Math-32 Spring 2020 Final Review Problems

Vectors and the Geometry of Space

1. Let $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + \hat{\mathrm{k}}$ and $\vec{b} = 2\hat{\jmath} + 5\hat{\mathrm{k}}$. Calculate the following:
   (a) $\vec{a} + \vec{b}$
   (b) $\vec{b} - \vec{a}$
   (c) $\vec{a} \cdot \vec{b}$
   (d) $\vec{b} \times \vec{a}$
   (e) $|\vec{a}|$
   (f) $\hat{a}$
   (g) $\mathrm{comp}_{\vec{a}}(\vec{b})$
   (h) $\mathrm{proj}_{\vec{b}}(\vec{a})$
   (i) Are $\vec{a}$ and $\vec{b}$ orthogonal? Give your reason.
   (j) Are $\vec{a}$ and $\vec{b}$ parallel? Give your reason.
   (k) Assuming that $\vec{a}$ and $\vec{b}$ are in the standard position, what is the measure of the angle between them (in radians).
2. Consider the four points:

   $$A(-1,-1,-1)$$
   $$B(2,2,0)$$
   $$C(-2,2,1)$$
   $$D(1,1,3)$$

   What is the volume of the parallelepiped defined by $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$?
3. Consider the tetrahedron with vertices $(0,0,2)$, $(0,1,0)$, $(0,0,0)$, and $1,2,0$.
   (a) Sketch the tetrahedron.
   (b) Determine the volume of the tetrahedron.
4. Consider the line through the points $A(-1,3,2)$ and $B(5,0,1)$:

(a) Determine the vector equation for the line.

(b) Determine the parametric equations for the line.

(c) Determine the symmetric equations for the line.

(d) Find a third point on the line not the same as A or B.

(e) Determine the equation of the line orthogonal to this line through the third point that you found.

5. Consider the points $A(1, 1, 1)$, $B(5, 0, 6)$, and $C(-1, -2, -3)$:

    (a) Determine the scalar equation of the plane containing these three points.

    (b) Determine the x, y, and z intercepts of the plane.

    (c) Sketch the plane.

    (d) Determine the distance from the point $(10, 10, 10)$ to the plane.

6. Determine the scalar equation for the plane through the point $(1, 2, -2)$ that contains the line with the parametric equations:

$$x = 2t$$
$$y = 3 - t$$
$$z = 1 + 3t$$

7. Consider the following linear equations for two planes:

$$3x + 5y - z = 3$$
$$2x - 3y + 5z = 4$$

    (a) Show that these planes are neither parallel nor orthogonal.

    (b) Determine the measure of the angle between the two planes (in radians, to 2 decimal places).

    (c) Determine the parametric equations for the line of intersection between the two planes.

    (d) Determine the symmetric equations for the line of intersection between the two planes.

8. Consider the following linear equations for two planes:

$$3x + y - 4z = 2$$
$$3x + y - 4z = 24$$

   (a) Show why these planes are parallel.

   (b) Determine the distance between the two planes.

Vector Functions

1. Consider the following parameterized equation for a curve:

$$\left.\begin{aligned} x &= \sin\theta \\ y &= \cos\theta \end{aligned}\right\} \quad 0 \le \theta \le \pi$$

   (a) Determine the Cartesion equation for the curve by eliminating the parameter.

   (b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

2. Determine the speed of a particle with the position function:

$$\vec{r}(t) = \frac{\sqrt{5}}{2} t\hat{\mathrm{i}} + e^{5t}\hat{\mathrm{j}} - e^{-5t}\hat{\mathrm{k}}$$

3. Consider the curve $\vec{r}(t) = \langle t^2, 0, t\rangle$.

   (a) Determine the length of the curve for $1 \le t \le 4$ (to four decimal places).

   (b) Determine the curvature.

4. Determine the curvature of the curve $\vec{r}(t) = \langle\sqrt{15}t, e^t, e^{-t}\rangle$ at the point $(0, 1, 1)$. Simplify your answer.

