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Math 161A
Homework #4

3.48

NBC News reported on May 2, 2013 that 1 in 20 children in the United States have a food allergy of some sort. Consider selecting a random sample of 25 children and let X be the number in the sample who have a food allergy. Then $X \sim B(25, 0.05)$.

a) Determine both $P(X \leq 3$ and $P(X < 3)$.

From the table:

$$P(X \leq 3) = 0.966$$
$$P(X < 3) = P(X \leq 2) = 0.873$$

By hand:

$$P(X = 0) = \binom{25}{0}(0.05)^0(0.95)^{25} = 0.277$$
$$P(X = 1) = \binom{25}{1}(0.05)^1(0.95)^{24} = 0.365$$
$$P(X = 2) = \binom{25}{2}(0.05)^2(0.95)^{23} = 0.231$$
$$P(X = 3) = \binom{25}{3}(0.05)^3(0.95)^{22} = 0.093$$

$$P(X < 3) = 0.277 + 0.365 + 0.231 = 0.873$$
$$P(X \leq 3) = 0.873 + 0.093 = 0.966$$

b) Determine $P(X \geq 4)$.

$$P(X \geq 4) = 1 - P(X \leq 3) = 1 - 0.966 = 0.034$$

c) Determine $P(1 \leq X \leq 3)$.

$$P(1 \leq X \leq 3) = P(X \leq 3) - P(X = 0) = 0.966 - 0.277 = 0.693$$

d) What are $E(X)$ and σ_X?

$$E(X) = np = 25 \cdot 0.05 = 1.25$$
$$V(X) = np(1 - p) = 25 \cdot 0.05 \cdot 0.95 = 1.1875$$
$$\sigma_X = \sqrt{1.1875} \approx 1.090$$

e) In a sample of 50 children, what is the probability that none has a food allergy?

$$X \sim \mathrm{B}(50, 0.05)$$

$$P(X = 0) = 0.95^{50} \approx 0.077$$

3.60

A toll bridge charges \$1.00 for passenger cars and \$2.50 for other vehicles. Suppose that during the daytime hours, 60% of all vehicles are passenger cars. If 25 vehicles cross the bridge during a particular daytime period, what is the resulting expected toll revenue? [Hint: Let X = the number of passenger cars; then the toll revenue $h(X)$ is a linear function of X.]

$$X \sim \mathrm{B}(25, 0.6)$$

$$h(X) = 1.00X + 2.50(25 - X) = X + 62.5 - 2.5X = 62.5 - 1.5X$$

$$\begin{aligned} E(h(X)) &= E(62.5 - 1.5X) \\ &= 62.5 - 1.5E(X) \\ &= 62.5 - 1.5np \\ &= 62.5 - 1.5(25)(0.6) \\ &= \$40.00 \end{aligned}$$

3.66

An airport limousine can accommodate up to four passengers on any one trip. The company will accept a maximum of six reservations for a trip, and a passenger must have a reservation. From previous records, 20% of all those making reservations do not appear for the trip. Answer the following questions, assuming independence wherever appropriate.

a) If six reservations are made, what is the probability that at least one individual with a reservation cannot be accommodated on the trip?

Let X = the number of passengers who do not show up. Then $X \sim \mathrm{B}(6, 0.2)$.

$$P(X = 0) = \binom{6}{0} 0.2^0 0.8^{6-0} = 0.8^6 = 0.262$$
$$P(X = 1) = \binom{6}{1} 0.2^1 0.8^{6-1} = 6 \cdot 0.2 \cdot 0.8^5 = 0.393$$

$$P(X < 2) = 0.262 + 0.393 = 0.655$$

b) If six reservations are made, what is the expected number of available places when the limousine departs?

$$E(X) = np = 6 \cdot 0.2 = 1.2$$

On average, 1.2 people do not show up. Thus, the limousine is always full (i.e., no available places).

