Metric Spaces

Definition: Metric

Let M be a set. A *metric* is a function $d : M \times M \to \mathbb{R}^+ \cup \{0\}$ such that for all $a, b, c \in M$ the following properties hold:

1. $d(a,b) \geq 0$ with $d(a,b) = 0$ iff $a = b$ (positive definite)
2. $d(a,b) = d(b,a)$ (symmetric)
3. $d(a,c) \leq d(a,b) + d(b,c)$ (triangle inequality)

Definition: Metric Space

A *metric space* (M, d) is a set M imbued with a metric d.

Examples

Show that the following are all metrics on $\mathbb{R}^n$:

1. The *Euclidean metric* defined by:

$$d(x,y) = \|x - y\| = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}$$

Positive Definition:

$$\begin{aligned}
&(x_k - y_k)^2 \geq 0 \\
&\sum (x_k - y_k)^2 \geq 0 \\
&\sqrt{\sum (x_k - y_k)^2} \geq 0 \\
&d(x,y) \geq 0
\end{aligned}$$

$$\begin{aligned}
d(x,y) = 0 &\iff \sqrt{\sum (x_k - y_k)^2} = 0 \\
&\iff \sum (x_k - y_k)^2 = 0 \\
&\iff (x_k - y_k)^2 = 0 \\
&\iff x_k - y_k = 0 \\
&\iff x_k = y_k \\
&\iff x = y
\end{aligned}$$

Symmetric:

$$d(x,y) = \sqrt{\sum (x_k - y_k)^2} = \sqrt{\sum (y_k - x_k)^2} = d(y,x)$$

Triangle Inequality:

$$\begin{aligned}[d(x,y)]^2 &= \sum (x_k - y_k)^2 \\ &= \sum |(x_k - z_k) + (z_k - y_k)|^2 \\ &\leq \sum (|x_k - z_k| + |z_k - y_k|)^2 \\ &= \sum (|x_k - z_k|^2 + |z_k - y_k|^2 + 2|x_k - z_k||z_k - y_k|) \\ &= \sum |x_k - z_k|^2 + \sum |z_k - y_k|^2 + 2\sum |x_k - z_k||z_k - y_k|\end{aligned}$$

Now, by the Cauchy-Schwarz inequality:

$$\begin{aligned}\sum |x_k - z_k||z_k - y_k| &\leq \sqrt{\left(\sum (x_k - z_k)^2\right)\left(\sum (z_k - y_k)^2\right)} \\ &= \sqrt{[d(x,z)]^2[d(z,y)]^2} \\ &= d(x,z)d(z,y)\end{aligned}$$

and so:

$$[d(x,y)]^2 \leq [d(x,z)]^2 + [d(z,y)]^2 + 2d(x,z)d(z,y) = [d(x,z) + d(z,y)]^2$$

Therefore $d(x,y) \leq d(x,z) + d(z,y)$.

2. The *box metric* defined by:

$$d(x,y) = \max_{1 \leq k \leq n} \{|x_k - y_k|\}$$

Positive Definition:

$$\begin{aligned}&|x_k - y_k| \geq 0 \\ &\max\{x_k - y_k\} \geq 0 \\ &d(x,y) \geq 0\end{aligned}$$

$$\begin{aligned}d(x,y) = 0 &\iff \max\{|x_k - y_k|\} = 0 \\ &\iff |x_k - y_k| = 0 \\ &\iff x_k - y_k = 0 \\ &\iff x_k = y_k \\ &\iff x = y\end{aligned}$$

Symmetric:

$$d(x,y) = \max\{|x_k - y_k|\} = \max\{|y_k - x_k|\} = d(y,x)$$

Triangle Inequality:

$$\begin{aligned} d(x,y) &= \max\{|x_k - y_k|\} \\ &= \max\{|(x_k - z_k) + (z_k - y_k)|\} \\ &\leq \max\{|x_k - z_k| + |z_k - y_k|\} \\ &\leq \max\{|x_k - z_k|\} + \max\{|z_k - y_k|\} \\ &= d(x,z) + d(z,y) \end{aligned}$$

