Algebraic Numbers

Definition

To say that a number is *algebraic* means that there exists a polynomial equation of the form $\sum_{k=0}^{n} c_k x^k$ such that $n \geq 1$, $c_k \in \mathbb{Z}$, and $c_n \neq 0$ for which the number is a solution. The set of all algebraic numbers is denoted by $\mathbb{A}$.

Theorem

$\mathbb{Q} \subset \mathbb{A}$

Proof

Assume $r \in \mathbb{Q}$.
$\exists p, q \in \mathbb{Z}, r = \frac{p}{q}, q \neq 0$
Consider the polynomial equation $qx - p = 0$.
r is a solution.
$\therefore r \in \mathbb{A}$

Example

Show that $\sqrt{2} \in \mathbb{A}$.

Let $x = \sqrt{2}$.
$x^2 = 2$
$x^2 - 2 = 0$
$\sqrt{2}$ is a solution to this polynomial equation.
$\therefore \sqrt{2} \in \mathbb{A}$

Example

Show that $\sqrt[3]{2 + \sqrt{5}} \in \mathbb{A}$.

Let $x = \sqrt[3]{2 + \sqrt{5}}$.
$x^3 = 2 + \sqrt{5}$
$x^3 - 2 = \sqrt{5}$
$(x^3 - 2)^2 = 5$
$x^6 - 4x^3 + 4 = 5$
$x^6 - 4x^3 - 1 = 0$
$\sqrt[3]{2 + \sqrt{5}}$ is a solution to this polynomial equation.
$\therefore \sqrt[3]{2 + \sqrt{5}} \in \mathbb{A}$