Linear Functionals

Definition: Linear Functional

Let H be a Hilbert space. A linear function $f : H \to \mathbb{C}$ is called a *linear functional*.

The vector space of all bounded linear functionals on H, denoted H' or H^*, is called the *dual space* of H.

Lemma

Let E be an inner product space and $\vec{x} \in E$:

$$\|\vec{x}\| = \sup_{\|\vec{y}\|=1} |\langle \vec{x}, \vec{y} \rangle|$$

Proof

$$\sup_{\|\vec{y}\|=1} |\langle \vec{x}, \vec{y} \rangle| \leq \sup_{\|\vec{y}\|=1} \|\vec{x}\| \, \|\vec{y}\| = \|\vec{x}\| \cdot 1 = \|\vec{x}\|$$

$$\sup_{\|\vec{y}\|=1} |\langle \vec{x}, \vec{y} \rangle| \geq \left| \left\langle \vec{x}, \frac{\vec{x}}{\|\vec{x}\|} \right\rangle \right| = \frac{1}{\|\vec{x}\|} \langle \vec{x}, \vec{x} \rangle = \frac{1}{\|\vec{x}\|} \|\vec{x}\|^2 = \|\vec{x}\|$$

$$\|\vec{x}\| \leq \sup_{\|\vec{y}\|=1} |\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\|$$

$$\therefore \|\vec{x}\| = \sup_{\|\vec{y}\|=1} |\langle \vec{x}, \vec{y} \rangle|$$

Examples

1). $H = \mathbb{C}^N$ and let $a \in H$. Define $f(x) = \sum_{k=1}^{N} a_k x_k$.

   $f \in H'$

   f is linear due to the linearity of the sum.

   Also, $f(x) = \langle x, \bar{a} \rangle$ and so $\|f\| = \|\bar{a}\| = \|a\|$.
   Thus, f is bounded.

2). In general, for a Hilbert space H and $f(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$ for some fixed $\vec{y} \in H$:

   $f \in H'$ and $\|f\| = \|\vec{y}\|$

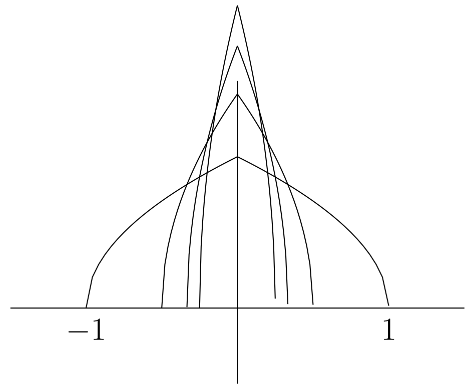
3). Let $H = \mathcal{C}[-1, 1]$ and $\langle \vec{x}, \vec{y} \rangle = \int_{-1}^{1} x\bar{y}$

   Define $f : H \to \mathbb{C}$ by $f(x) = x(0)$.

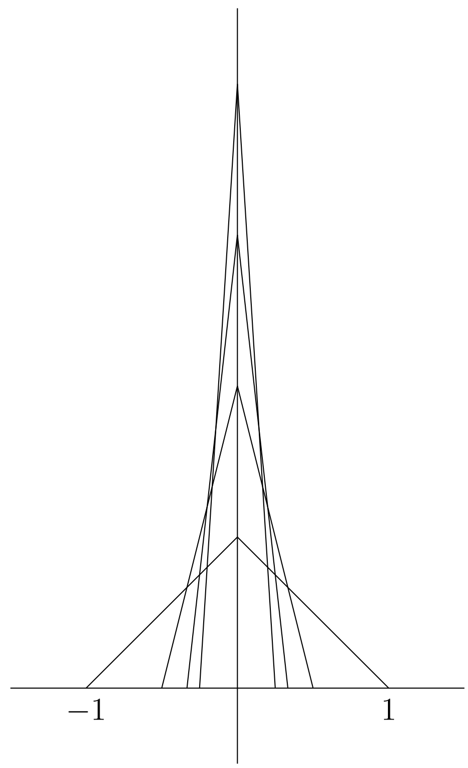
Assume $x, y \in H$ and $\alpha, \beta \in \mathbb{C}$:

$$f(\alpha x + \beta y) = (\alpha x + \beta y)(0) = \alpha x(0) + \beta y(0) = \alpha f(x) + \beta f(y)$$

Therefore f is linear. But is it bounded?



Let $f_n(t) = \begin{cases} 0, & -1 \leq t \leq -\dfrac{1}{n} \\ \sqrt{n + n^2 t}, & -\dfrac{1}{n} \leq t \leq 0 \\ \sqrt{n - n^2 t}, & 0 \leq t \leq \dfrac{1}{n} \\ 0, & \dfrac{1}{n} \leq t \leq -1 \end{cases}$



Let $|f_n(t)|^2 = \begin{cases} 0, & -1 \leq t \leq -\dfrac{1}{n} \\ n + n^2 t, & -\dfrac{1}{n} \leq t \leq 0 \\ n - n^2 t, & 0 \leq t \leq \dfrac{1}{n} \\ 0, & \dfrac{1}{n} \leq t \leq -1 \end{cases}$

$\|f_n\| = 1$ But $f_n \to \delta_0 \notin \mathcal{C}[-1, 1]$ (Dirac-Delta)