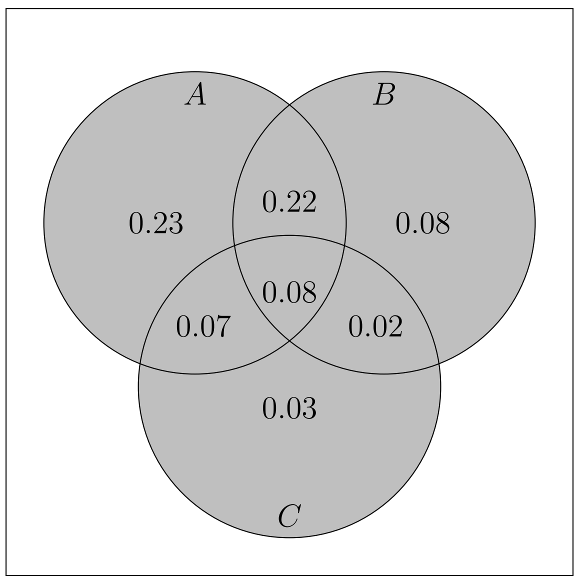
Cavallaro, Jeffery
Math 161A
Homework #2

2.47

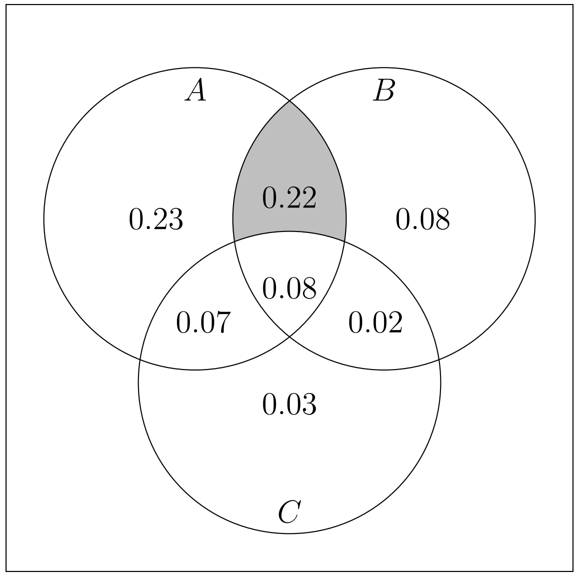
Return to the credit card scenario of Exercise 12 (Section 2.2), and let C be the event that the selected student has an American Express card. In addition to $P(A) = 0.6$, $P(B) = 0.4$, and $P(A \cap B) = 0.3$, suppose $P(C) = 0.2$, $P(A \cap C) = 0.15$, $P(B \cap C) = 0.1$, and $P(A \cap B \cap C) = 0.08$.

a) What is the probability that the selected student has at least one of the three types of cards?



$$
\begin{aligned}
P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) \\
&\quad + P(A \cap B \cap C) \\
&= 0.6 + 0.4 + 0.2 - 0.3 - 0.15 - 0.1 + 0.08 \\
&= 0.73
\end{aligned}
$$

b) What is the probability that the selected student has both a VISA card and a MasterCard but not an American Express card?



$$P(A \cap B \cap C^C) = 0.22$$

c) Calculate and interpret $P(B \mid A)$ and also $P(A \mid B)$.

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.6} = 0.5$$

Of those that have a VISA card, 50% also have a MasterCard.

$$P(A \mid B) = \frac{P(B \cap A)}{P(B)} = \frac{0.3}{0.4} = 0.75$$

Of those that have a MasterCard, 75% also have a VISA card.

d) If we learn that the selected student has an American Express card, what is the probability that she or he also has both a VISA card and a MasterCard?

$$P(A \cap B \mid C) = \frac{P((A \cap B) \cap C)}{P(C)} = \frac{0.08}{0.2} = 0.4$$

e) Given that the selected student has an American Express card, what is the probability that she or he has at least one of the other two types of cards?

