Subgroups

Definition

To say that a set H is a *subgroup* of a group G, denoted $H \leq G$, means:

1). $H \subseteq G$

2). H is a group using the induced operation of G

When $H = G$, H is called the *improper* subgroup of G.

When $H \subset G$, H is called a *proper* subgroup of G, denoted $H < G$.

Example

$$Z_4 = \{0, 1, 2, 3\}$$

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$$\{0\}$$
$$\{0, 2\}$$
$$\{0, 1, 2, 3\}$$

One proper, non-trivial subgroup

$$V = D_4 = \{e, a, b, c\}$$

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$$\{e\}$$
$$\{e, a\}$$
$$\{e, b\}$$
$$\{e, c\}$$
$$\{e, a, b, c\}$$

Three proper, non-trivial subgroup

$$\begin{tikzcd} \mathbb{Z}_4 \arrow[d, dash] \\ \{0, 2\} \arrow[d, dash] \\ \{0\} \end{tikzcd}$$

$$\begin{tikzcd} & D_4 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \{e, a\} \arrow[dr, dash] & \{e, b\} \arrow[d, dash] & \{e, c\} \arrow[dl, dash] \\ & \{e\} & \end{tikzcd}$$

Theorem

Let G be a group:

$$\{e\} \leq G$$

Proof

$\{e\} \subseteq G$

$(ee)e = ee = e$
$e(ee) = ee = e$
$\therefore \{e\}$ is associative.

$e \in \{e\}$
$\therefore \{e\}$ has identity.

$ee = ee = e$
$\therefore \{e\}$ has inverses.

$\therefore \{e\} \leq G$

Definition

$\{e\} \subseteq G$ is called the *trivial* subgroup of G. All other subgroups are referred to as *non-trivial*.

Theorem

Let G be a group and $H \subseteq G$. $H \leq G$ iff the following three properties hold:

1). H is closed under the induced operation of G

2). $e \in H$

3). $\forall\, a \in H, a^{-1} \in H$

Proof

$\implies$ Assume $H \leq G$.

H is a group, so it is closed under the induced operation and $\forall\, a \in H, a^{-1} \in H$. Also, by closure, $aa^{-1} = e \in H$.

$\therefore$ the three properties hold.

$\impliedby$ Assume the three properties hold.

Assume $a, b, c \in H$.
By closure, $(ab)c \in H$ and $a(bc) \in H$.
$a, b, c \in G$
$(ab)c = a(bc)$ in G, so this must also hold in H.
$\therefore H$ is associative.

$e \in H$, and since e is the identity for G, it must also be the identity for H.

$\forall\, a \in H, a^{-1} \in H$.

$\therefore H \leq G$.

Example

$G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$

$H = \{0, 2\}$

$+_4$	0	2
0	0	2
2	2	0

H is closed

$0 = e \in H$

$0^{-1} = 0 \in H$

$2^{-1} = 2 \in H$

$\therefore H \leq G$

Theorem: Subgroup Test

Let G be a group and $H \neq \emptyset, H \subseteq G$

$$H \leq G \iff \forall\, a, b \in H, ab^{-1} \in H \;\; (b^{-1} \in G)$$

Proof

$\implies$ Assume $H \leq G$

Assume $a, b \in H$
Since H is a group, $b^{-1} \in H$
But $H \subseteq G$, so $b^{-1} \in G$
By closure, $ab^{-1} \in H$

$\impliedby$ Assume $\forall\, a, b \in H, ab^{-1} \in H \;\; (b^{-1} \in G)$

$H \neq \emptyset$
Assume $a, b \in H$

$b = (b^{-1})^{-1} \in H$
But $H \subseteq G$, so $b = (b^{-1})^{-1} \in G$
By assumption, $a(b^{-1})^{-1} \in H$
$ab \in H$
$\therefore H$ is closed under the induced operation of G.

Since $H \subseteq G, a \in G$
But since G is a group, $a^{-1} \in G$
By assumption, $aa^{-1} \in H$
$\therefore e \in H$

$e \in H$ and $a^{-1} \in G$
So by assumption, $ea^{-1} \in H$
$\therefore a^{-1} \in H$

Example

Let $G = GL(n, \mathbb{R})$ and $H = \{A \in G \mid \det(A) = 1\}$
Prove: $H < G$

Assume $A, B \in H$
Clearly, $H \subset G$
So, $B \in G$
$\det(B) = 1 \neq 0$, so B is invertible
$B^{-1} \in G$
$\det(AB^{-1}) = \frac{\det(A)}{\det(B)} = \frac{1}{1} = 1$
$AB^{-1} \in H$
$\therefore H < G$

Theorem

Let G and G' be groups and $\phi : G \to G'$ be an isomorphism:

$$H \leq G \implies \phi[H] \leq G'$$

Isomorphisms map subgroups to subgroups.

Proof

Assume $H \leq G$
Assume $x, y \in \phi[H]$
$\exists\, a, b \in H, \phi(a) = x$ and $\phi(b) = y$
$\phi[H] \subseteq G'$
So, $y \in G'$
But G' is a group, so $y^{-1} \in G'$
ϕ is a homomorphism
$xy^{-1} = \phi(a)\phi(b)^{-1} = \phi(a)\phi(b^{-1}) = \phi(ab^{-1})$
H is a group, so $b^{-1} \in H$
By closure, $ab^{-1} \in H$
ϕ is well-defined, so $\phi(ab^{-1}) \in \phi[H]$
$xy^{-1} \in \phi[H]$
$\therefore$ by the subgroup test, $\phi[H] \leq G'$

Corollary

Let $G \simeq G'$:

$$\forall\, H \leq G, \exists\, H' \leq G', H \simeq H'$$

Theorem

Let G be a group:

$$H, K \leq G \implies H \cap K \leq G$$

Proof

Assume $H, K \leq G$
Assume $a, b \in H \cap K$
$a, b \in H$ and $a, b \in K$
But H and K are groups, so $b^{-1} \in H$ and $b^{-1} \in K$
$H \cap K \subseteq H, K \subseteq G$
So $b^{-1} \in G$
$ab^{-1} \in H$ and $ab^{-1} \in K$
$ab^{-1} \in H \cap K$
$\therefore$ by the subgroup test, $H \cap K \leq G$