Math-08 Homework #13 Solutions

Reading

- Text book section 3.3, 3.4, 4.1

Problems

Note that all sketches of graphs must have all found intercepts and discontinuities labeled. All domains and ranges must be expressed in interval notation. Remember, sketches do not have to be to scale!

1). Let $p(x)$ be a polynomial whose graph contains the points $(-2,1)$ and $(3,0)$.

a). What is the remainder with $p(x)$ is divided by $(x+2)$?

By the remainder theorem: $p(-2)=1$

b). What is the remainder with $p(x)$ is divided by $(x-3)$?

By the remainder theorem: $p(3)=0$

2). Let:

$$p(x) = 2x^4 + x^3 - 9x^2 + 8x - 2$$

a). Fully factor $p(x)$. Note that you may end up with irrational zeros! You must show all work, including the possible candidates for zeros, and long (or synthetic) divisions that lead you to the final answer. There is *no* credit for simply stating an answer.

First, we determine the leading and constant coefficients and then list all of their factors (positive and negative!):

$a_n = 2 : \pm 1, \pm 2$
$a_0 = -2 : \pm 1, \pm 2$

Next, we determine all possible candidates: $\frac{a_0}{a_n}$:

$\pm 1, \pm 2, \pm \frac{1}{2}$

Now, look for actual zeros using the remainder/factor theorem:

$p(1) = 2 + 1 - 9 + 8 - 2 = 0$ it works! so divide out $(x-1)$

$$
\begin{array}{r}
2x^3 + 3x^2 - 6x + 2 \\
x-1 \overline{\big)\, 2x^4 \;\; + x^3 - 9x^2 + 8x - 2} \\
\underline{-2x^4 + 2x^3} \qquad\qquad\qquad\quad \\
3x^3 - 9x^2 \qquad\qquad \\
\underline{-3x^3 + 3x^2} \qquad\qquad \\
-6x^2 + 8x \qquad \\
\underline{6x^2 - 6x} \qquad \\
2x - 2 \\
\underline{-2x + 2} \\
0
\end{array}
$$

At this point we have:

$p(x) = (x-1)(2x^3 + 3x^2 - 6x + 2)$

We now repeat the process with the third degree polynomial:

$a_n = 2 : \pm 1, \pm 2$
$a_0 = 2 : \pm 1, \pm 2$

$\pm 1, \pm 2, \pm \frac{1}{2}$

We need to try 1 again, since it may be a repeated zero:

$p(1) = 2 + 3 - 6 + 2 \neq 0$
$p(-1) = -2 + 3 + 6 + 2 \neq 0$
$p(2) = 16 + 12 - 12 + 2 \neq 0$
$p(-2) = -16 + 12 + 12 + 2 \neq 0$
$p(\frac{1}{2}) = \frac{1}{4} + \frac{3}{4} - 3 + 2 = 0$ works!

Instead of dividing by $(x - \frac{1}{2})$ we can divide out the $\frac{1}{2}$ and instead divide by $(2x-1)$. This is OK because the leading coefficient of the polynomial is 2:

$$
\begin{array}{r}
x^2 + 2x - 2 \\
2x-1 \overline{\big)\, 2x^3 + 3x^2 - 6x + 2} \\
\underline{-2x^3 \;\; + x^2} \qquad\qquad\quad \\
4x^2 - 6x \qquad \\
\underline{-4x^2 + 2x} \qquad \\
-4x + 2 \\
\underline{4x - 2} \\
0
\end{array}
$$

So, now we have:

$p(x) = (x-1)(2x-1)(x^2 + 2x - 2)$

We are down to a quadratic that does not factor by inspection, so we need to use the quadratic formula:

$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2(1)} = \frac{-2 \pm \sqrt{12}}{2} = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$

Thus, the final factorization is:

$p(x) = (x-1)(2x-1)[x - (-1+\sqrt{3})][x - (-1-\sqrt{3})]$

b). Determine all x-intercepts (if any).

