Compact Operators

Definition

Let H be a Hilbert space and let A be an operator on H. To say that A is *compact* (*completely continuous*) means for all bounded subsequences $(\vec{x}_n)$, the sequence $(A\vec{x}_n)$ in H has a convergent subsequence.

Examples

1). I is not compact.

Consider $(\vec{e}_n)$ be an orthonormal sequence in H.

$\forall\, n \in \mathbb{N}, \|\vec{e}_n\| = 1$ and so $(\vec{e}_n)$ is bounded.

But $\forall\, j, k, \|e_j - e_k\| = \sqrt{2} \not\to 0$

Therefore $(I\vec{e}_n)$ does not have a convergent subsequence.

2). Let H be finite dimensional and let T be a linear operator on H.

Claim: T is compact.

H is isomorphic to $\mathbb{C}^n$.
Since H is finite dimensional, T is bounded.
Assume $(\vec{x}_n)$ in H is bounded and let $\|\vec{x}_n\| \leq M$.
$\|T\vec{x}_n\| \leq \|T\| \, \|\vec{x}_n\| \leq M \, \|T\| < \infty$
And so $(T\vec{x}_n)$ is a bounded sequence in $\mathbb{C}^n$.

Therefore, by the Bolzano-Weierstrass theorem, $(T\vec{x}_n)$ has a convergent subsequence and thus T is compact.

3). Let H be a Hilbert space and let T be a linear operator on H defined by:

$$T\vec{x} = \langle \vec{x}, \vec{y}_0 \rangle \, \vec{z}_0$$

for fixed $\vec{y}_0, \vec{z}_0 \in H$.

Claim: T is compact.

Assume $(\vec{x}_n)$ in H is bounded.
Let $\|\vec{x}_n\| \leq M$.
$|\langle \vec{x}_n, \vec{y}_0 \rangle| \leq \|\vec{x}_n\| \, \|\vec{y}_0\| \leq M \, \|\vec{y}_0\|$
Thus $\langle \vec{x}_n, \vec{y}_0 \rangle$ is a bounded sequence in $\mathbb{C}$.
And so by Bolzano-Weierstrass, $\exists\, (\vec{x}_{n_k})$ such that $\langle \vec{x}_{n_k}, \vec{y}_0 \rangle \to \alpha \in \mathbb{C}$
And so $T\vec{x}_{n_k} = \langle \vec{x}_{n_k}, \vec{y}_0 \rangle \, \vec{z}_0 \to \alpha \vec{z}_0$.

Therefore T is compact.

Definition: Equicontinuous

Let $f \in L^2[a,b]$. To say that f is *equicontinuous* means:

$$\forall \epsilon > 0, \exists \delta > 0, \forall n \geq 1, \forall x, y \in [a,b], |x-y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Theorem: Arzelá-Ascoli

Let H be a Hilbert space and let (q_n) be a sequence in H:

(q_n) uniformly bounded and equicontinuous $\implies$ (q_n) has a convergent subsequence.

Theorem

Let $H = L^2[a,b]$ and let T be a linear operator on H defined by:

$$(Tf)(s) = \int_a^b K(s,t) f(t) dt$$

K continuous $\implies$ T compact.

Proof

Assume K is continuous on $Q = [a,b] \times [a,b]$.
Assume (f_n) is bounded.

Applying Hölder:

$$\begin{aligned} |(Tf_n)(s)| &= \left| \int_a^b K(s,t) f_n(t) dt \right| \\ &\leq \int_a^b |K(s,t) f_n(t)| \, dt \\ &= \int_a^b |K(s,t)| \, |f_n(t)| \, dt \\ &\leq \left(\int_a^b |K(s,t)|^2 \right)^{\frac{1}{2}} \left(\int_a^b |f_n(t)|^2 \right)^{\frac{1}{2}} dt \\ &= \left(\int_a^b |K(s,t)|^2 \right)^{\frac{1}{2}} \|f_n\| \end{aligned}$$

But K is continuous on compact Q and thus is bounded so $K(s,t) \leq C$.
Furthermore, (f_n) is bounded and so $\|f_n\| \leq M$. And so:

$$|(Tf_n)(s)| \leq CM \left(\int_a^b dt \right) = CM\sqrt{b-a}$$

For all $s \in [a,b]$ and all $n \in \mathbb{N}$.

