Bessel's Inequality

Theorem: Bessel

Let E be an inner product space and let $(\vec{x}_n)$ be an orthonormal sequence in E. $\forall\,\vec{x} \in E$:

$$\sum_{n=1}^{\infty} |\langle \vec{x}, \vec{x}_n \rangle|^2 \leq \|\vec{x}\|^2$$

Proof

Assume $\{\vec{x}_{n_1}, \dots, \vec{x}_{n_r}\}$ is a finite subset of $(\vec{x}_n)$.
Let $S = \mathrm{Span}\{\vec{x}_{n_1}, \dots, \vec{x}_{n_r}\}$.
Let $y = \mathrm{proj}_S\, \vec{x} = \displaystyle\sum_{k=1}^{r} \langle \vec{x}, \vec{x}_{n_k} \rangle\, \vec{x}_{n_k}$.

$$\begin{aligned}
\|\vec{x}\|^2 &= \|(\vec{x} - \vec{y}) + \vec{y}\|^2 \\
&= \|\vec{x} - \vec{y}\|^2 + \|\vec{y}\|^2 \\
&\geq \|\vec{y}\|^2 \\
&= \left\| \sum_{k=1}^{r} \langle \vec{x}, \vec{x}_{n_k} \rangle\, \vec{x}_{n_k} \right\|^2 \\
&= \left\langle \sum_{k=1}^{r} \langle \vec{x}, \vec{x}_{n_k} \rangle\, \vec{x}_{n_k}, \sum_{j=1}^{r} \left\langle \vec{x}, \vec{x}_{n_j} \right\rangle \vec{x}_{n_j} \right\rangle \\
&= \sum_{k=1}^{r} |\langle \vec{x}, \vec{x}_{n_k} \rangle|^2
\end{aligned}$$

Now, let $r \to \infty$.

$\therefore \displaystyle\sum_{n=1}^{\infty} |\langle \vec{x}, \vec{x}_n \rangle|^2 \leq \|\vec{x}\|^2$

Corollary

Let E be an inner product space and let $(\vec{x}_n)$ be an orthonormal sequence in E. $\forall\,\vec{x} \in E$:

$(\langle \vec{x}, \vec{x}_n \rangle)$ is a sequence in ℓ^2.

Proof

$$\sum_{n=1}^{\infty} |\langle \vec{x}, \vec{x}_n \rangle|^2 \leq \|\vec{x}\|^2 < \infty$$

Therefore, by definition, $(\langle \vec{x}, \vec{x}_n \rangle)$ is in ℓ^2.

Corollary

Let E be an inner product space and let $(\vec{x}_n)$ be an orthonormal sequence in E:

$$\vec{x} \xrightarrow{w} \vec{0}$$

Proof

Assume $\vec{x} \in E$.
$(\langle \vec{x}, \vec{x}_n \rangle)$ is a sequence in ℓ^2.
And so $\langle \vec{x}, \vec{x}_n \rangle \to 0 = \left\langle \vec{x}, \vec{0} \right\rangle$.

$\therefore \vec{x} \xrightarrow{w} \vec{0}$

But note that for an orthonormal sequence $(\vec{x}_n)$, $\vec{x}_n \not\to \vec{0}$:

$$\left\| \vec{x}_n - \vec{0} \right\| = \|\vec{x}_n\| = 1 \not\to 0$$