Vector Space

Definition: Vector Space

A *vector (linear) space* is an algebraic structure that consists of:

1). A set of objects V called vectors.

2). A field $\mathbb{F}$ called scalars.

3). An operation of vector addition $(\vec{x} + \vec{y})$.

4). An operation of scalar multiplication $(c\vec{x})$.

such that $\forall\, \vec{x}, \vec{y}, \vec{z} \in V$ and $\forall\, a, b \in F$ the following ten properties hold:

1). Additive Closure: $\vec{x} + \vec{y} \in V$

2). Additive Commutativity: $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

3). Additive Associativity: $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$

4). Additive Identity: $\exists\, \vec{0} \in V, \vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x}$

5). Additive Inverse: $\exists\, (-\vec{x}) \in V, \vec{x} + (-\vec{x}) = (-\vec{x}) + \vec{x} = \vec{0}$

6). Multiplicative Closure: $a\vec{x} \in V$

7). Multiplicative Inverse: $1\vec{x} = \vec{x}$

8). Multiplicative Associativity $(ab)\vec{x} = a(b\vec{x})$

9). Scalar (Left) Distributivity: $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$

10). Vector (Right) Distributivity: $(a + b)\vec{x} = a\vec{x} + b\vec{x}$

Typical choices for scalar fields are the infinite fields: $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$; or a finite field like $\mathbb{Z}_2$, but not $\mathbb{Z}$, which is only a ring.

Example

1). $\mathbb{F}^n$, where the scalars are from $\mathbb{F}$ and the vectors are n-tuples of elements from $\mathbb{F}$, usually represented by column vectors. Addition and multiplication are component-wise:

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \qquad c\vec{x} = c \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$$

2). $M_{m \times n}(\mathbb{F})$, where the scalars are from $\mathbb{F}$ and the vectors are $m \times n$ matrices whose components are also from $\mathbb{F}$. Addition and multiplication are the standard matrix operations:

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

$$(cA)_{ij} = cA_{ij}$$

3). $\mathcal{F}(S, \mathbb{F})$, where the scalars are from $\mathbb{F}$ and the vectors are functions with domain S and codomain $\mathbb{F}$. Addition and multiplication are the standard function operations:

$$(f+g)(s) = f(s) + g(s)$$

$$(cf)(s) = cf(s)$$

4). $\mathbb{F}[x]$, where the scalars are from $\mathbb{F}$ and the vectors are polynomials with coefficients from $\mathbb{F}$. Note that this is an example of $\mathcal{F}(S, \mathbb{F})$, so addition and multiplication are the standard function operations as well.

Properties

Theorem: Cancellation Rules

Let V be a vector space over a field F. The cancellation rules hold in V:

$\forall\, \vec{x}, \vec{y}, \vec{z} \in V$ and $\forall\, a \in \mathbb{F} - \{0\}$:

1). Right: $\vec{x} + \vec{z} = \vec{y} + \vec{z} \implies \vec{x} = \vec{y}$

2). Left: $\vec{z} + \vec{x} = \vec{z} + \vec{y} \implies \vec{x} = \vec{y}$

3). Scalar: $a\vec{x} = a\vec{y} \implies \vec{x} = \vec{y}$

Proof

Assume $\vec{x}, \vec{y}, \vec{z} \in V$ and $a \in \mathbb{F} - \{0\}$:

1). Assume $\vec{x} + \vec{z} = \vec{y} + \vec{z}$

$$\begin{aligned}
&\exists\, (-z) \in V \\
&(\vec{x} + \vec{z}) + (-\vec{z}) = (\vec{y} + \vec{z}) + (-\vec{z}) \\
&\vec{x} + [\vec{z} + (-\vec{z})] = \vec{y} + [\vec{z} + (-\vec{z})] \\
&\vec{x} + \vec{0} = \vec{y} + \vec{0}
\end{aligned}$$

