Heredity

Definition: Heredity

Let P be a topological property. To say that a topological space X is hereditarily P means that each subspace Y of X has property P when Y is given the relative topology from X.

Theorem

Every T_2 space is hereditarily T_2.

Proof. Assume that X is a T_2 topological space and assume that $Y \subset X$. Now assume that $a, b \in Y$. Thus $a, b \in X$ and, since X is T_2, there exists $U, V \in \mathscr{T}_X$ such that $a \in U$, $b \in V$, and $U \cap V = \emptyset$. Furthermore, $a \in U \cap Y \in \mathscr{T}_Y$ and $b \in V \cap Y \in \mathscr{T}_Y$. And so:

$$(Y \cap U) \cap (Y \cap V) = Y \cap (U \cap V) = Y \cap \emptyset = \emptyset$$

Therefore Y is also T_2. ∎

Theorem

Every regular space is hereditarily regular.

Proof. Assume that X is a regular topological space and assume that $Y \subset X$. Assume that $p \in Y$. This means that there exists some $U_Y \in \mathscr{T}_Y$ such that $p \in U_Y$, and hence there exists $U_X \in \mathscr{T}_X$ such that $U_Y = U_X \cap Y$ and so $p \in U_X$. Now, since X is regular, there exists $V_X \in \mathscr{T}_X$ such that $p \in V_X \subset \overline{V_X} \subset U_X$, and hence $p \in V_X \cap Y = V_Y \in \mathscr{T}_Y$. Furthermore, since $\overline{V_X}$ is closed in X, $\overline{V_X} \cap Y = W_Y$ is closed in Y. Finally, since $\overline{V_Y}$ is the smallest closed set in Y containing V_Y:

$$p \in V_Y \subset \overline{V_Y} \subset W_Y \subset U_Y$$

Therefore Y is regular. ∎

Lemma

Let X be a normal topological space and let $Y \subset X$ such that Y is closed in X. For all $A \subset Y$, if A is closed in Y then A is closed in X.

Proof. Assume $A \subset Y$ such that A is closed in Y. This means that $Y - A \in \mathscr{T}_Y$, and so there exists $W \in \mathscr{T}_X$ such that $W \cap Y = Y - A$. Furthermore, $X - W$ is closed in X. Now:

$$(X - W) \cap Y = (X \cap Y) - (W \cap Y) = Y - (Y - A) = A$$

But $X - W$ and Y are closed in X and therefore A is also closed in X. ∎

Theorem

Let X be a normal topological space and let $Y \subset X$ such that Y is closed in X. Y is normal when given the relative topology.

Proof. Assume $A, B \subset Y$ such that A and B are closed in Y and $A \cap B = \emptyset$. This means that A and B are also closed in X. Since X is normal, there exists $U, V \in \mathscr{T}_X$ such that $A \in U, B \in V$, and $U \cap V = \emptyset$. Finally, since $A \subset (U \cap Y) \in \mathscr{T}_Y$ and $B \subset (V \cap Y) \in \mathscr{T}_Y$:

$$(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \emptyset \cap Y = \emptyset$$

Therefore Y is normal. ∎