Subspaces

Definition: Subspace

Let $(X, \mathscr{T})$ be a topological space and let $Y \subset X$. The set:

$$\mathscr{T}_Y = \{V \cap Y \mid V \in \mathscr{T}\}$$

is a topology on Y called the *subspace* topology or the *relative* topology on Y *inherited* from X. The topological space $(Y, \mathscr{T}_Y)$ is called a *subspace* of $(X, \mathscr{T})$.

Theorem

Let $(X, \mathscr{T})$ be a topological space and let $Y \subset X$. $\mathscr{T}_Y$ is a topology on Y.

Proof. $\emptyset \cap Y = \emptyset \in \mathscr{T}_Y$ and $X \cap Y = Y \in \mathscr{T}_Y$.

Assume $U, V \in \mathscr{T}_Y$. Then there exists $U', V' \in \mathscr{T}$ such that $U = U' \cap Y$ and $V = V' \cap Y$. So:

$$U \cap V = (U' \cap Y) \cap (V' \cap Y) = (U' \cap V') \cap Y$$

But $U' \cap V' \in \mathscr{T}$. Therefore $U \cap V \in \mathscr{T}_Y$.

Now, assume that $\{U_\alpha : \alpha \in \lambda\}$ such that $U_\alpha \in \mathscr{T}_Y$. Then for each U_α there exists a $U'_\alpha \in \mathscr{T}$ such that $U_\alpha = U'_\alpha \cap Y$. So:

$$U = \bigcup_{\alpha \in \lambda} U_\alpha = \bigcup_{\alpha \in \lambda} (U'_\alpha \cap Y) = \left(\bigcup_{\alpha \in \lambda} U'_\alpha \right) \cap Y$$

But $\bigcup_{\alpha \in \lambda} U'_\alpha \in \mathscr{T}$. Therefore, $U \in \mathscr{T}_Y$.

Therefore $\mathscr{T}_Y$ is a topology on Y. ■

Example

Consider $Y = [0, 1)$ as a subspace of $\mathbb{R}_{std}$. In Y, is the set $\left[\frac{1}{2}, 1\right)$ open, closed, neither, or both?

There is no open set in X that will result in a closed endpoint at $\frac{1}{2}$ so the set is not open. However, $[0, 1) \cap \left(\frac{1}{2}, 1\right) = \left(\frac{1}{2}, 1\right) \in \mathscr{T}_Y$ and $\frac{1}{2}$ serves as a limit point in Y so $\left[\frac{1}{2}, 1\right)$ is closed in Y. Hence it is not neither and not both.

Theorem

Let $(Y, \mathscr{T}_Y)$ be a subspace of $(X, \mathscr{T})$. $C \subset Y$ is closed in $(Y, \mathscr{T}_Y)$ iff there exists $D \subset X$, closed in $(X, \mathscr{T})$, such that $C = D \cap Y$.

Proof.

$\implies$ Assume $C \subset Y$ is closed in $(Y, \mathscr{T}_Y)$.

Since C in closed in Y, $Y - C$ is open in Y. So there exists some $U \in \mathscr{T}$ such that $Y - C = U \cap Y$. Let $D = X - U$, which is closed in X:

$$D \cap Y = (X - U) \cap Y = (X \cap Y) - (U \cap Y) = Y - (Y - C) = C$$

Therefore there exists $D \subset X$, closed in $(X, \mathscr{T})$, such that $C = D \cap Y$.

$\impliedby$ Assume there exists $D \subset X$, closed in $(X, \mathscr{T})$, such that $C = D \cap Y$.

Since D is closed in X, $X - D$ is open in X and $(X - D) \cap Y$ is open in Y:

$$(X - D) \cap Y = (X \cap Y) - (D \cap Y) = Y - C$$

Therefore C is closed in $(Y, \mathscr{T}_Y)$.

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