Word Problems

Units

There is nothing special about a number — it is just syntax. Numbers become meaningful when they are used to measure something, and the type of thing that is being measured is indicated by the units attached to a number.

There are three primitive unit types:

1). length (L)

2). mass (not weight!) (M)

3). time (T)

All other units are combinations of these three primitive types:

- Area is L^2 - square mile, square meter, acre
- Volume is L^3 - cubic inch, gallon, liter
- Speed/velocity is $L/T = LT^{-1}$ - mph, miles/hour, meters/second

There are three systems of units:

1). English

2). Meter-Kilogram-Second (mks)

3). Centimeter-Gram-Second (cgs)

In general, problems are solved within a single unit system. This is especially important when working with derived units:

Weight:

English: $slug\text{-}ft/s^2$=lb (pound)
mks: $kg\text{-}m/s^2$=N (newton)
cgs: $g\text{-}cm/s^2$=dyn (dyne)

To convert between units of the same type, use a conversion factor:

$$(6ft)\left(\frac{12in}{1ft}\right) = 72in$$

$$(6ft)\left(\frac{1yard}{3ft}\right) = 2yards$$

$$(10in)\left(\frac{2.54cm}{1in}\right) = 25.4cm$$

In equations:

1). Adding like units

$$4ft + 3in = 4ft + 3in\left(\frac{1ft}{12in}\right) = 4ft + 0.25ft = 4.25ft$$

2). Combining a rate by an amount to get a total amount

$$(3ft/s)(10s) = 30ft$$

3). Combining values with different units to get a different type of unit

$$(5slugs)(32ft/s^2) = 160lbs$$

These are the three ways that mathematics is used to describe reality. Take the example of an electomagnet. We have a current (moving electrons) flowing through a wire (mass) that results in a magnetic field.

Significant Figures

Don't go crazy with calculator digits. In general, only use as many digits as are presented in the original problem:

3.1 meter
1.6 seconds
1.9375 m/s

But each input value was only measured to one decimal place, so the 375 digits are worthless.

In general, you will be asked to provide an answer to a certain number of decimal points.

Problem Types

1). Percentages (Interest)

2). Distance

3). Mixture

4). Shared Work

5). Triangles

6). Circumference, Area, Volume

7). Falling Object

Example

A community theatre group is staging a play in a 100 seat theatre. Adult tickets are \$25 and child tickets (12 and under) are \$8. Receipts from opening night to a full house are \$2126. How many tickets of each type were sold?

1). Be clear on what the answer should look like.

   n adult tickets
   m child tickets

2). Determine what is important and what can be ignored

   - 100 seat theatre
   - ticket prices
   - full house
   - Total receipts

3). Select a key unknown element of the problem and let it be represented by a variable:

   Let x be the number of adult tickets sold

4). Describe other unknown elements in terms of the key element:

   $100 - x$ is the number of child tickets sold

5). Build an equation relating the unknown elements to any restrictions using the three types of unit combinations:

   $25x + 8(100 - x) = 2126$

6). Double-check the units

   adults x \$/adult + kids x \$/kid = \$

7). Solve for the key unknown:

$$25x + 800 - 8x = 2126$$

$$17x = 1326$$

$$x = 78 adults$$

8). Determine other unknowns in terms of the key unknown:

$$kids = 100 - 78 = 22$$

9). Clearly state the answer:

   78 adult tickets were sold
   22 child tickets were sold

Percentages

A percentage is a fraction of a whole, expressed as parts per 100:

3% of 10 is $\left(\frac{3}{100}\right) 10 = 0.03 \cdot 10 = 0.3$

Percentage problems usually include and increase or decrease from an original value:

$$r = \frac{n-o}{o}$$

$$ro = n - o$$

$$n = ro + o$$

$$n = o(1+r)$$

$$p = 100 * r$$

You have an off-campus job that pays $180 per week. But you are such a good worker that your boss gives you a raise and your next paycheck is $192. What is the percentage of your raise?

$$180(1+r) = 192$$

$$1 + r = \frac{192}{180}$$

$$r = \frac{192}{180} - 1$$

$$r = 0.0\bar{6} = 6.\bar{6}\%$$

About a 7% raise

You want to buy a pair of basketball shoes that normally go for $125, but you wait until there is a 25% sale. The sales associate rings you up and says that the shoes cost $100. Is this correct?

$$125(1-r) = 100$$

$$1 - r = \frac{100}{125}$$

$$r = 1 - \frac{100}{125}$$

$$r = 0.20 = 20\%$$

Distance

rate = distance / time distance = rate x time time = distance / rate

Two travelers start from cities 100 km apart. One drives 80 km/h and the other drives 60 km/h. When and where do they pass each other?

