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Math 275A
Homework #8

Theorem: 6.2

Let $A \subset \mathbb{R}_{\text{std}}$. If A is compact then A has a maximum point.

Proof. If A is finite then trivial, so assume that A is infinite. ABC that A has no maximum point. This means that for all $a \in A$ there exists $b_a \in A$ such that $b_a > a$. So let $\{(-\infty, b_a) : a \in A\}$ be an open cover for A. Since A is compact, there exists a finite subcover $U = \{(-\infty, b_{a_k}) : 1 \leq k \leq n\}$. Let $c = \max\{b_{a_k}\}$, and so $\bigcup U = (-\infty, c)$. Thus $c \in A$ but $c \notin U$, contradicting the assumption that U is a finite subcover.

Therefore A has a maximum point. ■

Theorem: 6.9

Every compact subspace of a Hausdorff space is closed.

Proof. Assume that X is Hausdorff and A is a compact subspace of X. Assume that $b \in A^C$. Since X is Hausdorff, for every $a \in A$ there exists $U_a, V_a \in \mathscr{T}_X$ such that $a \in U_a$, $b \in V_a$, and $U_a \cap V_a = \emptyset$. So let the $\{U_a : a \in A\}$ be an open cover of A in X. Thus $\{U_a \cap A : a \in A\}$ for $U_a \cap A \in \mathscr{T}_Y$ is an open cover of A in A. Now, since A is a compact subspace of X, there exists a finite subcover $(U_{a_1} \cap A) \cup \cdots \cup (U_{a_n} \cap A)$ of A in A, and hence a finite subcover $U_{a_1} \cup \cdots \cup U_{a_n}$ of A in X. Let $V = V_{a_1} \cap \cdots \cap V_{a_n}$. Note that $b \in V$ and $V \in \mathscr{T}_X$. Furthermore, since all the $U_a \cap V_a = \emptyset$, it must be the case that $V \cap (U_{a_1} \cup \cdots \cup U_{a_n}) = \emptyset$. But since $U_{a_1} \cup \cdots \cup U_{a_n} \supset A$ it must be the case that $V \subset A^C$. So b is an interior point in A^C, meaning that all the points in A^C are interior, and so $A^C \in \mathscr{T}_X$. Therefore A is closed in X. ■

Lemma

Every compact, Hausdorff space is regular.

Proof. Assume that X is compact and Hausdorff. Assume that $A \subset X$ is closed. Thus, by previous theorem, A is also compact. So assume $p \in A^C$. This means that $p \notin A$ and so, by the previous proof, there exists $U, V \in \mathscr{T}$ such that $A \subset U$ and $p \in V$ and $U \cap V = \emptyset$.

Therefore X is regular. ■

Theorem: 6.12

Every compact, Hausdorff space is normal.

Proof. Assume $A, B \subset X$ are closed. Since X is regular (by the previous lemma), for all $b \in B$ there exists $U_b, V_b \in \mathscr{T}$ such that $A \subset U_b$ and $b \in V_b$ and $U_b \cap V_b = \emptyset$. So let $V = \{V_b : b \in B\}$ be an open cover for B. But, by previous theorem, B is also compact, and so there exists a finite subcover $V_{b_1} \cup \cdots \cup V_{b_n} \supset B$. So let $U = U_{b_1} \cap \cdots \cap U_{b_n} \in \mathscr{T}$. Note that $A \subset U$ and, since all the $U_b \cap V_b = \emptyset$, $U \cap V = \emptyset$. Therefore, X is normal. ∎