Free Abelian Groups

Definition

Let G be an abelian group and $X \subseteq G$. To say that X is a basis for G means:

1). $G = \langle X \rangle$

2). $\forall\, x_k \in X$ distinct and $n_k \in \mathbb{N}$:

$$\sum_{k=1}^{n} n_k x_k = 0 \iff \forall\, n_k = 0$$

Example

$G = \bigoplus_{i=1}^{n} Z$

Let $x_k = \begin{cases} 1, i = k \\ 0, i \neq k \end{cases}$

Theorem

Let F be an abelian group. TFAE:

1). F has a non-empty basis.

2). F is the internal direct sum of a family of infinite cyclic groups.

3). F is isomorphic to a direct sum of copies of $\mathbb{Z}$.

4). There exists a non-empty set X and function $\iota : X \to F$ such that given an abelian group G and a function $f : X \to G$, there exists a unique homomorphism $\phi : F \to G$ such that $f = \phi\iota$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota"'] & G \\ F \arrow[ur, "\phi"'] & \end{tikzcd}$$

A group F that satisfies these properties is called a *free* abelian group.

Proof

$1 \to 2$: Assume F has a non-empty basis X

Assume $x \in X$
$nx = 0 \implies n = 0$
Thus, $\langle x \rangle$ is infinite cyclic
Since F is abelian, $\langle x \rangle \triangleleft F$

Since X is a basis, there exists a set $\{x_k\} \subseteq X$ such that $F = \langle \bigcup_{k=1}^{n} \langle x_k \rangle \rangle$
Assume $z \in X$
ABC: $\langle z \rangle \cap \left\langle \bigcup_{x_k \neq z} \langle x_k \rangle \right\rangle \neq \{0\}$
$\exists\, n \in \mathbb{N}, nz = \sum_{k=1}^{n} n_k x_k$
$nz + \sum_{k=1}^{n} n_k x_k = 0$; however, $n \neq 0$
Contradiction (of basis)!
So $\langle z \rangle \cap \left\langle \bigcup_{x_k \neq z} \langle x_k \rangle \right\rangle = \{0\}$
$\therefore F$ is an internal direct sum of the $\langle x_k \rangle$.

$2 \rightarrow 3$: Assume F is the internal direct sum of a family of infinite cyclic groups

Let $F = \sum F_k$ be such a sum
$F \simeq \bigoplus F_k$
$F_k \simeq Z$
There exists an isomorphism $\phi_k : F_k \rightarrow Z$
Therefore there exists an isomorphism $\phi : F \rightarrow \bigoplus \mathbb{Z}$.

$3 \rightarrow 1$ Assume F is isomorphic to a direct sum of copies of $\mathbb{Z}$

Let $\{u_k\}$ be the standard basis for $\bigoplus \mathbb{Z}$
There exists isomorphism $\phi : \bigoplus \mathbb{Z} \rightarrow F$
Let $x_k = \phi_k(u_k)$
Let $X = \{x_k\}$
Let $f_i = \phi(z_i)$
$f_i = \phi(\sum n_k u_k) = \sum \phi_k(n_k u_k) = \sum n_k \phi_k(u_k) = \sum n_k x_x$
Thus, $F = \langle X \rangle$

Now, assume $\sum n_k x_k = 0$
$\phi^{-1}(\sum n_k x_k) = \sum \phi_k^{-1}(n_k x_k) = \sum n_k \phi^{-1}(x_k) = \sum n_k u_k = 0$
This, $\forall\, n_k = 0$

Therefore, X is a basis for F.

$1 \rightarrow 4$: Assume F has a non-empty basis X

Let $\iota : X \rightarrow F$ be the canonical injection homomorphism

Assume G is an abelian group and $f : F \rightarrow G$
Assume $u \in F$
$u = \sum_{k=1}^{n} n_k x_k$
Assume $u = \sum_{k=1}^{n} m_k x_k$
$\sum_{k=1}^{n} n_k x_k - \sum_{k=1}^{n} m_k x_k = \sum_{k=1}^{n} (n_k - m_k) x_k = 0$
But $n_k - m_k = 0$ since X is a basis for F
So the representation for each $u \in F$ is unique wrt basis X

Let $\phi : F \rightarrow G$ be defined by $\phi(u) = \phi(\sum_{k=1}^{n} n_k x_k) = \sum_{k=1}^{n} n_k f(x_k)$
Since the representation for each $u \in F$ is unique, ϕ is well-defined

Assume $u, v \in F$

$$\begin{aligned}
\phi(u+v) &= \phi(\sum_{k=1}^{n} n_k x_k + \sum_{j=1}^{n} m_j x_j) \\
&= \phi(\sum_{k=1}^{n} (n_k + m_k) x_k) \\
&= \sum_{k=1}^{n} (n_k + m_k) f(x_k) \\
&= \sum_{k=1}^{n} n_k f(x_k) + \sum_{k=1}^{n} m_k f(x_k) \\
&= \phi(u) + \phi(v)
\end{aligned}$$

Therefore ϕ is a homomorphism

Assume that there is another such homomorphism ψ
Since X generates F
So the action of ψ on F is completely determined by ψ on X
Assume $x \in X$
$\psi(x) = \psi(\iota(x)) = (\psi\iota)(x) = f(x) = (\phi\iota)(x) = \phi(\iota(x)) = \phi(x)$