Roots

Definition

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. To say that $z_1 = z_2$ means:

1). $r_1 = r_2$

2). $\theta_1 = \theta_2 + 2\pi k, k \in \mathbb{Z}$

Theorem

The equation $z^n = z_0$ has n roots all placed on the circle with center at 0 and radius $\sqrt[n]{|z_0|}$, starting at $\frac{\theta_0}{n}$ and evenly spaced by $\frac{2\pi}{n}$.

Proof

Let $z = re^{i\theta}$ and $z_0 = r_0 e^{1\theta_0}$.

$$\begin{array}{rcl} z^n & = & z_0 \\ \left(re^{i\theta}\right)^n & = & r_0 e^{1\theta_0} \\ r^n e^{i(n\theta)} & = & r_0 e^{i\theta_0} \\ r^n = r_0 & \text{and} & n\theta = \theta_0 + 2\pi k \\ r = \sqrt[n]{r_0} & \text{and} & \theta = \dfrac{\theta_0}{n} + k\dfrac{2\pi}{n} \end{array}$$

Note that unique roots are obtained for $0 \leq k < n$.

Definition

The n^{th} roots of $z = re^{i\theta}$, denoted c_k, are given by:

$$c_k = \sqrt[n]{r} e^{i(\frac{\theta}{n} + \frac{2\pi k}{n})}$$

To say that c_0 is the *principle root* means $\theta = \operatorname{Arg} z$.

Let $\omega_n = e^{i\frac{2\pi}{n}}$.

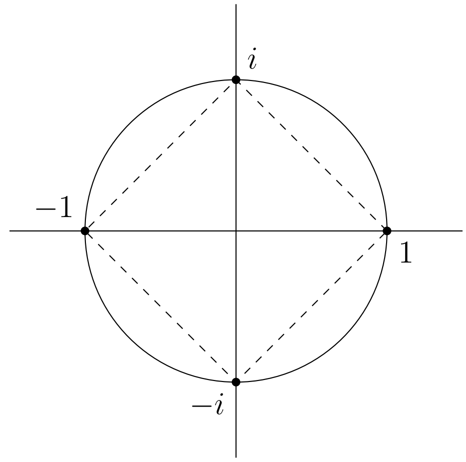
$$\omega_n^k = e^{i\frac{2\pi k}{n}}$$

$$c_k = \sqrt[n]{r} e^{i(\frac{\theta}{n} + \frac{2\pi k}{n})} = \sqrt[n]{r} e^{i(\frac{\theta}{n})} e^{i(\frac{2\pi k}{n})} = c_0 \omega_n^k$$

Example

Find the fourth roots of 1.

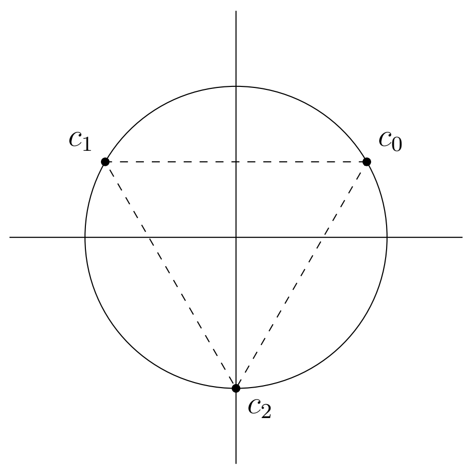
$$
\begin{aligned}
z^4 &= 1 \\
z^4 &= e^{i(0+2\pi k)} \\
z &= \left[e^{i(2\pi k)}\right]^{\frac{1}{4}} \\
z &= e^{i(k\frac{\pi}{2})}, k = 0, 1, 2, 3 \\
z &= 1, i, -1, -i \\
\omega &= e^{i\frac{2\pi}{4}} = e^{i\frac{\pi}{2}} = i \\
c_0 &= 1 \\
c_1 &= c_0 \cdot i = 1 \cdot i = i \\
c_2 &= c_1 \cdot i = i \cdot i = -1 \\
c_3 &= c_2 \cdot i = -1 \cdot i = -i
\end{aligned}
$$



Example

Find the cubed roots of i.

$$
\begin{aligned}
z^3 &= i \\
z^3 &= e^{i(\frac{\pi}{2}+2\pi k)} \\
z &= \left[e^{i(\frac{\pi}{2}+2\pi k)}\right]^{\frac{1}{3}} \\
z &= e^{i(\frac{\pi}{6}+\frac{2}{3}\pi k)}, k = 0, 1, 2 \\
z &= e^{i\frac{\pi}{6}}, e^{i\frac{5\pi}{6}}, e^{i\frac{9\pi}{6}} = e^{i\frac{3\pi}{2}} = -i \\
\omega &= e^{i\frac{2\pi}{3}} \\
c_0 &= e^{i\frac{\pi}{6}} \\
c_1 &= e^{i\frac{\pi}{6}} e^{i\frac{2\pi}{3}} = e^{i\frac{5\pi}{6}} \\
c_2 &= e^{i\frac{5\pi}{6}} e^{i\frac{2\pi}{3}} = e^{i\frac{3\pi}{2}} = -i
\end{aligned}
$$



Note that the roots are the vertices of an n-regular polygon.

