Order

Definition

The *order* of a group G, denoted $|G|$ is the cardinality of the set of G.

$\underline{|G| = 1}$

$G = \{e\}$

$*$	e
e	e

$G \simeq \mathbb{Z}_1$
(abelian)

$\underline{|G| = 2}$

$G = \{e, a\}$

$*$	e	a
e	e	a
a	a	e

$G \simeq \mathbb{Z}_2$
(abelian)

$\underline{|G| = 3}$

$G = \{e, a, b\}$

$*$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$G \simeq \mathbb{Z}_3$
(abelian)

Note that each element may appear only once in each row; otherwise, the linear equation $ax = b$ would have more than one solution. Likewise, each element may appear only once in each column; otherwise, the linear equation $xa = b$ would have more than one solution.

$\underline{|G| = 4}$

$G = \{e, a, b, c\}$

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$G \simeq \mathbb{Z}_4$
(abelian)

$*$	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

$b \leftrightarrow c$

$*$	e	a	c	b
e	e	a	c	b
a	a	b	e	c
c	c	e	b	a
b	b	c	a	e

$\rightarrow$

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$G \simeq \mathbb{Z}_4$
(abelian)

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

$a \leftrightarrow b$

$*$	e	b	a	c
e	e	b	a	c
b	b	e	c	a
a	a	c	b	e
c	c	a	e	b

$\rightarrow$

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$G \simeq \mathbb{Z}_4$
(abelian)

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

This group is also abelian; however, $G \not\simeq \mathbb{Z}_4$ because:

$$\forall\, x \in G, xx = e$$

Thus, it is structurally different from $\mathbb{Z}_4$. It is referred to as the Klein-4 group, denoted by V or K_4.

So, every group of 4 elements is isometric to either $\mathbb{Z}_4$ or K_4. Thus, there are only 2 distinct groups *up to isomorphism*.