Cavallaro, Jeffery
Math 275A
Homework #5

Theorem: Exercise 3.34

Let X, Y be topological spaces. If $A \subset X$ and $B \subset Y$ are closed sets then $A \times B$ is closed in $X \times Y$.

Proof. Since A and B are closed, $X - A$ and $Y - B$ are open. And so:

$$(X - A) \times (X - B) = (X \times Y) - (A \times B)$$

is open. Therefore $A \times B$ is closed. ∎

Theorem: Exercise 3.35

Let X and Y be topological spaces. The product topology on $X \times Y$ is the same as the topology generated by the subbasis of inverse images of open sets under the projection functions, that is, the basis is given by:

$$\mathcal{B} = \left\{ \pi_X^{-1}(U) \,\middle|\, U \in \mathscr{T}_X \right\} \cup \left\{ \pi_Y^{-1}(V) \,\middle|\, V \in \mathscr{T}_Y \right\}$$

Proof. Assume $U \in \mathscr{T}_X$ and $V \in \mathscr{T}_y$:

$$\pi_X^{-1}(U) = \{(x, y) \,|\, x \in U, y \in Y\} = U \times Y$$
$$\pi_Y^{-1}(V) = \{(x, y) \,|\, x \in X, y \in V\} = X \times V$$

$$\pi_X^{-1}(U) \cap \pi_Y^{-1}(V) = (U \times Y) \cap (X \times V) = (U \cap X, V \cap Y) = (U, V)$$

∎

Theorem: 4.1

Let X be a topological space. X is T_1 iff every point in X is a closed set.

Proof. Assume $x, y \in X$ such that $x \neq y$.

$\implies$ Assume X is T_1.

So there exists $U \in \mathscr{T}$ such that $x \notin U$ and $y \in U$. This means that $U \cap \{x\} = \emptyset$ and so y is not a limit point of $\{x\}$.

Therefore, $\{x\}$ is closed.

$\Longleftarrow$ Assume that every point in X is a closed set.

So x is not a limit point of $\{y\}$ and y is not a limit point of $\{x\}$. This means that there exists $U, V \in \mathscr{T}$ such that $x \in U$ and $U \cap \{y\} = \emptyset$ and likewise $y \in V$ and $V \cap \{x\} = \emptyset$. Hence $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.

Therefore X is T_1.

■

Theorem: Exercise 4.2

Let X be a topological space. If X is cofinite then X is T_1.

Proof. Assume that X is cofinite and assume that $x \in X$. But $X - \{x\}$ is open in the cofinite topology, and so $\{x\}$ is closed. Therefore, by the previous theorem, X is T_1. ■

Example: Exercise 4.6

Consider $\mathbb{R}^2$ with the standard topology.

1. Let $p \in \mathbb{R}^2$ and let $A \subset \mathbb{R}^2$ be a closed set such that $p \notin A$. Show that:

   $$\inf \{d(a, p) \mid a \in A\} > 0$$

   Since A is closed and $p \notin A$, p is not a limit point of A. Thus, there exists $\epsilon > 0$ such that $B(p, \epsilon) \cap A = \emptyset$ and so for all $a \in A$ the distance from p to a is at least ϵ.

   Therefore, $\inf \{d(a, p) \mid a \in A\} > \epsilon > 0$.

2. Show that $\mathbb{R}^2$ with the standard topology is regular.

   Assume that $p \in \mathbb{R}^2$ and $A \subset \mathbb{R}^2$ such that $p \notin A$ and A is closed. By (1), there exists some $\epsilon > 0$ such that for all $a \in A$, $d(p, a) > \epsilon$. Let $\delta = \frac{\epsilon}{3}$ and consider $U = B(p, \delta)$ and open set V generated by $\{B(a, \delta_a) \mid a \in A, \delta_a < \delta\}$. Thus, for every point $x \in U$ and $y \in V$, $d(x, y) \geq \delta$ and so $U \cap V = \emptyset$.

   Therefore R^2 is regular.

3. Find two disjoint closed sets $A, B \subset \mathbb{R}^2$ with the standard topology such that:

   $$\inf \{d(a, b) \mid a \in A, b \in B\} = 0$$

   Any two asymptotic functions in R^2 will do. So let:

   $$A = \{(x, 0) \mid x \in [1, \infty)\}$$
   $$B = \left\{ \left(x, \frac{1}{x}\right) \middle| \, x \in [1, \infty) \right\}$$

4. Show that $\mathbb{R}^2$ with the standard topology is regular.

   Assume that $A, B \in \mathbb{R}^2$ such that A and B are closed and $A \cap B = \emptyset$. By (2), for every $a \in A$ there exists $B(a, \epsilon_a)$ such that $B(a, \epsilon_a) \cap B = \emptyset$. Likewise, for every $b \in B$ there exists $B(b, \epsilon_b)$ such that $B(b, \epsilon_b) \cap A = \emptyset$. So let $\delta_a = \frac{\epsilon_a}{3}$ and let $\delta_b = \frac{\epsilon_b}{3}$ and consider the families of open sets $U_a = B(a, \delta_a)$ and $V_b = B(b, \delta_b)$. Let:

   $$U = \bigcup_{a \in A} U_a \supset A$$
   $$V = \bigcup_{b \in B} V_b \supset B$$

   Now, assume that $a \in A$ and $b \in B$:

   $$d(a, b) \geq \max\{\epsilon_a, \epsilon_b\} > \max\{\delta_a, \delta_b\}$$

   Thus $U_a \cap V_b = \emptyset$ and hence $U \cap V = \emptyset$.

