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Math 221a
Homework #4

1.4.2

a) Let H be the cyclic subgroup (of order 2) of S_3 generated by (12).
Prove: No left coset of H (except H itself) is also a right coset.

$S_3 = \{(), (12), (13), (23), (123), (132)\}$
$H = \{(), (12)\}$

$()H = \{(), (12)\} = H$	$H() = \{(), (12)\} = H$
$(13)H = \{(13), (123)\}$	$H(13) = \{(13), (132)\}$
$(23)H = \{(23), (132)\}$	$H(23) = \{(23), (123)\}$

Thus, no left coset matches a right coset (other than H).

Prove: $\exists\, a \in S_3, aH \cap Ha = \{a\}$

Let $a = (13)$
$(13)H \cap H(13) = (13)$

b) Prove: If K is the cyclic group (of order 3) generated by (123) then every left coset of K is also a right coset of K

$H = \{(), (123), (132)\}$

$()H = \{(), (123), (132)\} = H$	$H() = \{(), (123), (132)\} = H$
$(12)H = \{(12), (13), (23)\}$	$H(12)H = \{(12), (13), (23)\}$

$()H = H()$ and $(12)H = H(12)$

1.4.3

Let G be a finite group and p be a prime number. Prove: TFAE:

1). $|G| = p$

2). $G \neq \langle e \rangle$ and G has no proper (non-trivial) subgroups

3). $G \simeq \mathbb{Z}_p$

$1 \implies 2$: Assume $|G| = p$

$p > 1$ so $G \neq \langle e \rangle = \{e\}$.

ABC: G has a proper (non-trivial) subgroup H
$|H|$ divides $|G|$ (Lagrange)
But $|H| \neq 1$
So $|H| = p = |G|$
$H = G$
CONTRADICTION!
$\therefore G$ has no proper (non-trivial) subgroups.

$2 \implies 3$: Assume $G \neq \langle e \rangle$ and G has no proper (non-trivial) subgroups

Assume $a \in G$
$\langle a \rangle = G$
G is finite cyclic of order p
$\therefore G \simeq \mathbb{Z}_p$.

$3 \implies 1$: Assume $G \simeq \mathbb{Z}_p$

$\therefore G$ is cyclic finite of order p.

1.4.11

Let G be a group of order $2n$

a) Prove: G contains an element of order 2.

ABC: G does not contain an element of order 2
Thus, no element other than e is its own inverse
Since inverses are unique, there is a one-to-one correspondence between a non-identity element and its inverse, resulting in an even number of elements
But $|G - \{e\}| = 2n - 1$, which is odd
CONTRADITION!
$\therefore G$ must contain at least one element of order 2.

b) Prove: n is odd and G abelian $\implies$ G has exactly one element of order 2.

Assume G has more than one element of order 2, say a and b
Since G is abelian, $\langle a, b \rangle = \{a^s b^t \mid s, t \in \mathbb{Z}^+ \cup \{0\}\}$
But since a is order 2:
$a^s = \begin{cases} e, & s \text{ even} \\ a, & s \text{ odd} \end{cases}$
Likewise for b^t
So $\langle a, b \rangle = \{e, a, b, ab\}$
ABC: $ab = e$
$a = b^{-1} = b$
CONTRADICTION!
ABC: $ab = a$
$b = e$

CONTRADICTION!
ABC: $ab = b$
$a = e$
CONTRADICTION!
So $|\langle a, b\rangle| = 4$
But $\langle a, b\rangle \leq G$, and thus by Lagrange $|\langle a, b\rangle|$ must divide $|G|$
$4 \nmid 2$ and $4 \nmid n$, which is odd
So $4 \nmid 2n$
CONTRADICTION!
So G has at most one element of order 2
But by part (a), there is at least one
$\therefore G$ has exactly one element of order 2.

