Math 275A: Topology

Spring 2020

Instructor: Slobodan Simić

Welcome

Welcome!

Outline for today

1. Logistics
2. What is topology?
3. Sets and functions

Acknowledgement

I would like to thank Francis Su for permission to borrow from the web site for his IBL topology course Harvey Mudd.

Course information

- Prerequisite: Math 131A with a grade of B or better or instructor consent.
- Textbook: Michael Starbird and Francis Su, *Topology Through Inquiry*, AMS/MAA Textbooks Volume: 58; 2019
- Office hours: Mondays 10:30–11:30 AM and 2:45–3:15 PM, Wednesdays 10:00–11:30 AM, or by appointment (in MH 318A)
- Exams: None
- Class web page: `https://sites.google.com/sjsu.edu/slobodan-simic/home/teaching/spring20/math-275a`

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- You will take turns presenting proofs of theorems in class, while other students will determine if they are correct.
- I will provide perspective on the material and motivating examples if you get stuck.

No outiside sources

- As is customary in any IBL course, you are not allowed to consult any outside sources, including textbooks or the internet, to solve these problems.
- The exceptions are sources that you are required to use for other courses.

Notebook

- You should acquire a loose-leaf binder, in which you will save all course notes. This will become your Notebook.
- As you prove theorems in this course (or see them proved in class), you will write up these proofs and add them to your Notebook.
- In a sense, you are writing your own book on the subject, filled with your own proofs.

Reading assignments and homework

- Weekly homework assignments will consists of a list of theorems and exercises from the sections covered by the reading assignment.
- You will be asked to write up proofs/solutions of the assigned theorems/exercises and hand them in each Wednesday.
- For each class, we will have a target, consisting of a list of theorems and exercises from the homework to be proved/solved in that class.
- You will be able to revise and rewrite your homework based on the class discussion and my feedback. It is important to pay attention to proofs of theorems presented in class, since you will write these up for credit.

How to write your homework

- You should write your proofs using full English sentences, as if you were writing a paper or a book.
- Follow Some Guidelines for Good Mathematical Writing by Francis Su, which can be found on Piazza and the class web page.

Honor code

- Cooperation is encouraged, but solutions should be written up individually.
- You may not consult outside mathematical sources without my permission, unless required for some other course.

Class mechanics

Here is how each week will be organized:

1. On Wednesday of the previous week I will assign reading, homework, and targets for the coming week.
2. On Monday we will discuss the targeted theorems. That means: you will be asked to present your proofs, while other students will determine if they are correct.
3. Same on Wednesday.
4. You will turn in your homework on Wednesday.

Class mechanics, continued

5. I will grade the homework and return it the following Monday.
6. Next, you will revise and rewrite your homework based on my feedback and class discussion.
7. The final version of your homework goes into your Notebook.

Notebook

Here is what should be in your Notebook:

1. A summary sheet to track your progress;
2. Proofs or sketches for Theorems and Exercises;
3. Any other notes you have taken;
4. Returned homework assignments.

- How you organize the notebook is up to you.
- The rule of thumb is that this notebook should be something you look back on with pride 10 years from now. It should contain enough details for you to be able to reconstruct your thoughts.

Keeping track of progress

As we discuss the theorems in class, please record on the summary sheet (using the code provided below), whether a theorem was:

- P Presented in class;
- S Solved and prepared to present [mostly correct];
- C Proved with some collaboration and prepared to present [mostly correct, worked with others, or others supplied small fixes for your proof];
- L Learned in class, or with substantial collaboration.

Suggestions for proofs

There will be many times when you are stuck on a problem. This is where the real learning occurs. Here are several ideas:

- Draw pictures!
- Make sure you thoroughly understand the definitions involved, and work out examples!
- Work on a simpler special case if you cannot solve the whole problem.
- Is every hypothesis necessary? Construct examples to show why the theorem fails if a hypothesis is missing. This will often show you what is needed for the proof.

