Space-filling Curves
Play the Peano in One Easy Lesson

History

- Up until the late 19^{th} century, mathematics was still limited by the Hellenistic world view: an extreme existentialism grounded in intuition and an open hostility to the concept of non-intuitive logic and especially infinity.
- The problem was that most of the "intuitive" problems had been solved based on earlier work by notables such as Euler and Gauss, and now people were starting to come up with a whole class of problems whose solutions were not solvable using the past intuitive methods. We talked about some of these at the beginning of the semester.
- The popular reaction was to dismiss such problems, usually with a near-religious zeal, and probably not altogether disconnected with the nationalism and drive for empire that was sweeping across Europe at the time.
- Enter Georg Cantor (1845–1918), Russia with a formalized, logical, non-intuitive set theory and a formal treatment of infinity that could be used to solve some of the new problems, and provide a guide on how to approach the other non-intuitive problems.
- As a result, Cantor was lambasted and spent most of his later life in a sanatorium.
- However, Cantor had a soulmate in the person of Guiseppe Peano (1858–1932), Italy, who attempted to formalize mathematical logic in education as part of the so-called *Formulario Project*.
- Recall that one of the first things taught in analysis is Peano's Axioms on the natural numbers, especially N5 that sets the stage for mathematical induction.
- Peano was so well-liked by his students (no tests!) and peers that he didn't receive the same scorn as Cantor; however, most of his work was ignored — until . . .:
- David Hilbert (1862–1943), Prussia, starts solving many of the non-intuitive problems using Peano and Cantor's methods.
- Other mathematicians such as Borel and Lebesque in France, and Riemann and Dedekind in Germany (Gauss's students) were making similar discoveries.
- One of Peano's discoveries that so epitomizes this movement is his formulation of a continuous, "space-filling" curve in 1890.
- It was then left to Hilbert to come up with a method for generating such a curve. And this is what we are going to look at today.

Goal

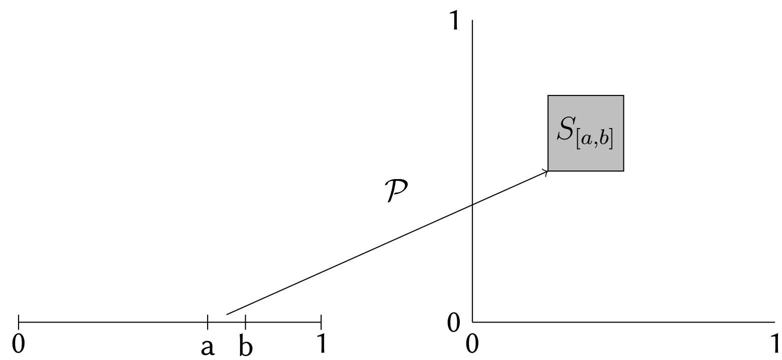
The goal is to find a curve $t \mapsto \mathcal{P}(t)$ such that:

1). $\mathcal{P} : [0,1] \to [0,1] \times [0,1]$.

2). $\mathcal{P}$ is continuous.

3). $\mathcal{P}$ is onto (surjective).

4). The image under $\mathcal{P}$ of any $[a,b] \subset [0,1]$ is a square $S_{[a,b]} \subset [0,1] \times [0,1]$ such that $m_2(S_{[a,b]}) = b - a$.

