Orthogonal Systems

Definition: Orthogonal System

Let E be an inner product space and let $S \subset E - \{\vec{0}\}$. To say that S is an *orthogonal system* means $forall\, \vec{x}, \vec{y} \in E$:

$$\vec{x} \neq \vec{y} \implies \vec{x} \perp \vec{y}$$

To say that S is an *orthonormal system* means that S is an orthogonal system and:

$$\forall\, \vec{x} \in S, \|\vec{x}\| = 1$$

A sequence of vectors that form an orthonormal system is called an *orthonormal sequence*.

Theorem

Let E an inner product space over a field $\mathbb{F}$ and left S be an orthogonal system in E:

S is a linearly independent set.

Proof

Assume $X = \{\vec{x}_1, \ldots, \vec{x}_n\} \subseteq S$.

Assume $\displaystyle\sum_{k=1}^{n} \lambda_k \vec{x}_k = 0$ for $\lambda_k \in \mathbb{F}$.

$$\left\|\sum_{k=1}^{n} \lambda_k \vec{x}_k\right\|^2 = \left\langle \sum_{j=1}^{n} \lambda_j \vec{x}_j, \sum_{k=1}^{n} \lambda_k \vec{x}_k \right\rangle = \sum_{k=1}^{n} \langle \lambda_k \vec{x}_k, \lambda_k \vec{x}_k \rangle = \sum_{k=1}^{n} |\lambda_k|^2 \|\vec{x}_k\|^2 = 0$$

But none of the $\vec{x}_k = \vec{0}$, so $|\lambda_k| = 0$.

Therefore $\lambda_k = 0$ and S is a linearly independent set.

Examples

1). $E = \ell^2$ and $\langle (x_n), (y_n) \rangle = \displaystyle\sum_{n=1}^{\infty} x_k \overline{y_k}$

Let $S = \{e_n \mid n \in \mathbb{N}\}$

$$\langle e_n, e_n \rangle = \sum_{k=1}^{\infty} e_{n,k} \overline{e_{n,k}} = 1$$

$$\langle e_n, e_m \rangle = \sum_{k=1}^{\infty} e_{n,k} \overline{e_{m,k}} = 0$$

$$\langle e_n, e_m \rangle = \delta_{nm}$$

2). $E = L^2[-\pi,\pi]$ and $\langle f,g\rangle = \int_{-\pi}^{\pi} f\overline{g}$

Let $\varphi_n(t) = \frac{1}{\sqrt{2\pi}}e^{int}$

$$\begin{aligned}
\langle \varphi_n,\varphi_n\rangle &= \left\langle \frac{1}{\sqrt{2\pi}}e^{int}, \frac{1}{\sqrt{2\pi}}e^{int}\right\rangle \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\overline{e^{int}}dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}e^{-int}dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} dt \\
&= \frac{1}{2\pi}\cdot 2\pi \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\langle \varphi_n,\varphi_m\rangle &= \left\langle \frac{1}{\sqrt{2\pi}}e^{int}, \frac{1}{\sqrt{2\pi}}e^{imt}\right\rangle \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\overline{e^{imt}}dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}e^{-imt}dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)t}dt \\
&= \frac{1}{2\pi}\left[\frac{1}{i(n-m)}\right] e^{i(n-m)t}\Big|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}\left[\frac{1}{i(n-m)}\right][\cos(n-m)\pi - \cos(-(n-m)\pi)] \\
&= \frac{1}{2\pi}\left[\frac{1}{i(n-m)}\right][\cos(n-m)\pi - \cos(n-m)\pi] \\
&= 0
\end{aligned}$$

$\langle \varphi_n,\varphi_m\rangle = \delta_{nm}$

3). The Legendre Polynomials

$E = L^2[-1,1]$ and $\langle f,g\rangle = \int_{-1}^{1} f\overline{g}$

Let $P_0(x) = 1$ and $P_n(x) = \frac{1}{2^n n!}\frac{d}{dx}(x^2-1)^n$

The polynomials $P_n(x)$ form an orthogonal system in $L^2[-1,1]$.

The polynomials $\sqrt{n+\frac{1}{2}}P_n(x)$ form an orthonormal system in $L^2[-1,1]$.

4). The Hermite Polynomials

$E = L^2(\mathbb{R})$ and $\langle f, g\rangle = \int_{-1}^{1} f\overline{g}$

Let $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$

The polynomials $H_n(x)$ form an orthogonal system in $L^2(\mathbb{R})$.

The functions $\frac{1}{\sqrt{2^n n!\sqrt{\pi}}}e^{-\frac{x^2}{2}}H_n(x)$ form an orthonormal system in $L^2(\mathbb{R})$.