Sets

We will be using sets to identify parts of the real number line. We start with a somewhat complicated and unfortunately incomplete and circular definition:

Definition: Set

A set is a *well-defined*, *distinct*, and *unordered* collection of members called elements, where the elements are selected from an all-inclusive set called a universe.

The definition is poor because it uses "set" in the definition, and incomplete because it appeals to the reader's intuition of what it means to be a collection and a member of a collection. There are branches of set theory that attempt to resolve these problems; however, that is way beyond our needs here. As it is, this approach to sets is often referred to as naive set theory.

How a set differs from just a collection is explained by the three emphasized adjectives in the definition:

well-defined: Each element in the universe is unambiguously either an element or not an element of the set. For example, let's assume that the universe is all integers and then define the set:

$$\{1, 2, 3\}$$

This set unambiguously contains the elements 1, 2, and 3, and nothing else. If you are asked if 2 is in the set, you can say yes. If you are asked if 5 is in the set you can say no. If you are asked if π is in the set you can also say no, because π isn't even in the universe. However, the set:

$$\{x | x \text{ is a large number}\}$$

is ambiguous. If you are asked if 100 is in the set, it is not clear whether or not 100 qualifies as "large" in the mind of the person that devised the set.

distinct: Elements are not replicated. For example, the sets:

$$\{1, 2, 3\}$$

$$\{1, 2, 3, 2\}$$

are the same set. In fact, the latter set is poorly formed; we never list duplicated elements.

unordered: Elements do not exist in any particular order in the set. For example, the sets:

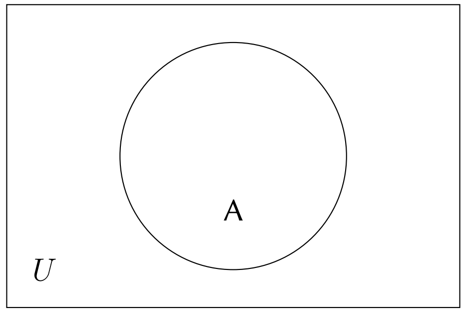
$$\{1, 2, 3\}$$

$$\{1, 3, 2\}$$

are the same set.

Universe

We can use a *Venn Diagram* to help us visualize a set and its relation to a universe:



The diagram shows a set A contained in a universe U. We typically use capital letters to represent sets. Everything within the circle is an element of the set. Everything in the box but not in the circle is not an element of the set.

Note that the choice of a universe is arbitrary (as long as it can contain the desired set) and is usually driven by the problem at hand.

Elements

The notation to show that a is an element in a set A is as follows:

$$a \in A$$

To say that a is not an element in set A:

$$a \notin A$$

Note that since sets are well-defined, $a \in A \; or \; a \notin A$ must be a true statement (something is unambiguously either in or not in the set).

For example, if we choose our universe to be the set of natural numbers $\mathbb{N}$ and let $A = \{1, 2, 3\}$, then $2 \in A$ and $4 \notin A$. A statement like $\pi \notin A$ is true, but a bit nonsensical because π is not even in our selected universe.

The empty set is the set with no elements, denoted as follows:

$$\emptyset = \{\}$$

For all elements a in a universe, $a \notin \emptyset$.

Specification

Sets can be specified in any of the following ways:

- Description
- Roster
- Setbuilder Notation

Description

We can describe a set using just words. Some examples are:

- The set of natural numbers from 2 to 5
- The set of irrational numbers
- The set of students in Math-19 for Spring 2016 at SJSU

We typically only use this method when we want to conceptually describe a set. When we want to be mathematically precise, we will use one of the other methods.

Roster

To describe a set by roster, start with a '$\{$', list the elements one by one, and then finish with a '$\}$'. When a set is small and finite, we can usually list all of the elements. For example:

$$A = \{1, 2, 3\}$$

is the set consisting of the three numbers 1, 2, and 3. For larger, finite sets with a pattern, we can establish the pattern and use an ellipsis to represent the omitted elements. For example, the set of natural numbers from 1 to 10 can be represented by:

$$A = \{1, 2, 3, \ldots, 10\}$$

For infinite sets, we can just leave off the terminating value. So, for the set of natural numbers we have:

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

For the set of integers we can say:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Setbuilder Notation

Most of the time we will use setbuilder notation:

$$\{\textit{universe or pattern} | \textit{qualifier}\}$$

Note that there are two forms, depending on what comes before the vertical bar. In the first form, we specify a particular universe that we are interested in, and then we use the qualifier to pick specific elements from that universe. For example, one way to pick out all the irrational numbers from the real numbers would be:

$$I = \{x \in \mathbb{R} | x \notin \mathbb{Q}\}$$

Here, $x \in \mathbb{R}$ tells us that we are going to select elements x from the universe of all real numbers, and $x \notin \mathbb{Q}$ tells us that we only what those that are not in the set of rational numbers - hence, only the irrational numbers.

In the second form, the part before the vertical bar gives us a pattern for the desired elements and the qualifier defines things in the pattern. The selection of the universe is often obvious from the pattern and qualifier. For example:

$$E = \{2n | n \in \mathbb{Z}\}$$

Here, the patterns tells us that we want numbers of the form $2n$ and the qualifier tells us that n can be any integer. Hence, E is the set of all even integers, and we can infer that the selected universe is the set of all integers.