Suggestions for class presentations

- Sketch. Begin by giving a brief outline of the argument, before giving details.
- Be prepared to justify the details if asked. Speak loudly.
- Get together with others in the class, and practice presenting your proofs to each other.
- Knowing a proof and presenting it are two very different things!
- An ideal presentation (in the sense of learning) is not necessarily a perfect presentation!

What to expect

- Expect this course to be challenging, but also very rewarding.
- The value of the IBL format is that when you prove theorems by yourself, you will never forget the proofs you came up with and you will gain confidence in your abilities as mathematicians!

Expository paper and presentation

- Each student will be required to write a short expository paper on a topic of his or her choice related to topology.
- I will post (on Piazza) a list of potential topics. You are welcome to suggest some too.
- Papers need to be typeset in LaTeX.
- At the end of the semester, each student will be required to present her/his expository paper.
- Presentations will be around 10 minutes long with a few minutes for questions.

Grading policy

- Homework 20%, notebook 20%, participation 20%, expository paper 30%, presentation 10%
- Class participation will include points for "productive failures".
- I would rather you not worry about grades but concentrate on learning.

Piazza

For most out of class Q&A we will be using Piazza: https://piazza.com.

- Piazza is a free online gathering place where students can ask, answer, and explore 24/7, under the guidance of their instructors.
- Students can post questions and collaborate Wikipedia-style to edit responses to these questions.
- Instructors can answer questions, endorse student answers, and edit or delete any posted content.
- Instead of emailing me questions, I encourage you to post them on Piazza instead.

Outline of the course

Questions

ANY QUESTIONS?

What is topology?

Let us answer a related question:

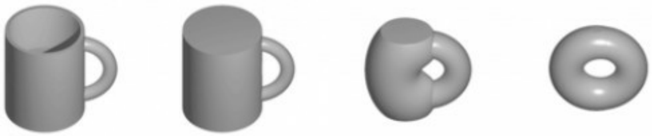
What makes someone a topologist?

A topologist is someone who cannot distinguish between these objects:

Figure: A cup and a bagel.

Cup = bagel

Reason:

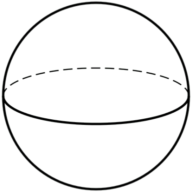


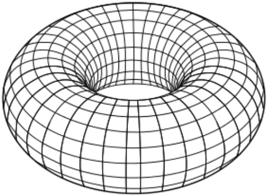
So, what is topology?

One definition of topology is:

Topology is a branch of mathematics that is concerned with properties of geometric objects preserved under continuous deformations, such as stretching, twisting, crumpling and bending, but not tearing or gluing.

For a topologist the following objects are not the same:

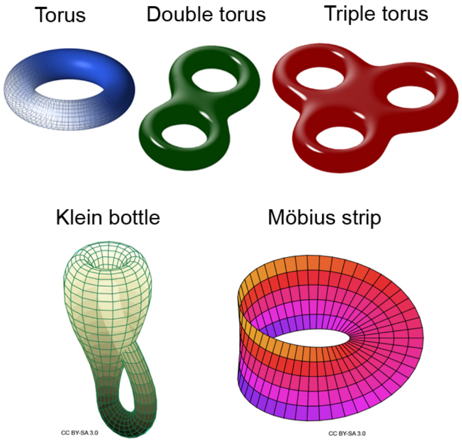




Topological equivalence

- In topology two spaces are the same (or topologically equivalent) if there is a homeomorphism between them, i.e., a continuous map with a continuous inverse.
- **Question**: How does one define continuity between two spaces?
- Even more **basic question**: what do we mean by a "space"?

Examples of (topological) spaces



But also this...

