Components

Definition: Component

Let X be a topological space and let $p \in X$. The *(connected) component* of p in X is the union of all connected subsets of X that contain p.

Theorem

Let X be a topological space. Each component of X is connected, closed, and not contained in any strictly larger connected subset of X.

Proof. Assume that U is a component of X and that $p \in U$. By definition, $U = \bigcup_{\alpha \in \lambda} U_\alpha$ where U_α is a connected subset of X containing p. Now, for each U_α, $U_\alpha \cap \{p\} \neq \emptyset$ and $\{p\}$ is trivially connected. Therefore U is connected.

Assume that V is a connected component of X such that $U \subset V$. This means that $p \in V$ and so, by definition, $U = U \cup V$. But this is ony true if $V \subset U$. Therefore $U = V$.

Now, since U is connected, $\bar{U}$ is connected. But $p \in \bar{U}$ and so, by definition, $U = U \cup \bar{U}$. But this is only true if $\bar{U} \subset U$. Therefore $U = \bar{U}$ and hence U is closed. ∎

Theorem

Let X be a topological space. The set of components of X are a partition of X.

Theorem

Let X be a topological space with a finite number of components. The components are clopen in X.

Proof. Assume that $X = U_1 \sqcup \cdots \sqcup U_n$. All of the U_k are closed. But $U_k = X - (U_1 \sqcup \cdots \sqcup U_{k-1} \sqcup U_{k+1} \sqcup \cdots \sqcup U_n)$ is then open. Therefore, all of the U_k are clopen. ∎

This result does not hold for an infinite number of components. Consider $Q \subset \mathbb{R}$ with the subspace topology. Q is disconnected with components $\{\{r\} \,|\, r \in \mathbb{Q}\}$. But $\{r\}$ is closed and not open.