Cauchy-Schwarz

Theorem

$\forall f, g \in L^2, |\langle f, g\rangle| \leq \|f\|\|g\|$

Proof

Assume $f, g \in L^2$

case 1: $\|f\| = 0$ or $\|g\| = 0$

$f = 0 \ \ a.e.$ or $g = 0 \ \ a.e.$
$f\bar{g} = 0 \ \ a.e.$
$|\langle f, g\rangle| = \left|\int f\bar{g}\right| = 0$
$\|f\|\|g\| = 0$
$\therefore |\langle f, g\rangle| = \|f\|\|g\|$

case 2: $\|f\| = \|g\| = 1$

$|\langle f, g\rangle| = \left|\int f\bar{g}\right| \leq \int |f\bar{g}| \leq \frac{1}{2}\left(\int |f|^2 + \int |g|^2\right) = \frac{1}{2}\left(\|f\|^2 + \|g\|^2\right) = \frac{1}{2}(1+1) = 1$
$\|f\|\|g\| = 1 \cdot 1 = 1$
$\therefore |\langle f, g\rangle| = \|f\|\|g\|$

case 3: otherwise

Let $\hat{f} = \frac{f}{\|f\|}$ and $\hat{g} = \frac{g}{\|g\|}$
$\|\hat{f}\| = \|\hat{g}\| = 1$
$\left|\langle \hat{f}, \hat{g}\rangle\right| = \left|\langle \frac{f}{\|f\|}, \frac{g}{\|g\|}\rangle\right| = \frac{1}{\|f\|\|g\|}\,|\langle f, g\rangle| \leq 1$
$\therefore |\langle f, g\rangle| \leq \|f\|\|g\|$

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