Kernel

Definition

Let $\phi: G \to G'$ be a homomorphism of groups. The *kernel* of ϕ , denoted $\ker(\phi)$, is given by:

$$\ker(\phi) = \{ g \in G \mid \phi(g) = e' \}$$

The kernel contains all elements that map to the identity.

Theorem

Let $\phi:G\to G'$ be a homomorphism of groups:

$$\ker(\phi) \leq G$$

Proof

Assume $a, b \in \ker(\phi)$

 $a, b \in G$

By closure, $ab \in G$

$$\phi(ab) = \phi(a)\phi(b) = e'e' = e'$$

So $ab \in \ker(\phi)$

 $\therefore \ker(\phi)$ is closed under the operation.

$$\phi(e) = e'$$

So $e \in \ker(\phi)$

 $\therefore \ker(\phi)$ has an identity.

Assume $a \in \ker(\phi)$

 $a \in G$

$$a^{-1} \in G$$

$$\phi(a^{-1}) = \phi(a)^{-1} = (e')^{-1} = e'$$

So $a^{-1} \in \ker(\phi)$

 $\therefore \ker(\phi)$ is closed under inverses.

$$\therefore \ker(\phi) \leq G$$

Theorem

Let $\phi:G\to G'$ be a homomorphism of groups:

$$\phi$$
 is one-to-one $\iff \ker(\phi) = e$

Proof

Assume
$$\phi$$
 is one-to-one
$$\phi(e) = e'$$
Assume $a, b \in G, \phi(a) = \phi(b)$
Assume $\exists a \in G, \phi(a) = e'$

$$\phi(e) = \phi(a)$$

$$e = a$$

$$\therefore \ker(\phi) = \{e\}$$
Assume $\ker(\phi) = \{e\}$

$$\phi(a)\phi(b)^{-1} = \phi(a)\phi(b^{-1}) = \phi(ab^{-1}) = e'$$

$$ab^{-1} = e$$

$$a = b$$

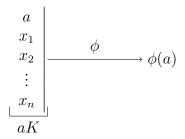
$$\therefore \phi \text{ is one-to-one}$$

Theorem

Let $\phi: G \to G'$ be a homomorphism of groups and $K = \ker(\phi)$:

$$\forall a \in G, aK = \{x \in G \mid \phi(x) = \phi(a)\}\$$

A homomorphism collapses all of the elements in the coset to the same element in G'.



Proof

Assume $a \in G$

Assume
$$y \in aK$$
 $\exists k \in K, y = ak$ $\phi(y) = \phi(ak) = \phi(a)\phi(k) = \phi(a)e' = \phi(a)$ $\therefore y \in \{x \in G \mid \phi(x) = \phi(a)\}$ Assume $y \in \{x \in G \mid \phi(x) = \phi(a)\}$ $\phi(y) = \phi(a)$ $\phi(a)^{-1}\phi(y) = \phi(a^{-1})\phi(y) = \phi(a^{-1}y) = e'$ So $a^{-1}y \in K$ and thus $\exists k \in K, a^{-1}y = k$ $y = ak$ $\therefore y \in aK$ $\therefore aK = \{x \in G \mid \phi(x) = \phi(a)\}$

Theorem

Let $\phi:G\to G'$ be a homomorphism of groups and $K=\ker(\phi)$:

$$\forall a \in G, Ka = \{x \in G \mid \phi(x) = \phi(a)\}\$$

Proof

Assume
$$a \in G$$

Assume
$$y \in Ka$$
 $\exists k \in K, y = ka$ $\phi(y) = \phi(ka) = \phi(k)\phi(a) = e'\phi(a) = \phi(a)$ $\therefore y \in \{x \in G \mid \phi(x) = \phi(a)\}$ Assume $y \in \{x \in G \mid \phi(x) = \phi(a)\}$ $\phi(y) = \phi(a)$ $\phi(y)\phi(a)^{-1} = \phi(y)\phi(a^{-1}) = \phi(ya^{-1}) = e'$ So $ya^{-1} \in K$ and thus $\exists k \in K, ya^{-1} = k$ $y = ka$ $\therefore y \in Ka$ $\therefore Ka = \{x \in G \mid \phi(x) = \phi(a)\}$

Corollary

Let $\phi: G \to G'$ be a homomorphism of groups and $\ker(\phi) = K$:

$$\forall a \in G, aK = Ka$$

Theorem

Let $\phi: G \to G'$ be a homomorphism of groups and $\ker(\phi) = K$:

$$\forall g_1, g_2 \in G, \phi[g_1 K g_2] = \{\phi(g_1 g_2)\}\$$

Proof

Assume $g_1,g_2\in G$ Assume $k\in K$ $\phi(g_1kg_2)=\phi(g_1)\phi(k)\phi(g_2)=\phi(g_1)e'\phi(g_2)=\phi(g_1)\phi(g_2)=\phi(g_1g_2)$