Equivalent Norms

Definition: Equivalence

Let E be a normed vector space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on E. To say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are *equivalent* on E means $\forall\,(\vec{x}_n), x \in E$:

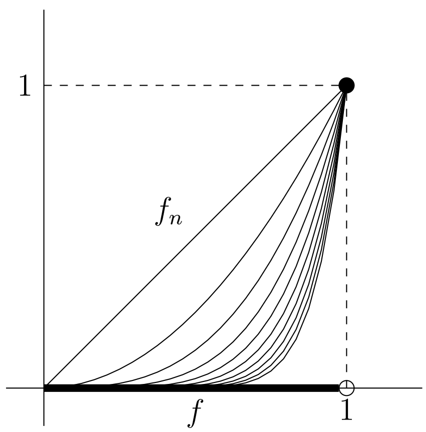
$$\|x_n - x\|_1 \to 0 \iff \|x_n - x\|_2 \to 0$$

Example

Let $E = \mathcal{C}[0,1]$:

$\|\cdot\|_\infty$ is not equivalent to $\|\cdot\|_{L_1}$.

Consider $f_n(t) = t^n$:



$$f_n(t) \to f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t = 1 \end{cases}$$

Comparing the two norms:

$$\|f_n - 0\|_{L_1} = \|f_n\|_{L_1} = \int_0^1 t^n dt = \left.\frac{t^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1} \to 0$$

But:

$$\|f_n - 0\|_\infty = \|f_n\|_\infty = \max_{0\le t\le 1}\{t^n\} = 1 \ne 0$$

Theorem

Let E be a normed vector space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on E. $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent iff $\exists\,\alpha, \beta > 0$ such that $\forall\,\vec{x} \in E$:

$$\alpha\,\|\vec{x}\|_1 \le \|\vec{x}\|_2 \le \beta\,\|\vec{x}\|_1$$

Proof

$\Longrightarrow$ Assume $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

$\forall\,(\vec{y}_n), \vec{y} \in E, \|\vec{y}_n - \vec{y}\|_1 \to 0 \iff \|\vec{y}_n - \vec{y}\|_2 \to 0$
Let $\vec{y} = \vec{0}$

$\forall\,(\vec{y}_n) \in E, \left\|\vec{y}_n - \vec{0}\right\|_1 \to 0 \iff \left\|\vec{y}_n - \vec{0}\right\|_2 \to 0$
$\forall\,(\vec{y}_n) \in E, \|\vec{y}_n\|_1 \to 0 \iff \|\vec{y}_n\|_2 \to 0$

ABC: $\forall\,\alpha, \beta > 0, \exists\,\vec{x} \in E, \alpha\,\|\vec{x}\|_1 > \|\vec{x}\|_2$ or $\|\vec{x}\|_2 > \beta\,\|\vec{x}\|_1$.

Let $\alpha = \dfrac{1}{n}$ and $\beta = n$.
$\exists\,\vec{x}_n \in E$ such that $\dfrac{1}{n}\,\|\vec{x}_n\|_1 > \|\vec{x}_n\|_2$ (and so $\|\vec{x}_n\|_1 > n\,\|\vec{x}_n\|_2$) or $\|\vec{x}_n\|_2 > n\,\|\vec{x}_n\|_1$.

Case 1: $\|\vec{x}_n\|_1 > n\,\|\vec{x}_n\|_2$

Let $\vec{y}_n = \dfrac{1}{\sqrt{n}}\dfrac{\vec{x}_n}{\|\vec{x}_n\|_2}$:

$$\|\vec{y}_n\|_2 = \left\|\frac{1}{\sqrt{n}}\frac{\vec{x}_n}{\|\vec{x}_n\|_2}\right\|_2 = \frac{1}{\sqrt{n}}\frac{\|\vec{x}_n\|_2}{\|\vec{x}_n\|_2} = \frac{1}{\sqrt{n}} \to 0$$

But:

$$\|\vec{y}_n\|_1 = \left\|\frac{1}{\sqrt{n}}\frac{\vec{x}_n}{\|\vec{x}_n\|_2}\right\|_1 = \frac{1}{\sqrt{n}}\frac{\|\vec{x}_n\|_1}{\|\vec{x}_n\|_2} > \frac{1}{\sqrt{n}}\frac{n\,\|\vec{x}_n\|_2}{\|\vec{x}_n\|_2} = \sqrt{n} \to \infty$$

So $\|\vec{y}_n\|_2 \to 0$ but $\|\vec{y}_n\|_1 \to \infty$.