Therefore (Tf_n) is uniformly bounded.

Assume $\epsilon > 0$.
Since Q is compact, Q is uniformly continuous on Q.
So $\exists\, \delta > 0, \forall\, x, y \in [a,b], |x-y| < \delta \implies |K(x,t) - K(y,t)| < \dfrac{\epsilon}{M\sqrt{b-a}}$.
And so, once again applying Hölder:

$$\begin{aligned}
|(Tf_n)(x) - (Tf_n)(y)| &= \left| \int_a^b K(x,t) f_n(t) dt - \int_a^b K(y,t) f_n(t) dt \right| \\
&= \left| \int_a^b [K(x,t) - K(y,t)] f_n(t) dt \right| \\
&\leq \int_a^b |[K(x,t) - K(y,t)] f_n(t) dt| \\
&= \int_a^b |K(x,t) - K(y,t)|\, |f_n(t)|\, dt \\
&\leq \left[\int_a^b |K(x,t) - K(y,t)|^2\, dt \right]^{\frac{1}{2}} \left(\int_a^b |f_n(t)|^2\, dt \right)^{\frac{1}{2}} \\
&= \left[\int_a^b |K(x,t) - K(y,t)|^2\, dt \right]^{\frac{1}{2}} \|f_n\| \\
&< M \left[\int_a^b \left(\frac{\epsilon}{M\sqrt{b=a}} \right) dt \right]^{\frac{1}{2}} \\
&= \frac{\epsilon}{\sqrt{b-a}} \sqrt{b-a} \\
&= \epsilon
\end{aligned}$$

Therefore T is equicontinuous.

Therefore, by Arzelá-Ascoli, (Tf_n) has a convergent subsequence and thus T is compact.

Note that T is compact iff it maps bounded sets to sets with compact closure:

$\forall\, B \subset H$ where B is bounded, $\overline{T[B]}$ is compact.

Notation

Let H be a Hilbert space. The set of all compact operators on H is denoted by $\mathcal{K}(H)$.

Theorem

Let H be a Hilbert space:

$$\mathcal{K}(H) \subset \mathcal{B}(H)$$

In other words, all compact operators on H are also bounded.

Proof

Assume T is compact.
ABC: T is not bounded.
$\forall\, M > 0, \exists\, \vec{x} \in H, \|\vec{x}\| = 1, \|T\vec{x}\| > M$
Let $(\vec{x}_n)$ in H such that $\|\vec{x}_n\| = 1$.
Let $M = n$.
$\|T\vec{x}_n\| > n \to \infty$
And so $(T\vec{x}_n)$ does not have a convergent subsequence.
CONTRADICTION!

Therefore T is bounded.

Theorem

Let H be a Hilbert space:

$\mathcal{K}(H)$ is an ideal in $\mathcal{B}(H)$.

Proof

Clearly, $\mathcal{K}(H)$ is a subspace of $\mathcal{B}(H)$.

Assume $T \in \mathcal{K}(H)$.
Assume $(\vec{x}_n)$ is a bounded sequence in H.
$(T\vec{x}_n)$ has a convergent subsequence $(T\vec{x}_{n_k}) \to \vec{y}$.
Assume $B \in \mathcal{B}(H)$.
$B(T\vec{x}_n) \to B(\vec{y})$ and so $BT \in \mathcal{K}(H)$.
Also, $(\vec{x}_n)$ bounded $\implies$ $(B\vec{x}_n)$ bounded.
Thus $TB(\vec{x}_n)$ has a convergent subsequence and thus $TB \in \mathcal{K}(H)$.

Therefore $\mathcal{K}(H)$ is an ideal of $\mathcal{B}(H)$.