3. The *taxi-cab metric* defined by:

$$d(x,y) = \sum_{k=1}^{n} |x_k - y_k|$$

Positive Definition:

$$\begin{aligned} &|x_k - y_k| \geq 0 \\ &\sum |x_k - y_k| \geq 0 \\ &d(x,y) \geq 0 \end{aligned}$$

$$\begin{aligned} d(x,y) = 0 &\iff \sum |x_k - y_k| = 0 \\ &\iff |x_k - y_y| = 0 \\ &\iff x_k - y_k = 0 \\ &\iff x_k = y_k \\ &\iff x = y \end{aligned}$$

Symmetric:

$$d(x,y) = \sum |x_k - y_k| = \sum |y_k - x_k| = d(y,x)$$

Triangle Inequality:

$$\begin{aligned} d(x,y) &= \sum |x_k - y_k| \\ &= \sum |(x_k - z_k) + (z_k - y_k)| \\ &\leq \sum (|x_k - z_k| + |z_k - y_k|) \\ &= \sum |x_k - z_k| + \sum |z_k - y_k| \\ &= d(x,z) + d(z,y) \end{aligned}$$

Show that when $n \geq 2$, these metrics are different.

Consider $(0,0),(3,4) \in \mathbb{R}^2$:

$$\begin{aligned}
d_E &= \sqrt{(3-0)^2 + (4-0)^2} = 5 \\
d_B &= \max\{(3-0),(4-0)\} = 4 \\
d_T &= (3-0) + (4-0) = 7
\end{aligned}$$

Example

Let X be a compact topological space and let $\mathcal{C}(X)$ denote the set of continuous functions $f : X \to \mathbb{R}$. We can endow $\mathcal{C}(X)$ with a metric:

$$d(f,g) = \sup_{x \in X}\{|f(x) - g(x)|\}$$

This distance is sometimes denoted $\|f - g\|$. Check that d is a well-defined metric on $\mathcal{C}(X)$.

Note that for any $f \in \mathcal{C}(X)$, since X is compact and $f : X \to f(X)$ is surjective, $f(X)$ is compact and thus bounded. Therefore, all sups are finite.

Positive Definition:

$$\begin{aligned}
&|f(x) - g(x)| \geq 0 \\
&\sup\{|f(x) - g(x)|\} \geq 0 \\
&d(f,g) \geq 0
\end{aligned}$$

$$\begin{aligned}
d(f,g) = 0 &\iff \sup\{|f(x) - g(x)|\} = 0 \\
&\iff |f(x) - g(x)| = 0 \\
&\iff f(x) - g(x) = 0 \\
&\iff f(x) = g(x) \\
&\iff f = g
\end{aligned}$$

Symmetric:

$$d(f,g) = \sup\{|f(x) - g(x)|\} = \sup\{|g(x) - f(x)|\} = f(g,f)$$

Triangle Inequality:

$$\begin{aligned}
d(f,g) &= \sup\{|f(x) - g(x)|\} \\
&= \sup\{|(f(x) - h(x)) + (h(x) - g(x))|\} \\
&\leq \sup\{|f(x) - h(x)| + |h(x) - g(x)|\} \\
&\leq \sup\{|f(x) - h(x)|\} + \sup\{|h(x) - g(x)|\} \\
&= d(f,h) + d(h,g)
\end{aligned}$$

Definition: d-Metric Topology

Let (X, d) be a metric space. The collection of open balls:

$$\mathcal{B} = \{B(p, \epsilon) \,|\, p \in X, \epsilon > 0\}$$

is called the *d-metric topology* on X.

Lemma

Let (X, d) be a metric space. For all $x \in X$ and for all $\epsilon > 0$, for every $y \in B(x, \epsilon)$, there exists a $\delta > 0$ such that $B(y, \delta) \subset B(x, \epsilon)$.

