Cavallaro, Jeffery
Math 279b
Homework #5

1). $G(n, \frac{1}{2})$
X = number of vertices with degree $< \frac{n}{2}$
Compute $E(X)$

Let $a = \begin{cases} \frac{n-2}{2}, n \text{ even} \\ \frac{n-1}{2}, n \text{ odd} \end{cases}$

Let $X_v = deg(v) \leq a$

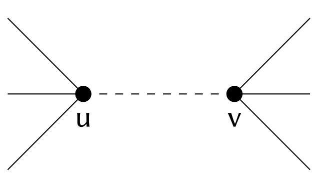
$$\begin{aligned} E(X) &= \sum_v P(X_v = 1) \\ &= \sum_v \left[\sum_{k=0}^{a} \binom{n-1}{k} \left(\frac{1}{2}\right)^a \left(\frac{1}{2}\right)^{n-1-a} \right] \\ &= n \left(\frac{1}{2}\right)^{n-1} \sum_{k=0}^{a} \binom{n-1}{k} \end{aligned}$$

2). Compute $E(X^2)$

$$E(X^2) = E(X) + \sum_{u \neq v} P(X_u = 1 \text{ and } X_v = 1)$$

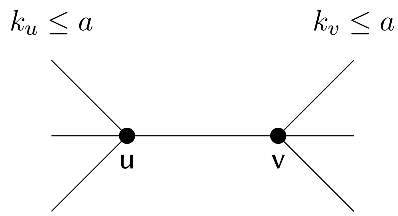
case 1: $uv \notin E(G)$

$k_u \leq a$ $k_v \leq a$



$$\begin{aligned} P(X_u = 1 \text{ and } X_v = 1) &= \left[\sum_{k=0}^{a} \binom{n-2}{k} \right]^2 \frac{1}{2} \left(\frac{1}{2}\right)^{k_u} \left(\frac{1}{2}\right)^{n-2-k_u} \left(\frac{1}{2}\right)^{k_v} \left(\frac{1}{2}\right)^{n-2-k_v} \\ &= \left[\sum_{k=0}^{a} \binom{n-2}{k} \right]^2 \left(\frac{1}{2}\right)^{2n-3} \end{aligned}$$

case 2: $uv \in E(G)$



$$P(X_u = 1 \text{ and } X_v = 1) = \left[\sum_{k=1}^{a} \binom{n-2}{k-1}\right]^2 \frac{1}{2} \left(\frac{1}{2}\right)^{k_u} \left(\frac{1}{2}\right)^{n-2-k_u} \left(\frac{1}{2}\right)^{k_v} \left(\frac{1}{2}\right)^{n-2-k_v}$$

$$= \left[\sum_{k=1}^{a} \binom{n-2}{k-1}\right]^2 \left(\frac{1}{2}\right)^{2n-3}$$

$$= \left[\sum_{k=0}^{a-1} \binom{n-2}{k}\right]^2 \left(\frac{1}{2}\right)^{2n-3}$$

$$E(X^2) = n\left(\frac{1}{2}\right)^{n-1} \sum_{k=0}^{a} \binom{n-1}{k} + n(n-1)\left(\frac{1}{2}\right)^{2n-3} \left\{\left[\sum_{k=0}^{a} \binom{n-2}{k}\right]^2 + \left[\sum_{k=0}^{a-1} \binom{n-2}{k}\right]^2\right\}$$

3). Prove $\frac{1}{2^{2m}}\binom{2m}{m} \to 0$ as $m \to \infty$

Lemma

$$\frac{1}{2^{2m}}\binom{2m}{m} = \prod_{k=1}^{m}\left(1 - \frac{1}{2k}\right)$$

Proof (by induction on m)

Base: $m = 1$

$$\frac{1}{2^{2\cdot 1}}\binom{2\cdot 1}{1} = \frac{1}{4} \cdot 2 = \frac{1}{2}$$

$$\prod_{k=1}^{1}\left(1 - \frac{1}{2k}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

Assume $\frac{1}{2^{2m}}\binom{2m}{m} = \prod_{k=1}^{m}\left(1 - \frac{1}{2k}\right)$