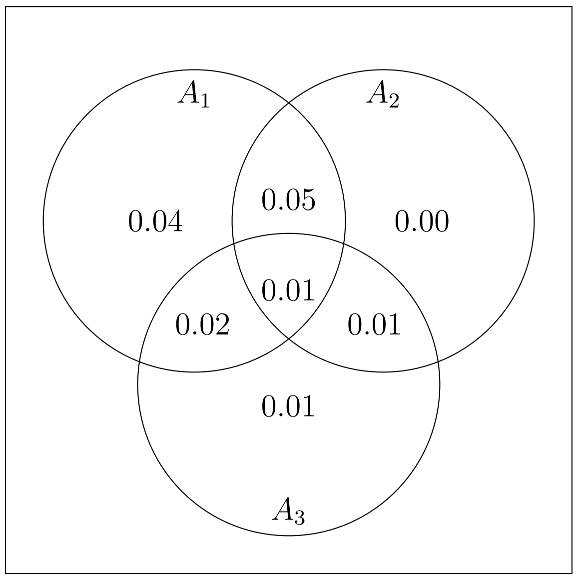
$$P(A \cup B \mid C) = \frac{P((A \cup B) \cap C)}{P(C)} = \frac{0.07 + 0.08 + 0.02}{0.2} = \frac{0.17}{0.2} = 0.85$$

2.48

Reconsider the system defect situation described in Exercise 26 (Section 2.2).

$$
\begin{array}{lll}
P(A_1) = 0.12 & P(A_2) = 0.07 & P(A_3) = 0.05 \\
P(A_1 \cup A_2) = 0.13 & P(A_1 \cup A_3) = 0.14 & P(A_2 \cup A_3) = 0.10 \\
P(A_1 \cap A_2 \cap A_3) = 0.01 & &
\end{array}
$$

$$
\begin{aligned}
P(A_1 \cap A_2) &= P(A_1) + P(A_2) - P(A_1 \cup A_2) = 0.12 + 0.07 - 0.13 = 0.06 \\
P(A_1 \cap A_3) &= P(A_1) + P(A_3) - P(A_1 \cup A_3) = 0.12 + 0.05 - 0.14 = 0.03 \\
P(A_2 \cap A_3) &= P(A_2) + P(A_3) - P(A_2 \cup A_3) = 0.07 + 0.05 - 0.10 = 0.02
\end{aligned}
$$



a) Given that the system has a type 1 defect, what is the probability that it has a type 2 defect?

$$P(A_2 \mid A_1) = \frac{P(A_2 \cap A_1)}{P(A_1)} = \frac{0.06}{0.12} = 0.05$$

b) Given that the system has a type 1 defect, what is the probability that it has all three types of defects?

$$P(A_1 \cap A_2 \cap A_3 \mid A_1) = \frac{P((A_1 \cap A_2 \cap A_3) \cap A_1)}{P(A_1)} = \frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1)} = \frac{0.01}{0.12} = 0.083$$

c) Given that the system has at least one type of defect, what is the probability that it has exactly one type of defect?

$$
\begin{aligned}
P\left(\begin{matrix}\text{exactly}\\\text{one}\end{matrix}\,\middle|\,\begin{matrix}\text{at least}\\\text{one}\end{matrix}\right) &= \frac{P\left(\begin{matrix}\text{exactly}\\\text{one}\end{matrix} \cap \begin{matrix}\text{at least}\\\text{one}\end{matrix}\right)}{P\left(\begin{matrix}\text{at least}\\\text{one}\end{matrix}\right)} \\
&= \frac{P\left(\begin{matrix}\text{exactly}\\\text{one}\end{matrix}\right)}{P\left(\begin{matrix}\text{at least}\\\text{one}\end{matrix}\right)} \\
&= \frac{0.04 + 0.00 + 0.01}{0.04 + 0.00 + 0.01 + 0.05 + 0.01 + 0.02 + 0.01} \\
&= \frac{0.05}{0.14} \\
&= 0.357
\end{aligned}
$$

d) Given that the system has both of the first two types of defects, what is the probability that it does not have the third type of defect?

$$P(A_3^C \mid A_1 \cap A_2) = \frac{P(A_3^C \cap (A_1 \cap A_2))}{P(A_1 \cap A_2)} = \frac{0.05}{0.06} = 0.833$$

2.56

For any two events A and B with $P(B) > 0$, show that $P(A \mid B) + P(A^C \mid B) = 1$.

