Weak Direct Product

Definition

Let $\{G_i \mid i \in I\}$ be a family of groups. The *weak direct product* of $\{G_i \mid i \in I\}$ is given by:

$$\prod_{i\in I}^{w} G_i = \{g \in \prod_{i\in I} G_i \mid g(i) = e_i \text{ except for a finite number of } i \in I\}$$

If the G_i are additive/abelian then $\prod_{i\in I}^{w} G_i$ is called the *direct sum* and is denoted $\sum_{i\in I} G_i$.

Note that if I is finite then the weak direct product and the direct product are the same.

Theorem

Let $\{G_i \mid i \in I\}$ be a family of groups:

$$\prod_{i\in I}^{w} G_i \lhd \prod_{i\in I} G_i$$

Proof

Clearly, $\prod_{i\in I}^{w} G_i \subseteq \prod_{i\in I} G_i$

Assume $g, h \in \prod_{i\in I}^{w} G_i$
Let $I_g = \{i \in I \mid g(i) \neq e_i\}$
Let $I_h = \{j \in I \mid h(j) \neq e_j\}$
By definition, I_g and I_h are finite sets
$\forall\, i \in I, (gh)(i) \neq e_i \implies i \in I_g \cap I_h$
But $I_g \cap I_h$ is finite
So $gh \in \prod_{i\in I}^{w} G_i$
$\therefore \prod_{i\in I}^{w} G_i$ is closed under the operation.

$\forall\, i \in I, e(i) = e_i$
So $e(i) \neq e_i$ for 0 (a finite number) of $i \in I$
$\therefore e \in \prod_{i\in I}^{w} G_i$.

Assume $g \in \prod_{i\in I}^{w} G_i$
$g(i) \neq e_i$ for a finite number of $i \in I$
$g^{-1} \in \prod_{i\in I} G_i$
$g^{-1}(i) \neq e_i$ for a finite number of $i \in I$
So $g^{-1} \in \prod_{i\in I}^{w} G_i$
$\therefore \prod_{i\in I}^{w} G_i$ is closed under inverses.

$\therefore \prod_{i\in I}^{w} G_i \leq \prod_{i\in I} G_i$

Assume $g \in \prod_{i\in I}^{w} G_i$
Let $I_g = \{i \in I \mid g(i) \neq e_i\}$
By definition, I_g is finite

Assume $h \in \prod_{i \in I} G_i$
Assume $i \in I$
$(hgh^{-1})(i) = h(i)g(i)h^{-1}(i) = \begin{cases} h_i g_i h_i^{-1}, i \in I_g \\ h_i e_i h_i^{-1} = e_i, i \notin I_g \end{cases}$
So $hgh^{-1} = e_i$ for all but a finite number of $i \in I$
$hgh^{-1} \in \prod_{i \in I}^{w} G_i$
$\therefore \prod_{i \in I}^{w} G_i \triangleleft \prod_{i \in I} G_i$

Definition

Let $\prod_{i \in I}^{w} G_i$ be a weak product of groups and $\forall\, k \in I$ define the map $\iota_k : G_k \to \prod_{i \in I}^{w} G_i$ by:

$$\iota_k(g_k) = g$$

such that $\forall\, i \in I$:

$$g(i) = \begin{cases} g_k, & i = k \\ e_k, & i \neq k \end{cases}$$

The ι_k are called the *canonical injections* of the weak direct product.

Theorem

Let $\prod_{i \in I}^{w} G_i$ be a weak direct product of groups and let ι_k be the canonical injections for the weak direct product. i_k is a one-to-one homomorphism.

Proof

Assume $k \in I$
Assume $g_k, h_k \in G_k$
$\iota_k(g_k h_k) = \ell$ where $\ell(i) = \begin{cases} g_k h_k, & i = k \\ e_k, & i \neq k \end{cases}$
$\iota_k(g_k) = g$ where $g(i) = \begin{cases} g_k, & i = k \\ e_k, & i \neq k \end{cases}$
$\iota_k(h_k) = h$ where $h(i) = \begin{cases} h_k, & i = k \\ e_k, & i \neq k \end{cases}$
$\iota_k(g_k)\iota_k(h_k) = gh$ where $(gh)(i) = \begin{cases} g_k h_k, & i = k \\ e_k, & i \neq k \end{cases}$
$\ell = gh$
$\iota_k(g_k h_k) = \iota_k(g_k)\iota_k(h_k)$
$\therefore \iota_k$ is a homomorphism.

Assume $\iota_k(g_k) = \iota_k(h_k)$
$g = h$
$g(k) = h(k)$

$g_k = h_k$
$\therefore \iota_k$ is a one-to-one.

Theorem

Let $\prod_{i \in I} G_i$ be a direct product of groups:

$$\forall\, k \in I, \iota_k[G_k] \lhd \prod_{i \in I} G_i$$

Proof

Assume $k \in I$

G_k is a group
ι_k is a homomorphism
So $\iota_k[G_k] \leq \prod_{i \in I}^{w} G_i$
But $\prod_{i \in I}^{w} G_i \leq \prod_{i \in I} G_i$
$\therefore \iota_k[G_k] \leq \prod_{i \in I} G_i$

Assume $g \in \iota_k[G_k]$
Assume $h \in \prod_{i \in I} G_i$
Assume $i \in I$

$(hgh^{-1})(i) = h(i)g(i)h^{-1}(i) = \begin{cases} h_i g_i h_i^{-1}, i = k \\ h_i e_i h_i^{-1} = e_i, i \neq k \end{cases}$

$hgh^{-1} \in \iota_k[G_k]$

$\therefore \iota_k[G_k] \lhd \prod_{i \in I} G_i$