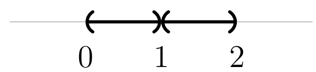
Connectedness

Intuitively, a topological space X exists as a single piece.

Example: Disconnected

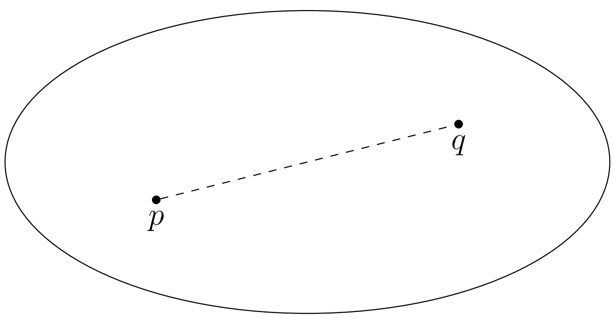
$$X = (0,1) \cup (2,3)$$



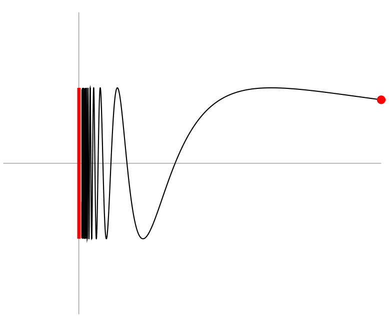
$$X = (0,1) \cup (1,2)$$

There are two notions of connectedness:

1. Connected: One piece
2. Path Connected: The ability to *walk* from any point to any other.



Example: The Topologists Sine Curve



$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \,\middle|\, x \in (0,1) \right\}$$

$$\bar{S} = S \cup \{(1, \sin(1))\} \cup \{(0,y) \,|\, y \in [-1,1]\}$$

S is connected and path connected.

$\bar{S}$ is connected but not path connected.

Definition: Connected

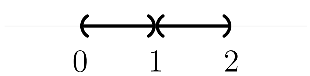
Let X be a topological space. To say that X is *connected* means that X is not the union of two disjoint non-empty open sets.

Note that if $X = U \sqcup V$ such that $U, V \in \mathscr{T}$ then $U = X - V$ and $V = X - U$, so both U and V are clopen.

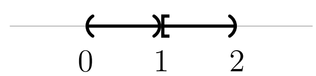
Definition: Separated

Let X be a topological space and $A, B \subset X$. To say that A and B are *separated* means that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$. Thus, A and B do no contain each other's limit points.

Example: Separated



$X = (0,1) \cup (1,2)$
Disjoint/Separated



$X = (0,1) \cup [1,2)$
Disjoint/Not Separated

Notation

$X = A|B$ means that $X = A \cup B$ and A and B are separated sets.

Definition: Reachable

Let X be a topological space and let $p, q \in X$. To say that q is reachable from p means that for every open cover $\{U_\alpha : \alpha \in \lambda\}$ of X, there exists a finite subset $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ such that $p \in U_{\alpha_1}$, $q \in U_{\alpha_n}$, and for all $1 \leq k < n$, $U_{\alpha_k} \cap U_{\alpha_{k+1}} \neq \emptyset$.

Lemma

Reachable is an equivalence relation.

Proof. Assume that X is a topological space, $\{U_\alpha : \alpha \in \lambda\}$ is an open cover for X, and $x, y, z \in X$.

R: There exists some U_{a_k} such that $x \in U_{\alpha_k}$. Thus x is trivially reachable from x.

S: Assume that x is reachable from y. This means that there exists a finite subset $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ that links x and y. Taking those sets in reverse order links y and x. Therefore y is reachable from x.

T: Assume that y is reachable from x and z is reachable from y. Then there exists a finite subset $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ linking x and y and a finite subset $\{U_{\beta_1}, \ldots, U_{\beta_m}\}$ linking y and z. Let

$U_{\alpha_j} = U_{\beta_k}$ be the first common subset in the two paths. Then $\{U_{\alpha_1}, \ldots, U_{\alpha_j}, U_{\beta_{k+1}}, \ldots, U_{\beta_m}\}$ is a finite subset linking x and z. Therefore z is reachable from x.

