Hypothesis testing

– Math 161a, Spring 2019, San Jose State University

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May 7, 2019

Introduction

Consider the brown eggs problem again.

Suppose the weights of the eggs produced at the farm (population) are normally distributed with unknown mean μ but known standard deviation $\sigma = 2$ g.

It is claimed by the manufacturer that $\mu = 65$ g.

You bought a carton of 12 eggs, with an average weight of 61.5 g.

Question. Is such a difference purely due to randomness or significant evidence against the claim?

The formal procedure of hypothesis testing

First, we set up the following hypothesis test:

$$H_0 : \mu = 65 \quad \text{vs} \quad H_1 \, (\text{or } H_a) : \mu \neq 65$$

in which

- H_0: **null hypothesis** (statement which we intend to reject)
- H_1: **alternative hypothesis** (statement we suspect to be true)

The goal is to make a decision, based on a random sample $X_1, \ldots, X_n$ from the population, whether or not to reject H_0 so as to correspondingly establish H_1.

There are two kinds of decisions:

- If the sample "strongly" contradicts H_0, then we reject H_0 and correspondingly accept H_1;
- If the sample "does not strongly" contradict H_0, then we fail to reject H_0, or equivalently we **retain** H_0.

Remark. This is essentially a proof by contradiction approach.

Remark. There is a perfect analogy to **courtroom trial**. In this scenario, the following two hypotheses are tested:

- H_0: *Defendant is innocent*;
- H_a: *Defendant is guilty*.

The prosecutor presents evidence to the court, examined by the jury:

- If the jury thinks the evidence is strong enough (significant), the defendant will be convicted (H_0 is rejected and H_a is then accepted);
- Otherwise, the defendant is not found guilty and will be acquitted (the prosecutor has thus failed to convict the defendant due to insufficient evidence).

Remark. It is also possible to use a one-sided alternative:

$$H_0 : \mu = 65 \quad \text{vs} \quad H_a : \mu < 65.$$

In this case, the null is understood as "μ is *at least* 65 ($\mu \geq 65$)".

For example, the FDA's main interest is to know whether the eggs are lighter than 65 g (on average). It is not an issue if they are actually heavier (good for customers).

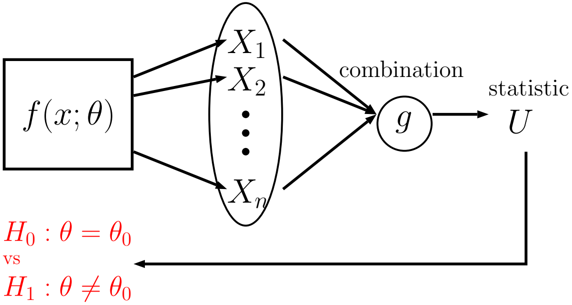
Similarly, for some other consideration, we might want to test

$$H_0 : \mu = 65 \quad \text{vs} \quad H_a : \mu > 65,$$

where the null is understood as "μ is *at most* 65 ($\mu \leq 65$)".

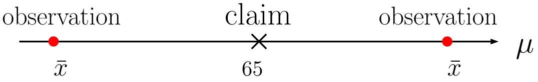
Test statistic

Typically, a test statistic needs to be specified to assist in making a decision. It is often a point estimator for the parameter being tested.

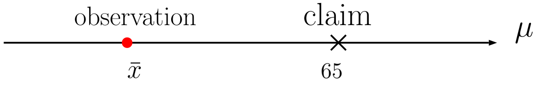


In the brown eggs example, we can use $\bar{X}$ as a test statistic to test $H_0 : \mu = 65$ against

- $H_1 : \mu \neq 65$: "very small or large" values of $\bar{X}$ are evidence against the null and correspondingly in favor of the alternative hypothesis.



- $H_1 : \mu < 65$: **only** "very small" values of $\bar{X}$ are evidence against the null and correspondingly in favor of the alternative hypothesis.

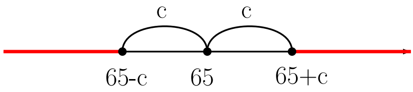


Decision rules

Clearly, a rule needs to be specified in order to decide **when to reject the null** $H_0 : \mu = 65$. This leads to a rejection region for the test.

