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Math 231b
Homework #7

4.12.48

Show that the projection onto a closed subspace F of a Hilbert space H is a compact operator if and only if F is finite-dimensional.

Since F is closed, F is also Hilbert (and separable).

$\implies$ Assume P_F is compact.

ABC: F is infinite-dimensional.
Assume $(\vec{e}_n)$ is a complete orthonormal sequence in F.
$\|\vec{e}_k\| = 1$ for all $k \in \mathbb{N}$ and so $(\vec{e}_n)$ is a bounded sequence in H.
Since $(\vec{e}_n)$ is also a sequence in F, $P_F\vec{e}_k = \vec{e}_k$ for all $k \in \mathbb{N}$.
But $(P_F\vec{e}_n) = (\vec{e}_n)$ has no convergent subsequence.
CONTRADICTION!

Therefore F is finite-dimensional.

$\impliedby$ Assume F is finite-dimensional.

P_F is onto F and so $\mathcal{R}(P_F) = F$.
So P_F is a finite rank operator on a Hilbert space.

Therefore, P_F is compact.

4.12.49

Show that the operator $T : \ell^2 \to \ell^2$ defined by $T(x_n) = (2^{-n}x_n)$ is compact.

Assume (x_n) is a bounded sequence in ℓ^2.

$\exists\, M > 0$ such that $\|(x_n)\|^2 = \sum_{k=1}^{\infty} |x_{n,k}|^2 \leq M < \infty$.

Now, since $\frac{1}{2^k} < 1$ for all $k \in \mathbb{N}$:

$$\|T(x_n)\|^2 = \sum_{k=1}^{\infty} \left|2^{-k}x_{n,k}\right|^2 < \sum_{k=1}^{\infty} |1 \cdot x_{n,k}|^2 = \sum_{k=1}^{\infty} |x_{n,k}|^2 = \|x_n\|^2 \leq M$$

Thus $(T(x_n))$ is also a bounded sequence in ℓ^2.

Let $T_N(x_n)$ be defined by:

$$T_N(x_n)_k = \begin{cases} T(x_n)_k, & 1 \leq k \leq N \\ 0, & otherwise \end{cases}$$

Thus, T_n applies T and then sets all but the first N terms to 0.
Note that if $T(x_n) \in \ell^2$ then certainly $T_n(x_n) \in \ell^2$.
Furthermore, note that $T_n(x_n)$ is a finite rank operator.

$$\begin{aligned}
\|(T_N - T)(x)\|^2 &= \sum_{k=N+1}^{\infty} \left|2^{-k} x_{n,k}\right|^2 \\
&= \sum_{k=N+1}^{\infty} \left|2^{-2k}\right| |x_{n,k}|^2 \\
&\leq \sum_{k=N+1}^{\infty} \left|2^{-2k}\right| \|(x_n)\|^2 \\
&= M \sum_{k=N+1}^{\infty} 2^{-2k} \\
&< M \sum_{k=N+1}^{\infty} 2^{-k} \\
&= M \sum_{k=0}^{\infty} 2^{-(k+N+1)} \\
&= M \sum_{k=0}^{\infty} 2^{-(N+1)} 2^{-k} \\
&= \frac{M}{2^{N+1}} \sum_{k=0}^{\infty} 2^{-k} \\
&= \frac{M}{2^{N+1}}(2) \\
&= \frac{M}{2^N} \\
&\to 0
\end{aligned}$$

And so $T_n \to T$ as $N \to \infty$.
But $T_n \in \mathcal{K}(H)$, which is closed.

Therefore $T \in \mathcal{K}(H)$ and thus T is compact.

4.12.50

Show that a self-adjoint operator T is compact if and only if there exists a sequence of finite-dimensional operators strongly convergent to T.

Assume H is a Hilbert space.
Assume T is a self-adjoint operator on H.

$\implies$ Assume T is compact.

By the spectral theorem for compact, self-adjoint operators:

$$T = \sum_{n=1}^{\infty} \lambda_n P_n$$

where the λ_n are distinct, non-zero eigenvalues of T and each P_n is the projection operator onto a corresponding orthonormal eigenvector for λ_n. Furthermore, $\lambda_n \to 0$.
Let $(\vec{e}_n)$ be the corresponding orthonormal eigenvector sequence.
Assume $\vec{x} \in H$:

$$T\vec{x} = \sum_{n=1}^{\infty} \lambda_n \langle \vec{x}, \vec{e}_n \rangle \vec{e}_n$$

Now, define the sequence of finite-rank (and hence compact) operators $(T_n\vec{x})$ where:

$$T_n\vec{x} = \sum_{k=1}^{n} \lambda_n \langle \vec{x}, \vec{e}_n \rangle \vec{e}_n$$

Since the $\vec{e}_k$ are orthonormal we can apply Parseval:

$$\begin{aligned}
\|(T_n - T)\vec{x}\|^2 &= \left\| \sum_{k=n+1}^{\infty} \lambda_k \langle \vec{x}, \vec{e}_k \rangle \vec{e}_k \right\|^2 \\
&= \sum_{k=n+1}^{\infty} |\lambda_k \langle \vec{x}, \vec{e}_k \rangle|^2 \\
&= \sum_{k=n+1}^{\infty} |\lambda_k|^2 |\langle \vec{x}, \vec{e}_k \rangle|^2 \\
&\leq \sum_{k=n+1}^{\infty} |\lambda_k|^2 \|\vec{x}\|^2 \|\vec{e}_k\|^2 \\
&= \sum_{k=n+1}^{\infty} |\lambda_k|^2 \|\vec{x}\|^2 \|\vec{e}_k\|^2 \\
&= \|\vec{x}\|^2 \sum_{k=n+1}^{\infty} |\lambda_k|^2 \\
&\to 0
\end{aligned}$$

$\therefore T_n \to T$.

$\impliedby$ Assume $T_n \to T$ where T_n is finite dimensional.

T_n is finite-dimensional $\implies$ T_n is compact.
Furthermore, $\mathcal{K}(H)$ is a closed subspace of $\mathcal{B}(H)$.

Therefore $T \in \mathcal{K}(H)$ and thus T is compact.

4.12.51

Show that the space of all eigenvectors corresponding to a nonzero eigenvalue of a compact operator is finite-dimensional.

Assume A is a compact operator on a Hilbert space H.
Assume λ is an eigenvalue of A such that $\lambda \neq 0$.
ABC: E_λ is infinite-dimensional.
Since $E_\lambda = \ker(A - \lambda I)$, E_λ is a closed subspace of H and is thus also Hilbert (and separable).
So there exists a complete orthonormal sequence $(\vec{x}_n)$ in E_λ.
Since A is compact it maps orthonormal sequences to sequences that converge to 0.
And so $A\vec{x}_n \to 0$.
Thus $A\vec{x}_n = \lambda\vec{x}_n \to 0$.
But $\lambda \neq 0$ and so $\vec{x}_n \to 0$.
CONTRADICTION!

Therefore E_λ is finite-dimensional.