L^2 Separability

Definition

Let V be a vector space equipped with a norm $\|\cdot\|$. To say that V is *separable* means there exists a countable set $S \subset V$ such that $span(S)$ is dense in V.

Theorem

L^2 is separable.

Proof

Assume $f \in L^2$
Let:

$$g_n(x) = \begin{cases} f(x), & |x| \leq n \text{ and } |f(x)| \leq n \\ 0, & otherwise \end{cases}$$

Since $f \in L^2, f < \infty \;\; a.e.$, and so $g \to f \;\; a.e.$

$$|f - g_n| \leq |f| + |g_n| \leq |f| + |f| = 2\,|f|$$

$$|f - g_n|^2 \leq 4\,|f|^2$$

Thus, by the DCT:

$$\lim \int |f - g_n|^2 = \int \lim |f - g_n|^2 = 0$$

$\therefore \|f - g_n\| \to 0$

Assume $\epsilon > 0$
$\exists N, \|f - g_N\| < \frac{\epsilon}{2}$
Let $g = g_N$
Since g is a bounded function supported on a bounded set,
$g \in L^1$
Since the step functions are dense in L^1, there exists a step function ϕ such that:
$|\phi| \leq N$ and $\int |g - \phi| < \frac{\epsilon^2}{16N}$

Now, consider the family of functions $S = \{r\chi_R\}$, where r is a complex number with rational real and imaginary parts, and R is a rectangle in $\mathbb{R}^d$ with rational coordinates.

Let ψ be a linear combination of step functions from this family with real and imaginary parts arbitrarily close to those of ϕ so that $|\psi| \leq N$ and $\int |\phi - \psi| < \frac{\epsilon^2}{16N}$.

$$\int |g - \psi| \leq \int |(g - \phi) + (\phi - \psi)| \leq \int |g - \phi| + \int |\phi - \psi| < \frac{\epsilon^2}{16N} + \frac{\epsilon^2}{16N} = \frac{\epsilon^2}{8N}$$

$$|g - \psi|^2 = |g - \psi|\,|g - \psi| \leq (|g| + |\psi|)\,|g - \psi| \leq (N + N)\,|g - \psi| = 2N\,|g - \psi|$$

$$\int |g-\psi|^2 \leq 2N \int |g-\psi| < 2N\left(\frac{\epsilon^2}{8N}\right) = \frac{\epsilon^2}{4}$$

so:

$$\|g-\psi\| < \frac{\epsilon}{2}$$

and finally:

$$\|f-\psi\| = \|(f-g)+(g-\psi)\| \leq \|f-g\| + \|g-\psi\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore$ $span(S)$ is dense in L^2 and thus L^2 is separable.