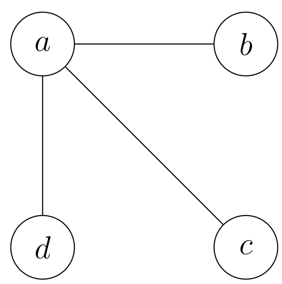
Connected Graphs

Definition

Let G be a graph and let $u, v \in V(G)$:

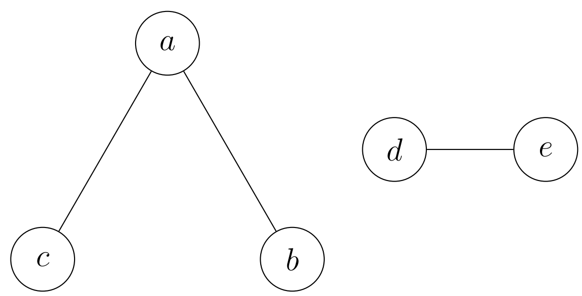
- To say that u and v are *connected* means that G contains a $u - v$ path.
- To say that G is *connected* means that $\forall\, u, v \in G$, u and v are connected. Otherwise, G is said to be *disconnected*.
- By definition, the trivial graph is connected.

Examples



(a, b)
(a, c)
(a, d)
(b, a, c)
(b, a, d)
(c, a, d)

CONNECTED



No path from any of a, b, c to any of d, e

DISCONNECTED

Theorem

Let G be a graph with $n(G) \geq 3$:

G is connected $\iff$ $\exists\, u, v \in V(G), u \neq v$ such that $G - u$ and $G - v$ are connected.

Proof.

$\implies$ Assume G is connected.

Let $u, v \in V(G)$ such that $d(u, v) = \operatorname{diam}(G)$.

ABC/WLOG: $G - v$ is disconnected.

Since $n(G) \geq 3$, there exists distinct $x, y \in V(G - v)$ such that x and y are not connected in $G - v$. However, G is connected and so x and y are connected in G. So let P_1 be a $x - u$

geodesic in G and let P_2 be a $u - y$ geodesic in G. Since v cannot appear in P_1 or P_2, P_1 and P_2 are paths in $G - v$ as well. Thus $P_1 \cup P_2$ is a $x - y$ walk in $G - v$, and so there exists a $x - y$ path in $G - v$, contradicting the disconnectedness of x and y. And so $G - v$ is connected. Likewise, $G - u$ is connected.

$\therefore \exists\, u, v \in V(G), u \neq v$ such that $G - u$ and $G - v$ are connected.

$\Longleftarrow$ Assume $\exists\, u, v \in V(G), u \neq v$ such that $G - u$ and $G - v$ are connected.

Assume $x, y \in V(G)$

Case 1: $\{x, y\} \neq \{u, v\}$

AWLOG: $u \notin \{x, y\}$

By assumption, x and y are connected in $G - u$, and thus are connected in G.

Case 2: $\{x, y\} = \{u, v\}$

Since, by assumption, G contains at least three vertices, there exists a third distinct vertex $w \in V(G)$. Also by assumption, u and w are connected in $G - v$ and hence G. Likewise, v and w are connected in $G - u$ and hence G. So let P_1 be a $u - w$ path in G and let P_2 be a $v - w$ path in G. $P_1 \cup P_2$ is a $u - v$ walk in G, and so G must contain a $u - v$ path, and thus u and v are connected in G.

$\therefore G$ is connected.

■

Theorem

Let G be a connected graph and let P and Q be two longest paths in G, both of length k:

P and Q have at least one vertex in common.

Proof. ABC: P and Q have no vertices in common.

Let $P = (u_0, u_1, \ldots, u_k)$ and $Q = (v_0, v_1, \ldots, v_k)$. Since G is connected, every u_i in P is connected to every v_j in Q. Let $R = (u_i = w_1, w_2, \ldots, w_\ell = v_j)$ be the shortest such path and AWLOG that $i \geq j$. Note that no other vertices in P or Q can exist in R, otherwise the minimality of $|R|$ is contradicted. Now, consider the path $S = (u_0, \ldots, u_i, \ldots v_j, \ldots v_k)$:

$$\begin{aligned} |S| &= i + \ell + (k - j) \\ &= k + \ell + (i - j) \\ &> k \end{aligned}$$

since $\ell > 0$ and $i - j \geq 0$, thus contradicting the maximality of $|P|$ and $|Q|$.

$\therefore$, P and Q share at least one vertex in common. ■