MATH 231B, FALL 2017
HOMEWORK 3 SOLUTIONS

1. (Sec. 3.8, ex. 19) (a) This statement is correct. **Proof**: For every z, we have

$$\begin{aligned}\langle x_n + y_n - (x+y), z\rangle &= \langle x_n - x, z\rangle + \langle y_n - y, z\rangle \\ &\to 0,\end{aligned}$$

as $n \to \infty$.

(b) This statement is correct. **Proof**: for all z, we have

$$\begin{aligned}|\langle \alpha_n x_n \alpha x, z\rangle| &= |\langle \alpha_n x_n - \alpha x_n, z\rangle + \langle \alpha x_n - \alpha x, z\rangle| \\ &\le |\alpha_n - \alpha|\,\|x_n\| + |\alpha\langle x_n - x, z\rangle| \\ &\to 0,\end{aligned}$$

as $n \to \infty$. Here we used the fact that a weakly convergent sequence is bounded, so $\|x_n\| \le M$, for some $M > 0$ and all $n \ge 1$, guaranteeing that $|\alpha_n - \alpha|\,\|x_n\| \to 0$.

(c) This statement is incorrect. Take $E = \ell^2$ and $x_n = y_n = e_n$. Then $e_n \to 0$ weakly, but $\langle e_n, e_n\rangle = 1 \not\to 0$.

(d) This statement is incorrect. The same example as in (c) works.

(e) This statement is correct. Suppose a sequence (x_n) converges weakly to both x and y. Let z be arbitrary. Then by the continuity of the inner product:

$$\begin{aligned}\langle x - y, z\rangle &= \langle x, z\rangle - \langle y, z\rangle \\ &= \lim_{n\to\infty}\langle x_n, z\rangle - \lim_{n\to\infty}\langle x_n, z\rangle \\ &= 0. \quad \square\end{aligned}$$

2. (Sec. 3.8, ex. 20) Assume $\dim H = N$. Then H is Hilbert space isomorphic to $\mathbb{C}^N$ with the usual inner product given by

$$\langle x, y\rangle = \sum_{k=1}^{N} x_k \overline{y}_k.$$

So it suffices to show that weak convergence implies strong convergence in $\mathbb{C}^N$. Let (x^n) be a sequence in $\mathbb{C}^N$ weakly convergent to $x = (x_1, \ldots, x_N)$. Write $x^n = (x_1^n, \ldots, x_N^n)$. Then for each element e_k of the standard basis of $\mathbb{C}^N$, we have

$$x_k^n = \langle x^n, e_k\rangle \to \langle x, e_k\rangle = x_k,$$

as $n \to \infty$. In other words, each component of x^n converges to the corresponding component of x. Therefore:

$$\begin{aligned}\|x^n - x\|^2 &= \sum_{k=1}^{N} |x_k^n - x_k|^2 \\ &\to 0,\end{aligned}$$

as $n \to \infty$, i.e., $x^n \to x$ in the strong sense. $\square$

3. (Sec. 3.8, ex. 23) (a) If $m \neq n$, then

$$\begin{aligned}
\langle x_m, x_n \rangle &= \int_{-\pi}^{\pi} \sin mt \cdot \sin nt \, dt \\
&= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m-n)t - \cos(m+n)t] \, dt \\
&= 0,
\end{aligned}$$

since $\int_{-\pi}^{\pi} \cos kt \, dt = 0$ for every non-zero integer k. Therefore, (x_n) is an orthogonal sequence.

(b) For all $m \neq n$, we have:

$$\begin{aligned}
\langle y_m, y_n \rangle &= \int_{-\pi}^{\pi} \cos mt \cdot \cos nt \, dt \\
&= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m+n)t + \cos(m-n)t] \, dt \\
&= 0.
\end{aligned}$$

Therefore, (y_n) is an orthogonal sequence. $\square$

4. (Sec. 3.8, 34) The expression

$$\int_{-1}^{1} \left| x^3 - a - bx - cx^2 \right|^2 \, dx$$

is minimized when $a + bx + cx^2$ is the orthogonal projection P of $f(x) = x^3$ to the subspace S of $C[-1, 1]$ (equipped with the usual L^2-inner product) spanned by the functions $f_0(x) = 1, f_1(x) = x$, and $f_2(x) = x^2$. We will apply the Gram-Schmidt process to f_0, f_1, f_2 to obtain an orthonormal basis e_0, e_1, e_2 of S. Then

$$P = \langle f, e_0 \rangle e_0 + \langle f, e_1 \rangle e_1 + \langle f, e_2 \rangle e_2.$$

Carefully applying the Gram-Schmidt process to f_0, f_1, f_3, we obtain

$$e_0(x) = \frac{1}{\sqrt{2}}, \quad e_1(x) = \sqrt{\frac{3}{2}} x, \quad e_2(x) = \frac{1}{2} \sqrt{\frac{5}{2}} (3x^2 - 1).$$

It is not hard to check that $\langle f, e_0 \rangle = \langle f, e_2 \rangle = 0$ (since the integral of an odd function over $[-1, 1]$ equals zero). Next,

$$\begin{aligned}
\langle f, e_1 \rangle &= \int_{-1}^{1} \sqrt{\frac{3}{2}} x^4 \, dx \\
&= 2 \sqrt{\frac{3}{2}} \int_{0}^{1} x^4 \, dx \\
&= 2 \sqrt{\frac{3}{2}} \frac{1}{5} \\
&= \frac{\sqrt{6}}{5}.
\end{aligned}$$

Therefore:

$$\begin{aligned} P(x) &= \langle f, e_0\rangle e_0(x) + \langle f, e_1\rangle e_1(x) + \langle f, e_2\rangle e_2(x) \\ &= \frac{\sqrt{6}}{5} \cdot \sqrt{\frac{3}{2}} x \\ &= \frac{3}{5} x. \end{aligned}$$

It follows that

$$\begin{aligned} \min_{a,b,c} \int_{-1}^{1} \left| x^3 - a - bx - cx^2 \right|^2 \, dx &= \int_{-1}^{1} \left| x^3 - \frac{3}{5} x \right|^2 \, dx \\ &= \frac{8}{175}. \quad \square \end{aligned}$$