   Therefore R^2 is normal.

<u>Theorem: 4.7</u>

1. A T_2-space (Hausdorff) is a T_1-space.
2. A T_3-space (regular and T_1) is a T_2-space (Hausdorff).
3. A T_4-space (normal and T_1) is a T_3-space (regular and T_1).

Proof. Let X be a topological space.

1. Assume that X is T_2.

   Assume $x, y \in X$ such that $x \neq y$. Since X is T_2, there exists $U, V \in \mathscr{T}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Thus, $x \in U$, $y \notin U$, $x \notin V$, and $y \in V$.

   Therefore X is T_1.

2. Assume that X is T_3.

   Assume $x, y \in X$ such that $x \neq y$. Since X is T_1, $\{y\}$ is closed, and since X is T_3, there exists $U, V \in \mathscr{T}$ such that $x \in U$, $\{y\} \subset V$ ($y \in V$), and $U \cap V = \emptyset$.

   Therefore X is T_2.

3. Assume that X is T_4.

   Assume $x \in X$ and $A \subset X$ such that A is closed and $x \notin A$. Assume $y \in A$. Since X is T_1, $\{x\}$ and $\{y\}$ are closed, and since X is normal, there exists $U_x, V_y \in \mathscr{T}$ such that $\{x\} \subset U_x$ and $\{y\} \subset V_y$ and $U_x \cap V_y = \emptyset$. So use the V_y to generate a set $V_A \in \mathscr{T}$:

   $$V_A = \bigcup_{y \in A} V_y \supset A$$

Since $U_x \cap V_y = \emptyset$, it must be the case that $U_x \cap V_A = \emptyset$. Hence, $x \in U_x$, $A \subset V_A$ and closed, and $U_x \cap V_A = \emptyset$.

Therefore X is regular and T_1 and hence T_3.

■

Theorem: 4.8

Let X be a topological space. X is regular iff for all $p \in X$ and $U \in \mathcal{U}_p$, there exists $V \in \mathcal{U}_p$ such that $\bar{V} \subset U$.

Proof.

$\Longrightarrow$ Assume that X is regular.

Assume $p \in X$ and assume $U \in \mathcal{U}_p$. Since U is open, $X - U$ is closed. So, since X is regular, there exists $V, W \in \mathscr{T}$ such that $p \in V$, $X - U \subset W$, and $V \cap W = \emptyset$. Now, since $X - U \subset W$:

$$X - (X - U) \supset X - W$$

and so $X - W \subset U$. Next, since $V \cap W = \emptyset$, it must be the case that $V \subset X - W$. But since W is open, $X - W$ is closed. Therefore:

$$\bar{V} \subset \overline{X - W} = X - W \subset U$$

$\Longleftarrow$ Assume that $\forall\, p \in X, \forall\, U \in \mathcal{U}_p, \exists\, V \in \mathcal{U}_p, \bar{V} \subset U$.

Assume $p \in X$ and $A \subset X$ such that A is closed and $p \notin A$. This means that p is not a limit point of A and so there exists $U \in \mathcal{U}_p$ such that $U \cap A = \emptyset$. Furthermore, there exists $V \in \mathcal{U}_p$ such that $V \subset \bar{V} \subset U$, and so $\bar{V} \cap A = \emptyset$. This means that $A \subset X - \bar{V}$, with $X - \bar{V}$ open. But $V \cap X - \bar{V} = \emptyset$.

Therefore X is regular.

■

Theorem: 4.9

Let X be a topological space. X is normal iff for all closed sets $A \subset X$ and for all $U \in \mathcal{U}_A$ there exists $V \in \mathcal{U}_A$ such that $\bar{V} \subset U$.

Proof.

$\Longrightarrow$ Assume that X is normal.

Assume $A \subset X$ and assume $U \in \mathcal{U}_A$. Since U is open, $X - U$ is closed. So, since X is normal, there exists $V, W \in \mathscr{T}$ such that $A \subset V$, $X - U \subset W$, and $V \cap W = \emptyset$. Now, since $X - U \subset W$:

$$X - (X - U) \supset X - W$$

and so $X - W \subset U$. Next, since $V \cap W = \emptyset$, it must be the case that $V \subset X - W$. But since W is open, $X - W$ is closed. Therefore:

$$\bar{V} \subset \overline{X - W} = X - W \subset U$$

$\Longleftarrow$ Assume that for all closed sets $A \subset X$ and for all $U \in \mathcal{U}_A$ there exists $V \in \mathcal{U}_A$ such that $\bar{V} \subset U$.

Assume $A, B \subset X$ such that A and B are closed and $A \cap B = \emptyset$. This means that for all $p \in A$, p is not a limit point of B and so there exists $U_p \in \mathscr{T}$ such that $p \in U_p$ and $U_p \cap B = \emptyset$. Let $U \supset A$ be the open set generated by these U_p:

$$U = \bigcup_{p \in A} U_p \supset A$$

Now, since $U_p \cap B = \emptyset$ for all $p \in A$, it must be the case that $U \cap B = \emptyset$. Furthermore, there exists $V \in \mathcal{U}_A$ such that $V \subset \bar{V} \subset U$, and so $\bar{V} \cap B = \emptyset$. This means that $B \subset X - \bar{V}$, with $X - \bar{V}$ open. But $V \cap X - \bar{V} = \emptyset$.

Therefore X is normal.

$\blacksquare$