Equality and Operators

Definition

To say that two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal $(z_1 = z_2)$ means that $x_1 = x_2$ and $y_1 = y_2$.

Definition

The following two binary operators are defined on $\mathbb{C}$:

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

Note that these operators are closed and well-defined because each component in the reals is closed and well-defined.

Every complex number z can be expressed as follows:

$$z = (x, y) = (x, 0) + (0, 1)(0, y) = x + iy$$

where $i^2 = (0, 1)(0, 1) = (-1, 0) = -1$, or $i = \sqrt{-1}$.

Note that the powers of i cycle every four:

$$\begin{aligned} i^0 &= 1 \\ i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \end{aligned}$$

We can now redefine addition and multiplication as follows:

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned}$$

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + ix_1 y_2 + ix_2 y_1 + i^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

Theorem

Let $z = x + iy$.

1). $Re(iz) = -Im(z)$

2). $Im(iz) = Re(z)$

Proof

$iz = i(x + iy) = ix + i^2y = -y + ix$
$Re(iz) = -y = -Im(z)$
$Im(iz) = x = Re(z)$