5. Consider the following parameterized equation for a curve:

$$x = 2\sin(3t)$$
$$y = t$$
$$z = 2\cos(3t)$$

   at the point $(0, \pi, -2)$.

(a) Determine the unit vector $\vec(T)(t)$.

(b) Determine the unit normal vector $\vec{N}(t)$.

(c) Determine the unit binormal vector $\vec{B}(t)$.

(d) Determine the equation of the normal plane to the curve at the point.

(e) Determine the equation of the osculating plane to the curve at the point.

6. Two particles travel along the space curves:

$$\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$$
$$\vec{r}_2(s) = \langle 1+2s, 1+6s, 1+14s \rangle$$

   (a) Determine all point of intersection.

   (b) Do the two particles collide? Explain why or why not.

   (c) Determine the angle between the curves at any one point of intersection (in radians, rounded to two decimal places).

7. Reparameterize the curve:

$$\vec{r}(t) = e^{7t}\cos(7t)\hat{\imath} + 6\hat{\jmath} + e^{7t}\sin(7t)\hat{\mathrm{k}}$$

   with respect to arc length s measured from $t = 0$ in the direction of increasing t.

Partial Derivatives

1. Consider the function $f(x, y) = \ln(x + y + 1)$.

   (a) Determine an expression for the domain.

   (b) Sketch the domain.

2. Let $N(u, v, w) = \dfrac{p+q}{p+r}$ where:

$$p = u + vw$$
$$q = v + uw$$
$$r = w + uv$$

(a) Using proper chain rule notation, determine all three first order partial derivatives.

(b) Evaluate the partial derivatives at $u = 2$, $v = 3$, and $w = 4$.

3. How many critical points does the following function have?

$$g(x, y) = -32xy + (x + y)^4$$

4. Find all extrema and saddle points for the function:

$$f(x, y) = x^2 + x^2y + 2y^2 + 3$$

5. Use the second partial derivative test to determine all extrema for the function:

$$f(x, y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$$

6. Find the absolute minimum and maximum value of the function:

$$f(x, y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$$

over the region $R = [0, 4] \times [0, 2]$.

7. Determine the absolute maximum value of $f(x, y) = 2x^3 + y^4$ on the disk $D = \{(x, y) \,|\, 0 \le x \le 3, 0 \le y \le 2\}$.

8. Determine the gradient of the function $f(x, y, z) = z^2 e^{x\sqrt{y}}$.

9. Consider the function $f(x, y) = xe^{xy}$.

(a) Determine the linearization $L(x, y)$ of the function at the point $(1, 0)$.

(b) Use $L(x, y)$ to approximate $f(1.1, -0.1)$.

10. Find three positive numbers whose sum is 12 and whose squares is as small as possible.

11. At what point on the paraboloid $y = x^2 + z^2$ is the tangent plane parallel to the plane $7x + 4y + 5z = 4$?

12. The electric potential V at any point (x, y) is given by $V = \ln \sqrt{x^2 + y^2}$.

(a) Determine the rate of change in V at the point $(3, 4)$ in the direction of the point $(2, 6)$.

(b) What is the direction of maximum change in V from the point $(3, 4)$.

(c) What is the maximum rate of change?

13. The temperature in $^\circ C$ of a flat plate is a function of the distances x [mm] and y [mm] from the center of the plate. The following table samples some values of the function $T(x, y)$ in the vicinity of the point $(-2, 1)$. Assume that $T(x, y)$ is continuous and differentiable everywhere (i.e., is smooth).

