DeMoivre's Formula

Theorem

Let $n \in \mathbb{Z}$:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Proof

$$(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{i(n\theta)} = \cos(n\theta) + i\sin(n\theta)$$

Theorem

Let $n \in \mathbb{N}$ and:

$$m = \begin{cases} \frac{n-1}{2}, & n \text{ odd} \\ \frac{n}{2}, & n \text{ even} \end{cases}$$

$$\cos(n\theta) = \sum_{k=0}^{m} \binom{n}{2k} (-1)^k \cos^{n-2k}\theta \sin^{2k}\theta$$

$$\sin(n\theta) = \sum_{k=0}^{m} \binom{n}{2k+1} (-1)^k \cos^{n-(2k+1)}\theta \sin^{2k+1}\theta$$

Proof

$$\begin{aligned} \cos(n\theta) + i\sin(n\theta) &= (\cos\theta + i\sin\theta)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta (i\sin\theta)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} i^k \cos^{n-k}\theta \sin^k\theta \end{aligned}$$

Note that $\cos(n\theta)$ is the real terms, occurring at even k:

$$\cos(n\theta) = \sum_{k=0}^{m} \binom{n}{2k} (-1)^k \cos^{n-2k}\theta \sin^{2k}\theta$$

Similarly, $\sin(n\theta)$ is the imaginary terms, occurring at odd k:

$$\sin(n\theta) = \sum_{k=0}^{m} \binom{n}{2k+1} (-1)^k \cos^{n-(2k+1)}\theta \sin^{2k+1}\theta$$

Example

$$\begin{aligned}\cos(2\theta) &= \sum_{k=0}^{1}\binom{2}{2k}(-1)^k\cos^{2-2k}\theta\sin^{2k}\theta\\ &= \binom{2}{0}\cos^2\theta-\binom{2}{1}\sin^2\theta\\ &= \cos^2\theta-\sin^2\theta\end{aligned}$$

$$\begin{aligned}\sin(2\theta) &= \sum_{k=0}^{1}\binom{2}{2k+1}(-1)^k\cos^{2-(2k+1)}\theta\sin^{2k+1}\theta\\ &= \binom{2}{1}\cos\theta\sin\theta-0\\ &= 2\sin\theta\cos\theta\end{aligned}$$

$$\begin{aligned}\cos(3\theta) &= \sum_{k=0}^{1}\binom{3}{2k}(-1)^k\cos^{3-2k}\theta\sin^{2k}\theta\\ &= \binom{3}{0}\cos^3\theta-\binom{3}{2}\cos\theta\sin^2\theta\\ &= \cos^3\theta-3\cos\theta\sin^2\theta\end{aligned}$$

$$\begin{aligned}\sin(3\theta) &= \sum_{k=0}^{1}\binom{3}{2k+1}(-1)^k\cos^{3-(2k+1)}\theta\sin^{2k+1}\theta\\ &= \binom{3}{1}\cos^2\theta\sin\theta-\binom{3}{3}\sin^3\theta\\ &= 3\cos^2\theta\sin\theta-\sin^3\theta\end{aligned}$$