Definition: Finite Rank

Let H be a Hilbert space and let T be an operator on H. To sat that T is a *finite rank* operator means $\dim \mathcal{R}(T) < \infty$.

Theorem

Finite rank bounded operators are compact.

Proof

Assume H is a Hilbert space and assume $T \in \mathcal{B}(H)$.
Let $\mathcal{R}(T) = S$ such that $\dim S = n$.
Assume $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is an orthonormal basis for S.
Define: $T_n\vec{x} = \langle T\vec{x}, \vec{e}_n\rangle\, \vec{e}_n$. But $T_n\vec{x} = \langle \vec{x}, T^*\vec{e}_n\rangle\, \vec{e}_n$, which is compact by above example. Furthermore, $\displaystyle\sum_{k=1}^{n} \langle T\vec{x}, \vec{e}_k\rangle\, \vec{e}_k = T\vec{x} \in S$.

Also $\mathcal{R}(T)$ is a vector space.

Therefore T is compact.

Theorem

Let H be a Hilbert space:

$\mathcal{K}(H)$ is closed in $\mathcal{B}(H)$.

Proof

Assume (T_n) is a sequence in $\mathcal{K}(H)$ such that $\|T_n - T\| \to 0$.
Assume $(\vec{x}_n)$ is a bounded sequence in H such that $\|\vec{x}_n\| \leq M$.
Since T_1 is compact, there exists a subsequence $(\vec{x}_{1_n})$ of $(\vec{x}_n)$ such that $(T_1\vec{x}_{1_n})$ converges.
Similarly, the sequence $(T_2\vec{x}_{1_n})$ contains a convergent subsequence $(T_2\vec{x}_{2_n})$.
In general, for $k \geq 2$, let $(\vec{x}_{k_n})$ be a subsequence of $(\vec{x}_{(k-1)_n})$ such that $(T_k\vec{x}_{k_n})$ converges.
Let $(\vec{x}_{n_n}) = (\vec{x}_{p_n})$, where (p_n) is increasing positive.
$\forall\, k \in \mathbb{N}, (T_k\vec{x}_{p_n})$ converges.
Assume $\epsilon > 0$.
$\exists\, N > 0, k > N \implies \|T_n - T\| < \dfrac{\epsilon}{3}$
Assume $n, m > N$.

$$\begin{aligned}
\|T\vec{x}_{p_n} - T\vec{x}_{p_m}\| &= \|T\vec{x}_{p_n} - T_k\vec{x}_{p_n} + T_k\vec{x}_{p_n} - T_k\vec{x}_{p_m} + T_k\vec{x}_{p_m} - T\vec{x}_{p_m}\| \\
&\leq \|T\vec{x}_{p_n} - T_k\vec{x}_{p_n}\| + \|T_k\vec{x}_{p_n} - T_k\vec{x}_{p_m}\| + \|T_k\vec{x}_{p_m} - T\vec{x}_{p_m}\| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

Thus $(T\vec{x}_{p_n})$ is Cauchy, and by completeness of H, converges.

Therefore T is compact and $T \in \mathcal{K}(H)$, and thus $\mathcal{K}(H)$ is closed.

Theorem

The adjoint of a compact operator is compact.

Proof

Assume H is a Hilbert space.
Assume $T \in \mathcal{K}(H)$, and thus $T \in \mathcal{B}(H)$.
Hence $T^* \in \mathcal{B}(H)$.
Assume $(\vec{x}_n)$ is a bounded sequence in H such that $\|\vec{x}_n\| \leq M$.
Let $\vec{y}_n = T^*\vec{x}_n$.
Since T^* is bounded, $\vec{y}_n$ is bounded.

