Math-08 Homework #1 Solutions

Reading

- If necessary, skim through the documents in the resources module on canvas dealing with mathematical logic, sets, rational numbers, and prime factorization. The documents have more information than you will need; however, if you have something in your notes that you do not understand then you can find more description in the documents.
- Text book sections 0.1 and 0.2.

Problems

1). Let:

$$\begin{aligned} P &:= 0 \text{ is a positive number} \\ Q &:= 0 \text{ is a rational number} \end{aligned}$$

Determine whether the following are true or false:

a). P

0 is neither positive nor negative, so the statement is FALSE.

b). Q

0 can be written as $\frac{0}{1}$, a ratio of two integers where the denominator is not 0. Thus, the statement is TRUE.

c). not P

not FALSE = TRUE

d). not Q

not TRUE = FALSE

e). P and Q

For an AND statement to be true, both statements must be true. Since P is false, the statement is FALSE:

FALSE and TRUE = FALSE

f). P or Q

For an OR statement to be true, at least one of the statements must be true. Since Q is true, the statement is TRUE:

FALSE or TRUE = TRUE

2). Decimal to rational form conversion.

a). Convert $0.14\overline{23}$ to rational form.

Let $x = 0.14\overline{23}$

Multiply x by 100 to capture all of the non-repeating digits to the left of the decimal point:

$100x = 14.\overline{23}$

Now multiply x by 10000 to capture the non-repeating digits and one set of the repeating digits to the left of the decimal point:

$10000x = 1423.\overline{23}$

Now subtract so that the repeating digits on the right of the decimal point cancel and then solve for x:

$$\begin{aligned} 10000x - 100x &= 1423.\overline{23} - 14.\overline{23} \\ 9900x &= 1409 \\ x &= \frac{1409}{9900} \end{aligned}$$

b). Show that $0.\overline{1} = \frac{1}{9}$.

$$\begin{aligned} x &= 0.\overline{1} \\ 10x &= 1.\overline{1} \\ 10x - x &= 1.\overline{1} - 0.\overline{1} \\ 9x &= 1 \\ x &= \frac{1}{9} \end{aligned}$$

c). If this is so, then $\frac{2}{9}$ should equal $0.\overline{2}$, $\frac{3}{9}$ should equal $0.\overline{3}$, and so on until $\frac{8}{9}$ should equal $0.\overline{8}$. So, what do you think that $0.\overline{9}$ should equal?

According to the pattern, $0.\overline{9}$ should equal $\frac{9}{9}$, which equals 1. This demonstrates the fact that when an infinite decimal value is arbitrarity close to an exact value, then the two values are considered to be equal.

d). Show that this is so by converting $0.\overline{9}$ to rational form.

$$\begin{aligned} x &= 0.\overline{9} \\ 10x &= 9.\overline{9} \\ 10x - x &= 9.\overline{9} - 0.\overline{9} \\ 9x &= 9 \\ x &= \frac{9}{9} \\ x &= 1 \end{aligned}$$

e). Take a guess at what $25.3\overline{9}$ equals.

Note that this value gets arbitrarily close to 25.4.

3). Rational numbers and closure.

a). Write down the definition of $\mathbb{Q}$ using setbuilder notation.

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

b). Prove that $\mathbb{Q}$ is closed under addition (Hint: Assume that two numbers are in $\mathbb{Q}$, use the definition to express them as a ratio of integers, then add then and show why the result must be rational).

Assume $a, b \in \mathbb{Q}$
Per the definition, let $a = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$
Also, per the definition, let $b = \frac{r}{s}$ where $r, s \in \mathbb{Z}$ and $s \neq 0$

$$a + b = \frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs}$$

Now remember that the integers are closed under addition and multiplication (integers make integers), so:

$$ps + rq \in \mathbb{Z} \text{ and } qs \in \mathbb{Z}$$

Furthermore, $q \neq 0$ and $s \neq 0$, so $qs \neq 0$

Thus, the result meets the definition of a rational number. Since our choice of a and b was arbitrary, this result holds for any two rational numbers. Therefore, $\mathbb{Q}$ is closed under addition.

c). Prove that $\mathbb{Q}$ is closed under multiplication (Hint: same as above, but multiply the two numbers).

$$a + b = \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

$pr \in \mathbb{Z}$ and $qs \in \mathbb{Z}$

Furthermore, $q \neq 0$ and $s \neq 0$, so $qs \neq 0$

Thus, the result meets the definition of a rational number. Therefore, $\mathbb{Q}$ is closed under multiplication.

d). Give a counterexample showing that $\mathbb{R} - \mathbb{Q}$ is not closed under addition.

The statement that we wish to test is:

$$\forall\, a, b \in \mathbb{R} - \mathbb{Q}, a + b \in \mathbb{R} - \mathbb{Q}$$

Consider the following:

$$\pi + (-\pi) = 0$$

Both pi and $-pi$ are irrational, but adding them results in 0, a rational number. This is a counterexample to the statement that the irrationals are closed under addition. Therefore, the irrationals are not closed under addition.

e). Give a counterexample showing that $\mathbb{R} - \mathbb{Q}$ is not closed under multiplication.