CONTRADICTION!

Case 2: $\|\vec{x}_n\|_2 > n\,\|\vec{x}_n\|_1$

Let $\vec{y}_n = \dfrac{1}{\sqrt{n}}\dfrac{\vec{x}_n}{\|\vec{x}_n\|_1}$:

$$\|\vec{y}_n\|_1 = \left\|\frac{1}{\sqrt{n}}\frac{\vec{x}_n}{\|\vec{x}_n\|_1}\right\|_1 = \frac{1}{\sqrt{n}}\frac{\|\vec{x}_n\|_1}{\|\vec{x}_n\|_1} = \frac{1}{\sqrt{n}} \to 0$$

But:

$$\|\vec{y}_n\|_2 = \left\|\frac{1}{\sqrt{n}}\frac{\vec{x}_n}{\|\vec{x}_n\|_1}\right\|_2 = \frac{1}{\sqrt{n}}\frac{\|\vec{x}_n\|_2}{\|\vec{x}_n\|_1} > \frac{1}{\sqrt{n}}\frac{n\,\|\vec{x}_n\|_1}{\|\vec{x}_n\|_1} = \sqrt{n} \to \infty$$

So $\|\vec{y}_n\|_1 \to 0$ but $\|\vec{y}_n\|_2 \to \infty$.

CONTRADICTION!

$\therefore \exists\,\alpha, \beta > 0, \forall\,\vec{x} \in E, \alpha\,\|\vec{x}\|_1 \le \|\vec{x}\|_2 \le \beta\,\|\vec{x}\|_1$

$\Longleftarrow$ Assume $\exists\,\alpha, \beta > 0$ such that $\forall\,\vec{x} \in E, \alpha\,\|\vec{x}\|_1 \le \|\vec{x}\|_2 \le \beta\,\|\vec{x}\|_1$

Assume $(\vec{x}_n), \vec{x} \in E$.
Assume $\alpha, \beta > 0$.
$\alpha\,\|\vec{x}_n - \vec{x}\|_1 \le \|\vec{x}_n - \vec{x}\|_2 \le \beta\,\|\vec{x}_n - \vec{x}\|_1$

$\implies$ Assume $\|\vec{x}_n - \vec{x}\|_1 \to 0$

$\alpha \|\vec{x}_n - \vec{x}\|_1 \to 0$
$\beta \|\vec{x}_n - \vec{x}\|_1 \to 0$

Therefore, by the squeeze theorem, $\|\vec{x}_n - \vec{x}\|_2 \to 0$.

$\impliedby$ Assume $\|\vec{x}_n - \vec{x}\|_1 \not\to 0$

$\alpha \|\vec{x}_n - \vec{x}\|_1 \to x > 0$
So $\|\vec{x}_n - \vec{x}\|_2 \to y \geq x > 0$

Therefore, $\|\vec{x}_n - \vec{x}\|_2 \not\to 0$.

Therefore, $\|\cdot\|_1$ is equivalent to $\|\cdot\|_1$.

<u>Theorem</u>

Let E be a normed space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on E. Also, let B be the unit sphere with respect to some norm $\|\cdot\|$ on E:

$$B = \{\vec{x} \in E | \, \|\vec{x}\| = 1\}$$

$\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent (everywhere) iff they are equivalent on B.

<u>Proof</u>

Assume $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Therefore they must be equivalent on B.

Assume $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on B.

Assume $\vec{x} \in E$.
$\vec{x} = \|\vec{x}\| \dfrac{\vec{x}}{\|\vec{x}\|}$
Let $\vec{x}_0 = \dfrac{\vec{x}}{\|\vec{x}\|}$
$\|\vec{x}_0\| = \left\| \dfrac{\vec{x}}{\|\vec{x}\|} \right\| = \dfrac{\|\vec{x}\|}{\|\vec{x}\|} = 1$
Thus, $\vec{x}_0 \in B$.
Let $\lambda = \|\vec{x}\|$.
Thus, $\lambda > 0$ and $\vec{x} = \lambda \vec{x}_0$.