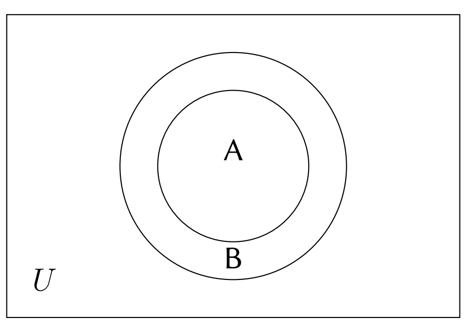
When the universe is obvious or previously stated, then sometimes the part before the vertical bar is just an element variable. For example:

$$\mathbb{R} = \{x | x \text{ is a real number}\}$$

is another way to say that $\mathbb{R}$ is the set of all real numbers.

Subsets

A set A is called a *subset* of a set B, denoted $A \subseteq B$, when all of the elements in A are also in B. Mathematically, we would state this as: $\forall a \in A, a \in B$. We can visualize this with the following Venn diagram:



Note that for a set A selected from a universe U:

$$A \subseteq U$$

Instead of using the quantifier definition for subset, we typically use an implication form:

$$A \subseteq B \coloneqq a \in A \rightarrow a \in B$$

in other words, if a is an element of A then it must also be an element of B.

By definition, a set is always a subset of itself: $A \subseteq A$. In fact, A is referred to as the *improper* subset of A. If we want A to be a subset of B but not the same as B, then we write: $A \subset B$. In this case, A is called a *proper* subset of B. Also by definition, the empty set is a subset of any set: $\emptyset \subseteq A$. So, any set other than the empty set always has at least two subsets: itself and the empty set. The empty set only has one subset: itself.

Equality

Two sets A and B are equal if they have the same elements; everything in A is in B and everything in B is in A. In other words, they are subsets of each other:

$$A = B \coloneqq A \subseteq B \; and \; B \subseteq A$$

In terms of elements:

$$A = B \coloneqq a \in A \text{ iff } a \in B$$

In other words, if an element a is in A then it must also be in B (and vice-versa), and if it is not in A then it cannot be in B.

Operations

Set operations are used to construct new sets from existing sets. We will be concerned with three such operations:

- Union
- Intersection
- Difference

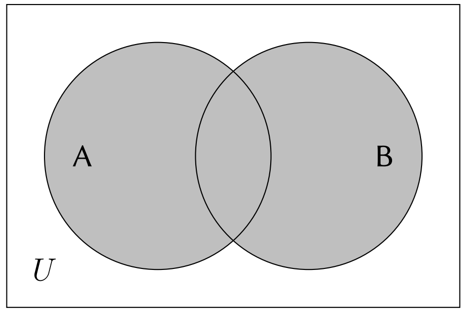
Although these operators are defined for two sets, they can be extended to include more. In some cases in mathematics, the number of sets may even be infinite (though not for our purposes).

Union

The *union* of two sets A and B, denoted $A \cup B$, is the set consisting of everything in A or B; an element only needs to be in one of the sets to be in the union:

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

The following Venn diagram demonstrates a union between two sets.



For example, let:

$$\begin{aligned} A &= \{1, 2, 3\} \\ B &= \{3, 4, 5\} \end{aligned}$$

The resulting union would be:

$$A \cup B = \{1, 2, 3, 4, 5\}$$

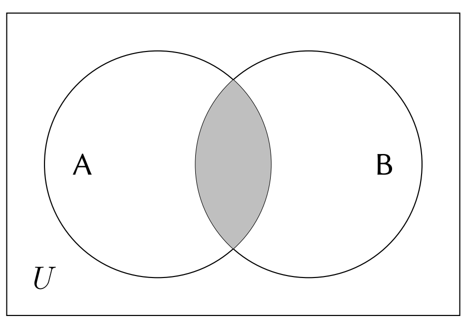
Note that even though the element ‘3’ is in both A and B, it is only included in the union once because set elements are distinct.

Intersection

The *intersection* of two sets A and B, denoted $A \cap B$, is the set consisting of everything that is in both A and B; an element must be in both to be in the intersection:

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

The following Venn diagram demonstrates an intersection between two sets.



Continuing with the previous example:

$$A \cap B = \{3\}$$

since the element ‘3’ is the only element that is in both A and B.

Sometimes, the intersection is the empty set; there are no elements in common. For example, let:

$$\begin{aligned} A &= \{1, 2, 3\} \\ B &= \{4, 5, 6\} \end{aligned}$$

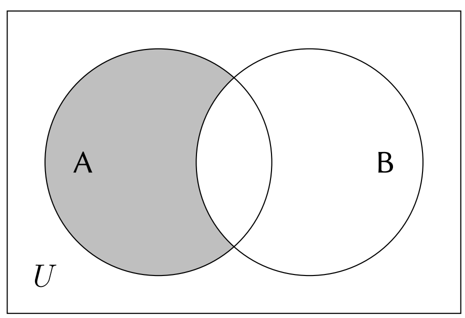
Now, $A \cap B = \emptyset$.

Difference

You have probably encountered set unions and intersections before; however, set difference may be new to you. The *difference* between two sets A and B, denoted $A - B$, is the set consisting of everything that is in A but not in B; we start with the set A and then remove any of its elements that are also in B. Elements in B that are not in A are ignored:

$$A - B = \{x | x \in A \text{ and } x \notin B\}$$

The following Venn diagram demonstrates the difference between A and B:



For example, let:

$$\begin{aligned} A &= \{1, 2, 3, 4, 5\} \\ B &= \{3, 5, 6, 7\} \end{aligned}$$

$$A - B = \{1, 2, 4\}$$

We start with A and then remove the elements '3' and '5' which are also in B. We ignore the other elements in B that are not in A ('6' and '7').

Note the union and intersection operators are commutative, the difference operator is not:

$$\begin{aligned} A \cup B &= B \cup A \\ A \cap B &= B \cap A \\ A - B &\neq B - A \end{aligned}$$