t Distributions

Previously for random variables with normal distributions, $1-\alpha$ confidence intervals for μ were determined when μ is unknown but σ is known. It is now assumed that neither μ nor σ are known.

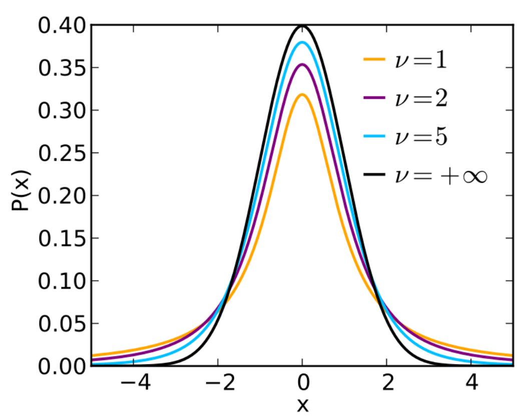
Definition: t Distribution

The *t distribution* with ν degrees of freedom is a continuous distribution whose pdf has the form:

$$f(x) = C\left(1+\frac{x^2}{\nu}\right)^{-\frac{v+1}{2}}$$

for all $x \in \mathbb{R}$.

Properties: t Distributions



1. Symmetric, unimodal, and bell-shaped
2. $E(X) = 0$
3. $V(X) = \frac{v}{v-2} \quad v > 2$
4. $t(v) \to \mathrm{N}(0,1)$ as $v \to \infty$
5. Thicker tails than $\mathrm{N}(0,1)$

Since σ is unknown, S will be used as a point estimate.

Theorem

Let $X_i \overset{\text{iid}}{\sim} \mathrm{N}(\mu, \sigma^2)$ such that μ and σ are unknown:

$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} \sim T$$

Furthermore, the $1-\alpha$ confidence interval for $\bar{X}$ is given by:

$$\bar{x} \pm t_{\frac{\alpha}{2},n-1}\frac{s}{\sqrt{n}}$$

where $t_{\frac{\alpha}{2},n-1}$ is the $\frac{\alpha}{2}$ critical point for the t distribution with $n-1$ degrees of freedom (i.e., $\nu = n-1$).

Example

A sample carton of brown eggs from a farm has $\bar{x} = 65.5$ and $s^2 = 4.69$. Assuming a normal population with unknown variance, obtain the 95% confidence interval.

$$1 - \alpha = 0.95$$

$$\alpha = 1 - 0.95 = 0.05$$

$$\frac{\alpha}{2} = \frac{0.05}{2} = 0.025 t_{0.025,11} = 2.201$$

$$\bar{x} \pm t_{0.025,11} \frac{s}{\sqrt{n}} = 65.5 \pm 2.201 \sqrt{\frac{4.69}{12}} = 65.5 \pm 1.4 = (64.1, 66.9)$$