Binary Operators

Definition

A *binary operator* '$*$' on a non-empty set S is a function $* : S \times S \to S$, where $*(a, b)$ is typically denoted by $a * b$, or even ab (juxtaposition) when there is no ambiguity.

Thus, a binary operator '$*$' on a set S must be:

1). Closed: $\forall\, a, b \in S, a * b \in S$.

2). Well-defined: $\forall\, a, b, c, d \in S, a * b = c$ and $a * b = d \implies c = d$.

Definition

Let '$*$' be a binary operator on a set S and let $H \subset S$. To say that '$*$' is an *induced* operation on H means that H is closed under '$*$': $\forall\, a, b \in H, a * b \in H$.

To count the number of possible operators for a set S, consider the following:

$S = \{a, b\}$

$*$	a	b
a	aa	ab
b	ba	bb

$*$	a	b
a	2	2
b	2	2

$2^4 = 2^{2^2} = 16$ possibilities

$S = \{a, b, c\}$

$*$	a	b	c
a	aa	ab	ac
b	ba	bb	bc
c	ca	cb	cc

$*$	a	b	c
a	3	3	3
b	3	3	3
c	3	3	3

$3^9 = 3^{3^2}$ possibilities

$S = \{a, b, c, d\}$

$*$	a	b	c	d
a	aa	ab	ac	ad
b	ba	bb	bc	bd
c	ca	cb	cc	cd
d	da	db	dc	dd

$*$	a	b	c	d
a	4	4	4	4
b	4	4	4	4
c	4	4	4	4
d	4	4	4	4

$4^{16} = 4^{4^2}$ possibilities

In general, for $|S| = n$, there are n^{n^2} possible operations.

Definition

To say that a binary operator '$*$' on a set S is *commutative* means:

$$\forall\, a, b \in S, a * b = b * a$$

The table for a commutative binary operator must be symmetric:

$*$	a	b
a	2	2
b	1	2

$2^3 = 2^{\left(\frac{2\cdot 3}{2}\right)} = 8$ possibilities

$*$	a	b	c
a	3	3	3
b	1	3	3
c	1	1	3

$3^6 = 3^{\left(\frac{3\cdot 4}{2}\right)}$ possibilities

$*$	a	b	c	d
a	4	4	4	4
b	1	4	4	4
c	1	1	4	4
d	1	1	1	4

$4^{10} = 4^{\left(\frac{4\cdot 5}{2}\right)}$ possibilities

In general, for $|S| = n$, there are $n^{\left[\frac{n(n+1)}{2}\right]}$ possible commutative operations.

Definition

To say that a binary operator '$*$' on a set S has an *identity* element e means:

$$\exists\, e \in S, \forall\, a \in S, e * a = a * e = a$$

$*$	e	a
e	e	a
a	a	2

$2 = 2^{(2-1)^2}$ possibilities

$*$	e	a	b
e	e	a	b
a	a	3	3
b	b	3	3

$3^4 = 3^{(3-1)^2} = 81$ possibilities

$*$	e	a	b	c
e	e	a	b	c
a	a	4	4	4
b	b	4	4	4
c	c	4	4	4

$4^9 = 4^{(4-1)^2}$ possibilities

In general, for $|S| = n$, there are $n^{(n-1)^2}$ possible operations when there is an identity element.

Combining cummutativity and identity:

$*$	e	a
e	e	a
a	a	2

$2 = 2^{\left(\frac{2\cdot 1}{2}\right)}$ possibilities

$*$	e	a	b
e	e	a	b
a	a	3	3
b	b	1	3

$3^3 = 3^{\left(\frac{3\cdot 2}{2}\right)} = 27$ possibilities

$*$	e	a	b	c
e	e	a	b	c
a	a	4	4	4
b	b	1	4	4
c	c	1	1	4

$4^6 = 4^{\left(\frac{4\cdot 3}{2}\right)}$ possibilities

In general, for $|S| = n$, there are $n^{\left[\frac{n(n-1)}{2}\right]}$ possible communtative operations when there is an identity element.

Definition

To say that a binary operator '$*$' on a set S is *associative* means:

$$\forall\, a, b, c \in S, (a * b) * c = a * (b * c)$$

Determining associativity is a bit more tedius:

Example

Let $S = \{e, a\}$. The two possible operations are:

$*$	e	a
e	e	a
a	a	e

$\cdot$	e	a
e	e	a
a	a	a

a	b	c	$(a * b) * c$	$a * (b * c)$	$(a \cdot b) \cdot c$	$a \cdot (b \cdot c)$
e	e	e	e	e	e	e
e	e	a	a	a	a	a
e	a	e	a	a	a	a
e	a	a	e	e	a	a
a	e	e	a	a	a	a
a	e	a	e	e	a	a
a	a	e	e	e	a	a
a	a	a	a	a	a	a

So an operator on a set with identity is always associative.

Theorem

Composition is associative.

Proof

Assume that f, g, h are binary operators on a set S.
Assume $x \in S$.

$$[(f \circ g) \circ h](x) = (f \circ g)(h(x)) = f(g(h(x)))$$

$$[f \circ (g \circ h)](x) = f((g \circ h)(x)) = f(g(h(x)))$$

However, composition is not necessarily commutative.

Example

Let $S = \{a, b\}$ and define the following functions:

	E	A	B	C
a	a	a	b	b
b	b	a	b	a

$\circ$	E	A	B	C
E	E	A	B	C
A	A	A	A	A
B	B	B	B	B
C	C	B	A	E

The table is not symmetric, and thus the composition is not commutative.