Theorem

$\forall\, z \in \mathbb{C}$:

$$z^n - 1 = \prod_{k=0}^{n-1} \left(1 - e^{i\frac{2\pi k}{n}}\right)$$

Proof

Assume $z \in \mathbb{C}$

The roots of $z^n - 1 = 0$ are the n^{th} roots of unity:

$$z = e^{i\frac{2\pi k}{n}}, \qquad 0 \leq k < n$$

$$\therefore z^n - 1 = \prod_{k=0}^{n-1}\left(1 - e^{i\frac{2\pi k}{n}}\right)$$

Corollary

$\forall\, n \in \mathbb{N}$:

$$\prod_{k=1}^{n-1}\left|1 - e^{i\frac{2\pi k}{n}}\right| = n$$

Proof

Assume $n \in \mathbb{N}$

$$z^n - 1 = \prod_{k=0}^{n-1}\left(1 - e^{i\frac{2\pi k}{n}}\right) = (z-1)\prod_{k=1}^{n-1}\left(1 - e^{i\frac{2\pi k}{n}}\right)$$

Assume $z \neq 1$

$$\frac{z^n - 1}{z - 1} = \prod_{k=1}^{n-1}\left(1 - e^{i\frac{2\pi k}{n}}\right)$$

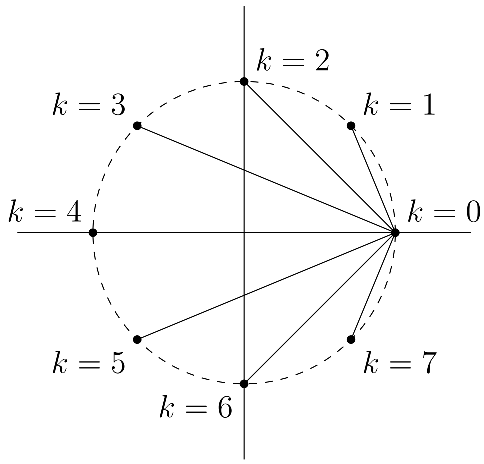
$$\left|\frac{z^n - 1}{z - 1}\right| = \left|\prod_{k=1}^{n-1}\left(1 - e^{i\frac{2\pi k}{n}}\right)\right| = \prod_{k=1}^{n-1}\left|1 - e^{i\frac{2\pi k}{n}}\right|$$

Let $z \to 1$

$$\begin{aligned} \lim_{z\to 1}\left|\frac{z^n - 1}{z - 1}\right| &= \left|\lim_{z\to 1}\frac{z^n - 1}{z - 1}\right| \\ &= \left|\lim_{z\to 1}\frac{nz^{n-1}}{1}\right| \qquad \text{(L'Hospital)} \\ &= |n| \\ &= n \end{aligned}$$

$$\therefore \prod_{k=1}^{n-1}\left|1 - e^{i\frac{2\pi k}{n}}\right| = n$$

Geometrically, this is the product of the line segments from 1 to each of the other $n-1$ roots. For example, for $n = 8$:



<u>Corollary</u>

$\forall\, n \in \mathbb{N}$:

$$2^{n-1} \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = n$$

<u>Proof</u>

Assume $n \in \mathbb{N}$

$$\begin{aligned}
\prod_{k=1}^{n-1} \left| 1 - e^{i\frac{2\pi k}{n}} \right| &= n \\
\prod_{k=1}^{n-1} \left| e^{i\frac{k\pi}{n}} \right| \left| e^{-\frac{k\pi}{n}} - e^{i\frac{\pi k}{n}} \right| &= n \\
\prod_{k=1}^{n-1} (1) \left| 2i \sin \frac{k\pi}{n} \right| &= n \\
2^{n-1} \prod_{k=1}^{n-1} \left| \sin \frac{k\pi}{n} \right| &= n
\end{aligned}$$

But $0 < \frac{k\pi}{n} < \pi$ for $0 < k < n$, so only in QI and QII

$$\therefore 2^{n-1} \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = n$$