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Math 161A
Homework #5

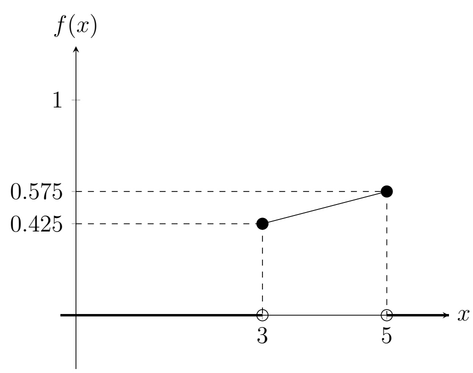
4.1

The current in a certain circuit as measured by an ammeter is a continuous random variable X with the following density function:

$$f(x) = \begin{cases} 0.075x + 0.2 & 3 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

a) Graph the pdf and verify that the total area under the density curve is indeed 1.

$$f(3) = 0.075(3) + 0.2 = 0.425$$
$$f(5) = 0.075(5) + 0.2 = 0.575$$



$$A = 0.5(0.425 + 0.575)(5 - 3) = 0.5(1)(2) = 1$$

b) Calculate $P(X \le 4)$. How does this probability compare to $P(X < 4)$?

$$f(4) = 0.075(4) + 0.2 = 0.5$$

$$P(X \le 4) = 0.5(0.425 + 0.5)(4 - 3) = 0.5(0.925)(1) = 0.4625$$

Since endpoints don't matter, $P(X < 4) = P(X \le 4) = 0.4625$.

c) Calculate $P(3.5 < X < 4.5)$ and also $P(4.5 < X)$.

$$f(3.5) = 0.075(3.5) + 0.2 = 0.4625$$
$$f(4.5) = 0.075(4.5) + 0.2 = 0.5375$$

$$P(3.5 < X < 4.5) = 0.5(0.4625 + 0.5375)(4.5 - 3.5) = 0.5(1)(1) = 0.5$$
$$P(4.5 < X) = 0.5(0.5375 + 0.575)(5 - 4.5) = 0.5(1.1125)(0.5) = 0.2781$$

4.5

A college professor never finishes his lecture before the end of the hour and always finishes his lectures within 2 minutes after the hour. Let $X =$ the time that elapses between the end of the hour and the end of the lecture and suppose the pdf for X is:

$$f(x) = \begin{cases} kx^2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of k and draw the corresponding density curve. [Hint: Total area under the graph of $f(x)$ is 1.]

$$\int_0^2 kx^2 dx = 1$$
$$\frac{1}{3}kx^3\Big|_0^1 = 1$$
$$\frac{8}{3}k = 1$$
$$k = \frac{3}{8} = 0.375$$

b) What is the probability that the lecture ends within 1 minute of the end of the hour?

$$P(X \le 1) = 0.375\int_0^1 x^2 dx = 0.125x^3\big|_0^1 = 0.125(1^3 - 0^3) = 0.125$$

c) What is the probability that the lecture continues beyond the hour for between 60 and 90 seconds?

$$P(1 \le X \le 1.5) = 0.125x^3\big|_1^{1.5} = 0.125(1.5^3 - 1^3) = 0.297$$

d) What is the probability that the lecture continues for at least 90 seconds beyond the end of the hour?

$$P(1.5 < X) = 1 - P(X \le 1) - P(1 \le X \le 1.5) = 1 - 0.125 - 0.297 = 0.578$$

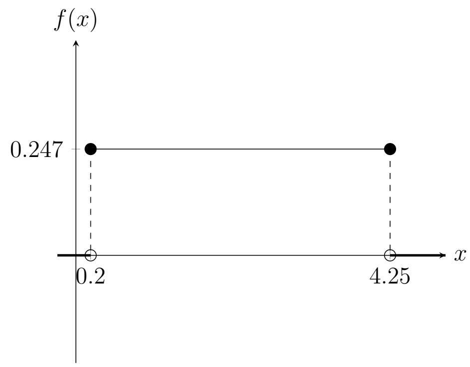
4.7

The article *Second Moment Reliability Evaluation vs. Monte Carlo Simulations for Weld Fatigue Strength* (Quality and Reliability Engr. Intl., 2012: 887–896) considered the use of a uniform distribution with $A = 0.20$ and $B = 4.25$ for the diameter X of a certain type of weld (mm).

