# Jordan Canonical Form

## Definition: Jordan Block

A *Jordan block* of size $k$ for a value $\lambda$, denoted $J_k(\lambda)$, is a $k \times k$ upper triangular matrix with $\lambda$ on the diagonal, $1$ on the super-diagonal, and $0$ everywhere else:

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & 1 & 0 & \cdots & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \lambda \end{bmatrix}$$

## Example

$$J_2(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad J_3(0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

## Notation: Direct Sum

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

## Definition: Jordan Canonical Form

A *Jordan Matrix* is a direct sum of Jordan blocks.

The *Jordan Canonical Form* for a matrix $A \in M_n$ is a Jordan matrix where each block corresponds to $\lambda \in \mathrm{Sp}(A)$:

$$J_A = J_{n_1}(\lambda_1) \oplus J_{n_2}(\lambda_2) \oplus \cdots \oplus J_{n_k}(\lambda_k)$$

where the $n_i$ and the $\lambda_i$ are not necessarily distinct.

## Properties: $J_k(\lambda)$

1). $\mathrm{tr}(J_k(\lambda)) = k\lambda$

2). $\det(J_k(\lambda) = \lambda^k$

3). $p_{J_k(\lambda)} = (t-\lambda)^k$

4). $\sigma(J_k(\lambda)) = \{\lambda\}$

5). $\mathrm{rank}(J_k(\lambda) - \lambda I)^m = \mathrm{rank}(J_k(0))^m = \begin{cases} 0, & m \geq k \\ k-m, & m < k \end{cases}$

6). $J_k(\lambda) \sim J_k(\mu) \iff \lambda = \mu$

**Lemma**

Let $T \in UT(n)$ such that $T$ is of the form:

$$T = \begin{bmatrix} T_1 & & & R \\ & T_2 & & \\ & & \ddots & \\ 0 & & & T_k \end{bmatrix}$$

where $T_i$ is triangular with constant diagonal $\lambda_i$, the $\lambda_i$ are distinct, and $R$ is any matrix:

$$T \sim T_1 \oplus T_2 \oplus \cdots \oplus T_k$$

*Proof.* Proof by induction $k$, the number of distinct diagonal values.

Base case: $k = 1$

$T = \begin{bmatrix} \lambda \end{bmatrix}$ Done.

Assume the statement is true for $k-1$ distinct diagonal values.

Consider $T = \left[\begin{array}{c|c} T' & R \\ \hline 0 & T_k \end{array}\right]$

By the inductive assumption: $T' \sim T_1 \oplus T_2 \oplus \cdots \oplus T_{k-1}$
Note that by construction, $\sigma(T') \cup \sigma(T_k) = \emptyset$
So by the Sylvester equation: $T'X - XT_k = R$ has a solution for all $R$. Let $X$ be such a solution and let:

$$S = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$$

Note that:

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

So $S$ is invertible and:

$$\begin{aligned} T &\sim \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} T' & R \\ 0 & T_k \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} T' & R + XT_k \\ 0 & T_k \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} T' & -T'X + R + XT_k \\ 0 & T_k \end{bmatrix} \end{aligned}$$

But $T'X - XT_k = R$ and so $-T'X + R + XT_k = 0$ and so:

$$T \sim \begin{bmatrix} T' & 0 \\ 0 & T_k \end{bmatrix} = T' \oplus T_k$$

$\therefore T \sim T_1 \oplus T_2 \oplus \cdots \oplus T_k$

$\square$

**Lemma**

Let $S \in M_n$ be a strictly upper triangular matrix. $S$ is similar to a direct sum of Jordan blocks with respect to $0$:

$$S \sim J_{n_1}(0) \oplus J_{n_2}(0) \oplus \cdots \oplus J_{n_m}(0)$$

Proof

Proof by strong induction on $n$, the size of $S$:

Base case: $n = 1$

$S = \begin{bmatrix} 0 \end{bmatrix} = J_1(0)$
Done.

Assume the statement is true for size $n - 1$

Consider $S \in M_n$ where $S = \begin{bmatrix} 0 & \vec{a}^T \\ 0 & S_1 \end{bmatrix}$ where $S_1$ is strictly upper triangular.

