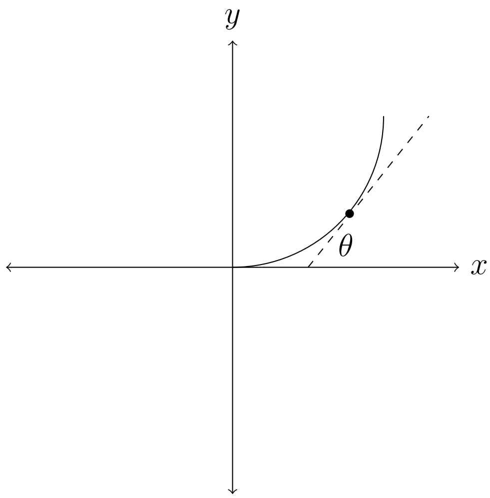
Conformal Mapping



$$z(t) = x(t) + iy(t)$$
$$z'(t) = x'(t) + iy'(t)$$
$$\tan\theta = \frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$
$$\theta = \arg z'(t)$$

Definition

To say that a curve C represented by $z(t) = x(t) + iy(t)$ is *continuously differentiable* in an interval $[a, b]$ means that $z'(t)$ exists and is continuous $\forall\, t \in [a, b]$.

To say that C is a *regular curve* means that:

1). C is continuously differentiable

2). $z'(t) \neq 0$, except at perhaps the endpoints

Theorem

Let $f(z) = u + iv$ be analytic in a domain D. The Jacobian of $f(z)$ is given by:

$$J = |f'(z)|^2$$

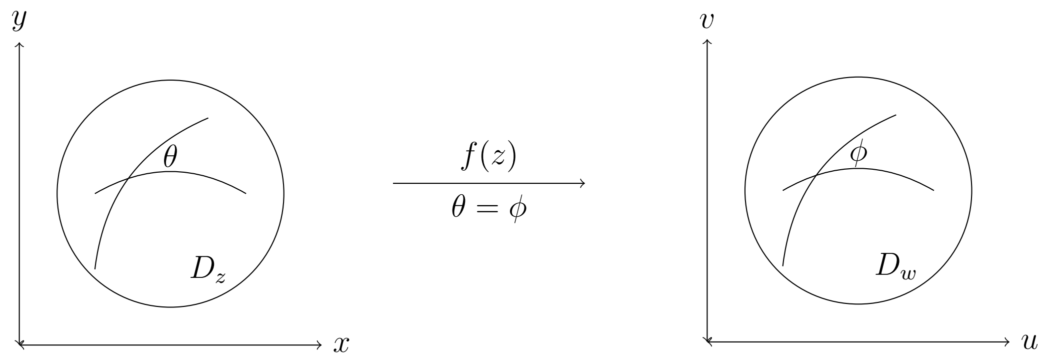
Proof

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - v_x u_y = u_x u_x = v_x(-v_x) = u_x^2 + v_x^2$$

$$f'(z) = f_x = u_x + iv_x$$
$$|f'(z)| = \sqrt{u_x^2 + v_x^2}$$
$$|f'(z)|^2 = u_x^2 + v_x^2$$
$$J = |f'(z)|^2$$

Definition

To say that the mapping $u = u(x, y)$ and $v = v(x, y)$ defined on a domain D is *conformal* in D means that the angle between any two intersecting regular curves in z is preserved in w under the map.



Theorem

Let $f(z) = u(x, y) + iv(x, y)$ be a continuously differentiable mapping on a domain D such that $J \neq 0$ in D:

$f(z)$ is analytic in D $\iff$ the mapping is conformal on D

Proof

$\Longrightarrow$ Assume $f(z)$ is analytic in D

Let $C_1 : p_1(t)$ and $C_2 : p_2(t)$ be two regular curves in D on $[a, b]$ intersecting at $z_0 \in D$
$z_0 = p_1(t_0) = p_2(t_0)$ for some $t_0 \in [a, b]$
Let θ be the angle between C_1 and C_2 at z_0
$\theta = \arg \frac{{p_1}'(t_0)}{{p_2}'(t_0)} = \arg {p_2}'(t_0) - \arg {p_1}'(t_0)$

