Distance

Definition: Distance

Let G be a graph and let $u, v \in V(G)$ such that u and v are connected:

- The *distance* between u and v, denoted by $d_G(u,v)$ or $d(u,v)$, is the length of the shortest $u-v$ path:

$$d_G(u,v) = d(u,v) = \min\{|P| \mid P \text{ is a } u-v \text{ path in } G\}$$

- To say that a $u-v$ path P in G is *geodesic* means that:

$$|P| = d_G(u,v)$$

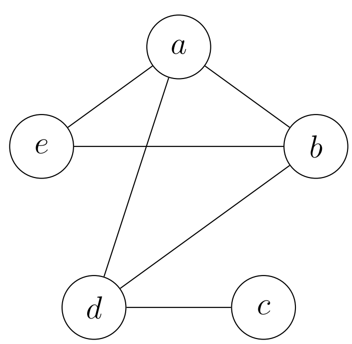
- The *diameter* of G is the greatest distance between any two vertices in G:

$$\operatorname{diam}(G) = \max\{d_G(u,v) \mid u, v \in V(G)\}$$

For a graph G or order n:

- $d_G(u,v) = 0 \iff u = v$
- $d_G(u,v) = 1 \iff uv \in E(G)$
- $0 \le d(u,v) < n$

Example



G

$\operatorname{diam}(G) = 3$

$* =$ geodesic

$a-b$	$a-c$	$a-d$	$a-e$	$b-c$	$b-d$	$b-e$	$c-d$	$c-e$	$d-e$
$(a,b)^*$ (a,d,b) (a,e,b)	(a,b,d,c) $(a,d,c)^*$ (a,e,b,d,c)	(a,b,d) $(a,d)^*$ (a,e,b,d)	(a,b,e) (a,d,b,e) $(a,e)^*$	(b,a,d,c) $(b,d,c)^*$ (b,e,a,d,c)	(b,a,d) $(b,d)^*$ (b,e,a,d)	(b,a,e) (b,d,a,e) $(b,e)^*$	$(c,d)^*$	(c,d,a,b,e) $(c,d,a,e)^*$ (c,d,b,a,e) $(c,d,b,e)^*$	(d,a,b,e) $(d,a,e)^*$ (d,b,a,e) $(d,b,e)^*$
$d=1$	$d=2$	$d=1$	$d=1$	$d=2$	$d=1$	$d=1$	$d=1$	$d=3$	$d=2$

Theorem

Let G be a graph with $u, v \in V(G)$ and let $P = (u = v_0, v_1, \ldots, v_k = v)$ be a $u - v$ geodesic in G. For all i such that $0 \leq i \leq k$:

$$d(u, v_i) = i$$

Proof. Assume $0 \leq i \leq k$.

Since $(u = v_0, v_1, \ldots, v_i)$ is a $u - v_i$ path of length i in G, it must be that case that $d(u, v_i) \leq i$.

ABC: There exists a shorter $u - v_i$ path in G: $(u = w_0, w_1, \ldots, w_\ell = v_i)$ for $\ell < i$.

Let $W = (u = w_0, w_1, \ldots, w_\ell = v_i, \ldots v_k = v)$. W is a $u - v$ walk in G of length:

$$\ell + (k - i) = k - (i - \ell) < k$$

So there exists a $u - v$ path in G of length $< k$, contradicting the minimality of P.

$\therefore d(u, v_i) = i$ ∎

Theorem

Let G be a graph with $u, v \in V(G)$ such that $d(u, v) = \mathrm{diam}(G)$. $\forall\, w \in V(G), w \neq u, v$:

v appears in no $u - w$ geodesic in G.

Proof. Assume $w \in V(G), w \neq u, v$ and P is a $u - w$ geodesic in G.

ABC: v is contained in P.

Then $P = (u, \ldots, v, \ldots, w)$ and the $u - v$ subpath must be geodesic. But $u, v \neq w$, and so $d(u, w) > d(u, v)$, contradicting the maximality of $d(u, v)$.

$\therefore v$ cannot appear in any $u - w$ geodesic in G. ∎