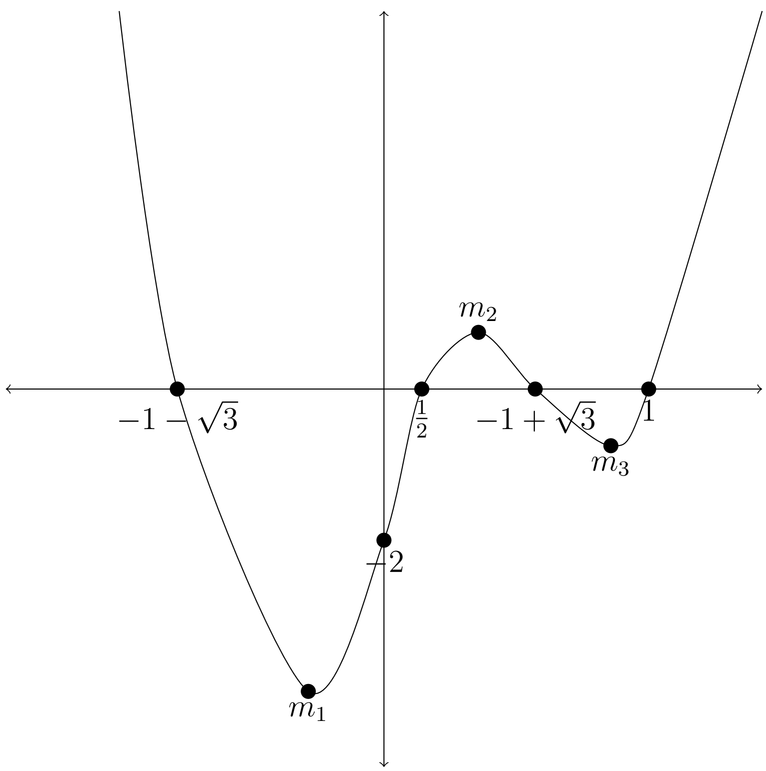
The x-intercepts are just the zeros:

$(1, 0), (\frac{1}{2}, 0), (-1 \pm \sqrt{3}, 0)$

c). Determine all y-intercepts (if any).

$f(0) = -2$, so $(0, -2)$

d). Sketch the graph. Be sure to label all intercepts.



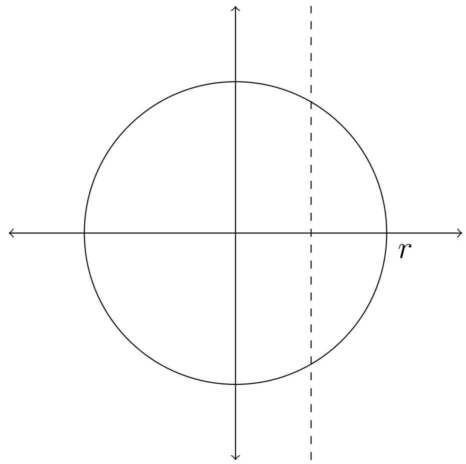
e). Use a calculator to determine all extrema. Just state the points; you don't need to attach screenshots this time.

minima: $m_1(-1.87, -30.51)$ and $m_3(0.89, -0.05)$
maxima: $m_2(0.60, 0.04)$

3). Consider a circle: $x^2 + y^2 = r^2$.

a). Solve for y.

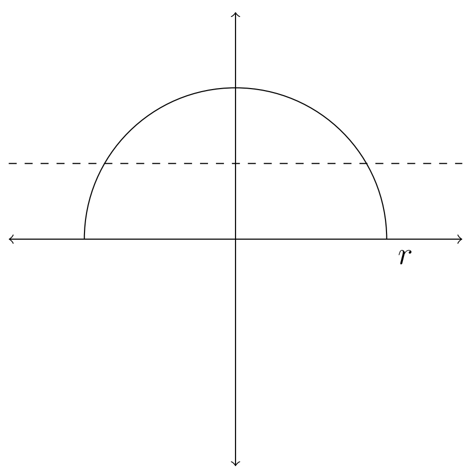
$y = \pm\sqrt{r^2 - x^2}$



b). Limit the range so that you get a function of x.

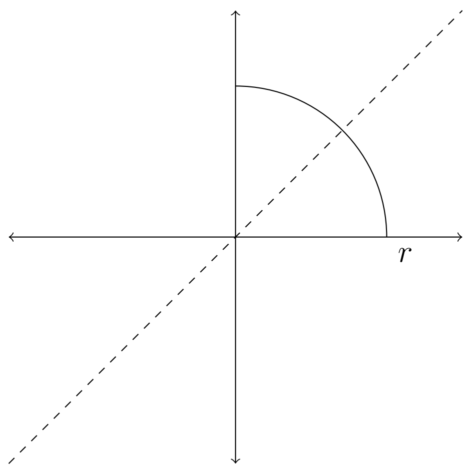
Since a circle fails the vertical line test, we need to take either the top half or the bottom half. Normally, we take the top half:

$y = \sqrt{r^2 - x^2}$



c). Limit the domain so that you get a one-to-one function of x.

Since the half-circle fails the horizontal line test, we need to eliminate half of it in order to make the function one-to-one. We normally take the quarter circle in QI:



d). Determine the inverse function based on your limited domain by solving for x.

Upon observing the graph of the final function, we see that it is reflected across the line $y = x$. Thus, it is its own inverse:

$x = \sqrt{r^2 - y^2}$

4). Consider a function $f(x)$ whose graph contains the following points:

x	-5	-3	0	1	-2
y	1	-1	-3	-2	6

Evaluate the following:

$$2f^{-1}(-2) - f(-3) + [f(1)]^{-1}$$

From the table, we see that:

$f^{-1}(-2) = 1$
$f(-3) = -1$
$f(1) = -2$

So: $2(1) - (-1) + (-2)^{-1} = 2 + 1 - \frac{1}{2} = \frac{5}{2}$