$$
\begin{aligned}
P(A \mid B) + P(A^C \mid B) &= \frac{P(A \cap B)}{P(B)} + \frac{P(A^C \cap B)}{P(B)} \\
&= \frac{P(A \cap B) + P(A^C \cap B)}{P(B)} \\
&= \frac{P((A \cap B) \cup (A^C \cap B)) + P((A \cap B) \cap (A^C \cap B))}{P(B)} \\
&= \frac{P((A \cup A^C) \cap B) + P(\emptyset)}{P(B)} \\
&= \frac{P(\mathcal{S} \cap B) + 0}{P(B)} \\
&= \frac{P(B)}{P(B)} \\
&= 1
\end{aligned}
$$

2.59

At a certain gas station, 40% of the customers use regular gas (A_1), 35% use plus gas (A_2), and 25% use premium gas (A_3). Of those customers using regular gas, only 30% fill their tanks (event B). Of those customers using plus, 60% fill their tanks. whereas of those using premium, 50% fill their tanks.

a) What is the probability that the next customer will request plus gas and fill the tank $(A_2 \cap B)$?

$$P(A_2 \cap B) = P(A_2)P(B \mid A_2) = 0.35 \cdot 0.60 = 0.21$$

b) What is the probability that the next customer fills the tank?

$$\begin{aligned} P(B) &= P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3) \\ &= 0.40 \cdot 0.30 + 0.35 \cdot 0.60 + 0.25 \cdot 0.50 \\ &= 0.455 \end{aligned}$$

c) If the next customer fills the tank, what is the probability that regular gas is requested? Plus? Premium?

$$\begin{aligned} P(A_1|B) &= \frac{P(A_1)P(B \mid A_1)}{P(B)} = \frac{0.40 \cdot 0.30}{0.455} = 0.264 \\ P(A_2|B) &= \frac{P(A_2)P(B \mid A_2)}{P(B)} = \frac{0.35 \cdot 0.60}{0.455} = 0.462 \\ P(A_3|B) &= \frac{P(A_3)P(B \mid A_3)}{P(B)} = \frac{0.25 \cdot 0.50}{0.455} = 0.275 \end{aligned}$$

2.68

A friend who lives in Los Angeles makes frequent consulting trips to Washington, D.C.; 50% of the time she travels on airline #1, 30% of the time on airline #2, and the remaining 20% of the time on airline #3. For airline #1, flights are late into D.C. 30% of the time and late into L.A. 10% of the time. For airline #2, these percentages are 25% and 20%, whereas for airline #3 the percentages are 40% and 25%. If we learn that on a particular trip she arrived late at exactly one of the two destinations, what are the posterior probabilities of having flown on airlines #1, #2, and #3? Assume that the chance of a late arrival in L.A. is unaffected by what happens on the flight to D.C. [Hint: From the tip of each first-generation branch on a tree diagram, draw three second-generation branches labeled, respectively, 0 late, 1 late, and 2 late.]

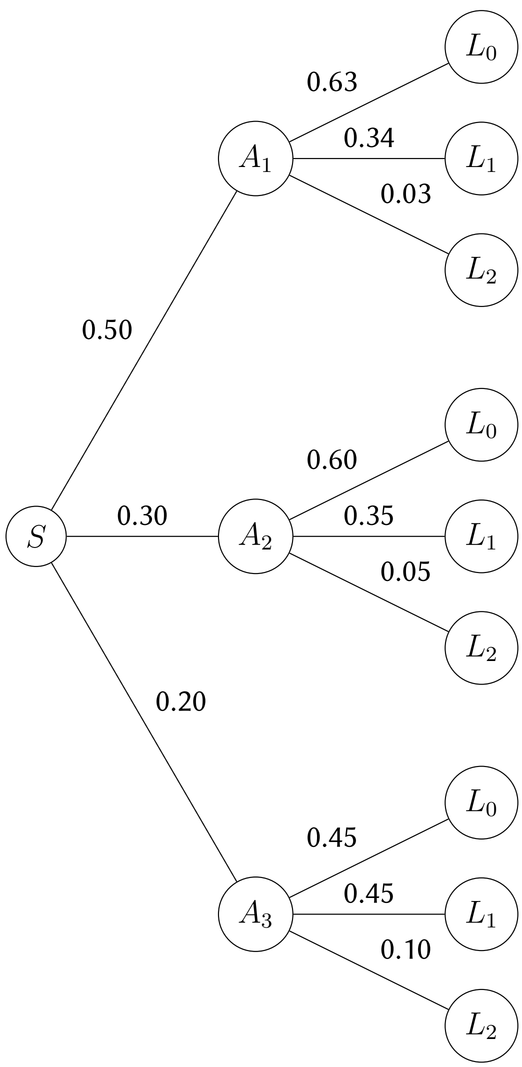
Based on the hint, it appears that the intent of the the problem is that she travels on the same airline for both legs of a round trip. So, let $A_i, i = 1, 2, 3$ be the selection of airline #i, and let

