Normal Matrices

Definition

Let $A \in M_n$. To say that A is *normal* means:

$$AA^* = A^*A$$

Examples:

1). Unitary ($UU^* = I$)

2). Hermitian ($H^* = H$)

3). Skew-Hermitian ($H^* = -H$)

4). Positive Definite ($\forall \vec{x} \in \mathbb{C}^n - \{\vec{x}\}, x^*Ax > 0$)

5). Positive Semidefinite ($\forall \vec{x} \in \mathbb{C}^n, x^*Ax \geq 0$)

Lemma

Let $T \in UT(n)$:

$$T \text{ normal} \implies T \text{ diagonal}$$

Proof

Proof by induction on n

Base Case: $n = 1$

$T = \left[\lambda\right]$ is diagonal.

Assume that $T \in UT(n-1)$ normal $\implies$ T diagonal.

Assume $T \in UT(n)$ is normal

Let $T = \left[\begin{array}{c|c} S & \vec{x} \\ \hline 0 & a \end{array}\right]$ where $S \in UT(n-1)$, $\vec{x} \in \mathbb{C}^{n-1}$, and $a \in \mathbb{C}$

Now, since T is normal:

$$\begin{aligned} TT^* &= T^*T \\ \left[\begin{array}{c|c} S & \vec{x} \\ \hline 0 & a \end{array}\right] \left[\begin{array}{c|c} S^* & 0 \\ \hline \vec{x}^* & \bar{a} \end{array}\right] &= \left[\begin{array}{c|c} S^* & 0 \\ \hline \vec{x}^* & \bar{a} \end{array}\right] \left[\begin{array}{c|c} S & \vec{x} \\ \hline 0 & a \end{array}\right] \\ \left[\begin{array}{c|c} SS^* + \vec{x}\vec{x}^* & \bar{a}\vec{x} \\ \hline a\vec{x}^* & |a|^2 \end{array}\right] &= \left[\begin{array}{c|c} S^*S & S^*\vec{x} \\ \hline \vec{x}^*S & \vec{x}^*\vec{x} + |a|^2 \end{array}\right] \end{aligned}$$

From the lower right quadrant we get:

$$|a|^2 = \vec{x}^*\vec{x} + |a|^2$$

And so $\vec{x}^*\vec{x} = 0$, and thus $\vec{x} = 0$:

$$T = \left[\begin{array}{c|c} S & 0 \\ \hline 0 & a \end{array}\right]$$

Now, from the upper left quadrant we get:

$$SS^* + \vec{x}\vec{x}^* = S^*S$$

and so $SS^* = S^*S$, indicating that S is normal. Thus, by the inductive assumption, S is also diagonal.

Therefore, T is diagonal.

<u>Theorem</u>

Let $A \in M_n$. TFAE:

1). A is normal ($AA^* = A^*A$)

2). A is unitary diagonalizable ($A = UDU^*$)

3). $\mathrm{tr}(A^*A) = \sum_{k=1}^{n} |\lambda_k|^2$, where $\lambda_k \in \mathrm{Sp}(A)$

<u>Proof</u>

$1 \implies 2$: Assume A is normal

There exists unitary U such that $A = UTU^*$ for some $T \in UT(n)$ (Schur)
$T = U^*AU$
Now, using the fact that A is normal:

$$\begin{array}{rcl} AA^* & = & A^*A \\ U^*AUU^*A^*U & = & U^*A^*UU^*AU \\ (U^*AU)(U^*AU)^* & = & (UAU^*)^*(U^*AU) \\ TT^* & = & T^*T \end{array}$$

T is triangular and normal and thus, by the above lemma, T is diagonal

Therefore A is unitary diagonalizable.

$2 \implies 1$: Assume A is unitary diagonalizable

$A = UDU^*$ for some diagonal matrix D

$$\begin{array}{rcl} AA^* & = & (UDU^*)(UDU^*)^* \\ & = & UDU^*UD^*U^* \\ & = & UDD^*U^* \\ & = & UD^*DU^* \\ & = & UD^*U^*UDU^* \\ & = & (UDU^*)^*(UDU^*) \\ & = & A^*A \end{array}$$

Therefore A is normal.

$2 \implies 3$: Assume A is unitary diagonalizable

There exists unitary U such that:

$$A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*$$

$$\begin{aligned}
\operatorname{tr}(A^*A) &= \operatorname{tr}\left(\left(U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*\right)^* \left(U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*\right)\right) \\
&= \operatorname{tr}\left(U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}^* U^*U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*\right) \\
&= \operatorname{tr}\left(U \begin{bmatrix} \overline{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*\right) \\
&= \operatorname{tr}\left(U^*U \begin{bmatrix} \overline{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\right) \\
&= \operatorname{tr}\left(\begin{bmatrix} \overline{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\right) \\
&= \operatorname{tr}\left(\begin{bmatrix} |\lambda_1|^2 & & 0 \\ & \ddots & \\ 0 & & |\lambda_n|^2 \end{bmatrix}\right) \\
&= \sum_{k=1}^{n} |\lambda_k|^2
\end{aligned}$$

$3 \implies 2$: Assume $\operatorname{tr}(A^*A) = \sum_{k=1}^{n} |\lambda_k|^2$

There exists unitary U and $T \in UT(n)$ such that:

$$A = UTU^*$$

such that $T = \begin{bmatrix} \lambda_1 & & t_{ij} \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$$\begin{aligned}
\operatorname{tr}(A^*A) &= \operatorname{tr}((UTU^*)^*(UTU^*)) \\
&= \operatorname{tr}(UT^*U^*UTU^*) \\
&= \operatorname{tr}(UT^*TU^*) \\
&= \operatorname{tr}(U^*UT^*T) \\
&= \operatorname{tr}(T^*T) \\
\sum_{k=1}^{n} |\lambda_k|^2 &= \sum_{k=1}^{n} |\lambda_k|^2 + \sum_{i<j} |t_{ij}|^2
\end{aligned}$$

So $\sum |t_{ij}|^2 = 0$ and thus $t_{ij} = 0$

Therefore, T is diagonal and A is unitary diagonalizable.