Product Spaces and Continuity

Definition: Projection

Let X and Y be topological spaces. The *projection maps* $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are defined by $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$.

Theorem

Let X and Y be topological spaces. The projection maps π_X and π_Y are continuous, surjective, and open.

Proof. Assume $U \in \mathscr{T}_X$. $\pi_X^{-1}(U) = U \times Y \in \mathscr{T}_{X \times Y}$. Therefore π_X is continuous.

Next, assume that $x \in X$. Now, assume that $y \in Y$, and so $(x, y) \in X \times Y$. Thus, $\pi_X(x, y) = X$. Therefore π_X is surjective.

Assume $W \in \mathscr{T}_{X \times Y}$. Then $W = \bigcup_{\alpha \in \lambda} U_\alpha \times V_\alpha$, where $U_\alpha \in \mathscr{T}_X$ and $V_\alpha \in \mathscr{T}_Y$. Now:

$$\pi_X(W) = \pi_X(\bigcup_{\alpha \in \lambda} U_\alpha \times V_\alpha) = \bigcup_{\alpha \in \lambda} \pi_X(U_\alpha \times V_\alpha) = \bigcup_{\alpha \in \lambda} U_\alpha \in \mathscr{T}_X$$

Thus, π_X is open.

A similar argument is used for π_Y.

Therefore, π_X and π_Y are continuous, surjective, and open. ■

Example

Let X and Y be topological spaces. $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ need not be closed. Consider $X = Y = \mathbb{R}$ and $A = \{(x, y) \mid xy = 1\}$. Since all points of $X - A$ are interior points, $X - A$ is open and so A is closed. But $\pi_X(A) = \pi_Y(A) = \mathbb{R} - \{0\}$, which is not closed.

Theorem

Let X, Y, and Z be topological spaces. A function $g : Z \to X \times Y$ is continuous iff $\pi_X \circ g$ and $\pi_Y \circ g$ are both continuous.

Proof.

$\Longrightarrow$ Assume that $g : Z \to X \times Y$ is continuous.

Since π_X and π_Y are continuous, and since the composition of continuous functions is continuous, $\pi_X \circ g$ and $\pi_Y \circ g$ are both continuous.

$\Longleftarrow$ Assume that $\pi_X \circ g$ and $\pi_Y \circ g$ are both continuous.

Assume that $W \in \mathscr{T}_{X\times Y}$. So $W = \bigcup_{\alpha\in\lambda} U_\alpha \times V_\alpha$ where $U_\alpha \in \mathscr{T}_X$ and $V_\alpha \in \mathscr{T}_Y$. Then:

$$\begin{aligned}
g^{-1}(W) &= g^{-1}(\bigcup_{\alpha\in\lambda} U_\alpha \times V_\alpha) \\
&= g^{-1}(\bigcup_{\alpha\in\lambda}((U_\alpha \times Y) \cap (X \times V_\alpha))) \\
&= g^{-1}(\pi_X^{-1}(\bigcup_{\alpha\in\lambda} U_\alpha) \cap \pi_Y^{-1}(\bigcup_{\alpha\in\lambda} V_\alpha)) \\
&= g^{-1}(\pi_X^{-1}(\bigcup_{\alpha\in\lambda} U_\alpha)) \cap g^{-1}(\pi_Y^{-1}(\bigcup_{\alpha\in\lambda} V_\alpha)) \\
&= (\pi_X^{-1} \circ g^{-1})(\bigcup_{\alpha\in\lambda} U_\alpha) \cap (\pi_Y \circ g^{-1})(\bigcup_{\alpha\in\lambda} V_\alpha)
\end{aligned}$$

Now, since $\pi_X^{-1} \circ g^{-1}$ is continuous and $\bigcup_{\alpha\in\lambda} U_\alpha \in \mathscr{T}_X$, $(\pi_X^{-1} \circ g^{-1})(\bigcup_{\alpha\in\lambda} U_\alpha) \in \mathscr{T}_X$. Similarly, $(\pi_Y^{-1} \circ g^{-1})(\bigcup_{\alpha\in\lambda} V_\alpha) \in \mathscr{T}_Y$. Thus, $g^{-1}(W) \in \mathscr{T}_Z$.

Therefore $g : Z \to X \times Y$ is continuous.

■

The previous theorem generalizes to arbitrary products. In fact, $X = \prod_{\alpha\in\lambda} X_\alpha$ is the smallest topology that makes each π_{X_α} continuous.

Example: The Cantor Set

$$\begin{aligned}
&C_0 = [0, 1] \\
&C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \\
&C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \\
&\quad\vdots \\
&C = \bigcap_{n=0}^{\infty} C_n
\end{aligned}$$

The Cantor set is:

- Perfect (closed with no isolated points)
- Totally disconnected (no open intervals)
- Measure zero
- Uncountable
- Homeomorphic to $\{0, 1\}^N$ with the discrete topology

The last condition indicates C is isomorphic to trinary digit strings $0.a_1a_2a_3\ldots$ such that $a_k \neq 2$.