Trace

Definition: Trace

Let $A \in M_n$. The *trace* of A is given by:

$$\operatorname{tr}(A) = \sum_{k=1}^{n} a_{kk}$$

In other words, the trace is the sum of the diagonal entries.

Properties: Trace

1). $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$

2). $\operatorname{tr}(A) = \operatorname{tr}(A^T)$

3). $\operatorname{tr}(cA) = c \operatorname{tr}(A)$

Theorem

Let $f : M_n \to \mathbb{C}$ be a linear transformation such that $f(AB) = f(BA)$:

$$f(I_n) = n \iff f \text{ is the trace}$$

Proof

Assume $f(I_n) = n$

Assume $A \in M_n$
Let $f(A) = f\left(\sum_{i,j} a_{ij} E_{ij}\right) = \sum_{i,j} a_{ij} f(E_{ij})$

Case 1: $i \neq j$

$$f(E_{ij}) = f(E_{i1}E_{1j}) = f(E_{1j}E_{i1}) = f(0) = 0$$

Case 2: $i = j$

$$f(E_{ii}) = f(E_{i1}E_{1i}) = f(E_{1i}E_{i1}) = f(E_{11}) = \alpha$$

So, after discarding the zero $f(E_{ij})$ entries and replacing the $f(E_{ii})$ entries with α:

$$f(A) = \alpha \sum_{i=1}^{n} a_{ii}$$

But $f(I) = \alpha n = n$
So $\alpha = 1$

$\therefore f(A) = \sum_{i=1}^{n} a_{ii} = \operatorname{tr}(A)$

Assume f is the trace

$$f(I_n) = tr(I_n) = n \cdot 1 = n$$