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Math 221a
Homework #6

2.2.1

Prove: A finite abelian group G that is not cyclic contains a subgroup which is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

Let $|G| = n$
In order for finite abelian group G to not be cyclic:

1). $n \geq 4$

2). n is not prime

So, by Corollary II.2.4, G contains a subgroup of order $p^r q^s$ where p and q are primes
But if all p and q are distinct then $G \simeq \bigoplus_{k=1}^{m} \mathbb{Z}_{{p_k}^{r_k}}$, which would make G cyclic and thus a contradiction
Thus, G must contain a subgroup isomorphic to $\mathbb{Z}_{p^r} \oplus \mathbb{Z}_{p^s}$
But both $\mathbb{Z}_{p^r}$ and $\mathbb{Z}_{p^s}$ contain subgroups isomorphic to $\mathbb{Z}_p$
So $\mathbb{Z}_{p^r} \oplus \mathbb{Z}_{p^s}$ contains a subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$

Therefore, G contains a subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

2.2.12

a) What are the invariant factors and elementary divisors of:

i) $\mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{35}$

$2 = 2$
$9 = 3^2$
$35 = 5 \cdot 7$

$$\begin{array}{c} 2 \\ 3^2 \\ 5 \\ 7 \end{array}$$

$2 \cdot 3^2 \cdot 5 \cdot 7 = 630$

Invariant factors: 630
Elementary divisors: $2, 3^2, 5, 7$

ii) $\mathbb{Z}_{26} \oplus \mathbb{Z}_{42} \oplus \mathbb{Z}_{49} \oplus \mathbb{Z}_{200} \oplus \mathbb{Z}_{1000}$

$26 = 2 \cdot 13$
$42 = 2 \cdot 3 \cdot 7$
$49 = 7^2$
$200 = 2^3 \cdot 5^2$
$1000 = 2^3 \cdot 5^3$

$$\begin{array}{cccc}
2 & 2 & 2^3 & 2^3 \\
 & & & 3 \\
 & & 5^2 & 5^3 \\
 & & 7 & 7^2 \\
 & & & 13
\end{array}$$

2
2
$2^3 \cdot 5^2 \cdot 7 = 1400$
$2^3 \cdot 3 \cdot 5^3 \cdot 7^2 \cdot 13 = 191100$

Invariant factors: $2, 2, 1400, 1911000$
Elementary divisors: $2, 2, 2^3, 2^3, 3, 5^2, 5^3, 7, 7^2, 13$

b) Determine up to isomorphism all abelian groups of the following orders:

i) $64 = 2^6$

$\mathbb{Z}_{64}$
$\mathbb{Z}_{32} \oplus \mathbb{Z}_2$
$\mathbb{Z}_{16} \oplus \mathbb{Z}_4$
$\mathbb{Z}_{16} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\mathbb{Z}_8 \oplus \mathbb{Z}_8$
$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$
$\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$
$\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

ii) $96 = 2^5 \cdot 3$

$\mathbb{Z}_{96}$
$\mathbb{Z}_{32} \oplus \mathbb{Z}_3$
$\mathbb{Z}_{16} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$
$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$
$\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$
$\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$
$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$

c) Determine all abelian groups of order n for $n \leq 20$

$1 = 1$	$\{0\}$				
$2 = 2$	$\mathbb{Z}_2$				
$3 = 3$	$\mathbb{Z}_3$				
$4 = 2^2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$			
$5 = 5$	$\mathbb{Z}_5$				
$6 = 2 \cdot 3$	$\mathbb{Z}_6$	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$			
$7 = 7$	$\mathbb{Z}_7$				
$8 = 2^3$	$\mathbb{Z}_8$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$		
$9 = 3^2$	$\mathbb{Z}_9$	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$			
$10 = 2 \cdot 5$	$\mathbb{Z}_{10}$	$\mathbb{Z}_5 \oplus \mathbb{Z}_2$			
$11 = 11$	$\mathbb{Z}_{11}$				
$12 = 2^2 \cdot 3$	$\mathbb{Z}_{12}$	$\mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$		
$13 = 13$	$\mathbb{Z}_{13}$				
$14 = 2 \cdot 7$	$\mathbb{Z}_{14}$	$\mathbb{Z}_7 \oplus \mathbb{Z}_2$			
$15 = 3 \cdot 5$	$\mathbb{Z}_{15}$	$\mathbb{Z}_5 \oplus \mathbb{Z}_3$			
$16 = 2^4$	$\mathbb{Z}_{16}$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_4$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
$17 = 17$	$\mathbb{Z}_{17}$				
$18 = 2 \cdot 3^2$	$\mathbb{Z}_{18}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_9$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$		
$19 = 19$	$\mathbb{Z}_{19}$				
$20 = 2^2 \cdot 5$	$\mathbb{Z}_{20}$	$\mathbb{Z}_5 \oplus \mathbb{Z}_4$	$\mathbb{Z}_5 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$		

2.2.13

Show that the invariant factors of $\mathbb{Z}_m \oplus \mathbb{Z}_n$ are:

a) mn if $(m, n) = 1$

Assume $(m, n) = 1$
m and n have no common prime factors
Let $m = \prod p_i^{r_i}$ and $n = \prod q_j^{s_j}$ where $p_i \neq q_j$
So no combinations of factors of p_i and q_j will result in a product that divides another product

Therefore, the only invariant factor is mn

b) (m, n) and $[m, n]$ if $(m, n) > 1$

Assume $(m, n) = d > 1$
Let $m = m'd$ and $n = n'd$
Let $d = \prod p_i^{a_i}$, $m' = \prod q_j^{b_j}$, and $n' = \prod r_k^{c_k}$ where $q_j \neq r_k$

Arranging the elementary factors:

$$
\begin{array}{cc}
p_1^{a_1} & p_1^{a_1} \\
p_2^{a_2} & p_2^{a_2} \\
\vdots & \vdots \\
 & q_1^{b_1} \\
 & q_2^{b_2} \\
 & \vdots \\
 & r_1^{c_1} \\
 & r_2^{c_2} \\
 & \vdots
\end{array}
$$

Therefore, there are two invariants, (m, n) and $[m, n]$.