$\therefore \vec{x} = \vec{y}$

2). Assume $\vec{z} + \vec{x} = \vec{z} + \vec{y}$

$\vec{x} + \vec{z} = \vec{y} + \vec{z}$

$\therefore \vec{x} = \vec{y}$

3). Assume $a\vec{x} = a\vec{y}$

Since $a \neq 0$, $\exists\, a^{-1} \in \mathbb{F}$

$$\begin{aligned}
&a^{-1}(a\vec{x}) = a^{-1}(a\vec{y}) \\
&(a^{-1}a)\vec{x} = (a^{-1}a)\vec{y} \\
&1\vec{x} = 1\vec{y}
\end{aligned}$$

$\therefore \vec{x} = \vec{y}$

<u>Theorem: Zero</u>

Let V be a vector space over a field $\mathbb{F}$:

$\forall\, \vec{x} \in V$ and $\forall\, c \in \mathbb{F}$:

1). $\vec{0} \in V$ is unique

2). $(-\vec{0}) = \vec{0}$

3). $0\vec{x} = \vec{0}$

4). $c\vec{0} = \vec{0}$

<u>Proof</u>

Assume $\vec{x} \in V$ and $c \in \mathbb{F}$

1). Assume $\vec{0}, \vec{0}' \in V$ are both additive identities in V

$$\begin{aligned} &\vec{x} + \vec{0} = \vec{x} \\ &\vec{x} + \vec{0}' = \vec{x} \\ &\vec{x} + \vec{0} = \vec{x} + \vec{0}' \end{aligned}$$

$\therefore \vec{0} = \vec{0}'$

2). $$\begin{aligned} &\vec{0} + \vec{0} = \vec{0} \\ &\vec{0} + (-\vec{0}) = \vec{0} \\ &\vec{0} + (-\vec{0}) = \vec{0} + \vec{0} \end{aligned}$$

$\therefore (-\vec{0}) = \vec{0}$

3). $$\begin{aligned} &0\vec{x} = (0+0)\vec{x} = 0\vec{x} + 0\vec{x} \\ &0\vec{x} = 0\vec{x} + \vec{0} \\ &0\vec{x} + 0\vec{x} = 0\vec{x} + \vec{0} \end{aligned}$$

$\therefore 0\vec{x} = \vec{0}$

4). $$\begin{aligned} &c\vec{0} = c(\vec{0} + \vec{0}) = c\vec{0} + c\vec{0} \\ &c\vec{0} = c\vec{0} + \vec{0} \\ &c\vec{0} + c\vec{0} = c\vec{0} + \vec{0} \end{aligned}$$

$\therefore c\vec{0} = \vec{0}$

Theorem: Inverses

Let V be a vector space over a field $\mathbb{F}$:

$\forall\, \vec{x} \in V$ and $\forall\, c \in \mathbb{F}$:

1). $(-\vec{x})$ is unique

2). $(-1)\vec{x} = (-\vec{x})$

3). $(-c)\vec{x} = -(c\vec{x}) = c(-\vec{x})$

Proof

Assume $\vec{x} \in V$ and $c \in \mathbb{F}$

1). Assume $(-\vec{x}), (-\vec{x'}) \in V$ are both additive inverses for $\vec{x}$

$\vec{x} + (-\vec{x}) = \vec{0}$
$\vec{x} + (-\vec{x'}) = \vec{0}$
$\vec{x} + (-\vec{x}) = \vec{x} + (-\vec{x'})$

$\therefore (-\vec{x}) = (-\vec{x'})$

2). $(-1)\vec{x} + \vec{x} = (-1)\vec{x} + 1\vec{x} = [(-1) + 1]\vec{x} = 0\vec{x} = \vec{0}$
But inverses are unique

$\therefore (-1)\vec{x} = (-\vec{x})$

3). $(-c)\vec{x} + c\vec{x} = [(-c) + c]\vec{x} = 0\vec{x} = \vec{0}$
But additive inverses are unique

$\therefore (-c)x = -(cx)$

In particular, let $c = 1$
$c(-\vec{x}) = c[(-1)\vec{x}] = [c \cdot (-1)]\vec{x} = (-c)\vec{x}$

$\therefore (-c)\vec{x} = -(c\vec{x}) = c(-\vec{x})$