Cavallaro, Jeffery
Math 275A
Homework #13

Theorem: 8.3

$\mathbb{R}_{\text{std}}$ is connected.

Proof. Since $\mathbb{R}$ is homeomorphic to $(0,1)$, it is sufficient to show that $(0,1)$ is connected. So ABC that $(0,1)$ is disconnected. This means that there exists $A \subset (0,1)$ such that $A \neq \emptyset, (0,1)$ and A is clopen. Since A is bounded, it has a $\sup$, so let $a = \sup A$. But A is closed, so $a \in A$. But A is also open, so there exists $\epsilon > 0$ such that $B(a, \epsilon) \subset A$, violating the fact that $a = \sup A$. Therefore $(0,1)$ is connected, and so $\mathbb{R}$ is connected. ■

Theorem: Exercise 8.7

The closure of the topologist's sine curve in $\mathbb{R}^2$ is connected.

Proof. Let:

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \,\middle|\, x \in (0,1) \right\}$$

$$\bar{S} = S \cup \{(1, \sin(1))\} \cup \{(0, y) \,|\, y \in [-1, 1]\}$$

ABC that S is not connected. This means that there exists $g : S \to \{0,1\}$ such that g is continuous and surjective. But $f : (0,1) \to S$ defined by $f(x) = (x, \sin\frac{1}{x})$ is also continuous and surjective. This means that $g \circ f : (0,1) \to \{0,1\}$ is also continuous and surjective, indicating that $(0,1)$ is not connected, contradicting the connectedness of the interval. Therefore S is connected, and by previous corollary, $\bar{S}$ is connected. ■