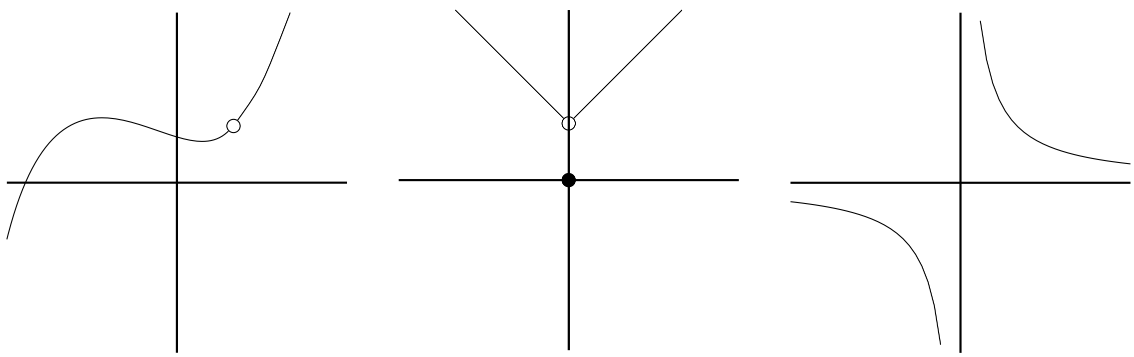
Problems The Need Calculus

We have made the statement that everything in real number algebra can be performed using definitions and theorems based upon the ten axioms, the axioms of equality, and the substitution principle. But there are some questions about functions that cannot be easily answered with these basic algebra techniques:

1. Is a function continuous?

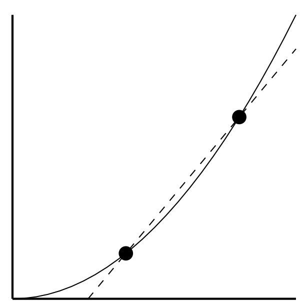
   Up until now we have been doing this visually, looking for holes, jumps, and breaks:

   HOLE JUMP BREAK

2. What is the (instantaneous) rate of change of a function at a point?

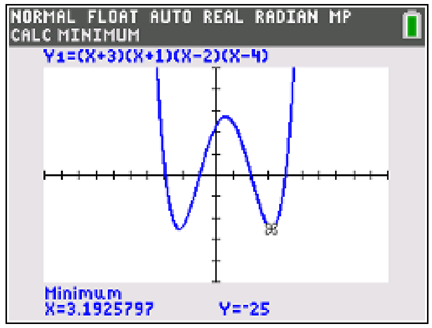
   Up until now all we can do is make an estimate that we call the *average* rate of change around a point based on the slope of a chord drawn between two selected points:

   But this has the drawback of not being exact and the fact that the function can do weird stuff in between the two selected points.

3. Where are the maxima (peaks) and minima (valleys) of a function?

   Up until now we have been plotting the function on a graphing calculator and using the min/max operations:



4. How do we calculate total amounts when the rate of change is not constant?

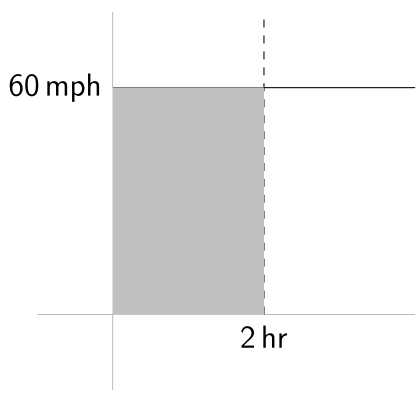
   Recall constrant rate of change problems like:

   $$distance = rate \times time$$

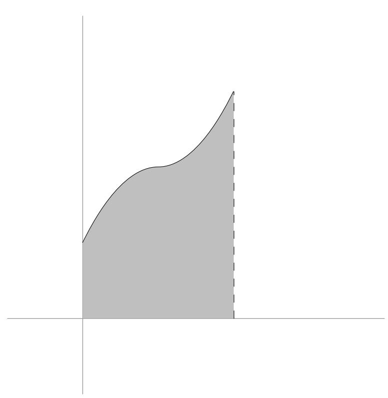
   When the rate and time are constant values, then we can just multiply them to obtain the total distance. For example, if a car is traveling at 60 mph and travels for 2 hours, we can calculate the total distance traveled by:

   $$60 \text{ miles/hour} \cdot 2 \text{ hours} = 120 \text{ miles}$$

   Geometrically, we are calculating the area of the rectangle under the constant rate-of-change curve:

   But what happens when the rate-of-change is a non-constant function of time?

It turns out that the total is still the area under the curve; however, there is no simple geometric formula for calculating that area.

To answer these questions we need a new concept: *arbitrarily close*. Our strategy is going to be to look at *arbitrarily small* intervals around a point and to see how the function is behaving in that interval. This will allow us to make determinations about how the function behaves at the point in question as we get arbitrarily close to that point.