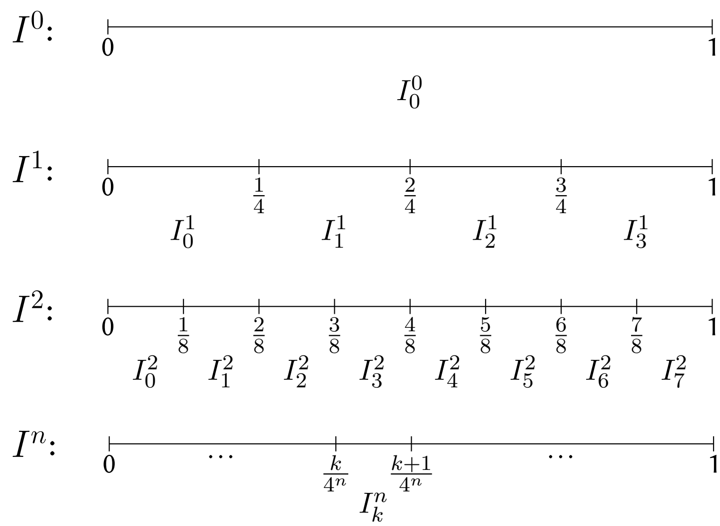


Such a mapping is called a *Peano Mapping* and its corresponding curve is called a *Peano Curve*.

Note that this is a parameterized curve from domain $\mathbb{R}$ to co-domain $\mathbb{R}^2$. It is not a mapping *on* $\mathbb{R}$ and thus is not subject to our normal intuitive notions of a curve in $\mathbb{R}^2$.

Quartic Intervals

Start with $[0, 1]$ and with each generation, subdivide each sub-interval into four equal parts. Designate each sub-interval as I_k^n, where n is the generation and k is the relative position of the interval (starting from 0).



Each generation has 4^n intervals of length (measure): $m_1(I_k^n) = \frac{1}{4^n}$.

Definition

A *chain* of quartic intervals is a decreasing sequence:

$$I^0 \subset I^1 \subset I^2 \subset \ldots \subset I^k \subset \ldots$$

where I^k is a quartic interval of the k^{th} generation.

Properties

1). If (I^k) is a chain of quartic intervals then there exists a unique $t \in [0, 1]$ such that $t \in \bigcap_k I^k$.

2). Conversely, given $t \in [0, 1]$, there is a chain (I^k) of quartic intervals such that $t \in \bigcap_k I^k$.

3). The set of t for which the chain in (ii) is not unique is a countable set of measure 0.

Note that the non-uniqueness occurs at the boundaries of each quartic interval, at the points $\{\frac{j}{4^k} | 0 < j < 4^k\}$, the set of *dyadic rationals*, which is countable and has measure 0.

Similar to the Cantor-Lebesgue function, we can represent each chain with a base-4 string of the form $0.a_1 a_2 \cdots a_k \cdots$ where each digit selects one of the 4 sub-intervals in an interval from the previous generation. Thus, each $t \in [0, 1]$ can be expressed as:

