Continuity

Definition

To say that $f(z)$ is *continuous* at a point z_0 means:

$$\forall\, \epsilon > 0, \exists\, \delta > 0, 0 < |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$$

In other words:

$$\lim_{z \to z_0} f(z) = f(z_0)$$

To say that $f(z)$ is continuous in a region R means $\forall\, z \in R$, $f(z)$ is continuous at z.

Theorem

Let $f(z) = u + iv$ and $z_0 = x_0 + iy_0$:
$f(z)$ is continuous at $z_0 \iff u$ and v are continuous at (x_0, y_0)

Proof

$\implies$ Assume $f(z)$ is continuous at z_0

Let $u(x_0, y_0) = u_0$ and $v(x_0, y_0) = v_0$
$\lim_{z \to z_0} f(z) = f(z_0) = u(x_0, y_0) + iv(x_0, y_0) = u_0 + iv_0$
$\lim_{z \to z_0} f(z) = \lim_{(x,y) \to (x_0,y_0)} u(x, y) + i \lim_{(x,y) \to (x_0,y_0)} v(x, y)$
$\lim_{(x,y) \to (x_0,y_0)} u(x, y) + i \lim_{(x,y) \to (x_0,y_0)} v(x, y) = u_0 + iv_0$
$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0$
$\therefore u$ and v are continuous at (x_0, y_0)

$\impliedby$ Assume u and v are continuous at (x_0, y_0)

Let $u(x_0, y_0) = u_0$ and $v(x_0, y_0) = v_0$
Let $\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0$
$\lim_{z \to z_0} f(z) = \lim_{(x,y) \to (x_0,y_0)} u(x, y) + i \lim v(x, y) = u_0 + iv_0 = f(z_0)$
$\therefore f(z)$ is continuous at z_0

The following theorem also follows directly from the limit laws:

Theorem

Assume $f(z)$ and $g(z)$ are continuous at a point z_0:

1). $|f(z)|$ is continuous at z_0

2). $-f(z)$ is continuous at z_0

3). $f(z) + g(z)$ is continuous at z_0

4). $f(z) - g(z)$ is continuous at z_0

5). $f(z)g(z)$ is continuous at z_0

6). $f(z_0) \neq 0 \implies \frac{1}{f(z)}$ is continuous at z_0

7). $g(z_0) \neq 0 \implies \frac{f(z)}{g(z)}$ is continuous at z_0

Theorem

$f(z)$ and $g(z)$ continuous at $z_0 \implies (f \circ g)(z)$ continuous at z_0.

Proof

Assume $f(z)$ and $g(z)$ are continuous at z_0
Assume $\epsilon > 0$
$\exists\, \delta_1 > 0, 0 < |z - z_0| < \delta_1 \implies |f(z) - f(z_0)| < \epsilon$
$\exists\, \delta_2 > 0, 0 < |f(z) - f(z_0)| < \delta_2 \implies |g[f(z)] - g[f(z_0)]| < \epsilon$
Let $\delta = \min\{\delta_1, \delta_2\}$
Assume $0 < |z - z_0| < \delta$
$|f(z) - f(z_0)| < \epsilon$
$|g[f(z)] - g[f(z_0)]| < \epsilon$

Theorem

Let $f(z)$ be continuous at z_0 and $f(z_0) \neq 0$:

$$\exists\, \epsilon > 0, \forall\, z \in N_\epsilon(z_0), f(z) \neq 0$$

Proof

Let $\epsilon = \frac{|f(z_0)|}{2} > 0$
$\exists\, \delta > 0, 0 < |z - z_0| < \delta \implies |f(z) - f(z_0)| < \frac{|f(z_0)|}{2}$
ABC: $\exists\, z^*, 0 < |z^* - z_0| < \delta$ and $f(z^*) = 0$
$|f(z^*) - f(z_0)| = |f(z_0)| < \frac{|f(z_0)|}{2}$
But $f(z_0) \neq 0$
Contradiction!
$\therefore f(z^*) \neq 0$

Theorem

Let R be a closed and bounded region:
$f(z)$ continuous on $R \implies \exists\, M > 0, \forall\, z \in R, |f(z)| \leq M$.
Furthermore, equality occurs for at least one $z \in R$.

Proof

Assume $f(z)$ is continuous on R
Let $f(z) = u + iv$
u and v are bounded and continuous in R
$\sqrt{u^2 + v^2} \in \mathbb{R}$ is bounded and continuous and achieves some maximum value M in R
But $|f(z)| = \sqrt{u^2 + v^2}$
$\therefore \forall\, z \in R, |f(z)| \leq M$, with equality somewhere in R.