

# Subgroups

## Definition

To say that a set  $H$  is a *subgroup* of a group  $G$ , denoted  $H \leq G$ , means:

- 1).  $H \subseteq G$
- 2).  $H$  is a group using the induced operation of  $G$

When  $H = G$ ,  $H$  is called the *improper* subgroup of  $G$ .

When  $H \subset G$ ,  $H$  is called a *proper* subgroup of  $G$ , denoted  $H < G$ .

## Example

$$Z_4 = \{0, 1, 2, 3\}$$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$$\begin{aligned} &\{0\} \\ &\{0, 2\} \\ &\{0, 1, 2, 3\} \end{aligned}$$

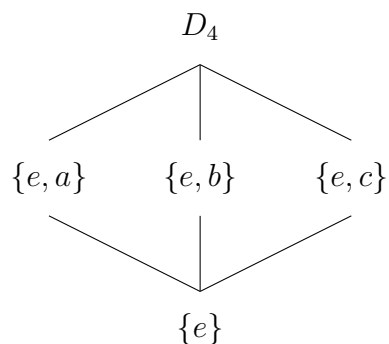
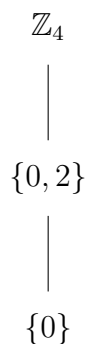
One proper, non-trivial subgroup

$$V = D_4 = \{e, a, b, c\}$$

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$$\begin{aligned} &\{e\} \\ &\{e, a\} \\ &\{e, b\} \\ &\{e, c\} \\ &\{e, a, b, c\} \end{aligned}$$

Three proper, non-trivial subgroup



## Theorem

Let  $G$  be a group:

$$\{e\} \leq G$$

### Proof

$$\{e\} \subseteq G$$

$$(ee)e = ee = e$$

$$e(ee) = ee = e$$

$\therefore \{e\}$  is associative.

$$e \in \{e\}$$

$\therefore \{e\}$  has identity.

$$ee = ee = e$$

$\therefore \{e\}$  has inverses.

$$\therefore \{e\} \leq G$$

### **Definition**

$\{e\} \subseteq G$  is called the *trivial* subgroup of  $G$ . All other subgroups are referred to as *non-trivial*.

### **Theorem**

Let  $G$  be a group and  $H \subseteq G$ .  $H \leq G$  iff the following three properties hold:

- 1).  $H$  is closed under the induced operation of  $G$
- 2).  $e \in H$
- 3).  $\forall a \in H, a^{-1} \in H$

### Proof

$\implies$  Assume  $H \leq G$ .

$H$  is a group, so it is closed under the induced operation and  $\forall a \in H, a^{-1} \in H$ . Also, by closure,  $aa^{-1} = e \in H$ .

$\therefore$  the three properties hold.

$\longleftarrow$  Assume the three properties hold.

Assume  $a, b, c \in H$ .

By closure,  $(ab)c \in H$  and  $a(bc) \in H$ .

$a, b, c \in G$

$(ab)c = a(bc)$  in  $G$ , so this must also hold in  $H$ .

$\therefore H$  is associative.

$e \in H$ , and since  $e$  is the identity for  $G$ , it must also be the identity for  $H$ .

$\forall a \in H, a^{-1} \in H$ .

$\therefore H \leq G$ .

### Example

$$G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$$

$$H = \{0, 2\}$$

$+_4$	$\left  \begin{array}{cc} 0 & 2 \end{array} \right.$
$0$	$\left  \begin{array}{cc} 0 & 2 \end{array} \right.$
$2$	$\left  \begin{array}{cc} 2 & 0 \end{array} \right.$

$H$  is closed

$$0 = e \in H$$

$$0^{-1} = 0 \in H$$

$$2^{-1} = 2 \in H$$

$$\therefore H \leq G$$

### Theorem: Subgroup Test

Let  $G$  be a group and  $H \neq \emptyset, H \subseteq G$

$$H \leq G \iff \forall a, b \in H, ab^{-1} \in H \quad (b^{-1} \in G)$$

#### Proof

$\implies$  Assume  $H \leq G$

Assume  $a, b \in H$

Since  $H$  is a group,  $b^{-1} \in H$

But  $H \subseteq G$ , so  $b^{-1} \in G$

By closure,  $ab^{-1} \in H$

$\Leftarrow$  Assume  $\forall a, b \in H, ab^{-1} \in H \quad (b^{-1} \in G)$

$H \neq \emptyset$

Assume  $a, b \in H$

$b = (b^{-1})^{-1} \in H$

But  $H \subseteq G$ , so  $b = (b^{-1})^{-1} \in G$

By assumption,  $a(b^{-1})^{-1} \in H$

$ab \in H$

$\therefore H$  is closed under the induced operation of  $G$ .

Since  $H \subseteq G, a \in G$

But since  $G$  is a group,  $a^{-1} \in G$

By assumption,  $aa^{-1} \in H$

$\therefore e \in H$

$e \in H$  and  $a^{-1} \in G$   
 So by assumption,  $ea^{-1} \in H$   
 $\therefore a^{-1} \in H$

### **Example**

Let  $G = GL(n, \mathbb{R})$  and  $H = \{A \in G \mid \det(A) = 1\}$   
 Prove:  $H < G$

Assume  $A, B \in H$   
 Clearly,  $H \subset G$   
 So,  $B \in G$   
 $\det(B) = 1 \neq 0$ , so  $B$  is invertible  
 $B^{-1} \in G$   
 $\det(AB^{-1}) = \frac{\det(A)}{\det(B)} = \frac{1}{1} = 1$   
 $AB^{-1} \in H$   
 $\therefore H < G$

### **Theorem**

Let  $G$  and  $G'$  be groups and  $\phi : G \rightarrow G'$  be an isomorphism:

$$H \leq G \implies \phi[H] \leq G'$$

Isomorphisms map subgroups to subgroups.

### **Proof**

Assume  $H \leq G$   
 Assume  $x, y \in \phi[H]$   
 $\exists a, b \in H, \phi(a) = x$  and  $\phi(b) = y$   
 $\phi[H] \subseteq G'$   
 So,  $y \in G'$   
 But  $G'$  is a group, so  $y^{-1} \in G'$   
 $\phi$  is a homomorphism  
 $xy^{-1} = \phi(a)\phi(b)^{-1} = \phi(a)\phi(b^{-1}) = \phi(ab^{-1})$   
 $H$  is a group, so  $b^{-1} \in H$   
 By closure,  $ab^{-1} \in H$   
 $\phi$  is well-defined, so  $\phi(ab^{-1}) \in \phi[H]$   
 $xy^{-1} \in \phi[H]$   
 $\therefore$  by the subgroup test,  $\phi[H] \leq G'$

### **Corollary**

Let  $G \simeq G'$ :

$$\forall H \leq G, \exists H' \leq G', H \simeq H'$$

### **Theorem**

Let  $G$  be a group:

$$H, K \leq G \implies H \cap K \leq G$$

### **Proof**

Assume  $H, K \leq G$

Assume  $a, b \in H \cap K$

$a, b \in H$  and  $a, b \in K$

But  $H$  and  $K$  are groups, so  $b^{-1} \in H$  and  $b^{-1} \in K$

$H \cap K \subseteq H, K \subseteq G$

So  $b^{-1} \in G$

$ab^{-1} \in H$  and  $ab^{-1} \in K$

$ab^{-1} \in H \cap K$

$\therefore$  by the subgroup test,  $H \cap K \leq G$