1.2	36.12	35.45	34.80	34.17	33.56
1.1	35.18	34.53	33.90	33.29	32.70
1.0	34.24	33.61	33.00	32.41	31.84
0.9	33.30	32.69	32.10	31.53	30.98
0.8	32.36	31.77	31.20	30.65	30.12
y \ x	-2.2	-2.1	-2.0	-1.9	-1.8

Select the best estimate for $\|\nabla T(-2, 1)\|$:

(a) 10.8

(b) 15.0

(c) 4.0

(d) 12.6

(e) 20.8

and select the best estimate for the direction of $\nabla T(-2, 1)$:

(a) $\langle -3, -2 \rangle$

(b) $\langle 3, 3 \rangle$

(c) $\langle -3, 2 \rangle$

(d) $\langle 2, -3 \rangle$

(e) $\langle -2, 3 \rangle$

14. The temperature over a particular surface is given by:

$$T(x, y, z) = 300e^{-x^2-3y^2-7z^2}$$

where T is measured in $^\circ C$ and x, y, x is measured in meters. Consider an ant on the surface at position $P(2, -1, 5)$ who starts crawling toward the point $Q(5, -2, 6)$. For each of the following questions, all answers must be in exact form (i.e., no decimals).

   (a) Determine the rate of change of temperature in the indicated direction.

   (b) Is the temperature getting hotter or cooler in the indicated direction.

   (c) In what direction does the temperature increase the fastest?

   (d) What is the maximum rate of change at P.

15. Use the Lagrange multiplier technique to determine the extreme values of the function $f(x, y) = xe^y$ subject to the constraint $x^2 + y^2 = 3$.

Multiple Integrals

1. Evaluate the integral $\displaystyle\iint_D xy\, dA$ where D is the triangle with vertices $(0, 2)$, $(3, -1)$, and $(3, 2)$.

2. Consider the integral $\displaystyle\iint_D (x^2 + 2y)dA$ where D is the region bounded by $y = x$, $y = x^3$, and $x \geq 0$.

   (a) Set up (but do not evaluate) the integral as a type 1 region.

   (b) Set up (but do not evaluate) the integral as a type 2 region.

   (c) Evaluate once using either order of integration.

3. Evaluate $\displaystyle\iint_D (x + y)\, dA$ where D is the region that lies to the left of the y-axis between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

4. Determine the area of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 4$.

5. Consider the lamina that occupies the region D bounded by the parabolas $y = x^2$ and $x = y^2$ with density $\rho(x, y) = \sqrt{x}$.

   (a) Determine the total mass of the lamina.

   (b) Determine the position of the lamina's center of mass.

6. Determine the volume of the solid under the paraboloid $z = x^2 + 4y^2$ and above the rectangle $R = [0, 2] \times [1, 4]$.

7. Evaluate $\displaystyle\iiint_E z\,dV$ where E is the region enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

8. Which of the following describes the boundary of E in the following integral:

$$\iiint_E f(x, y, z)dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{2-x^2-y^2}$$

   (a) A paraboloid and a plane

   (b) A sphere

   (c) A cone and a half-sphere

   (d) A paraboloid and a half-sphere

   (e) A cone and a plane

   (f) An ellipsoid

   (g) Two cones

   (h) A cone and a paraboloid

9. Evaluate the integral $\displaystyle\iint_D (x^2+y^2)y\,dA$ where D is the region bounded by the circle $x^2 + y^2 = 2x$ using polar coordinates.

10. Use cylindrical coordinates to determine the volume of the solid that is enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 50$. Your answer must be in exact form (i.e., no decimals).

11. Use cylindrical coordinates to evaluate $\iint_E (x^3 + xy^2)dV$ where E is the solid in the first octant the lies beneath the paraboloid $z = 1 - x^2 - y^2$.

12. Use spherical coordinates to determine the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 36$, above the xy-plane, and before the cone $z = \sqrt{x^2 + y^2}$. Your answer must be in exact form (i.e., no decimals).

13. Evaluate $\iiint_E xe^{x^2+y^2+z^2}\, dV$ where E is the portion of the unit ball $x^2 + y^2 + z^2 \leq 1$ that lies in the first octant.

14. Change $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\, dzdydx$ to spherical coordinates and evaluate it.