Cayley-Hamilton

Theorem: Cayley-Hamilton

Let $A \in M_n$ and $p_A(t)$ be its characteristic polynomial:

$$p_A(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

Example

$A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$

$p_A(t) = t^2 - 3t + 2$

$\begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}^2 - 3\begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} + 2I = \begin{bmatrix} 7 & 3 \\ -6 & -2 \end{bmatrix} + \begin{bmatrix} -9 & -3 \\ 6 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Lemma

Let $S_1, \dots, S_m \in UT(n)$ such that $(S_k)_{kk} = 0$:

$$\prod_{k=1}^{n} S_k = 0$$

Example

$\begin{bmatrix} 0 & 4 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 7 & 8 \\ 0 & 0 & 9 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 9 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 46 \\ 0 & 0 & 33 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & 9 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Proof

Proof by induction on n:

Base Case: $n = 1$:

$$S_1 = \begin{bmatrix} 0 \end{bmatrix} \quad \text{(done).}$$

Assume true for $n - 1$

Consider $S_1, \dots, S_n$

Let $T_1, \ldots, T_n$ be the corresponding leading principal $(n-1)x(n-1)$ matrices:

$$\begin{aligned}
\prod_{k=1}^{n} S_k &= \left[\begin{array}{c|c} T_1 & * \\ \hline 0 & * \end{array}\right] \left[\begin{array}{c|c} T_2 & * \\ \hline 0 & * \end{array}\right] \cdots \left[\begin{array}{c|c} T_{n-1} & * \\ \hline 0 & * \end{array}\right] \left[\begin{array}{c|c} T_n & * \\ \hline 0 & 0 \end{array}\right] \\
&= \left[\begin{array}{c|c} \prod_{k=1}^{n-1} T_k & * \\ \hline 0 & * \end{array}\right] \left[\begin{array}{c|c} T_n & * \\ \hline 0 & 0 \end{array}\right] \\
&= \left[\begin{array}{c|c} 0 & * \\ \hline 0 & * \end{array}\right] \left[\begin{array}{c|c} T_n & * \\ \hline 0 & 0 \end{array}\right] \quad \text{by inductive assumption} \\
&= \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right]
\end{aligned}$$

<u>Proof (Cayley-Hamilton)</u>

By Schur Triangularization, $\exists T \in UT(n)$ such that:

$$A = UTU^*$$

for some unitary matrix U

Let $T = \begin{bmatrix} t_{11} & & * \\ & \ddots & \\ 0 & & t_{nn} \end{bmatrix}$

Now, since characteristic polynomials for similar matrices are the same:

$$p_A(t) = p_T(t) = \prod_{k=1}^{n} (t - t_{kk})$$

$$p_A(A) = p_A(UTU^*) = Up_A(T)U* = U\left[\prod_{k=1}^{n}(T - t_{kk}I_n)\right]U^* = 0 \quad \text{(by lemma)}$$

A consequence of Cayley-Hamilton is that given any $A \in M_n$ and any $k \in \mathbb{Z}^+$, A^k is a polynomial of A of degree atmost $n-1$.

For example:

$A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$

$p_A(t) = t^2 - 3t + 2$

$p_A(A) = A^2 - 3A + 2I = 0$ so $A^2 = 3A - 2I$

$A^3 = 3A^2 - 2A = 3(3A - 2I) - 2A = 7A - 6I$