$$60t + 80t = 100$$

$$140t = 100$$
$$t = \frac{100}{140} \cdot 60 = 43min$$
$$80 \cdot \frac{100}{140} = 57km$$

This time, one is traveling 10 mph faster than the other. In the same amount of time, the first goes 140 miles but the second only 120 miles. How fast is each driver going?

$$\frac{120}{r} = \frac{140}{r+10}$$

$$120(r+10) = 140r$$

$$120r + 1200 = 140r$$

$$20r = 1200$$

$$r = 60mph$$

Driver 1: 70 mph
Driver 2: 60 mph

Mixtures

The ticket and the first driving problem are all examples of *mixture* problems:

$$r_1a_1 + r_2a_2 = r_3(a_1 + a_2)$$

$r = \frac{miles}{hour}, a = hours$
$r = \frac{kg}{L}, a = L$

Peanuts cost \$5 per pound and cashews cost \$7.50 per pound Mix 10 lb of peanuts and 5 lb of cashews What should the price per pound be for the mixture?

$r_1 = 5\$/lb, a_1 = 10lb$
$r_2 = 7.50\$/lb, a_2 = 5lb$

$x =$ price per pound of mixture

$$5(10) + 7.5(5) = (10 + 5)x$$
$$50 + 37.50 = 15x$$
$$15x = 87.50$$
$$x = 5.84$$

The mixture should be sold at \$5.84 per pound.

Shared Work

$$r_1 + r_2 = r_3$$

A hose from one spigot can fill a small pool in 40 min. A hose from a different spigot can fill the same pool in 65 min. How long does it take to fill the pool if both sources of water are used?

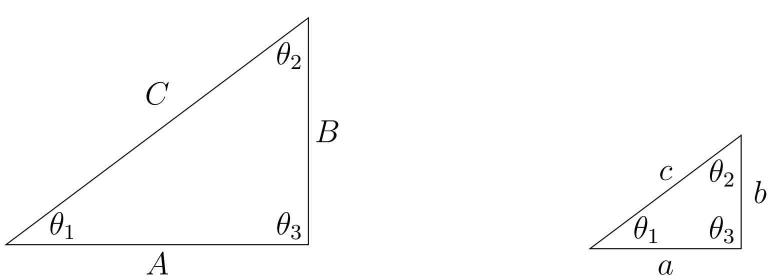
The rates here are jobs per minutes, where the number of jobs is typically 1.

$$\frac{1}{40} + \frac{1}{65} = \frac{1}{x}$$
$$\frac{65 + 40}{40(65)} = \frac{1}{x}$$
$$\frac{105}{2600} = \frac{1}{x}$$
$$105x = 2600$$
$$x = 24.8$$

About 25 minutes

Triangles

Similar triangles:



Ratios match:

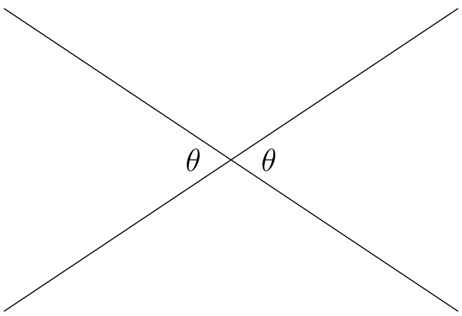
$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$$

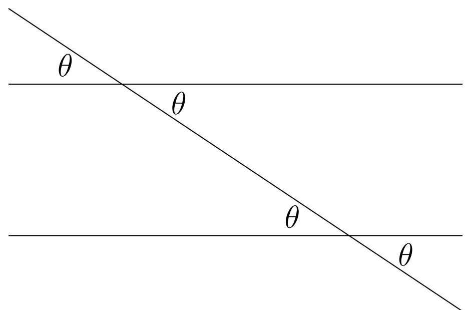
Can also mix:

$$aB = Ab$$

$$\frac{a}{b} = \frac{A}{B}$$

A building casts a 25 foot shadow. You place a 5 foot pole at the end of the shadow. The pole casts a 2 foot shadow. How tall is the building?





The sun is so far away that all of its rays hitting the earth can be assumed to be parallel. So, the building/shadow and pole/shadow make similar triangles:

$$\frac{x}{25} = \frac{5}{2}$$

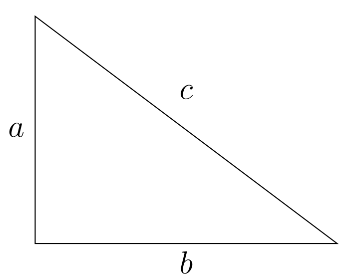
$$2x = 125$$

$$x = 62.5$$

The building is 62.5 feet tall.