Consider $m + 1$:

$$\begin{aligned}
\frac{1}{2^{2(m+1)}}\binom{2(m+1)}{m+1} &= \frac{1}{2^{2m+2}}\binom{2m+2}{m+1} \\
&= \left(\frac{1}{2^2}\right)\left(\frac{1}{2^{2m}}\right)\left[\frac{(2m+2)!}{(m+1)!(m+1)!}\right] \\
&= \frac{1}{2^2}\left[\frac{(2m+2)(2m+1)}{(m+1)(m+1)}\right]\left(\frac{1}{2^{2m}}\right)\left[\frac{(2m)!}{m!m!}\right] \\
&= \left(\frac{2m+1}{2m+2}\right)\left(\frac{1}{2^{2m}}\right)\binom{2m}{m} \\
&= \left(\frac{2m+2-1}{2m+2}\right)\left(\frac{1}{2^{2m}}\right)\binom{2m}{m} \\
&= \left[1 - \frac{1}{2(m+1)}\right]\prod_{k=1}^{m}\left(1 - \frac{1}{2k}\right) \\
&= \prod_{k=1}^{m+1}\left(1 - \frac{1}{2k}\right)
\end{aligned}$$

We will also need the following anti-derivative:

$$\int \log\left(1 - \frac{a}{x}\right) dx \qquad \text{(by parts)}$$

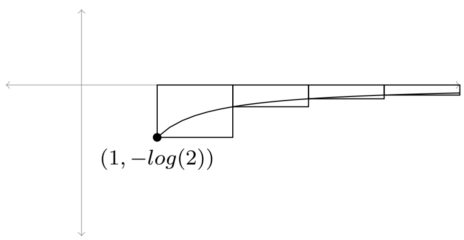
$$\begin{aligned}
u &= \log\left(1 - \frac{a}{x}\right) \\
du &= \frac{1}{1 - \frac{a}{x}}\left(\frac{a}{x^2}\right) dx = \frac{a}{x^2 - ax} dx \\
dv &= dx \\
v &= x
\end{aligned}$$

$$\begin{aligned}
\int \log\left(1 - \frac{a}{x}\right) dx &= x\log\left(1 - \frac{a}{x}\right) - a\int \frac{dx}{x - a} \\
&= x\log\left(1 - \frac{a}{x}\right) - a\log(x - a)
\end{aligned}$$

Now, let $y = \frac{1}{2^{2m}}\binom{2m}{m}$:

$$
\begin{aligned}
y &= \prod_{k=1}^{m} \left(1 - \frac{1}{2k}\right) \\
\log y &= \sum_{k=1}^{m} \log \left(1 - \frac{1}{2k}\right)
\end{aligned}
$$

Consider the function $f(x) = \log\left(1 - \frac{1}{2x}\right)$ on $[1, m]$ in relation to the sum:



Since the function is monotonically increasing, the terms of the series represent a lower sum for the integral:

$$
\begin{aligned}
\log y &\leq \int_{1}^{m} \log\left(1 - \frac{1}{2x}\right) dx \\
&= \left[\log\left(1 - \frac{1}{2x}\right) - \frac{1}{2}\log\left(x - \frac{1}{2}\right)\right]_{1}^{m} \\
&= \left[m \log\left(1 - \frac{1}{2m}\right) - \frac{1}{2}\log\left(m - \frac{1}{2}\right)\right] - \left[\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2}\right] \\
&= \log\left(1 - \frac{1}{2m}\right)^{m} - \log\sqrt{m - \frac{1}{2}} - \log\sqrt{2}
\end{aligned}
$$

