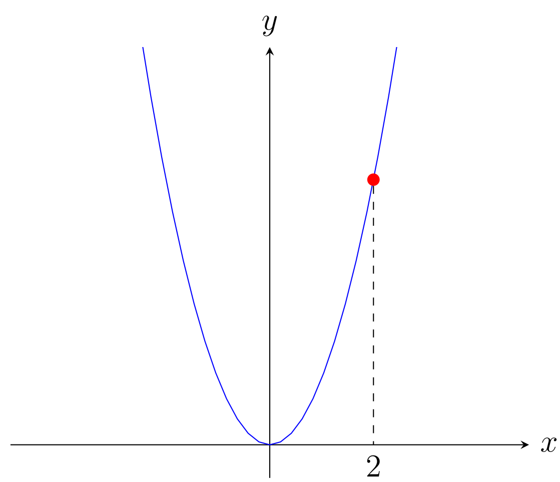
Limits

Example

Consider the standard function $f(x) = x^2$:



What happens to $f(x)$ as $x \to 2$ (but $x \neq 2$)?

x	$f(x)$
2.1	4.41
2.01	4.0401
2.001	4.004001
2.0001	4.00040001
2	???
1.9999	3.99960001
1.999	3.996001
1.99	3.9601
1.9	3.61

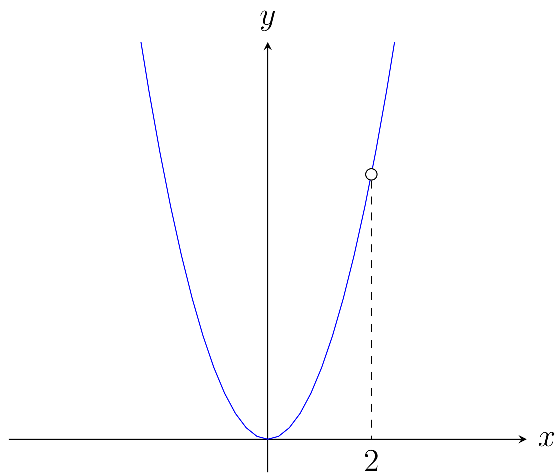
It appears that $f(x) \to 4$ as $x \to 2$ (from either direction).

In the previous example, it turns out that $f(x)$ is actually defined at $x = 2$ and furthermore, $f(2) = 4$. This special case will be used later as a formal definition of *continuity*. However, as previously stated, we don't actually care about the function value at $x = 2$. In fact, the function might not even be defined at the x value in question.

Example

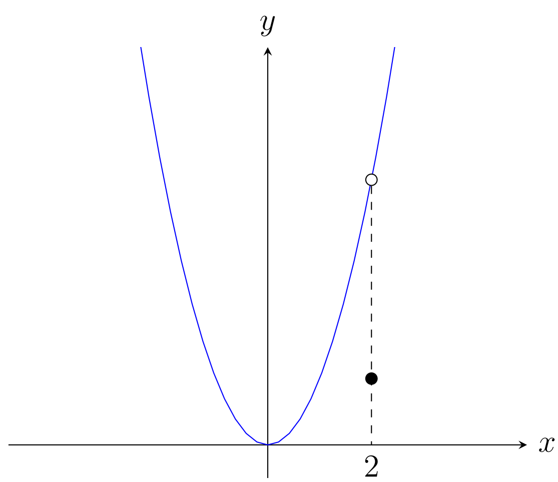
Consider the rational function:

$$f(x) = \frac{x^2(x-2)}{x-2}$$



Now, as $x \to 2$, the above table of values still applies and so it appears that $f(x) \to 4$ as $x \to 2$ (from either direction) even though $f(2)$ is not defined. To reiterate, we do not care what actually happens at $x = 2$. In fact, let's define $f(2) = 1$:

$$f(x) = \begin{cases} \frac{x^2(x-2)}{x-2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

