MATH 231B, FALL 2017
HOMEWORK 6 SOLUTIONS

1. (Sec. 4.12, ex. 20) ($\Rightarrow$) Assume A is unitary and let (e_n) be a complete orthonormal sequence. Since A preserves the inner product, it follows that

$$\langle Ae_m, Ae_n\rangle = \langle e_m, e_n\rangle = \delta_{mn},$$

which means that (Ae_n) is an orthonormal sequence. Let us show that it is complete. Let $y \in H$ be arbitrary. Since A is unitary, it is surjective, hence $y = Ax$, for some $x \in H$. Since (e_n) is a complete orthonormal sequence, we have

$$x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n,$$

hence

$$y = Ax = \sum_{n=1}^{\infty} \langle x, e_n\rangle Ae_n,$$

proving that (Ae_n) is complete.

($\Leftarrow$) Now assume that A maps complete orthonormal sequences to complete orthonormal sequences. Let (e_n) be a complete orthonormal sequence in H and let $x \in H$ be arbitrary. Then

$$x = \sum_{n=1}^{\infty} \alpha_n e_n,$$

where $\alpha_n = \langle x, e_n\rangle$. Since A is continuous, it follows that

$$Ax = \sum_{n=1}^{\infty} \alpha_n Ae_n.$$

By Parseval's identity we have

$$\|x\|^2 = \sum_{n=1}^{\infty} |\alpha_n|^2$$

as well as

$$\|Ax\|^2 = \sum_{n=1}^{\infty} |\alpha_n|^2$$

Thus A preserves the norm of every vector and is therefore an isometry. To show that A is unitary it remains to show that A is surjective. Let $y \in H$ be arbitrary. Choose a complete orthonormal sequence (e_n) in H. Then (Ae_n) is a complete orthonormal sequence and therefore

$$y = \sum_{n=1}^{\infty} \beta_n Ae_n,$$

where $\beta_n = \langle y, Ae_n\rangle$. It therefore follows that $y = Ax$, where $x = \sum_{n=1}^{\infty} \beta_n e_n$, proving that A is surjective. Thus A is unitary. This completes the proof. $\square$

2. (Sec. 4.12, ex. 23) To show that e^A is a well-defined bounded linear operator, denote the N^{th} partial sum of the given series by S_N; i.e.,

$$S_N = \sum_{n=0}^{N} \frac{A^n}{n!}.$$

For all $M > N$ we have:

$$\begin{aligned}
\|S_M - S_N\| &= \left\| \sum_{n=N+1}^{M} \frac{A^n}{n!} \right\| \\
&\leq \sum_{n=N+1}^{M} \frac{\|A\|^n}{n!} \\
&\leq \sum_{n=N+1}^{\infty} \frac{\|A\|^n}{n!} \\
&\to 0,
\end{aligned}$$

as $N \to \infty$. This follows from the fact that the series $\sum_0^\infty t^n/n!$ converges for every $t \in \mathbb{R}$. Therefore, (S_N) is a Cauchy sequence in $\mathscr{B}(H)$, hence convergent since $\mathscr{B}(H)$ is complete. This proves that $e^A \in \mathscr{B}(H)$.

(d) Assume $AB = BA$. Then Newton's binomial formula holds:

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}.$$

We therefore have:

$$\begin{aligned}
e^{A+B} &= \sum_{n=0}^{\infty} \frac{1}{n!}(A+B)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A^k B^{n-k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{A^k}{k!} \frac{B^{n-k}}{(n-k)!} \\
&= \sum_{n=0}^{\infty} \sum_{k+\ell=n} \frac{A^k}{k!} \frac{B^\ell}{\ell!} \\
&= \sum_{k=0}^{\infty} \frac{A^k}{k!} \cdot \sum_{\ell=0}^{\infty} \frac{B^\ell}{\ell!} \\
&= e^A e^B.
\end{aligned}$$

(a) This follows by repeated application of (d) with $B = A$.

(b) It is clear that $e^0 = I$ since all terms in the defining series except the zeroth one vanish.

(c) Follows by taking $B = -A$ in (d).

(e) Assume $A^* = A$. It is easy to see that $(e^T)^* = e^{T^*}$, for all bounded operators T, so

$$\left(e^{iA}\right)^* = e^{(iA)^*} = e^{-iA^*} = e^{-iA},$$

which equals $\left(e^{iA}\right)^{-1}$ by (c). Thus e^{iA} is unitary. $\square$

3. (Sec. 4.12, ex. 28) No, $T^*T = I$ does **not** imply that $TT^* = I$. Take the shift operator $T : \ell^2 \to \ell^2$ defined by

$$T(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots).$$

To compute T^*, start with arbitrary vectors $x, y \in H$. Then:

$$\begin{aligned} \langle Tx, y\rangle &= \sum_{n=2}^{\infty} x_{n-1}\overline{y}_n \\ &= \sum_{m=1}^{\infty} x_m \overline{y}_{m+1} \\ &= \langle x, T^*y\rangle, \end{aligned}$$

where we did a simple index change $m = n - 1$. Thus

$$T^*(y_1, y_2, y_3, \ldots) = (y_2, y_3, \ldots).$$

It is easy to verify that $T^*T = I$. It is also not hard to see that $T^*e_1 = 0$, so $TT^*e_1 = T0 = 0 \neq e_1$ and thus $TT^* \neq I$. $\square$

4. (Sec. 4.12, ex. 31) Since $A + B = 0$, we have for every $x \in H$:

$$\begin{aligned} 0 &= \langle (A + B)x, x\rangle \\ &= \langle Ax + Bx, x\rangle \\ &= \langle Ax, x\rangle + \langle Bx, x\rangle. \end{aligned}$$

Since A and B are positive operators, both $\langle Ax, x\rangle$ and $\langle Bx, x\rangle$ are non-negative, so since they add up to zero, they both have to equal zero. It was shown in an earlier homework that this implies $A = 0$ and $B = 0$. $\square$

5. (Sec. 4.12, ex. 54) Consider the multiplication operator T on $L^2[0, 1]$ defined by

$$(Tf)(x) = xf(x).$$

It was shown in class that T is self-adjoint.

Assume λ is an eigenvalue of T. Then $Tf = \lambda f$, for some $f \neq 0$, i.e.,

$$xf(x) = \lambda f(x),$$

for almost every $x \in [0, 1]$. Since $f \neq 0$ (in the L^2 sense), it follows that $f(x) \neq 0$ on a set A of positive measure. For every $x \in A$, $xf(x) = \lambda f(x)$ implies $x = \lambda$, which is of course impossible. This shows that T has no eigenvalues. $\square$