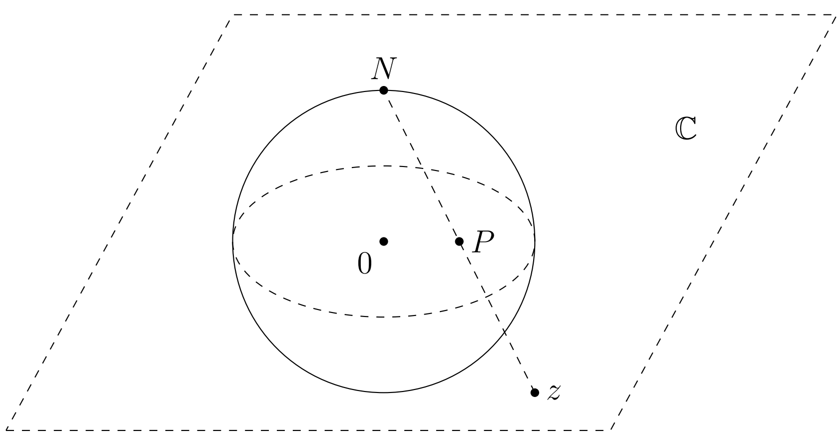
Limits at Infinity

Definition

The *extended complex plane*, denoted $\mathbb{C}_\infty$, is given by:

$$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$$

Consider the unit sphere with the complex plane passing through the equator, thus intersecting along the unit circle. For each point $z \in \mathbb{C}$, a line through z and the north pole N defines a single point of intersection on the surface of the upper hemisphere of the sphere. Note that all points in the interior of the unit circle ($0 \le |z| < 1$) correspond to N. All boundary points ($|z| = 1$) correspond to themselves. All points in the exterior ($|z| > 1$) and close to the boundary correspond to points near the equator. As $|z|$ increases, P moves arbitrarily close to N. Thus, there is a correspondence between N and ∞.



Such a sphere is referred to as a *Riemann sphere*.

Definition

For all small $\epsilon > 0$, $|z| > \frac{1}{\epsilon}$ is referred to as a neighborhood of infinity.

In other words, $|z|$ small is closer to 0 and $|z|$ large is closed to ∞.

Definition

To say that:

$$\lim_{z \to z_0} f(z) = \infty$$

means:

$$\forall \epsilon > 0, \exists \delta > 0, 0 < |z - z_0| < \delta \implies |f(z)| > \frac{1}{\epsilon}$$

Theorem

$$\lim_{z \to z_0} f(z) = 0 \iff \lim_{z \to z_0} \frac{1}{f(z)} = \infty$$

Proof

Assume $\epsilon > 0$

$$\begin{aligned} \lim_{z \to z_0} f(z) = 0 \quad &\iff \quad \exists\, \delta > 0, 0 < |z - z_0| < \delta \implies |f(z)| < \epsilon \\ &\iff \quad \exists\, \delta > 0, 0 < |z - z_0| < \delta \implies \left|\frac{1}{f(z)}\right| > \frac{1}{\epsilon} \\ &\iff \quad \lim_{z \to z_0} \frac{1}{f(z)} = \infty \end{aligned}$$

Corollary

$$\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$$

Proof

$$\lim_{z \to z_0} \frac{1}{f(z)} = 0 \iff \lim_{z \to z_0} \frac{1}{\frac{1}{f(z)}} = \infty \iff \lim_{z \to z_0} f(z) = \infty$$

Definition

To say that:

$$\lim_{z \to \infty} f(z) = w_0$$

means:

$$\forall\, \epsilon > 0, \exists\, \delta > 0, |z| > \delta \implies |f(z) - w_0| < \epsilon$$

Theorem

$$\lim_{z \to \infty} f(z) = w_0 \iff \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$$

Proof

Assume $\epsilon > 0$

$$\begin{aligned}
\lim_{z\to\infty} f(z) = w_0 \quad &\iff \quad \exists\, \delta > 0, |z| > \delta \implies |f(z) - w_0| < \epsilon \\
&\iff \quad \exists\, \delta > 0, 0 < \left|\frac{1}{z}\right| < \delta \implies |f(z) - w_0| < \epsilon \\
&\iff \quad \exists\, \delta > 0, 0 < |z| < \delta \implies \left|f\left(\frac{1}{z}\right) - w_0\right| < \epsilon \\
&\iff \quad \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0
\end{aligned}$$

Definition

To say that:

$$\lim_{z\to\infty} f(z) = \infty$$

means:

$$\forall\, \epsilon > 0, \exists\, \delta > 0, |z| > \delta \implies |f(z)| > \frac{1}{\epsilon}$$

Theorem

$$\lim_{z\to\infty} f(z) = \infty \iff \lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$$

Proof

Assume $\epsilon > 0$

$$\begin{aligned}
\lim_{z\to\infty} f(z) = \infty \quad &\iff \quad \exists\, \delta > 0, |z| > \delta \implies |f(z)| > \frac{1}{\epsilon} \\
&\iff \quad \exists\, \delta > 0, 0 < \left|\frac{1}{z}\right| < \delta \implies |f(z)| > \frac{1}{\epsilon} \\
&\iff \quad \exists\, \delta > 0, 0 < |z| < \delta \implies \left|f\left(\frac{1}{z}\right)\right| > \frac{1}{\epsilon} \\
&\iff \quad \exists\, \delta > 0, 0 < |z| < \delta \implies \left|\frac{1}{f\left(\frac{1}{z}\right)}\right| < \epsilon \\
&\iff \quad \lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0
\end{aligned}$$