# Partite Graphs

**Definition: Partite Set**

Let $G$ be a graph and let $\{S_i : 1 \le i \le k\}$ be a partition of $V(G)$. The $S_i$ are called the *partite sets* of the partition.

**Definition: Independent Set**

Let $G$ be a graph and let $S \subseteq V(G)$. To say that $S$ is an *independent set* means that none of the vertices in $S$ are adjacent:

$$\forall\, u, v \in S, uv \notin E(G)$$

**Definition: Bipartite**

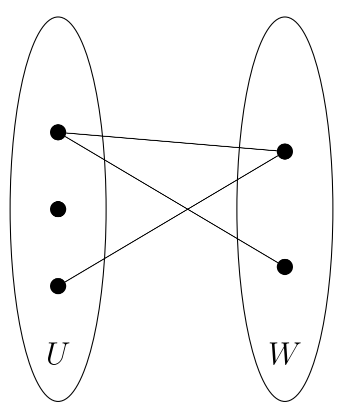
Let $G$ be a graph whose vertices are partitioned into partite sets $U$ and $W$. To say that $G$ is *bipartite*, denoted by $G = B(U, W)$, means that $U$ and $W$ are independent sets:

$$\forall\, uw \in E(G), u \in U \text{ and } w \in W$$

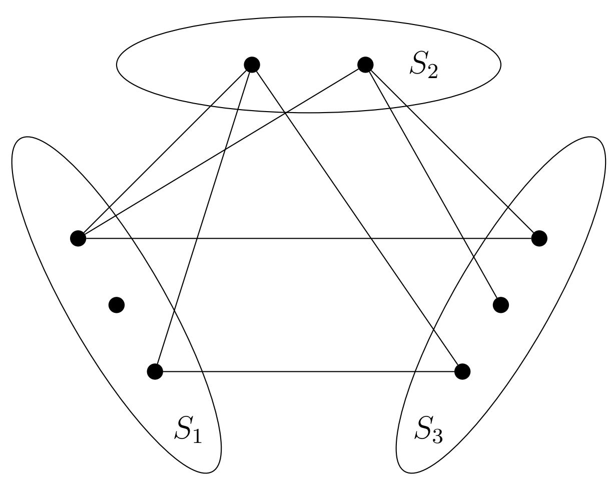
**Definition: $k$-Partite**

Let $G$ be a graph and let $\{S_i : 1 \le i \le k\}$ be a partition of $V(G)$. To say that $G$ is *$k$-partite*, denoted by $G(S_1, S_2, \ldots, S_k)$, means that every induced subgraph $G[S_i \cup S_j], i \neq j$ is bipartite.

**Examples**




BIPARTITE




3-PARTITE

**Definition: Complete Bipartite**

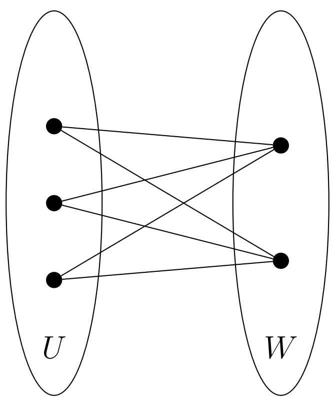
Let $B(U, W)$ be a bipartite graph such that $|U| = r$ and $|W| = s$. To say that $B$ is *complete* *bipartite*, denoted $K_{r,s}$, means that every vertex in $U$ is adjacent to every vertex in $W$:

$$E(B) = \{uw \mid u \in U \text{ and } w \in W\}$$

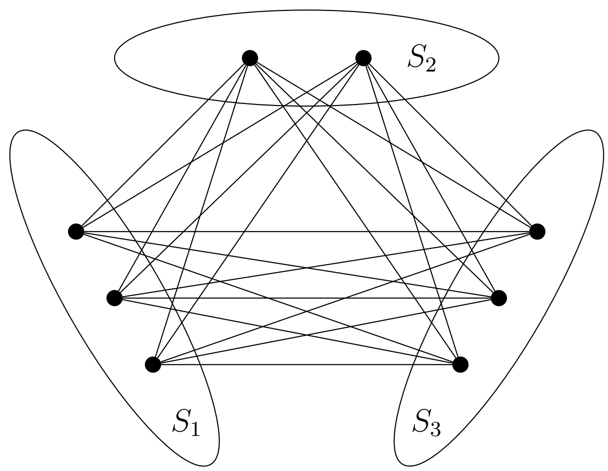
**Definition: Complete $k$-Partite**

Let $G(S_1, S_2, \ldots, S_k)$ be a $k$-partite graph such that $|S_i| = n_i$. To say that $G$ is *complete $k$-partite*, denoted $K_{n_1,n_2,\ldots,n_k}$, means that every induced subgraph $G[S_i \cup S_j], i \neq j$ is complete bipartite $K_{n_i,n_j}$.