a) Determine the pdf of X and graph it.

$$\frac{1}{4.25 - 0.2} = \frac{1}{4.05}$$

$$f(x) = \begin{cases} \frac{1}{4.05} & 0.2 \leq x \leq 4.25 \\ 0 & \text{otherwise} \end{cases}$$



b) What is the probability that diameter exceeds 3mm?

$$P(3 \leq X) = \frac{4.25 - 0.2}{4.05} = 0.309$$

c) What is the probability that diameter is within 1mm of the mean diameter?

$$\mu = \frac{0.2 + 4.25}{2} = \frac{4.45}{2} = 2.225$$

$$P(1.225 \leq X \leq 3.225) = \frac{3.225 - 1.225}{4.05} = \frac{2}{4.05} = 0.494$$

d) For any value a satisfying $0.20 < a < a + 1 < 4.25$, what is $P(a < X < a + 1)$?

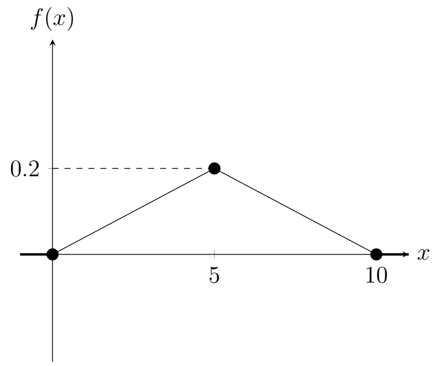
$$P(a < X < a + 1) = \frac{(a + 1) - a}{4.05} = \frac{1}{4.05} = 0.247$$

4.8

In commuting to work, a professor must first get on a bus near her house and then transfer to a second bus. If the waiting time (in minutes) at each stop has a uniform distribution with $A = 0$ and $B = 5$, then it can be shown that the total waiting time Y has the pdf:

$$f(y) = \begin{cases} \frac{1}{25}y & 0 \leq y < 5 \\ \frac{2}{5} - \frac{1}{25}y & 5 \leq y \leq 10 \\ 0 & y < 0 \text{ or } y > 10 \end{cases}$$

a) Sketch a graph of the pdf.



b) Verify that $\int_{-\infty}^{\infty} f(y)dy = 1$.

$$\int_{-\infty}^{\infty} f(y)dy = 0.5(10)(0.2) = 1$$

c) What is the probability that total waiting time is at most 3 minutes?

$$P(X \leq 3) = \int_0^3 \frac{1}{25} y dy = \left. \frac{1}{50} y^2 \right|_0^3 = \frac{9}{50} = 0.18$$

d) What is the probability that total waiting time is at most 8 minutes?

$$\begin{aligned}
P(X \leq 8) &= \int_0^5 \frac{1}{25} y dy + \int_5^8 \left(\frac{2}{5} - \frac{1}{25} y \right) dy \\
&= \left. \frac{1}{50} y^2 \right|_0^5 + \left. \left(\frac{2}{5} y - \frac{1}{50} y^2 \right) \right|_5^8 \\
&= \frac{1}{2} + \left[\left(\frac{16}{5} - \frac{32}{25} \right) - \left(2 - \frac{1}{2} \right) \right] \\
&= 0.5 + [(3.2 - 1.28) - 1.5] \\
&= 0.5 + 1.92 - 1.5 \\
&= 0.92
\end{aligned}$$

e) What is the probability that total waiting time is between 3 and 8 minutes?

$$P(3 \leq Y \leq 8) = P(Y \leq 8) - P(Y \leq 3) = 0.92 - 0.18 = 0.74$$

f) What is the probability that total waiting time is either less than 2 minutes or more than

6 minutes?