Applying limit laws to the last line:

$$\left(1 - \frac{1}{2m}\right)^{m} = \left\{\left[1 + \left(-\frac{1}{2m}\right)\right]^{(-2m)}\right\}^{-\frac{1}{2}} \to e^{-\frac{1}{2}}$$

$$\log\left(1 - \frac{1}{2m}\right)^{m} \to \log e^{-\frac{1}{2}} = -\frac{1}{2}$$

$$\log\sqrt{m - \frac{1}{2}} \to \infty$$

$$\log\sqrt{2} \to \log\sqrt{2}$$

$\therefore \log y \to -\infty$ and $y \to 0$

4). Prove that $\frac{E(X^2)}{E(X)^2} \to 1$ as $n \to \infty$

Recall the binomial identity: $\sum_{k=0}^{m} \binom{m}{k} = 2^m$, and the fact that the binomial coefficients are symmetric:

m odd:

$$\begin{aligned}
\sum_{k=0}^{m} \binom{m}{k} &= \binom{m}{0} + \ldots + \binom{m}{\frac{m-1}{2}} + \binom{m}{\frac{m+1}{2}} + \ldots + \binom{m}{m} \\
2^m &= 2\sum_{k=0}^{\frac{m-1}{2}} \binom{m}{k} \\
\sum_{k=0}^{\frac{m-1}{2}} \binom{m}{k} &= 2^{m-1}
\end{aligned}$$

m even:

$$\begin{aligned}
\sum_{k=0}^{m} \binom{m}{k} &= \binom{m}{0} + \ldots + \binom{m}{\frac{m-2}{2}} + \binom{m}{\frac{m}{2}} + \binom{m}{\frac{m+2}{2}} + \ldots + \binom{m}{m} \\
2^m &= 2\sum_{k=0}^{\frac{m}{2}} \binom{m}{k} - \binom{m}{\frac{m}{2}} \\
\sum_{k=0}^{\frac{m}{2}} \binom{m}{k} &= 2^{m-1} + \frac{1}{2}\binom{m}{\frac{m}{2}}
\end{aligned}$$

We can use this to replace the sums in the expressions for $E(X^2)$ and $E(X)^2$, being careful to use the correct even/odd case:

case 1: n **even**

$$
\begin{aligned}
\frac{E(X^2)}{E(X^2)} &= \frac{n\left(\frac{1}{2}\right)^{n-1}\sum_{k=0}^{\frac{n-2}{2}}\binom{n-1}{k} + n(n-1)\left(\frac{1}{2}\right)^{2n-3}\left\{\left[\sum_{k=0}^{\frac{n-2}{2}}\binom{n-2}{k}\right]^2 + \left[\sum_{k=0}^{\frac{n-4}{2}}\binom{n-2}{k}\right]^2\right\}}{\left[n\left(\frac{1}{2}\right)^{n-1}\sum_{k=0}^{\frac{n-2}{2}}\binom{n-1}{k}\right]^2} \\
&= \frac{n\left(\frac{1}{2}\right)^{n-1}\sum_{k=0}^{\frac{n-2}{2}}\binom{n-1}{k} + n(n-1)\left(\frac{1}{2}\right)^{2n-3}\left\{\left[\sum_{k=0}^{\frac{n-2}{2}}\binom{n-2}{k}\right]^2 + \left[\sum_{k=0}^{\frac{n-2}{2}}\binom{n-2}{k} - \binom{n-2}{\frac{n-2}{2}}\right]^2\right\}}{\left[n\left(\frac{1}{2}\right)^{n-1}\sum_{k=0}^{\frac{n-2}{2}}\binom{n-1}{k}\right]^2} \\
&= \frac{n\left(\frac{1}{2}\right)^{n-1}\left(2^{n-2}\right) + n(n-1)\left(\frac{1}{2}\right)^{2n-3}\left\{\left[2^{n-3} + \frac{1}{2}\binom{n-2}{\frac{n-2}{2}}\right]^2 + \left[2^{n-3} + \frac{1}{2}\binom{n-2}{\frac{n-2}{2}} - \binom{n-2}{\frac{n-2}{2}}\right]^2\right\}}{\left[n\left(\frac{1}{2}\right)^{n-1}\left(2^{n-2}\right)\right]^2} \\
&= \frac{\frac{n}{2} + n(n-1)\left(\frac{1}{2}\right)^{2n-3}\left\{\left[2^{n-3} + \frac{1}{2}\binom{n-2}{\frac{n-2}{2}}\right]^2 + \left[2^{n-3} - \frac{1}{2}\binom{n-2}{\frac{n-2}{2}}\right]^2\right\}}{\left(\frac{n}{2}\right)^2} \\
&= \frac{2}{n} + \frac{4n(n-1)}{n^2}\left(\frac{1}{2}\right)^{2n-3}\left\{\left[2^{n-3} + \frac{1}{2}\binom{n-2}{\frac{n-2}{2}}\right]^2 + \left[2^{n-3} - \frac{1}{2}\binom{n-2}{\frac{n-2}{2}}\right]^2\right\} \\
&= \frac{2}{n} + \left(\frac{n-1}{n}\right)\left(\frac{1}{2}\right)^{2n-5}\left[2\cdot 2^{2n-6} + 2\cdot\frac{1}{4}\binom{n-2}{\frac{n-2}{2}}^2\right] \\
&= \frac{2}{n} + \left(\frac{n-1}{n}\right)\left[1 + \left(\frac{1}{2}\right)^{2n-4}\binom{n-2}{\frac{n-2}{2}}^2\right] \\
&= \frac{2}{n} + \left(\frac{n-1}{n}\right)\left\{1 + \left[\left(\frac{1}{2}\right)^{n-2}\binom{n-2}{\frac{n-2}{2}}\right]^2\right\} \\
&\to 0 + 1(1 + 0^2) \\
&= 1
\end{aligned}
$$

