Bounded Linear Maps

Definition: Bounded

Let $L : E_1 \to E_2$ be a linear map of normed spaces. To say that L is *bounded* means $\exists M > 0$ such that $\forall \vec{x} \in E_1$:

$$\|L\vec{x}\| \leq M \|\vec{x}\|$$

Theorem

Let L be a linear map on a normed, finite dimensional space E:

L is bounded.

Proof

Assume $\dim E = n < \infty$.
Assume $\vec{e}_1, \ldots, \vec{e}_n$ is an orthonormal basis for E.
Assume $\vec{x} \in E_1$.

$$\|L\vec{x}\| = \left\| L \sum_{k=1}^{n} x_k \vec{e}_k \right\| = \left\| \sum_{k=1}^{n} x_k L\vec{e}_k \right\| \leq \sum_{k=1}^{n} |x_k| \, \|L\vec{e}_k\| \leq \max_{1 \leq k \leq n} |x_i| \sum_{k=1}^{n} \|L\vec{e}_k\|$$

Let $M = \displaystyle\sum_{k=1}^{n} \|L\vec{e}_k\| < \infty$.
Also note that $\displaystyle\max_{1 \leq k \leq n} |x_i| \leq \|\vec{x}\|$.
And so $\|L\vec{x}\| \leq M \|\vec{x}\|$.

Therefore L is bounded.

Theorem

Let $L : E_1 \to E_2$ be a linear map of normed spaces:

L is bounded iff L is bounded on the unit sphere.

Proof

$$\vec{x} \in E_1 \iff \frac{\vec{x}}{\|\vec{x}\|} \in S_1(\vec{0}, 1)$$

$$\begin{aligned}
L \text{ is bounded} \quad &\iff \quad \exists\, M > 0, \forall\, \vec{x} \in E_1, \|L\vec{x}\| < M \|\vec{x}\| \\
&\iff \quad \frac{1}{\|\vec{x}\|} \|L\vec{x}\| < \frac{1}{\|\vec{x}\|} M \|\vec{x}\| \\
&\iff \quad \left\| L \frac{\vec{x}}{\|\vec{x}\|} \right\| < M
\end{aligned}$$

Examples

1). $f_a : \mathbb{R}^N \to \mathbb{R}$ where $a \in \mathbb{R}^N$ and $f_a(x) = a \cdot x = \sum_{k=1}^{N} a_k x_k$.

$$\begin{aligned} f_a(\alpha x + \mathcal{B}y) &= f_a\left(\sum_{k=1}^{N}(\alpha x_k + \mathcal{B}x_k\right) \\ &= \sum_{k=1}^{N} a_k(\alpha x_k + \mathcal{B}y_k) \\ &= \alpha \sum_{k=1}^{N} a_k x_k + \mathcal{B}\sum_{k=1}^{N} a_k y_k \\ &= \alpha f_a(x) + \mathcal{B} f_a(y) \end{aligned}$$

Therefore, f_a is linear.

Assume $x \in \mathbb{R}^N$:

$$|f_a(x)| = |a \cdot x| \leq \|a\| \|x\| = M \|x\|$$

with $M = \|a\|$.

2). $\Phi : \mathcal{C}[0,1] \to \mathbb{R}$ where $\Phi(f) = \int_0^1 f(t)dt$

Φ is linear due to linearity of the integral.

Assume $f \in \mathcal{C}[0,1]$:

$$|\Phi(f)| = \left|\int_0^1 f(t)dt\right| \leq \int_0^1 |f(t)|\, dt \leq \int_0^1 \max_{t\in[0,1]} |f(t)|\, dt = \int_0^1 \|f\|\, dt = \|f\|$$

with $M = 1$.

3). Differentiation is an unbounded linear map.

Let $D : \mathcal{C}^1[-1,1] \to \mathcal{C}[-1,1]$ where $D(f) = f'$.

D is linear due to linearity of differentiation.

WTS: $\forall\, M > 0, \exists f \in \mathcal{C}^1[-1,1], \|Df\| > M\|f\|$

Let $f_n = \sin(nx)$ for $n \geq 2$.
$f_n'(x) = n\cos(nx)$
$\|D(f_n)\| = \max_{x\in[-1,1]} |n\cos(nx)| = n$ which occurs at $x = 0$.
$\|f_n\| = \max_{x\in[-1,1]} |\sin(nx)| = 1$ which occurs at $x \in \frac{\pi}{2n}$.

Assume $M > 0$.
Let $n = \lceil M \rceil + 1$.
Let $f = \sin(nx)$.
$\|D(f)\| = n > M$.

Therefore, D is unbounded.

Notation

Let E_1 and E_2 be normed spaces:

$$\mathcal{B}(E_1, E_2) = \{T : E_1 \to E_2 \mid T \text{ is linear and bounded}\}$$

Definition

Let E_1 and E_2 be normed spaces and $T \in \mathcal{B}(E_1, E_2)$:

$$\|T\| = \sup_{\|\vec{x}\|=1} \|T\vec{x}\|$$

This is a measure of the distortion of the unit sphere by T.

Theorem

Let E_1 and E_2 be normed spaces and $T \in \mathcal{B}(E_1, E_2)$:

$M = \|T\|$ is the tightest bound.

Proof

Assume $\vec{x} \in E_1$.

$$\|T\| = \sup_{x \in E_1 - \{\vec{0}\}} \left\| T \frac{\vec{x}}{\|\vec{x}\|} \right\|$$

And so:

$$\|T\| \, \|\vec{x}\| = \sup_{x \in E_1 - \{\vec{0}\}} \|T\vec{x}\|$$

Thus:

$\|T\vec{x}\| \leq \|T\| \, \|\vec{x}\|$ with equality at $\vec{x} = \vec{0}$ and $M = \|T\|$.

