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Math 275A
Homework #12

Theorem: 7.32

Let X and Y be topological spaces. The projection maps π_X and π_Y are continuous, surjective, and open.

Proof. Assume $U \in \mathscr{T}_X$. $\pi_X^{-1}(U) = U \times Y \in \mathscr{T}_{X \times Y}$. Therefore π_X is continuous.

Next, assume that $x \in X$. Now, assume that $y \in Y$, and so $(x, y) \in X \times Y$. Thus, $\pi_X(x, y) = X$. Therefore π_X is surjective.

Assume $W \in \mathscr{T}_{X \times Y}$. Then $W = \bigcup_{\alpha \in \lambda} U_\alpha \times V_\alpha$, where $U_\alpha \in \mathscr{T}_X$ and $V_\alpha \in \mathscr{T}_Y$. Now:

$$\pi_X(W) = \pi_X(\bigcup_{\alpha \in \lambda} U_\alpha \times V_\alpha) = \bigcup_{\alpha \in \lambda} \pi_X(U_\alpha \times V_\alpha) = \bigcup_{\alpha \in \lambda} U_\alpha \in \mathscr{T}_X$$

Thus, π_X is open.

A similar argument is used for π_Y.

Therefore, π_X and π_Y are continuous, surjective, and open. ∎

Theorem: 7.36

Let X, Y, and Z be topological spaces. A function $g : Z \to X \times Y$ is continuous iff $\pi_X \circ g$ and $\pi_Y \circ g$ are both continuous.

Proof.

$\Longrightarrow$ Assume that $g : Z \to X \times Y$ is continuous.

Since π_X and π_Y are continuous, and since the composition of continuous functions is continuous, $\pi_X \circ g$ and $\pi_Y \circ g$ are both continuous.

$\Longleftarrow$ Assume that $\pi_X \circ g$ and $\pi_Y \circ g$ are both continuous.

Assume that $W \in \mathscr{T}_{X\times Y}$. So $W = \bigcup_{\alpha\in\lambda} U_\alpha \times V_\alpha$ where $U_\alpha \in \mathscr{T}_X$ and $V_\alpha \in \mathscr{T}_Y$. Then:

$$\begin{aligned}
g^{-1}(W) &= g^{-1}(\bigcup_{\alpha\in\lambda} U_\alpha \times V_\alpha) \\
&= g^{-1}(\bigcup_{\alpha\in\lambda}((U_\alpha \times Y) \cap (X \times V_\alpha))) \\
&= g^{-1}(\pi_X^{-1}(\bigcup_{\alpha\in\lambda} U_\alpha) \cap \pi_Y^{-1}(\bigcup_{\alpha\in\lambda} V_\alpha)) \\
&= g^{-1}(\pi_X^{-1}(\bigcup_{\alpha\in\lambda} U_\alpha)) \cap g^{-1}(\pi_Y^{-1}(\bigcup_{\alpha\in\lambda} V_\alpha)) \\
&= (\pi_X^{-1} \circ g^{-1})(\bigcup_{\alpha\in\lambda} U_\alpha) \cap (\pi_Y \circ g^{-1})(\bigcup_{\alpha\in\lambda} V_\alpha)
\end{aligned}$$

Now, since $\pi_X^{-1} \circ g^{-1}$ is continuous and $\bigcup_{\alpha\in\lambda} U_\alpha \in \mathscr{T}_X$, $(\pi_X^{-1} \circ g^{-1})(\bigcup_{\alpha\in\lambda} U_\alpha) \in \mathscr{T}_X$. Similarly, $(\pi_Y^{-1} \circ g^{-1})(\bigcup_{\alpha\in\lambda} V_\alpha) \in \mathscr{T}_Y$. Thus, $g^{-1}(W) \in \mathscr{T}_Z$.

Therefore $g : Z \to X \times Y$ is continuous.