case 2: n **odd**

$$
\begin{aligned}
\frac{E(X^2)}{E(X^2)} &= \frac{n\left(\frac{1}{2}\right)^{n-1}\sum_{k=0}^{\frac{n-1}{2}}\binom{n-1}{k} + n(n-1)\left(\frac{1}{2}\right)^{2n-3}\left\{\left[\sum_{k=0}^{\frac{n-1}{2}}\binom{n-2}{k}\right]^2 + \left[\sum_{k=0}^{\frac{n-3}{2}}\binom{n-2}{k}\right]^2\right\}}{\left[n\left(\frac{1}{2}\right)^{n-1}\sum_{k=0}^{\frac{n-1}{2}}\binom{n-1}{k}\right]^2} \\
&= \frac{n\left(\frac{1}{2}\right)^{n-1}\sum_{k=0}^{\frac{n-1}{2}}\binom{n-1}{k} + n(n-1)\left(\frac{1}{2}\right)^{2n-3}\left\{\left[\sum_{k=0}^{\frac{n-3}{2}}\binom{n-2}{k} + \binom{n-2}{\frac{n-1}{2}}\right]^2 + \left[\sum_{k=0}^{\frac{n-3}{2}}\binom{n-2}{k}\right]^2\right\}}{\left[n\left(\frac{1}{2}\right)^{n-1}\sum_{k=0}^{\frac{n-1}{2}}\binom{n-1}{k}\right]^2} \\
&= \frac{n\left(\frac{1}{2}\right)^{n-1}\left[2^{n-2} + \frac{1}{2}\binom{n-1}{\frac{n-1}{2}}\right] + n(n-1)\left(\frac{1}{2}\right)^{2n-3}\left\{\left[2^{n-3} + \binom{n-2}{\frac{n-1}{2}}\right]^2 + \left[2^{n-3}\right]^2\right\}}{\left\{n\left(\frac{1}{2}\right)^{n-1}\left[2^{n-2} + \frac{1}{2}\binom{n-1}{\frac{n-1}{2}}\right]\right\}^2} \\
&= \frac{\frac{n}{2}\left[1 + \left(\frac{1}{2}\right)^{n-1}\binom{n-1}{\frac{n-1}{2}}\right] + n(n-1)\left(\frac{1}{2}\right)^{2n-3}\left\{\left[2^{n-3} + \binom{n-2}{\frac{n-1}{2}}\right]^2 + \left[2^{n-3}\right]^2\right\}}{\left\{\frac{n}{2}\left[1 + \left(\frac{1}{2}\right)^{n-1}\binom{n-1}{\frac{n-1}{2}}\right]\right\}^2} \\
&= \frac{\frac{n}{2}\left[1 + \left(\frac{1}{2}\right)^{n-1}\binom{n-1}{\frac{n-1}{2}}\right] + \frac{n(n-1)}{8}\left(\frac{1}{2}\right)^{2n-6}\left\{\left[2^{n-3} + \binom{n-2}{\frac{n-1}{2}}\right]^2 + \left[2^{n-3}\right]^2\right\}}{\left\{\frac{n}{2}\left[1 + \left(\frac{1}{2}\right)^{n-1}\binom{n-1}{\frac{n-1}{2}}\right]\right\}^2} \\
&= \frac{\frac{n}{2}\left[1 + \left(\frac{1}{2}\right)^{n-1}\binom{n-1}{\frac{n-1}{2}}\right] + \frac{n(n-1)}{8}\left\{\left[1 + \left(\frac{1}{2}\right)^{n-3}\binom{n-2}{\frac{n-1}{2}}\right]^2 + 1^2\right\}}{\left\{\frac{n}{2}\left[1 + \left(\frac{1}{2}\right)^{n-1}\binom{n-1}{\frac{n-1}{2}}\right]\right\}^2} \\
&= \frac{\frac{2}{n}\left[1 + \left(\frac{1}{2}\right)^{n-1}\binom{n-1}{\frac{n-1}{2}}\right] + \left(\frac{n-1}{2n}\right)\left\{\left[1 + \left(\frac{1}{2}\right)^{n-3}\binom{n-2}{\frac{n-1}{2}}\right]^2 + 1\right\}}{\left[1 + \left(\frac{1}{2}\right)^{n-1}\binom{n-1}{\frac{n-1}{2}}\right]^2}
\end{aligned}
$$

