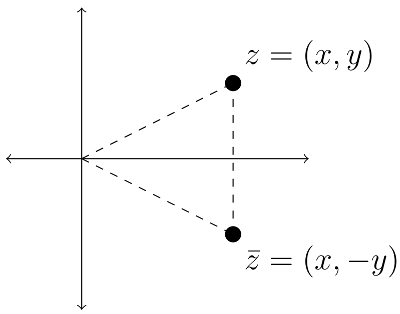
Conjugates

Definition

Let $z = x + iy \in \mathbb{C}$. The conjugate of z, denoted $\bar{z}$, is given by:

$$\bar{z} = x - iy$$



Properties

1). $\bar{\bar{z}} = z$

2). $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

3). $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

4). $\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$

5). $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

Proof

1).

$$\begin{aligned} \bar{\bar{z}} &= \overline{\overline{x + iy}} \\ &= \overline{x - iy} \\ &= x + iy \\ &= z \end{aligned}$$

2).

$$\begin{aligned} \overline{z_1 \pm z_2} &= \overline{(x_1 + iy_1) \pm (x_2 + iy_2)} \\ &= \overline{(x_1 \pm x_2) + i(y_1 \pm y_2)} \\ &= (x_1 \pm x_2) - i(y_1 \pm y_2) \\ &= (x_1 - iy_1) \pm (x_2 - iy_2) \\ &= \bar{z}_1 \pm \bar{z}_2 \end{aligned}$$

3).

$$
\begin{aligned}
\overline{z_1 z_2} &= \overline{(x_1+iy_2)(x_2+iy_2)} \\
&= \overline{(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)} \\
&= (x_1x_2-y_1y_2)-i(x_1y_2+x_2y_1) \\
&= x_1(x_2-iy_2)-y_1(y_2+ix_2) \\
&= x_1(x_2-iy_2)-iy_1(x_2-iy_2) \\
&= (x_1-iy_1)(x_2-iy_2) \\
&= \bar{z}_1\bar{z}_2
\end{aligned}
$$

4).

$$
\begin{aligned}
\overline{\left(\frac{1}{z}\right)} &= \overline{\left(\frac{x-iy}{x^2+y^2}\right)} \\
&= \frac{x+iy}{x^2+y^2} \\
&= \frac{x-i(-y)}{x^2+(-y)^2} \\
&= \frac{1}{x-iy} \\
&= \frac{1}{\bar{z}}
\end{aligned}
$$

5).

$$
\begin{aligned}
\overline{\left(\frac{z_1}{z_2}\right)} &= \overline{z_1\cdot\frac{1}{z_2}} \\
&= \bar{z}_1\overline{\left(\frac{1}{z_2}\right)} \\
&= \bar{z}_1\left(\frac{1}{\bar{z}_2}\right) \\
&= \frac{\bar{z}_1}{\bar{z}_2}
\end{aligned}
$$

Theorem

Let $z = x + iy \in \mathbb{C}$:

- $Re(z) = \frac{z+\bar{z}}{2}$
- $Im(z) = \frac{z-\bar{z}}{2}$

Proof

$z + \bar{z} = (x + iy) + (x - iy) = 2x = 2Re(z)$
$\therefore Re(z) = \frac{z+\bar{z}}{2}$

$z - \bar{z} = (x + iy) - (x - iy) = 2iy = 2iIm(z)$
$\therefore Im(z) = \frac{z-\bar{z}}{2i}$

Example

Let $Z = \frac{1}{z+i}$.

$$
\begin{aligned}
Im(Z) &= \frac{z - \bar{z}}{2i} \\
&= \frac{1}{2i}\left[\frac{1}{z+i} - \overline{\left(\frac{1}{z+i}\right)}\right] \\
&= \frac{1}{2i}\left(\frac{1}{z+i} - \frac{1}{\overline{\overline{z+i}}}\right) \\
&= \frac{1}{2i}\left(\frac{1}{z+i} - \frac{1}{\bar{z}-i}\right) \\
&= \frac{1}{2i}\left[\frac{(\bar{z}-i)-(z+i)}{(z+i)\overline{(z+i)})}\right] \\
&= \frac{1}{2i}\left(\frac{\bar{z}-z-2i}{|z+i|^2}\right) \\
&= -\frac{\frac{z-\bar{z}}{2i}+1}{|z+i|^2} \\
&= -\frac{y+1}{|(x+iy)+i|} \\
&= -\frac{y+1}{|x+i(y+1)|} \\
&= -\frac{y+1}{x^2+(y+1)^2}
\end{aligned}
$$