By previous theorem, there exists $\alpha, \beta > 0$ such that:

$$\alpha \|\vec{x}_0\|_1 \leq \|\vec{x}_0\|_2 \leq \beta \|\vec{x}_0\|_1$$

Then:

$$\lambda\alpha \|\vec{x}_0\|_1 \leq \lambda \|\vec{x}_0\|_2 \leq \lambda\beta \|\vec{x}_0\|_1$$

$$\alpha \|\lambda\vec{x}_0\|_1 \leq \|\lambda\vec{x}_0\|_2 \leq \beta \|\lambda\vec{x}_0\|_1$$

$\alpha \|\vec{x}\|_1 \leq \|\vec{x}\|_2 \leq \beta \|\vec{x}\|_1$

Therefore, the norms are equivalent everywhere.

Theorem

Let E be a finite dimensional vector space over a field $\mathbb{F}$ and let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for E. Define $\|\cdot\|_0$ on E as follows:

$$\|\vec{x}\|_0 = \left\| \sum_{k=1}^{n} \alpha_k \vec{v}_k \right\|_0 = \sum_{k=1}^{n} |\alpha_k|$$

$\|\cdot\|_0$ is a norm on E.

Proof

Assume $\vec{x}, \vec{y} \in E$ and $\lambda \in \mathbb{F}$.

1). Positivity

$$\exists, \alpha_k \in \mathbb{F}, \vec{x} = \sum_{k=1}^{n} \alpha_k \vec{v}_k$$

$\implies$ Assume $\vec{x} = 0$.

$$\vec{x} = \sum_{k=1}^{n} \alpha_k \vec{v}_k = \vec{0}.$$

But the $\vec{v}_k$ are linearly independent, and thus all the $\alpha_k = 0$.

$$\therefore \|\vec{x}\|_0 = \sum_{k=1}^{n} |0| = 0.$$

$\impliedby$ Assume $\|\vec{x}\|_0 = 0$.

$$\left\| \sum_{k=1}^{n} \alpha_k \vec{v}_k \right\|_0 = 0$$

$$\sum_{k=1}^{n} |\alpha_k| = 0$$

But $|\alpha_k| \geq 0$, so all the $\alpha_k = 0$.

$$\therefore \vec{x} = \sum_{k=0}^{n} 0 \vec{v}_k = \vec{0}.$$

2). Homogeneity

$$\exists, \alpha_k \in \mathbb{F}, \vec{x} = \sum_{k=1}^{n} \alpha_k \vec{v}_k$$

$$\|\lambda \vec{x}\|_0 = \left\| \lambda \sum_{k=1}^{n} \alpha_k \vec{x}_k \right\|_0 = \left\| \sum_{k=1}^{n} \lambda \alpha_k \vec{x}_k \right\|_0 = \sum_{k=1}^{n} |\lambda \alpha_l| = |\lambda| \sum_{k=1}^{n} |\alpha_k| = |\lambda| \|\vec{x}\|_0$$

3). Subadditivity

$$\exists, \alpha_k \in \mathbb{F}, \vec{x} = \sum_{k=1}^{n} \alpha_k \vec{v}_k$$

$$\exists, \beta_k \in \mathbb{F}, \vec{y} = \sum_{k=1}^{n} \beta_k \vec{v}_k$$

$$\begin{aligned}
\|\vec{x} + \vec{y}\|_0 &= \left\| \sum_{k=0}^{n} \alpha_k \vec{v}_k + \sum_{k=0}^{n} \beta_k \vec{v}_k \right\|_0 \\
&= \left\| \sum_{k=1}^{n} (\alpha_k + \beta_k) \vec{v}_k \right\|_0 \\
&= \sum_{k=1}^{n} |\alpha_k + \beta_k| \\
&\leq \sum_{k=1}^{n} (|\alpha_k| + |\beta_k|) \\
&= \sum_{k=1}^{n} |\alpha_k| + \sum_{k=1}^{n} |\beta_k| \\
&= \|\vec{x}\|_0 + \|\vec{y}\|_0
\end{aligned}$$

Theorem

Let E be a finite dimensional vector space over a field $\mathbb{F}$ and let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for E. All norms on E are continuous with respect to $\|\cdot\|_0$.

