Banach Spaces

Definition: Banach

Let E be a normed space. To say that E is *complete* means that every Cauchy sequence in E converges to some element of E.

A complete normed space is called a *Banach* space.

Examples

1). $E = \mathcal{P}[a, b]$ with the sup (uniform convergence) norm is not Banach.

As a counterexample, consider $f_n = \displaystyle\sum_{k=1}^{n} \frac{t^k}{k!} \in \mathcal{P}[0, 1]$

AWLOG: $n < m$.

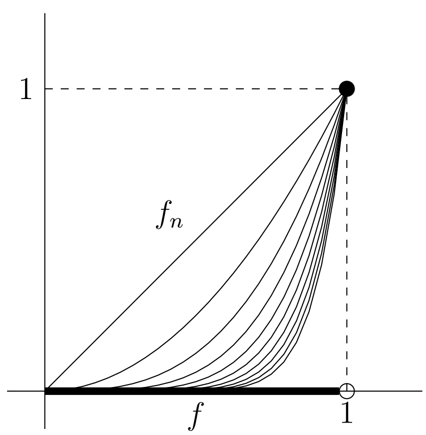
$$\|f_n - f_m\| = \left\| \sum_{k=1}^{m} \frac{t^k}{k!} - \sum_{k=1}^{n} \frac{t^k}{k!} \right\| = \left\| \sum_{k=n+1}^{m} \frac{t^k}{k!} \right\| = \sum_{k=n+1}^{m} \frac{1}{k!} \to 0$$

Thus, f_n is Cauchy; however, $f_n \to f = e^t \notin \mathcal{P}[0, 1]$.

Therefore, $\mathcal{P}[0, 1]$ is not Banach.

2). $E = \mathcal{C}[0, 1]$ with $\|f\| = \displaystyle\int_0^1 |f(t)|\, dt$ is not Banach.

As a counterexample, consider $f_n = t^n \in \mathcal{C}[0, 1]$.



Claim: f_n is Cauchy in the norm.

AWLOG: $n < m$

$$
\begin{aligned}
\|f_n - f_m\| &= \int_0^1 |f_n - f_m| \\
&= \int_0^1 (t^n - t^m)dt \\
&= \left[\frac{1}{n+1}t^{n+1} - \frac{1}{m+1}t^{m+1}\right]_0^1 \\
&= \frac{1}{n+1} - \frac{1}{m+1} \\
&\to 0
\end{aligned}
$$

Claim: $f_n \to f$ where $f = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$

$$
\begin{aligned}
\|f_n - f\| &= \|f_n - 0\| \\
&= \|f_n\| \\
&= \int_0^1 t^n dt \\
&= \frac{1}{n+1}t^{n+1}\Big|_0^1 \\
&= \frac{1}{n+1} \\
&\to 0
\end{aligned}
$$

Thus, f_n is Cauchy in the norm and $f_n \to f$ in the norm; however, f is discontinuous and thus $f \notin \mathcal{C}[0,1]$.

Therefore, $\mathcal{C}[0,1]$ is not complete, and thus not Banach.

3). $E = \mathcal{C}[a,b]$ with the sup (uniform convergence) norm is Banach.

Assume (f_n) in $\mathcal{C}[a,b]$ is Cauchy.
Thus, $\forall\, \epsilon > 0, \exists\, N > 0, n, m > N \implies \|f_n - f_m\| < \epsilon$.

$|f_n(x) - f_m(x)| \le \max_{x\in[a,b]} |f_n - f_m| = \|f_n - f_m\| < \epsilon$

Thus, $\forall\, x \in [a,b], (f_n(x))$ is Cauchy.
So by completeness of $\mathbb{R}$, $\forall\, x \in [a,b], f_n(x) \to f(x)$.
By letting $m \to \infty$, $\forall\, x \in [a,b], |f_n(x) - f(x)| < \epsilon$.
Thus, $f_n \rightrightarrows f$ and is f_n continuous, so f is also continuous and $f \in \mathcal{C}[a,b]$.

Therefore, $\mathcal{C}[a,b]$ is Banach.

4). ℓ^p with $\|x\|_p = \left(\sum_{k=1}^{\infty}(x_n)^p\right)^{\frac{1}{p}}$ is Banach for $1 \leq p < \infty$.

