Rayleigh-Ritz Quotient

Definition

Let $A \in M_n$ be Hermitian and assume $\vec{x} \neq \vec{0}$. The *Rayleigh-Ritz Quotient* is given by:

$$\frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}}$$

Q: What is the maximum (or minimum) value of the RR quotient over a specific subspace $S - \{\vec{0}\}$?

Lemma: Key Lemma

Let $A \in M_n$ be Hermitian with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and orthonormal eigenvectors $\{\vec{u}_1, \ldots, \vec{u}_n\}$ and let $S = \operatorname{span}\{\vec{u}_{i_1}, \ldots, \vec{u}_{i_m}\}$ where $i_1 \leq \cdots \leq i_m$. $\forall\, \vec{x} \in S - \{\vec{0}\}$:

$$\lambda_{i_1} \leq \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} \leq \lambda_{i_m}$$

Proof

Assume $\vec{x} \in S - \{\vec{0}\}$
$\vec{x} = \sum_{k=1}^{m} \alpha_k \vec{u}_{i_k}$
$\vec{x}^* \vec{x} = \sum_{k=1}^{m} |\alpha_k|^2$, since the $\vec{u}_{i_k}$ are orthonormal
Likewise:

$$\begin{aligned}
\vec{x}^* A \vec{x} &= \left(\sum_{k=1}^{n} \overline{\alpha_k \vec{u}_{i_k}}\right)\left(A \sum_{k=1}^{n} \alpha_k \vec{u}_{i_k}\right) \\
&= \left(\sum_{k=1}^{n} \overline{\alpha_k \vec{u}_{i_k}}\right)\left(\sum_{k=1}^{n} \alpha_k A \vec{u}_{i_k}\right) \\
&= \left(\sum_{k=1}^{n} \overline{\alpha_k \vec{u}_{i_k}}\right)\left(\sum_{k=1}^{n} \alpha_k \lambda_{i_k} \vec{u}_{i_k}\right) \\
&= \sum_{k=1}^{m} \lambda_{i_k} |\alpha_k|^2
\end{aligned}$$

And so:

$$\frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} = \frac{\sum_{k=1}^{m} \lambda_{i_k} |\alpha_k|^2}{\sum_{k=1}^{m} |\alpha_k|^2}$$

Therefore:

$$\lambda_{i_1} \leq \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} \leq \lambda_{i_m}$$

Lemma

Let $A \in M_n$ be Hermitian with $\mathrm{Sp}(A) = \{\lambda_1, \dots, \lambda_n\}$. $\forall \vec{x} \in \mathbb{C}^n$ such that $\vec{x} \neq \vec{0}$:

$$\lambda_i = \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} \iff \vec{x} \in \mathrm{Eig}_A(\lambda_i)$$

Proof

$\implies$ Assume $\lambda_i = \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}}$

$$\begin{aligned} \vec{x}^* A \vec{x} &= \lambda_i \vec{x}^* \vec{x} \\ \vec{x}^* A \vec{x} &= \vec{x}^* \lambda_i \vec{x} \\ A \vec{x} &= \lambda_i \vec{x} \end{aligned}$$

Therefore, since $\vec{x} \neq \vec{0}$, $\vec{x} \in \mathrm{Eig}_A(\lambda_i)$

$\impliedby$ Assume $\vec{x} \in \mathrm{Eig}_A(\lambda_i)$

$$\frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} = \frac{\vec{x}^* \lambda_i \vec{x}}{\vec{x}^* \vec{x}} = \lambda_i \frac{\vec{x}^* \vec{x}}{\vec{x}^* \vec{x}} = \lambda_i$$

Corollary

Let $A \in M_n$ be Hermitian with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and orthonormal eigenvectors $\{\vec{u}_1, \dots, \vec{u}_n\}$ and let $S = \mathrm{span}\{\vec{u}_{i_1}, \dots, \vec{u}_{i_m}\}$ where $i_1 \leq \cdots \leq i_m$:

$$\lambda_{i_1} = \min_{\vec{x} \in S - \{\vec{0}\}} \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}}$$

and:

$$\lambda_{i_m} = \max_{\vec{x} \in S - \{\vec{0}\}} \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}}$$

Proof

From the key lemma:

$$\lambda_{i_1} \leq \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} \leq \lambda_{i_m}$$

From the subsequent lemma, equality occurs at $\vec{x} = \vec{u}_{i_1}$ and $\vec{x} = \vec{u}_{i_m}$