# Simple Rings

## Definition: Simple

Let $R$ be a ring. To say that $R$ is *simple* means that $R$ contains no proper, non-trivial ideals.

## Theorem

Let $R$ be a ring with $1 \neq 0$ and $I \trianglelefteq R$:

$$I = R \iff \exists\, r \in I, r \text{ is a unit in } R$$

Proof

$\implies$ Assume $I = R$

$1 \in R$ and $I = R$, so $1 \in I$
$1 \cdot 1 = 1$
So $1$ is a unit in $R$
Let $r = 1$

$\therefore \exists\, r \in I, r$ is a unit in $R$.

$\impliedby$ Assume $\exists\, r \in I, r$ is a unit in $R$

$\exists\, s \in R, rs = sr = 1$
But $I \trianglelefteq R$ so $1 \in I$
Assume $a \in R$
$1a = a \in I$
So $R \subseteq I$
But clearly, $I \subseteq R$

$\therefore I = R$

## Theorem

Let $R$ be a commutative ring with $1 \neq 0$:

$$R \text{ is simple} \iff R \text{ is a field}$$

Proof

$\implies$ Assume $R$ is simple

Assume $a \in R, a \neq 0$
Since $R$ is simple, $(a) = R$, and in particular, $1 \in (a)$
So $1 = ba$ for some $b \in R$, and thus $a$ is a unit in $R$

Therefore $R$ is a field.

$\Longleftarrow$ Assume $R$ is a field

Assume $I \trianglelefteq R$ such that $I$ is not the zero ideal
There exists $u \in I, u \neq 0$
But $R$ is a field, so $u$ is a unit in $R$
So $I = R$ and thus $R$ has no proper, non-trivial ideals

Therefore $R$ is simple.