■

Theorem

Let X be a topological space. TFAE:

1. X is connected.
2. There are no continuous functions $f : X \to \mathbb{R}$ such that $f(X) = \{0, 1\}$.
3. X is not the union of two disjoint non-empty separated sets.
4. X is not the union of two disjoint non-empty closed sets.
5. The only clopen sets of X are $\emptyset$ and X.
6. For all $p, q \in X$, q is reachable from p.

Proof.

$(1 \implies 2)$ Assume that X is connected.

ABC that there exists a continuous function $f : X \to \mathbb{R}$ such that $f(X) = \{0, 1\}$. Let $U = f^{-1}(\{0\})$ and $V = f^{-1}(\{1\})$. Since $\{0\}$ and $\{1\}$ are closed in $\mathbb{R}$ and f is continuous, U and V are closed in X. But $U \sqcup V = X$, so $U = X - V$ and $V = X - U$ meaning that $U, V \in \mathscr{T}$ also, contradicting the connectedness of X.

Therefore there are no continuous functions $f : X \to \mathbb{R}$ such that $f(X) = \{0, 1\}$.

$(2 \implies 3)$ Assume that there are no continuous functions $f : X \to \mathbb{R}$ such that $f(X) = \{0, 1\}$.

ABC that there exists $A, B \subset X$ such that $X = A|B$ and consider $f : X \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 0, & x \in A \\ 1, & x \in B \end{cases}$$

Now, $\{0\}$ is closed in $\mathbb{R}$ and $f^{-1}(\{0\}) = A$. But $X = A|B$ and $\bar{A} \cap B = \emptyset$, so A must contain all of its own limit points, hence $A = \bar{A}$, meaning that A is closed in X. Similarly, B is closed in X. Thus, f is continuous, contradicting the assumption of non-existence.

Therefore X is not the union of two disjoint non-empty separated sets.

$(3 \implies 4)$ (CP) Assume that X is the union of two disjoint non-empty closed sets.

This means that there exists A and B that are closed in X such that $X = A \sqcup B$ and $A, B \neq \emptyset$. But $A = \bar{A}$ and so $\bar{A} \cap B = \emptyset$. Similarly, $A \cap \bar{B} = \emptyset$. Thus, $X = A|B$.

Therefore X is the union of two disjoint non-empty separated sets.

$(4 \implies 5)$ (CP) Assume that there exists A clopen in X such that $A \neq \emptyset$ and $A \neq X$.

This means that $X - A$ is also clopen. So A and $X - A$ are closed in X, $X \sqcup (X - A) = X$, and $X, X - A \neq \emptyset$.

Therefore X is the union of two disjoint non-empty closed sets.

$(5 \implies 6)$ Assume that the only clopen sets of X are $\emptyset$ and X.

Assume that $p \in X$ and define $U = \{q \in X \mid q \text{ is reachable from } p\}$.

WTS: $U = X$

First, note that $p \in U$ and so $U \neq \emptyset$.

Claim: $U \in \mathscr{T}$

Assume that $q \in U$. This means that for any open cover $\{U_\alpha : \alpha \in \lambda\}$ of X there exists some finite subset $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ linking p and q such that $q \in U_{\alpha_n}$. But every other point in U_{α_n} is reachable from p, and so $U_{\alpha_n} \subset U$. Thus all $q \in U$ are interior points and therefore U is open.

Claim: U is closed in X

Assume that $q \in \bar{U}$. This means that for all $U_q \in \mathscr{T}$ such that $q \in U_q$, it must be the case that $U_q \cap U \neq \emptyset$. So any open cover of X must include some such U_q. Let $r \in U_q \cap U$. The r is reachable from p and r is trivially reachable from q, and so q is reachable from p. Therefore $q \in U$ and so $U = \bar{U}$, hence U is closed.

Thus U is clopen and $U \neq \emptyset$, so $U = X$.

Therefore, for all $p, q \in X$, q is reachable from p.

$(6 \implies 1)$ Assume that for all $p, q \in X$, q is reachable from p.

ABC that X is not connected. This means that there exists $U, V \in \mathscr{T}$ such that $U \sqcup V = X$, and so $\{U, V\}$ is an open cover for X. Now, assume that $p \in U$ and $q \in V$. There is no finite subset of this two-set cover that allows q to be reachable from p, contradicting the assumption.

Therefore X is connected.

■

Example

Determine whether the following topological spaces are connected or disconnected:

1. $\mathbb{R}$ with the discrete topology.

