Basics of Probability Theory

– Math 161a, Spring 2019, San José State University

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Outline

Section 2.1 Sample Spaces and Events

Section 2.2 Axioms, Interpretations, and Properties of Probability

Introduction

To study a random phenomenon (such as flipping a coin, rolling a die), we need to define the following basic concepts:

- **Sample space**
- **Events**
- **Probability**

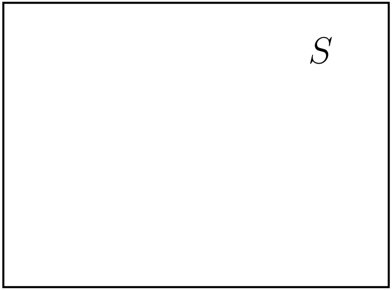
We'll go through them one by one.

Sample space

Definition 0.1. The **set of all possible outcomes** of a random phenomenon is called the sample space for that experiment.

Notation and diagram:

- We often denote a sample space by S (or sometimes Ω).
- We illustrate a sample space by using a rectangle.



Example 0.1. Write down the sample space of each of the following experiments:

- Tossing a coin: $S = \{H,\ T\}$.
- Rolling a die: $S = \{1, 2, 3, 4, 5, 6\}$.
- Drawing a card from an ordinary deck of 52: $S = \{$All 52 cards$\}$.

Example 0.2. Write down the sample space of each of the following experiments:

- Throw a coin twice. The sample space is

$$S = \{(H,H),(H,T),(T,H),(T,T)\}.$$

- Roll two dice:

$$\begin{aligned} S &= \{(1,1),(1,2),\ldots,(1,6),(2,1),\ldots,(6,6)\} \longleftarrow \text{by enumeration} \\ &= \{(i,j) : 1 \le i \le 6, 1 \le j \le 6\} \qquad \longleftarrow \text{by formula} \end{aligned}$$

- Throw a coin repeatedly until a head first appears:

$$S = \{H,\ TH,\ TTH,\ TTTH, \ldots\}$$

The sample spaces in the previous example are countable sets (i.e., sets with finite or countably infinite number of objects).

In the following example, the sample spaces are continuous intervals.

Example 0.3.

- Life time of a new light bulb. The sample space is an interval $S = (0, \infty)$.
- Waiting time (in minutes) to talk to a customer service representative: $S = (0, 60)$

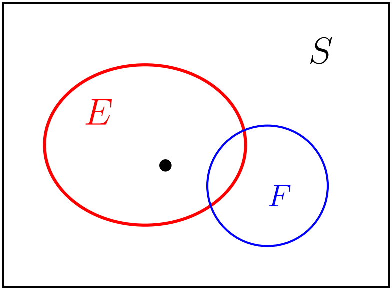
Events

Consider the following probability questions about certain events:

- (**Toss two fair dice**) What is the probability of getting a sum of 8?
- (**Toss two fair dice**) What is the probability of getting two even numbers?
- (**Toss two fair dice**) What is the probability of getting two identical numbers?
- (**Toss a fair coin repeatedly until a head first appears**) What is the probability that at most 3 tails are observed?

Definition 0.2. Mathematically, an event is just a subset E of outcomes in the sample space S.

- In particular, $S, \emptyset$ are events.
- We say an event E **occurs** if the actual outcome of the experiment lies in E.
- We often only consider events whose outcomes have a common characteristic.



Example 0.4 (Roll a single die). The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. The following are events:

- $A = \{1\}$ $\longleftarrow$ simple event
- $B = \{6\}$ $\longleftarrow$ simple event
- $C = \{2, 4, 6\} = \{\text{An even number}\}$ $\longleftarrow$ compound event
- $D = \{1, 3, 5\} = \{\text{An odd number}\}$ $\longleftarrow$ compound event

If an outcome of 1 was observed when performing the experiment, then which events occurred (and which events did not occur)?