$$t = \sum_{k=1}^{\infty} \frac{a_k}{4^k}$$

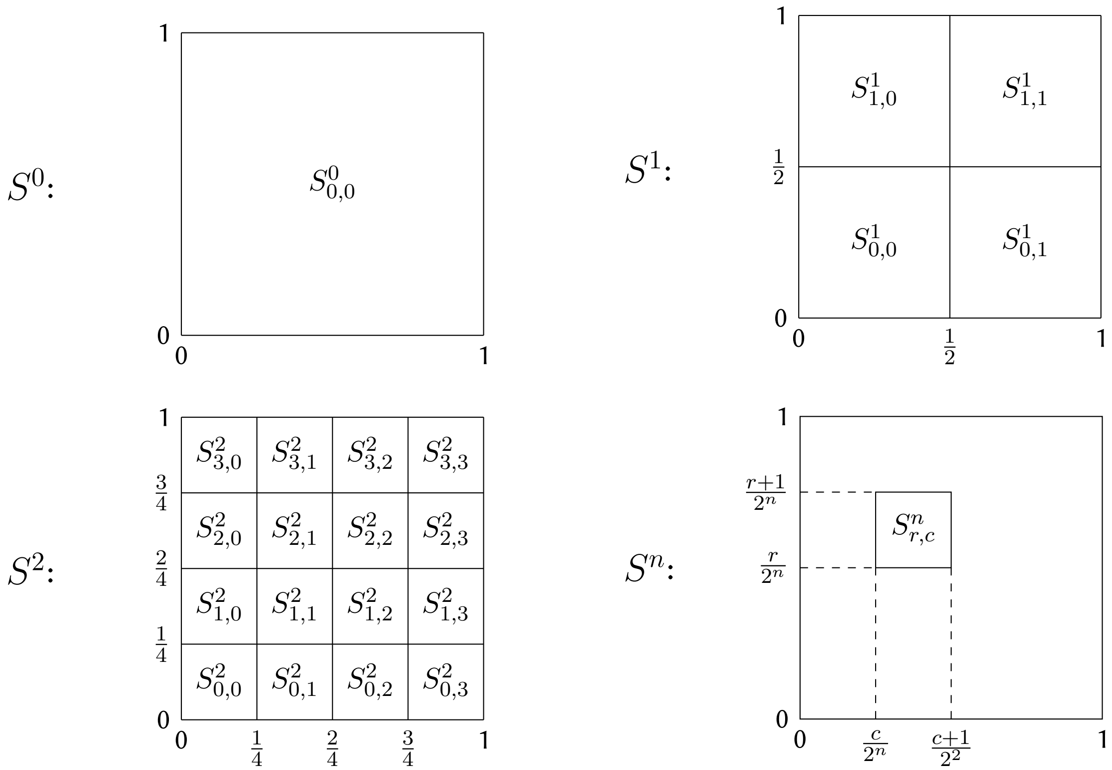
which is well-defined except at the dyadic rationals, which correspond to numbers of the form:

$$0.a_1 a_2 \cdots a_k 0000 \cdots = 0.a_1 a_2 \cdots (a_k - 1)3333 \cdots$$

on the interval boundaries.

Dyadic Squares

Start with $[0,1] \times [0,1]$ and with each generation, subdivide each sub-square into four equal parts. Designate each sub-interval as $I_{r,c,}{}^{n}$, where n is the generation, r is the row, and c is the column. Row and column numbering starts from 0, with $(0,0)$ being in the lower left-hand corner.



Each generation has 4^n squares ($2^n \times 2^n$) of area (measure): $m_2(S^n_{r,c}) = \frac{1}{4^n}$.

Definition

A *chain* of dyadic squares is a decreasing sequence:

$$S^0 \subset S^1 \subset S^2 \subset \ldots \subset S^k \subset \ldots$$

where S^k is a dyadic square of the k^{th} generation.

Properties

1). If (S^k) is a chain of dyadic squares then there exists a unique $x \in [0,1] \times [0,1]$ such that $x \in \bigcap_k S^k$.

2). Conversely, given $x \in [0,1] \times [0,1]$, there is a chain (S^k) of dyadic squares such that $x \in \bigcap_k S^k$.

3). The set of x for which the chain in (ii) is not unique is a countable set of measure 0.

Note that the non-uniqueness occurs at all points (x_1, x_2), where at least one of the coordinates is a dyadic rational ($4^k = 2^{2k}$), corresponding to the countable set of vertical and horizontal line sequences of measure 0.

Similar to the Cantor-Lebesgue function, we can represent each chain with a base-4 string of the form $0.b_1 b_2 \cdots b_k \cdots$ where each digit selects one of the 4 sub-squares in a square from the previous generation. Thus, each $x \in [0,1] \times [0,1]$ can be expressed as:

$$x = \sum_{k=1}^{\infty} \frac{\bar{b}_k}{4^k}$$

where

$$\bar{b}_k = \begin{cases} (0,0) & b_k = 0 \\ (0,1) & b_k = 1 \\ (1,0) & b_k = 2 \\ (1,1) & b_k = 3 \end{cases}$$

Dyadic Correspondence

Note that: $|I^n| = |S^n|$ and $m_1(I^n_k) = m_2(S^n_{r,c})$. So, we would like to establish a bijection between the intervals in the domain and the squares in the co-domain.

