Cardinality

Definition

The *cardinality* of a set A, denoted |A|, is the number of elements in A.

This is fairly intuitive for finite sets, but what about infinite sets like \mathbb{Z} and \mathbb{R} ?

Definition

To say that two sets A and B are equivalent, denoted $A \approx B$, means there exists a bijection $\phi: A \to B$. When two sets are equivalent they are said to have the same cardinality, denoted |A| = |B|.

Notation

$$[0] = \emptyset$$

 $[n] = \{1, 2, 3, ..., n\}$

Definition

To say that a set A is finite means $\exists n \in \mathbb{Z}^+, A \approx [n]$. Otherwise, A is said to be infinite.

To say that A is countable means A is finite or $A \approx Z^+$. Otherwise, A is said to be uncountable.

Note that by definition, \mathbb{Z}^+ is countably infinite. We denote the cardinality of \mathbb{Z}^+ by:

$$\left|\mathbb{Z}^{+}\right|=\aleph_{0}$$

Theorem

 \mathbb{Z} is countable, and in fact:

$$|\mathbb{Z}| = \aleph_0$$

Proof

Consider the following bijection mapping \mathbb{Z} to \mathbb{Z}^+ :

Thus, $\mathbb{Z} \approx \mathbb{Z}^+$ and is therefore countable. Furtherfore:

$$|\mathbb{Z}| = \left|\mathbb{Z}^+\right| = \aleph_0$$

So, even though $\mathbb{Z}^+ \subset \mathbb{Z},$ they have the same cardinality. In fact:

Theorem

Every subset of a countable set is countable. Furthermore, if the set and the subset are infinite then they have the same cardinality.

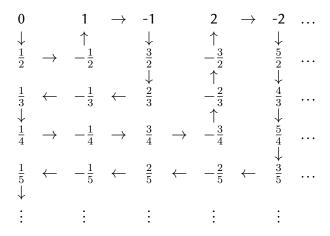
Theorem

 \mathbb{Q} is countable, and in fact:

$$|\mathbb{Q}| = \aleph_0$$

Theorem

Arrange and traverse the elements of \mathbb{Q} as follows:



Note that every rational number is counted once; none are missed.

Therefore, $\mathbb{Q} \approx \mathbb{Z}^+$ and $|\mathbb{Q}| = |\mathbb{Z}^+| = \aleph_0$.

Theorem: Cantor Diagonalization

[0,1] is uncountable, and thus $\mathbb R$ is uncountable.

Proof

ABC that \mathbb{R} is countable.

Consider $[0,1] \subset \mathbb{R}$; it must also be countable.

Let the following represent an exhaustive list of the elements in [0,1]:

$$0.\underline{a_{1,1}}a_{1,2}a_{1,3}a_{1,4}a_{1,5}\dots$$

$$0.a_{2,1}\underline{a_{2,2}}a_{2,3}a_{2,4}a_{2,5}\dots$$

$$0.a_{3,1}a_{3,2}\underline{a_{3,3}}a_{3,4}a_{3,5}\dots$$

$$0.a_{4,1}a_{4,2}a_{4,3}\underline{a_{4,4}}a_{4,5}\dots$$

$$0.a_{5,1}a_{5,2}a_{5,3}a_{5,4}a_{5,5}\dots$$

Now, consider a real number $x \in [0,1]$ whose n^{th} digit differs from the n^{th} digit in the n^{th} number in the list. Such a number is not in the original list, and thus the list can never be exhaustive. Thus, [0,1] is not countable, and therefore neither is \mathbb{R} .

The uncountable cardinality of \mathbb{R} is simply denoted by $|\mathbb{R}|$, and the cardinality of \mathbb{R}^n is denoted by $|\mathbb{R}|^n$, where:

$$\aleph_0 < |\mathbb{R}| < |\mathbb{R}|^2 < \dots$$