Schwarz's Lemma

Theorem

Let $f(z)$ be analytic in $|z| < R$, $f(0) = 0$, and $|f(z)| \leq M$:

1). $|f(z)| \leq |z| \frac{M}{R}$

2). $|f'(0)| \leq \frac{M}{R}$

3). Equality holds for $f(z) = cz\frac{M}{R}$ where $|c| = 1$

Proof

Let $g(z) = \begin{cases} \frac{f(z)}{z}, & 0 < z < R \\ f'(0), & z = 0 \end{cases}$
Since $f(z)$ is analytic $|z| < R$, $f'(z)$ exists in $|z| < R$
So, by L'Hospital, $\lim_{z \to 0} g(z) = \frac{f'(z)}{1} = f'(0)$
Thus, $f(z)$ is continuous at $z = 0$
Assume $0 < r < R$
On $|z| = r$,$|g(z)| = \frac{f(z)}{z} \leq \frac{M}{r}$
Since $g(z)$ analytic in $|z| \leq r$, by the maximum principle: $|g(z)| \leq \frac{M}{R}$
Let $r \to R$
Therefore, $|f(z)| \leq |z| \frac{M}{R}$

$|f'(0)| = |g(0)| \leq \frac{M}{R}$

For equality, $|g(z)|$ reaches its maximum value of $\frac{M}{R}$
Let $c \in \mathbb{C}, |c| = 1$
$|g(z)| = \frac{M}{R} = \left|\frac{f(z)}{z}\right|$
$|f(z)| = |z| \frac{M}{R}$
$|f(z)| = |c| |z| \frac{M}{R} = |cz| \frac{M}{R}$
$\therefore f(z) = cz\frac{M}{R}$