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Example: Exercise 7.44

Construct a Möbius band explicitly as an identification space of $X = [0, 8] \times [0, 1]$.

$$X^* = \{\{(x, y)\} \mid x \in (0, 8), y \in [0, 1]\} \cup \{(0, y), (8, 1 - y) \mid y \in [0, 1]\}$$

Example: Exercise 7.45

Construct a torus explicitly as:

1. An identification space of a cylinder C.

$$C = \{(R\sin\theta, R\cos\theta, \ell) \mid \theta \in [0, 2\pi), \ell \in [0, L]\}$$

$$\begin{aligned}
C^* = &\{\{(R\sin\theta, R\cos\theta, \ell)\} \mid \theta \in [0, 2\pi), \ell \in (0, L)\} \cup \\
&\{\{(R\sin\theta, R\cos\theta, 0), (R\sin\theta, R\cos\theta, L)\} \mid \theta \in [0, 2\pi)\}
\end{aligned}$$

2. An identification space of $X = [0, 1] \times [0, 1]$.

$$\begin{aligned}
X^* = &\{\{(x, y)\} \mid x \in (0, 1), y \in (0, 1)\} \cup \\
&\{\{(x, 0), (x, 1)\} \mid x \in (0, 1)\} \cup \\
&\{\{(0, y), (1, y)\} \mid y \in [0, 1]\}
\end{aligned}$$

3. An identification space of $\mathbb{R}^2$.

$$(x, y) \sim (u, v) \iff x - u \in \mathbb{Z} \text{ and } y - v \in \mathbb{Z}$$

Theorem: 7.47

The quotient topology actually defines a topology.

Proof. Assume X is a topological space, Y is a set, and $f : X \to Y$ is surjective.

1. $f^{-1}(\emptyset) = \emptyset \in \mathscr{T}_X$. Therefore $\emptyset \in \mathscr{T}_Y$.
2. $f^{-1}(Y) = X \in \mathscr{T}_X$. Therefore $Y \in \mathscr{T}_Y$.
3. Assume that $U, V \in \mathscr{T}_Y$. This means that $f^{-1}(U), f^{-1}(V) \in \mathscr{T}_X$ and so:

   $$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) \in \mathscr{T}_X$$

   Therefore $U \cap V \in \mathscr{T}_Y$.
4. Assume that $\{U_\alpha : \alpha \in \lambda\} \subset \mathscr{T}_Y$. This means that for all $a \in \lambda$, $f^{-1}(U_\alpha) \in \mathscr{T}_X$ and so:

   $$\bigcup_{\alpha \in \lambda} f^{-1}(U_\alpha) = f^{-1}(\bigcup_{\alpha \in \lambda} U_\alpha) \in \mathscr{T}_X$$

   Therefore $\bigcup_{\alpha \in \lambda} U_\alpha \in \mathscr{T}_Y$.

Therefore, the quotient topology on Y defines a topology. ∎

Theorem: 7.48

Let X be a topological space and Y be a set, and let $f : X \to Y$ be surjective. The quotient topology on Y is the finest topology that makes f continuous.

Proof. ABC there exists some topology $\mathscr{T}$ on T that is finer than T_Y. Thus, there exists $U \in \mathscr{T}$ but $U \notin \mathscr{T}_Y$. This would mean that $f^{-1}(U)$ is not open in X, contradicting the continuity of f.

Therefore $\mathscr{T} = \mathscr{T}_Y$. ∎

Theorem: 7.49

Let X and Y be topological spaces and let $f : X \to Y$ be a continuous, surjective, open map. f is a quotient map.

Proof. Let $\mathscr{T}_Y^f = \{U \subset Y \mid f^{-1}(U) \in \mathscr{T}_X\}$. Since $\mathscr{T}_Y^f$ is the finest topology that makes f continuous, it must be the case that $\mathscr{T}_Y \subset \mathscr{T}_Y^f$.

WTS: $\mathscr{T}_Y^f \subset \mathscr{T}_Y$.

Assume $U \in \mathscr{T}_Y^f$. Then, by definition, $f^{-1}(U) \in \mathscr{T}_X$. But f is open and surjective, so:

$$f(f^{-1}(U)) = U \in \mathscr{T}_Y$$

Therefore $\mathscr{T}_Y^f \subset \mathscr{T}_Y$ and hence $T_Y^f = T_Y$. ∎

Theorem: 7.53

Let X, Y, and Z be topological spaces and let $f : X \to Y$ be a quotient map. The map $g : Y \to Z$ is continuous iff $g \circ f$ is continuous.

Proof.

$\implies$ Assume $g : Y \to Z$ is continuous.

But the composition of continuous functions is continuous.

Therefore $g \circ f$ is continuous.

$\impliedby$ Assume $g \circ f$ is continuous.

Assume $W \in \mathscr{T}_Z$, and thus $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \in \mathscr{T}_X$. But f is a quotient map, and so by definition, $g^{-1}(W) \in \mathscr{T}_Y$.

Therefore g is continuous.

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