Strong Convergence

Definition: Strong Convergence

Let E be an inner product space and let $(\vec{x}_n)$ be a sequence of vectors in E. To say that $(\vec{x}_n)$ converges to $\vec{x} \in E$ *strongly*, denoted $\vec{x}_n \to \vec{x}$, means:

$$\|\vec{x}_n - \vec{x}\| \to 0$$

Theorem

Let E be an inner product space and let $(\vec{x}_n)$ be a sequence of vectors in E:

$$\vec{x}_n \to \vec{x} \implies \|\vec{x}_n\| \to \|\vec{x}\|$$

Proof

Assume $\vec{x}_n \to \vec{x}$.

$0 \leq |\|\vec{x}_n\| - \|\vec{x}\|| \leq \|\vec{x}_n - \vec{x}\|$

But $\|\vec{x}_n - \vec{x}\| \to 0$.

$\therefore |\|\vec{x}_n\| - \|\vec{x}\|| \to 0$ and thus $\|\vec{x}_n\| \to \|\vec{x}\|$.

Theorem

Let E be an inner product space and let $(\vec{x}_n)$ and $(\vec{y}_n)$ be sequences of vectors in E:

$$\vec{x}_n \to \vec{x} \text{ and } \vec{y}_n \to \vec{y} \implies \langle \vec{x}_n, \vec{y}_n \rangle \to \langle \vec{x}, \vec{y} \rangle$$

Proof

Assume $\vec{x}_n \to \vec{x}$ and $\vec{y}_n \to \vec{y}$.

This means that $(\vec{x}_y)$ is bounded.
So $\exists\, M > 0$ such that $\forall\, n \in \mathbb{N}$, $\|\vec{y}_n\| \leq M$.

$$\begin{aligned}
|\langle \vec{x}_n, \vec{y}_n \rangle - \langle \vec{x}, \vec{y} \rangle| &= |\langle \vec{x}_n, \vec{y}_n \rangle - \langle \vec{x}, \vec{y}_n \rangle + \langle \vec{x}, \vec{y}_n \rangle - \langle \vec{x}, \vec{y} \rangle| \\
&= |(\langle \vec{x}_n, \vec{y}_n \rangle - \langle \vec{x}, \vec{y}_n \rangle) + (\langle \vec{x}, \vec{y}_n \rangle - \langle \vec{x}, \vec{y} \rangle)| \\
&\leq |\langle \vec{x}_n, \vec{y}_n \rangle - \langle \vec{x}, \vec{y}_n \rangle| + |\langle \vec{x}, \vec{y}_n \rangle - \langle \vec{x}, \vec{y} \rangle| \\
&= |\langle \vec{x}_n - \vec{x}, \vec{y}_n \rangle| - |\langle \vec{x}, \vec{y}_n - \vec{y} \rangle| \\
&\leq \|\vec{x}_n - \vec{x}\| \, \|\vec{y}_n\| - \|\vec{x}\| \, \|\vec{y}_n - \vec{y}\| \\
&\leq \|\vec{x}_n - \vec{x}\| \, M - \|\vec{x}\| \, \|\vec{y}_n - \vec{y}\| \\
&\to 0
\end{aligned}$$

$\therefore \langle \vec{x}_n, \vec{y}_n \rangle \to \langle \vec{x}, \vec{y} \rangle$