Definition

A *dyadic correspondence* is a mapping Φ from quartic intervals to dyadic squares such that:

1). Φ is bijective.

2). Φ respects generations: $I \in I^k \implies \Phi(I) \in S^k$.

3). Φ respects inclusion: $I \subset J \implies \Phi(I) \subset \Phi(J)$.

The trivial correspondence is given by $a_k = b_k$ in the base-4 representations of each chain; however, this results in a discontinuous mapping of intervals to rectangles.

We can now define the *induced mapping* $\Phi^* : [0,1] \to [0,1] \times [0,1]$. Let $\{t\} = \bigcap_{k=1}^{\infty} I^k$ where (I^k) is a chain of quartic intervals. Then, by definition, $(\Phi(I^k))$ is a chain of dyadic squares such that $\bigcap_{k=1}^{\infty} \Phi(I^k) = \{x\}$ and:

$$\Phi^*(t) = x$$

Note that if I is a quartic interval of the k^{th} generation then $\Phi^*[I] = \Phi(I)$, where $\Phi(I)$ is a dyadic square of the k^{th} generation and: $m_1(I) = m_2(\Phi^*[I])$.

The problem with Φ^* is that it is not well-defined on the interval boundary points, since each is representable by two different quartic chains, resulting in two different dyadic chains.

Key Theorem

Theorem

Given a dyadic correspondence Φ, there exists sets $Z_1 \subset [0,1]$ and $Z_2 \subset [0,1] \times [0,1]$, both of measure 0, such that:

1). $\Phi^* : [0,1] - Z_1 \to [0,1] \times [0,1] - Z_2$ is a bijection.

2). $E \subset [0,1]$ is measurable iff $\Phi^*[E]$ is measurable.

3). $m_1(E) = m_2(\Phi^*[E])$.

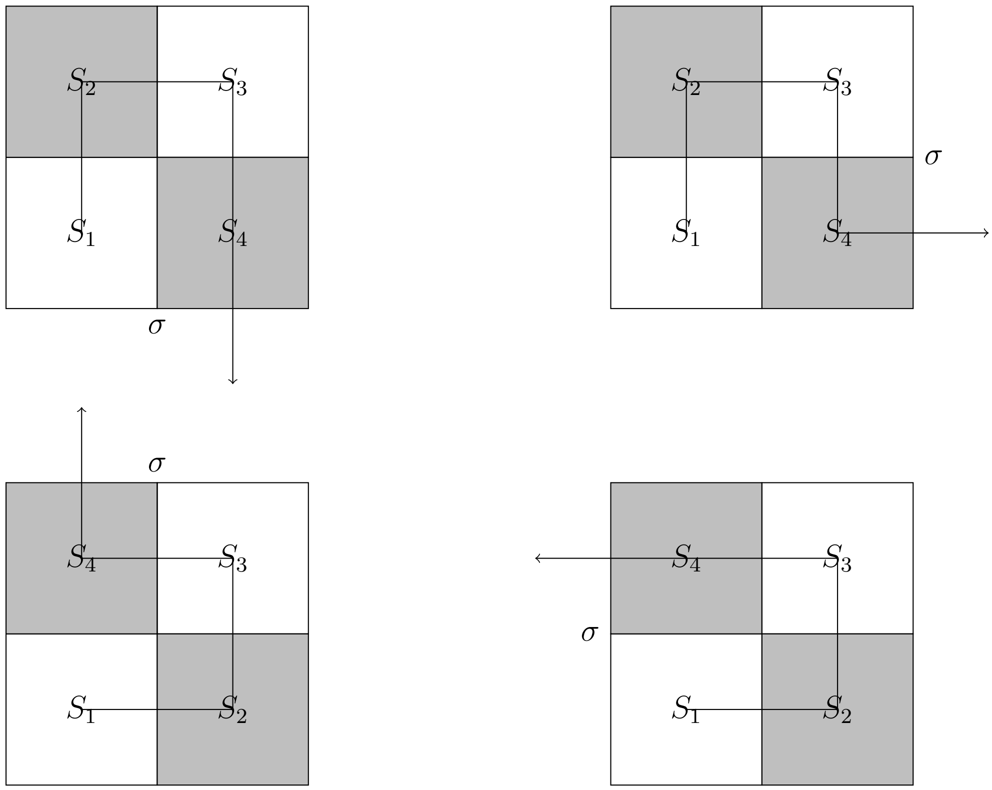
First part follows by excluding all points that do not have unique chains.