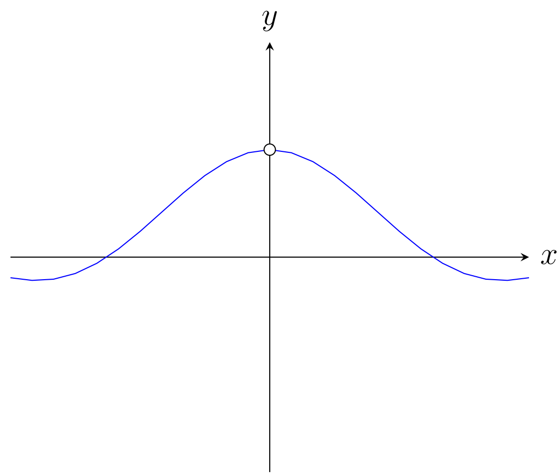


Still, $f(x) \to 4$ as $x \to 2$, regardless of the fact that $f(2) = 1$. Once again, we do not care about the function at $x = 2$; we only care what happens arbitrarily close to $x = 2$.

Example

Consider the function:

$$f(x) = \frac{\sin x}{x}$$



As $x \to 0$:

x	$f(x)$
1	0.841471
0.1	0.998334
0.01	0.999983
0	$???$
-0.01	0.999983
-0.1	0.998334
-1	0.841471

It appears that $f(x) \to 1$ as $x \to 0$.

In the previous two examples, when the functions are evaluated at the point in question the result is $\frac{0}{0}$, which is one of the so-called *indeterminate forms* ($\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 1^\infty$). When the resulting form is indeterminate, additional effort is required to determine the actual behavior arbitrarily close to the point.

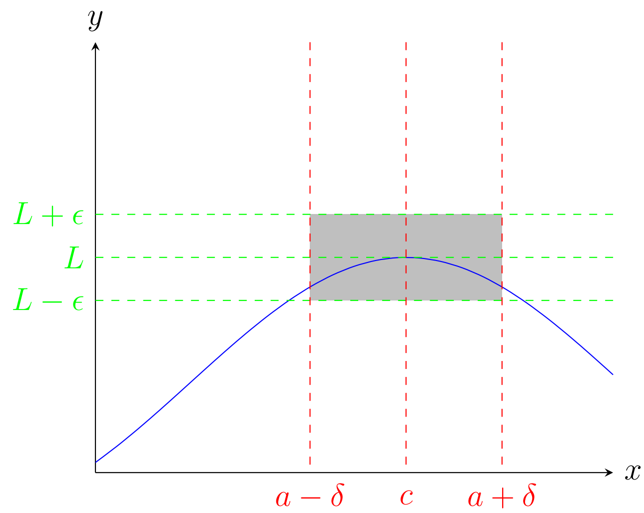
Definition: Limit of a Function at a Point

To say that $L \in \mathbb{R}$ is the *limit* of a function $f(x)$ at $x = a$, denoted by $\lim\limits_{x \to a} f(x) = L$, means that $f(x) \to L$ as $x \to a$ (but $x \neq a$):

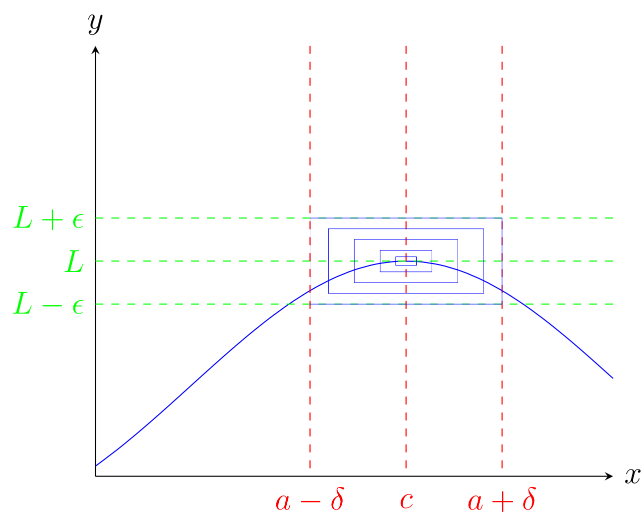
$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

An alternate syntax is: $f(x) \to 0$ as $x \to a$; however, note that $f(x) = L$ is allowed.