Proof. Assume that $x \in X$ and $\epsilon > 0$. Now, assume $y \in B(x, \epsilon)$ and let $\delta = \epsilon - d(x, y)$. Assume $z \in B(y, \delta)$. This means that $d(y, z) < \delta = \epsilon - d(x, y)$, and so $d(x, y) + d(y, z) < \epsilon$. Thus $d(x, z) < e$. Therefore $B(y, \delta) \subset B(x, \epsilon)$. ■

Theorem

Let X, d be a metric space. The d-metric topology is a basis for a topology on X

Proof. Assume that $p \in X$. There exists $\epsilon > 0$ such that $p \in B(p, \epsilon) \in \mathcal{B}$.

Now, assume that B_1 and B_2 are open balls such that $B_1 \cap B_2 \neq \emptyset$. Assume that $y \in B_1 \cap B_2$. This means that there exists $\delta_1, \delta_2 > 0$ such that $B(y, \delta_1) \subset B_1$ and $B(y, \delta_2) \subset B_2$. So let $\delta = \min\{\delta_1, \delta 2\}$. Thus, $B(y, \delta) \subset B_1 \cap B_2$.

Therefore the d-metric is a basis for a topology on X. ■

Lemma

Let X be a metric space with metrics d_1 and d_2. If there exists $\alpha, \beta > 0$ such that for all $x, y \in X$:

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y)$$

then d_1 and d_2 generate the same topology.

Proof. Let B_1 denote a ball using d_1 and let B_2 denote a ball using d_2. Assume that $x \in X$ and $\epsilon > 0$.

First, assume that $y \in B_2(x, \epsilon)$. This means that $d_2(x, y) < \epsilon$, and so $d_1(x, y) < \frac{\epsilon}{\alpha}$. Hence $y \in B_1\left(x, \frac{\epsilon}{\alpha}\right)$, and so $B_2(x, \epsilon) \subset B_1\left(x, \frac{\epsilon}{\alpha}\right)$.

Next, assume that $y \in B_1(x, \epsilon)$. This means that $d_1(x, y) < \epsilon$, and so $d_2(x, y) < \beta\epsilon$. Hence $y \in B_2(x, \beta\epsilon)$, and so $B_1(x, \epsilon) \subset B_2(x, \beta\epsilon)$.

Now, assume that $B_1 \in \mathscr{T}_1$. For every $x \in B_1$ there exists $B_{2_x} \in \mathscr{T}_2$ such that $B_{2_x} \subset B_1$. Thus, $\mathscr{T}_2$ generates $\mathscr{T}_1$. Likewise, assume that $B_{2_x} \in \mathscr{T}_2$. For every $x \in B_2$ there exists $B_{1_x} \in \mathscr{T}_1$ such that $B_{1_x} \in \mathscr{T}_1$. Thus $\mathscr{T}_1$ generates $\mathscr{T}_2$.

Therefore $\mathscr{T}_1 = \mathscr{T}_2$. ■

Example

Show that the Euclidean metric, box metric, and taxicab metric generate the same topology as the product topology on n copies of $\mathbb{R}$.

$$\begin{aligned} d_E(x,y) &= \sqrt{\sum (x_k - y_k)^2} \\ &\leq \sqrt{\sum \max\{(x_k - y_k)^2\}} \\ &= \sqrt{n \cdot \max\{(x_k - y_k)^2\}} \\ &= \sqrt{n} \max\{|x_k - y_k|\} \\ &= \sqrt{n} \cdot d_B(x,y) \end{aligned}$$

Also:

$$\begin{aligned} d_E(x,y) &= \sqrt{\sum (x_k - y_k)^2} \\ &\geq \sqrt{\max\{(x_k - y_k)^2\}} \\ &= \max\{\sqrt{(x_k - y_k)^2}\} \\ &= \max\{|x_k - y_k|\} \\ &= d_B(x,y) \end{aligned}$$

So $d_B(x,y) \leq d_E(x,y) \leq \sqrt{n} d_B(x,y)$ and thus $\mathscr{T}_B = \mathscr{T}_E$.