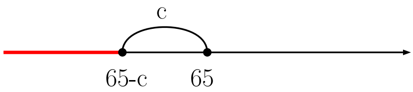
- For $H_1 : \mu \neq 65$:

  $$|\bar{x} - 65| > c$$



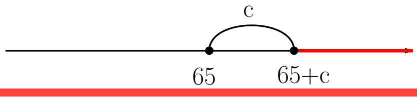
- For $H_1 : \mu < 65$:

  $$\bar{x} < 65 - c$$



- For $H_1 : \mu > 65$:

  $$\bar{x} > 65 + c$$



Test errors

There are two kinds of test errors depending on whether H_0 is true or not.

		Decision	
		Retain H_0	Reject H_0
H_0	**true**	Correct decision	Type I error
	false	Type II error	Correct decision

Remark. In the courtroom trial scenario, a type I error is convicting an innocent person, while a type II error is acquitting a guilty person.

Calculating the type-I error probability

Example 0.1. In the brown eggs problem, suppose the true population standard deviation is $\sigma = 2$ grams. A person decides to use the following decision rule (for a sample of size $n = 12$, i.e., a carton of eggs)

$$|\bar{x} - 65| > 1 \qquad \longleftarrow \text{rejection region of the test}$$

to conduct the two-sided test

$$H_0 : \mu = 65 \qquad \text{vs} \quad H_1 : \mu \neq 65.$$

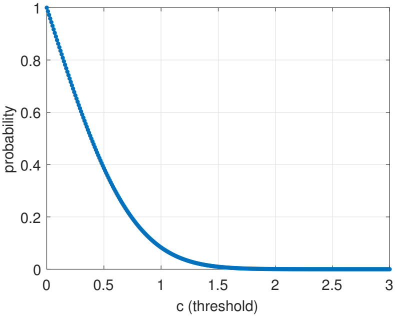
What is the probability of making a type-I error? (Answer: 0.0833)

Example 0.2. Redo the above example for a different decision rule:

$$|\bar{x} - 65| > 2$$

(Answer: 0.000532)

Type I error probabilities for using $|\bar{x} - 65| > c$ as rejection regions:



Observation. The larger the threshold (c), the smaller the rejection region (the harder to reject H_0), the smaller the type-I error.

Example 0.3. Redo the previous two examples for the one-sided test $H_1 : \mu < 65$ with corresponding decision rule

- $\bar{x} < 65 - 1 = 64$ $(\bar{x} - 65 < -1)$, or
- $\bar{x} < 65 - 2 = 63$ $(\bar{x} - 65 < -2)$

(Answers: $0.0416,\ 0.000266$)

Too easy, too good?

It seems that by increasing the threshold c (which would shrink the rejection region), we can make the type-I error probability arbitrarily small.

This seems a bit too easy and too good to be true.

This is indeed true, as far as only type-I error is concerned, but is this perhaps at the expense of something else?

How is the type-II error affected?

It turns out that reducing the rejection region will cause the probability of making a type-II error to increase:

- Making it hard to reject H_0 (by using a small rejection region) is good when H_0 is true (this corresponds to type-I errors).
- But it would be bad when H_0 is false (we actually want to reject H_0 in this case).

The thing is that we don't know which hypothesis is true, so we have to choose a rejection region carefully such that both errors are small.

Calculating the type-II error probabilities

Consider the two sided test

$$H_0 : \mu = 65 \qquad \text{vs} \quad H_1 : \mu \neq 65.$$

When $H_0 : \mu = 65$ is false (H_1 is correspondingly true), μ could be 64, or 68, or any other value.

Thus, there is a separate type-II error probability at each $\mu \neq 65$.

For any fixed decision rule $|\bar{x} - 65| > c$ (with c given), the probability of making a type-II error depends on the true value of μ:

$$\beta(\mu) = P(\text{Fail to reject } H_0 \mid H_0 \text{ false}) = P(|\bar{X} - 65| < c \mid H_1 \text{ true})$$

Remark.

- $1 - \beta(\mu)$ is the probability of making a correct decision by rejecting H_0 when it is false:

$$1 - \beta(\mu) = P(\text{Reject } H_0 \mid H_0 \text{ false}) = P(|\bar{X} - 65| > c \mid H_1 \text{ true})$$

- It is called the **power** (function) of the test.
- We would like
  - the type-II error probability $\beta(\mu)$ for a given μ to be small, and
  - the power of the test at the given μ to be large (80% or bigger).

