1)s^2}{\chi^2_{\frac{a}{2}, n-1}}, \frac{(n-1)s^2}{\chi^2_{1-\frac{a}{2}, n-1}} \right)$$

Proof.

$$P\left(a < \frac{(n-1)S^2}{\sigma^2} < b\right) = 1 - \alpha$$

Let $a = \chi^2_{1-\frac{a}{2},n-1}$ and $b = \chi^2_{\frac{a}{2},n-1}$.

$$P\left(\chi^2_{1-\frac{a}{2},n-1} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{\frac{a}{2},n-1}\right) = 1 - \alpha$$
$$P\left(\frac{(n-1)S^2}{\chi^2_{\frac{a}{2},n-1}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{1-\frac{a}{2},n-1}}\right) = 1 - \alpha$$

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<u>Example</u>

A sample carton of brown eggs from a farm has $s^2 = 4.69$. Assuming a normal population with unknown variance, obtain a 95% confidence interval for σ^2.

$$1 - \alpha = 0.95$$
$$\alpha = 1 - 0.95 = 0.05$$
$$\frac{\alpha}{2} = \frac{0.05}{2} = 0.025$$
$$1 - \frac{\alpha}{2} = 1 - 0.025 = 0.975$$

$$\chi^2_{0.025,11} = 21.920$$
$$\chi^2_{0.975,11} = 3.816$$

$$(n-1)s^2 = 11(4.69) = 51.59$$

$$\frac{(n-1)s^2}{\chi^2_{\frac{a}{2},n-1}} = \frac{51.59}{21.92} \approx 2.35$$
$$\frac{(n-1)s^2}{\chi^2_{1-\frac{a}{2},n-1}} = \frac{51.59}{3.816} \approx 13.52$$

Thus, we are 95% confident that the true value of σ^2 is contained in $(2.35, 13.52)$.