Lagrange's Identity

Theorem

$$\sum_{k=0}^{n} \cos k\theta = \frac{1}{2} + \frac{\sin\left[(2n+1)\frac{\theta}{2}\right]}{2\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\left[(n+1)\frac{\theta}{2}\right]\cos\left(n\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

$$\sum_{k=0}^{n} \sin k\theta = \frac{\cot\left(\frac{\theta}{2}\right)}{2} - \frac{\cos\left[(2n+1)\frac{\theta}{2}\right]}{2\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\left[(n+1)\frac{\theta}{2}\right]\sin\left(n\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

Proof

Let $R = \sum_{k=0}^{n} \cos k\theta$
Let $I = \sum_{k=0}^{n} \sin k\theta$

$$\begin{aligned}
R + iI &= \sum_{k=0}^{n} \cos k\theta + i\sum_{k=0}^{n} \sin k\theta \\
&= \sum_{k=0}^{n} (\cos k\theta + i\sin k\theta) \\
&= \sum_{k=0}^{n} \left(e^{i\theta}\right)^k \\
&= \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1} \\
&= \frac{e^{-i\frac{\theta}{2}}\left[e^{i(n+1)\theta} - 1\right]}{e^{-i\frac{\theta}{2}}\left(e^{i\theta} - 1\right)} \\
&= \frac{\left[e^{i(n\theta+\theta-\frac{\theta}{2})} - e^{-i\frac{\theta}{2}}\right]}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}} \\
&= \frac{\left[e^{i(n\theta+\frac{\theta}{2})} - e^{-i\frac{\theta}{2}}\right]}{2i\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{\left[\cos\left(n\theta+\frac{\theta}{2}\right) + i\sin\left(n\theta+\frac{\theta}{2}\right)\right] - \left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)}{2i\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{\left[\cos\left(n\theta+\frac{\theta}{2}\right) - \cos\frac{\theta}{2}\right] + i\left[\sin\left(n\theta+\frac{\theta}{2}\right) + \sin\frac{\theta}{2}\right]}{2i\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{\left[\sin\left(n\theta+\frac{\theta}{2}\right) + \sin\frac{\theta}{2}\right] - i\left[\cos\left(n\theta+\frac{\theta}{2}\right) - \cos\frac{\theta}{2}\right]}{2\sin\left(\frac{\theta}{2}\right)}
\end{aligned}$$

$$\sum_{k=0}^{n} \cos(k\theta) = Re(R+iI) = \frac{\sin\frac{\theta}{2} + \sin\left(n\theta + \frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} = \frac{1}{2} + \frac{\sin\left[(2n+1)\frac{\theta}{2}\right]}{2\sin\left(\frac{\theta}{2}\right)}$$

$$\sum_{k=0}^{n} \sin(k\theta) = Im(R+iI) = \frac{\cos\frac{\theta}{2} - \cos\left(n\theta + \frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} = \frac{\cot\left(\frac{\theta}{2}\right)}{2} - \frac{\cos\left[(2n+1)\frac{\theta}{2}\right]}{2\sin\left(\frac{\theta}{2}\right)}$$

Continuing:

$$\begin{aligned}
R+iI &= \frac{\left[\sin\left(n\theta+\frac{\theta}{2}\right) + \sin\frac{\theta}{2}\right] - i\left[\cos\left(n\theta+\frac{\theta}{2}\right) - \cos\frac{\theta}{2}\right]}{2\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{2\sin\left(\frac{n\theta+\frac{\theta}{2}+\frac{\theta}{2}}{2}\right)\cos\left(\frac{n\theta+\frac{\theta}{2}-\frac{\theta}{2}}{2}\right) + i2\sin\left(\frac{n\theta+\frac{\theta}{2}+\frac{\theta}{2}}{2}\right)\sin\left(\frac{n\theta+\frac{\theta}{2}-\frac{\theta}{2}}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{\sin\left(\frac{n\theta+\frac{\theta}{2}+\frac{\theta}{2}}{2}\right)\cos\left(\frac{n\theta+\frac{\theta}{2}-\frac{\theta}{2}}{2}\right) + i\sin\left(\frac{n\theta+\frac{\theta}{2}+\frac{\theta}{2}}{2}\right)\sin\left(\frac{n\theta+\frac{\theta}{2}-\frac{\theta}{2}}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{\sin\left(\frac{n\theta+\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right) + i\sin\left(\frac{n\theta+\theta}{2}\right)\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}
\end{aligned}$$

$$\sum_{k=0}^{n} \cos k\theta = Re(R+iI) = \frac{\sin\left(\frac{n\theta+\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\left[(n+1)\frac{\theta}{2}\right]\cos\left(n\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

$$\sum_{k=0}^{n} \sin k\theta = Im(R+iI) = \frac{\sin\left(\frac{n\theta+\theta}{2}\right)\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\left[(n+1)\frac{\theta}{2}\right]\sin\left(n\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$