$$\begin{aligned}
P(Y \le 2 \text{ or } 6 \le Y) &= \int_0^2 \frac{1}{25} y dy + \int_6^{10} \left(\frac{2}{5} - \frac{1}{25} y \right) dy \\
&= \frac{1}{50} y^2 \Big|_0^2 + \left(\frac{2}{5} y - \frac{1}{50} y^2 \right) \Big|_6^{10} \\
&= \frac{2}{25} + \left[(4-2) - \left(\frac{12}{5} - \frac{18}{25} \right) \right] \\
&= \frac{2}{25} + 2 - \frac{42}{25} \\
&= \frac{10}{25} \\
&= \frac{2}{5} \\
&= 0.4
\end{aligned}$$

4.11

Let X denote the amount of time a book on two-hour reserve is actually checked out, and suppose the cdf is:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{4} & 0 \le x < 2 \\ 1 & 2 \le x \end{cases}$$

a) Calculate $P(X \le 1)$.

$$P(X \le 1) = F(1) = \frac{1}{4} = 0.25$$

b) Calculate $P(0.5 \le x \le 1)$.

$$P(0.5 \le x \le 1) = F(1) - F(0.5) = \frac{1}{4} - \frac{1}{16} = \frac{3}{16} = 0.1875$$

c) Calculate $P(X > 1.5)$.

$$P(X > 1.5) = 1 - F(1.5) = 1 - \frac{9}{16} = \frac{7}{16} = 0.4375$$

d) What is the median checkout duration $\tilde{\mu}$? [solve $0.5 = F(\tilde{\mu})$].

$$\begin{aligned}
&\frac{1}{2} = F(\tilde{\mu}) = \frac{\tilde{\mu}^2}{4} \\
&\tilde{\mu}^2 = 2 \\
&\tilde{\mu} = \pm\sqrt{2}
\end{aligned}$$

Honoring the domain:

$$\tilde{\mu} = \sqrt{2} = 1.4142$$

e) Obtain the density function $f(x)$.

$$f(x) = F'(x) = \begin{cases} \frac{1}{2}x & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

f) Calculate $E(X)$.

$$E(X) = \int_0^2 x\left(\frac{1}{2}x\right)dx = \frac{1}{2}\int_0^2 x^2 dx = \frac{1}{6}x^2\bigg|_0^2 = \frac{8}{6} = \frac{4}{3} = 1.3333$$

g) Calculate $V(X)$ and σ_X.

$$E(X^2) = \int_0^2 x^2\left(\frac{1}{2}x\right)dx = \frac{1}{2}\int_0^2 x^3 dx = \frac{1}{8}x^4\bigg|_0^2 = \frac{16}{8} = 2$$

$$V(X) = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \frac{2}{9} = 0.2222$$

$$\sigma = \sqrt{\frac{2}{9}} = \frac{1}{3}\sqrt{2} = 0.4714$$

h) If the borrower is charged an amount $h(X) = X^2$ when checkout duration is X, compute the expected charge $E(h(X))$.

$$E(h(X)) = \int_0^2 x^2\left(\frac{1}{2}x\right)dx = \frac{1}{2}\int_0^2 x^3 dx = \frac{1}{8}x^4\bigg|_0^2 = \frac{16}{8} = 2$$

4.13

Example 4.5 introduced the concept of time headway in traffic flow and proposed a particular distribution for X = the headway between two randomly selected consecutive cars (seconds). Suppose that in a different traffic environment, the distribution of time headway has the form:

$$f(x) = \begin{cases} \frac{k}{x^4} & x > 1 \\ 0 & x \leq 1 \end{cases}$$

a) Determine the value of k for which $f(x)$ is a legitimate pdf.

$$\int_1^\infty \frac{k}{x^4} dx = 1$$
$$-\frac{k}{3x^3}\bigg|_1^\infty = 1$$
$$\frac{k}{3x^3}\bigg|_\infty^1 = 1$$
$$\frac{k}{3} = 1$$
$$k = 3$$

b) Obtain the cumulative distribution function.