Example 0.4. Consider the two-sided test:

$$H_0 : \mu = 65 \qquad \text{vs} \quad H_1 : \mu \neq 65$$

along with the following decision rule:

$$|\bar{x} - 65| > c.$$

Find the probability of making a type-II error when $\mu = 64$ for each value of $c = .5, 1, 2$.

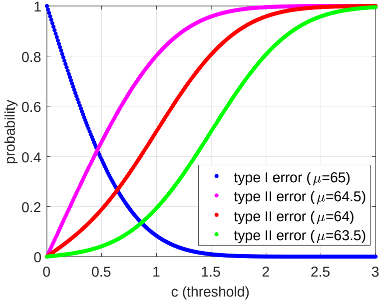
(Answer: $\beta(64) = P(|\bar{X} - 65| < c \mid \mu = 64) = 0.1886, 0.4997, 0.9584$)

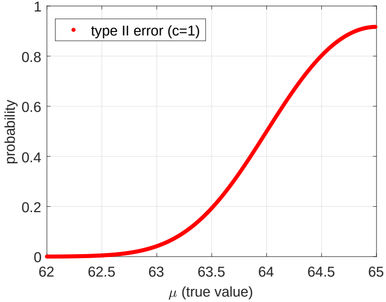
What about other values of $\mu \neq 65$?

Type-II errors	$c = 0.5$	1	2
$\mu = 63.5$	.0414	.1932	.8068
$\mu = 64$	.1886	.4997	.9584
$\mu = 64.5$	.4584	.8021	.9953

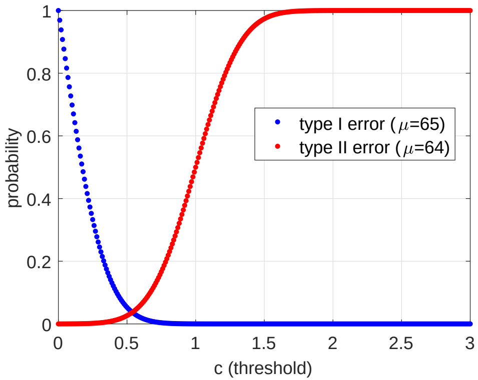
Observations.

- **For fixed value** μ: the larger c (the smaller the rejection region, and thus the harder to reject H_0), the larger the type-II error.
- **For fixed test** (c): the closer μ is to the value in H_0 (65), the larger the type II error.

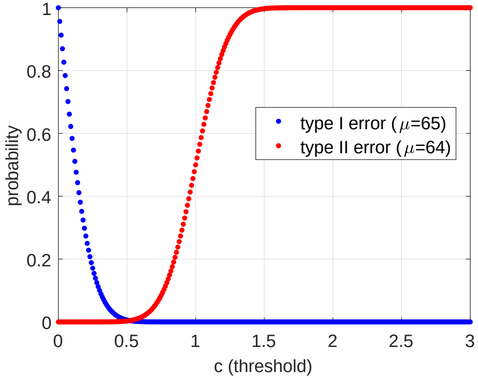

1
0.8
0.6
0.4
0.2
0
probability
0
0.5
1
1.5
2
2.5
3
c (threshold)
type I error (μ=65)
type II error (μ=64.5)
type II error (μ=64)
type II error (μ=63.5)


1
0.8
0.6
0.4
0.2
0
probability
62
62.5
63
63.5
64
64.5
65
μ (true value)
type II error (c=1)

How can you do better in both errors?



Increase the sample size!



How to control both errors together

Previously we assumed that both sample size n and test threshold c are fixed so as to evaluate the type-I and type-II errors of the test

$$H_0 : \mu = 65 \quad \text{vs} \quad H_a : \mu \neq 65$$

Here we consider the inverse design problem by assuming the two types of error probabilities are given first:

- type-I error probability α (called **level of the test**) $\longleftarrow$ typically 5%
- type-II error probability β (at specified location μ) $\longleftarrow$ typically 20%

and then trying to determine the required n and c as follows:

1. For the given level of the test i.e., α, solve

$$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ true}) = P(|\bar{X} - 65| > c \mid \mu = 65)$$

to determine the required threshold c (dependent on n).

