Bases

Definition: Finite Basis

Let E be a vector space over a scalar field $\mathbb{F}$ and let B be a non-empty, finite subset of E. To say that B is a *finite basis* of E means:

1). B is a linearly independent set.

2). $\operatorname{Span} B = E$.

A vector space with such a finite basis is called a *finite dimensional* vector space. If no such finite basis exists then the vector space is called an *infinite dimensional* vector space.

$\mathbb{R}^n$ and C^n are finite-dimensional vector spaces. In fact, all other finite-dimensional vector spaces are isomorphic to these.

Now, consider a space like ℓ^2 and the set $S = \{e_k | k \in \mathbb{N}\}$. Since linear combinations are of a finite number of elements, the span of S consists of sequences with a finite number of non-zero elements. This is clearly not all of ℓ^2.

Definition: Closure

Let E be a vector space over a scalar field $\mathbb{F}$ and let $S \subset E$. The *closure* of the span of S, denoted $\overline{S}$, is the union of the $\operatorname{Span}(S)$ and the limits of all partial sums in $\operatorname{Span}(S)$.

Definition: Basis

Let E be a vector space over a scalar field $\mathbb{F}$ and let B be a non-empty subset of E. To say that B is a *basis* of E means:

1). B is a linearly independent set.

2). $\overline{\operatorname{Span}(B)} = E$.

Examples of infinite-dimensional vector spaces are: $C^k(\Omega)$, $P(\Omega)$, and ℓ^p.