Linear Operators on Hilbert Spaces

Recall for a linear operator L on a Hilbert space H:

1). $L : \mathcal{D}(L) \to \mathcal{R}(L)$ where $\mathcal{D}(L)$ and $\mathcal{R}(L)$ are subspaces of H.

2). $\|L\| = \sup_{\|\vec{x}\|=1} \|L\vec{x}\|$

3). For L bounded: $\forall\, \vec{x} \in \mathcal{D}(L), \|L\vec{x}\| \leq M \|\vec{x}\|$

4). $\|L\vec{x}\| \leq \|L\| \|\vec{x}\|$

5). L is bounded iff L is continuous.

Examples

1). All finite dimensional Hilbert spaces are isomorphic to $\mathbb{C}^N$, so let $H = \mathbb{C}^N$ and let $A : \mathbb{C}^N \to \mathbb{C}^N$ be linear.

Consider the standard basis: $e = \{\vec{e}_1, \ldots, \vec{e}_N\}$.

Assume $\vec{x} \in H$.

Since $\vec{x} = \sum_{k=1}^{N} x_k \vec{e}_k = \langle \vec{x}, \vec{e}_k \rangle \vec{e}_k$, define the notation:

$$[\vec{x}]_e = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

$$\begin{aligned} A\vec{x} &= \sum_{i=1}^{N} \langle A\vec{x}, \vec{e}_i \rangle \vec{e}_i \\ &= \sum_{i=1}^{N} \left\langle A \sum_{j=1}^{N} x_j \vec{e}_j, \vec{e}_i \right\rangle \vec{e}_i \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} x_j \langle A\vec{e}_j, \vec{e}_i \rangle \vec{e}_i \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} x_j \vec{e}_i \end{aligned}$$

where $a_{ij} = \langle A\vec{e}_j, \vec{e}_i \rangle$. Let $[A]_e = [a_{ij}]$:

$$[A\vec{x}]_e = [A]_e [\vec{x}]_e$$

Thus, A is represented by matrix multiplication.

2). The Differential operator: $Df = f'$

This was previously shown to be unbounded.

3). Fredholm (Integral) Operator

Define $T : L^2[a,b] \to L^2[a,b]$ as:

$$(Tf)(s) = \int_a^b K(s,t)f(t)dt$$

where $K \in L^2(Q = [a,b] \times [a,b])$ and:

$$\iint_Q K(s,t)dsdt < \infty$$

This is clearly linear, but is it bounded?

$$\begin{aligned}
\|Tf\|^2_{L^2[a,b]} &= \int_a^b \left| \int_a^b K(s,t)f(t)dt \right|^2 ds \\
&= \int_a^b \left| \int_a^b K_s(t)f(t)dt \right|^2 ds \\
&= \int_a^b \left| \int_a^b K_s(t)f(t)dt \right|^2 ds \\
&= \int_a^b |\langle K_s, f\rangle|^2 ds \\
&\leq \int_a^b \|K_s\|^2_{L^2[a,b]} \|f\|^2_{L^2[a,b]} ds \\
&= \|f\|^2_{L^2[a,b]} \int_a^b \|K_s\|^2_{L^2[a,b]} ds \\
&= \|f\|^2_{L^2[a,b]} \int_a^b \left(\int_a^b |K(s,t)|^2 dt \right) ds \\
&= \|f\|^2_{L^2[a,b]} \int_a^b \left(\int_a^b |K(s,t)|^2 ds \right) dt \\
&= \|f\|^2_{L^2[a,b]} \int_Q |K(s,t)|^2 dsdt \\
&= \|f\|^2_{L^2[a,b]} \|K\|^2_{L^2(Q)}
\end{aligned}$$

$\therefore \|T\| \leq \|K\|$ and thus T is bounded.

4). Multiplication Operators

Define $M : L^2[a,b] \to L^2[a,b]$.
Fix a function $f_0 \in \mathcal{C}[a,b]$ such that:

$$(Mf)(t) = f_o(t)f(t)$$

This is clearly linear, but is it bounded?

$$\begin{aligned}
\|Mf\|_{L_2} &= \int_a^b |(Mf)(t)|^2\,dt \\
&= \int_a^b |(f_o f)(t)|^2\,dt \\
&= \int_a^b |f_o(t)|^2\,|f(t)|^2\,dt \\
&\leq \max_{t\in[a,b]} |f_o(t)|^2 \int_a^b |f(t)|^2\,dt \\
&= \max_{t\in[a,b]} |f_o(t)|^2\,\|f\|
\end{aligned}$$

Therefore M is bounded.

Notation

Let H be a Hilbert space. The Banach space of linear and bounded operators on H is denoted by $\mathcal{B}(H)$.

Multiplication within $\mathcal{B}(H)$ is composition:

$$\forall\, A, B \in \mathcal{B}(H), (AB)\vec{x} = (A \circ B)\vec{x} = A(B\vec{x})$$

Composition in $\mathcal{B}(H)$ is generally not commutative:

Examples

$H = \mathbb{C}^2$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore AB \neq BA$.

$H = \mathcal{C}^1[a,b]$

$(Af)(x) = xf(x) \qquad (Bf)(x) = f'(x)$

$(AB)f(x) = A(f'(x)) = xf'(x) \qquad (BA)f(x) = B(xf(x)) = f(x) + xf'(x)$

$\therefore AB \neq BA$.

Theorem

Let H be a Hilbert space and $A, B \in \mathcal{B}(H)$:

$$\|AB\| \leq \|A\| \, \|B\|$$

Thus, the product of two bounded operators is bounded.

Proof

$\|(AB)x\| = \|A(Bx)\| \leq \|A\| \, \|Bx\| \leq \|A\| \, \|B\| \, \|x\|$

Theorem

Let H be an infinite dimensional and separable Hilbert space and let $A \in \mathcal{B}(H)$. A can be represented by matrix multiplication with an infinite matrix.

Proof

There exists a countable orthonormal basis for H.
Let the basis be $e = \{\vec{e}_n \mid n \in \mathbb{N}\}$.
Assume $\vec{x} \in H$.
$\vec{x} = \sum_{n=0}^{\infty} x_n \vec{e}_n$
Let $[\vec{x}]_e = (x_1, x_2, \ldots)^T$.

$$\begin{aligned}
A\vec{x} &= \sum_{i=1}^{\infty} \langle A\vec{x}, \vec{e}_i \rangle \, \vec{e}_i \\
&= \sum_{i=1}^{\infty} \left\langle A \sum_{j=1}^{\infty} x_j \vec{e}_j, \vec{e}_i \right\rangle \vec{e}_i \\
&= \lim_{N \to \infty} \sum_{i=1}^{N} \left\langle A \sum_{j=1}^{N} x_j \vec{e}_j, \vec{e}_i \right\rangle \vec{e}_i \\
&= \lim_{N \to \infty} \sum_{i=1}^{N} \sum_{j=1}^{N} x_j \, \langle A\vec{e}_j, \vec{e}_i \rangle \, \vec{e}_i \\
&= \lim_{N \to \infty} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} x_j \vec{e}_i \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} x_j \vec{e}_i
\end{aligned}$$

where $a_{ij} = \langle A\vec{e}_j, \vec{e}_i \rangle$. Let $[A]_e = [a_{ij}]$:

$$[A\vec{x}]_e = [A]_e [\vec{x}]_e$$

Thus, A is represented by multiplication by an infinite matrix.