2. Choose sample size n to achieve type-II error probability β at given location μ ($\mu \neq 65$):

$$\beta = P(\text{Fail to reject } H_0 \mid H_0 \text{ false}) = P(|\bar{X} - 65| < c \mid \mu)$$

Example 0.5. Assume the setting of the brown eggs example (with known $\sigma = 2$, but sample size n TBD). Consider the following two-sided test

$$H_0 : \mu = 65 \quad \text{vs} \quad H_a : \mu \neq 65$$

with decision rule

$$|\bar{x} - 65| > c$$

Choose n, c so that the test has level 5% and power 80% (at $\mu = 64$).

Answer: $c = z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 0.693,\ n \approx \left(\frac{\sigma(z_{\alpha/2}+z_\beta)}{\mu_0-\mu'}\right)^2 = 32$

Connection to confidence intervals

In the last example, the rejection region (for $\alpha = 5\%$) is

$$\bar{x} > 65 + z_{.025}\frac{\sigma}{\sqrt{n}}, \qquad \text{or} \quad \bar{x} < 65 - z_{.025}\frac{\sigma}{\sqrt{n}}$$

which is equivalent to

$$65 \notin (\bar{x} - z_{.025}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{.025}\frac{\sigma}{\sqrt{n}}) = \bar{x} \pm z_{.025}\frac{\sigma}{\sqrt{n}} \quad (95\% \text{ CI})$$

That is, we reject the null at level α if and only if the $1-\alpha$ confidence interval fails to capture the claimed value 65.

One can thus use a $1-\alpha$ confidence interval to conduct the hypothesis test at level α:

- Confidence interval captured $\mu = 65$: Do not reject H_0
- Confidence interval failed to capture $\mu = 65$: Reject H_0

Note the relationship between and interpretation of:

$1-\alpha$ (confidence level) and α (level of the test).

Remark. For a one-sided test

$$H_0 : \mu = 65 \quad \text{vs} \quad H_a : \mu < 65$$

with corresponding decision rule

$$\bar{x} < 65 - c$$

the two equations (for determining n, c) become

$$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ true}) = P(\bar{X} < 65 - c \mid \mu = 65)$$
$$\beta = P(\text{Fail to reject } H_0 \mid H_0 \text{ false}) = P(\bar{X} > 65 - c \mid \mu)$$

Example 0.6. Redo the preceding example but instead for a one-sided test

$$H_0 : \mu = 65 \quad \text{vs} \quad H_a : \mu < 65$$

with corresponding decision rule

$$\bar{x} < 65 - c$$

Answer: $c = z_\alpha \frac{\sigma}{\sqrt{n}} = 0.658,\ n = \left(\frac{\sigma(z_\alpha + z_\beta)}{\mu_0 - \mu'}\right)^2 = 25$

i-Clicker quiz 10 (extra credit)

Consider the decision rule $\bar{x} > 65 + c$ for a one-sided test

$$H_0 : \mu = 65 \quad \text{vs} \quad H_a : \mu > 65$$

As c is increased, which one of the following statements is WRONG?

A. The rejection region becomes smaller.

B. It gets harder to reject H_0.

C. Type-I error probability becomes smaller.

D. Type-II error probability becomes smaller.

E. None of the above

Summary

A hypothesis test has the following components:

- **Population**: e.g., all brown eggs produced by the farm, whose weights have a normal distribution with unknown mean μ but known variance σ^2
- **Null and alternative hypotheses**: $H_0 : \mu = \mu_0 \text{ vs } H_a : \mu \neq \mu_0$;
- **Random sample** from the population: $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$
- **Test statistic**: e.g., $\bar{X}$
- **Decision rule** (based on a specified **rejection region**): $|\bar{x} - \mu_0| > c$

Evaluation of the test:

- **Type-I error**:

$$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ true}) = P(|\bar{X} - \mu_0| > c \mid \mu = \mu_0)$$

If α is specified first as the level of the test, then set $c = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ (for a two sided test)

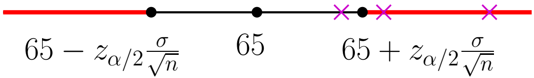
- **Type-II errors** (at any $\mu \neq \mu_0$)

$$\beta(\mu) = P(\text{Fail to reject } H_0 \mid H_0 \text{ false}) = P(|\bar{X} - \mu_0| < c \mid H_1 \text{ true})$$

To control both errors, we first choose c (dependent on n) to attain level α, then choose sample size n to achieve certain power $1 - \beta(\mu)$.

