Span

<u>**Definition: Span**</u>

Let E be a vector space over a scalar field $\mathbb{F}$ and let X be non-empty subset of E. The *span* of X, denoted $\mathrm{Span}(X)$, is the set of all (finite) linear combinations of X.

<u>**Theorem**</u>

Let E be a vector space over a scalar field $\mathbb{F}$ and let $X \subset E$:

$\mathrm{Span}(X)$ is a subspace of E.

This subspace is called the subspace *spanned* by X.

<u>Proof</u>

Assume $\vec{x}, \vec{y} \in \mathrm{Span}(X)$ and $\alpha, \beta \in \mathbb{F}$.

$\exists\, a_k \in \mathbb{F}$ and $X_1 = \{\vec{x}_1, \ldots, \vec{x}_n\} \subset X$ such that $\vec{x} = \displaystyle\sum_{k=1}^{n} a_k \vec{x}_k$.

$\exists\, b_k \in \mathbb{F}$ an $X_2 = \{\vec{y}_1, \ldots, \vec{y}_m\} \subset X$ such that $\vec{y} = \displaystyle\sum_{k=1}^{n} b_k \vec{y}_k$.

$$\alpha\vec{x} + \beta\vec{y} = \alpha \sum_{k=1}^{n} a_k \vec{x}_k + \beta \sum_{k=1}^{n} b_k \vec{y}_k = \sum_{k=1}^{n} (\alpha a_k)\vec{x}_k + \sum_{k=1}^{n} (\beta b_k)\vec{y}_k = \sum_{k=1}^{r} \lambda_k \vec{z}_k$$

where $\lambda_k \vec{z}_k = \begin{cases} \alpha a_i \vec{x}_i, & \vec{x}_i \in X_1 - X_2 \\ \beta b_i \vec{y}_i, & \vec{x}_i \in X_2 - X_1 \\ (\alpha a_i + \beta b_i)\vec{x}_i, & \vec{x}_i \in X_1 \cap X_2 \end{cases}$

But $\displaystyle\sum_{k=1}^{r} \lambda_k \vec{z}_k \in \mathrm{Span}(X)$.

Therefore, by the subspace test, $\mathrm{Span}(X)$ is a subspace of E

<u>**Theorem**</u>

Let E be a vector space let X be a non-empty subset of E. $\mathrm{Span}(X)$ is the smallest subspace of E containing X.

<u>Proof</u>

Assume S is a subspace of E and $X \subseteq S$.
Assume $\vec{x} \in \mathrm{Span}(X)$.
But $X \subseteq S$ and so $\vec{x} \in S$.

$\therefore \mathrm{Span}(X) \subseteq S$.

Theorem

Let E be a vector space over a field F and let $X = \{\vec{x}_1, \ldots, \vec{x}_n\}$ be a linearly independent subset of E. $\forall\, r < n$ and $\forall\, \lambda_k \in \mathbb{F}$:

$$X' = \{\vec{x}, \vec{x}_{r+1}, \ldots, \vec{x}_n\}$$

where $\vec{x} \in \mathrm{Span}\{\vec{x}_1, \ldots, \vec{x}_r\} - \{\vec{0}\}$ is a linearly independent set.

Proof

Assume $r < n$.
Assume $\vec{x} \in \mathrm{Span}\{\vec{x}_1, \ldots, \vec{x}_r\}$.
$\exists\, \lambda_k \in \mathbb{F}$ such that $\vec{x} = \displaystyle\sum_{k=1}^{r} \lambda_k \vec{x}_k$.

Assume $\alpha\vec{x} + \displaystyle\sum_{k=r+1}^{n} \alpha_k \vec{x}_k = 0$.

$$\alpha \sum_{k=1}^{r} \vec{x}_k + \sum_{k=r+1}^{n} \alpha_k \vec{x}_k = 0$$
$$\sum_{k=1}^{r} \alpha\vec{x}_k + \sum_{k=r+1}^{n} \alpha_k \vec{x}_k = 0$$

But X is linearly dependent, and so $\alpha, \alpha_k = 0$.

Therefore X' is a linearly independent set.