Homework #10 Solutions

Problem

You work for a large conglomerate of 100 associated companies. In order to avoid antitrust issues with the DOJ, you need to divide the 100 companies into 4 different groups such that:

1. Companies within the same group cannot do business with each other.

2. Companies in different groups can do business with each other.

What organization of the 100 companies into the 4 groups maximimizes the number of business opportunities?

a) Start by labeling the groups X, Y, Z, and W. Let $x =$ the number of companies assigned to group X and so on for the other groups. Construct an equation in x for the number of business opportunities for a company in group X — i.e., how many companies can that company do business with?

   $$100 - x$$

b) Now build an equation in x for the total number of business opportunities for all companies in group X.

   $$x(100 - x)$$

c) Do likewise for the remaining groups and construct a function $f(x, y, z, w)$ that gives the total number of business opportunities across all the groups.

   $$f(x, y, z, w) = x(100 - x) + y(100 - y) + z(100 - z) + w(100 - w)$$

d) What is the constraint on x, y, z, and w?

   $$x + y + z + w = 100$$

e) Introduce a Lagrange multiplier λ and determine $f_x = \lambda g_x$, where g is the function constructed from the above constraint.

   $$f_x = 100 - 2x$$
   $$g_x = 1$$

   $$100 - 2x = \lambda$$

f) Do likewise for f_y, f_z, and f_w, and combine them with the constraint so that you have 5 equations in 5 unknowns.

$$100 - 2x = \lambda$$
$$100 - 2y = \lambda$$
$$100 - 2z = \lambda$$
$$100 - 2w = \lambda$$
$$x + y + z + w = 100$$

g) Use substitution to determine a value for λ.

$$x = \frac{1}{2}(100 - \lambda) = 50 - \frac{\lambda}{2}$$
$$y = 50 - \frac{\lambda}{2}$$
$$z = 50 - \frac{\lambda}{2}$$
$$w = 50 - \frac{\lambda}{2}$$

$$(50 - \frac{\lambda}{2}) + (50 - \frac{\lambda}{2}) + (50 - \frac{\lambda}{2}) + (50 - \frac{\lambda}{2}) = 100$$
$$4(50 - \frac{\lambda}{2}) = 100$$
$$50 - \frac{\lambda}{2} = 25$$
$$\frac{\lambda}{2} = 25$$
$$\lambda = 50$$

h) Substitute the value for λ into the other equations to determine the optimal distribution of companies into the 4 groups.

$$x = 50 - \frac{50}{2} = 50 - 25 = 25$$
$$y = 25$$
$$z = 25$$
$$w = 25$$

Thus, the best allocation is an even allocation among the groups:

$$x = y = z = w = 25$$