Random variables and their distributions

– Math 161a, Spring 2019, San Jose State University

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Section 3.1 Random variables

Section 3.2 Probability distributions for discrete random variables

i-Clicker activity 3 (extra credit)

Which of the following statements about a random variable is incorrect?

A. It is a numerical description of outcomes

B. It is a rule to assign numbers to outcomes

C. It is a function whose domain is the sample space

D. The set of all the values the random variable can take is called its range

E. Its value must be nonnegative

Introduction

Consider the following experiments:

- Flip a coin once;
- Flip a coin 5 times;
- Toss two dice;
- Select four numbers from 1:20, without replacement;
- Toss a coin repeatedly until a head first appears.

What are the outcomes of each experiment?

Some likely outcomes of each experiment:

- Flip a coin once; $\longrightarrow$ H, T
- Flip a coin 5 times; $\longrightarrow$ HHTTH, HHHHT
- Toss two dice; $\longrightarrow$ (3,1), (5,5), (2,6)
- Select four numbers from 1:20, without replacement; $\xrightarrow{\text{unordered}}$ $\{6,9,17,2\}$, $\{20,7,12,16\}$
- Toss a coin repeatedly until a head first appears. $\longrightarrow$ H, TTH, TTTTTTH

It is often desirable to convert the outcomes to numbers in some way.

Informally, a random variable is a **numerical description** of the outcomes.

For example,

- Flip a coin once; $\longrightarrow X = 1$ (H), 0 (T)
- Flip a coin 5 times; $\longrightarrow X = \#$heads, $Y = \#$tails
- Toss two dice; $\longrightarrow X =$ sum, $Y =$ absolute value of difference
- Select four numbers from 1:20 at random, without replacement; $\longrightarrow$ $X =$ maximum of the 4 numbers
- Toss a coin repeatedly until a head first appears. $\longrightarrow X =$ total $\#$trials needed, $Y = \#$tails before the first head

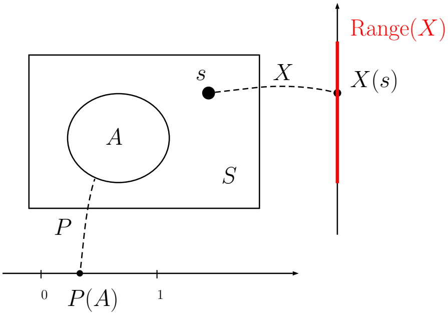
Definition of random variables

Def 0.1. A **random variable (r.v.)** associated to a sample space S is a **rule** that assigns a real number to each outcome of S.

$$X : S \mapsto \mathbb{R}.$$

Def 0.2. The set of all possible values of X is called its **range**:

$$\mathrm{Range}(X) = \{X(s) \mid s \in S\}.$$



Ex 0.1. Find the range of the following random variables.

- Flip a coin once; $\longrightarrow$ $X=1$ (H), 0 (T)
- Flip a coin 5 times; $\longrightarrow$ $X=\#$heads
- Toss two dice; $\longrightarrow$ $X=$ sum, $Y=$ absolute value of difference
- Select four numbers from 1:20 at random, without replacement; $\longrightarrow$ $X=$ maximum of the 4 numbers
- Toss a coin repeatedly until a head first appears. $\longrightarrow$ $X=$ total $\#$trials needed, $Y=\#$tails before the first head

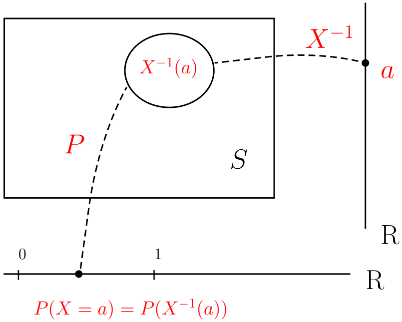
Answers:

- $\{0,1\}$
- $\{0,1,2,3,4,5\}$
- Range$(X) = \{2,3,\ldots,12\}$, Range$(Y) = \{0,1,\ldots,5\}$
- $\{4,5,\ldots,20\}$
- Range$(X) = \{1,2,3,\ldots\}$, Range$(Y) = \{0,1,2,\ldots\}$

Preimages of a random variable are events

Def 0.3. Let $X : S \mapsto \mathbb{R}$. For any $a \in \mathbb{R}$, its preimage is defined as

$$X^{-1}(a) = \{s \in S \mid X(s) = a\}$$



Remarks.