Limitation of the rejection region approach

The rejection region approach to conducting a hypothesis test at a given level makes sense, but the decision is discrete (reject or retain the null).



It does not reflect the strength of the evidence against H_0 (when rejecting it) or the closeness to the rejection region (when failing to reject it).

Another way of performing the hypothesis test is to assign a **score of extremeness** (relative to the null), called p-**value**, to any observed value of the test statistic in a continuous way.

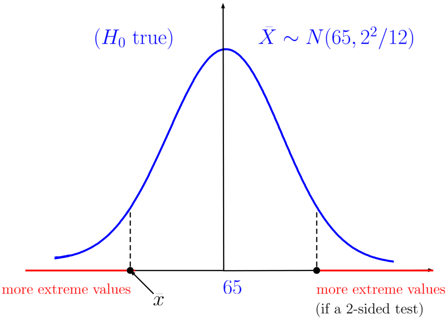
Logic behind the p-value approach to hypothesis testing

Consider the two-sided test again (in same setting but with a fresh mind):

$$H_0 : \mu = 65 \quad \text{vs} \quad H_a : \mu \neq 65 \text{ (or } H_a : \mu < 65)$$

We adopt a proof-by-contradiction procedure:

- **Assume H_0 is true**. Then $\mu = 65$ and $\bar{X} \sim N(65, 2^2/12)$.
- Intuitively, most observed values of $\bar{X}$ should be "around 65", while "extreme" values should be rare.
- For every observation $\bar{x}$ of $\bar{X}$, we assign an **extremeness score**, called p-value (e.g., most extreme 5%):



$$\text{pval}(\bar{x}) = \begin{cases} \text{left tail area only}, & \text{for } H_a : \mu < 65 \\ \text{total area of both tails}, & \text{for } H_a : \mu \neq 65 \end{cases}$$

- If for a specific sample, $\bar{x}$ is extreme (with small p-value), we have two possible explanations: **bad luck** or **wrong assumption** (H_0 does not hold true).
- If "very bad luck" is needed to explain the extreme observation, we choose to believe instead that the assumption must be wrong, and consequently H_0 should be rejected.
- Thus, small p-values lead to rejections of the null.
- Apparently, such a decision possesses a risk of making a type-I error (when H_1 is actually true).

The formal definition of p-value

Definition 0.1. The p-**value** of an observed value $\bar{x}$ of the test statistic $\bar{X}$ is the probability of observing $\bar{x}$, or values that are "more contradictory" to H_0, when assuming H_0 is true:

$$\text{pval}(\bar{x}) = P(\bar{X} \text{ is at least as contradictory as } \bar{x} \mid H_0 \text{ true})$$

We will reject H_0 if and only if the observed value of $\vec{X}$ corresponding to a sample is "very extreme".

Remark. The more extreme the observation, the smaller the p-value, the stronger the evidence against H_0.

Example 0.7. In the brown eggs example, suppose we observed $\bar{x} = 63.8$.

- $H_1 : \mu \neq 65$: The more contradictory values are $\bar{x} < 63.8$ and $\bar{x} > 66.2$ (mirror point). Thus, for a 2-sided test,

$$\begin{aligned} \text{pval}(63.8) &= 2 \cdot P(\bar{X} \leq 63.8 \mid H_0 \text{ true}) \\ &= 2 \cdot P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq \frac{63.8 - 65}{2/\sqrt{12}} \mid \mu = 65 \right) \\ &= 2 \cdot P(Z \leq -2.08) = 2 \cdot .019 = .038 \end{aligned}$$

- $H_1 : \mu \neq 65$: The more contradictory values are only $\bar{x} < 63.8$. In this case, the p-value is

$$\text{pval}(63.8) = P(\bar{X} \leq 63.8 \mid H_0 \text{ true}) = .019$$

Significance level

Definition 0.2. The cutoff p-value at which we choose to reject the null is called the **significance level** of the test. We denote it by α.

p-values that are smaller than the significance level (α) are said to be **significant** and will lead to the rejection of the null:

$$\text{Reject } H_0 \text{ if and only if } p\text{-value} \le \alpha.$$

Example 0.8. In the previous example, what is your conclusion if $\alpha = 5\%$? 1%?