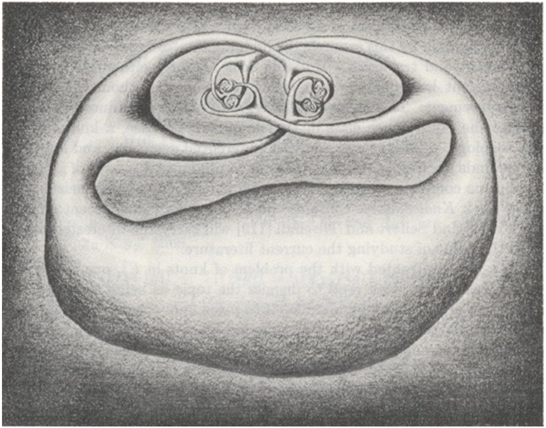


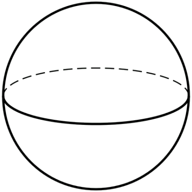
Figure: Alexander horned sphere

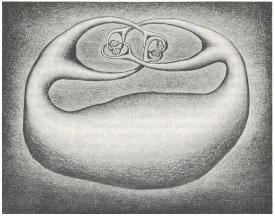
Ultimate goal of topology

The ultimate goal of topology is:

Given two spaces, decide if they are topologically equivalent.

For example, are these topologically equivalent?





Continuity

- To make this question well-defined, we need to figure out what continuity really means.
- For that we need to go back to analysis, where continuity first came from.
- Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point a if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

- f is continuous if it is continuous at every point.

Continuity (continued)

- This definition depends strongly on the notion of absolute value in $\mathbb{R}$.
- How do we generalize it to spaces other than $\mathbb{R}$?
- Note that the same definition works in $\mathbb{R}^n$: $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous at $a \in \mathbb{R}^m$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\, \|x - a\| < \delta \Rightarrow \|f(x) - f(a)\| < \varepsilon.$$

where

$$\|v\| = \left(\sum_{i=1}^{n} v_i^2 \right)^{1/2}.$$

Continuity (continued further)

- Note that $\|x - a\|$ is the distance between a and x.
- We can use the same definition for any map between sets on which we can define the notion of distance.
- If $f : X \to Y$ is a map between sets on which we can measure distance between points, then we can define f to be continuous at $a \in X$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in X)\, d(x, a) < \delta \Rightarrow d(f(x), f(a)) < \varepsilon.$$

- Conclusion: we can define continuity on metric spaces.

Origins of topology

- At the end of the 19th century, faced with paradoxes, mathematicians embarked on a program to axiomatize all of mathematics.
- The goal was to emulate Euclidean geometry.
- The fundamental objects in this program were sets.
- Axioms for set theory were proposed and the goal was to express all known mathematics in set-theoretic terms.
- The challenge was to recast the familiar concepts (real number, convergence, continuity) in terms of sets.
- Thus arose the concept of a topological space and the field of topology was born.

Some important early personalities

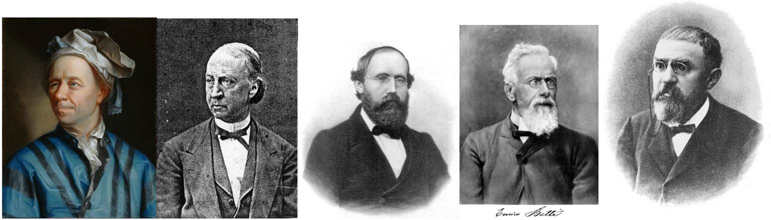


Figure: Euler, Listing, Riemann, Betti, and Poincaré

Continuity recast in terms of sets

- Recall that $f : \mathbb{R} \to \mathbb{R}$ is continuous if for every $a \in \mathbb{R}$,

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

- Equivalently:

$$(\forall \varepsilon > 0)(\exists \delta > 0) f(B(a, \delta)) \subset B(f(a), \varepsilon),$$

where $B(x, r) = (x - r, x + r)$ is the r-neighborhood of x.

- Same thing holds for arbitrary unions of neighborhoods. Those are exactly the open sets in $\mathbb{R}$.
- Equivalently: for every open set V (in the codomain) and every a (in the domain) such that $f(a) \in V$, there is a neighborhood $B(a, \delta)$ of a which f maps into V.