Proof

Assume $\|\cdot\|$ is a norm on E.
Assume $\vec{x}, \vec{y} \in E$.

$$\exists, \alpha_k \in \mathbb{F}, \vec{x} = \sum_{k=1}^{n} \alpha_k \vec{v}_k$$

$$\exists, \beta_k \in \mathbb{F}, \vec{y} = \sum_{k=1}^{n} \beta_k \vec{v}_k$$

Assume $\epsilon > 0$. Let $\delta = \dfrac{\epsilon}{\max\limits_{1 \leq k \leq n} \|\vec{v}_k\|}$.

Assume $\|\vec{x} - \vec{y}\|_0 < \delta$.

$$\|\vec{x} - \vec{y}\|_0 = \left\| \sum_{k=1}^{n} \alpha_k \vec{v}_k - \sum_{k=1}^{n} \beta_k \vec{v}_k \right\|_0 = \left\| \sum_{k=1}^{n} (\alpha_k - \beta_k) \vec{v}_k \right\|_0 = \sum_{k=1}^{n} |\alpha_k - \beta_k| < \delta$$

$$
\begin{aligned}
\|\|\vec{x}\| - \|\vec{y}\|\|_0 &= |\|\vec{x}\| - \|\vec{y}\|| \\
&\leq \|\vec{x} - \vec{y}\| \\
&= \left\| \sum_{k=1}^{n} \alpha_k \vec{v}_k - \sum_{k=1}^{n} \beta_k \vec{v}_k \right\| \\
&= \left\| \sum_{k=1}^{n} (\alpha_k - \beta_k) \vec{v}_k \right\| \\
&\leq \sum_{k=1}^{n} \|(\alpha_k - \beta_k)\vec{v}_k\| \\
&= \sum_{k=1}^{n} |\alpha_k - \beta_k| \, \|\vec{v}_k\| \\
&\leq \sum_{k=1}^{n} |\alpha_k - \beta_k| \max_{1 \leq k \leq n} \|\vec{v}_k\| \\
&= \max_{1 \leq k \leq n} \|\vec{v}_k\| \sum_{k=1}^{n} |\alpha_k - \beta_k| \\
&\leq \max_{1 \leq k \leq n} \|\vec{v}_k\| \, \delta \\
&= \max_{1 \leq k \leq n} \|\vec{v}_k\| \frac{\epsilon}{\max_{1 \leq k \leq n} \|\vec{v}_k\|} \\
&= \epsilon
\end{aligned}
$$

Theorem

Let E be a finite-dimensional vector space. Any two norms on E are equivalent.

Proof

Assume $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on E.
Assume $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for E and define $\|\cdot\|_0$ as above.
By previous theorem, $\|\cdot\|_1$ and $\|\cdot\|_2$ are continuous with respect to $\|\cdot\|_0$.
Let $f(\vec{x}) = \dfrac{\|\vec{x}\|_1}{\|\vec{x}\|_2}$.
f is continuous on the unit sphere with respect to $\|\cdot\|_0$.
But the unit sphere is compact and thus f achieves a minimum and a maximum value.
Let α be the minimum value and β be the maximum value.
Assume $\vec{x}$ is on the unit sphere.

$\alpha \leq f(\vec{x}) \leq \beta$

$\alpha \leq \dfrac{\|\vec{x}\|_1}{\|\vec{x}\|_2} \leq \beta$

$\alpha \|\vec{x}\|_2 \leq \|\vec{x}\|_1 \leq \beta \|\vec{x}\|_2$

Thus, $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on the unit sphere, and therefore, $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent (everywhere).

Example

Let $E = \mathbb{R}^N$ or $\mathbb{C}^N$:

1). $\|(z_1, \ldots, z_N)\|_p = \left(\sum_{k=1}^{N} |z_n|^p \right)^{\frac{1}{p}}$ for $1 \leq p < \infty$

2). $\|(z_1, \ldots, z_N)\|_\infty = \max_{1 \leq k < N} \{|z_k|\}$

Any two of these norms is equivalent.