Remark. For a p-value test at significance level α, the following three are the same

- significance level
- probability of making a type-I error
- level of the test.

They all equal α.

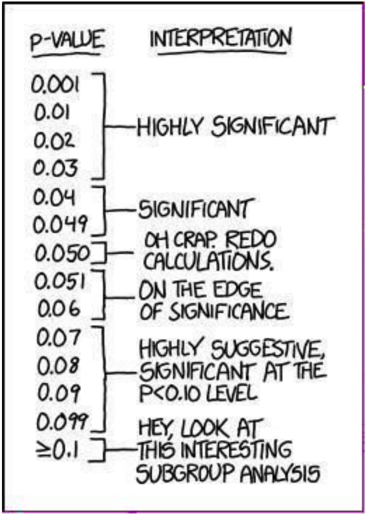
In theory, the p value is a continuous measure of evidence, but in practice it is typically trichotomized approximately into

- highly significant ($p \leq 0.01$)
- moderately significant ($0.01 < p \leq 0.03$)
- marginally significant ($p \approx 0.05$), and
- not statistically significant ($p > 0.06$)

Joke. What does a statistician call it when the heads of 10 rats are cut off and 1 survives?

Nonsignificant.

Hypothesis testing



When population variance is unknown

How do we conduct a hypothesis test for each of them?

- Population mean μ

- Population variance σ^2

Testing for μ with unknown variance

Recall that in the case of a normal population $N(\mu, \sigma^2)$ (with unknown μ and known σ^2), to conduct the two-sided test

$$H_0 : \mu = 65 \qquad vs \qquad H_1 : \mu \neq 65$$

at level α, one can use the following decision rule

$$|\bar{x} - 65| > z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \quad \text{or equivalently } \left| \frac{\bar{x} - 65}{\sigma/\sqrt{n}} \right| > z_{\alpha/2}$$

The test statistic $\frac{\bar{X}-65}{\sigma/\sqrt{n}}$ is correctly standardized (when H_0 is true), which has a standard normal distribution.

For the above reasons, the above test is called a (two-sided) z-test.

When σ is unknown, we can use the sample standard deviation S in place of σ (like the construction of confidence interval), yielding a t-test:

$$\left|\frac{\bar{x}-65}{s/\sqrt{n}}\right| > t_{\alpha/2,n-1}$$

Similarly, for a one-sided test, we can use a one-sided z-test (when σ known) or a one-sided t-test (when σ unknown).

Additionally, when σ is unknown, we can use the t distribution to calculate the p-value of a specific sample in order to conduct the hypothesis test at certain level α.

Example 0.9. Consider the egg-weight example again. Conduct the following test at level 95%

$$H_0 : \mu = 65 \qquad vs \qquad H_1 : \mu \neq 65$$

for a specific sample of 12 eggs with $\bar{x} = 64$ and $s^2 = 4.69$. Conduct the test at level $\alpha = .05$. What is the p-value of the sample?

(Answer: $|\frac{\bar{x}-65}{s/\sqrt{n}}| = 1.6 < t_{\alpha/2,n-1} = 2.201$, thus failing to reject the null. p-value=.138)

Testing for population variance

For population variance we are often only interested in a one-sided test of the form

$$H_0 : \sigma^2 = \sigma_0^2 \qquad vs \qquad H_1 : \sigma^2 > \sigma_0^2$$

Following previous reasoning, we write down a decision rule as follows

$$\frac{(n-1)s^2}{\sigma_0^2} > c$$

For a given level α, the cutoff c is determined from the following equation

$$\alpha = P\left(\frac{(n-1)s^2}{\sigma_0^2} > c \mid \sigma^2 = \sigma_0^2\right) \longrightarrow c = \chi^2_{\alpha,n-1}$$

Example 0.10 (Continuation of previous example). Conduct the following test at level 5%:

$$H_0 : \sigma^2 = 2^2 \qquad vs \qquad H_1 : \sigma^2 > 2^2$$

What is the p-value?

(Answer: $\frac{(n-1)s^2}{\sigma_0^2} = 12.9 < \chi^2_{\alpha,n-1} = 19.7$, thus failing to reject the null. $p$val$=.3$)