Math-42 Practice Final

1. Let $A = \{2, 3, 4\}$ and $B = \{15, 20\}$ and defined the relations U and V between A and B as:

   $$(x, y) \in U \iff x|yV = \{(2, 15), (3, 15), (2, 20)\}$$

   (a) Draw a bubble diagram for U.

   (b) Is U a function?

   (c) Draw a bubble diagram for V.

   (d) Is V a function?

2. Consider the following statements:

   $p :=$ Max is a cat.

   $q :=$ Max is black.

   $r :=$ Max is asleep.

   Construct logic expressions for the following compound statements:

   (a) Max is a black cat but he is not asleep.

   (b) Max is neither a cat, black, nor asleep.

   (c) Max is a cat but he is not both black and asleep.

3. Construct a truth table for the logic expression: $\neg p \vee q \wedge r$.

4. Find a counterexample to show that the following statement is false.

   $$\forall\, n \in \mathbb{N}, n^2 < 2^n$$

5. Let C be the set of all cats in a shelter and define the following predicates:

   $B(c) := c$ has black coloring.

   $W(c) := c$ has white coloring.

   $T(c) := c$ has tan coloring.

   Express each of the following statements using quantifiers:

   (a) There is a white cat who is also tan.

   (b) Every black cat is also white.

   (c) No cats have all three colors.

6. Which of the following is a negation of: "all pigs are smart."

   (a) Some pigs are smart.

   (b) No pigs are smart.

(c) There is at least one dump pig.

(d) All pigs are dumb.

(e) There is at least one smart pig.

(f) Some pigs are dumb.

7. Let $A = \{0, 2, 4, 5, 6\}$.

   (a) Negate the statement: $\forall\, a \in A, \exists, b \in A, a|b$.

   (b) Is the original statement true or false?

   (c) Is the negation true or false?

8. Prove: The sum of any two odd integers is even.

9. Show that the following sum is rational by writing it as the ratio two integers: $\frac{3}{4} + \frac{2}{5}$.

10. Are the following true or false?

    (a) $10|0$

    (b) $0|10$

    (c) $10|10$

    (d) $0|0$

11. Find the division algorithm of the following integers using 9 as the divisor:

    (a) 256

    (b) -256

12. Prove: $5n + 9$ is not divisible by 5.

13. Find a closed form for the sequence:

$$\frac{1}{2} - \frac{1}{3}, \frac{1}{4} - \frac{1}{5}, \frac{1}{6} - \frac{1}{7}, \frac{1}{8} - \frac{1}{9} \dots$$

14. Find the first four terms of the sequence $a_n = 3a_{n-1} + 1$ with the initial condition $a_0 = 5$.

15. Define the family of intervals $\{A_i = [0, i) : i \in N\}$ and evaluate the following:

    (a) $\displaystyle\bigcup_{i=1}^{4} A_i$

    (b) $\displaystyle\bigcap_{i=1}^{4} A_i$

    (c) $\displaystyle\bigcup_{i=1}^{n} A_i$

(d) $\bigcap_{i=1}^{n} A_i$

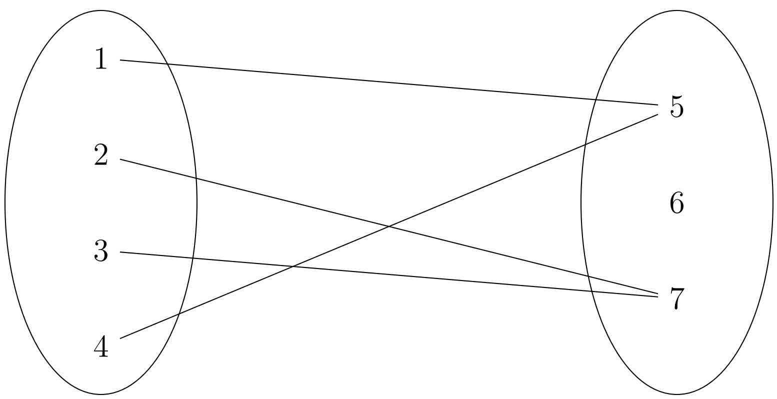
(e) $\bigcup_{i=1}^{\infty} A_i$

(f) $\bigcap_{i=1}^{\infty} A_i$

(g) Are the A_i mutually disjoint, and if so then why specifically?

