Elementary Row Operations

Definition

A *matrix* is a rectangular collection of objects organized into *rows* and *columns*. Matrices are determined by the type of their objects and their size: an $m \times n$ matrix has m rows and n columns. When the objects are from $\mathbb{R}$ or $\mathbb{C}$ then matrices are useful for representing and solving SOLEs.

Example

$$\begin{aligned} 2x - y &= 4 \\ x + 2y &= 8 \end{aligned} \quad \Longrightarrow \quad \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}}_{\text{Coefficient Matrix}} \quad \Longrightarrow \quad \underbrace{\left[\begin{array}{cc:c} 2 & -1 & 4 \\ 1 & 2 & 8 \end{array}\right]}_{\text{Augmented Matrix}}$$

An $m \times n$ augmented matrix represents a SOLE with m equations and $n - 1$ unknowns.

Definition

The following are called the *elementary row operations* (EROs):

1). Interchange: $R_i \leftrightarrow R_j$

2). Scaling: $cR_i \to R_i$ $(c \neq 0)$

3). Replacement: $cR_i + R_j \to R_j$

EROs are reversible and do not change the solution set of an SOLE:

1). $R_j \leftrightarrow R_i$

2). $\frac{1}{c}(cR_i) \to R_i$

3). $(-cR_i) + (cR_i + R_j) \to R_j$

Definition

To say that two matrices (SOLEs) are row equivalent means that there exists a sequence of EROs that transform one matrix into the other.

Thus, row equivalent matrices (SOLEs) have the same solution set.

Example

$$\begin{aligned} 2x + y &= 4 \\ -x + 2y &= 3 \end{aligned} \qquad \left[\begin{array}{cc:c} 2 & 1 & 4 \\ -1 & 2 & 3 \end{array}\right]$$

$\frac{1}{2}R_1 \rightarrow R_1$

$$x + \frac{1}{2}y = 2 \qquad \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 2 \\ -1 & 2 & 3 \end{array}\right]$$
$$-x + 2y = 3$$

$(1)R_1 + R_2 \rightarrow R_2$

$$x + \frac{1}{2}y = 2 \qquad \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 2 \\ 0 & \frac{5}{2} & 5 \end{array}\right]$$
$$\frac{5}{2}y = 5$$

$\frac{2}{5}R_2 \rightarrow R_2$

$$x + \frac{1}{2}y = 2 \qquad \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 2 \end{array}\right]$$
$$y = 2$$

$-\frac{1}{2}R_2 + R_1 \rightarrow R_1$

$$x = 1 \qquad \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$
$$y = 2$$

Consistent with one (unique) solution: $(1, 2)$

Check:

$$2(1) + 2 = 2 - 2 = 4\checkmark$$

$$-1 + 2(2) = -1 + 4 = 3\checkmark$$

Example

$$x_2 - 4x_3 = 8 \qquad \left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right]$$
$$2x_1 - 3x_2 + 2x_3 = 1$$
$$5x_1 - 8x_2 + 7x_3 = 1$$

$R_1 \leftrightarrow R_3$

$$5x_1 - 8x_2 + 7x_3 = 1 \qquad \left[\begin{array}{ccc|c} 5 & -8 & 7 & 1 \\ 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \end{array}\right]$$
$$2x_1 - 3x_2 + 2x_3 = 1$$
$$x_2 - 4x_3 = 8$$

$-2R_2 + R_1 \rightarrow R_1$

$$x_1 - 2x_2 + 3x_3 = -1 \qquad \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \end{array}\right]$$
$$2x_1 - 3x_2 + 2x_3 = 1$$
$$x_2 - 4x_3 = 8$$

$-2R_1 + R_2 \to R_2$

$$\begin{aligned} &x_1 - 2x_2 + 3x_3 = -1 \\ &x_2 - 4x_3 = 3 \\ &x_2 - 4x_3 = 8 \end{aligned} \qquad \left[\begin{array}{ccc:c} 1 & -2 & 3 & -1 \\ 0 & 1 & -4 & 3 \\ 0 & 1 & -4 & 8 \end{array}\right]$$

$-R_2 + R_3 \to R_3$

$$\begin{aligned} &x_1 - 2x_2 + 3x_3 = -1 \\ &x_2 - 4x_3 = 3 \\ &0 = 5 \end{aligned} \qquad \left[\begin{array}{ccc:c} 1 & -2 & 3 & -1 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

Inconsistent, because $0 \neq 5$