1.4.13

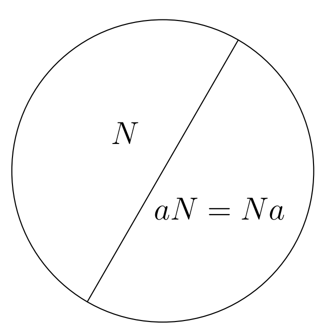
Let p and q be prime numbers such that $p > q$ and let G be a group of order pq
Prove: G has at most one subgroup of order p

ABC: G has more than one subgroup of order p, say H and K
Since $H \cap K \leq H$, $|H \cap K|$ must divide $|H| = p$ (Lagrange)
Thus, $|H \cap K| = 1$ or p
But $|H \cap K| \neq p$, otherwise $H = K$, but it was assumed that H and K are distinct
So $|H \cap K| = 1$, meaning $H \cap K = \{e\}$
$|HK| = \frac{|H||K|}{|H \cap K|} = \frac{p^2}{1} = p^2$
Thus $|H \vee K| \geq p^2$
But $H \vee K \leq G$ and so $|H \vee K|$ must divide $|G|$
But $p^2 > pq$
CONTRADICTION!
$\therefore G$ has at most one subgroup of order p.

1.5.1

Prove: $N \leq G$ and $(G : N) = 2 \implies N \triangleleft G$

Assume $(G : N) = 2$
Assume $a \in G, a \notin N$
N and aN are the two distinct left cosets
N and Na are the two distinct right cosets
$aN = Na$
$\therefore N \triangleleft G$



1.5.2

Let $\{N_i \mid i \in I\}$ be a family of normal subgroups of G.
Prove: $\bigcap_{i \in I} N_i \lhd G$

Let $N = \bigcap_{i \in I} N_i$
Assume $g \in G$
Assume $n \in N$
$\forall\, i \in I, n \in N_i$
$\forall\, i \in I, gng^{-1} \in N_i$, since $N_i \lhd G$
So $gng^{-1} \in N$
$\therefore N \lhd G$

1.5.5

Let $N = \{\sigma \in S_4 \mid \sigma(4) = 4\}$
Is $N \lhd G$?

No. Here is a counterexample:

Let $\sigma = (12) \in N$
Let $g = (14) \in S_4$
$g^{-1} = (14)$
$g\sigma g^{-1} = (14)(12)(14) = (24) \notin N$

1.5.6

Let $H < G$.
Prove: $\forall\, a \in G, aHa^{-1} < G$ and $H \simeq aHa^{-1}$

Assume $a \in G$

Assume $h_1, h_2 \in H$
By closure, $h_1 h_2 \in H$
$(ah_1a^{-1})(ah_2a^{-1}) = ah_1h_2a^{-1} \in aHa^{-1}$
$\therefore aHa^{-1}$ is closed under the operation.

$e \in H$
$aea^{-1} = aa^{-1} = e$
$e \in aHa^{-1}$
$\therefore aHa^{-1}$ has the identity.

Assume $h \in H$
$h^{-1} \in H$
$ah^{-1}a^{-1} \in aHa^{-1}$
$(aha^{-1})(ah^{-1}a^{-1}) = ahh^{-1}a^{-1} = aa^{-1} = e$

$(ah^{-1}a^{-1})(aha^{-1}) = ah^{-1}ha^{-1} = aa^{-1} = e$
$\therefore aHa^{-1}$ is closed under inverses.

$\therefore aHa^{-1} < G$

Now, let $\phi_a : H \to aHa^{-1}$ be defined by $\phi_a(h) = aha^{-1}$

Assume $\phi_a(h_1) = \phi_a(h_2)$
$ah_1a^{-1} = ah_2a^{-1}$
So, by left and right cancellation, $h_1 = h_2$
$\therefore \phi_a$ is one-to-one.

Assume $g \in aHa^{-1}$
$\exists\, h \in H, g = aha^{-1}$
$a^{-1}ga = h$
So $a^{-1}ga \in H$
$\phi_a(a^{-1}ga) = a(a^{-1}ga)a^{-1} = g$
$\therefore \phi_a$ is onto and thus a bijection.

Assume $h_1, h_2 \in H$
By closure, $h_1h_2 \in H$
$\phi_a(h_1h_2) = ah_1h_2a^{-1} = (ah_1a^{-1})(ah_2a^{-1}) = \phi_a(h_1)\phi_a(h_2)$
$\therefore \phi$ is a homomorphism and thus an isomorphism.

