Prime Ideals

Definition: Prime

Let R be a commutative ring and P be a proper ideal in R. To say that P is a *prime ideal* in R means $\forall\, a, b \in R$:

$$ab \in P \implies a \in P \text{ or } b \in P$$

Theorem

Let R be a commutative ring with $1 \neq 0$ and $P \trianglelefteq R$:

$$P \text{ is a prime ideal} \iff R/P \text{ is an integral domain}$$

Proof

Since $P \trianglelefteq R$ and R is commutative with $1 \neq 0$, R/P is a commutative ring with additive identity $0 + P = P$ and multiplicative identity $1 + P \neq 0 + P$
Furthermore, $a + P = 0 + P = P \iff a \in P$

$\implies$ Assume P is a prime ideal in R

Assume $(a + P)(b + P) = ab + P = 0 + P$
So $ab \in P$
But P is prime so $a \in P$ or $b \in P$
Thus $a + P = 0 + P$ or $b + P = 0 + P$

Therefore R/P is an integral domain.

$\impliedby$ Assume R/P is an integral domain

Assume $a, b \in R$ such that $ab \in P$
$ab + P = (a + P)(b + P) = 0 + P$
If $a \in P$ then done, so AWLOG: $a \neq P$
So $a + P \notin 0 + P$
But R/P is an integral domain, so $b + P = 0 + P$
Thus $b \in P$

Therefore P is prime in R.

Corollary

Maximal ideals are prime ideals.

Proof

Assume R is a commutative ring with $1 \neq 0$ and $P \trianglelefteq R$ is maximal
So R/I is a field, and thus an integral domain

Therefore P is prime.

Example

Let $R = \mathbb{Z}[x]$

The principal ideal (x) is prime, but it is not maximal:

$$\mathbb{Z}[x]/(x) \simeq \mathbb{Z}$$

But $\mathbb{Z}$ is a ring, not a field.

However, we know that there should be a maximal ideal containing (x).
Let $p \in \mathbb{Z}$ be prime.
$(p, x) = (p) + (x)$ is maximal, but not principle:

$$Z[x]/(x, p) \simeq \mathcal{F}_p$$

But $\mathcal{F}_p$ is a field.