   Let $U = (-\infty, 0)$ and $V = [0, \infty)$. $U, V \in \mathscr{T}$ and $U \sqcup V = \mathbb{R}$.

   Disconnected

2. $\mathbb{R}$ with the indiscrete topology.

   $\mathscr{T} = \{\emptyset, \mathbb{R}\}$, so there are no open non-empty $U, V \in \mathscr{T}$ such that $U \sqcup V = \mathbb{R}$.

   Connected

3. $\mathbb{R}$ with the cofinite topology.

   ABC there exists non-empty $U, V \in \mathscr{T}$ such that $U \sqcup V = \mathbb{R}$. The $V = \mathbb{R} - U$ is finite and $U = \mathbb{R} - V$ is finite, meaning that $U \sqcup V = \mathbb{R}$ is finite, a contradiction.

   Connected.

4. $\mathbb{R}_{\mathsf{LL}}$

   Let $U = (-\infty, 0)$ and $V = [0, \infty)$. $U, V \in \mathscr{T}$ and $U \sqcup V = \mathbb{R}$.

   Disconnected

5. $\mathbb{Q}$ as a subspace of R

   Let $U = (-\infty, \pi) \cap \mathbb{Q}$ and $V = (\pi, \infty) \cap \mathbb{Q}$. $U, V \in \mathscr{T}_{\mathbb{Q}}$ and $U \sqcup V = \mathbb{Q}$.

   Disconnected

6. $\mathbb{R} - \mathbb{Q}$ as a subspace of R

   Let $U = (-\infty, 0) \cap (\mathbb{R} - \mathbb{Q})$ and $V = (0, \infty) \cap (\mathbb{R} - \mathbb{Q})$. $U, V \in \mathscr{T}_{\mathbb{R}-\mathbb{Q}}$ and $U \sqcup V = \mathbb{R} - \mathbb{Q}$.

   Disconnected

<u>Theorem</u>

$\mathbb{R}_{\mathsf{std}}$ is connected.

Proof. Since $\mathbb{R}$ is homeomorphic to $(0, 1)$, it is sufficient to show that $(0, 1)$ is connected. So ABC that $(0, 1)$ is disconnected. This means that there exists $A \subset (0, 1)$ such that $A \neq \emptyset, (0, 1)$ and A is clopen. Since A is bounded, it has a $\sup$, so let $a = \sup A$. But A is closed, so $a \in A$. But A is also open, so there exists $\epsilon > 0$ such that $B(a, \epsilon) \subset A$, violating the fact that $a = \sup A$. Therefore $(0, 1)$ is connected, and so $\mathbb{R}$ is connected. ∎

<u>Corollary</u>

An open interval in R is connected.

<u>Definition: Interval</u>

To say that $I \subset \mathbb{R}$ is an *interval* means that for all $a, b \in I$, $[a, b] \subset I$.

<u>Theorem</u>

$C \subset \mathbb{R}$ is connected iff C is an interval.

Proof. Assume $C \subset \mathbb{R}$.

$\Longrightarrow$ Assume that C is connected.

ABC that C is not an interval. So there exists $a, b \in C$ such that $[a, b] \not\subset C$. So there exists $z \in [a, b]$ such that $z \notin C$. Let $U = (-\infty, z) \cap C$ and $V = (z, \infty) \cap C$. $U, V \in \mathscr{T}_C$, $U, V \neq \emptyset, C$, and $U \sqcup V = C$. Thus, C is disconnected, contradicting the assumption that C is connected. Therefore C is an interval.

$\Longleftarrow$ Assume that C is an interval.

Already proved by previous theorem.

■

Note that the union of connected sets need not be connected. For example, $U = (0, 1)$ and $V = (1, 2)$ are both connected; however, $U \sqcup V$ is not an interval and hence is not connected.

Theorem

Let X be a topological space and let $A, B \subset X$ be separated. If $C \subset A \cap B$ is connected then either $C \subset A$ or $C \subset B$ (but not both).

Theorem

Let X be a topological space and let $\{C_\alpha : \alpha \in \lambda\}$ be a collection of connected subsets of X. Furthermore, let $E \subset X$ also be connected such that for all $\alpha \in \lambda$, $C_\alpha \cap E \neq \emptyset$. $E \cup \bigcup_{\alpha \in \lambda} U_\alpha$ is connected.

