Eigenvalues and Eigenvectors

Definition: Eigenvalue

Let E be a complex vector space and let A be an operator on E. To say that $\lambda \in \mathbb{C}$ is an *eigenvalue* of A means $\exists\, \vec{x} \in E$ such that $\vec{x} \neq \vec{0}$ and:

$$A\vec{x} = \lambda\vec{x}$$

All such $\vec{x}$ are called the *eigenvectors* of A corresponding to λ.

When E is a function space the eigenvectors are referred to as *eigenfunctions*.

The *eigenspace* corresponding to λ, denoted E_λ, is given by:

$$E_\lambda = \{\vec{x} \in H - \{\vec{0}\} \mid A\vec{x} = \lambda\vec{x}\}$$

Theorem

Let E be a complex vector space and let A be an operator on E with an eigenvalue λ:

> E_λ is a vector space.

Proof

Assume $\vec{x} \in E_\lambda$.
$A\vec{x} = \lambda\vec{x}$
$A\vec{x} - \lambda\vec{x} = \vec{0}$
$(A - \lambda I)\vec{x} = \vec{0}$

Therefore $E_\lambda = \ker(A - \lambda I)$, which is a subspace of H.

Thus, λ is an eigenvalue of A iff $\ker(A - \lambda I)$ is nontrivial.

Definition: Multiplicity

Let E be a complex vector space and let A be an operator on E with an eigenvalue λ. The *multiplicity* of λ is the dimension of the corresponding eigenspace E_λ.

An eigenvalue with a multiplicity of 1 is called *simple*.

Example

Let $E = L^2[0, 2\pi]$ and let A be an operator on E defined by:

$$Au = \cos \star u$$

and so:

$$(Au)(t) \int_0^{2\pi} \cos(t - x)u(x)dx$$

First, assume $\lambda \neq 0$:

$Au = \lambda u$

$$\int_0^{2\pi} \cos(t-x)u(x)dx = \lambda u(t)$$

$$\int_0^{2\pi} [cos(t)cos(x) + sin(t)sin(x)]u(x)dx = \lambda u(t)$$

$$\left[\int_0^{2\pi} cos(x)u(x)dx\right] cos(t) + \left[\int_0^{2\pi} sin(x)u(x)dx\right] sin(t) = \lambda u(t)$$

And so $u(t) = \alpha\cos(t) + \beta\sin(t) \in \mathrm{Span}\{\cos, \sin\}$ and $\dim E_\lambda = 2$.

Let:

$$a = \int_0^{2\pi} cos(x)u(x)dx$$

$$b = \int_0^{2\pi} sin(x)u(x)dx$$

Now solve for a and b:

$$\begin{aligned}
a &= \int_0^{2\pi} cos(x)[\alpha\cos(x) + \beta\sin(x)]dx \\
&= \alpha\int_0^{2\pi} cos^2(x)dx + \beta\int_0^{2\pi} \cos(x)\sin(x)dx \\
&= \frac{\alpha}{2}\int_0^{2\pi} [1 + cos(2x)]dx + \frac{\beta}{2}\int_0^{2\pi} \sin(2x)dx \\
&= \frac{\alpha}{2}\int_0^{2\pi} [1 + cos(2x)]dx + 0 \\
&= \frac{\alpha}{2}\left[\int_0^{2\pi} dx + \int_0^{2\pi} cos(2x)dx\right] \\
&= \frac{\alpha}{2}(2\pi + 0) \\
&= \alpha\pi
\end{aligned}$$

$$\begin{aligned}
b &= \int_0^{2\pi} sin(x)[\alpha\cos(x) + \beta\sin(x)]dx \\
&= \alpha\int_0^{2\pi} sin(x)cos(x)dx + \beta\int_0^{2\pi} \sin^2(x)dx \\
&= \frac{\alpha}{2}\int_0^{2\pi} \sin(2x)dx + \frac{\beta}{2}\int_0^{2\pi} [1 - \cos(2x)]dx \\
&= 0 + \frac{\beta}{2}\int_0^{2\pi} [1 - cos(2x)]dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\beta}{2}\left[\int_0^{2\pi} dx - \int_0^{2\pi} cos(2x)dx\right] \\
&= \frac{\beta}{2}(2\pi + 0) \\
&= \beta\pi
\end{aligned}
$$

And so:

$$\alpha\pi\cos(t) + \beta\pi\sin(t) = \lambda(\alpha\cos(t) + \beta\sin(t)$$

and: $\alpha\pi = \lambda\alpha$ and $\beta\pi = \lambda\beta$ And therefore $\lambda = \pi$.

Now assume $\lambda = 0$.

$a\cos(t) + b\sin(t) = 0 \iff a = b = 0$ and so:

$$E_0 = \{u \in E \mid u \perp \cos \text{ and } u \perp \sin\} = \{\cos, \sin\}^{\perp}$$

and $\dim E_0 = \infty$.

And so $L^2[0, 2\pi] = E_\pi \oplus E_0$.