Linear Fractional Transformations

Definition

A *linear fractional transformation*, also known as a *mobius* or *bilinear* transformation, if a function of the form:

$$w = f(z) = \frac{az+b}{cz+d}$$

where:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$$

Theorem

Let $f(z) = \frac{az+b}{cz+d}$ be a LFT on a domain D:

$$ad - bc = 0 \implies f(z) \text{ is constant in } D$$

Proof

Assume $ad - bc = 0$
Assume $z_1, z_2 \in D$

$$\begin{aligned}
f(z_1) - f(z_2) &= \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d} \\
&= \frac{(az_1+b)(cz_2+d) - (az_2+b)(cz_1+d)}{(cz_1+d)(cz_2+d)} \\
&= \frac{acz_1z_2 + adz_1 + bcz_2 + bd - acz_1z_2 - adz_2 - bcz_1 - bd}{(cz_1+d)(cz_2+d)} \\
&= \frac{(ad-bc)z_1 - (ad-bc)z_2}{(cz_1+d)(cz_2+d)} \\
&= 0
\end{aligned}$$

So, $\forall\, z_1, z_2 \in D, f(z_1) = f(z_2)$

$\therefore f(z)$ is constant on D.

Theorem

Let $\mathcal{S}$ be the binary algebraic structure of all LFT's under composition and let $\phi : \mathcal{S} \to M_2(\mathbb{C})$ be defined by $\phi(s) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. ϕ is an isomorphism.

Proof

Assume $s_1, s_2 \in \mathcal{S}$

$$\begin{aligned}
(s_1 \circ s_2)(z) &= \frac{a_1\left(\frac{a_2z+b}{c_2z+d}\right)+b_1}{c_1\left(\frac{a_2z+b}{c_2z+d}\right)+d_1} \\
&= \frac{a_1(a_2z+b_2)+b_1(c_2z+d_2)}{c_1(a_2z+b_2)+d_1(c_2z+d_2)} \\
&= \frac{(a_1a_2+b_1c_2)z+(a_1b_2+b_1d_2)}{(c_1a_2+d_1c_2)z+(c_1b_2+d_1d_2)} \\
&\in \mathcal{S}
\end{aligned}$$

$\therefore \phi$ is well-defined and closed.

Clearly, ϕ is bijective.

$$\phi(s_1 \circ s_2) = \begin{bmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2 \\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \phi(s_1)\phi(s_2)$$

$\therefore \phi$ is a homomorphism.

$\therefore \phi$ is an isomorphism.