16. Draw the Venn diagram showing the included regions in $A \cap (B \cup C)$.

17. Consider a function $f : A \to B$ represented by the following diagram:



(a) Write f as a set of ordered pairs.

(b) What is the domain of f?

(c) What is the codomain of f?

(d) What is $f(3)$?

(e) What is the range of f?

(f) What is the image of $\{1, 4\}$?

(g) What is the preimage of $\{5, 6\}$?

18. US phone numbers are of the form NDD–NDD–DDDD where N is a digit from 2 to 9 and D is a digit from 0–9.

(a) How many possible phone numbers are there?

(b) How many possible phone numbers are there that start with 800?

(c) How many possible phone numbers are there if the digits within each part are distinct?

19. Suppose that you have 10 pairs of socks of different colors all jumbled up in your drawer. What is the minimum number of socks that you need to select from the drawer to ensure that you have a pair of the same color?

20. A committee of 4 people must be selected from a staff of 10 people: 6 men and 4 women.

    (a) How many ways are there to select a committee?

    (b) How many ways to select a committee of 2 men and 2 women?

    (c) How many ways to select a committee with at least 1 woman?

    (d) How many ways to select a committee with at most 2 men?

    (e) How many ways are there to select a committee if Bill and Ted refuse to work together?

    (f) How many ways are there to select a committee if Bill and Joy must either both serve or not serve?

21. You take 10 cards out of a standard deck: 6 red (D,H) and 4 black (C,S). You shuffle the 10 cards well and then select 2 at random.

    (a) Construct a decision tree for the probabilities associated with this experiment.

    (b) What is the probability that both cards are red?

    (c) What is the probability that both cards are black?

    (d) What is the probability that a black followed by a red is selected?

    (e) What is the probability that second card is black?

22. What is the minimum number of integers that must be selected between 0 and 10 to ensure that at least one is odd?

23. Consider the numbers from 1 to 100.

    (a) How many are multiples of 5 or 6?

    (b) If a number is selected randomly, what is the probability that it is a multiple of 5 or 6?

    (c) If a number is selected randomly, what is the probability that it is not a multiple of 5 or 6?

24. Consider the word "ONWARD":

    (a) How many ways are there to arrange the letters?

    (b) How many ways are there to arrange the letters if WR must remain together in that order?

    (c) How many ways are there to arrange the letters if WR must remain together in either order?

25. Suppose you flip a fair coin 4 times.

    (a) Describe the sample space using setbuilder notation.

    (b) Describe the sample space using roster notation.

    (c) What is the probability of exactly one heads?

(d) What is the probability of at least two heads?

26. Let $f, g : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = n \bmod 2$ and $g(n) = n + 3$. Evaluate:

    (a) $(f \circ g)(5)$

    (b) $(g \circ f)(5)$

    (c) $(f \circ f)(5)$

    (d) $(g \circ g)(5)$

27. Consider the function of problem 17.

    (a) Is f injective (one-to-one)?

    (b) Is f surjective (onto)?

28. Prove: $f(x) = e^{2x+1}$ is injective (one-to-one).

29. Negative the following statement:

$$\forall\, \epsilon > 0, \exists\, N \in \mathbb{N}, \forall\, n \in \mathbb{N}, (n > N \implies |x_n - x| < \epsilon)$$

30. The following is a rigorous proof of $(p \vee \bar{q}) \wedge (p \vee q) \equiv p$. Justify each of the steps.

$$\begin{aligned}(p \vee \bar{q}) \wedge (p \vee q) &\equiv p \vee (\bar{q} \wedge q) \\ &\equiv p \vee F \\ &\equiv p\end{aligned}$$