Adjugate

Definition

Let $A \in M_n$. The *adjugate* of A is the matrix given by:

$$\operatorname{adj}(A) = [(-1)^{i+j} \det(A_{ji})]$$

Thus, each entry in the adjugate of A is the corresponding cofactor of A.

Lemma

Let $A \in M_n$:

$$A(\operatorname{adj}(A)) = (\operatorname{adj}(A))A = (\det(A))I$$

Proof

$[A(\operatorname{adj}(A))]_{ii} = \sum_{k=1}^{n} a_{ik}[\operatorname{adj}(A)]_{ki} = \sum_{k=1}^{n} a_{ik}(-1)^{k+i} \det(A_{ik}) = \det(A)$

Now, consider a matrix $\tilde{A}$ where the i^{th} row is replaced by the j^{th} row and expand along the j^{th} row. Since $\tilde{A}$ has two dependent rows, its determinant is 0:

$$\det(\tilde{A}) = \sum_{k=1}^{n} (-1)^{j+k} \tilde{a}_{jk} \det(\tilde{A}_{jk}) = 0$$

But $\tilde{a}_{jk} = a_{ik}$ and $\det(\tilde{A}_j k) = \det(A_{jk})$, so:

$$\sum_{k=1}^{n} (-1)^{j+k} a_{ik} \det(A_{jk}) = [A \operatorname{adj}(A)]_{ij} = 0$$

$\therefore A(\operatorname{adj}(A)) = (\det(A))I$

$A^T(\operatorname{adj}(A^T)) = (\det(A^T))I$
$A^T(\operatorname{adj}(A)^T) = (\det(A^T))I$
$(\operatorname{adj}(A)A)^T = (\det(A^T))I$

$\therefore (\operatorname{adj}(A))A = (\det(A))I$

Corollary

Let $A \in M_n$ be invertible:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Proof

$A(\operatorname{adj}(A)) = (\det(A))I$
But A is invertible and so $\det(A) \neq 0$

$\therefore A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

Lemma

There exists invertible matrices A_ϵ such that:

$$\lim_{\epsilon \to 0} A_\epsilon = A$$

Theorem

Let $A \in M_n$, and $x, y \in M_{n,1}$:

$$\det(A + \vec{x}\vec{y}^T) = \det(A) + \vec{y}^T(\operatorname{adj}(A))\vec{x}$$

Example

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned}
\det(A + \vec{x}\vec{y}^T) &= \det\left(\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} 2 & 5 & 7 \\ 2 & 4 & 10 \\ 3 & 3 & 6 \end{bmatrix}\right) \\
&= 2(24 - 30) - 5(12 - 30) + 7(6 - 12) \\
&= -12 + 90 - 42 \\
&= 36
\end{aligned}$$

$$\begin{aligned}
\det(A) + \vec{y}^T(\operatorname{adj}(A))\vec{x} &= \det\left(\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}\right) + \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \operatorname{adj}(a) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= 0 + \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 14 \\ 0 & 0 & -6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= 36
\end{aligned}$$

Proof

Claim: $\det(x + \vec{x}\vec{y}^T) = \det \begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix}$

$\det \begin{bmatrix} I_n & 0_{n\times 1} \\ -\vec{y}^T & 1 \end{bmatrix} = \det(I_n)\det(\begin{bmatrix}1\end{bmatrix}) = 1 \cdot 1 = 1$

$\det \begin{bmatrix} A + \vec{x}\vec{y}^T & -\vec{x} \\ 0 & 1 \end{bmatrix} = \det(A + \vec{x}\vec{y}^T)\det(\begin{bmatrix}1\end{bmatrix}) = \det(A + \vec{x}\vec{y}^T) \cdot 1 = \det(A + \vec{x}\vec{y}^T)$

$$\begin{aligned}
\det\left(\begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix}\begin{bmatrix} I_n & 0_{n\times 1} \\ -\vec{y}^T & 1 \end{bmatrix}\right) &= \det \begin{bmatrix} A + \vec{x}\vec{y}^T & -\vec{x} \\ 0 & 1 \end{bmatrix} \\
\left(\det \begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix}\right)\left(\det \begin{bmatrix} I_n & 0_{n\times 1} \\ -\vec{y}^T & 1 \end{bmatrix}\right) &= \det \begin{bmatrix} A + \vec{x}\vec{y}^T & -\vec{x} \\ 0 & 1 \end{bmatrix} \\
\left(\det \begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix}\right)(1) &= \det(A + \vec{x}\vec{y}^T) \\
\therefore \det(A + \vec{x}\vec{y}^T) &= \det \begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix}
\end{aligned}$$

Claim: $\det \begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix} = \det(A) + \vec{y}^T(\operatorname{adj}(A))\vec{x}$

Case 1: A is invertible

$$\begin{aligned}
\begin{bmatrix} I & 0 \\ -\vec{y}^T A^{-1} & 1 \end{bmatrix}\begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix} &= \begin{bmatrix} A & -\vec{x} \\ 0 & y^T A^{-1}\vec{x} + 1 \end{bmatrix} \\
\det \begin{bmatrix} I & 0 \\ -\vec{y}^T A^{-1} & 1 \end{bmatrix} \det \begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix} &= \det \begin{bmatrix} A & -\vec{x} \\ 0 & \vec{y}^T A^{-1}\vec{x} + 1 \end{bmatrix} \\
1 \cdot \det \begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix} &= \det(A)(\vec{y}^T A^{-1}\vec{x} + 1) \\
\cdot \det \begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix} &= \det(A) + \vec{y}^T(\det(A))A^{-1}\vec{x} \\
&= \det(A) + \vec{y}^T \operatorname{adj}(A)\vec{x}
\end{aligned}$$

Case 2: A is not invertible

There exists invertible matrices A_ϵ such that by case 1:

$$\det \begin{bmatrix} A_\epsilon & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix} = \det(A_\epsilon) + \vec{y}^T(\operatorname{adj}(A_\epsilon))\vec{x}$$

Take the limit as $\epsilon \to 0$.

$\therefore \det(A + \vec{x}\vec{y}^T) = \det(A) + \vec{y}^T(\operatorname{adj}(A))\vec{x}$