The fact that these points form a set of measure 0 follows from the lemma that if a_k differs from some fixed f_k for k sufficiently large then the resulting set has measure 0.

Last part is a repeat of Theorem 1.4, only this time use dyadic squares for the covering of the open subset of $\mathbb{R}^2$.

Traversal

Given a dyadic square S in generation k, consider its four sub-squares in generation $k+1$. Assuming we can enter at any square S_1, we can always traverse adjacent squares in a sequence (S_1, S_2, S_3, S_4) and exit at any edge σ. The four possibilities are:



Note that each entry square is colored white and each exit square is colored black.

Dyadic Correspondence with Adjacency

Theorem

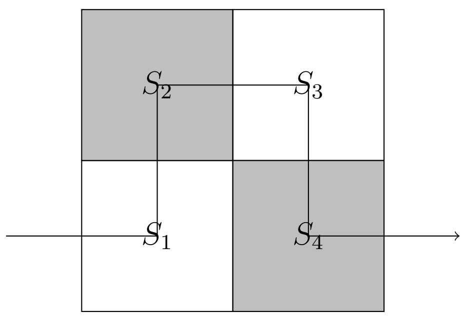
There exists a dyadic correspondence Φ such that:

1). In generation k, Let I_- be the leftmost interval and I_+ be the rightmost interval. $\Phi(I_-)$ is the lower left square and $\Phi(I_+)$ is the lower right square.

2). If I and J are two adjacent intervals in generation k then $\Phi(I)$ and $\Phi(J)$ are adjacent squares in generation k.

Proof (by Induction on k)