Right Triangles:

Pythagorean Theorem



$a^2 + b^2 = c^2$
$3, 4, 5$
$5, 12, 13$
$6, 8, 10$
$8, 15, 17$
$7, 24, 25$

You wish to measure how high up a window is in the side of a building. You have a 13 foot ladder and a measuring tape. By placing the top of the ladder at the bottom of the window, the bottom of the ladder is 5 feet away from the bottom of the building. How high is the window?

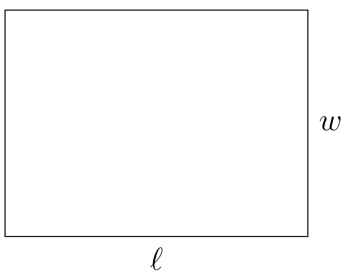
$$5^2 + h^2 = 13^2$$
$$25 + h^2 = 169$$
$$h^2 = 144$$
$$h = 12$$

The window is 12 feet up the side of the building.

Circumference, Area, Volume

See page 86 for some common formulas.

A farmer has 1200ft of fencing. He wants to build a horse enclosure that is twice as long as it is wide. What are the dimensions of the enclosure?



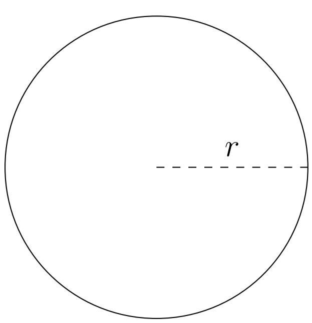
$$C = 2\ell + 2w$$

$w = x$
$\ell = 2x$
$2x + 2(2x) = 1200$
$6x = 1200$
$x = 200$

$w = 200$
$\ell = 400$

The enclosure should be 400 feet by 200 feet.

A car tire has a radius of 16 inches. Tires are to be replaced after 50,000 miles. How many rotations before replacement?



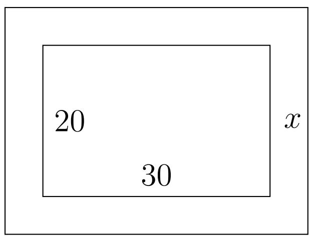
$$C = 2\pi r$$

$n =$ number of rotations

$(2\pi r \text{ in/rev})(n \text{ revs}) = (50000 \text{ mi})(5280 \text{ ft/mi})(12 \text{ in/ft})$

$n = \frac{50000(5280)(10)}{2\pi(16)} = 31.5$ million

A rectangular garden is 30 feet long and 20 feet wide. It is surrounded on all sides by a rock path of equal width. The total area of the garden and path is 1200 square feet. How wide is the path?



$A = w\ell = 1200$ sq ft

$x =$ width of path

$$(2x + 20)(2x + 30) = 1200$$

$$4(x + 10)(x + 15) = 1200$$
$$(x + 10)(x + 15) = 300$$
$$x^2 + 25x + 150 = 300$$
$$x^2 + 25x - 150 = 0$$
$$(x + 30)(x - 5) = 0$$
$$x = -30, 5$$

The path is 5 feet wide.

Falling Object

Set up a coordinate system, establishing a reference point (0) and the direction of positive motion.

s=displacement
v=velocity=change in displacement per unit time
a=acceleration=change in speed per unit time

Motion under constant acceleration:

$$s = s_0 + v_0 t + \frac{1}{2}at^2$$

Gravity, near the surface of the earth, results in constant acceleration: objects moving with gravity go continually faster and objects moving against gravity move continually slower.

$$g = 32ft/s^2$$

$$g = 9.8m/s^2$$

A man stands on a 256 foot cliff with a red ball:

1). simply drops the ball

2). throws the ball up with an initial velocity of 5 ft/s

3). throws the ball down with an initial velocity of 5 ft/s

$$s = 256 + v_0t + \frac{1}{2}(-32)t^2)$$

$$s = 256 + v_0t - 16t^2)$$

Case 1: $v_0 = 0$

$$0 = 256 - 16t^2$$

$$16t^2 = 256$$
$$t^2 = 16$$
$$t = \pm 4$$

Case 2: $v_0 = 5$ ft/s

$$0 = 256 + 5t - 16t^2$$

$$16t^2 - 5t - 256 = 0$$
$$t = \frac{5 \pm \sqrt{5^2 - 4(16)(-256)}}{2(16)}$$
$$t = -3.85, 4.16$$

Case 3: $v_0 = -5$ ft/s

$$0 = 256 - 5t - 16t^2$$

$$16t^2 + 5t - 256 = 0$$
$$t = \frac{-5 \pm \sqrt{(-5)^2 - 4(16)(-256)}}{2(16)}$$
$$t = -4.16, 3.85$$