Binary Algebraic Structures

Definition

A *binary algebraic structure* is a non-empty set S equipped with a binary operator '$*$' and is denoted $\langle S, * \rangle$.

When the binary operation is understood, the structure is simply referred to as S and operations are written using the more convenient ab form (juxtaposition) instead of $a * b$.

Definition

Let $\langle S, * \rangle$ and $\langle T, *' \rangle$ be two binary algebraic structures and $\phi : S \to T$. To say that S is *homomorphic* to T means there exists $\phi : S \to T$ such that:

$$\forall\, a, b \in S, \phi(a * b) = \phi(a) *' \phi(b)$$

If such a ϕ exists then it is referred to as a *homomorphism*.

When the binary operations are understood, the statement of homomorphism is written using the shorted $\phi(ab) = \phi(a)\phi(b)$ form. Note that the ab operation takes place is structure S with its equipped binary operator, and the $\phi(a)\phi(b)$ operation takes place in structure T with its equipped binary operator.

Definition

Let $\langle S, * \rangle$ and $\langle T, *' \rangle$ be two binary algebraic structures. To say that S is *isomorphic* to T, denoted $S \simeq T$, means there exists $\phi : S \to T$ such that:

1). ϕ is a bijection

2). ϕ is a homomorphism

If such a ϕ exists then it is referred to as an *isomorphism*.

Example

It was previously shown that $\langle U, \cdot \rangle \simeq \langle \mathbb{R}_{2\pi}, +_{2\pi} \rangle$. In particular:

$$\phi\left(e^{i\theta}\right) = \theta$$

$$\begin{aligned}
\phi(u_1 u_2) &= \phi\left(e^{i\theta_1} e^{1\theta_2}\right) \\
&= \phi\left(e^{i(\theta_1 +_{2\pi} \theta_2)}\right) \\
&= \theta_1 +_{2\pi} \theta_2 \\
&= \phi\left(e^{i\theta_1}\right) +_{2\pi} \phi\left(e^{i\theta_2}\right) \\
&= \phi(u_1) +_{2\pi} \phi(u_2)
\end{aligned}$$

Similarly, $\langle U_n, \cdot\rangle \simeq \langle \mathbb{Z}_n, +_n\rangle$:

$$\phi\left(e^{i\left[\frac{2\pi k}{n}\right]}\right) = k$$

$$\begin{aligned}
\phi(\zeta^h \zeta^k) &= \phi\left(e^{i\left[\frac{2\pi h}{n}\right]} e^{i\left[\frac{2\pi k}{n}\right]}\right) \\
&= \phi\left(e^{i\left[\frac{2\pi (h +_n k}{n}\right]}\right) \\
&= h +_n k \\
&= \phi\left(e^{i\left[\frac{2\pi h}{n}\right]}\right) +_n \phi\left(e^{i\left[\frac{2\pi k}{n}\right]}\right) \\
&= \phi(\zeta^h) +_n \phi(\zeta^k)
\end{aligned}$$

Let $n = 4$:

U_4	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	i
-1	-1	-i	1	i
-i	-i	1	i	-1

$\mathbb{Z}_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

ζ	$\phi(\zeta)$
1	0
i	1
-1	2
-i	3

$$\phi((-1)(-i)) = \phi(i) = 1$$

$$\phi((-1)(-i)) = \phi(-1) +_4 \phi(-i)) = 2 +_4 3 = 1$$

<u>Example</u>

Prove: $\langle \mathbb{R}, +\rangle \simeq \langle \mathbb{R}^+, \cdot\rangle$

Let $\phi : \mathbb{R} \to \mathbb{R}^+$ be defined by $\phi(x) = e^x$

<u>one-to-one</u>
Assume $\phi(x) = \phi(y)$
$e^x = e^y$
$x = y$
$\therefore \phi$ is one-to-one

<u>onto</u>
Assume $y \in \mathbb{R}^+$
Let $x = \ln y \in \mathbb{R}$
$e^x = y$
$\therefore \phi$ is onto

