Quadratic Equations

Definition

The general form of a quadratic equation is given by:

$$ax^2 + bx + c = 0$$

where $a \neq 0$

Note that if $a = 0$ we have a linear equation.

Before we tackle such equations, we need to be clear on two points:

1). $|x| = c$

Ask for a value whose absolute value is c. From either the definition: the high school one or the notion of distance, $x = \pm c$.

$|x| = 2$
$x = \pm 2$

$|x| = 0$
$x = 0$

$|x| = -1$
no solutions

2). $a = b \iff a^r = b^r$?

It depends! Don't do it without thinking!

3). $x^n = c$, n odd

$x^3 = 8$
$(x^3)^{\frac{1}{3}} = 8^{\frac{1}{3}}$
$x = 2$

$x^3 = -8$
$(x^3)^{\frac{1}{3}} = (-8)^{\frac{1}{3}}$
$x = -2$

4). $x^n = c$, n even

$x^2 = 4$
$(x^2)^{\frac{1}{2}} = 4^{\frac{1}{2}}$
$|x| = 2$
$|x| = \pm 2$

$x^2 = -4$
$(x^2)^{\frac{1}{2}} = (-4)^{\frac{1}{2}}$
no solution in $\mathbb{R}$

Don't mix up $\sqrt{a}$, which asks for the principle (positive) root, compared to $x^2 = c$, asking for solutions - they are two different questions!

5). $(x+d)^2 = c$

$(x+1)^2 = 4$
$|x+1| = 2$
$x+1 = \pm 2$
$x+1 = 2$ and $x+1 = -2$
$x = 1, -3$

Case 1: $b = 0$

This is the case above:

$$\begin{aligned} 2x^2 - 8 &= 0 \\ 2x^2 &= 8 \\ x^2 &= 4 \\ (x^2)^{\frac{1}{2}} &= 4^{\frac{1}{2}} \\ |x| &= 2 \\ x &= \pm 2 \end{aligned}$$

Case 2: $b \neq 0$

If we can factor by inspection, then we get an easy answer:

Two solutions:

$x^2 - 5x + 6 = 0$
$(x-2)(x-3) = 0$
$x = 2, 3$

Repeated solution:

$x^2 + 2x + 1 = 0$
$(x+1)^2 = 0$
$x = -1, -1$

Otherwise, complete the square!

Start with $a = 1$:

$$x^2 + bx$$

Want to know what constant to add so that it can be rewritten as $(x+d)^2$, because once we do that, we can use the rules above to get at the variable.

$$x^2 + bx + c = (x+d)^2 = x^2 + 2d + d^2$$

So $c = d^2$ and $d = \frac{b}{2}$, so divide by 2 and then square.

Example

$x^2 + 6x$
$b = 6$
$d = \frac{6}{2} = 3$ $c = 3^2 = 9$ $x^2 + 6x + 9 = (x+3)^2$

$x^2 - 2x$
$b = -2$
$d = \frac{-2}{2} = -1$
$c = (-1)^2 = 1$
$x^2 - 2x + 1 = (x-1)^2$

$x^2 - x$
$b = -1$
$d = -\frac{1}{2}$
$c = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$
$x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$

When $a \neq 1$, divide it out first:

Example

$2x^2 + 3x = 2\left(x^2 + \frac{3}{2}x\right)$
$b = \frac{3}{2}$
$d = \frac{3}{4}$
$c = \frac{9}{16}$
$2\left(x^2 + \frac{3}{2}x + \frac{9}{16}\right) = 2\left(x + \frac{3}{4}\right)^2$

$$
\begin{aligned}
2x^2+4x-3 &= 0\\
2x^2+4x &= 3\\
2(x^2+2x) &= 3\\
x^2+2x &= \frac{3}{2}\\
x^2+2x+1 &= \frac{3}{2}+1\\
(x+1)^2 &= \frac{5}{2}\\
|x+1| &= \sqrt{\frac{5}{2}}\\
x+1 &= \pm\sqrt{\frac{5}{2}}\\
x &= -1\pm\sqrt{\frac{5}{2}}
\end{aligned}
$$

$$
\begin{aligned}
ax^2+bx+c &= 0\\
ax^2+bx &= -c\\
x^2+\frac{b}{a}x &= -\frac{c}{a}\\
x^2+\frac{b}{a}x+\frac{b^2}{4a^2} &= -\frac{c}{a}+\frac{b^2}{4a^2}\\
\left(x+\frac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2}\\
x+\frac{b}{2a} &= \pm\frac{\sqrt{b^2-4ac}}{2a}\\
x &= -\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}\\
x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}
$$

And viola! We have the quadratic formula! So, we can solve quadratics in one of three ways (in order of desiredness):

1). Inspection

2). Completing the square

3). Quadratic formula

Definition

The discriminant of the quadratic equation $ax^2+bx+c=0$ is given by:

$$D = b^2 - 4ac$$

4 cases:

1). $D > 0$ and a perfect square:

Two rational solutions (probably by inspection):

x^2+4x+3
$a=1, b=4, c=3$
$x = \frac{-4\pm\sqrt{4^2-4(1)(3)}}{2(1)} = \frac{-4\pm 2}{2} = -1, -3$

Note that this provides a factoring: $(x-r_1)(x-r_2)=0$

$(x+1)(x+3)=0$

Another example:

$3x^2 + 10x + 3$
$a = 3, b = 10, c = 3$
$x = \frac{-10 \pm \sqrt{10^2 - 4(3)(3)}}{2(3)} = \frac{-10 \pm 8}{6} = -3, -\frac{1}{3}$

$\left(x + \frac{1}{3}\right)(x + 3) = 0$

But if you foil this, you don't quite get the original, need to adjust so that the F works:

$3\left(x + \frac{1}{3}\right)(x + 3) = 3(0)$
$(3x + 1)(x + 3) = 3(0)$

which is what we would get if we factored by inspection.

2). $D > 0$ and not a perfect square:

Two matching irrational roots, like the previous example.

$2x^2 + 4x - 3 = 0$
$x = -1 \pm \sqrt{\frac{5}{2}}$
$[x - (-1 + \sqrt{\frac{5}{2}})][x - (-1 - \sqrt{\frac{5}{2}})] = 0$
$[2x - 2(-1 + \sqrt{\frac{5}{2}})][x - (-1 - \sqrt{\frac{5}{2}})] = 0$
$[\sqrt{2}x - (-\sqrt{2} + \sqrt{5})][\sqrt{2}x - (-\sqrt{2} - \sqrt{5})] = 0$

3). $D = 0$

One repeated rational root:

$x^2 - 2x + 1 = 0$
$a = 1, b = 2, c = 1$
$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)} = \frac{2}{2} = 1$
$(x - 1)^2 = 0$

4). $D < 0$

No real solutions:

$x^2 + x + 1 = 0$
$a = 1, b = 1, c = 1$
$1^2 - 4(1)(1) = -3$

$x^2 + x = -1$
$x^2 + x + \frac{1}{4} = -\frac{3}{4}$
$\left(x + \frac{1}{2}\right)^2 = -\frac{3}{4}$

stuck!