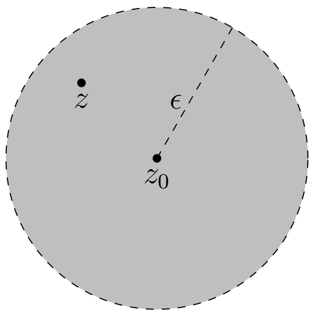
Regions

Definition

A *neighborhood* of $z_0 \in \mathbb{C}$ is the locus given by:

$$|z - z_0| < \epsilon$$

For some $\epsilon > 0$.

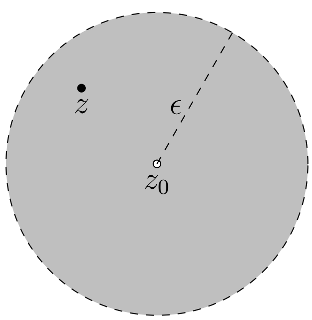


Definition

A *deleted neighborhood* of $z_0 \in \mathbb{C}$ is the locus given by:

$$0 < |z - z_0| < \epsilon$$

For some $\epsilon > 0$.

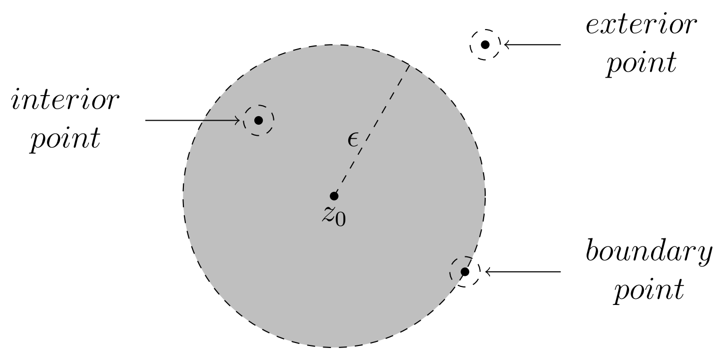


Definition

Let $S \subseteq \mathbb{C}$:

- To say that $z_0 \in \mathbb{C}$ is an *interior point* of S means that there exists a neighborhood N of z_0 such that $N \subseteq S$.
- To say that $z_0 \in \mathbb{C}$ is an *exterior point* of S means that there exists a neighborhood N of z_0 such that $N \cap S = \emptyset$.
- To say that $z_0 \in \mathbb{C}$ is a *boundary point* of S means that for all neighborhoods N of z_0, N contains both interior and exterior points:

  $N \cap S \neq \emptyset$ and $N - S \neq \emptyset$



Given a set $S \subseteq \mathbb{C}$, a point $z_0 \in \mathbb{C}$ is either an interior, exterior, or boundary point for S.

Definition

Let $S \subseteq \mathbb{C}$. The *boundary* of S is the set:

$$B = \{b \in \mathbb{C} \mid b \text{ is a boundary point for } S\}$$

Definition

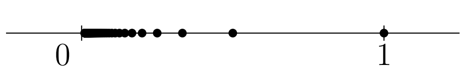
Let $S \subseteq \mathbb{C}$ with boundary B:

- To say that S is *open* means $S \cap B = \emptyset$.
- To say that S is *closed* means $B \subseteq S$.
- To say that S is *clopen* means that S is both open and closed.
- The *closure* of S is given by:

$$\bar{S} = S \cup B$$

Example

$$S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$$



S contains no interior points; they are all boundary points.

Example

$\mathbb{C}$ is clopen, since $B = \emptyset$:

$$\mathbb{C} \cap B = \mathbb{C} \cap \emptyset = \emptyset$$

$$B = \emptyset \subset \mathbb{C}$$

Example

The punctured disk $0 < |z - z_0| \leq \epsilon$ is neither open nor closed; it contains the boundary points on the circle; however, it does not include the boundary point at the excluded center.

Definition

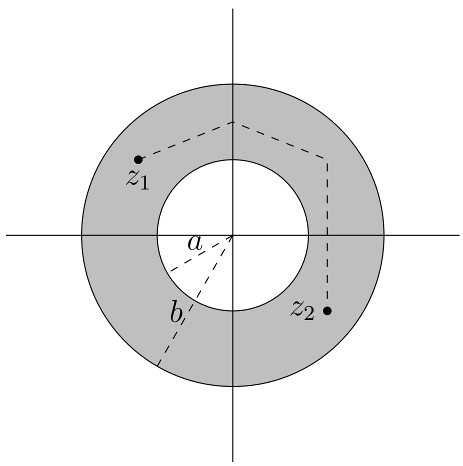
To say that a set $S \subseteq \mathbb{C}$ is connected means:

1). S is open.

2). $\forall\, z_1, z_2 \in S$, there exists a path that connects the two points consisting of a finite number of line segments $L = \bigcup_{k=1}^{n} \ell_k$ such that $L \subseteq S$.

Example

Consider the (open) annulus $a \leq |z - z_0| \leq b$:



Definition

Let $S \subseteq C$. To say that S is a *domain* means:

1). $S \neq \emptyset$

2). S is open

3). S is connected

A set consisting of a domain and zero or more of its boundary points is called a *region*.

Definition

Let $S \subseteq \mathbb{C}$. To say that S is bounded means:

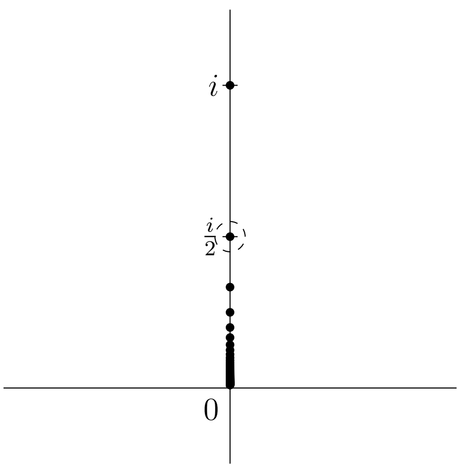
$$\exists\, r \in \mathbb{R} \mid \forall\, z \in S, |z| \leq r$$

Definition

Let $S \subseteq C$. To say that $z_0 \in S$ is an accumulation (limit) point of S means for all deleted neighborhoods N of $z_0, N \cap S \neq \emptyset$.

Example

$$S = \left\{ \frac{i}{n} \mid n \in \mathbb{N} \right\}$$



$\frac{i}{2}$ is not a limit point for S. In fact, the only limit point for S is 0.