3.68

Eighteen individuals are scheduled to take a driving test at a particular DMV office on a certain day, eight of whom will be taking the test for the first time. Suppose that six of these individuals are randomly assigned to a particular examiner, and let X be the number among the six who are taking the test for the first time.

a) What kind of distribution does X have (name and values of all parameters)?

$$X \sim \text{HyperGeom}(18, 8, 6)$$

b) Compute $P(X = 2)$, $P(X \leq 2)$, and $P(X \geq 2)$.

$$P(X = x) = \frac{\binom{8}{x}\binom{10}{6-x}}{\binom{18}{6}}$$

$$P(X = 0) = \frac{\binom{8}{0}\binom{10}{6}}{\binom{18}{6}} = \frac{1 \cdot 210}{18564} = 0.0113$$

$$P(X = 1) = \frac{\binom{8}{1}\binom{10}{5}}{\binom{18}{6}} = \frac{8 \cdot 252}{18564} = 0.1086$$

$$P(X = 2) = \frac{\binom{8}{2}\binom{10}{4}}{\binom{18}{6}} = \frac{28 \cdot 210}{18564} = 0.3167$$

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 0.0113 + 0.1086 + 0.3167 = 0.4366$$

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - 0.0113 - 0.1086 = 0.8801$$

3.75

The probability that a randomly selected box of a certain type of cereal has a particular prize is 0.2. Suppose you purchase box after box until you have obtained two of these prizes.

Let X = number of boxes purchased:

$$X \sim \text{NB}(0.2, 2)$$

a) What is the probability that you purchase x boxes that do not contain the desired prize?

I am interpreting this question as meaning that $x+2$ total boxes are purchased.

$$P(X = x+2) = \binom{(x+2)-1}{2-1}(0.2)^2(1-0.2)^{(x+2)-2} = \binom{x+1}{1}(0.04)(0.8)^x = 0.04(x+1)(0.8)^x$$

b) What is the probability that you purchase four boxes?

From part (a) with $x = 2$:

$$P(X = 4) = 0.04(2+1)(0.8)^2 = 0.04(3)(0.64) = 0.0768$$

c) What is the probability that you purchase at most four boxes?

$$P(X = 2) = 0.04(0+1)(0.8)^0 = 0.04(1)(1) = 0.04$$
$$P(X = 3) = 0.04(1+1)(0.8)^1 = 0.04(2)(0.8) = 0.064$$

$$P(X \leq 4) = P(X = 2) + P(X = 3) + P(X = 4) = 0.04 + 0.064 + 0.0768 = 0.1808$$

d) How many boxes without the desired prize do you expect to purchase? How many boxes do you expect to purchase?

$$E(X) = \frac{r}{p} = \frac{2}{0.2} = 10$$

Therefore, we would expect to purchase a total of 10 boxes, 8 of which do not have the desired prize.

3.76

A family decides to have children until it has three children of the same gender. Assuming $P(B) = P(G) = 0.5$, what is the pmf of $X =$ the number of children in the family.

The possibilities are as follows:

BOYS	GIRLS	TOTAL
3	0	3
3	1	4
3	2	5
0	3	3
1	3	4
2	3	5

Thus, the range for X is $\{3, 4, 5\}$ and each value appears twice: once for boys and once for girls. Since each gender is $\sim \mathrm{NB}(0.5, 3)$:

$$f_X(x) = 2\binom{x-1}{3-1}(0.5)^3(1-0.5)^{x-3} = 2\binom{x-1}{2}(0.5)^3(0.5)^{x-3} = 2\binom{x-1}{2}(0.5)^x$$

$$f_X(3) = 2\binom{3-1}{2}(0.5)^3 = 2\binom{2}{2}(0.5)^3 = 2(1)(0.125) = 0.250$$
$$f_X(4) = 2\binom{4-1}{2}(0.5)^4 = 2\binom{3}{2}(0.5)^4 = 2(3)(0.0625) = 0.375$$
$$f_X(5) = 2\binom{5-1}{2}(0.5)^5 = 2\binom{4}{2}(0.5)^5 = 2(6)(0.03125) = 0.375$$

And therefore:

$$f_X(x) = \begin{cases} 0.250 & x = 3 \\ 0.375 & x = 4 \\ 0.375 & x = 5 \\ 0 & \text{otherwise} \end{cases}$$

As expected, the total probability $= 1$.

