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Math 275A
Homework #1

Theorem: Exercise 1.3

For a function $f : X \to Y$ and sets $A, B \subset Y$:

- $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
- $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Proof.

$$\begin{aligned} x \in f^{-1}(A \cup B) &\iff f(x) \in A \cup B \\ &\iff f(x) \in A \text{ or } f(x) \in B \\ &\iff x \in f^{-1}(A) \text{ or } x \in f^{-1}(B) \\ &\iff x \in f^{-1}(A) \cup f^{-1}(B) \end{aligned}$$

Therefore, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

$$\begin{aligned} x \in f^{-1}(A \cap B) &\iff f(x) \in A \cap B \\ &\iff f(x) \in A \text{ and } f(x) \in B \\ &\iff x \in f^{-1}(A) \text{ and } x \in f^{-1}(B) \\ &\iff x \in f^{-1}(A) \cap f^{-1}(B) \end{aligned}$$

Therefore, $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. ■

Theorem: Restatement of 1.8

Let A and B be sets such that $A \subset B$. If B is countable then A is countable.

Lemma

Let $\{U_i : i \in N\}$ be a countably infinite number of countably infinite sets such that the U_i are pairwise disjoint. Then:

$$U = \bigcup_{i \in \mathbb{N}} U_i$$

is countable.

Proof. Let $U_i = \{u_{ij} : j \in \mathbb{N}\}$ and arrange the U_i as the rows of a matrix. Note that the u_{ij} are distinct and in one-to-one correspondence with the elements of U.

Now, enumerate the u_{ij} along the diagonals as follows:

$$
\begin{array}{cccccc}
u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & \cdots \\
u_{21} & u_{22} & u_{23} & u_{24} & u_{25} & \cdots \\
u_{31} & u_{32} & u_{33} & u_{34} & u_{35} & \cdots \\
u_{41} & u_{42} & u_{43} & u_{44} & u_{45} & \cdots \\
u_{51} & u_{52} & u_{53} & u_{54} & u_{55} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

This is a one-to-one correspondence between the u_{ij} and $\mathbb{N}$ and hence the u_{ij} are countable.

Therefore, U is countable. ■

Theorem: 1.12

The union of countably many countable sets is countable.

Proof. Let $A = \bigcup_{i \in I} A_i$ be a union of countably many countable sets. In order to remove duplicates from the A_i (elements in the intersections of two or more A_i), let:

$$A'_1 = A_1$$

$$A'_i = A_i - \bigcup_{j=1}^{i-1} A_j$$

Note that the A'_i are pairwise disjoint and $A = \bigcup_{i \in I} A'_i$

Now arrange the A'_i as the rows of a matrix B. This means that the b_{ij} are distinct and in one-to-one correspondence with the elements of A. Let U be a matrix consisting of a countably infinite number of rows and columns as described in the preceding lemma. There is an injection between the rows in B and the rows in U. Furthermore, there is an injection between the columns of each row B_i and its corresponding row U_i. Thus, there is a one-to-one correspondence between the elements of B, and hence the elements of A, and the elements of some subset C of the elements of U. But the subset of a countable set is countable (Theorem 1.8) and so C is countable.

Therefore A is countable. ■

Theorem: 1.13

The set $\mathbb{Q}$ is countable.

Proof. Let $\{Q_i : i \in \mathbb{N}\}$ be a family of sets where $Q_i = \left\{ \frac{p}{i} \,\middle|\, p \in \mathbb{Z} \right\}$. Note that:

$$\mathbb{Q} = \bigcup_{i \in \mathbb{N}} Q_i$$

But $\{Q_i : i \in \mathbb{N}\}$ is a countable number of countable sets, and hence is countable (Theorem 1.2).

Therefore, Q is countable. ■

Theorem: 1.16

The set of real numbers $\mathbb{R}$ is uncountable.

Proof. ABC that $(0, 1)$ is countable. This means that there exists some bijection $f : \mathbb{N} \to (0, 1)$. Let a_{ij} be j^{th} decimal digit of the i^{th} number:

$$\begin{aligned} f(1) &= 0.a_{11}a_{12}a_{13}a_{14}a_{15}\cdots \\ f(2) &= 0.a_{21}a_{22}a_{23}a_{24}a_{25}\cdots \\ f(3) &= 0.a_{31}a_{32}a_{33}a_{34}a_{35}\cdots \\ f(4) &= 0.a_{41}a_{42}a_{43}a_{44}a_{45}\cdots \\ f(5) &= 0.a_{51}a_{52}a_{53}a_{54}a_{55}\cdots \\ \vdots &= \vdots \end{aligned}$$

If $f(n)$ is rational with more than one representation, for example: $0.4\bar{9} = 0.5\bar{0}$, then the repeating 0 case is selected.

Now, let $b = b_1b_2b_3b_4b_5\cdots$ where:

$$b_i = \begin{cases} 1, & a_{ii} \neq 1 \\ 2, & a_{ii} = 1 \end{cases}$$

So b never contains a 0 or 9 digit and thus the non-unique cases are avoided. This means that $b \in (0, 1)$ but $b \notin f(\mathbb{N})$, contradicting the bijectiveness of f. Thus, $(0, 1)$ is uncountable. But $(0, 1) \subset \mathbb{R}$.

Therefore, by the contrapositive of Theorem 1.8, $\mathbb{R}$ is uncountable. ∎