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Math 221a
Homework #7

Exercise 1

Given groups H and K and homomorphism $\phi : K \to \mathrm{Aut}(H)$, prove that $H \rtimes K$ is a group.

Assume $(h_1, k_1), (h_2, k_2) \in H \rtimes K$ and consider the binary operation:

$$(h_1, k_1)(h_2, k_2) = (h_1\phi(k_1)(h_2), k_1k_2)$$

The binary operations for the groups H and K are well-defined and closed
ϕ is well-defined
$\phi(k_1)$ is an automorphism of H
$\phi(k_1)(h_2) \in H$
$h_1\phi(k_1)(h_2) \in H$
$k_1k_2 \in K$
So $(h_1\phi(k_1)(h_2), k_1k_2) \in H \rtimes K$
Therefore the binary operation is well-defined and closed.

Assume $(h_1, k_1), (h_2, k_2), (h_3, k_3) \in H \rtimes K$

$$\begin{aligned}
[(h_1, k_1)(h_2, k_2)](h_3, k_3) &= (h_1\phi(k_1)(h_2), k_1k_2)(h_3, k_3) \\
&= (h_1\phi(k_1)(h_2)\phi(k_1k_2)(h_3), (k_1k_2)k_3) \\
&= (h_1\phi(k_1)(h_2)\phi(k_1)(\phi(k_2)(h_3)), k_1(k_2k_3)) \\
&= (h_1\phi(k_1)(h_2\phi(k_2)(h_3)), k_1(k_2k_3)) \\
&= (h_1, k_1)(h_2\phi(k_2)(h_3), k_2k_3) \\
&= (h_1, k_1)[(h_2, k_2)(h_3, k_3)]
\end{aligned}$$

Therefore the operation is associative.

Consider $(e_H, e_K) \in H \rtimes K$
Assume $(h, k) \in H \rtimes K$
$\phi(k)$ is an isomorphism, so $\phi(k)(e_H) = e_H$
ϕ is a homomorphism, so $\phi(e_K) = \iota_H$

$$(h, k)(e_H, e_K) = (h\phi(k)(e_H), ke_K) = (he_H, k) = (h, k)$$

$$(e_H, e_K)(h, k) = (e_H\phi(e_K)(h), e_Kk) = (\phi(e_k)(h), k) = (\iota_H(h), k) = (h, k)$$

Therefore (e_H, e_K) is the identity element for $H \rtimes K$.

Assume $(h, k) \in H \rtimes K$
$h^{-1} \in H$
$k^{-1} \in K$
$\phi(k^{-1}) \in \mathrm{Aut}(H)$

$\phi(k^{-1})(h^{-1}) \in H$
$(\phi(k^{-1})(h^{-1}), k^{-1}) \in H \rtimes K$

Consider $(\phi(k^{-1})(h^{-1}), k^{-1})$:

$$\begin{aligned}
(h,k)(\phi(k^{-1})(h^{-1}), k^{-1}) &= (h\phi(k)(\phi(k^{-1})(h^{-1})), kk^{-1}) \\
&= (h\phi(kk^{-1})(h^{-1}), e_K) \\
&= (h\phi(e_K)(h^{-1}), e_K) \\
&= (h\iota_H(h^{-1}), e_K) \\
&= (hh^{-1}, e_K) \\
&= (e_H, e_K)
\end{aligned}$$

$$\begin{aligned}
(\phi(k^{-1})(h^{-1}), k^{-1})(h,k) &= (\phi(k^{-1})(h^{-1})\phi(k^{-1})(h), k^{-1}k) \\
&= (\phi(k^{-1})(h^{-1}h), e_K) \\
&= (\phi(k^{-1})(e_H), e_K) \\
&= (e_H, e_K)
\end{aligned}$$

Therefore $H \rtimes K$ is closed under inverses.

Therefore $H \rtimes K$ is a group.

Exercise 2

Let $\phi : K \to \mathrm{Aut}(H)$ be the trivial homomorphism. Prove $H \rtimes K$ is the same as $H \times K$.

Assume $(h_1, k_1), (h_2, k_2) \in H \rtimes K$
$(h_1, k_1)(h_2, k_2) = (h_1\phi(k_1)(h_2), k_1k_2) = (h_1\iota_H(h_2), k_1k_2) = (h_1h_2, k_1k_2)$

Therefore $H \rtimes K$ is the same as $H \times K$.