Convex Sets

Definition: Line Segment

Let E be a vector space and $\vec{x}, \vec{y} \in E$. The *line segment* from $\vec{x}$ to $\vec{y}$, denoted $\overline{xy}$, is given by:

$$\overline{xy} = \{(1-t)x + ty \mid t \in [0,1]\}$$

Definition: Convex

Let E be a vector space. To say that $S \subset E$ is *convex* means:

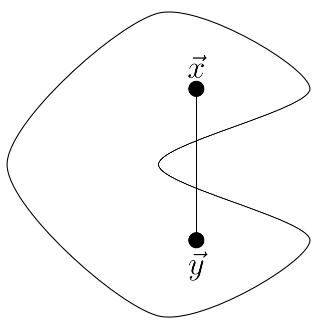
$$\forall\, \vec{x}, \vec{y} \in S, \overline{xy} \subset S$$

Examples

1). Vector spaces

2). Closed balls

3). Not convex:

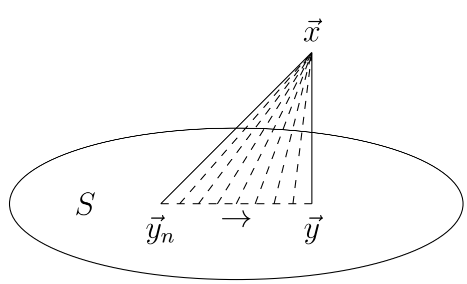


Theorem: Closest Point Property

Let H be a Hilbert space and let S be a closed and convex subset of H. $\forall\, \vec{x} \in H$, there exists a unique closest point $\vec{y} \in S$ to $\vec{x}$:

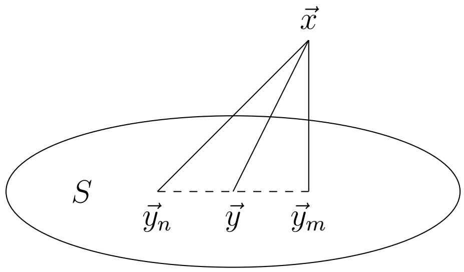
$$\exists!\, \vec{y} \in S, d(\vec{x}, S) = \|\vec{x} - \vec{y}\| = d(x, y)$$

Proof



Assume $\vec{x} \in H$.
$d(\vec{x}, S) = \inf_{\vec{y} \in S} \|\vec{x} - \vec{y}\|$
So $\exists\, (\vec{y}_n)$ in S such that $\|\vec{x} - \vec{y}_n\| \to d$.

Claim: $(\vec{y}_n)$ is Cauchy.



Let $\vec{y} = \dfrac{\vec{y}_n + \vec{y}_m}{2}$.
Note that $\vec{y} \in S$ because S is convex.

Apply the parallelogram law with $\vec{x} - \vec{y}_n$ and $\vec{x} - \vec{y}_m$:

$$\begin{aligned} \|(\vec{x}-\vec{y}_n)+(\vec{x}-\vec{y}_m)\|^2 + \|(\vec{x}-\vec{y}_n)-(\vec{x}-\vec{y}_m)\|^2 &= 2\|\vec{x}-\vec{y}_n\|^2 + 2\|\vec{x}-\vec{y}_m\|^2 \\ \|2\vec{x}-(\vec{y}_m+\vec{y}_n)\|^2 + \|\vec{y}_m-\vec{y}_n\|^2 &= 2\|\vec{x}-\vec{y}_n\|^2 + 2\|\vec{x}-\vec{y}_m\|^2 \\ 4\left\|\vec{x}-\frac{\vec{y}_m+\vec{y}_n}{2}\right\|^2 + \|\vec{y}_m-\vec{y}_n\|^2 &= 2\|\vec{x}-\vec{y}_n\|^2 + 2\|\vec{x}-\vec{y}_m\|^2 \end{aligned}$$

And so:

$$\|\vec{y}_m-\vec{y}_n\|^2 = 2\|\vec{x}-\vec{y}_n\|^2 + 2\|\vec{x}-\vec{y}_m\|^2 - 4\left\|\vec{x}-\frac{\vec{y}_m+\vec{y}_n}{2}\right\|^2 \to 2d^2+2d^2-4d^2 = 0$$

Therefore $(\vec{y}_n)$ is Cauchy.

Now, H is Hilbert and thus complete, so $\vec{y}_n \to \vec{y} \in H$.
But S is closed, and so $\vec{y} \in S$.

Therefore such a $\vec{y} \in S$ exists.

Now, assume that there are two such points: $\vec{y}$ and $\vec{y}\,'$.
Applying the parallelogram law with $\vec{x} - \vec{y}$ and $\vec{x} - \vec{y}\,'$:

$$\begin{aligned} \|(\vec{x}-\vec{y})+(\vec{x}-\vec{y}\,')\|^2 + \|(\vec{x}-\vec{y})-(\vec{x}-\vec{y}\,')\|^2 &= 2\|\vec{x}-\vec{y}\|^2 + 2\|\vec{x}-\vec{y}\,'\|^2 \\ \|2\vec{x}-(\vec{y}\,'+\vec{y})\|^2 + \|\vec{y}\,'-\vec{y}\|^2 &= 2\|\vec{x}-\vec{y}\|^2 + 2\|\vec{x}-\vec{y}\,'\|^2 \\ 4\left\|\vec{x}-\frac{\vec{y}\,'+\vec{y}}{2}\right\|^2 + \|\vec{y}\,'-\vec{y}\|^2 &= 2\|\vec{x}-\vec{y}\|^2 + 2\|\vec{x}-\vec{y}\,'\|^2 \end{aligned}$$

And so:

$$\|\vec{y}\,'-\vec{y}\|^2 = 2\|\vec{x}-\vec{y}\|^2 + 2\|\vec{x}-\vec{y}\,'\|^2 - 4\left\|\vec{x}-\frac{\vec{y}\,'+\vec{y}}{2}\right\|^2 \le 2d^2+2d^2-4d^2 = 0$$

Therefore $\vec{y}\,' - \vec{y} = 0$ and thus $\vec{y} = \vec{y}\,'$, proving uniqueness.