Approximating Values

Definition

Let $x \in \mathbb{R}$:

- The greatest integer (floor) function, denoted $\lfloor x \rfloor$, yields the greatest integer that is less than or equal to x:

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$

- The least integer (ceiling) function, denoted $\lceil x \rceil$, yields the least integer that is greater than or equal to x:

$$\lceil x \rceil - 1 < x \leq \lceil x \rceil$$

- The fractional part of x is given by:

$$\{x\} = x - \lfloor x \rfloor$$

Theorem

$$\forall\, x \in \mathbb{R}, 0 \leq \{x\} < 1$$

Proof

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$
$$0 \leq x - \lfloor x \rfloor < 1$$
$$\therefore 0 \leq \{x\} < 1$$

Example

$$\left\lfloor \frac{3}{2} \right\rfloor = 1$$
$$\left\{ \frac{3}{2} \right\} = \frac{3}{2} - 1 = \frac{1}{2}$$

$$\left\lfloor -\frac{3}{2} \right\rfloor = -2$$
$$\left\{ -\frac{3}{2} \right\} = -\frac{3}{2} - (-2) = \frac{1}{2}$$

A real number is always within $\frac{1}{2}$ of some integer. The following theorem provides a more general approximation for real numbers:

Theorem: Dirichlet Approximation

Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{Z}^+$:

$$\exists\, a, b \in \mathbb{Z}, |a\alpha - b| < \frac{1}{n}$$

Given an α and n, some multiple of α is within $\frac{1}{n}$ of some other integer.

Proof

Let $S = \{\{k\alpha\} \mid 0 \le k \le n\}$
S contains the fractional parts of $n+1$ multiples of α
$0 \le \{k\alpha\} < 1$
Let $T = \left\{\left[\frac{k}{n}, \frac{k+1}{n}\right) \mid 0 \le k < n\right\}$
T is a partition of $[0, 1)$ into n mutually disjoint intervals of length $\frac{1}{n}$
By the pigeonhole principle, at least one of the intervals in T must contain at least two of the fractional parts in S
$|\{k\alpha\} - \{j\alpha\}| < \frac{1}{n}, 0 \le j < k \le n$
$|(k\alpha - \lfloor k\alpha \rfloor) - (j\alpha - \lfloor j\alpha \rfloor)| < \frac{1}{n}$
$|(k-j)\alpha - (\lfloor k\alpha \rfloor - \lfloor j\alpha \rfloor)| < \frac{1}{n}$
Let $a = (k-j)$ and $b = (\lfloor k\alpha \rfloor - \lfloor j\alpha \rfloor)$
$|a\alpha - b| < \frac{1}{n}$
$1 \le a \le n$

Example

$\alpha = \sqrt[3]{3} \approx 1.44225$
$n = 10$
$\frac{1}{n} = \frac{1}{10} = 0.1$

a	$a\alpha$	b	$\lvert a\alpha - b\rvert$	
1	1.44225	1	0.44225	
2	2.88450	3	0.11550	
3	4.32675	4	0.32675	
4	5.76900	6	0.23100	
5	7.21225	7	0.21225	
6	8.65350	9	0.34650	
7	10.0957	10	0.09575	✓
8	11.5380	12	0.46200	
9	12.9802	13	0.01975	✓
10	14.4225	14	0.42250	

Corollary

Given $\alpha \in \mathbb{R} - \mathbb{Q}$, a rational approximation $\frac{p}{q}$ can be found within $\frac{1}{q^2}$ of α.

Proof

$\exists\, p, q \in \mathbb{Z}, |q\alpha - p| < \frac{1}{n}$ with $1 \leq q \leq n$

$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{nq} \leq \frac{1}{q^2}$

Example

$\alpha = \sqrt{2} \approx 1.41421$

q	$\frac{1}{q}$	$\frac{1}{q^2}$	p	$\left\|\alpha - \frac{p}{q}\right\|$		
1	1	1	1	0.414214	✓	✓
			2	0.585786	✓	✓
2	0.5000	0.2500	3	0.085786	✓	✓
3	0.3333	0.1111	4	0.080880	✓	✓
			5	0.252453	✓	
4	0.2500	0.0625	5	0.164214	✓	
5	0.2000	0.0400	6	0.214214		
			7	0.014214	✓	✓
			8	0.185786	✓	