By the inductive assumption, $S_1$ is similar to a direct sum of Jordan blocks with respect to 0. So there exists invertible $P$ such that:

$$S_1 = P_1(J_{k_1}(0) \oplus L)P^{-1}$$

where $J_{k_1}(0)$ is the Jordan block with the largest size and $L$ is the direct sum of the remaining Jordan blocks.
Note that:

$$\begin{bmatrix} 1 & 0 \\ 0 & P_1^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & P_1 \end{bmatrix} = I$$

and so:

$$\begin{aligned}
S &\sim \begin{bmatrix} 1 & 0 \\ 0 & P_1^{-1} \end{bmatrix} \begin{bmatrix} 0 & \vec{a}^T \\ 0 & S_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & P_1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \vec{a}^T \\ 0 & P_1^{-1}S_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & P_1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \vec{a}^T P_1 \\ 0 & P_1^{-1}S_1P \end{bmatrix} \\
&= \left[\begin{array}{c|cc} 0 & \vec{a}_1^T & \vec{a}_2^T \\ \hline & J_{k_1}(0) & 0 \\ 0 & 0 & L \end{array}\right] \\
&\sim \begin{bmatrix} 1 & -\vec{a}^T J_{k_1}(0) & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & \vec{a}_1^T & \vec{a}_2^T \\ 0 & J_{k_1}(0) & 0 \\ 0 & 0 & L \end{bmatrix} \begin{bmatrix} 1 & \vec{a}^T J_{k_1}(0) & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} 0 & (\vec{a}_1^T\vec{e}_1)\vec{e}_1^T & \vec{a}_2^T \\ 0 & J_{k_1}(0) & 0 \\ 0 & 0 & L \end{bmatrix}$$

Case 1: $\vec{a}_T\vec{e}_1 = 0$

$$S \sim \begin{bmatrix} 0 & 0 & \vec{a}_2^T \\ 0 & J_{k_1}(0) & 0 \\ 0 & 0 & L \end{bmatrix} \sim \left[\begin{array}{c|cc} J_{k_1}(0) & 0 & 0 \\ \hline 0 & 0 & \vec{a}_2^T \\ 0 & 0 & L \end{array}\right]$$

But note that the lower quarter is a strictly upper triangular array of size less than $n$, and so by the inductive assumption is similar to a direct sum of Jordan blocks.

Therefore, $S$ is similar to a direct sum of Jordan blocks.

Case 2: $\vec{a}_T\vec{e}_1 \neq 0$

$$\begin{aligned}
S &\sim \begin{bmatrix} \frac{1}{\vec{a}_1^T\vec{e}_1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \frac{1}{\vec{a}_1^T\vec{e}_1}I \end{bmatrix} \begin{bmatrix} 0 & (\vec{a}_1^T\vec{e}_1)\vec{e}_1^T & \vec{a}_2^T \\ 0 & J_{k_1}(0) & 0 \\ 0 & 0 & L \end{bmatrix} \begin{bmatrix} \vec{a}_1^T\vec{e}_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \vec{a}_1^T\vec{e}_1 I \end{bmatrix} \\
&= \begin{bmatrix} 0 & \vec{e}_1^T & \frac{1}{\vec{a}_1^T\vec{e}_1}\vec{a}_2^T \\ 0 & J_{k_1}(0) & 0 \\ 0 & 0 & \frac{1}{\vec{a}_1^T\vec{e}_1}L \end{bmatrix} \begin{bmatrix} \vec{a}_1^T\vec{e}_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \vec{a}_1^T\vec{e}_1 I \end{bmatrix} \\
&= \begin{bmatrix} 0 & \vec{e}_1^T & \vec{a}_2^T \\ 0 & J_{k_1}(0) & 0 \\ 0 & 0 & L \end{bmatrix} \\
&= \begin{bmatrix} J_{k_1+1}(0) & \vec{e}_1\vec{a}_2^T \\ 0 & L \end{bmatrix} \\
&\sim \begin{bmatrix} I & \vec{e}_2\vec{a}_2^T \\ 0 & I \end{bmatrix} \begin{bmatrix} J_{k_1+1}(0) & \vec{e}_1\vec{a}_2^T \\ 0 & L \end{bmatrix} \begin{bmatrix} I & -\vec{e}_2\vec{a}_2^T \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} J_{k_1+1}(0) & \vec{e}_1\vec{a}_2^T + \vec{e}_2\vec{a}_2^T L \\ 0 & L \end{bmatrix} \begin{bmatrix} I & -\vec{e}_2\vec{a}_2^T \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} J_{k_1+1}(0) & -J_{k_1+1}(0)\vec{e}_2\vec{a}_2^T + \vec{e}_1\vec{a}_2^T + \vec{e}_2\vec{a}_2^T L \\ 0 & L \end{bmatrix} \\
&= \begin{bmatrix} J_{k_1+1}(0) & -\vec{e}_1\vec{a}_2^T + \vec{e}_1\vec{a}_2^T + \vec{e}_2\vec{a}_2^T L \\ 0 & L \end{bmatrix} \\
&= \begin{bmatrix} J_{k_1+1}(0) & \vec{e}_2\vec{a}_2^T L \\ 0 & L \end{bmatrix} \\
&\sim \begin{bmatrix} I & \vec{e}_2\vec{a}_2^T L \\ 0 & I \end{bmatrix} \begin{bmatrix} J_{k_1+1}(0) & \vec{e}_2\vec{a}_2^T L \\ 0 & L \end{bmatrix} \begin{bmatrix} I & -\vec{e}_2\vec{a}_2^T L \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} J_{k_1+1}(0) & \vec{e}_3\vec{a}_2^T L^2 \\ 0 & L \end{bmatrix} \\
&\sim \begin{bmatrix} J_{k_1+1}(0) & \vec{e}_4\vec{a}_2^T L^3 \\ 0 & L \end{bmatrix}
\end{aligned}$$