<u>homo</u>

Assume $x, y \in \mathbb{R}^+$
$\phi(x+y) = e^{x+y} = e^x e^y = \phi(x)\phi(y)$
$\therefore \phi$ is a homomorphism

Thus, ϕ is an isomorphism and therefore $\langle \mathbb{R}, + \rangle \simeq \langle \mathbb{R}^+, \cdot \rangle$

<u>Theorem</u>

Let S and T be binary algebraic structures.

$$\phi : S \to T \text{ is an isomorphism } \iff \phi^{-1} : T \to S \text{ is an isomorphism}$$

<u>Proof</u>

$\implies$: Assume $\phi : S \to T$ is an isomorphism

ϕ is a bijection
ϕ^{-1} also exists and is a bijection
Assume $t_1, t_2 \in T$
ϕ is onto
$\exists\, s_1, s_2 \in S, \phi(s_1) = t_1$ and $\phi(s_2) = t_2$
ϕ is a homomorphism
$\phi^{-1}(t_1 t_2) = \phi^{-1}(\phi(s_1)\phi(s_2)) = \phi^{-1}(\phi(s_1 s_2)) = (\phi^{-1}\phi)(s_1 s_2) = s_1 s_2 = \phi^{-1}(t_1)\phi^{-1}(t_2)$
ϕ^{-1} is a homomorphism

$\therefore \phi^{-1} : T \to S$ is an isomorphism

$\impliedby$: Assume $\phi^{-1} : T \to S$ is an isomorphism

ϕ^{-1} is a bijection
ϕ also exists and is a bijection
Assume $s_1, s_2 \in S$
ϕ^{-1} is onto
$\exists\, t_1, t_2 \in T, \phi^{-1}(t_1) = s_1$ and $\phi^{-1}(t_2) = s_2$
ϕ^{-1} is a homomorphism
$\phi(s_1 s_2) = \phi(\phi^{-1}(t_1)\phi^{-1}(t_2)) = \phi(\phi^{-1}(t_1 t_2)) = (\phi\phi^{-1})(t_1 t_2) = t_1 t_2 = \phi(s_1)\phi(s_2)$
ϕ is a homomorphism

$\therefore \phi : S \to T$ is an isomorphism

<u>Theorem</u>

Let S, T, U be binary algebraic structures such that $\phi : S \to T$ is an isomorphism and $\gamma : T \to U$ is an isomorphism.

$$\gamma\phi : S \to U \text{ is an isomorphism}$$

Proof

ϕ and γ are bijections and homomorphisms
$\gamma\phi$ is a bijection
Assume $s_1, s_2 \in S$
$(\gamma\phi)(s_1 s_2) = \gamma(\phi(s_1 s_2)) = \gamma(\phi(s_1)\phi(s_2)) = \gamma(\phi(s_1))\gamma(\phi(s_2)) = (\gamma\phi)(s_1)(\gamma\phi)(s_2)$
$\gamma\phi$ is a homomorphism
$\therefore \gamma\phi$ is an isomorphism

Theorem

Let $\mathscr{S}$ be the set of all binary algebraic structures.
Isomorphism is an equivalence relation on $\mathscr{S}$.

Proof

- Reflexive

  Assume $S \in \mathscr{S}$
  The identity function i_S is clearly bijective and homomorphic
  $i_S : S \to S$ is an isomorphism
  $\therefore S \simeq S$

- Symmetric

  Assume $S \simeq T$
  There exists isomorphism $\phi : S \to T$
  So there exists isomorphism $\phi^{-1} : T \to S$
  $\therefore T \simeq S$

- Transitive

  Assume $S \simeq T$ and $T \simeq U$
  There exists isomorphisms $\phi \to T$ and $\gamma : T \to U$
  So $\gamma\phi : S \to U$ is an isomorphism
  $\therefore S \simeq U$