Theorem

Let X be a topological space and let $C \subset X$ be connected. If $D \subset X$ such that $C \subset D \subset \bar{C}$ then D is connected.

Proof. Assume $D \subset X$ such that $C \subset D \subset \bar{C}$ and ABC that D is disconnected. This means that there exists $A, B \subset D$ such that $A, B \neq \emptyset$ and $D = A|B$. Now, since $C \subset D$, it must be the case that either $C \subset A$ or $C \subset B$ (but not both). So AWLOG that $C \subset A$, and hence $\bar{C} \subset \bar{A}$. But $\bar{A} \cap B = \emptyset$ and so $\bar{C} \cap B = \emptyset$. And since $D \subset \bar{C}$, $D \subset B = \emptyset$. But this can only be the case if $B = \emptyset$, contradicting the assumption that B is not empty. Therefore D is connected. ■

Corollary

Let X be a topological space and $C \subset X$. If C is connected then $\bar{C}$ is connected.

Proof. Assume that C is connected. But $C \subset \bar{C} \subset \bar{C}$. Therefore, by previous theorem, $\bar{C}$ is connected. ■

Theorem

The closure of the topologist's sine curve in $\mathbb{R}^2$ is connected.

Proof. Let:

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \middle| \, x \in (0,1) \right\}$$

$$\bar{S} = S \cup \{(1, \sin(1))\} \cup \{(0,y) \,|\, y \in [-1,1]\}$$

ABC that S is not connected. This means that there exists $g : S \to \{0,1\}$ such that g is continuous and surjective. But $f : (0,1) \to S$ defined by $f(x) = (x, \sin \frac{1}{x})$ is also continuous and surjective. This means that $g \circ f : (0,1) \to \{0,1\}$ is also continuous and surjective, indicating that $(0,1)$ is not connected, contradicting the connectedness of the interval. Therefore S is connected, and by previous corollary, $\bar{S}$ is connected. ■

Theorem

Let X and Y be a topological spaces and let $f : X \to Y$ be continuous and surjective. If X is connected then Y is connected.

Proof. Assume that X is connected and ABC that Y is disconnected. This means that there exists $U, V \in \mathscr{T}_Y$ such that $U, V \neq \emptyset$ and $U \sqcup V = Y$. Now, since f is continuous, $f^{-1}(U), f^{-1}(V) \in \mathscr{T}_X$. Furthermore, since f is surjective, $f^{-1}(U), f^{-1}(V) \neq \emptyset$ and $f^{-1}(U) \sqcup f^{-1}(V) = X$. Thus, X is disconnected, violating the assumption. Therefore Y is connected. ■

Corollary

Let X and Y be homemorphic topological spaces. X is connected iff Y is connected.

Proof. It is sufficent to prove one direction, so assume that X is connected. This means that there exists a homeomorphism $f : X \to Y$. But homeomorphism are continuous and surjective. Therefore Y is connected. ■

Theorem: IVT

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $(a,b) \subset \mathbb{R}$. If $f(a) < r < f(B)$ then there exists $c \in (a,b)$ such that $f(c) = r$.

Proof. Assume that $f(a) < r < f(b)$. Since $[a,b]$ is connected and f is continuous, $f([a,b])$ is connected, and hence $f([a,b])$ must be an interval. But $f(a), f(b) \in f([a,b])$ and $f(a) < r < f(b)$, so $r \in f((a,b))$. Therefore, there must exist some $c \in (a,b)$ such that $f(c) = r$. ■

Theorem

Let X and Y be topological spaces. $X \times Y$ is connected iff X and Y are connected.

Proof.

$\implies$ Assume that $X \times Y$ is connected.

π_X and π_Y are continuous and surjective. Therefore X and Y are connected.

$\impliedby$ Assume that X and Y are connected.

Assume $x_0 \in X$ and consider $\{x_0\} \times Y$. Since $\{x_0\} \times Y$ is homeomorphic to Y and Y is connected, $\{x_0\} \times Y$ is connected. Similarly, for all $y \in Y$, $X \times \{y\}$ is connected. Note that $X \times Y = \bigcup_{y \in Y} X \times \{y\}$. Furthermore, for all $y \in Y$:

$$(\{x_0\} \times Y) \cap (X \times \{y\}) = \{(x_0, y)\} \neq \emptyset$$

Therefore, by previous theorem, $X \times Y$ is connected.

∎