Example 0.5 (Throw two dice)**.** The sample space is $S = \{(i,j) \mid 1 \le i,j \le 6\}$. The following are events:

$$\begin{aligned} A &= \{\text{Sum equals } 6\} \\ &= \{(1,5),(2,4),(3,3),(4,2),(5,1)\} \end{aligned}$$

$$\begin{aligned} B &= \{\text{Two identical numbers}\} \\ &= \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\} \end{aligned}$$

$$\begin{aligned} C &= \{\text{Both even}\} \\ &= \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}. \end{aligned}$$

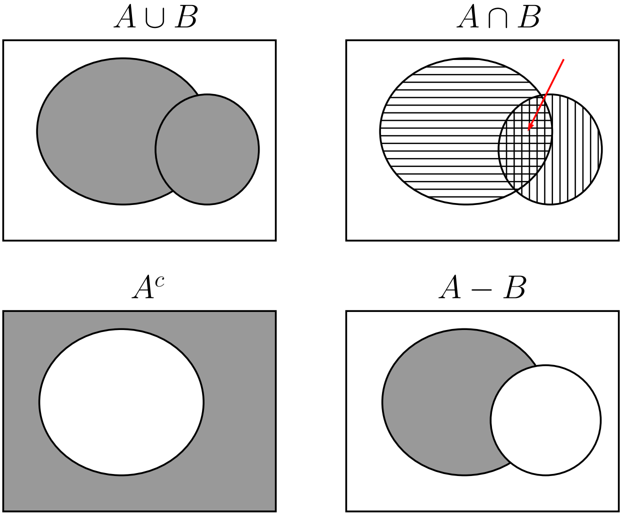
Example 0.6. Consider the experiment where you repeatedly toss a coin until you see the first head. The following is an event:

$$E = \{\text{At most 4 tails occurred}\} = \{\text{H, TH, TTH, TTTH, TTTTH}\}.$$

Event operations

Definition 0.3. Let $A, B \subseteq S$ be two events. We define

- **Set size** $|A|$: $\#$ outcomes in A
- **Complement** A^c: set of all outcomes not in A
- **Union** $A \cup B$: set of all outcomes in A or B (or both)
- **Intersection** $A \cap B$: set of all outcomes in both A and B
- **Difference** $A - B = A \cap B^c$: set of all outcomes in A and not in B



$\longleftarrow$ Venn Diagram

Example 0.7 (Throw two dice)**.** Let

- $A = \{\text{Sum equals 6}\}$
- $B = \{\text{Two identical numbers}\}$
- $C = \{\text{Both even}\}$

Compute $|C|, A \cap B, A \cup B, B^c, A - C$

Proposition 0.1. *Two useful set laws.*

- *Distributive law:*

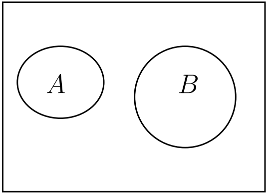
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- *De Morgan's Laws*

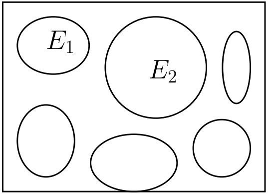
$$(A \cup B)^c = A^c \cap B^c,$$
$$(A \cap B)^c = A^c \cup B^c$$

Disjoint events

Definition 0.4. Two events A, B are said to be **disjoint**, or **mutually exclusive**, if their intersection is empty: $A \cap B = \emptyset$.



A sequence of events $E_1, E_2, \ldots$ are said to be **pairwise disjoint** (or **mutually exclusive**) if $E_i \cap E_j = \emptyset$ for all $i \neq j$.



Example 0.8 (Toss two fair dice)**.** Are the following two events disjoint?

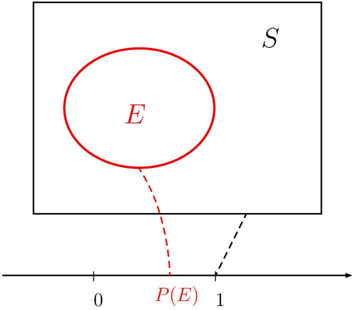
- $A = \{\text{Sum equals 7}\}$.
- $B = \{\text{Two identical numbers}\}$.

Probability

Definition 0.5. Probability is a function defined on the space of events that satisfies the following Axioms of Probability:

1. $P(E) \geq 0$ for any $E \subseteq S$.
2. $P(S) = 1$.
3. If an infinite sequence of events $E_1, E_2, \ldots$ are pairwise disjoint, then

$$P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i).$$



Theorem 0.2. *The Axioms of Probability imply* that*

- $P(\emptyset) = 0$.
- *If* $E_1, E_2, \ldots, E_k$ *are pairwise disjoint, then* $P(\cup_{i=1}^k E_i) = \sum_{i=1}^k P(E_i)$
- $P(E^c) = 1 - P(E)$. *This implies that* $P(E) \leq 1$.
- *If* $A \subseteq B$, *then* $P(A) \leq P(B)$.