Select an $\epsilon > 0$ and then find a $\delta > 0$ such that $f(x)$ is completely contained in the bounding box $(a - \delta, a + \delta) \times (L - \epsilon, L + \epsilon)$.



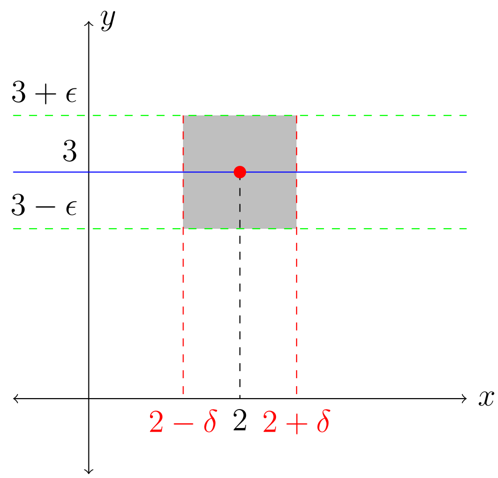
Since this must be the case for all possible ϵ, as $\epsilon \to 0$ arbitrarily small values of δ can be selected such that the bounding box converges on the limit point.



It is tempting to think that $\epsilon \to 0$ *forces* $\delta \to 0$; however, this is not always the case.

Example

Consider the constant function $f(x) = 3$ and $a = 2$:



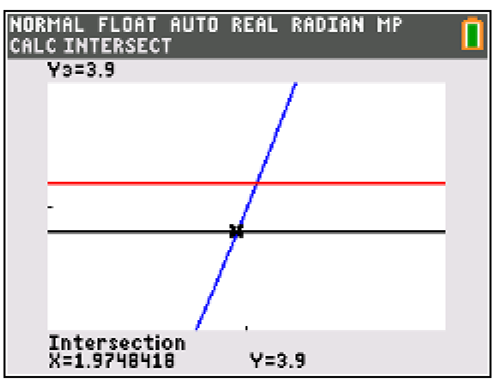
$$\lim_{x \to 2} f(x) = 3$$

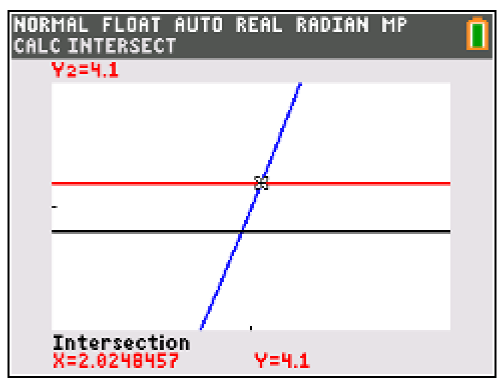
For any ϵ, any δ is sufficient. Also note that $f(x) = L$ within every bounding box.

Example

Let $f(x) = x^2$ and assume that $\lim_{x \to 2} f(x) = 4$. Find a suitable δ (to four decimal places) for $\epsilon = 0.1$.

Using a calculator:





$$\delta_1 = 2 - 1.9748418 = 0.0251582$$
$$\delta_2 = 2.0248457 - 2 = 0.0248457$$

$$\delta = \min\{\delta_1, \delta_2\} = 0.0248457$$

$$\delta = 0.0248$$

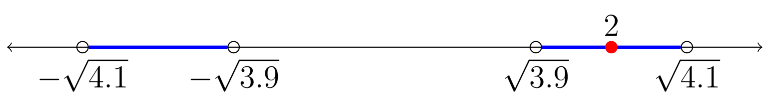
Analytically:

$$|f(x) - L| < 0.1$$
$$|x^2 - 4| < 0.1$$
$$-0.1 < x^2 - 4 < 0.1$$

$$-0.1 < x^2 - 4$$
$$x^2 > 3.9$$
$$\pm x > \sqrt{3.9}$$
$x < -\sqrt{3.9}$ or $x > \sqrt{3.9}$
$$x \in (-\infty, -\sqrt{3.9}) \cup (\sqrt{3.9}, \infty)$$

$$x^2 - 4 < 0.1$$
$$x^2 < 4.1$$
$$\pm x < \sqrt{4.1}$$
$$-\sqrt{4.1} < x < \sqrt{4.1}$$
$$x \in (-\sqrt{4.1}, \sqrt{4.1})$$

$$x \in (-\sqrt{4.1}, -\sqrt{3.9}) \cup (\sqrt{3.9}, \sqrt{4.1})$$



$$\delta_1 = 2 - \sqrt{3.9} = 0.0251582$$
$$\delta_2 = \sqrt{4.1} - 2 = 0.0248456$$

