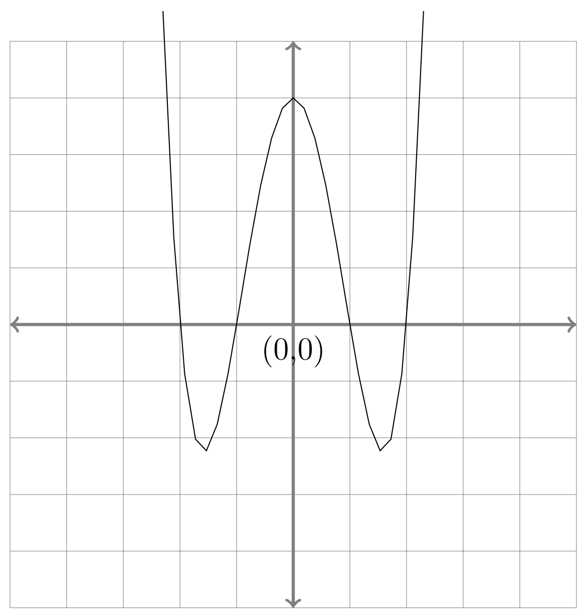
San José State University
Fall 2015
Math-8: College Algebra
Section 03: MW noon–1:15pm
Section 05: MW 4:30–5:45pm

Quiz #11 Solutions)

You may use your book, notes, and homework, but please do not work together or ask for help from others.

1. Consider the following graph of a polynomial function, where each grid line represents 1 unit:



What is the remainder when the polynomial is divided by $(x - 2)$? Why?

The only important piece of information from this graph is the fact that $f(2) = 0$. This tells us that (2,0) is a zero/x-intercept for $f(x)$, that $x - 2$ divides $f(x)$, and thus the remainder is 0.

Some people started with $x^4 - 5x^2 + 4$ with no justification why they started there. At best, they left off the justification or at worst copied this from someone (cheating). Such efforts received 0 points.

Others started with $f(x) = (x + 2)(x + 1)(x - 1)(x + 2)$, which may or may not be true, but you can't assume a general equation from a sketch, unless the sketch is something specific like a parabola, circle, or line. Sometimes, people leave detail out of sketches. But at least this showed that people were thinking about the intercepts, so partial credit. But then, incredibly, some people multiplied this out to get $x^4 - 5x^2 + 4$. and then divided by $x - 2$ to show no remainder!? But they should have already known this since it is $x - 2$ was a factor in what they started with!!!

2. Without doing the long (or synthetic) division, what is the remainder when $x^4 - 2x^3 - 7x^2 + 8x + 12$ is divided by $(x + 2)$? Why?

Just apply the remainder theorem:

$$\begin{aligned} f(-2) &= (-2)^4 - 2(-2)^3 - 7(-2)^2 + 8(-2) + 12 \\ &= 16 + 16 - 28 - 16 + 12 \\ &= 0 \end{aligned}$$

Thus, the remainder is 0.

3. Divide $x^2 + 1$ into $x^5 - 3x^2 + 2x - 1$. Be sure to express the answer completely.

When doing the division by hand, be sure to include placeholders for missing terms. Thus, the divisor should become x^2+0x+1 and the dividend should be $x^5+0x^4+0x^3-3x^2+2x-1$. Then do the division. Note that the formatting package that I am using uses blanks instead of the placeholders, so please pretend that they are there.

$$\begin{array}{rrrrrr}
 & & x^3 & & -x & -3 \\ \cline{2-6}
x^2+1\big) & x^5 & & -3x^2 & +2x & -1 \\
 & -x^5 & -x^3 & & & \\ \cline{2-3}
 & & -x^3 & -3x^2 & +2x & \\
 & & x^3 & & +x & \\ \cline{3-5}
 & & & -3x^2 & +3x & -1 \\
 & & & 3x^2 & & +3 \\ \cline{4-6}
 & & & & 3x & +2
\end{array}$$

The correct form for the answer here is: $x^3 - x - 3 + \frac{3x+2}{x^2+1}$.

4. Express the answer in (4) per the division algorithm.

$$f(x) = (x^2 + 1)(x^3 - x - 3) + (3x + 2)$$

5. Sketch the graph of $f(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$, showing all x and y intercepts, and behavior as $x \to \pm\infty$. For full credit, show how you determined the intercepts and behavior, and how you determined the sign of the function in between the x-intercepts.