- $X^{-1}(a) \subseteq S$, thus an event.
- We define the probability that $X = a$ as

$$P(X = a) = P(X^{-1}(a)).$$

Ex 0.2. Determine the following events:

- Flip a coin once; define $X = 1$ (H), 0 (T). $X^{-1}(1)$
- Toss two dice; define $X =$ sum. $X^{-1}(7)$
- Select four numbers from 1:20 at random, without replacement; define $X =$ maximum of the 4 numbers. $X^{-1}(3),\ X^{-1}(5)$
- Toss a coin repeatedly until a head first appears; define $X =$ total #trials needed. $X^{-1}(3)$

Ex 0.3. Find the following probabilities:

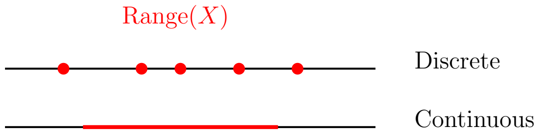
- Flip a fair coin once; define $X = 1$ (H), 0 (T). $P(X = 1)$=?
- Toss two fair dice; define $X =$ sum. $P(X = 7)$=?
- Select four numbers from 1:20 at random, without replacement; define $X =$ maximum of the 4 numbers. $P(X = 3) =?$, $P(X = 5) =?$
- Toss a fair coin repeatedly until a head first appears; define $X =$ total #trials needed. $P(X = 3)$=?

Ex 0.4. Find the following probabilities:

- Toss two fair dice; define $X =$ sum. $P(X \leq 3)$=?, $P(X \geq 10)$=?
- Select four numbers from 1:20 at random, without replacement; define $X =$ maximum of the 4 numbers. $P(X \leq 5) =$?
- Toss a fair coin repeatedly until a head first appears; define $X =$ total #trials needed. $P(X \leq 3)$=?

Classification of random variables

Def 0.4. A random variable X is said to be **discrete** if it takes only a countable number of possible values, i.e., Range(X) is a finite or countably infinite set. Otherwise, it is said to be **continuous**.



Ex 0.5. All the random variables we have seen so far are discrete.

Remark. Chapter 3 is about discrete random variables.

Ex 0.6. Below are some examples of continuous random variables:

- Waiting time for your bus to come,
- Life time of electronic products
- A randomly selected SJSU student's height/weight/temperature
- Throwing a dart toward a board. Let X be the distance to the center, and Y the angle relative to the positive x-axis

Remark. Chapter 4 is about continuous random variables.

Distribution of random variables

Informally speaking, the **probability distribution** of a random variable X refers to both

- the set of values it can take (range), and
- how often it takes those values (frequency).

The distribution of a discrete random variable can be fully characterized by a **probability mass function (pmf)**.

Def 0.5. Let X be a discrete random variable with range $\{x_1, x_2, \ldots\}$. The *probability mass function (pmf)* of X, denoted $f_X : \mathbb{R} \to \mathbb{R}$, is defined as

$$f_X(x) = \begin{cases} P(X = x_i), & \text{if } x = x_i, \text{ for } i = 1, 2, \ldots \\ 0, & \text{for all other } x. \end{cases}$$

For example, let X be the numerical outcome of a single toss of a fair die (0 for tails and 1 for heads). Then its pmf is

$$f_X(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 0 \\ \frac{1}{2}, & \text{if } x = 1 \\ 0, & \text{for all other } x. \end{cases}$$

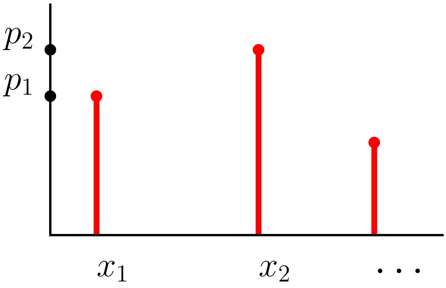
Displaying pmf

We may display the distribution of a discrete random variable using either a table or a plot consisting of spikes (line graph).

x	x_1	x_2	$\cdots$
$P(X=x)$	p_1	p_2	$\cdots$

(Notation: $p_i = f_X(x_i)$ for all i)

Important reminder:
f_X is defined everywhere on $\mathbb{R}$ (it takes the value 0 at locations not indicated in the table or plot).



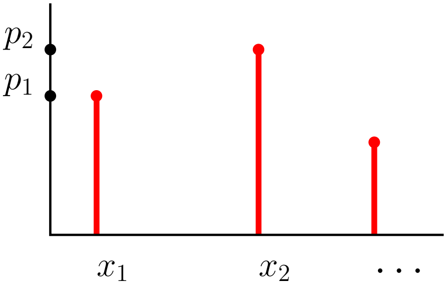
Find the pmf of X in each question below and display it in both ways.

Ex 0.7 (Flip a fair coin once)**.** Let $X = 1$ (heads), and 0 (tails).

Ex 0.8 (Roll two fair dice)**.** Let $X=$ their sum.

Properties of a pmf f_X:

- It is nonnegative on $\mathbb{R}$: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$
- It is positive (i.e., $f_X(x) > 0$) only in a countable number of locations, say $x_1, x_2, \ldots$
- The total sum is 1: $\sum_i f_X(x_i) = 1$.



Conversely, any function satisfying all 3 conditions above is a pmf.

