Path Connectedness

Definition: Path

Let X be a topological space and let $x_0, x_1 \in X$. A *path* from x_0 to x_1 is a continuous function $f : [0, 1] \to X$ such that $f(0) = x_0$ an $f(1) = x_1$. x_0 is called the initial point of the path and x_1 is called the final point of the path.

Definition: Path Connected

Let X be a topological space. To say that X is *path connected* means that between every $x_0, x_1 \in X$ there exists some path.

Definition: Convex

Let $A \subset \mathbb{R}^n$. To say that A is *convex* means that for all $x, y \in A$:

$$\{(1 - t)x + ty \,|\, t \in [0, 1]\} \subset A$$

Thus, every convex subset of $\mathbb{R}^n$ is path connected.

Theorem

A path connected topological space is connected.

Proof. Assume that X is a path connected topological space and ABC that X is disconnected. This means that there exists $A, B \subset X$ such that $A \sqcup B = X$ where A, B are open and non-empty. So assume that $x \in A$ and $y \in B$. Since X is path connected, there exists some continuous $f : [0, 1] \to X$ such that $f(0) = x \in A$ and $f(1) = y \in B$. This mean that $[0, 1] = f^{-1}(A) \cup f^{-1}(B)$ where neither $f^{-1}(A)$ nor $f^{-1}(B)$ are empty. Furthermore, since A and B are disjoint, $f^{-1}(A)$ and $f^{-1}(B)$ must also be disjoint, contradicting the connectedness of $[0, 1]$. Therefore X is connected. ∎

Example

The closure of the topologist's sine curve is connected but not path connected.

The topologists sine curve is given by:

$$S = \left\{ \left(x, \sin\frac{1}{x} \right) \,\middle|\, x \in (0, 1) \right\}$$

and its closure is given by:

$$\bar{S} = S \cup \{(1, \sin(1)\} \cup \{(0, y) \,|\, y \in [-1, 1]\}$$

Note that $\bar{S}$ was already shown to be connected.

ABC that $\bar{S}$ is path connected and assume that $p \in S$. This means that there exists a path in $\bar{S}$ such that $f(0) = p$ and $f(1) = (0,0)$. Let $f(t) = (x(t), y(t))$. Note that since f is continuous, $x(t)$ and $y(t)$ are also continuous. Now, defined $U = \{t \in [0,1] \,|\, x(t) > 0\}$. Thus, for all $t \in U$, $f(t) \in S$ and $y(t) = \frac{1}{x(t)}$.

Next, since $U \subset [0,1]$, U is bounded and thus has a sup. So let $t_* = \sup U$. Note that t_* is the final value of t at which the path jumps to the y-axis part of $\bar{S}$ and stays there on the way to $(0,0)$. So $x(t_*) = 0$. Let $b = y(t_*)$ and let select $\epsilon > 0$ such that:

$$\epsilon < \begin{cases} 1-b, & b < 1 \\ \frac{1}{2}, & b = 1 \end{cases}$$

Now, since f is continuous, there exists $\delta > 0$ such that for all $t \in [0,1]$, if $|t - t_*| < \delta$ then $\|f(t) - f(t_*)\| < \epsilon$. Note that $[t_* - \delta, t_*]$ is connected and compact. Furthermore, f is continuous. Hence $f[t_* - \delta, t_*]$ is connected and compact, and thus must be an interval. So let $x([t_* - \delta, t_*) = [0, x_0]$ for some $x_0 \in (0,1]$. This means that for every $x \in (0, x_0]$ there exists some $t \in [t_* - \delta, t_*]$ such that $f(t) \in S$, meaning $f(t) = (x, \sin \frac{1}{x})$.

Define a sequence x_n in [0,1] by:

$$x_n = \frac{1}{2n\pi + \frac{\pi}{2}}$$

Note that $x_n \to 0$ and:

$$\sin \frac{1}{x_n} = \sin \left(2n\pi + \frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 1$$

But since $x_n \to 0$, there exists $N \in \mathbb{N}$ such that for all $x_n < x_0$ for all $n > N$. And so there exists $t_n \in [t_* - \delta, t_*)$ such that:

$$f(t_n) = \left(x_n, \sin \frac{1}{x_n}\right) = (x_n, 1)$$

Thus:

$$\|f(t_n) - f(t_*)\| = \|(x_n, 1) - (0, b)\| \geq 1 - b > \epsilon$$

This contradicts the continuity of f. Therefore $\bar{S}$ is not path connected.

Theorem

Let X and Y be topological spaces. If X and Y are path connected then $X \times Y$ is path connected.

Proof. Assume that X and Y are path connected and assume that $(x_1, y_1), (x_2, y_2) \in X \times Y$. This means that there must exist a path f from x_1 to x_2 and a path g from y_1 to y_2. Now, defined $h : [0,1] \to X \times Y$ as $h(t) = (f(t), g(t))$. But $\pi_X \circ h = f$ and $\pi_Y \circ h = g$ are by definition continuous, and thus h is continuous. Furthermore, $h(0) = (f(0), g(0)) = (x_1, y_1)$ and $h(1) = (f(1), g(1)) = (x_2, y_2)$, and so h is a path between (x_1, y_1) and (x_2, y_2). Therefore $X \times Y$ is path connected. ∎