Continuity in $\mathbb{R}$ in terms of sets

- Conclusion: $f : \mathbb{R} \to \mathbb{R}$ is continuous if for every open set $V \subset \mathbb{R}$ (codomain),

$$f^{-1}(V) = \{x \in \mathbb{R} : f(x) \in V\}$$

  is open.
- Therefore: to generalize continuity we need to generalize the notion of an open set to more general spaces.
- We also need to explore basic properties of sets and functions/maps between them.

Sets

- We adopt (as most mathematicians do) the naive point of view regarding sets.
- We will assume that it is intuitively clear what is meant by a set.
- Deeper analysis of the concept of a set belongs to foundations of mathematics and mathematical logic.
- However, beware Russell's paradox! We must be careful.
- One of the reasons for the axiomatization of set theory was to formulate rules that will avoid paradoxes and contradictions.
- Although we will not study axioms of set theory, the rules we will follow derive from them.

Operations on sets

- If a is an element of a set A, we write $a \in A$.
- $A \subset B$ if $(\forall a)\ a \in A \Rightarrow a \in B$
- Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- Intersection: $A \cup B = \{x : x \in A \text{ and } x \in B\}$
- Set difference: $A - B = \{x \in A : x \notin B\}$
- Complement: if $A \subset X$, the complement of A in X is $X - A$ (sometimes A^c).
- Unions and intersections can easily be extended to an arbitrary family of sets, $(A_i)_{i \in I}$.

Functions

- The Cartesian (or direct) product of sets:

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

- A relation between sets A and B is a subset of $A \times B$.
- A function $f : X \to Y$ is a relation such that

$$(\forall x \in X)(\exists! y \in Y) \quad (x, y) \in f.$$

  We write $y = f(x)$. ($\exists! =$ there exists a unique...)
- X is the domain and Y is the codomain of f.
- Function = map = mapping = transformation

Functions

Let $f : X \to Y$ be a function.

- If $A \subset X$, then

  $$f(A) = \{f(x) : x \in A\} \subset Y$$

  is the image of A under f.
- If $B \subset Y$, then

  $$f^{-1}(B) = \{x \in X : f(x) \in B\} \subset X$$

  is the preimage of B under f.

Functions

- $f : X \to Y$ is injective (or one-to-one) if

$$(\forall x_1, x_2 \in X)\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

- f is surjective (or onto) if

$$(\forall y \in Y)(\exists x \in X)\ f(x) = y.$$

- f is a bijection if it is injective and surjective.

Cardinality

- A and B have the same cardinality ($|A| = |B|$) if there is a bijection from A to B.
- A is finite if it is either empty or has the same cardinality with $\{1, 2, \ldots, n\}$ for some $n \geq 1$, in which case we write $|A| = n$.
- Otherwise, A is called infinite.
- A is countable if it is finite or it has the same cardinality as $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- Otherwise, it is uncountable.

Countable sets

- $\mathbb{Z}$ (the set of integers) and $\mathbb{N}$ have the same cardinality. Can you prove it?
- Every infinite set has a countably infinite subset.
- Is $\mathbb{R}$ countable?

Your homework

- (**Exercise 1.3**) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
- (**Theorem 1.12**) The union of countably many sets is countable. That is, if A is countable and X_α is countable for each $\alpha \in A$, then

$$X = \bigcup_{\alpha \in A} X_\alpha$$

  is countable.
- (**Theorem 1.13**) The set $\mathbb{Q}$ (of rational numbers) is countable.
- (**Theorem 1.16**, Cantor) $\mathbb{R}$ is uncountable.

Reading assignment for this week

Sections 1.1, 1.2 and 1.3.
Sections 1.4 and 1.5 are recommended but optional.

Target for 1/29: In class be prepared to present **Theorems 1.12, 1.13, and 1.16**.