$\therefore H \simeq aHa^{-1}$

1.5.7

Let G be a finite group and $H < G$ where $|H| = n$.
Prove: H is the only subgroup of order $n \implies H \triangleleft G$

By problem (6): $\forall\, a \in G, aHa^{-1} < G$ and $H \simeq aHa^{-1}$
But H is the only subgroup of order n,
So $H = aHa^{-1}$
$\therefore H \triangleleft G$

1.5.9

a) Let G be a group and $H = Z(G)$. Prove: $H \triangleleft G$

   It was previously proven that $H \leq G$, so need to show normality.

   Assume $g \in G$
   Assume $h \in H$
   $gh = hg$
   $gH = Hg$
   $\therefore H \triangleleft G$

b) Prove: $Z(S_n) = \{()\}, n \geq 3$

$() \in S_n$ always commutes with everything, so $() \in Z(S_n)$

Assume $\sigma \in S_n, \sigma \neq ()$
$\exists\, i, j \in [n], i \neq j$ and $\sigma(i) = j$
Since σ is a bijection, $\sigma(j) \neq j$
Since $n \geq 3$, $\exists\, k \in [n], k \neq j$ and $k \neq \sigma(j)$
Let $\tau = (jk) \in S_n$
Let $\sigma(j) = \ell$
$\ell \neq j$ and $\ell \neq k$
$(\tau\sigma)(j) = \tau(\sigma(j)) = \tau(\ell) = \ell = \sigma(j)$
$(\sigma\tau)(j) = \sigma(\tau(j)) = \sigma(k)$
But σ is a permutation, and thus a bijection, and thus one-to-one
$j \neq k \implies \sigma(j) \neq \sigma(k)$
$\tau\sigma \neq \sigma\tau$
So $\sigma \notin Z(S_n)$

$\therefore Z(S_n) = \{e\}$

1.5.12

Let $H \triangleleft G$ such that H and G/H are finitely-generated.
Prove: G is finitely-generated

Let $H = \langle X \rangle$ where $X = \{x_1, \ldots, x_r\}$
Let $G/H = \langle Y \rangle$ where $Y = \{y_1 H, \ldots, y_s H\}$, such that the y_i are the selected representatives of each left coset in the generating set
Since the cosets partition G:
$G = \bigcup gH = \bigcup \prod (y_i H)^{n_i} = \bigcup \left(\prod y_i^{n_i}\right) H = \bigcup \left(\prod y_i^{n_i}\right) \left(\prod x_j^{m_j}\right)$
So $G = \langle X \cup Y \rangle$
But X and Y finite $\implies X \cup Y$ finite
Therefore G is finitely-generated

1.5.16

Let $f : G \to H$ be a homomorphism of groups, H is abelian, and $\ker(f) \leq N \leq G$.
Prove: $N \triangleleft G$

Let $K = \ker(f)$
$f[G] \leq H$ and so $f[G]$ is abelian
But by the FIT, $f[G] \simeq G/K$
So G/K is also abelian
Assume $g \in G$
Assume $n \in N$

$(gK)nK(g^{-1}K) = (gK)(g^{-1}K)nK = (eK)(nK) = nK$
Thus $N/K \triangleleft G/K$
Therefore, by Cor 5.12, $N \triangleleft G$

1.5.19

Let $N \triangleleft G$, $(G : N)$ finite, $H < G$, $|H|$ finite, and $((G : N), |H|) = 1$. Prove: $H \leq N$

Since $N \triangleleft G$, $HN \leq G$. This results in the following subgroup relationships:

$$\begin{tikzcd} & G \arrow[d, dash] & \\ & HN \arrow[dl, dash] \arrow[dr, dash] & \\ H \arrow[dr, dash] & & N \arrow[dl, dash] \\ & H \cap N \arrow[d, dash] & \\ & \{e\} & \end{tikzcd}$$

$(G : N) = (G : HN)(HN : N)$
$(HN : N) = (H : H \cap N)$ (prop I.4.8)
$(G : N) = (G : HN)(H : H \cap N)$

$|H| = (H : \{e\}) = (H : H \cap N)(H \cap N : \{e\})$

$((G : N), |H|) = ((G : HN)(H : H \cap N), (H : H \cap N)(H \cap N : \{e\})) = 1$
So $(H : H \cap N) = 1$, meaning $H = H \cap N$
$\therefore H \leq N$