Similarly:

$$\begin{aligned} d_T(x,y) &= \sum |x_k - y_k| \\ &\leq \sum \max\{x_k - y_k\} \\ &= n \cdot \max\{x_k - y_k\} \\ &= n \cdot d_B(x,y) \end{aligned}$$

Also:

$$\begin{aligned} d_T(x,y) &= \sum |x_k - y_k| \\ &\geq \max\{x_k - y_k\} \\ &= d_B(x,y) \end{aligned}$$

So $\frac{1}{n} d_T(x,y) \leq d_B(x,y) \leq d_T(x,y)$ and thus $\mathscr{T}_T = \mathscr{T}_B$.

Therefore $\mathscr{T}_E = \mathscr{T}_B = \mathscr{T}_T$.

Now, consider a basis element $U = \prod_{k=1}^n U_k \in \mathbb{R}^n$ and assume that $p \in U$. Then there exists some $\epsilon > 0$ such that $p \in \prod_{k=1}^n (p - \epsilon, p + \epsilon)$. But $B(p, \epsilon) \subset \prod_{k=1}^n (p - \epsilon, p + \epsilon)$ and so $\mathscr{T}_E$ generates $\mathscr{T}_{\mathbb{R}^n}$. Similarly, consider a basis element $B(p, r) \in \mathbb{R}^n$ and assume that $a \in B(p, r)$.

Then there exists some $\epsilon > 0$ such that $B(a, \epsilon) \in B(p, r)$. But $\prod_{k=1}^{n} \left(a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2}\right) \subset B(a, \epsilon)$ and so $\mathscr{T}_{\mathbb{R}^n}$ generates T_E.

Therefore $\mathscr{T}_E = \mathscr{T}_B = \mathscr{T}_T = \mathscr{T}_{\mathbb{R}^n}$.

Lemma

Let (X, d) be a metric space and let $p \in X$ and $A \subset X$ such that $p \notin A$ and A is closed:

$$\mathrm{dist}(p, A) = \inf \{d(a, p) \mid a \in A\} > 0$$

Proof. Since A is closed and $p \notin A$, p is not a limit point of A. Thus, there exists $\epsilon > 0$ such that $B(p, \epsilon) \cap A = \emptyset$ and so for all $a \in A$ the distance from p to a is at least ϵ.

Therefore, $\inf \{d(a, p) \mid a \in A\} > \epsilon > 0$. ∎

Theorem

A metric space is Hausdorff, regular, and normal.

Proof. Let (X, d) be a metric space and let $p \in X$ and $A \subset X$ such that $p \notin A$ and A is closed. Then there exists some $\epsilon > 0$ such that for all $a \in A$, $d(p, a) > \epsilon$. Let $\delta = \frac{\epsilon}{3}$ and consider $U = B(p, \delta)$ and open set V generated by $\{B(a, \delta_a) \mid a \in A, \delta_a < \delta\}$. Thus, for every point $x \in U$ and $y \in V$, $d(x, y) \geq \delta$ and so $U \cap V = \emptyset$.

Therefore (X, d) is regular, and hence also Hausdorff.

Now, assume that $A, B \subset (X, d)$ such that A and B are closed and $A \cap B = \emptyset$. Then for every $a \in A$ there exists $B(a, \epsilon_a)$ such that $B(a, \epsilon_a) \cap B = \emptyset$. Likewise, for every $b \in B$ there exists $B(b, \epsilon_b)$ such that $B(b, \epsilon_b) \cap A = \emptyset$. So let $\delta_a = \frac{\epsilon_a}{3}$ and let $\delta_b = \frac{\epsilon_b}{3}$ and consider the families of open sets $U_a = B(a, \delta_a)$ and $V_b = B(b, \delta_b)$. Let:

$$U = \bigcup_{a \in A} U_a \supset A$$
$$V = \bigcup_{b \in B} V_b \supset B$$

Now, assume that $a \in A$ and $b \in B$:

$$d(a, b) \geq \max\{\epsilon_a, \epsilon_b\} > \max\{\delta_a, \delta_b\}$$

Thus $U_a \cap V_b = \emptyset$ and hence $U \cap V = \emptyset$.

Therefore (X, d) is normal. ∎