$L_j, j = 0, 1.2$ be the number of late arrivals during a particular round trip.

$$P(L_0 \mid A_1) = 0.70 \cdot 0.90 = 0.63$$
$$P(L_1 \mid A_1) = 0.30 \cdot 0.90 + 0.70 \cdot 0.10 = 0.34$$
$$P(L_2 \mid A_1) = 0.30 \cdot 0.10 = 0.03$$

$$P(L_0 \mid A_2) = 0.75 \cdot 0.80 = 0.60$$
$$P(L_1 \mid A_2) = 0.25 \cdot 0.80 + 0.75 \cdot 0.20 = 0.35$$
$$P(L_2 \mid A_2) = 0.25 \cdot 0.20 = 0.05$$

$$P(L_0 \mid A_3) = 0.60 \cdot 0.75 = 0.45$$
$$P(L_1 \mid A_3) = 0.40 \cdot 0.75 + 0.60 \cdot 0.25 = 0.45$$
$$P(L_2 \mid A_3) = 0.40 \cdot 0.25 = 0.10$$



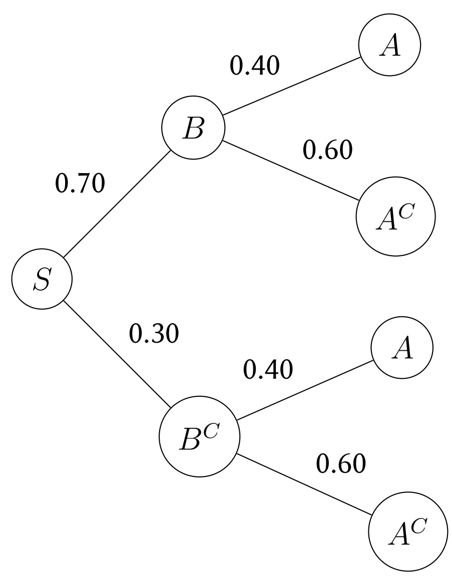
$$P(A_1)P(L_1 \mid A_1) = 0.5 \cdot 0.34 = 0.17$$
$$P(A_2)P(L_1 \mid A_2) = 0.3 \cdot 0.35 = 0.105$$
$$P(A_3)P(L_1 \mid A_3) = 0.2 \cdot 0.45 = 0.09$$

$$P(L_1) = P(A_1)P(L_1|A_1) + P(A_2)P(L_1|A_2) + P(A_3)P(L_1|A_3) = 0.17 + 0.105 + 0.09 = 0.365$$

$$P(A_1 \mid L_1) = \frac{P(A_1)P(L_1 \mid A_1)}{P(L_1)} = \frac{0.17}{0.365} = 0.466$$
$$P(A_2 \mid L_1) = \frac{P(A_2)P(L_1 \mid A_2)}{P(L_1)} = \frac{0.105}{0.365} = 0.288$$
$$P(A_3 \mid L_1) = \frac{P(A_3)P(L_1 \mid A_3)}{P(L_1)} = \frac{0.09}{0.365} = 0.247$$

2.71

An oil exploration company currently has two active projects, one in Asia and the other in Europe. Let A be the event that the Asian project is successful and B be the event hat the European project is successful. Suppose A and B are independent events with $P(A) = 0.4$ and $P(B) = 0.7$.



a) If the Asian project is not successful, what its the probability that the European project is also not successful? Explain your reasoning.

