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Math 229
Homework #6

7.1.1

Let $A = [a_{ij}] \in M_n$ be positive semidefinite. Why is $a_{ii}a_{jj} \geq |a_{ij}|^2$ for all distinct $i, j \in [n]$? If A is positive definite, why is $a_{ii}a_{jj} > |a_{ij}|^2$? If there is a pair of distinct indices i, j such that $a_{ii}a_{jj} = |a_{ij}|^2$, why is A singular.

Assume $i, j \in [n]$ and AWLOG $i \leq j$. Consider the $\{i, j\}$ principal submatrix for A:

$$A_{ij} = \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$$

Note that A_{ij} is also positive semidefinite and so $\det A_{ij} \geq 0$. Hence:

$$a_{ii}a_{jj} - a_{ij}a_{ji} \geq 0$$

But A is also Hermitian, so $a_{ij} = \overline{a_{ji}}$, and so:

$$a_{ii}a_{jj} - |a_{ij}|^2 \geq 0$$

and finally:

$$a_{ii}a_{jj} \geq |a_{ij}|^2$$

For the positive definite case, use the same proof, only replace '$\geq$' with '$>$'.

Now assume $a_{ii}a_{jj} = |a_{ij}|^2$. Thus, there exists a 2×2 principle submatrix with a zero determinant. Use permutation matrices to make this submatrix a leading principle submatrix (which does not affect the eigenvalues), and call it A_2. Since $\det A_2 = 0$, we know that $0 \in \sigma(A_2)$. Since A_2 is also positive semidefinite, $\sigma(A_2) \subseteq [0, \infty)$ and thus we can conclude that $\lambda_1(A_2) = 0$.

Now, using interlacing, it is the case that $0 \leq \lambda_1(A_3) \leq \lambda_1(A_2)$ and so $\lambda_1(A_3) = 0$. This can be continued all the way to A_n and thus A has a 0 eigenvector. Therefore, by the IMT, A is singular.

7.1.2

Let A be a positive semidefinite matrix. Prove:

A has a zero entry on its main diagonal $\iff$ the corresponding entire row and column are zero.

$\implies$ Assume $a_{ii} = 0$

By the previous problem, for all $1 \leq j \leq n$:

$$a_{ii}a_{jj} = 0 \geq |a_{ij}|^2$$

And thus $a_{ij} = 0$ for all $1 \leq j \leq n$. But A is Hermitian, so $a_{ij} = \overline{a_{ji}} = 0$ and so $a_{ji} = 0$ for all $1 \leq j \leq n$. Therefore, the corresponding row and column are all zeros.

$\impliedby$ Assume for given i and all $1 \leq j \leq n$ that $a_{ij} = a_{ji} = 0$

Then clearly $a_{ii} = 0$.

7.2.6

Let $A \in M_n$ for $n \geq 2$ be Hermitian and let $B \in M_{n-1}$ be a leading principal submatrix of A. Prove: B is positive semidefinite and $\mathrm{rank}(B) = \mathrm{rank}(A) \implies A$ is positive semidefinite.

Assume B is positive semidefinite and $\mathrm{rank}(B) = \mathrm{rank}(A)$.

Since the ranks are equal but $\dim(A) = \dim(B) + 1$, A has one more zero singular value than B. Since B is positive semidefinite, and thus, Hermitian, and thus normal, $s_k = |\lambda_k(B)|$ and so $a_A(0) = a_B(0) + 1$.

Now, by applying the interlacing theorem, we note that:

$$\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \lambda_2(B) \leq \cdots \leq \lambda_{n-1}(B) \leq \lambda_n(A)$$

However, since B is positive semidefinite, all of the $\lambda_k(B) \geq 0$, and thus all of the $\lambda_k(A) \geq 0$ for $2 \leq k \leq n$. And, sinc e $\mathrm{Sp}(A)$ has an additional 0, we can conclude $\lambda_1 = 0$

Thus A is Hermitian and $\mathrm{Sp}(A) \subseteq [0, \infty)$ and therefore A is positive semidefinite.

7.3.3

Let $A = M_n$. Prove: A has a zero singular value $\iff$ A has a zero eigenvector.

A has a zero singular value $\iff$ $\det(A) = 0 \iff$ A has a zero eigenvector.

Alternately, consider the proof done in class. Let S be the set of singular matrices and use the operator norm to measure distance. Given a matrix A, the distance between A and S is given by the smallest singular value. Thus, the smallest singular values is zero $\iff$ $A \in S$ $\iff$ A is singular $\iff$ A has a zero eigenvalue.

7.4.13

Let $\|\cdot\|$ be a self-adjoint norm and let H_n be the set of all $n \times n$ Hermitian matrices. Prove that the distance from a matrix A to H_n is given by:

$$d(A) = \frac{1}{2}\|A - A^*\|$$

Assume $H \in H_n$:

$$\|A - H\| = \frac{1}{2}\|A - H\| + \frac{1}{2}\|A - H\| = \frac{1}{2}\|A - H\| + \frac{1}{2}\|H - A\|$$

But $\|\cdot\|$ is self-adjoint by assumption, so:

$$\|A - H\| = \frac{1}{2}\|A - H\| + \frac{1}{2}\|(H - A)^*\| = \frac{1}{2}\|A - H\| + \frac{1}{2}\|H^* - A^*\|$$

But H is Hermitian, so:

$$\|A - H\| = \frac{1}{2}\|A - H\| + \frac{1}{2}\|H - A^*\|$$

Now, apply the triangle inequality:

$$\|A - H\| \geq \left\|\frac{1}{2}(A - H) + \frac{1}{2}(H - A^*)\right\| = \frac{1}{2}\|A - A^*\|$$

So $d(A) \geq \frac{1}{2}\|A - A^*\|$.

Now, consider $A + A^*$ and note that:

$$(A + A^*)^* = A^* + (A^*)^* = A^* + A = A + A^*$$

So $A + A^*$ is Hermitian, and so is $\frac{1}{2}(A + A^*)$, so let $H = \frac{1}{2}(A + A^*)$:

$$\left\|A - \frac{1}{2}(A + A^*)\right\| = \left\|\frac{1}{2}(A - A^*)\right\| = \frac{1}{2}\|A - A^*\|$$

$\therefore d(A) = \frac{1}{2}\|A - A^*\|$