**Examples**




$K_{3,2}$



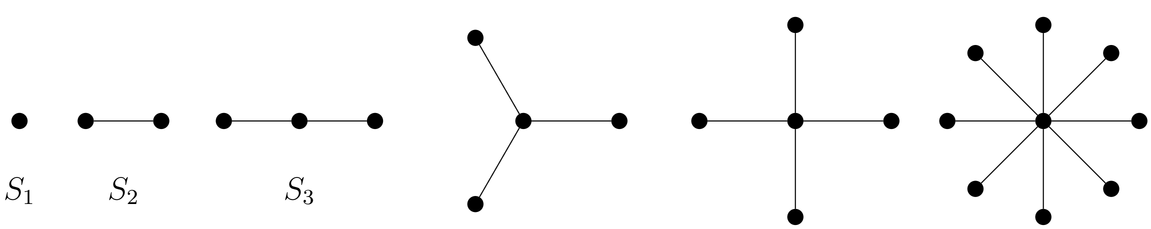

$K_{3,2,3}$

Note that $K_{1,1,\ldots,1}$ is the complete graph $K_k$.

**Definition: Star**

The complete graph $K_{1,n-1}$ is called a *star* graph, denote by $S_n$.

**Examples**



$S_1$ $S_2$ $S_3$ $S_4$ $S_5$ $S_9$

**Theorem**

Let $G$ be a a non-trivial graph:

$G$ is bipartite $\iff$ $G$ has no odd cycles.

*Proof.*

$\implies$ Assume $G$ is bipartite.

Let $G = B(U, W)$.

ABC: $G$ contains an odd cycle $(u_1, u_2, \ldots, u_{2k+1}, u_1)$ for some $k \in \mathbb{N}$.

AWLOG: $u_1 \in U$.

Since $G$ is bipartite, adjacent vertices must alternate between $U$ and $W$:

$$u_i \in \begin{cases} U, & i \text{ odd} \\ W, & i \text{ even} \end{cases}$$

But this means that $u_1, u_{2k+1} \in U$ and $u_1 u_{2k+1} \in E(G)$, contradicting the bipartiteness of $G$.

$\therefore$ $G$ contains no odd cycles.

$\impliedby$ Assume $G$ has no odd cycles.

**Case 1:** $G$ is connected.

Assume $u \in V(G)$ and let:

$$U = \{v \in V(G) \mid d(u, v) \text{ is even}\}$$
$$W = \{v \in V(G) \mid d(u, v) \text{ is odd}\}$$

Note that $u \in U$ since $d(u, u) = 0$ is even. Thus, $\{U, W\}$ is a partition of $V(G)$.

Claim: $G = B(U, W)$ is bipartite.

ABC: There exists $u_1, u_2 \in U$ such that $u_1 u_2 \in E(G)$ or there exists $w_1, w_2 \in W$ such that $w_1 w_2 \in E(G)$.

**Case a:** There exists $u_1, u_2 \in U$ such that $u_1 u_2 \in E(G)$.

Let $P$ be a $u - u_1$ geodesic and let $P'$ be a $u - u_2$ geodesic. Both are even paths. Let $u_i$ be the last vertex in common between $P$ and $P'$.

**Case i:** $u_i \in U$

And so $d(u, u_i)$ is even. This means that $d(u_i, u_1)$ and $d(u_i, u_2)$ are both even, and thus the cycle $u_i, \ldots, u_1, u_2, \ldots, u_i$ is an odd cycle, contradicting the assumption.

**Case ii:** $u_i \in W$

And so $d(u, u_i)$ is odd. This means that $d(u_i, u_1)$ and $d(u_i, u_2)$ are both odd, and thus the cycle $u_i, \ldots, u_1, u_2, \ldots, u_i$ is an odd cycle, contradicting the assumption.

**Case b:** There exists $w_1, w_2 \in W$ such that $w_1 w_2 \in E(G)$.

Let $P$ be a $u - w_1$ geodesic and let $P'$ be a $u - w_2$ geodesic. Both are odd paths. Let $u_i$ be the last vertex in common between $P$ and $P'$.

**Case i:** $u_i \in U$

And so $d(u, u_i)$ is even. This means that $d(u_i, w_1)$ and $d(u_i, w_2)$ are both odd, and thus the cycle $u_i, \ldots, w_1, w_2, \ldots, u_i$ is an odd cycle, contradicting the assumption.

**Case ii:** $u_i \in W$

And so $d(u, u_i)$ is odd. This means that $d(u_i, w_1)$ and $d(u_i, w_2)$ are both even, and thus the cycle $u_i, \ldots, w_1, w_2, \ldots, u_i$ is an odd cycle, contradicting the assumption.

$\therefore G = B(U, W)$ is bipartite.

**Case 2:** $G$ is disconnected.

This means that $G$ is composed of $k$ connected components $G_1, G_2, \ldots, G_k$. But since $G$ contains no odd cycles, none of the $G_i$ can contain any odd cycles, and so by the first case, each $G_i$ is bipartite. So let $G_i = B(U_i, W_i)$ be the $k$ bipartite components and let:

$$U = \bigcup_{1 \le i \le k} U_i$$

$$W = \bigcup_{1 \le i \le k} W_i$$

Now assume $u \in U$. Then $u \in U_i$ for some $i$. Furthermore, assume $uw \in E(G)$. Then $w \in W_i \subseteq W$ and thus $w \in W$.

$\therefore G = B(U, W)$ is bipartite.

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**Corollary**

Let $G$ be a graph with $k$ components:

$G$ is bipartite $\iff$ each component $G_i$ is bipartite.

*Proof.* $G$ is bipartite $\iff$ $G$ contains no odd cycles $\iff$ every component $G_i$ has no odd cycles $\iff$ each component $G_i$ is bipartite. ■