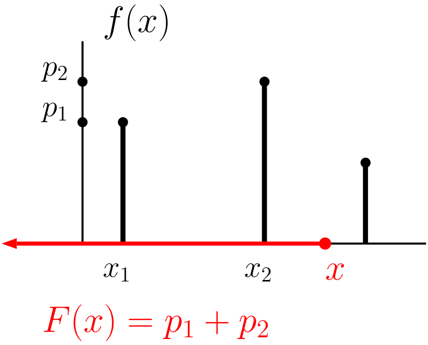
Cumulative distribution function (cdf)

A different way of characterizing the distribution of a random variable is through specifying all the cumulative probabilities.

Def 0.6. The *cdf* of a r.v. X, denoted $F_X : \mathbb{R} \to \mathbb{R}$, is defined by

$$F_X(x) = P(X \leq x), \quad \forall x \in \mathbb{R}.$$

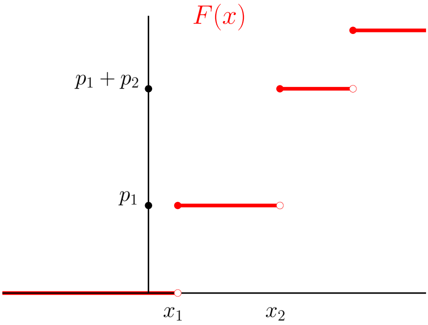
Remark. The cdf is also defined everywhere on $\mathbb{R}$.



The cdf can also be displayed as a table or graph.

cdf table

x	x_1	x_2	$\cdots$
$P(X \leq x)$	p_1	$p_1 + p_2$	$\cdots$



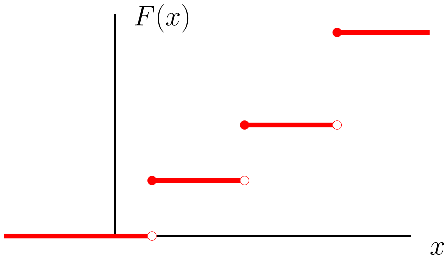
Note that the value of the cdf between two neighboring points is not zero, but determined by the left neighbor.

Ex 0.9. Find the cdf in the last two examples.

Properties of a cdf $F(x)$:

- $\lim_{x\to-\infty} F(x) = 0$,
  $\lim_{x\to\infty} F(x) = 1$.
- $F(x)$ is nondecreasing.
- $F(x)$ is right-continuous.

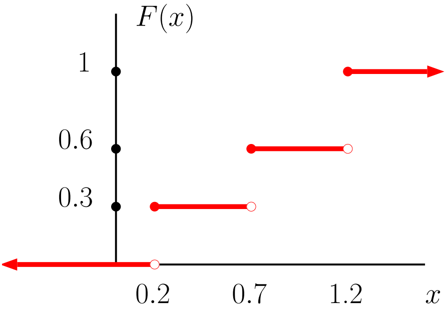
The converse is also true.

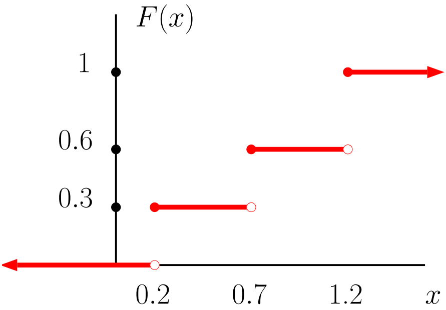


The cdf of a discrete random variable X is a step function.

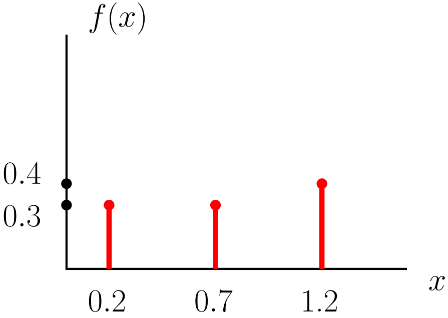
In next chapter we will see that the cdf of a continuous X is a continuous curve (satisfying the three conditions above).

Ex 0.10. Find the pmf corresponding to the cdf given below.





$F(x)$
1
0.6
0.3
0.2
0.7
1.2
x



$f(x)$
0.4
0.3
0.2
0.7
1.2
x

Remark. More properties of cdf:

- $F_X(x) = \sum_{i:\, x_i \le x} f_X(x_i), \quad x \in \mathbb{R}$
- $P(X > x) = 1 - F_X(x)$
- $P(x < X \le y) = F_X(y) - F_X(x)$.

Ex 0.11. For the pmf on the previous slide, find

- $P(X < 0.2), P(X \le 0.2), P(X > 0.2), P(X \ge 0.2)$
- $P(X \le 1), P(X < 1)$
- $P(0.2 < X \le 1.2)$

Ex 0.12 (Select four numbers from 1:20 at random, with replacement)**.** Let $X =$ the maximum. Find the cdf of X and then use it to find the pmf.