First Ring Isomorphism Theorem

Theorem

Let $\phi: R \to S$ be a homomorphism of rings:

$$\ker(\phi) \triangleleft R$$

Proof

From group theory, we know that $ker(\phi)$ is an additive subgroup of R

 $\text{Assume } r \in R$

Assume $k \in \ker(\phi)$

 $\phi(rk) = \phi(r)\phi(k) = \phi(r) \cdot 0 = 0$

 $rk \in \ker(\phi)$, so $\ker(\phi)$ is a left ideal in R

 $\phi(kr) = \phi(k)\phi(r) = 0 \cdot \phi(r) = 0$

 $kr \in \ker(\phi)$, so $\ker(\phi)$ is a right ideal in R

Therefore, by the ideal test, $\ker(\phi) \leq R$.

Theorem

Let R be a ring and $I \subseteq R$. R/I is a ring with operations:

$$(a+I) + (b+I) = (a+b) + I$$

 $(a+I)(b+I) = ab + I$

Proof

From group theory, we already know that R/I is an additive group

Assume $a, a' \in R$ are representative from the same coset

$$a - a' \in I$$

Similarly, assume $b,b'\in R$ are representative from the same coset

$$b - b' \in I$$

$$(a - a')b = ab - a'b \in I$$

So
$$ab + I = a'b + I$$

Likewise,
$$a'(b - b') = a'b - a'b' \in I$$

So
$$a'b + I = a'b' + I$$

Thus, by transitivity, ab + I = a'b' + I

Now, since all operations are on representative, we inherit all of the properties of the operations of R, including multiplicative associativity and the distributive rules

Therefore R/I is a ring.

Theorem

Let R be a ring and $I \subseteq R$, and let $\phi: I \to R/I$ be the canonical homomorphism:

$$\ker(\phi) = I$$

Thus, every ideal is the kernel of some homomorphism.

Proof

Note that I is the identity for R/I

$$x \in \ker(\phi) \iff \phi(x) = I \iff x + I = I \iff x \in I$$

$$\therefore \ker(\phi) = I$$

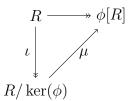
Theorem: First (Fundamental) Ring Isomorphism Theorem

Let $\phi: R \to S$ be a homomorphism of rings:

$$R/\ker\phi\simeq\phi[R]$$

Proof

From group theory, using the canonical injection homomorphism and the first (fundamental) group isomorphism theorem, we have:



So we already know that $\mu:R/\ker\phi\to\phi[R]$ is an isomorphism of groups where:

$$r + \ker(\phi) \mapsto \phi(r)$$

Assume $x, y \in R$

$$\mu((x + \ker(\phi))(y + \ker(\phi))) = \mu(xy + \ker(\phi))$$

$$= \phi(xy)$$

$$= \phi(x)\phi(y)$$

$$= \mu(x + \ker(\phi))\mu(y + \ker(\phi))$$

Thus μ preserves multiplication

Therefore μ is a ring isomorphism and $R/\ker\phi\simeq\phi[R]$.