Geometric Distribution

Definition: Geometric Distribution

To say that a random variable X has a *Geometric* distribution with parameter p, denoted:

$$X \sim \text{Geom}(p)$$

means that:

1. The underlying experiment is composed of repeated Bernoulli trials until the first success occurs.
2. The trials are independent.
3. Each of the trials has fixed probability p for success.
4. X counts the number of trials up to and including the first success.

Examples: Geometric Distributions

1. Flip a fair coin until the first head occurs: $X =$ the number trials.

$$X \sim \text{Geom}\left(\frac{1}{2}\right)$$

2. Select (with replacement) balls from an urn that has N balls, r of which are red, until the first red ball is selected: $Y =$ the number of balls selected.

$$Y \sim \text{Geom}\left(\frac{r}{N}\right)$$

Lemma

Assume $a \in \mathbb{R}$ such that $0 < |a| < 1$:

$$\sum_{x=1}^{\infty} x a^{x-1} = \frac{1}{(1-a)^2}$$
$$\sum_{x=2}^{\infty} x(x-1) a^{x-2} = \frac{2}{(1-a)^3}$$

Proof.

$$\sum_{x=0}^{\infty} a^x = \frac{1}{1-a}$$
$$\frac{d}{da}\left[\sum_{x=0}^{\infty} a^x\right] = \frac{d}{da}\left[\frac{1}{1-a}\right]$$
$$\sum_{x=1}^{\infty} x a^{x-1} = \frac{1}{(1-a)^2}$$

$$\frac{d}{da}\left[\sum_{x=1}^{\infty} x a^{x-1}\right] = \frac{d}{da}\left[\frac{1}{(1-a)^2}\right]$$
$$\sum_{x=2}^{\infty} x(x-1)a^{x-2} = \frac{2}{(1-a)^3}$$

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Theorem

Let X be a random variable with a Geometric distribution with parameter p:

- $f_X(x) = \begin{cases} p(1-p)^{x-1} & x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$
- $E(X) = \frac{1}{p}$
- $V(X) = \frac{1-p}{p^2}$

Proof. For $P(X = x)$, the first $x - 1$ failures have probability $(1-p)^{x-1}$ and the final success has probability p. Therefore:

$$f_X(x) = p(1-p)^{x-1}$$

To calculate the expected value:

$$E(X) = \sum_{x=1}^{\infty} xp(1-p)^{x-1} = p\sum_{x=1}^{\infty} x(1-p)^{x-1} = p \cdot \frac{1}{[1-(1-p)]^2} = p \cdot \frac{1}{p^2} = \frac{1}{p}$$

To calculate the variance:

$$\begin{aligned}
V(X) &= E(X^2) - E(X)^2 \\
&= E(X^2 - X + X) - E(X)^2 \\
&= E(X^2 - X) + E(X) - E(X)^2 \\
&= E(X(X-1)) + E(X) - E(X)^2 \\
&= \sum_{x=2}^{\infty} x(x-1)p(1-p)^{x-1} + \frac{1}{p} - \frac{1}{p^2} \\
&= p(1-p)\sum_{x=2}^{\infty} x(x-1)(1-p)^{x-2} + \frac{p-1}{p^2} \\
&= p(1-p)\frac{2}{[1-(1-p)]^3} + \frac{p-1}{p^2} \\
&= \frac{2(1-p)}{p^2} + \frac{p-1}{p^2} \\
&= \frac{2-2p+p-1}{p^2}
\end{aligned}$$

$$V(X) = \frac{1-p}{p^2}$$

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Example

Suppose X has a Geometric distribution with $p = \frac{1}{2}$.

$$X \sim \text{Geom}\left(\frac{1}{2}\right)$$

$$P(X = 4) = \frac{1}{2}\left(1 - \frac{1}{2}\right)^{4-1} = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

$$P(X \geq 4) = \sum_{x=4}^{\infty} \frac{1}{2}\left(1 - \frac{1}{2}\right)^{x-1} = \sum_{x=4}^{\infty} \left(\frac{1}{2}\right)^x = \frac{1}{1-\frac{1}{2}} - 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

$$E(X) = \frac{1}{p} = \frac{1}{\frac{1}{2}} = 2$$

$$V(X) = \frac{1-p}{p^2} = \frac{1-\frac{1}{2}}{\frac{1}{4}} = 2$$

$$\sigma = \sqrt{2} \approx 1.41$$