Series in a Normed Space

Definition: Infinite Series

Let E be a normed space and let $S = \sum_{n=1}^{\infty} \vec{x}_n$ be an *infinite series* in E. To say that S *converges* in E means the sequence of partial sums (S_N) where $S_N = \sum_{n=1}^{N} \vec{x}_n$ converges in the norm to some value $\vec{x} \in E$:

$$\|S_n - S\| = \left\| \sum_{n=1}^{N} \vec{x}_n - \vec{x} \right\| \to 0$$

To say that S converges *absolutely* in E means:

$$\sum_{n=1}^{\infty} \|\vec{x}_n\| < \infty$$

Examples

1). $E = \mathbb{R}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, but not absolutely.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln 2$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

2). $E = \mathcal{P}[0,1]$ and $f_n(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!}$ converges absolutely but does not converge in the norm.

$$f_n \to f = e^t \notin \mathcal{P}[0,1]$$

$$\sum_{n=1}^{\infty} \left\| \frac{t^n}{n!} \right\| = \sum_{n=1}^{\infty} \max_{t \in [0,1]} \left| \frac{t^n}{n!} \right| = \sum_{n=1}^{\infty} \frac{1}{n!} = e < \infty$$

Theorem

Let E be a normed space. E is Banach iff every ACV series in E converges in E.

Proof

$\Longrightarrow$ Assume E is Banach.

Assume $\sum_{n=1}^{\infty} \vec{x}_n$ is ACV in E.

Thus, $\sum_{n=1}^{\infty} \|\vec{x}_n\| < \infty$.

Let $S_N = \sum_{n=1}^{N} \|\vec{x}_n\|$.

Let $s_N = \sum_{n=1}^{N} \vec{x}_n$.

AWLOG: $N < M$.

$\|s_M - s_N\| = \left\| \sum_{n=1}^{M} \vec{x}_n - \sum_{n=1}^{N} \vec{x}_n \right\| = \left\| \sum_{n=N+1}^{M} \vec{x}_n \right\| \leq \sum_{n=N+1}^{M} \|\vec{x}_n\| = S_M - S_N \to 0$

Therefore, (s_n) is Cauchy in E.

But E is complete, therefore (s_n) converges in E.

$\Longleftarrow$ Assume every ACV series in E converges in E.

Assume $(\vec{x}_n)$ is Cauchy in E.

$\forall\, k \in \mathbb{N}, \exists\, N_k > 0, m, n > N_k \implies \|\vec{x}_n - \vec{x}_m\| < \dfrac{1}{2^k}$

Let (n_k) be a strictly increasing sequence in $\mathbb{N}$.

Thus, for all $n_k > N_k$:

$$\sum_{k=1}^{\infty} \|\vec{x}_{n_{k+1}} - \vec{x}_{n_k}\| \leq \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k < \infty$$

Thus $\sum_{k=1}^{\infty} (\vec{x}_{n_{k+1}} - \vec{x}_{n_k})$ is ACV, and by assumption, converges to some element $\vec{x} \in E$.

Let $S_N = \sum_{k=1}^{N} (\vec{x}_{n_{k+1}} - \vec{x}_{n_k})$.

Note that this sum is telescoping, so:

$$S_N = \vec{x}_{n_{N+1}} - \vec{x}_{n_1} \to \vec{x}$$

And so $\vec{x}_{n_{N+1}} \to \vec{x} + \vec{x}_{n_1}$.

This means that $(\vec{x}_{n_k})$ is a convergent subsequence of $(\vec{x}_n)$ that converges to $\vec{x} \in E$.

Therefore, by previous lemma, $\vec{x}_n \to \vec{x} \in E$ and thus E is complete.