Assume (α_n) is a Cauchy sequence (of sequences) in ℓ^p, where $\alpha_n = (\alpha_{n,1}, \alpha_{n,2}, \ldots)$.
Assume $\epsilon > 0$.

$$\exists\, N > 0, n, m > N \implies \|\alpha_n - \alpha_m\| = \left(\sum_{k=1}^{\infty} |\alpha_{n,k} - \alpha_{m,k}|^p\right)^{\frac{1}{p}} < \epsilon$$

And so:

$$\sum_{k=1}^{\infty} |\alpha_{n,k} - \alpha_{m,k}|^p < \epsilon^p$$

Thus, for each fixed k:

$$|\alpha_{n,k} - \alpha_{m,k}|^p \leq \sum_{k=1}^{\infty} |\alpha_{n,k} - \alpha_{m,k}|^p < \epsilon^p$$

And so:

$$|\alpha_{n,k} - \alpha_{m,k}| < \epsilon$$

Thus, for each fixed k, the sequence $(\alpha_{n,k})$ is Cauchy in $\mathbb{C}$.
But $\mathbb{C}$ is complete, so $\alpha_{n,k} \to \alpha_k \in \mathbb{C}$.
Let $\alpha = (\alpha_n)$, i.e., α is the sequence of the limits.

By letting $m \to \infty$ and assuming $n > N$:

$$\sum_{k=1}^{\infty} |\alpha_{n,k} - \alpha_k|^p < \epsilon^p < \infty$$

Furthermore, since $\alpha_N \in \ell^p$:

$$\sum_{k=1}^{\infty} |\alpha_N, k|^p < \infty$$

Now, applying Minkowski:

$$\begin{aligned}
\left(\sum_{k=1}^{\infty} |\alpha_k|^p\right)^{\frac{1}{p}} &= \left(\sum_{k=1}^{\infty}[(|\alpha_k| - |\alpha_{N,k}|) + |\alpha_{N,k}|]^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{k=1}^{\infty}(|\alpha_k| - |\alpha_{N,k}|)^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |\alpha_{N,k}|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{k=1}^{\infty} |\alpha_k - \alpha_{N,k}|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |\alpha_{N,k}|^p\right)^{\frac{1}{p}} \\
&< \infty
\end{aligned}$$

Therefore, $\alpha \in \ell^p$.

Moreover:

$$\|\alpha_n - \alpha\| = \left(\sum_{k=1}^{\infty} |\alpha_{n,k} - \alpha_k|^p \right)^{\frac{1}{p}} < \epsilon$$

And so $\|\alpha_n - \alpha\| \to 0$.

Thus, $\alpha_n \to \alpha \in \ell^p$, so ℓ^p is complete and therefore Banach.

5). ℓ^∞ with $\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|$ is Banach.

Assume (α_n) is a Cauchy sequence (of sequences) in ℓ^∞, where $\alpha_n = (\alpha_{n,1}, \alpha_{n,2}, \ldots)$.
Assume $\epsilon > 0$.

$$\exists\, N > 0, n, m > N \implies \|\alpha_n - \alpha_m\| = \sup_{k \in \mathbb{N}} |\alpha_{n,k} - \alpha_{m,k}| < \epsilon$$

And so:

$$|\alpha_{n,k} - \alpha_{m,k}| \leq \sup_{k \in \mathbb{N}} |\alpha_{n,k} - \alpha_{m,k}| < \epsilon$$

Thus, for each fixed k, the sequence $(\alpha_{n,k})$ is Cauchy in $\mathbb{C}$.
But $\mathbb{C}$ is complete, so $\alpha_{n,k} \to \alpha_k \in \mathbb{C}$.
Let $\alpha = (\alpha_n)$, i.e., α is the sequence of the limits.

By letting $m \to \infty$ and assuming $n > N$:

$$\sup_{k \in \mathbb{N}} |\alpha_{n,k} - \alpha_k| < \epsilon < \infty$$

Furthermore, since $\alpha_N \in \ell^\infty$:

$$\sup_{k \in \mathbb{N}} |\alpha_N, k| < \infty$$

Now, to show that $\alpha \in \ell^\infty$:

$$\begin{aligned}
\sup_{k \in \mathbb{N}} |\alpha_k| &= \sup_{k \in \mathbb{N}} \{(|\alpha_k| - |\alpha_{N,k}|) + |\alpha_{N,k}|\} \\
&\leq \sup_{k \in \mathbb{N}} \{|\alpha_k| - |\alpha_{N,k}|\} + \sup_{k \in \mathbb{N}} |\alpha_{N,k}| \\
&< \infty
\end{aligned}$$

Therefore, $\alpha \in \ell^\infty$.

Moreover:

$$\|\alpha_n - \alpha\| = \sup_{k \in \mathbb{N}} |\alpha_{n,k} - \alpha_k| < \epsilon$$

And so $\|\alpha_n - \alpha\| \to 0$.

Thus, $\alpha_n \to \alpha \in \ell^\infty$, so ℓ^∞ is complete and therefore Banach.

Theorem

Let E be a Banach space and F a closed subspace of E. F is also Banach.

Proof

Assume $(\vec{x}_n)$ is Cauchy in F.
Thus $(\vec{x}_n)$ is Cauchy in E and $\vec{x}_n \to \vec{x} \in E$, since E is complete.
But F is closed and thus contains all of its limit points, and so $\vec{x} \in F$.

Therefore F is complete, and thus Banach.