A and B independent $\iff$ A^C and B^C independent

$$P(B^C \mid A^C) = P(B^C) = 1.00 - P(B) = 1.00 - 0.70 = 0.30$$

b) What is the probability that at least one of the two projects will be successful?

$$\begin{aligned}P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A)P(B) \\ &= 0.4 + 0.7 - 0.4 \cdot 0.7 \\ &= 0.82\end{aligned}$$

c) Given that at least one of the two projects is successful, what is the probability that only the Asian project is successful?

$$P(A \cap B^C) = \frac{0.3 \cdot 0.4}{0.7 + 0.3 \cdot 0.4} = 0.146$$

2.74

The proportions of blood phenotypes in the U.S. population are as follows:

A	B	AB	O
0.40	0.11	0.04	0.45

Assuming that the phenotypes of two randomly selected individuals are independent of one another, what is the probability that both phenotypes are O?

$$P\,(\text{both O}) = 0.45 \cdot 0.45 = .203$$

What is the probability that the phenotypes of two randomly selected individuals match?

$$P\,(\text{both match}) = 0.40 \cdot 0.40 + 0.11 \cdot 0.11 + 0.04 \cdot 0.04 + 0.45 \cdot 0.45 = 0.376$$

2.82

Consider independently rolling two fair dice, one red and the other green. Let A be the event that the red die shows 3 dots, B be the event that the green die shows 4 dots, and C be the event that the total number of dots showing on the two dice is 7. Are these events pairwise independent (i.e., are A and B independent events, are A and C independent events, and are B

and C independent events)?

$$A = \{(3, x) \mid x = 1, 2, 3, 4, 5, 6\}$$
$$B = \{(x, 4) \mid x = 1, 2, 3, 4, 5, 6\}$$
$$C = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

$$P(A) = \frac{1}{6}$$
$$P(B) = \frac{1}{6}$$
$$P(C) = \frac{6}{36} = \frac{1}{6}$$

$$P(A \cap B) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = P(A)P(B)$$
$$P(A \cap C) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = P(A)P(C)$$
$$P(B \cap C) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = P(B)P(C)$$

Thus, A, B, and C are pairwise independent.

Are the three events mutually independent?

$$P(A \cap B \cap C) = \frac{1}{36}$$
$$P(A)P(B)P(C) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}$$

$$P(A \cap B \cap C) \neq P(A)P(B)P(C)$$

Thus, A, B, and C are *not* mutually independent.

2.84

Consider purchasing a system of audio components consisting of a receiver, a pair of speakers, and a CD player. Let A_1 be the event that the receiver functions properly throughout the warranty period, A_2 be the event that the speakers function properly throughout the warranty period, and A_3 be the event that the CD play functions properly throughout the warranty period. Suppose that these events are (mutually) independent with $P(A_1) = 0.95$, $P(A_2) = 0.98$, and $P(A_3) = 0.80$.

a) What is the probability that all three components function properly throughout the warranty period?

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3) = 0.95 \cdot 0.98 \cdot 0.80 = 0.745$$

b) What is the probability that at least one component needs service during the warranty period?

$$\begin{aligned}
P(A_1^C \cup A_2^C \cup A_3^C) &= P\left((A_1 \cap A_2 \cap A_3)^C\right) \\
&= 1 - P(A_1 \cap A_2 \cap A_3) \\
&= 1 - 0.745 \\
&= 0.255
\end{aligned}$$

c) What is the probability that all three components need service during the warranty period?

$$P(A_1^C \cap A_2^C \cap A_3^C) = P(A_1^C)P(A_2^C)P(A_3^C) = 0.05 \cdot 0.02 \cdot 0.20 = 0.0002$$

d) What is the probability that only the receiver needs service during the warranty period?

$$P(A_1^C \cap A_2 \cap A_3) = P(A_1^C)P(A_2)P(A_3) = 0.05 \cdot 0.98 \cdot 0.80 = 0.039$$

e) What is the probability that exactly one of the three components needs service during the warranty period?

$$\begin{aligned}
P(\text{exactly one}) &= P(A_1^C)P(A_2)P(A_3) + P(A_1)P(A_2^C)P(A_3) + P(A_1)P(A_2)P(A_3^C) \\
&= 0.05 \cdot 0.98 \cdot 0.80 + 0.95 \cdot 0.02 \cdot 0.80 + 0.95 \cdot 0.98 \cdot 0.20 \\
&= 0.241
\end{aligned}$$

f) What is the probability that all three components function properly throughout the warranty period but that at least one fails within a month after the warranty expires?

Since we have no data regarding the post-warranty period, any answer would be conjecture. The best that we can do is state that the probability is ≤ 0.745.