Negative Binomial Distribution

Definition: Negative Binomial Distribution

To say that a random variable X has a *Negative Binomial* distribution with parameters p and r, denoted:

$$X \sim \mathrm{NB}(p, r)$$

means that:

1. The underlying experiment is composed of repeated Bernoulli trials until r successes occur.
2. The trials are independent.
3. Each of the trials has fixed probability p for success.
4. X counts the number of trials up to and including the last success.

Examples: Negative Binomial Distributions

1. Flip a fair coin until 5 heads occur: $X =$ the number trials.

$$X \sim \mathrm{NB}\left(\frac{1}{2}, 5\right)$$

2. Select (with replacement) balls from an urn that has 30 red balls and 20 blue balls until 3 red balls are selected: $Y =$ the number of balls selected.

$$Y \sim \mathrm{NB}\left(0.6, 3\right)$$

Theorem

Let X be a random variable with a Negative Binomial distribution with parameters r and p:

- $f_X(x) = \begin{cases} \binom{x-1}{r-1} p^r (1-p)^{x-r} & x = r, r+1, r+2, \ldots \\ 0 & \text{otherwise} \end{cases}$
- $E(X) = \frac{r}{p}$
- $V(X) = \frac{r(1-p)}{p^2}$

Proof. For $P(X = x)$, fix a success in the x^{th} trial with probability p. The remaining $x - 1$ trials contain $r-1$ successes with probability p^{r-1} and $(x-1)-(r-1) = x-r$ failures with probability $(1-p)^{x-r}$. Therefore:

$$f_X(x) = \binom{x-1}{r-1} p^{r-1}(1-p)^{x-r} p = \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

Now, let $X_i =$ the number of trials since $i^{th} - 1$ success until the i^{th} success, with X_1 measured from the first trial. Note that each of these X_i are independent and $X_i \sim \mathrm{Geom}(p)$. Thus:

$$E(X) = E(\sum_{i=1}^{r} X_i) = \sum_{i=1}^{r} E(X_i) = \sum_{i=1}^{r} \frac{1}{p} = \frac{r}{p}$$
$$V(X) = V(\sum_{i=1}^{r} X_i) = \sum_{i=1}^{r} V(X_i) = \sum_{i=1}^{r} \frac{1-p}{p^2} = \frac{r(1-p)}{p^2}$$

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Example

Suppose X has a Negative Binomial distribution with $p = \frac{1}{2}$ and $r = 3$.

$$X \sim \mathrm{NB}\left(\frac{1}{2}, 3\right)$$

$$P(X = 2) = 0$$
$$P(X = 3) = \binom{3-1}{3-1}\left(\frac{1}{2}\right)^3\left(1 - \frac{1}{2}\right)^{3-3} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$
$$P(X = 4) = \binom{4-1}{3-1}\left(\frac{1}{2}\right)^3\left(1 - \frac{1}{2}\right)^{4-3} = \binom{3}{2}\left(\frac{1}{2}\right)^4 = 3 \cdot \frac{1}{16} = \frac{3}{16}$$
$$P(X = 5) = \binom{5-1}{3-1}\left(\frac{1}{2}\right)^3\left(1 - \frac{1}{2}\right)^{5-3} = \binom{4}{2}\left(\frac{1}{2}\right)^5 = 6 \cdot \frac{1}{32} = \frac{3}{16}$$
$$P(X \geq 5) = 1 - P(X = 3) - p(X = 4) = 1 - \frac{1}{8} - \frac{3}{16} = \frac{11}{16}$$

$$E(X) = \frac{r}{p} = \frac{3}{\frac{1}{2}} = 6$$

$$V(X) = \frac{r(1-p)}{p^2} = \frac{3(1-\frac{1}{2})}{\frac{1}{4}} = 6$$

$$\sigma = \sqrt{6} \approx 2.45$$