Ring Homomorphisms

Definition

Let R and R' be rings. To say that $\phi : R \to R'$ is a *ring homomorphism* means $\forall\, a, b \in R$:

1). $\phi(a + b) = \phi(a) + \phi(b)$

2). $\phi(ab) = \phi(a)\phi(b)$

Theorem

Let F be the set of real-values functions and let $\phi : F \to \mathbb{R}$ be defined by $\phi_a(f) = f(a)$. ϕ_a is a ring homomorphism, referred to as the *evaluation homomorphism*.

Proof

Assume $f, g \in F$
Assume $a \in \mathbb{R}$

$\phi_a(f + g) = (f + g)(a) = f(a) + g(a) = \phi_a(f) + \phi_a(g)$

$\phi_a(fg) = (fg)(a) = f(a)g(a) = \phi_a(f)\phi_a(g)$

Theorem

Let $\phi : \mathbb{Z} \to \mathbb{Z}_n$ be defined by $\phi(a) = a \bmod n$. ϕ is a ring homomorphism.

Proof

Assume $a, b \in \mathbb{Z}$
$a = nq_1 + r_1$
$b = nq_2 + r_2$

$\phi(a) = r_1$
$\phi(b) = r_2$

$\phi(a + b) = \phi(n(q_1 + q_2) + (r_1 + r_2)) = 0 + (r_1 + r_2) \bmod n = \phi(a) + \phi(b)$

$\phi(ab) = \phi(n^2 q_1 q_2 + nq_1 r_2 + nq_2 r_1 + r_1 r_2) = 0 + 0 + 0 + r_1 r_2 \bmod n = \phi(a)\phi(b)$