I was looking for the following steps for full credit:

1. Leading coefficient test

   Use this test to determine the behavior of $f(x)$ as $x \to \pm\infty$. Alternatively, you can use test points below; however, the leading coefficient test is preferred. In this problem, the degree is even and the leading coefficient is positive, so the end behavior is like x^2.

2. Determine candidate zeros

   I needed to see why you tried certain candidates. The justification is that you divide factors of the constant term coefficient by factors of the leading term coefficient. Since $a_4 = 1$ and $a_0 = 12$, the candidates are:

   $$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$$

   Simply starting to use these candidates without justification only received partial credit.

3. Find zeros from the candidates

   Just plug the candidates into $f(x)$. By the remainder theorem, actual zeros will have $f(k) = 0$, meaning $f(x)$ is divisible by $x - k$. There is no need to do long or synthetic division here to determine if a candidate is actually a zero.

4. Divide out candidates

   Once a candidate is identified by $f(k) = 0$ then $x - k$ should be divided out from $f(x)$ so that we get $f(x) = (x - k)g(x)$. The process should then be repeated recursively with $g(x)$ to find additional factors. Note that some people found all the factors first, without dividing out found ones immediately. This is OK; however, it is more work and it will miss repeated zeros - e.g., if $f(x)$ can be divided by $(x - 1)^2$. If you use this method, you must keep dividing by a found zero until it no longer divides. I can pretty much guarantee that a question on the exam will have such repeated zeros - so beware!

So here is how I proceeded to solve this problem.

$$\begin{aligned} f(1) &= 1 - 2 - 7 + 8 + 12 \neq 0 \\ f(-1) &= 1 + 2 - 7 - 8 + 12 = 0 \end{aligned}$$

So $x + 1$ divides $f(x)$. Doing the division:

$$
\begin{array}{rrrrrr}
 & & x^3 & -3x^2 & -4x & +12 \\
\cline{2-6}
x+1) & x^4 & -2x^3 & -7x^2 & +8x & +12 \\
 & -x^4 & -x^3 & & & \\
\cline{2-3}
 & & -3x^3 & -7x^2 & & \\
 & & 3x^3 & +3x^2 & & \\
\cline{3-4}
 & & & -4x^2 & +8x & \\
 & & & 4x^2 & +4x & \\
\cline{4-5}
 & & & & 12x & +12 \\
 & & & & -12x & -12 \\
\cline{5-6}
 & & & & & 0
\end{array}
$$

So, $f(x) = (x+1)g(x) = (x+1)(x^3 - 3x^2 - 4x + 12)$. We now continue the process with $g(x)$ to pull out more factors. Once again, $a_n = 1$ and $a_0 = 12$, so we have the same candidates. If $x = 1$ did not work before, then it certainly won't work now, so we don't need to consider it again; however, $x = -1$ might be a repeated root, so we need to try it again:

$$g(-1) = -1 - 3 + 4 + 12 \neq 0$$

So $x = -1$ is not repeated. Continuing:

$$g(2) = 8 - 12 - 8 + 12 = 0$$

So we need to divide out $x - 2$:

$$
\begin{array}{rrrrr}
 & x^2 & -x & -6 & \\
\cline{2-5}
x-2) & x^3 & -3x^2 & -4x & +12 \\
 & -x^3 & +2x^2 & & \\
\cline{2-3}
 & & -x^2 & -4x & \\
 & & x^2 & -2x & \\
\cline{3-4}
 & & & -6x & +12 \\
 & & & 6x & -12 \\
\cline{4-5}
 & & & & 0
\end{array}
$$

and now $f(x) = (x+1)(x-2)(x^2 - x - 6)$. We could continue in this fashion; however, we know how to finish the factoring:

$$f(x) = (x+1)(x-2)(x-3)(x+2)$$

We now have our x-intercepts for the graph.

5. Find the y-intercept

$$f(0) = 0 - 0 - 0 + 0 + 12 = 12$$

6. Use test points to determine sign in intervals between x-intercepts

	$(x+1)$	$(x+2)$	$(x-2)$	$(x-3)$	
-1.5	$-$	$+$	$-$	$-$	$-$
0	$+$	$+$	$-$	$-$	$+$
2.5	$+$	$+$	$+$	$-$	$-$

Alternatively, some people use the multiplicity of a factor to determine whether or not the graph passes through the zero to change sign: factors with odd exponents do and factors with even exponents do not.

7. Sketch the graph, labeling all intercepts