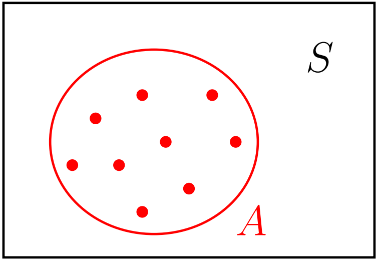
(*This is why we did not include these properties in the definition of probability)

Countable sample space

The following property implies that, to define the probability function P over a countable sample space, it suffices to specify only the probabilities of simple events.

Theorem 0.3. *If S contains at most a countable number of outcomes, then for any $A \subseteq S$,*

$$P(A) = \sum_{a \in A} P(\{a\}).$$



Example 0.9 (Fair coin model). Let $S = \{H, T\}$ with $P(\{H\}) = P(\{T\}) = .5$.

Example 0.10 (Biased coin model). Let $S = \{H, T\}$ with $P(\{H\}) = .55, P(\{T\}) = .45$.

Example 0.11 (Fair die model). Let $S = \{1, 2, \ldots, 6\}$ with $P(\{1\}) = P(\{2\}) = \cdots = P(\{6\}) = \frac{1}{6}$. The probability of getting an even number is

$$P(\{\text{An even number}\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$

i-Clicker activity 0 (no points)

How are you doing so far?

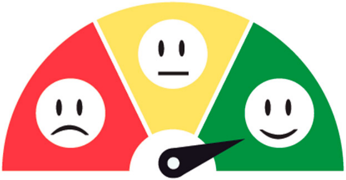
(A) Great!

(B) Still adjusting, but quite good

(C) Already having some difficulty

(D) Too early to say

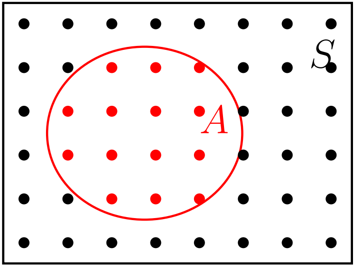
(E) Don't know



Finite sample space with equally likely outcomes

Theorem 0.4. *If $|S| < \infty$ (i.e., S is a finite set) and all the outcomes are equally likely to occur, then for any event $A \subseteq S$,*

$$P(A) = \frac{|A|}{|S|} = \frac{\#\ outcomes\ in\ A}{\#\ outcomes\ in\ S}.$$



Joke: What is a probability to meet a dinosaur?

A: What is a probability to meet a dinosaur on the street?

B: Well, 50x50!

A: How, why???

B: You either meet it or not!

So, i met it!

Example 0.12 (Throw a fair die). Find the following probabilities:

$$P(\{\text{An even number}\}) =$$

$$P(\{\text{At least 5}\}) =$$

$$P(\{\text{Not a 3}\}) =$$

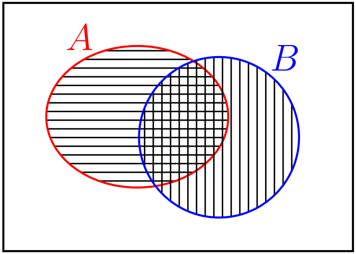
Example 0.13 (Toss a fair coin 5 times). What is the probability of getting at least one head?

Inclusive-exclusive formula (2 events)

Theorem 0.5. *For any events A, B,*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

In particular, if $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.



Example 0.14. In a large discrete math class, 55% of the students are math majors, 35% of the class are CS majors, and 5% are dual majors (in math and CS). What percentage of the class majors in neither of them?

Inclusive-exclusive formula (3 events)

Theorem 0.6. *For any three events $A, B, C \subseteq S$, we have*

$$P(A \cup B \cup C)$$
$$= P(A) + P(B) + P(C)$$
$$- P(A \cap B) - P(A \cap C) - P(B \cap C)$$
$$+ P(A \cap B \cap C).$$