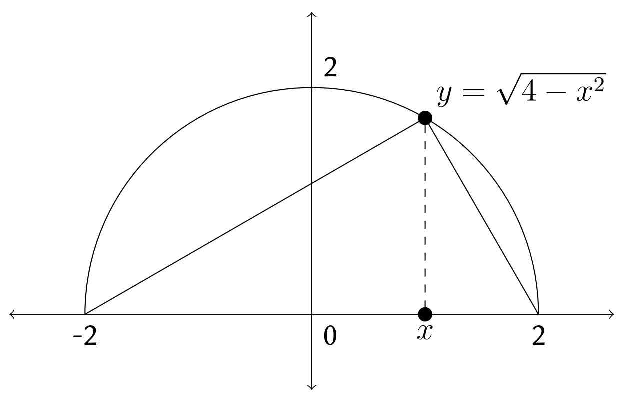
Math-71 Sections 9, 11, 12

Homework #6

Problem

A triangle is inscribed inside a semi-circle of radius 2 as shown below:



Find the maximum possible area of the inscribed triangle. Be sure to prove why it is a maximum by either a plausible explanation or by using the first or second derivative test.

The equation for a circle with center at the origin and radius 2 is given by:

$$x^2 + y^2 = 4$$

Solving for y and taking the upper semicircle only:

$$y = \sqrt{4 - x^2}$$

Recall that the area of a triangle is given by:

$$A = \frac{1}{2}bh$$

In this case, the base b is a constant value of 4. The height h for a given x value is $y(x) = \sqrt{4 - x^2}$. Thus, the equation for the area of the triangle as a function of x is given by:

$$A(x) = \frac{1}{2}(4)\sqrt{4 - x^2} = 2(4 - x^2)^{\frac{1}{2}}$$

where the domain for x is $[-2, 2]$.

We want to maximize the value of $A(x)$ with respect to x, so we determine the derivative:

$$A'(x) = 2\left[\frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x)\right] = -\frac{2x}{\sqrt{4 - x^2}}$$

The first derivative has critical points at $x = 0$ (zero) and $x = \pm 2$ (poles). Note that $x = \pm 2$ are both the endpoints and the zeros of the original function $A(x)$. Since $A(x) \geq 0$ these two points must be absolute minimums. Furthermore, since there is only one other critical point in the first derivative ($x = 0$), it must be an absolute maximum.

Therefore, the maximum area is:

$$A(0) = 2\sqrt{4 - 0^2} = 2 \cdot 2 = 4$$