Matrices

Definition: Operations

Addition: $A + B$

Multiplication: AB

Transpose: A^T

Conjugate: $\bar{A}$

Conjugate Transpose: $A^* = (\bar{A})^T$

Example

$$A = \begin{bmatrix} 1+i & 3 \\ 2i & 4 \end{bmatrix} \qquad \bar{A} = \begin{bmatrix} 1-i & 3 \\ -2i & 4 \end{bmatrix} \qquad A^* = \begin{bmatrix} 1-i & -2i \\ 3 & 4 \end{bmatrix}$$

Submatrices

Definition: Submatrix

Let $A \in M_{m,n}$. A *submatrix* of A, denoted $A[\alpha.\beta]$, is the matrix derived from A by selecting elements row-indexed by α and column-indexed by β, where $\alpha = \{i_1, i_2, \dots, \imath_r\}$ for $1 \leq r \leq m$ and $\beta = \{j_1, j_2, \dots, \jmath_s\}$ for $1 \leq s \leq n$.

In particular, when $\alpha = \beta$, $A[\alpha, \alpha]$ (also denoted $A[\alpha]$ is called a *principal* submatrix of A.

When $\alpha = \{1, 2, \dots, k\}$ for $1 \leq k \leq m, n$ then $A[\alpha]$ is called a *leading* principal submatrix of A.

Example

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{bmatrix}$$

$\alpha = \{1, 2, 4\}$ and $\beta = \{2, 5\}$ and $\gamma = \{1, 2, 3\}$

$$A[\alpha, \beta] = \begin{bmatrix} 2 & 5 \\ 7 & 10 \\ 17 & 20 \end{bmatrix} \qquad A[\alpha] = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 7 & 9 \\ 16 & 17 & 19 \end{bmatrix} \qquad A[\gamma] = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 11 & 12 & 13 \end{bmatrix}$$

Minors and Cofactors

Definition: Minor

Let $A \in M_n$. A *minor* of A is the determinant of a submatrix:

$$\det A[\alpha, \beta]$$

The minor generated by removing the i^{th} row and j^{th} column is denoted A_{ij}.

A *cofactor* of A is a signed minor given by:

$$(-1)^{i+j} \det A_{ij}$$

Example

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$A_{1,2} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$

$\det A_{1,2} = 36 - 42 = -6$

$(-1)^{1+2} \det A_{i,j} = (-1)^3(-6) = 6$