Let $\Gamma_1 : P_1(t) = f(p_1(t))$ and $\Gamma_2 : P_2(t) = f(p_2(t))$ be the corresponding image curves
Γ_1 and Γ_2 intersect at $w_0 = f(z_0)$
Let ϕ be the angle between Γ_1 and Γ_2 at w_0

$$\begin{aligned}
\phi &= \arg \frac{{P_1}'(t_0)}{{P_2}'(t_0)} \\
&= \arg {P_1}'(t_0) - \arg {P_2}'(t_0) \\
&= \arg[f'(p_1(t_0)){p_1}'(t_0)] - \arg[f'(p_2(t_0)){p_2}'(t_0)] \\
&= \arg[f'(z_0){p_1}'(t_0)] - \arg[f'(z_0){p_2}'(t_0)] \\
&= \arg f'(z_0) + \arg {p_1}'(t_0) - \arg f'(z_0) - \arg {p_2}'(t_0) \\
&= \arg {p_1}'(t_0) - \arg {p_2}'(t_0) \\
&= \theta
\end{aligned}$$

$\Longleftarrow$ Assume that the mapping is conformal

Example

Let D be $|z-1|<1$ and $f(z)=z^3$

$f(z)$ is entire, so it is analytic in D

$f'(z)=3z^2$
$f'(z)=0$ at $z=0\notin D$
$f'(z)\neq 0$ in D

$\therefore f(z)$ is conformal in D

Example

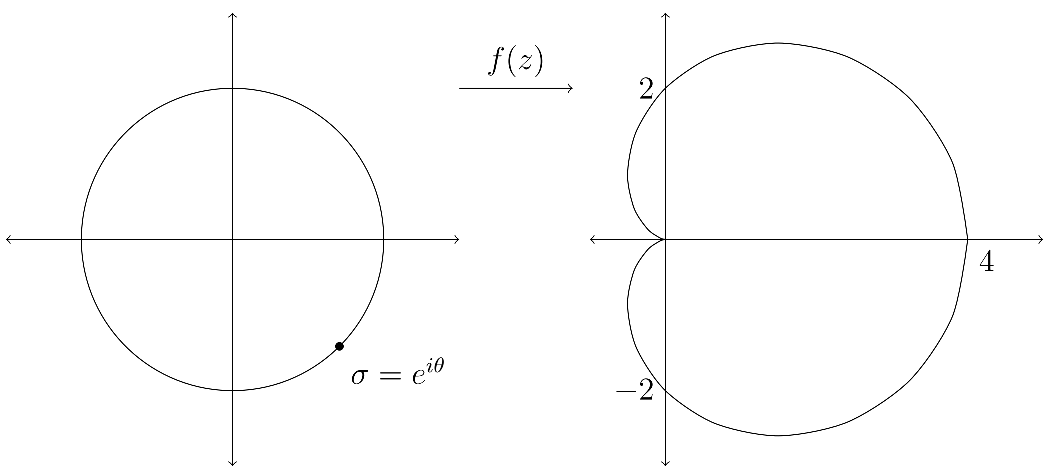
Let D be $|\zeta|<1$ and $z=w(\zeta)=(1+\zeta)^2$

$w(\zeta)$ is entire, so it is analytic in D

$w'(\zeta)=2(1+\zeta)$
$w'(\zeta)=0$ at $z=-1\notin D$
$w'(\zeta)\neq 0$ in D

$\therefore w(\zeta)$ is conformal in D

But what does the image look like?



$$\begin{aligned}
z &= (1+\sigma)^2 \\
&= (1+e^{i\theta})^2 \\
&= (1+\cos\theta+i\sin\theta)^2 \\
&= \left[2\cos^2\left(\frac{\theta}{2}\right)+i2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right]^2 \\
&= 4\cos^2\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{\theta}{2}\right)+i\sin\left(\frac{\theta}{2}\right)\right] \\
&= 4\cos^2\left(\frac{\theta}{2}\right)e^{i\theta}
\end{aligned}$$

Now, let $z = re^{i\phi}$ and let $\theta = \phi$:

$$\begin{aligned} re^{i\phi} &= 4\cos^2\left(\frac{\theta}{2}\right)e^{i\theta} \\ r &= 4\cos^2\left(\frac{\theta}{2}\right) \\ r &= 2(1+\cos\phi) \end{aligned}$$

Which is a cardiod.