Inner Product

Definition: Inner Product

Let V be a vector space over a field F. To say that a function $\langle\cdot,\cdot\rangle : V \times V \to F$ is an *inner product* on V means that it satisfies the following five properties $\forall\, \vec{x}, \vec{y} \in V$ and $\forall\, c \in F$:

1). Nonnegativity: $\langle \vec{x}, \vec{x} \rangle \geq 0$

2). Positivity: $\langle \vec{x}, \vec{x} \rangle = 0 \iff \vec{x} = \vec{0}$

3). Additivity: $\langle \vec{x} + \vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$

4). Homogeneity: $\langle c\vec{x}, \vec{y} \rangle = c \langle \vec{x}, \vec{y} \rangle$

5). Hermitian: $\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle}$

Properties: Inner Product

1). $\langle \vec{x}, c\vec{y} \rangle = \overline{c} \langle \vec{x}, \vec{y} \rangle$

2). $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

3). $\overline{\langle \vec{x}, \vec{x} \rangle} = \langle \vec{x}, \vec{x} \rangle$

Proof

1). $\langle \vec{x}, c\vec{y} \rangle = \overline{\langle c\vec{y}, \vec{x} \rangle} = \overline{c \langle \vec{y}, \vec{x} \rangle} = \overline{c}\overline{\langle \vec{y}, \vec{x} \rangle} = \overline{c} \langle \vec{x}, \vec{y} \rangle$

2). $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \overline{\langle \vec{y} + \vec{z}, \vec{x} \rangle} = \overline{\langle \vec{y}, \vec{x} \rangle + \langle \vec{z}, \vec{x} \rangle} = \overline{\langle \vec{y}, \vec{x} \rangle} + \overline{\langle \vec{z}, \vec{x} \rangle} = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

3). Trivial

Theorem: Cauchy-Schwarz

Let $\vec{x}, \vec{y} \in \mathbb{C}^n$:

$$|\langle \vec{x}, \vec{y} \rangle|^2 \leq \langle \vec{x}, \vec{x} \rangle \langle \vec{y}, \vec{y} \rangle$$

Proof

If $\vec{x} = \vec{y} = \vec{0}$ then there is nothing to prove, so AWLOG: $\vec{y} \neq \vec{0}$
Let $\vec{z} = \langle \vec{y}, \vec{y} \rangle \vec{x} - \langle \vec{x}, \vec{y} \rangle \vec{y}$:

$$\begin{aligned}
0 &\leq \langle \vec{z}, \vec{z} \rangle \\
&= \langle \langle \vec{y}, \vec{y} \rangle \vec{x} - \langle \vec{x}, \vec{y} \rangle \vec{y}, \langle \vec{y}, \vec{y} \rangle \vec{x} - \langle \vec{x}, \vec{y} \rangle \vec{y} \rangle \\
&= \langle \vec{y}, \vec{y} \rangle \overline{\langle \vec{y}, \vec{y} \rangle} \langle \vec{x}, \vec{x} \rangle - \langle \vec{y}, \vec{y} \rangle \overline{\langle \vec{x}, \vec{y} \rangle} \langle \vec{x}, \vec{y} \rangle - \langle \vec{x}, \vec{y} \rangle \overline{\langle \vec{y}, \vec{y} \rangle} \langle \vec{y}, \vec{x} \rangle + \langle \vec{x}, \vec{y} \rangle \overline{\langle \vec{x}, \vec{y} \rangle} \langle \vec{y}, \vec{y} \rangle \\
&= \langle \vec{y}, \vec{y} \rangle \langle \vec{y}, \vec{y} \rangle \langle \vec{x}, \vec{x} \rangle - \langle \vec{y}, \vec{y} \rangle \langle \vec{x}, \vec{y} \rangle \overline{\langle \vec{x}, \vec{y} \rangle} - \langle \vec{x}, \vec{y} \rangle \overline{\langle \vec{x}, \vec{y} \rangle} \langle \vec{y}, \vec{y} \rangle + \langle \vec{x}, \vec{y} \rangle \overline{\langle \vec{x}, \vec{y} \rangle} \langle \vec{y}, \vec{y} \rangle \\
&= \langle \vec{y}, \vec{y} \rangle^2 \langle \vec{x}, \vec{x} \rangle - \langle \vec{y}, \vec{y} \rangle |\langle \vec{x}, \vec{y} \rangle|^2 \\
&= \langle \vec{y}, \vec{y} \rangle (\langle \vec{y}, \vec{y} \rangle \langle \vec{x}, \vec{x} \rangle - |\langle \vec{x}, \vec{y} \rangle|^2)
\end{aligned}$$

But, by assumption, $\vec{y} \neq \vec{0}$, and so:

$$\langle \vec{x}, \vec{x} \rangle \langle \vec{y}, \vec{y} \rangle - |\langle \vec{x}, \vec{y} \rangle|^2 \geq 0$$

$\therefore |\langle \vec{x}, \vec{y} \rangle|^2 \leq \langle \vec{x}, \vec{x} \rangle \langle \vec{y}, \vec{y} \rangle$

Example

The standard inner product:

$$\langle \vec{x}, \vec{y} \rangle = \vec{y}^* \vec{x} = \sum_{k=1}^{n} \overline{y_k} x_k$$

Definition: Special Matrices

To say that a matrix $A \in M_n$ is *Hermitian* means $A = A^*$.

To say that a matrix $A \in M_n$ is *positive-definite* means:

1). A is Hermitian

2). $\forall \, \vec{x} \neq 0, \vec{x}^* A \vec{x} > 0$

Theorem

Any inner product on C^n is of the form:

$$\langle \vec{x}, \vec{y} \rangle = \vec{y}^* A \vec{x}$$

for some positive-definite matrix A.

Example

$A = \begin{bmatrix} \pi & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$

Note that A is real-diagonal and thus Hermitian.

Assume $\vec{x} \neq 0$:

$$\vec{x}^* A \vec{x} = \begin{bmatrix} \overline{x_1} & \overline{x_2} & \overline{x_3} \end{bmatrix} \begin{bmatrix} \pi & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \overline{x_1}\pi & \overline{x_2}e & \overline{x_3}\sqrt{2} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\vec{x}^* A \vec{x} = \pi |x_1|^2 + e |x_2|^2 + \sqrt{2} |x_3|^2 \geq 0$$

Thus A is positive-definite.

$\langle \vec{x}, \vec{y} \rangle = \pi \overline{y_1} x_1 + e \overline{y_2} x_2 + \sqrt{2} \overline{y_3} x_3$