Lab 9: Real Exponents

Consider the exponential expression a^b for $a, b \in \mathbb{R}$. There are three cases for b:

1). $b \in \mathbb{Z}$ (integer)

2). $b \in \mathbb{Q}$ (rational)

3). $b \in \mathbb{R} - \mathbb{Q}$ (irrational)

We have already studied what the first two cases mean: when b is an integer then we have $a \cdot a \cdots a$ a total of b times (or the reciprocal when $b < 0$) and when b is a rational number $\frac{p}{q}$ then $a^b = \sqrt[q]{a^p}$. But what does it mean when b is irrational? For example, what the heck does something like 2^π possibly mean?

If you enter 2^π into your calculator you will get an answer like: $2^\pi = 8.824977827$. Of course, 2^π is irrational, so this is just an approximation. But how did the calculator come by this answer? For this, we turn to our old friend *arbitrarily close*. Remember that π is the result of a sequence of fixed decimal (and therefore rational) approximations that get arbitrarily close to the exact value of π:

$$
\begin{array}{l}
3 \\
3.1 \\
3.14 \\
3.141 \\
3.1415 \\
3.14159 \\
3.141592 \\
3.1415926 \\
\vdots
\end{array}
$$

As such, we can calculate 2^p for each approximation of $\pi = p$:

π	2^π
3	8
3.1	8.574187700
3.14	8.815240927
3.141	8.821353305
3.1415	8.824411082
3.14159	8.824961595
3.141592	8.824973829
3.1415926	8.824977499

Note that as the approximation for π gets better and better, the approximation for 2^π also gets better. In fact, if you give me any $\epsilon > 0$, no matter how small, I can eventually find an approximation p of π such that 2^p is within ϵ of the exact value of 2^π.

Now you do it. Consider the irrational value $\pi^{\sqrt{2}}$. This time, both the base and exponent are irrational, so you will need to approximate both at each step. First get a value from your calculator. Be sure to list all of the decimal digits that your calculator provides:

$\pi^{\sqrt{2}} =$

Now get values for π and $\sqrt{2}$:

$\pi =$

$\sqrt{2} =$

Now complete the following table (I have done the first two rows for you):

π	$\sqrt{2}$	$\pi^{\sqrt{2}}$
3	1	3.000000000
3.1	1.4	4.874233962

Notice how your approximations for $\pi^{\sqrt{2}}$ converge to your calculator answer.