Theorem

Let E_1 and E_2 be normed spaces over a field $\mathbb{F}$:

$\mathcal{B}(E_1, E_2)$ is a normed space.

Proof

Assume $A, B \in \mathcal{B}(E_1, E_2)$, $\lambda \in \mathbb{F}$, and $\vec{x} \in E_1$:

$$\begin{aligned} \|(\lambda A + \mu B)(\vec{x})\| &= |\lambda| \, \|A\vec{x}\| + |\mu| \, \|B\vec{x}\| \\ &\leq |\lambda| \, M_A \, \|\vec{x}\| + |\mu| \, M_B \, \|\vec{x}\| \\ &\leq (|\lambda| \, M_A + |\mu| \, M_B) \, \|\vec{x}\| \end{aligned}$$

Let $M = (|\lambda|\, M_A + |\mu|\, M_B) > 0$.
$\|(\lambda A + \mu B)(\vec{x})\| \leq M\, \|\vec{x}\|$.
Thus $\lambda A + \mu B$ is bounded and so $\lambda A + \mu B \in \mathcal{B}(E_1, E_2)$.

Therefore, $\mathcal{B}(E_1, E_2)$ is a vector space.

Assume $L \in \mathcal{B}(E_1, E_2)$.

$$\|L\| = 0 \iff \sup_{\|\vec{x}\|=1} \|L\vec{x}\| = 0 \iff L\vec{x} = 0 \iff L = 0$$

$$\|\lambda L\| = \sup_{\|\vec{x}\|=1} \|\lambda L\vec{x}\| = |\lambda| \sup_{\|\vec{x}\|=1} \|L\vec{x}\| = |\lambda|\, \|L\|$$

Assume $L_1, L_2 \in \mathcal{B}(E_1, E_2)$.

$$\begin{aligned} \|L_1 + L_2\| &= \sup_{\|\vec{x}\|=1} \|(L_1 + L_2)\vec{x}\| \\ &= \sup_{\|\vec{x}\|=1} \|L_1\vec{x} + L_2\vec{x}\| \\ &\leq \sup_{\|\vec{x}\|=1} (\|L_1\vec{x}\| + \|L_2\vec{x}\|) \\ &\leq \sup_{\|\vec{x}\|=1} (\|L_1\|\, \|\vec{x}\| + \|L_2\|\, \|\vec{x}\|) \\ &= \|L_1\| + \|L_2\| \end{aligned}$$

Thus, $\|L\|$ is a proper norm on $B(E_1, E_2)$.

Therefore $\mathcal{B}(E_1, E_2)$ is a normed space.

<u>Theorem</u>

Let E_1 and E_2 be normed spaces over a field $\mathbb{F}$:

E_2 is Banach $\implies \mathcal{B}(E_1, E_2)$ is Banach.

<u>Proof</u>

Assume (L_n) is a Cauchy sequence in $\mathcal{B}(E_1, E_2)$.
Assume $\vec{x} \in E_1$:

$$\|L_n\vec{x} - L_m\vec{x}\| = \|(L_n - L_m)\vec{x}\| \leq \|L_n - L_m\|\, \|\vec{x}\| \to 0$$

Therefore, $(L_n\vec{x})$ is Cauchy in E_2.
But E_2 is Banach (complete), and so $\exists\, L\vec{x} \in E_2$ such that $L_n\vec{x} \to L\vec{x}$.

Assume $\vec{x}, \vec{y} \in E_1$ and $\alpha, \mathcal{B} \in \mathbb{F}$:

$$\begin{aligned} L(\alpha\vec{x} + \mathcal{B}\vec{y}) &= \lim_{n\to\infty} L_n(\alpha\vec{x} + \mathcal{B}\vec{y}) \\ &= \lim_{n\to\infty} (\alpha L_n\vec{x} + \mathcal{B}L_n\vec{y}) \\ &= \alpha \lim_{n\to\infty} L_n\vec{x} + \mathcal{B} \lim_{n\to\infty} L_n\vec{y} \\ &= \alpha L_n\vec{x} + \mathcal{B}L_n\vec{y} \end{aligned}$$

Therefore L linear.

Now, since all Cauchy sequences are bounded, $\exists\, M > 0$ such that $\|L_n\| \leq M$:

$$\|L\vec{x}\| = \left\|\lim_{n\to\infty} L_n\vec{x}\right\| = \lim_{n\to\infty} \|L_n\vec{x}\| \leq \lim_{n\to\infty} \|L_n\| \, \|\vec{x}\| \leq M \, \|\vec{x}\|$$

Therefore, L is linear and bounded and thus $L \in B(E_1, B_2)$.

Assume $\epsilon > 0$.
$\exists\, N > 0, m, n > N \implies \|L_n - L_m\| < \epsilon$
Assume $\vec{x} \in E_1$ such that $\|\vec{x}\| = 1$.
Assume $m, n > N$:

$\|L_n\vec{x} - L_m\vec{x}\| = \|(L_n - L_m)\vec{x}\| \leq \|L_n - L_m\| \, \|\vec{x}\| = \|L_n - L_m\| < \epsilon$

Now, let $m \to \infty$:

$\|L_n\vec{x} - L\vec{x}\| = \|(L_n - L)\vec{x}\| \leq \|L_n - L\| \, \|\vec{x}\| = \|L_n - L\| < \epsilon$

Therefore $L_n \to L \in \mathcal{B}(E_1, E_2)$ and so $\mathcal{B}(E_1, E_2)$ is complete (Banach).

Definition: Dual Space

Let E be a normed space. The *dual space* for E, denoted E' or E^*, is given by:

$$E' = \mathcal{B}(E, \mathbb{C})$$

Note that E' is always Banach because $\mathbb{C}$ is Banach.