Integers

Definition

The set of *Integers*, denoted $\mathbb{Z}$, are the positive and negative whole numbers and 0:

$$\mathbb{Z} = \{n|n \in \mathbb{N}\} \cup \{-n|n \in \mathbb{N}\} \cup \{0\} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Closure Property

$\forall n, m \in \mathbb{Z}$:

1). $n + m \in \mathbb{Z}$

2). $nm \in \mathbb{Z}$

Definition

To say that an integer n divides an integer m, denoted $n|m$, means $\exists k \in \mathbb{Z}, m = kn$.

Definition

To say that an integer n is *even* mean that $2|n$. Otherwise n is said to be *odd*. Thus, $\forall n \in \mathbb{Z}$:

1). n even $\iff \exists k \in \mathbb{Z}, n = 2k$

2). n odd $\iff \exists k \in \mathbb{Z}, n = 2k + 1$

Theorem

$\forall n, m \in \mathbb{Z}$:

1). n even and m even $\implies n + m$ even.

2). n odd and m odd $\implies n + m$ even

3). n even and m odd $\implies n + m$ odd

Proof

Assume $n, m \in \mathbb{Z}$.

1). Assume n even and m even
$\exists k \in \mathbb{Z}, n = 2k$
$\exists j \in \mathbb{Z}, m = 2j$
$n + m = 2k + 2j = 2(k + j)$
But by closure, $k + j \in \mathbb{Z}$.
$\therefore n + m$ is even.

2). Assume n odd and m odd
$\exists k \in \mathbb{Z}, n = 2k + 1$
$\exists j \in \mathbb{Z}, m = 2j + 1$
$n + m = (2k + 1) + (2j + 1) = 2k + 2j + 2 = 2(k + j + 1)$

But by closure, $k + j + 1 \in \mathbb{Z}$.
$\therefore n + m$ is even.

3). Assume n even and m odd
$\exists k \in \mathbb{Z}, n = 2k$
$\exists j \in \mathbb{Z}, m = 2j + 1$
$n + m = 2k + 2j + 1 = 2(k + j) + 1$
But by closure, $k + j \in \mathbb{Z}$.
$\therefore n + m$ is odd.

Theorem

$\forall n, m \in \mathbb{Z}$:

1). n even and m even $\implies nm$ even.

2). n odd and m odd $\implies nm$ odd

3). n even and m odd $\implies nm$ even

Proof

Assume $n, m \in \mathbb{Z}$.

1). Assume n even and m even
$\exists k \in \mathbb{Z}, n = 2k$
$\exists j \in \mathbb{Z}, m = 2j$
$nm = (2k)(2j) = 2(2kj)$
But by closure, $2kj \in \mathbb{Z}$.
$\therefore nm$ is even.

2). Assume n odd and m odd
$\exists k \in \mathbb{Z}, n = 2k + 1$
$\exists j \in \mathbb{Z}, m = 2j + 1$
$nm = (2k + 1)(2j + 1) = 4kj + 2k + 2j + 1 = 2(2kj + k + j) + 1$
But by closure, $2kj + k + j \in \mathbb{Z}$.
$\therefore nm$ is odd.

3). Assume n even and m odd
$\exists k \in \mathbb{Z}, n = 2k$
$\exists j \in \mathbb{Z}, m = 2j + 1$
$nm = (2k)(2j + 1) = 2(2kj + k)$
But by closure, $2kj + k \in \mathbb{Z}$.
$\therefore nm$ is even.

Theorem

$\forall n \in \mathbb{Z}$:

1). n even $\iff n^2$ even

2). n odd $\iff$ n^2 odd

Proof

1). Assume $n \in \mathbb{Z}$.

$\implies$ Assume n is even.
$\exists k \in \mathbb{Z}, n = 2k$
$n^2 = (2k)^2 = 4k^2 = 2(2kk)$
But by closure, $2kk \in \mathbb{Z}$.
$\therefore n^2$ is even.

$\impliedby$ Assume n^2 is even.
Contrapositive of (2).

2). Assume $n \in \mathbb{Z}$.

$\implies$ Assume n is odd.
$\exists k \in \mathbb{Z}, n = 2k + 1$
$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2kk + 2k) + 1$
But by closure, $2kk + 2k \in \mathbb{Z}$.
$\therefore n^2$ is odd.

$\impliedby$ Assume n^2 is odd.
Contrapositive of (1).