$$\begin{aligned} &\sim \vdots \\ &\sim \begin{bmatrix} J_{k_1+1}(0) & \vec{e}_{k_1+1}\vec{a}_2^T L^{k_1} \\ 0 & L \end{bmatrix} \\ &= \begin{bmatrix} J_{k_1+1}(0) & 0 \\ 0 & L \end{bmatrix} \end{aligned}$$

Therefore, $S$ is similar to a direct sum of Jordan blocks.

**Theorem: Jordan Decomposition**

Let $A \in M_n$. There exists a Jordan matrix $J_A$ such that:

$$A \sim J_A$$

Proof

Per Schur triangularization, $A$ is similar to an upper triangular matrix $T$ such that the eigenvalues of $A$ are on the diagonal of T (reflecting their algebraic multiplicity) and like eigenvalues are grouped together on the diagonal.

But by the first lemma, $T \sim T_1 \oplus T_2 \oplus \cdots \oplus T_k$ where each $T_i$ corresponds to a distinct $\lambda_i \in \sigma(A)$ and is of the form:

$$T_i = \begin{bmatrix} \lambda_i & & & R \\ & \lambda_i & & \\ & & \ddots & \\ 0 & & & \lambda_i \end{bmatrix} = \lambda_i I - \begin{bmatrix} 0 & & & R \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix} = \lambda_i I - S_i$$

where $S_i$ is strictly triangular.

But by the second lemma, $S_i$ is similar to a direct sum of Jordan blocks with respect to $0$, so there exists invertible $P_i$ such that:

$$\begin{aligned} S_i &= P_i \begin{bmatrix} J_{n_1}(0) & & 0 \\ & \ddots & \\ 0 & & J_{n_m}(0) \end{bmatrix} P_i^{-1} \\ \lambda_i I + S_i &= \lambda_i I + P_i \begin{bmatrix} J_{n_1}(0) & & 0 \\ & \ddots & \\ 0 & & J_{n_m}(0) \end{bmatrix} P_i^{-1} \\ &= \lambda_i P_i I P_i^{-1} + P_i \begin{bmatrix} J_{n_1}(0) & & 0 \\ & \ddots & \\ 0 & & J_{n_m}(0) \end{bmatrix} P_i^{-1} \\ &= P_i \left( \lambda_i I + \begin{bmatrix} J_{n_1}(0) & & 0 \\ & \ddots & \\ 0 & & J_{n_m}(0) \end{bmatrix} \right) P_i^{-1} \end{aligned}$$

$$= \quad P_i \begin{bmatrix} J_{n_1}(\lambda_i) & & 0 \\ & \ddots & \\ 0 & & J_{n_m}(\lambda_i) \end{bmatrix} P_i^{-1}$$

Thus, each $T_i$ is similar to a direct sum of Jordan blocks.

Now, note that:

$$(P_1 \oplus I \oplus \cdots \oplus I)(T_1 \oplus T_2 \oplus \cdots \oplus T_k)(P_1^{-1} \oplus I \oplus \cdots \oplus I) = PT_1P^{-1} \oplus T_2 \oplus \cdots \oplus T_k)$$

and likewise for each $T_i$, thus replacing each $T_i$ with its corresponding direct sum of Jordan blocks.

Therefore $A$ is similar to a direct sum of Jordan blocks.