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Math 231b
Homework #8

3.2

Suppose A and B are operators on a finite-dimensional Hilbert space and suppose that $AB - BA = cI$ for some constant c. Show that $c = 0$.

Let $\dim H = n$.
So $\operatorname{tr}(AB - BA) = \operatorname{tr}(cI) = nc$.

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$(BA)_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj}$$

$$(AB - BA)_{ij} = \sum_{k=1}^{n} (a_{ik} b_{kj} - b_{ik} a_{kj})$$

$$\begin{aligned}
\operatorname{tr}(AB - BA) &= \sum_{i=1}^{n} (AB - BA)_{ii} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} (a_{ik} b_{ki} - b_{ik} a_{ki}) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} - \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} a_{ki} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki} b_{ik} - \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} a_{ki} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki} b_{ik} - \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki} b_{ik} \\
&= 0
\end{aligned}$$

But $I \neq 0$.

$\therefore c = 0$

3.3

If A is a bounded operator on a Hilbert space H, then there exists a unique bounded operator A^* on H satisfying $\langle A\phi, \psi\rangle = \langle \phi, A^*\psi\rangle$ for all ϕ and ψ in H. The operator A^* is called the *adjoint* of A, and A is called *self-adjoint* if $A^* = A$.

(a) Show that for any bounded operator A and constant $c \in \mathbb{C}$, we have $(cA)* = \overline{c}A^*$, where $\overline{c}$ is the complex conjugate of c.

Note: $\langle \phi, A\psi\rangle = \langle \phi, Bt\rangle \implies A = B$

Assume $\phi, \psi \in H$.

$$\begin{aligned}
\langle \phi, (cA)^*\psi\rangle &= \langle (cA)\phi, \psi\rangle \\
&= \langle A(c\phi), \psi\rangle \\
&= \langle c\phi, A^*\psi\rangle \\
&= c\,\langle \phi, A^*\psi\rangle \\
&= \langle \phi, \overline{c}(A^*\psi)\rangle \\
&= \langle \phi, (\overline{c}A^*)\psi)\rangle
\end{aligned}$$

$\therefore (cA)^* = \overline{c}A^*$

(b) Show that if A and B are self-adjoint, then the operator $\dfrac{1}{i\hbar}[A, B]$ is also self-adjoint.

Note: $(A + B)^* = A^* + B^*$ and $(AB)^* = B^*A^*$

$$\begin{aligned}
(i\hbar[A, B])^* &= [i\hbar(AB - BA)]^* \\
&= \overline{i\hbar}(AB - BA)^* \\
&= -i\hbar[(AB)^* - (BA)^*] \\
&= -i\hbar(B^*A^* - A^*B^*) \\
&= -i\hbar(BA - AB) \\
&= i\hbar(AB - BA) \\
&= i\hbar[A, B]
\end{aligned}$$

Therefore $i\hbar[A, B]$ is self-adjoint.

3.5

Suppose that ψ is a unit vector in $L^2(\mathbb{R})$ such that the functions $x\psi(x)$ and $x^2\psi(x)$ also belong to $L^2(\mathbb{R})$. Show that:

$$\left\langle X^2\right\rangle_\psi > \left(\langle X\rangle_\psi\right)^2$$

Let $a = \langle X \rangle_\psi$ and consider the integral:

$$\int_{-\infty}^{\infty} (x-a)^2 \left|\psi(x)\right|^2 dx$$

Note that the integrand is positive (for non-constant x) and thus so is the integral.

$$\begin{aligned}
\int_{-\infty}^{\infty} (x-a)^2 \left|\psi(x)\right|^2 dx &= \int_{-\infty}^{\infty} (x^2 - 2ax + a^2) \left|\psi(x)\right|^2 dx \\
&= \int_{-\infty}^{\infty} x^2 \left|\psi(x)\right|^2 dx - 2a \int_{-\infty}^{\infty} x \left|\psi(x)\right|^2 dx + a^2 \int_{-\infty}^{\infty} \left|\psi(x)\right|^2 dx \\
&= \left\langle X^2 \right\rangle_\psi - 2 \langle X \rangle_\psi \langle X \rangle_\psi + \left(\langle X \rangle_\psi\right)^2 \cdot 1 \\
&= \left\langle X^2 \right\rangle_\psi - 2 \left(\langle X \rangle_\psi\right)^2 + \left(\langle X \rangle_\psi\right)^2 \\
&= \left\langle X^2 \right\rangle_\psi - \left(\langle X \rangle_\psi\right)^2 \\
&> 0
\end{aligned}$$

$\therefore \left\langle X^2 \right\rangle_\psi > \left(\langle X \rangle_\psi\right)^2$

3.6

Consider the Hamiltonian $\hat{H}$ for a quantum harmonic oscillator, given by:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{k}{2}x^2$$

where j is the spring constant of the oscillator. Show that the function

$$\psi_0(x) = e^{-\frac{\sqrt{km}}{2h}x^2}$$

is an eigenvector for $\hat{H}$ with eigenvalue $\dfrac{\hbar\omega}{2}$ where $\omega = \sqrt{\dfrac{k}{m}}$ is the classical frequency of the oscillator.

$$\frac{d}{dx}\psi_0(x) = \frac{d}{dx}e^{-\frac{\sqrt{km}}{2h}x^2} = -\frac{\sqrt{km}}{h}xe^{-\frac{\sqrt{km}}{2h}x^2}$$

$$\frac{d^2}{dx^2}\psi_0(x) = -\frac{\sqrt{km}}{h}\left(e^{-\frac{\sqrt{km}}{2h}x^2} - \frac{\sqrt{km}}{h}x^2e^{-\frac{\sqrt{km}}{2h}x^2}\right) = -\frac{\sqrt{km}}{h}\psi_0(x) + \frac{km}{\hbar^2}x^2\psi_0(x)$$

$$\begin{aligned}
\hat{H}\psi_0(x) &= \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{k}{2}x^2\right)\psi_0(x) \\
&= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_0(x) + \frac{k}{2}x^2\psi_0(x) \\
&= -\frac{\hbar^2}{2m}\left(-\frac{\sqrt{km}}{h}\psi_0(x) + \frac{km}{\hbar^2}x^2\psi_0(x)\right) + \frac{k}{2}x^2\psi_0(x) \\
&= \frac{\hbar}{2}\sqrt{\frac{k}{m}}\psi_0(x) - \frac{k}{2}x^2\psi_0(x) + \frac{k}{2}x^2\psi_0(x) \\
&= \frac{\hbar\omega}{2}\psi_0(x)
\end{aligned}$$