$$\int_1^x \frac{3}{t^4} dt = -\frac{1}{t^3}\bigg|_1^x = \frac{1}{t^3}\bigg|_x^1 = 1 - \frac{1}{x^3}$$

$$F(x) = \begin{cases} 1 - \frac{1}{x^3} & x > 1 \\ 0 & \text{otherwise} \end{cases}$$

c) Use the cdf from (b) to determine the probability that headway exceeds 2 seconds and also the probability that headway is between 2 and 3 seconds.

$$P(2 \leq X) = 1 - F(2) = 1 - \left(1 - \frac{1}{2^3}\right) = \frac{1}{8} = 0.125$$

$$P(2 \leq X \leq 3) = F(3) - F(2) = \left(1 - \frac{1}{3^3}\right) - \left(1 - \frac{1}{2^3}\right) = \frac{1}{8} - \frac{1}{27} = 0.088$$

d) Obtain the mean value of headway and the standard deviation of headway.

$$E(X) = \int_1^\infty x\left(\frac{3}{x^4}\right) dx = 3\int_1^\infty \frac{1}{x^3} dx = -\frac{3}{2x^2}\bigg|_1^\infty = \frac{3}{2x^2}\bigg|_\infty^1 = \frac{3}{2} = 1.5$$

$$E(X^2) = \int_1^\infty x^2\left(\frac{3}{x^4}\right) dx = 3\int_1^\infty \frac{1}{x^2} dx = -\frac{3}{x}\bigg|_1^\infty = \frac{3}{x}\bigg|_\infty^1 = 3$$

$$\sigma^2 = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{3}{4} = 0.75$$

$$\sigma = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} = 0.866$$

e) What is the probability that headway is within 1 standard deviation of the mean value?

$$P(0.634 < X < 2.366) = P(X < 2.366) = F(2.366) = 1 - \frac{1}{2.366^3} = 0.9245$$

4.20

Consider the pdf for total waiting time Y for two buses:

$$f(y) = \begin{cases} \frac{1}{25}y & 0 \le y < 5 \\ \frac{2}{5} - \frac{1}{25}y & 5 \le y \le 10 \\ 0 & \text{otherwiser} \end{cases}$$

introduced in Exercise 8.

a) Compute and sketch the cdf of Y. [Hint: Consider separately $0 \le y < 5$ and $5 \le y \le 10$ in computing $F(y)$. A graph of the pdf should be helpful.]

   Based on the pdf sketched above in Exercise 4.8:

   For $0 \le y < 5$:

$$F(y) = \int_0^y \frac{1}{25} t dt = \frac{1}{50} y^2$$

   At $y = 5$:

$$F(5) = \frac{1}{50}(5)^2 = \frac{1}{2}$$

   For $5 \le y \le 10$:

$$\begin{aligned} F(y) &= \frac{1}{2} + \int_5^y \left(\frac{2}{5} - \frac{1}{25}t\right) dt \\ &= \frac{1}{2} + \left(\frac{2}{5}t - \frac{1}{50}t^2\right)\bigg|_5^y \\ &= \frac{1}{2} + \left(\frac{2}{5}y - \frac{1}{50}y^2\right) - \left(2 - \frac{1}{2}\right) \\ &= -\frac{1}{50}y^2 + \frac{2}{5}y - 1 \end{aligned}$$

   And so:

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{50}y^2 & 0 \le y < 5 \\ -\frac{1}{50}y^2 + \frac{2}{5}y - 1 & 5 \le y \le 10 \\ 1 & y > 10 \end{cases}$$

b) Not assigned.

c) Compute $E(Y)$ and $V(Y)$.

