Homeomorphisms

Definition: Homeomorphism

Let X and Y be topological spaces and let $f : X \to Y$. To say that f is a *homeomorphism* means that f is a continuous bijection and f^{-1} is continuous. If such an f exists then X and Y are said to be *homeomorphic* or *topologically equivalent*.

Theorem

Homeomorphic is an equivalence relation.

Proof. Assume that X,Y, and Z are topological spaces.

R: Consider $i_X = i_X^{-1}$, which is continuous. Therefore X is homeomorphic to X.

S: Assume that X is homeomorphic to Y.

> Then there exists a homeomorphism $f : X \to Y$. Since f is a homeomorphism, it is invertible and its inverse is continuous. Thus, $f^{-1} : Y \to X$ is a continuous, invertible function and $(f^{-1})^{-1} = f$ is invertible. Therefore Y is homeomorphic to X.

T: Assume that X is homeomorphic to Y and Y is homeomorphic to Z.

> Then there exists homeomorphics $f : X \to Y$ and $g : Y \to Z$. So consider $g \circ f : X \to Z$. Since f and g are continuous and invertible, $g \circ f$ is continuous and invertible. Furthermore, since f^{-1} and g^{-1} are continuous, $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$ is continuous. Therefore X is homeomorphic to Z.

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Lemma

For all $a, b \in \mathbb{R}$ such that $a < b$, (a, b) is homeomorphic to $(0, 1)$.

Proof. Let $f : (0, 1) \to (a, b)$ be defined by $f(t) = a + t(a - b)$. f is linear, and thus continuous and invertible with $f^{-1}(s) = \frac{s-a}{b-a}$ which is also linear and thus continuous. Therefore (a, b) is homeomorphic to $(0, 1)$. ■

Corollary

All open intervals in $\mathbb{R}$ are homeomorphic.

Proof. Assume $(a, b), (c, d) \subset \mathbb{R}$. (a, b) is homeomorphic to $(0, 1)$ and $(0, 1)$ is homeomorphic to (c, d). Therefore, (a, b) is homeomorphic to (c, d). ■

Theorem

$(a, b) \subset \mathbb{R}$ is homeomorphic to R.

Proof. (a, b) is homeomorphic to $(-\frac{\pi}{2}, \frac{\pi}{2})$. Now, consider $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ defined by $f(x) = \tan x$. This is a continuous and invertible function whose inverse is also continuous. Thus, $(-\frac{\pi}{2}, \frac{\pi}{2})$ is $\mathbb{R}$. Therefore, (a, b) is homeomorphic to R. ■

Theorem

Let X and Y be topological spaces and let $f : X \to Y$ be continuous. TFAE:

1. f is a homeomorphism.
2. f is a closed bijection.
3. f is an open bijection.

Proof.

$(1 \implies 2)$ Assume that f is a homeomorphism.

This means that f is a bijection and its inverse is continuous. So assume that $A \subset X$ is closed in X. Since f is bijective, $f(A) = (f^{-1})^{-1}(A)$, and since $(f^{-1})^{-1}$ is continuous, $f(A)$ is also closed. Therefore f is a closed bijection.

$(2 \implies 3)$ Assume that f is a closed bijection.

Assume that $U \in \mathscr{T}_X$. This means that $X - U$ is closed in X, and since f is closed, $f(X - U)$ is closed in Y and so $Y - f(X - U) \in \mathscr{T}_Y$. But f is a bijection and so $Y - f(X - U) = f(U) \in \mathscr{T}_Y$. Therefore, f is an open bijection.

$(3 \implies 1)$ Assume that f is an open bijection.

Assume that $U \in \mathscr{T}_Y$. Since f is continuous, $f^{-1}(U) \in \mathscr{T}_X$. But f is open so $(f^{-1})^{-1}(U) \in \mathscr{T}_Y$. Therefore f^{-1} is continuous and hence f is a homeomorphism.

■

Theorem

Let X and Y be topological spaces such that X is compact and Y is Hausdorff and let $f : X \to Y$ be a continuous bijection. f is a homeomorphism.

Proof. Since X is compact, Y is Hausdorff, and f is a bijection, f is closed. Therefore, since f is a continuous closed bijection, f is a homeomorphism. ■

Example

Let X and Y be topological spaces and let f be continuous bijection.

1. Y is T_2 but X is not compact.

   Consider $X = [0, 2\pi]$ and $Y = S^1$ with $f : X \to Y$ defined by $f(t) = e^{it}$. But $f^{-1}(e^{it}) = t$ is not continuous.

2. X is compact but Y is not T_2.

   Consider $X = [0,1]_{\mathsf{std}}$ and $Y = [0,1]_{\mathsf{ind}}$ with $f(x) = x$. But f is not open.