Ideals

Definition

Let R be a ring and I an additive subgroup of R:

- To say that I is a *left ideal* in R means:

$$\forall r \in R, \forall i \in I, ri \in I$$

- To say that I is a *right ideal* in R means:

$$\forall r \in R, \forall i \in I, ir \in I$$

- To say that I is a (two-sided) *ideal* in R, denoted $I \trianglelefteq R$, means that I is both a left ideal and a right ideal in R.

Definition

$I = \{0\}$ is called the *zero ideal*.

Theorem

Let R be a ring and I be an ideal in R:

$$I \leq R$$

Proof

By definition, I is an additive subgroup of R
Assume $r, s \in I$
By definition, $rs \in I$

Therefore, by the subring test, $I \leq R$.

Theorem: Ideal Test

Let R be a ring and I a non-empty subset of R. $I \trianglelefteq R$ iff $\forall x, y \in I$ and $\forall z \in R$:

1). $x - y \in I$

2). $zx \in I$

3). $xz \in I$

Proof

Assume $x, y \in I$

$\implies$ Assume $I \trianglelefteq R$

I is an additive subgroup of R, so $(-y) \in I$
By closure, $x - y \in I$

Assume $z \in R$
I is a left ideal, so $zx \in I$
I is a right ideal, so $xz \in I$

Therefore, the three conditions hold.

$\impliedby$ Assume the three conditions hold

By condition 1, I is an additive subgroup of R

$I \subseteq R$ so $x, y \in R$
So by condition 2 (or 3), $xy \in I$

Thus, by the subring test, $I \leq R$

Moreover, condition 2 implies that I is a left ideal of R and condition 3 implies that I is a right ideal of R

$\therefore I \trianglelefteq R$

Theorem

Let R be a ring and $\{I_a \mid a \in A\}$ be a family of ideals in R:

$$I = \bigcap_{a \in A} I_a \trianglelefteq R$$

Proof

$I \leq R$
Assume $x \in I$ and $z \in R$
Assume $a \in A$
$x \in I_a$
But $I_a \trianglelefteq R$, so $zx \in I_a$ and $xz \in I_a$
$zx \in I$ and $xz \in I$

Therefore, by the ideal test, $I \trianglelefteq R$.

Unions of ideals are not generally ideas, except in the following condition:

Theorem

Let R be a ring with a chain of ideals $I_1 \subseteq I_2 \subseteq I_2 \subseteq \ldots$:

$$\bigcup_{k=1}^{\infty} I_k \trianglelefteq R$$

Proof

Clearly, I is a non-empty subset of R

Assume $a, b \in I$
$\exists\, i, j \in \mathbb{Z}^+, a \in I_i$ and $b \in I_j$
AWLOG: $I_i \subseteq I_j$
$a \in I_j$
$(-b) \in I_j$
$a - b \in I_j$
$a - b \in I$

Therefore, by the subgroup test, I is a subgroup of R. Furthermore, since R is an additive abelian group, so is I.

$a \in I$
Assume $r \in R$
$ar \in I_i$ and thus $ar \in I$
Likewise, $ra \in I$

Therefore, $I \trianglelefteq R$.

Theorem

$\forall\, n \in \mathbb{Z}, n\mathbb{Z} \trianglelefteq Z$

Proof

Assume $n \in \mathbb{Z}$

Case 1: $n = 0$

$$0\mathbb{Z} = \{0\} \trianglelefteq \mathbb{Z}$$

Case 2: $n > 0$

Assume $m \in n\mathbb{Z}$
$\exists\, k \in \mathbb{Z}, m = kn$
Assume $h \in= Z$
$hm = h(kn) = (hk)n \in n\mathbb{Z}$, so nZ is a left ideal in $\mathbb{Z}$
$mh = (kn)h = (kh)n \in n\mathbb{Z}$, so nZ is a right ideal in $\mathbb{Z}$

$\therefore n\mathbb{Z} \trianglelefteq \mathbb{Z}$

Case 3: $n < 0$

$(-n) > 0$

Assume $m \in n\mathbb{Z}$
$\exists\, k \in \mathbb{Z}, m = kn$
$m = kn = (-k)(-n)$
$m \in (-n)\mathbb{Z}$

Assume $m \in (-n)\mathbb{Z}$
$\exists\, k \in \mathbb{Z}, m = k(-n)$
$m = k(-n) = (-k)n$
$m \in n\mathbb{Z}$

Thus $n\mathbb{Z} = (-n)\mathbb{Z}$

$\therefore n\mathbb{Z} \trianglelefteq Z$

Theorem

$I \trianglelefteq \mathbb{Z} \implies \exists n \in \mathbb{Z}, I = n\mathbb{Z}$

Proof

Case 1: $I = \{0\}$

$I = 0Z$

Case 2: $\exists i \in I, i \neq 0$

I is an additive group, so $(-i) \in I$
Thus I contains a least positive element
Let the least positive element be n
Assume $m \in I$
By the DA, $m = qn + r$, where $0 \leq r < n$
$r = m - qn$
But I is an ideal, so $qn \in I$
So by closure, $r \in I$
But by the minimality of n, $r = 0$
$m = qn$

$\therefore I = n\mathbb{Z}$