Uniform Distribution

Definition: Uniform PDF

To say that a continuous random variable X has a *uniform distribution* with parameters a and b, denoted $X \sim \text{Unif}(a, b)$, means that it has the following pdf:

$$f(x) = \begin{cases} \frac{1}{a-b} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Note that:

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{b-a} \int_a^b dx = \frac{1}{b-a} x \Big|_a^b = \frac{1}{b-a}(b-a) = 1$$

Theorem: Uniform CDF

Let X be a continuous random variable such that $X \sim \text{Unif}(a, b)$. The cdf of X is given by:

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b < x \end{cases}$$

Proof.

$$F(x) = \int_{-\infty}^{x} f(x)dx = \frac{1}{b-a} \int_a^x dx = \frac{1}{b-a} x \Big|_a^x = \frac{x-a}{b-a}$$

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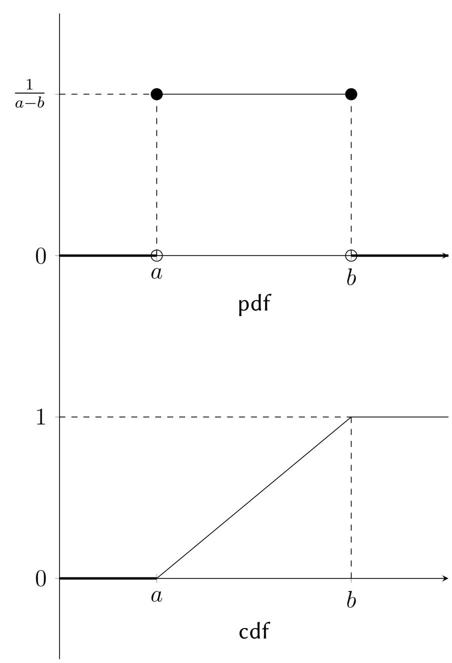

Theorem: Uniform Mean and Variance

Let X be a continuous random variable such that $X \sim \mathrm{Unif}(a, b)$:

- $E(X) = \dfrac{a+b}{2}$
- $V(X) = \dfrac{(b-a)^2}{12}$

Proof.

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{b-a} \int_a^b x dx = \frac{1}{2(b-a)} x^2 \Big|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \frac{1}{b-a} \int_a^b x^2 dx = \frac{1}{3(b-a)} x^3 \Big|_a^b = \frac{b^3 - a^3}{3(b-a)} = \frac{a^2 + ab + b^2}{3}$$

$$\begin{aligned} V(X) &= E(X^2) - E(X)^2 \\ &= \frac{a^2 + ab + b^2}{3} - \frac{(a+b)^2}{4} \\ &= \frac{4(a^2 + ab + b^2) - 3(a^2 + 2ab + b^2)}{12} \\ &= \frac{4a^2 + 4ab + 4b^2 - 3a^2 - 6ab - 3b^2}{12} \\ &= \frac{a^2 - 2ab + b^2}{12} \\ &= \frac{(b-a)^2}{12} \end{aligned}$$

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Example

Let $X \sim \mathrm{Unif}(2, 4)$:

$$f(x) = \frac{1}{4-2} = \frac{1}{2}$$

$$F(x) = \frac{x-2}{4-2} = \frac{x-2}{2} = \frac{1}{2}x - 1$$
$$E(X) = \frac{2+4}{2} = 3$$

$$V(X) = \frac{(4-2)^2}{12} = \frac{4}{12} = \frac{1}{3}$$

Example

A bus arrives at a stop uniformly random between noon and 12:15pm. A passenger arrives at the bus stop exactly at noon. What is the probability that the passenger will wait no more than 5 minutes, between 5 and 10 minutes, or more than 10 minutes for the bus?

Let $X =$ minutes waited. $X \sim \mathrm{Unif}(0, 15)$.

$$f(x) = \frac{1}{15}$$

$$F(x) = \frac{x}{15}$$

$$P(0 \leq X \leq 5) = F(5) = \frac{5}{15} = \frac{1}{3}$$
$$P(5 \leq X \leq 10) = F(10) - F(5) = \frac{10}{15} - \frac{5}{15} = \frac{5}{15} = \frac{1}{3}$$
$$P(10 \leq X \leq 15) = F(15) - F(10) = 1 - \frac{10}{15} = \frac{5}{15} = \frac{1}{3}$$