3.79

The article *"Expectation Analysis of the Probability of Failure for Water Supply Pipes"* (J. of Pipeline Systems Engr. and Practice, May 2012: 36–46) proposed using the Poisson distribution to model the number of failures in pipelines of various types. Suppose that for cast-iron pipe of a particular length, the expected number of failures is 1 (very close to one of the cases considered in the article). Then X, the number of failures, has a Poisson distribution with $\mu = 1$.

a) Obtain $P(X \leq 5$ by using Appendix Table A.2.

$$P(X \leq 5) = 0.999$$

b) Determine P(X=2) first from the pmf formula and then from Appendix Table A.2.

$$P(X = 2) = \frac{1^2}{2!}e^{-1} = \frac{1}{2e} = 0.184$$

$$P(X = 2) = P(X \leq 2) - P(X \leq 1) = 0.920 - 0.736 = 0.184$$

c) Determine $P(2 \leq X \leq 4)$.

$$P(2 \leq X \leq 4) = P(X \leq 4) - P(X \leq 1) = 0.996 - 0.736 = 0.260 \qquad \text{(from table)}$$

d) What is the probability that X exceeds its mean value by more than one standard deviation?

   Since $\mu = 1$ it is the case that $E(X) = V(X) = 1$ and $\sigma_X = \sqrt{1} = 1$. So, we want (from the table):

   $$P(X > 2) = 1 - P(X \leq 2) = 1 - 0.920 = 0.080$$

3.83

An article in the *Los Angeles Times* (Dec. 3, 1993) reports that 1 in 200 people carry the defective gene that causes inherited colon cancer. In a sample of 1000 individuals, what is the approximate distribution of the number who carry this gene?

Let $X =$ the number of people who carry the gene.

$$X \sim \text{Pois}(\mu = 1000 \cdot 0.005) = \text{Pois}(5)$$

Use this distribution to calculate the approximate probability that:

a) Between 5 and 8 (inclusive) carry the gene.

   From the table:

   $$P(5 \leq X \leq 8) = P(X \leq 8) - P(X \leq 4) = 0.932 - 0.440 = 0.492$$

b) At least 8 carry the gene.

   From the table:

   $$P(X \geq 8) = 1 - P(X \leq 7) = 1 - 0.867 = 0.133$$

3.84

The *Centers for Disease Control and Prevention* reported in 2012 that 1 in 88 American children had been diagnosed with an autism spectrum disorder (ASD).

a) If a random sample of 200 American children is selected, what are the expected value and standard deviation of the number who have been diagnosed with ASD?

   $$\mu = 200 \cdot \frac{1}{88} \approx 2.27$$

   $$E(X) = \mu = 2.27$$

   $$V(X) = \mu = 2.27$$
   $$\sigma_X = \sqrt{2.27} \approx 1.51$$

b) Referring back to (a), calculate the approximate probability that at least 2 children in the sample have been diagnosed with ASD.

$$P(X=0) = \frac{2.27^0}{0!}e^{-2.27} = e^{-2.27} = 0.1033$$
$$P(X=1) = \frac{2.27^1}{1!}e^{-2.27} = 2.27e^{-2.27} = 0.2345$$
$$P(X=2) = \frac{2.27^2}{2!}e^{-2.27} = \frac{2.27^2}{2}e^{-2.27} = 0.2662$$

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2) = 0.1033 + 0.2345 + 0.2662 = 0.6040$$

c) If the sample size is 352, what is the approximate probability that fewer than 5 of the selected children have been diagnosed with ASD?

$$\mu = 352 \cdot \frac{1}{88} = 4$$

From the table:

$$P(X \leq 4) = 0.440$$