Transpose

Definition: Transpose

Let A be an $m \times n$ matrix. The *transpose* of A, denoted A^T, is the $n \times m$ matrix with entries:

$$(A^T)_{ij} = A_{ji}$$

Thus, the i^{th} row becomes the i^{th} column and the j^{th} column becomes the j^{th} row.

Theorem

1). $(A^T)^T = A^T$

2). $(cA)^T = cA^T$

3). $(A + B)^T = A^T + B^T$

4). $(cA + dB)^T = cA^T + dB^T$

Proof

1). $((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}$

$\therefore (A^T)^T = A^T$

2). $((cA)^T)_{ij} = (cA)_{ji} = cA_{ji} = c(A^T)_{ij}$

$\therefore (cA)^T = cA^T$

3). $((A + B)^T)_{ij} = (A + B)_{ji} = A_{ji} + B_{ji} = (A^T)_{ij} + (B^T)_{ij} = (A^T + B^T)_{ij}$

$\therefore (A + B)^T = A^T + B^T$

4). $(cA + dB)^T = (cA)^T + (dB)^T = cA^T + dB^T$

Definition: Symmetric

To say that a matrix A is symmetry means $A = A^T$.

Thus, all symmetry matrices must be square matrices.

Theorem

Let S be the set of all symmetric $n \times n$ matrices over a field $\mathbb{F}$:

S is a subspace of $M_n(\mathbb{F})$

Proof

Clearly, $0^T = 0$, so the zero matrix is in S.

Assume $A, B \in S$
$(A + B)^T = A^T + B^T = A + B$
Thus, $A + B \in S$

Therefore S is closed under matrix addition.

Assume $c \in \mathbb{F}$
$(cA)^T = cA^T = cA$
Thus, $cA \in S$

Therefore S is closed under scalar multiplication.

Therefore, by the subspace test, S is a subspace of $M_n(\mathbb{F})$.