Thus $T\vec{y}_n$ has a convergent subsequence $(T\vec{y}_{n_k})$.

$$\begin{aligned}
\|\vec{y}_m - \vec{y}_n\|^2 &= \langle \vec{y}_m - \vec{y}_n, \vec{y}_m - \vec{y}_n \rangle \\
&= \langle T^*\vec{x}_m - T^*\vec{x}_n, T^*\vec{x}_m - T^*\vec{x}_n \rangle \\
&= \langle T^*(\vec{x}_m - \vec{x}_n), T^*(\vec{x}_m - \vec{x}_n) \rangle \\
&= \langle TT^*(\vec{x}_m - \vec{x}_n), \vec{x}_m - \vec{x}_n \rangle \\
&\leq \|TT^*(\vec{x}_m - \vec{x}_n)\| \, \|\vec{x}_m - \vec{x}_n\| \\
&\leq \|T(T^*\vec{x}_m - T^*\vec{x}_n)\| \, (\|\vec{x}_m\| + \|\vec{x}_n\|) \\
&= \|T(\vec{y}_m - \vec{y}_n)\| \, (M + M) \\
&= 2M \, \|T\vec{y}_m - T\vec{y}_n\|
\end{aligned}$$

Now, apply this to subsequences:

$$\left\|\vec{y}_{n_j} - \vec{y}_{n_k}\right\| \leq 2M \left\|T\vec{y}_{n_j} - T\vec{y}_{n_k}\right\| \to 0$$

Therefore $(\vec{y}_{n_k}) = (T^*\vec{x}_{n_k})$ converges and thus $T^* \in \mathcal{K}(H)$.

Theorem

T is compact iff T maps weakly-convergent sequences to strongly- convergent sequences.

Proof

Assume H is a Hilbert space.
Assume T is an operator on H.

$\implies$ Assume T is compact.

Assume $(\vec{x}_n)$ is a sequence in H such that $\vec{x}_n \xrightarrow{w} \vec{x}$.
ABC: $T\vec{x}_n \not\to T\vec{x}$.
$\exists\, \epsilon > 0, \forall\, N > 0, \exists\, n > N, \|T\vec{x}_n - T\vec{x}\| \geq \epsilon$
Similarly, $\|T\vec{x}_{p_n} - T\vec{x}\| \geq \epsilon$.

Since $\vec{x}_n \xrightarrow{w} \vec{x}$, $(\vec{x}_n)$ is bounded.
And since T is compact, $(T\vec{x})$ has a convergent subsequence $(T\vec{x}_{p_n})$.

Assume $\vec{y} \in H$.
$\langle T\vec{x}_n, \vec{y} \rangle = \langle \vec{x}_n, T^*\vec{y} \rangle \to \langle \vec{x}, T^*\vec{y} \rangle = \langle T\vec{x}, \vec{y} \rangle$
Thus $T\vec{x}_n \xrightarrow{w} T\vec{x}$.
Likewise, $T\vec{x}_{p_n} \xrightarrow{w} T\vec{x}$.
But $\vec{x}_n \to \vec{x} \implies \vec{x}_n \xrightarrow{w} \vec{x}$, and so $T\vec{x}_{p_n} \to T\vec{x}$.
CONTRADICTION!

Therefore $T\vec{x}_n \to T\vec{x}$.

$\impliedby$ Assume $\vec{x}_n \xrightarrow{w} \vec{x} \implies T\vec{x}_n \to T\vec{x}$.

Corollary

Compact operators map orthonormal sequences into sequences that strongly converge to 0.

Proof

Assume H is a Hilbert space.
Assume $T \in \mathcal{K}(H)$.
Assume $(\vec{e}_n)$ is an orthonormal sequence in H.
$\vec{e} \xrightarrow{w} \vec{0}$
Therefore $T\vec{e} \to 0$.

Corollary

Let H be a Hilbert space and $T \in \mathcal{K}(H)$ such that T is invertible:

T^{-1} is unbounded.

Proof

Assume H is a Hilbert space.
Assume $T \in \mathcal{K}(H)$.
Assume $(\vec{e}_n)$ is an orthonormal sequence in H.
$T\vec{e}_n \to \vec{0}$
$T^{-1}(T\vec{e}_n) = \vec{e}_n \not\to 0$.

Therefore T^{-1} is discontinuous, and thus unbounded.