By symmetry, $E(Y)$ should be 5. Check this:

$$
\begin{aligned}
E(Y) &= \int_0^{10} y f(y) dy \\
&= \int_0^5 y\left(\frac{1}{25}y\right) dy + \int_5^{10} y\left(\frac{2}{5} - \frac{1}{25}y\right) dy \\
&= \frac{1}{25}\int_0^5 y^2 dy + \int_5^{10}\left(\frac{2}{5}y - \frac{1}{25}y^2\right) dy \\
&= \frac{1}{75}y^3\Big|_0^5 + \left(\frac{1}{5}y^2 - \frac{1}{75}y^3\right)\Big|_5^{10} \\
&= \frac{125}{75} + \left(20 - \frac{1000}{75}\right) - \left(5 - \frac{125}{75}\right) \\
&= 15 - \frac{750}{75} \\
&= 15 - 10 \\
&= 5
\end{aligned}
$$

$$
\begin{aligned}
E(Y^2) &= \int_0^{10} y^2 f(y) dy \\
&= \int_0^5 y^2\left(\frac{1}{25}y\right) dy + \int_5^{10} y^2\left(\frac{2}{5} - \frac{1}{25}y\right) dy \\
&= \frac{1}{25}\int_0^5 y^3 dy + \int_5^{10}\left(\frac{2}{5}y^2 - \frac{1}{25}y^3\right) dy \\
&= \frac{1}{100}y^4\Big|_0^5 + \left(\frac{2}{15}y^3 - \frac{1}{100}y^4\right)\Big|_5^{10} \\
&= \frac{625}{100} + \left(\frac{2000}{15} - 100\right) - \left(\frac{250}{15} - \frac{625}{100}\right) \\
&= \frac{1250}{100} + \frac{1750}{15} - 100 \\
&= \frac{50}{4} + \frac{250}{15} \\
&= \frac{50}{4} + \frac{50}{3} \\
&= \frac{350}{12} \\
&= \frac{175}{6} && = 29.17
\end{aligned}
$$

$$V(X) = \frac{175}{6} - 5^2 = \frac{175}{6} - 25 = \frac{25}{6} = 4.17$$

How do these compare with the expected waiting time and variance for a single bus when the time is uniformly distributed on $[0,5]$?

Let X_1 = wait time for first bus and X_2 = wait time for second bus, each with a uniform distribution over $[0,5]$:

$$E(X_1) = E(X_2) = \frac{1}{5}\int_0^5 x dx = \frac{1}{10}x^2\Big|_0^5 = \frac{25}{10} = \frac{5}{2} = 2.5$$

$$E(X_1^2) = E(X_2^2) = \frac{1}{5}\int_0^5 x^2 dx = \frac{1}{15}x^3\Big|_0^5 = \frac{125}{15} = \frac{25}{3} = 8.33$$

$$V(X_1) = V(X_2) = \frac{25}{3} - \left(\frac{5}{2}\right)^2 = \frac{25}{3} - \frac{25}{4} = \frac{25}{12} = 2.08$$

And so:

$$E(Y) = E(X_1) + E(X_2)$$

and:

$$V(Y) = V(X_1) + V(X_2)$$

4.21

An ecologist wishes to mark off a circular sampling region having radius 10 meters. However, the radius of the resulting region is actually a random variable R with pdf:

$$f(r) = \begin{cases} \frac{3}{4}[1-(10-r)^2] & 9 \le r \le 11 \\ 0 & \text{otherwise} \end{cases}$$

What is the expected area of the resulting circular region?

Let $A = \pi r^2$:

$$\begin{aligned}
E(A) &= \int_9^{11} (\pi r^2) \left(\frac{3}{4}[1 - (10 - r)^2] \right) dr \\
&= \frac{3}{4}\pi \int_9^{11} r^2[1 - (100 - 20r + r^2)]dr \\
&= \frac{3}{4}\pi \int_9^{11} r^2(-r^2 + 20r - 99)dr \\
&= \frac{3}{4}\pi \int_9^{11} (-r^4 + 20r^3 - 99r^2)dr \\
&= \frac{3}{4}\pi \left(-\frac{1}{5}r^5 + 5r^4 - 33r^3 \right) \Bigg|_9^{11} \\
&= \frac{3}{4}\pi[-2928.2 - (-3061.8)] \\
&= \frac{3}{4}\pi(133.6) \\
&= 314.79\,\text{m}^2
\end{aligned}$$