Math-19 Homework #6 Solutions

Reading

Please read sections 2.8 and 4.1 through 4.7 and do all concept problems in the posted sections on webassign.

Problems

1). You use $1000 to open a savings account at your local bank on the first of February. The savings account has an interest rate of 1.5% per year and compounds monthly on the last day of the month. You set up an auto-deposit of $100 from your paycheck to occur on the first of each month, starting with the second month (March). During April, you withdraw $250 to purchase a new gameboy (gotta catch em all!).

a). Who is the lender and who is the borrower?

Lender: You
Borrower: The Bank

b). Calculate $x = 1 + \frac{r}{n}$

$$x = 1 + \frac{0.015}{12} = 1.00125$$

c). Construct a polynomial in x to determine the account value on July 2.

month	deposits	withdrawals	net
feb	1000	0	1000
mar	100	0	100
apr	100	250	-150
may	100	0	100
jun	100	0	100
jul	100	0	100

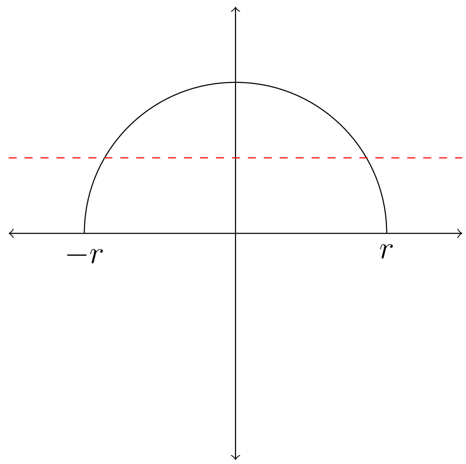
$$A(x) = 1000x^5 + 100x^4 - 150x^3 + 100x^2 + 100x + 100$$

d). What is the account value on July 2?

$$\$1256.58$$

2). Consider the circle $x^2 + y^2 = r^2$ and remember that we needed to restrict the range in order to obtain the function $y = \sqrt{r^2 - x^2}$.

a). Sketch the half-circle function and demonstrate why it is not one-to-one?

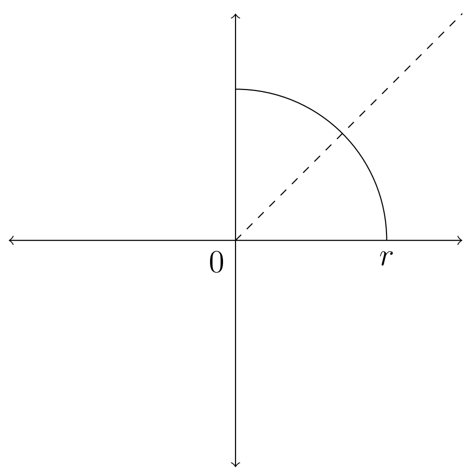


It fails the horizontal line test: each value in the range is mapped by two values: $+x$ and $-x$.

b). Suggest a how to limit the domain so that it is a one-to-one function.

Limit the domain for a quarter circle.

c). Sketch the new graph for the one-to-one function and state its domain and range.



Domain: $[0, r]$
Range: $[0, r]$

d). By observing the graph (and the line $y = x$), predict something about the inverse function.

Note that the graph is symmetry about the line $y = x$, and is therefore its own inverse.

e). Derive the inverse to prove your prediction.

$$
\begin{aligned}
f(x) &= \sqrt{r^2 - x^2} \\
y &= \sqrt{r^2 - x^2} \\
y^2 &= r^2 - x^2 \\
x^2 &= r^2 - y^2 \\
x &= \sqrt{r^2 - y^2} \\
f^{-1}(x) &= \sqrt{r^2 - x^2}
\end{aligned}
$$

$\therefore f(x) = f^{-1}(x)$

3). Consider the equation: $y = \log_a x$

a). Derive the change of base formula for some arbitrary base b.

$$
\begin{aligned}
y &= \log_a x \\
a^y &= x \\
\log_b a^y &= \log_b x \\
y \log_b a &= \log_b x \\
y &= \frac{\log_b x}{\log_b a}
\end{aligned}
$$

b). Use your formula with $b = e$ and your calculator to compute $\log_7 100$.

$$\log_7 100 = \frac{\ln 100}{\ln 7} = 2.36659$$

An easy way to check this is to use your calculator to show that:

$$7^{2.36659} = 100$$

c). Assume that you made a mistake and used the common log key instead of the natural log key in the above calculation. Would you get a different answer? Why or why not?

No, because the log rules do not depend on the base.

4). Researchers tend to prefer exponential (base e) equations. For example, the normal equation for the radioactive decay of Carbon-14, which has a half-life of 5730 years, would be:

$$A = A_0 \cdot 2^{-\frac{t}{5730}}$$

But the preferred exponential equations is:

$$A = A_o e^{-\frac{t}{8267}}$$

Prove that these two equations are equivalent.

In order for these two equations to be the same, the exponential factors must match:

$$\begin{aligned}
2^{-\frac{t}{5730}} &= e^{-\frac{t}{x}} \\
\ln 2^{-\frac{t}{5730}} &= \ln e^{-\frac{t}{x}} \\
-\frac{t}{5730} \ln 2 &= -\frac{t}{x} \\
x &= \frac{5730}{\ln 2} \\
x &= 8267
\end{aligned}$$

5). The San Francisco earthquake of 1906 was estimated at 7.6 on the Richter Scale. Current building codes have resulted in skyscrapers that can withstand an 8.0 earthquake. How much strong is this that the 1906 quake?

First, solve the Richter equation to find the intensity I of an earthquake given its rating M, where the reference $S = 10^{-4} cm$:

$$\begin{aligned}
M &= \log \frac{I}{S} \\
10^M &= \frac{I}{S} \\
I &= 10^M \cdot S
\end{aligned}$$

Given two ratings: M_1 and M_2:

$$\frac{I_1}{I_2} = \frac{10^{M_1} \cdot S}{10^{M_2} \cdot S} = 10^{M_1 - M_2}$$

Note that this is not dependent on the reference value. For our case:

$$\frac{I_1}{I_2} = 10^{8.0-7.6} = 10^{0.4} = 2.5$$

So, the buildings in SF can withstand an earthquake two and a half times the 1906 quake.