EXAM 1

Math 161a: Appl. Prob. & Stats.
Instructor: Guangliang Chen
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You have 75 minutes.

No books, but you are allowed to use a flash-card (provided by the instructor) as cheat sheet.

Please write legibly (unrecognizable work will receive zero credit).

You must show all necessary steps to receive full credit.

Good luck!

Name: ______________________________

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________

"I have adhered to the SJSU Academic Integrity Policy in completing this exam."

Signature: ____________________

Date: ____________________

Total score: ______________ (/50 points)

1. *(9 pts)* A small class has 4 boys and 5 girls.

   (a) In how many different ways can you arrange them along a line? What if the students of each gender must stand together?

   Answer. $9! = 362880$, and $2 \cdot 4! \cdot 5! = 5760$

   (b) In how many different ways can you select 2 boys and 2 girls to form a team of size 4 to work on some project?

   Answer. $\binom{4}{2} \cdot \binom{5}{2} = 60$

2. *(10 pts)* A poker hand of 5 cards is drawn from an ordinary deck of 52 cards at random. Consider the following two events: A = {All hearts}, B = {5 consecutive numbers} (i.e., $x, x+1, x+2, x+3, x+4$ where x represents only the face value; Ace can only be used as 14).

(a) Determine $P(B)$.

Answer.

$$P(B) = \frac{9 \cdot 4^5}{\binom{52}{5}} = 0.0035,$$

as there are 9 possible sequences (23456, ..., 10JQKA) and each of the 5 cards of a fixed sequence can be selected in 4 ways (heart, diamond, club, spade).

(b) Find $P(B \mid A)$. Are the events A and B independent?

Answer.

$$P(B \mid A) = \frac{9}{\binom{13}{5}} = 0.0070 \neq P(B)$$

. Therefore, they are not independent.

3. *(10 pts)* Suppose that 55% of the defendants are truly guilty. Suppose also that juries vote a guilty person innocent with probability 0.2 whereas the probability that a jury votes an innocent person guilty is 0.1.

(a) Find the probability that a defendant is convicted.

Answer. By the law of total probability,

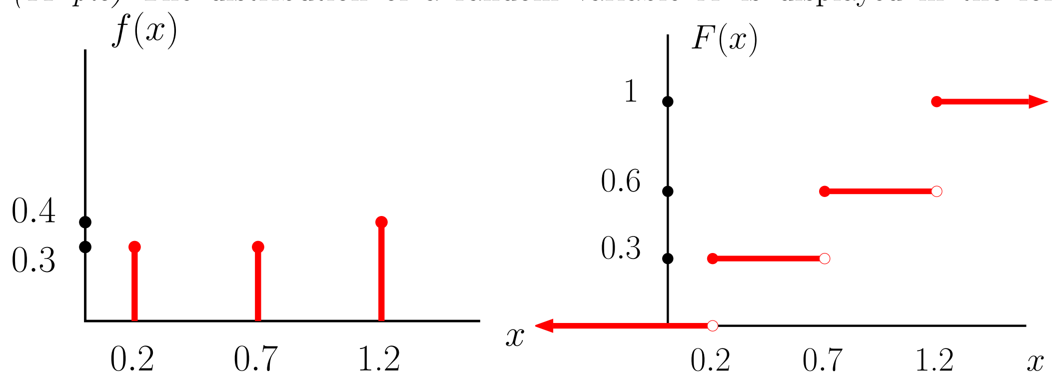
$$\begin{aligned} P(\text{convicted}) &= P(\text{convicted} \mid \text{truly guilty}) \cdot P(\text{truly guilty}) \\ &\quad + P(\text{convicted} \mid \text{truly innocent}) \cdot P(\text{truly innocent}) \\ &= (1 - 0.2) \cdot 0.55 + 0.1 \cdot (1 - 0.55) \\ &= 0.485 \end{aligned}$$

(b) What percentage of convicted defendants are actually innocent?

Answer. By Bayes rule,

$$\begin{aligned} P(\text{actually innocent} \mid \text{convicted}) &= \frac{P(\text{convicted} \mid \text{truly innocent}) \cdot P(\text{truly innocent})}{P(\text{convicted})} \\ &= \frac{0.1 \cdot (1 - 0.55)}{0.485} \\ &= 0.0928 \end{aligned}$$

4. *(11 pts)* The distribution of a random variable X is displayed in the following plot:



(a) What is the range of X?

Answer. Range(X)=$\{0.2, 0.7, 1.2\}$

(b) Find the following probabilities:

$$P(X = 0.3) = 0 \qquad P(X \leq 0.3) = 0.3$$
$$P(X = 0.7) = 0.3 \qquad P(X \leq 0.7) = 0.6$$

(c) Plot the cumulative distribution function (cdf) of X as a graph, to the right of the given graph. Make sure you mark everything clearly.

(see above)

(d) What are the expected value and standard deviation of X?

Answer. First,

$$\text{Exp}(X) = 0.2 \cdot 0.3 + 0.7 \cdot 0.3 + 1.2 \cdot 0.4 = 0.75.$$

To calculate the variance and standard deviation of X, we need to also compute

$$\text{Exp}(X^2) = 0.2^2 \cdot 0.3 + 0.7^2 \cdot 0.3 + 1.2^2 \cdot 0.4 = 0.735$$

From this, we get

$$\text{Var}(X) = 0.735 - 0.75^2 = 0.1725, \quad \text{Std}(X) = \sqrt{0.1725} = 0.415$$

(e) What is $\text{E}(2X - 3)$?

Answer. By linearity,

$$\text{E}(2X - 3) = 2\text{E}(X) - 3 = 2 \cdot 0.75 - 3 = -1.5$$

5. *(10 pts)* Toss two fair dice independently and let Y be the smaller number. Find the pmf of Y.

Answer. First, the range of Y is $\{1, 2, 3, 4, 5, 6\}$. The corresponding probabilities are

$$\begin{aligned} f(1) =& P(Y = 1) = \frac{11}{36} \\ f(2) =& P(Y = 2) = \frac{9}{36} \\ f(3) =& P(Y = 3) = \frac{7}{36} \\ f(4) =& P(Y = 4) = \frac{5}{36} \\ f(5) =& P(Y = 5) = \frac{3}{36} \\ f(6) =& P(Y = 6) = \frac{1}{36}, \end{aligned}$$

computed by counting the number of pairs that give each Y value. The pmf is zero elsewhere.

6. *(5 pts)* **Extra credit question**. *Your score earned for this question will be posted separately on Canvas under extra credit assignments.*

Consider the experiment of independently tossing two different coins with probabilities of getting heads equal to 0.5 and 0.6 respectively, and let X denote the total number of heads observed. Find the expected value and variance of X.

Answer. Let X_1, X_2 be the indicated variables for the two coins. Then $X_1 \sim \text{Bernoulli}(0.5)$ and $X_2 \sim \text{Bernoulli}(0.6)$, which are also independent. The total number of heads is $X = X_1 + X_2$.

By linearity,

$$E(X) = E(X_1) + E(X_2) = 0.5 + 0.6 = 1.1$$

and

$$V(X) = V(X_1) + V(X_2) = 0.5(1 - 0.5) + 0.6(1 - 0.6) = 0.49.$$