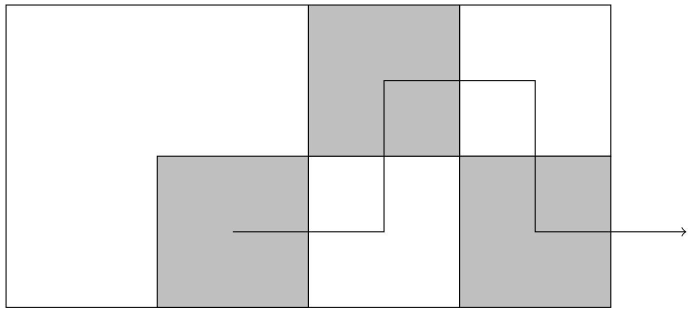
Base: $k = 1$

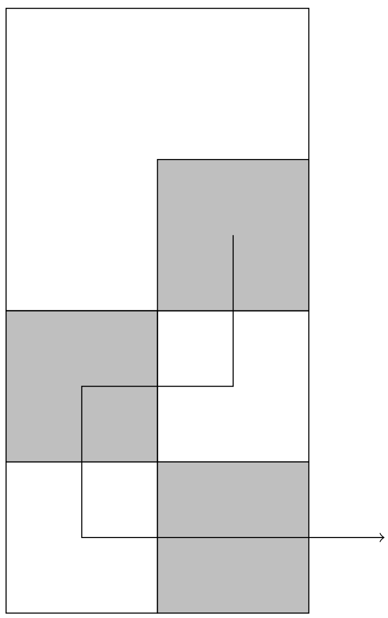


Assume Φ has been defined for all generations $\leq k$

Consider generation $k + 1$

Assume that S_j is a square from generation k that has been divided into squares $S_{j,1}, S_{j,2}, S_{j,3}, S_{j,4}$ in generation $k + 1$.
Square S_j is entered from adjacent square S_{j-1} by a traversal from a black square in S_{j-1} to a white square in S_j.
Let σ be the edge between S_j and S_{j+1}.
Traverse S_j using one of the four valid traversals to reach S_{j+1}.
By the inductive assumption S_{4^k} is in the lower-right corner, and must be entered from adjacent square S_{4^k-1} from either the top or left.
Since the lower left square in generation $k + 1$ is an exit square, it must be black. Since we must enter S_{4^k} on a white square, only one of the following two cases is possible:





$\therefore$, Φ is properly defined for generation $k+1$.

The Curve

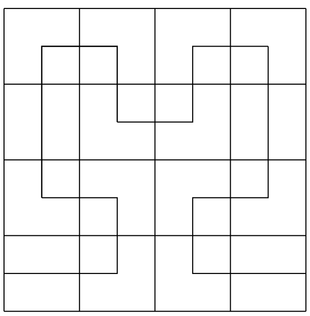
Let Φ be the dyadic correspondence described above. For generation k, let t_j be the center of the j^{th} quartic interval:

$$t_j = \frac{j - \frac{1}{2}}{4^k}, 1 \leq j \leq 4^k$$

and let x_j be the center of the j^{th} square per the ordering imposed by Φ. Define:

$$\mathcal{P}(t) = \begin{cases} x_j & t = t_j \\ (0, \frac{1}{2^{k+1}}) = x_0 & t = t_0 = 0 \\ (1, \frac{1}{2^{k+1}}) = x_{4^k+1} & t = t_{4^k+1} = 1 \end{cases}$$

Then linearly extend the values between the t_j to create a set of horizontal and vertical lines between the centers.



Theorem

$P_k(t)$ is continuous.

Proof

Assume $\epsilon > 0$
Let $\delta = min\{\frac{1}{4^k}, \frac{\epsilon}{2^k}\}$
Along each line segment, $P_k'(t) = \frac{\frac{1}{2^k}}{\frac{1}{4^k}} = 2^k$
Assume $|t_i - t_j| < \delta$

case 1: $\delta = \frac{1}{4^k}$

$$|\mathcal{P}(t_i) - \mathcal{P}(t_j)| \leq \frac{1}{2^k} < \epsilon$$

case 2: $\delta = \frac{\epsilon}{2^k}$

$$|\mathcal{P}(t_i) - \mathcal{P}(t_j)| \leq 2^k |t_i - t_j| < 2^k (\frac{\epsilon}{2^k}) = \epsilon$$

Theorem

$\mathcal{P}$ exists, is continuous, and is onto (surjective).

Proof

Assume $t \in [0,1]$
Assume $\epsilon > 0$
Let $N(\epsilon) = \log_2 \frac{\sqrt{2}}{\epsilon}$
Assume $k > N$

$$|\mathcal{P}_{k+1}(t) - \mathcal{P}_k(t)| \leq \frac{\sqrt{2}}{2^k} < \frac{\sqrt{2}}{2^{\log_2 \frac{\sqrt{2}}{\epsilon}}} = \epsilon$$

The continuous $\mathcal{P}_k$ converge uniformly to some $\mathcal{P}$, which is also continuous.
But $\mathcal{P}$ is also dense in $[0,1] \times [0,1]$ and is thus onto as well.

Theorem

Let Φ be the dyadic correspondence from above. $\Phi^*(t) = \mathcal{P}(t)$.

Proof

Assume $[a,b] \in [0,1]$
(a,b) can be written as $\bigcup_j I_j$, where the I_j are quartic intervals with disjoint interiors.

$$m_2(\mathcal{P}(a,b)) = \sum_{j=1}^{\infty} m_2(\mathcal{P}(I_j)) = \sum_{j=1}^{\infty} m_1(I_j) = m_1(a,b)$$