Separable Spaces

Definition: Dense

Let X be a topological space and let $A \subset X$. To say that A is *dense* in X means that $\bar{A} = X$.

Theorem

Let X be a topological space and let $A \subset X$. A is dense in X iff for all $U \in \mathscr{T}, U \neq \emptyset \implies U \cap A \neq \emptyset$.

Proof.

$\implies$ Assume that A is dense in X, and hence $\bar{A} = X$.

Assume that $U \in \mathscr{T}$ and assume that $U \neq \emptyset$. Since $\bar{A} = X$ it must be the case that $U \cap \bar{A} \neq \emptyset$. So assume that $x \in U \cap \bar{A}$, meaning that $x \in U$ and $x \in \bar{A}$. Therefore, since $x \in \bar{A}$, it must be the case that $U \cap A \neq \emptyset$.

$\impliedby$ Assume that $\forall U \in \mathscr{T}, U \neq \emptyset \implies U \cap A \neq \emptyset$.

Clearly, $\bar{A} \subset X$. So assume that $x \in X$. But by the assumption, $x \in \bar{A}$. Therefore $\bar{A} = X$ and hence A is dense in X.

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Definition: Separable

Let X be a topological space. To say that X is *separable* means that X has a countable dense subset.

Example

Show that $\mathbb{R}_{\text{std}}$ is separable. Which of the previously investigated topologies on $\mathbb{R}$ are not separable?

Consider $\mathbb{Q} \subset \mathbb{R}$ and assume that $x \in \mathbb{R} - \mathbb{Q}$. But x is the limit of some sequence in $\mathbb{Q}$ and hence $x \in \bar{\mathbb{Q}}$. This means that $\bar{\mathbb{Q}} = \mathbb{R}$ and thus $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$. Therefore $\mathbb{R}_{\text{std}}$ is separable.

For $\mathbb{R}_{LL}$, also consider $\mathbb{Q} \subset \mathbb{R}$. Assume $U \in \mathscr{T}$. This means that there exists some $a, b \in \mathbb{R}$ such that $[a, b) \subset U$. If $a \in \mathbb{Q}$ then done, so assume $a \in \mathbb{R} - \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}_{\text{std}}$, there exists $x \in \mathbb{Q}$ such that $x \in (a, b) \subset [a, b) \subset U$. Therefore $\mathbb{Q}$ is dense in $\mathbb{R}_{LL}$ as well and so $\mathbb{R}_{LL}$ is separable.

For $\mathbb{R}_{+00}$, consider $A = \{0', 0''\} \cup \mathbb{Q}^+$. Assume $U \in \mathscr{T}$. If $0' \in U$ or $0'' \in U$ then done, so assume that neither is in U. This means that there exists some $a, b \in \mathbb{R}^+$ such that $(a, b) \subset U$. Since $\mathbb{Q}$ is dense in $\mathbb{R}_{\text{std}}$, there exists $x \in \mathbb{Q}+$ such that $x \in (a, b)$. Therefore A is dense and countable in $\mathbb{R}_{+00}$ and so $\mathbb{R}_{+00}$ is separable.

Lemma

$(A \cap X) \times (B \cap Y) = (A \times B) \cap (X \times Y)$

Proof.

$$\begin{aligned}(a,b) \in (A \cap X) \times (B \cap Y) &\iff a \in A \cap X \text{ and } b \in B \cap Y \\ &\iff a \in X \text{ and } a \in X \text{ and } b \in B \text{ and } b \in Y \\ &\iff (a,b) \in A \times B \text{ and } (a,b) \in X \times Y \\ &\iff (a,b) \in (A \times B) \cap (X \times Y)\end{aligned}$$

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Theorem

Let X and Y be topological spaces. If X and Y are separable then $X \times Y$ is separable.

Proof. Assume that X and Y are separable. This means that there exists a countable dense $A \subset X$ and a countable dense $B \subset Y$.

Claim: $A \times B$ is countable and dense in $X \times Y$.

Since A and B are countable, $A \times B$ is countable.

Now, assume $W \in \mathscr{T}_{X \times Y}$. This means that there exists $U \in \mathscr{T}_X$ and $V \in \mathscr{T}_Y$ such that $U \times V \subset W$. But A is dense in X and so $U \cap A \neq \emptyset$. Likewise, B is dense in Y and so $V \cap B \neq \emptyset$. And so:

$$(U \cap A) \times (V \cap B) = (U \times V) \cap (A \times B) \neq \emptyset$$

Thus, $W \cap (A \times B) \neq \emptyset$ and so $A \times B$ is dense in $X \times Y$.

Therefore $A \times B$ is countable and dense in $X \times Y$ and hence $X \times Y$ is separable. ∎