Banach Fixed Point Theorem

Definition: Fixed Point

To say that x is a *fixed point* of a function f means:

$$f(x) = x$$

Note that any equation can be written in terms of a fixed point problem:

$$\begin{aligned} f(x) &= y \\ f(x) - y &= 0 \\ f(x) - y + x &= x \\ F(x) &= x \end{aligned}$$

where $F(x) = f(x) - y + x$.

Example

Let $y'(t) = f(t, y)$ and $y(t_0) = y_0$. This initial value problem has the unique solution:

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$$

So $y = T(y)$.

Definition: Contraction Mapping

Let E be a normed space and $A \subseteq E$. To say that a mapping $T : A \to E$ is a *contraction mapping* means $\exists\, \lambda \in (0, 1)$ such that $\forall\, \vec{x}, \vec{y} \in A$:

$$\|T\vec{x}) - T\vec{y}\| \leq \lambda \, \|\vec{x} - \vec{y}\|$$

Example

Let $f : \mathbb{R} \to \mathbb{R}$. By the MVT:

$$|f(x) - f(y)| = |f'(c)(x - y)|$$

for some $c \in (x, y)$.

Assume $|f'(x)| \leq \lambda < 1$

Then f is a contraction mapping.

Theorem: Banach Fixed Point Theorem

Let E be a Banach space and let X be a closed subset of E:

$T : X \to X$ is a contraction mapping $\implies$ T has a fixed point $Tx_* = x_*$.

Proof

Assume T is a contraction mapping.
So $\exists \lambda \in (0,1), \forall \vec{x}, \vec{y} \in X, \|T\vec{x} - T\vec{y}\| \leq \lambda \|\vec{x} - \vec{y}\|$.
Assume $x_0 \in X$.
Let $x_n = T^n(x_0)$ for $n \geq 1$:

$$\|\vec{x}_{n+1} - \vec{x}_n\| = \left\|T(T^n\vec{x}_0) - T(T^{n-1}\vec{x}_0)\right\| \leq \lambda \left\|T^n\vec{x}_0 - T^{n-1}\vec{x}_0)\right\| = \lambda \|\vec{x}_n - \vec{x}_{n-1}\|$$

And so:

$$\|\vec{x}_{n+1} - \vec{x}_n\| \leq \lambda^n \|\vec{x}_1 - \vec{x}_0\|$$

AWLOG: $n < m$

$$\begin{array}{rcl} \|\vec{x}_m - \vec{x}_n\| & = & \left\|\sum_{k=n+1}^{m} (\vec{x}_k - \vec{x}_{k-1})\right\| \\ & \leq & \sum_{k=n+1}^{m} \|\vec{x}_k - \vec{x}_{k-1}\| \\ & \leq & \sum_{k=n+1}^{m} \lambda^{k-1} \|\vec{x}_1 - \vec{x}_0\| \\ & = & \|\vec{x}_1 - \vec{x}_0\| \sum_{k=n}^{m-1} \lambda^k \\ & \leq & \|\vec{x}_1 - \vec{x}_0\| \sum_{k=n}^{\infty} \lambda^k \\ & = & \dfrac{\lambda^n}{1-\lambda} \|\vec{x}_1 - \vec{x}_0\| \\ & \to & 0 \end{array}$$

Thus, $\vec{x}_n$ is Cauchy.
Moreover, by assumption, E is Banach (complete), and so $\exists \vec{x}_* \in E$ such that $\vec{x}_n \to \vec{x}_*$.
But, by assumption, X is closed, and thus $\vec{x}_* \in X$.

Now, show that $\vec{x}_*$ is a fixed point:

$$\begin{array}{rcl} \|T\vec{x}_* - \vec{x}_*\| & = & \|(T\vec{x}_* - \vec{x}_n) + (\vec{x}_n - \vec{x}_*)\| \\ & \leq & \|T\vec{x}_* - \vec{x}_n\| + \|\vec{x}_n - \vec{x}_*\| \\ & = & \|T\vec{x}_* - T\vec{x}_{n-1}\| + \|\vec{x}_n - \vec{x}_*\| \\ & \leq & \lambda \|\vec{x}_* - \vec{x}_{n-1}\| + \|\vec{x}_n - \vec{x}_*\| \\ & \to & 0 \end{array}$$

But $\|T\vec{x}_* - \vec{x}_*\| = 0 \iff T\vec{x}_* - \vec{x}_* = \vec{0}$.

$\therefore T\vec{x}_* = \vec{x}_*$, in other words, $\vec{x}_*$ is a fixed point of T.

Now assume that there exists another fixed point $\vec{y}_*$.

$$\|\vec{y}_* - \vec{x}_*\| = \|T\vec{y}_* - T\vec{x}_*\| \leq \lambda \|\vec{y}_* - \vec{x}_*\|$$

But $\lambda \in (0,1)$ and so $\lambda \neq 0$.
And so $\|\vec{y}_* - \vec{x}_*\| = 0$.
But $\|\vec{y}_* - \vec{x}_*\| = 0 \iff \vec{y}_* - \vec{x}_* = \vec{0}$.

$\therefore \vec{y}_* = \vec{x}_*$, and so the fixed point is unique.

Note that $\|T\vec{x} - T\vec{y}\| < \|\vec{x} - \vec{y}\|$ does not guarantee a fixed point.

Consider $E = \mathbb{R}$, $X = [0, \infty)$, and $T(x) = x + e^{-x}$.
By the MVT: $|T(x) - T(y)| = |T'(c)(x - y)|$ for some $c \in (x, y)$.
$T'(x) = 1 - e^{-x} < 1$ for $x \in [0, \infty)$.
So $|T(x) - T(y)| < |x - y|$.
However, does $T(x) = x$?

$$\begin{aligned} x &= x + e^{-x} \\ e^{-x} &= 0 \end{aligned}$$

No solution. So there is no fixed point.

<u>Theorem</u>

Let E be a normed space, $X \subseteq E$, and let $T : X \to E$ be a linear mapping:

> T is a contraction mapping iff $\|T\| < 1$.

<u>Proof</u>

$\|T\vec{x} - T\vec{y}\| = \|T(\vec{x} - \vec{y})\| \leq \|T\| \, \|\vec{x} - \vec{y}\|$

Therefore T is a contraction mapping iff $\|T\| < 1$.