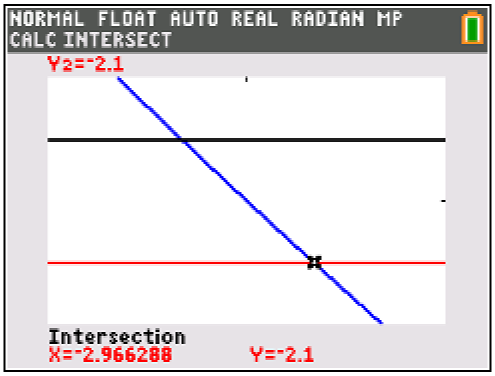
$$\delta = \min\{\delta_1, \delta_2\} = 0.0248456$$

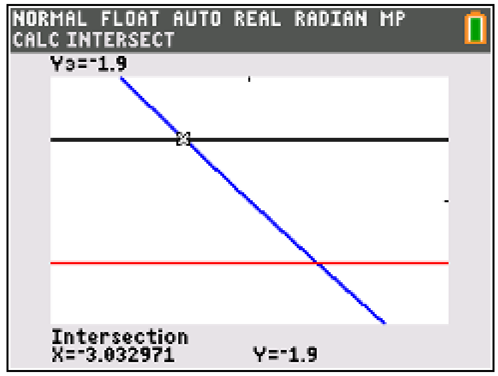
$$\delta = 0.0248$$

Example

Let $f(x) = x^2 + 3x - 2$ and assume that $\lim_{x \to -3} f(x) = -2$. Find a suitable δ (to four decimal places) for $\epsilon = 0.1$.

Using a calculator:





$$\delta_1 = -2.966288 - (-3) = 0.033712$$
$$\delta_2 = -3 - (-3.032971) = 0.032971$$

$$\delta = \min\{\delta_1, \delta_2\} = 0.032971$$

$$\delta = 0.0329$$

Analytically:

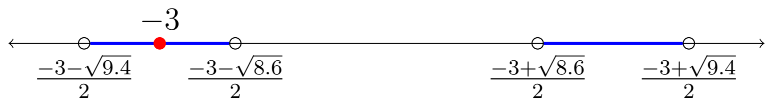
$$|f(x) - L| < 0.1$$
$$|x^2 + 3x - 2 - (-2)| < 0.1$$
$$-0.1 < x^2 + 3x < 0.1$$

$$-0.1 < x^2 + 3x$$
$$x^2 + 3x + 0.1 > 0$$
$$x = \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(0.1)}}{2(1)} = \frac{-3 \pm \sqrt{8.6}}{2}$$
$$x < \frac{-3 - \sqrt{8.6}}{2} \text{ or } x > \frac{-3 + \sqrt{8.6}}{2}$$

$$x^2 + 3x < 0.1$$
$$x^2 + 3x - 0.1 < 0$$
$$x = \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(-0.1)}}{2(1)} = \frac{-3 \pm \sqrt{9.4}}{2}$$

$$\frac{-3-\sqrt{9.4}}{2} < x < \frac{-3+\sqrt{9.4}}{2}$$

$$x \in \left(\frac{-3-\sqrt{9.4}}{2}, \frac{-3-\sqrt{8.6}}{2}\right) \cup \left(\frac{-3+\sqrt{8.6}}{2}, \frac{-3+\sqrt{9.4}}{2}\right)$$



$$\delta_1 = -3 - \frac{-3-\sqrt{9.4}}{2} = 0.0329709717$$
$$\delta_2 = \frac{-3-\sqrt{8.6}}{2} - (-3) = 0.0337121701$$

$$\delta = \min\{\delta_1, \delta_2\} = 0.0329709717$$

$$\delta = 0.0329$$