# Determinants

**Definition: Determinant**

Let $A \in M_n$. The *determinant* of $A$ is given by:

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\operatorname{sgn}(\sigma)} \prod_{k=1}^{n} a_{k,\sigma(k)}$$

where:

$$\operatorname{sgn}(\sigma) = \begin{cases} 0, & \sigma \text{ even} \\ 1, & \sigma \text{ odd} \end{cases}$$

**Definition: Laplace Expansion**

Let $A \in M_n$. The *determinant* of $A$ is given by:

$$\det(A) = \sum_{k=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

where the first expansion is across the $j^{th}$ column and the second expansion is across the $i^{th}$ row.

Recall that $(-1)^{i+j} \det(A_{ij})$ is a cofactor of $A$.

**Example**

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(A) = a \det \begin{bmatrix} d \end{bmatrix} - b \det \begin{bmatrix} c \end{bmatrix} = ad - bc$

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\begin{aligned} \det(A) &= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31}) \end{aligned}$$

**Properties**

1). $\det(A) = \det(A^T)$

2). $\det(\bar{A}) = \overline{\det(A)}$

3). Effects of elementary row operations:

Scale - scale determinant

Swap - negation (change sign)

Replace - no effect

4). Let $T$ be an upper-triangular matrix:

$$\det(T) = \prod_{k=1}^{n} t_{kk}$$

Use EROs to put a matrix in upper-triangular form (swap/replace) and then calculate by multiplying the diagonal entries.

5). $\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det(A)\det(B)$

$\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \to \det \begin{bmatrix} T_A & * \\ 0 & T_C \end{bmatrix}$

6). $\det(AB) = \det(A)\det(B)$

<u>Proof</u>

Case 1: $\det(A) = 0$

$A$ is not invertible
ABC: $AB$ is invertible
$\exists\, C, (AB)C = I$
$A(BC) = I$
$A^{-1} = BC$
Contradiction!
Thus, $AB$ is not invertible
Therefore $\det(AB) = 0$
Also $\det(A)\det(B) = 0 \cdot \det(B) = 0$

$\therefore \det(AB) = \det(A)\det(B)$

Case 2: $\det(A) \neq 0$

$A$ is invertible
There exists a sequence of ERO matrices $E_1 E_2 \cdot E_k$ such that $E_k \cdot E_2 E_1 A = I$
But ERO matrices are invertible, so $A = E_1^{-1} E_2^{-1} \cdot E_k^{-1} I$
$AB = E_1^{-1} E_2^{-1} \cdot E_k^{-1} B$
But $\det(EB) = \det(E)\det(B) = \pm \det(B)$, so:

$$\begin{aligned} \det(AB) &= \det(E_1^{-1})\det(E_2^{-1}) \cdot \det(E_k^{-1})\det(B) \\ &= \det(E_1^{-1} E_2^{-1} \cdot E_k^{-1})\det(B) \\ &= \det(A)\det(B) \end{aligned}$$