One-sided Axioms

The traditional definition of a group uses a two-sided identity and two-sided inverses. It is possible to give a weaker definition of a group using a one-sided identity and matching one-sided inverses:

Theorem

Let G be a semigroup. G is a group iff the following two properties hold:

1). G contains a left identity element:

$$\exists\, e \in G, \forall\, a \in G, ea = a$$

2). Each element in G contains a left inverse element that is also in G:

$$\forall\, a \in G, \exists\, a^{-1} \in G, a^{-1}a = e$$

Proof

$\Longrightarrow$ Assume G is a group.

G has a two-sided identity e, which is also one-sided.
$\forall\, a \in G$, a has a two-sided inverse, which is also one-sided.
$\therefore$ the two properties hold.

$\Longleftarrow$ Assume that G has a left-sided identity and left-sided inverses.

Assume $a \in G$:

$$\begin{aligned}
ee &= e \\
(a^{-1}a)e &= a^{-1}a \\
(a^{-1})^{-1}((a^{-1}a)e) &= (a^{-1})^{-1}(a^{-1}a) \\
((a^{-1})^{-1}a^{-1})(ae) &= ((a^{-1})^{-1}a^{-1})a \\
e(ae) &= ea \\
ae &= a
\end{aligned}$$

$\therefore$ e is a right identity as well.

$$\begin{aligned}
a^{-1}a &= e \\
(a^{-1}a)a^{-1} &= ea^{-1} \\
(a^{-1}a)a^{-1} &= a^{-1} \\
(a^{-1})^{-1}((a^{-1}a)a^{-1}) &= (a^{-1})^{-1}a^{-1} \\
((a^{-1})^{-1}a^{-1})(aa^{-1}) &= e \\
e(aa^{-1}) &= e \\
aa^{-1} &= e
\end{aligned}$$

$\therefore a^{-1}$ is a right inverse as well.

G is associative, has a two-sided identity, and has two-sided inverses.
$\therefore G$ is a group.

A similar theorem exists for a right identity and right inverses as well, but not for a mix of a left and a right.

<u>Theorem</u>

Let G be a semigroup. G is a group iff the following two properties hold:

1). $\forall\, a, b \in G, ax = b$ has a solution in G.

2). $\forall\, a, b \in G, xa = b$ has a solution in G.

<u>Proof</u>

$\Longrightarrow$ Assume G is a group.

Previously proven.

$\Longleftarrow$ Assume $\forall\, a, b \in G, ax = b$ and $xa = b$ have solutions in G.

Assume $a, b \in G$

Let $e \in G$ be a solution for $xa = a$
$ea = a$
Let c be a solution for $ax = b$
$ac = b$
$eb = e(ac) = (ea)c = ac = b$
$\therefore e$ is a left identity for G.

Let $a^{-1} \in G$ be a solution for $xa = e$
$a^{-1}a = e$
a^{-1} is a left inverse for a
$\therefore G$ has left inverses.

$\therefore$, by the one-sided test, G is a group.