Derivatives

Definition

Let $f(z)$ be defined on a domain D and let $z_0 \in D$. The derivative of f at z_0, denoted $f'(z_0)$, is given by:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Replacing $z - z_0$ with Δz:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

To say that f is differentiable at z_0 means that the limit exists, regardless of path from z to z_0.

When considering any $z \in D$, and letting $\Delta w = f(z + \Delta z) - f(z)$:

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

Example

$f(z) = z^2$

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} z^2 + 2z(\Delta z) + (\Delta z)^2 - z^2 \Delta z \\ &= \lim_{\Delta z \to 0} \frac{2z(\Delta z) + (\Delta z)^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} 2z + \Delta z \\ &= 2z \end{aligned}$$

Example

$f(z) = \bar{z}$

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} \end{aligned}$$

Consider a path along the x-axis, where $\Delta z = \Delta x$:

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta x \to 0} \frac{\overline{\Delta x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

Now consider a path along the y=axis, where $Dz = i\Delta y$:

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{i\Delta y \to 0} \frac{\overline{i\Delta y}}{i\Delta y} = \lim_{i\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

Thus, the limit DNE and $\therefore$ $f(z) = \bar{z}$ is not differentiable anywhere.

<u>Example</u>

$f(z) = |z|^2 = z\bar{z}$

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{(z + \Delta z)(\overline{z + \Delta z}) - z\bar{z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{(z + \Delta z)(\bar{z} + \overline{\Delta z}) - z\bar{z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{z\bar{z} + z\overline{\Delta z} + \bar{z}\Delta z + \Delta z\overline{\Delta z} - z\bar{z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \left(\bar{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z} \right) \end{aligned}$$

But this limit can only possibly exist at $z = 0$ and

$$\lim_{\Delta z \to 0} \left(0 + \overline{\Delta z} + 0\right) = \lim_{\Delta z \to 0} \overline{\Delta z} = 0$$

<u>Theorem</u>

Let $f(z)$ be a real-valued function:

f differentiable $\implies$ f is only differentiable at $f(z) = 0$

<u>Proof</u>

Assume f is differentiable
Let $L_R = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$
Let $L_I = \lim_{ih \to 0} \frac{f(z+ih) - f(z)}{ih}$
L_R and L_I must exist
$L_R \in \mathbb{R}$ and $L_I \in \mathbb{C}$
But $L_R = L_I$
This can only at $f(z) = 0$

Example

$f(z) = |z|$

Since $f(z)$ is real-valued, $f(z)$ can only be differentiable at: $f(z) = |z| = 0$, which only occurs at $z = 0$

$$\begin{aligned} f'(z) &= \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} \\ &= \lim_{z \to 0} \frac{|z| - |0|}{z - 0} \\ &= \lim_{z \to 0} \frac{|z|}{z} \\ &= \lim_{z \to 0} \frac{\sqrt{z\bar{z}}}{z} \\ &= \lim_{z \to 0} \sqrt{\frac{\bar{z}}{z}} \end{aligned}$$

But that limit DNE, so $f(z) = |z|$ is nowhere differentiable.

Properties: Consequences

Let $f(z) = u(x, y) + iv(x, y)$

1). u and v differentiable $\not\Longrightarrow$ f differentiable.

Example: $f(z) = \bar{z} = x - iy$

2). $f(z)$ may be differentiable at only one point and nowhere else.

Example: $f(z) = |z|^2$ is only differentiable at $z = 0$

3). Continuity does not imply differentiability

Example: $f(z) = |z|$ is continuous everywhere but differentiable nowhere.

Theorem

$f(z)$ differentiable at $z_0 \implies f(z)$ continuous at z_0

Proof

Assume $f(z)$ is differentiable at z_0
$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists
$\lim_{z \to z_0} (z - z_0) = 0$ exists

$$\begin{aligned}\lim_{z\to z_0}[f(z)-f(z_0)] &= \lim_{z\to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0}(z-z_0)\right]\\ &= \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\lim_{z\to z_0}(z-z_0)\\ &= f'(z)\cdot 0\\ &= 0\end{aligned}$$

$\therefore \lim_{z\to z_0} f(z) = f(z_0)$