Resolvent and Spectrum

Definition: Resolvent and Spectrum

Let E be a normed space and let A be an operator on E. The *resolvent* of A, denoted A_λ, is given by:

$$A_\lambda = (A - \lambda I)^{-1}$$

Note that λ is an eigenvalue of A iff A_λ is not defined.

To say that λ is a *regular value* of A means $A_\lambda \in \mathcal{B}(E)$.

The *resolvent set* of A, denoted $\rho(A)$, is the set of all regular values of A.

The *spectrum* of A, denoted $\sigma(A)$, is given by:

$$\sigma(A) = \mathbb{C} \setminus \rho(A)$$

The *spectral radius* of A, denoted $r(A)$, is given by:

$$r(A) = \sup\{|\lambda| \mid \lambda \in \sigma(A)\}$$

If λ is an eigenvalue of A then $\lambda \in \sigma(A)$. But note that the spectrum can contain non-eigenvalues and no eigenvalues.

Lemma

Let E be a Banach space and let $A \in \mathcal{B}(E)$ such that $\|A\| < 1$:

1). $I - A$ is invertible.

2). $(I - A)^{-1} = \displaystyle\sum_{n=0}^{\infty} A^n$

Proof

Let $S_n = \displaystyle\sum_{k=0}^{n} A^k$.
AWLOG: $n > m$.

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^{n} A^k \right\| \leq \sum_{k=m+1}^{n} \|A^k\| = \sum_{k=m+1}^{n} \|A\|^k \to 0$$

And so (S_n) is Cauchy.
But E Banach $\implies \mathcal{B}(E)$ Banach.
And $A \in \mathcal{B}(E) \implies S_n \in \mathcal{B}(E)$.

Thus (S_n) converges to $S \in \mathcal{B}(E)$ where $S = \displaystyle\sum_{n=0}^{\infty} A^n$.

$$S(I-A) = \left(\sum_{n=0}^{\infty} A^n\right)(I-A) = \sum_{n=0}^{\infty} A^n - \sum_{n=0}^{\infty} A^{n+1} = A^0 = I$$

$$(I-A)S = (I-A)\left(\sum_{n=0}^{\infty} A^n\right) = \sum_{n=0}^{\infty} A^n - \sum_{n=0}^{\infty} A^{n+1} = A^0 = I$$

$$\therefore S = (I-A)^{-1} = \sum_{n=0}^{\infty} A^n$$

Corollary

Let E be a Banach space and $T \in \mathcal{B}(E)$ such that $\|I-T\| < 1$:

1). T is invertible.

2). $T^{-1} = \sum_{n=0}^{\infty}(I-T)^n$

Proof

$$T^{-1} = [I-(I-T)]^{-1} = \sum_{n=0}^{\infty}(I-T)^n$$

Theorem

Let E be a Banach space and let $A \in \mathcal{B}(E)$ and $\|A\| \leq |\lambda|$:

1). $A_\lambda = -\sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} A^n$

2). $\|A_\lambda\| \leq \frac{1}{|\lambda| - \|A\|}$

Thus, $A_\lambda \in \mathcal{B}(E)$.

Proof

Let $B = \frac{A}{\lambda}$.

$\|B\| = \frac{\|A\|}{|\lambda|} < 1$

$\sum_{n=0}^{\infty} B^n = (I-B)^{-1}$

$\sum_{n=1}^{\infty} \left(\frac{A}{\lambda}\right)^n = \left(I - \frac{A}{\lambda}\right)^{-1} = -\frac{1}{\lambda}(A-\lambda I)^{-1}$

$$\therefore A_\lambda = (A-\lambda I)^{-1} = -\sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} A^n$$

$$\|A_\lambda\| = \left\| -\sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} A^n \right\| \le \sum_{n=0}^{\infty} \left\| \frac{1}{\lambda^{n+1}} A^n \right\| = \frac{1}{|\lambda|} \sum_{n=1}^{\infty} \left(\frac{\|A\|}{|\lambda|} \right)^n = \frac{1}{|\lambda|} \left(\frac{1}{1 - \frac{\|A\|}{|\lambda|}} \right)$$

$$\therefore \|A_\lambda\| \le \frac{1}{|\lambda| - \|A\|}$$

Thus, if λ is an eigenvalue of A, then $|\lambda| < \|A\|$ and $r(A) \le \|A\|$ and $\sigma(A)$ is an open set.