Note that the $\binom{n-2}{\frac{n-1}{2}}$ term is not quite in the right form, but we can adjust it:

$$
\begin{aligned}
\binom{n-2}{\frac{n-1}{2}} &= \frac{(n-2)!}{\left(\frac{n-1}{2}\right)!\left(n-2-\frac{n-1}{2}\right)!} \\
&= \left[\frac{n-1-\frac{n-1}{2}}{n-1}\right]\left[\frac{(n-1)!}{\left(\frac{n-1}{2}\right)!\left(n-1-\frac{n-1}{2}\right)!}\right] \\
&= \frac{1}{2}\binom{n-1}{\frac{n-1}{2}}
\end{aligned}
$$

Plugging in this result and continuing, we get:

$$
\begin{aligned}
\frac{E(X^2)}{E(X^2)} &= \frac{\frac{2}{n}\left[1+\left(\frac{1}{2}\right)^{n-1}\binom{n-1}{\frac{n-1}{2}}\right]+\left(\frac{n-1}{2n}\right)\left\{\left[1+\left(\frac{1}{2}\right)^{n-2}\binom{n-1}{\frac{n-1}{2}}\right]^2+1\right\}}{\left[1+\left(\frac{1}{2}\right)^{n-1}\binom{n-1}{\frac{n-1}{2}}\right]^2} \\
&= \frac{\frac{2}{n}\left[1+\left(\frac{1}{2}\right)^{n-1}\binom{n-1}{\frac{n-1}{2}}\right]+\left(\frac{n-1}{2n}\right)\left\{\left[1+2\left(\frac{1}{2}\right)^{n-1}\binom{n-1}{\frac{n-1}{2}}\right]^2+1\right\}}{\left[1+\left(\frac{1}{2}\right)^{n-1}\binom{n-1}{\frac{n-1}{2}}\right]^2} \\
&\to \frac{0(1+0)+\frac{1}{2}\left[(1+2\cdot 0)^2+1\right]}{(1+0)^2} \\
&= 1
\end{aligned}
$$

5). Prove that almost every graph has $\delta < \frac{n}{2}$

By Chebychev:

$$P(X=0) \leq 1 - \frac{E(X^2)}{E(X)^2} = 1 - 1 = 0$$

Therefore, there is zero probability that $X = 0$, meaning that at least one vertex must have degree $< \frac{n}{2}$.