The statement that we wish to test is:

$$\forall\, a, b \in \mathbb{R} - \mathbb{Q}, ab \in \mathbb{R} - \mathbb{Q}$$

Consider the following:

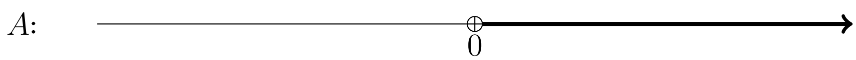
$$\pi \left(\frac{1}{\pi}\right) = 1$$

Both π and $\frac{1}{\pi}$ are irrational, but multiplying them results in 1, a rational number. This is a counterexample to the statement that the irrationals are closed under multiplication. Therefore, the irrationals are not closed under multiplication.

4). Let:

$$A = \text{the set of all positive real numbers}$$
$$B = \text{the set of real numbers between -3 (exclusive) and 3 (inclusive)}$$

a). Graph each set on the real number line.



b). Represent each set using set-builder notation.

$A = \{x \in \mathbb{R}, x > 0\}$

$B = \{x \in \mathbb{R}, -3 < x \leq 3\}$

c). Represent each set using interval notation.

$A = (0, \infty)$

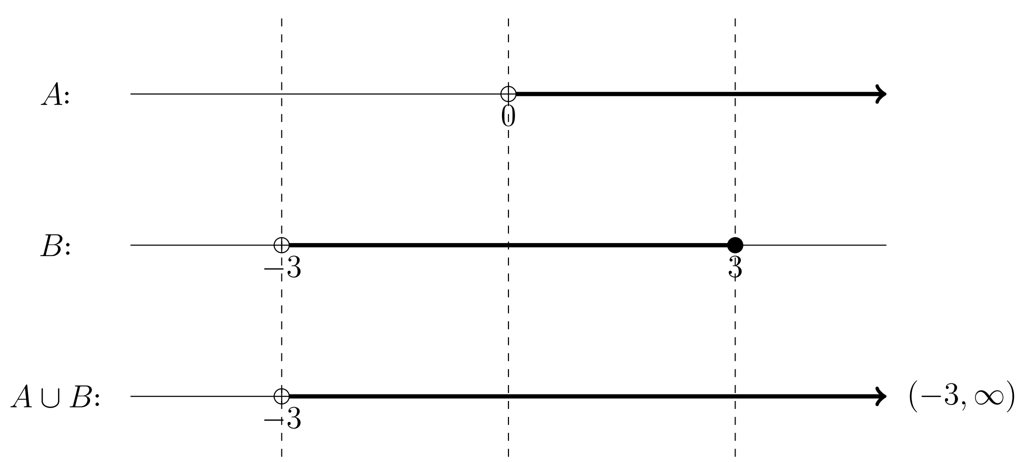
$B = (-3, 3]$

d). Graph $A \cup B$ and represent it in interval notation.

For an element to be in the union of A and B it must be in A or in B (it can be in both!):

$A \cup B = \{x \in \mathbb{R} \mid x \in A \text{ or } x \in B\}$

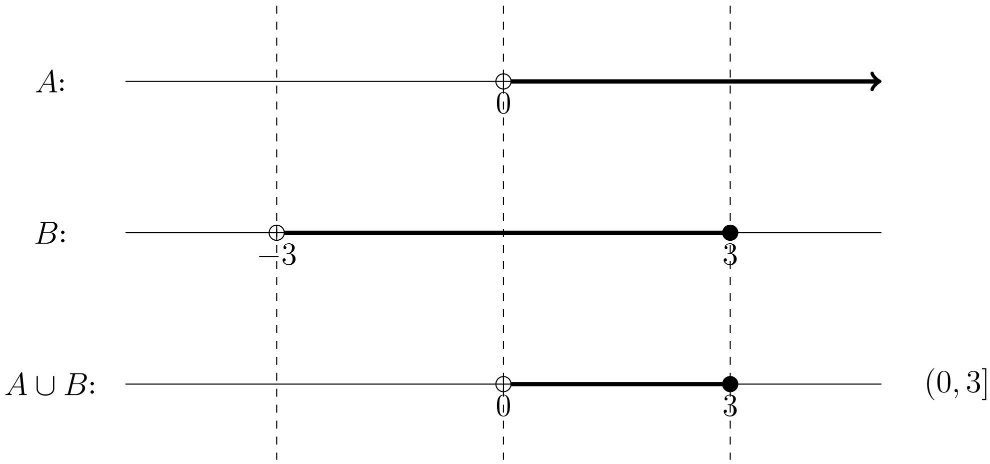
It helps to draw each graph on top of each other, draw dotted lines through all the endpoints, and then determine which regions are included:



e). Graph $A \cap B$ and represent it in interval notation.

For an element to be in the intersection of A and B it must be in both A and B:

$$A \cap B = \{x \in \mathbb{R} \mid x \in A \text{ and } x \in B\}$$



Note that the endpoint 0 is not included because even though it is in B, it is not in A. On the other hand, the endpoint 3 is included because it is in both A and B.