Weyl's Inequalities

Theorem

Let $A, B \in M_n$:

$$\sum_{k=1}^{n} \lambda_k(A+B) = \sum_{k=1}^{n} \lambda_k(A) + \sum_{k=1}^{n} \lambda_k(B)$$

Proof

$\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
$\mathrm{tr}(A+B) = \sum_{k=1}^{n} \lambda_k(A+B)$
$\mathrm{tr}(A) = \sum_{k=1}^{n} \lambda_k(A)$
$\mathrm{tr}(B) = \sum_{k=1}^{n} \lambda_k(B)$

$\therefore \sum_{k=1}^{n} \lambda_k(A+B) = \sum_{k=1}^{n} \lambda_k(A) + \sum_{k=1}^{n} \lambda_k(B)$

Theorem: Knutson-Tao

Given three sets of numbers: $\alpha_1 \leq \cdots \leq \alpha_n$, $\beta_1 \leq \cdots \leq \beta_n$, and $\gamma_1 \leq \cdots \leq \gamma_n$ satisfying some specific inequalities, there exists Hermitian matrices A, B, and C such that:

$$\begin{aligned} \mathrm{Sp}(A) &= \{\alpha_k\} \\ \mathrm{Sp}(B) &= \{\beta_k\} \\ \mathrm{Sp}(A+B) &= \{\gamma_k\} \end{aligned}$$

Example

$\mathrm{Sp}(A) = \{0, 1\}$

$\mathrm{Sp}(B) = \{2, 5\}$

$\mathrm{Sp}(A+B) = \{2, 6\}$

$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$

$\mathrm{Sp}(A) = \{0, 1\}$

$\mathrm{Sp}(B) = \{2, 5\}$

$\mathrm{Sp}(A+B) = \{3, 5\}$

$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \qquad A = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

$\mathrm{Sp}(A) = \{0, 1\}$

$\mathrm{Sp}(B) = \{2, 5\}$

$\mathrm{Sp}(A+B) = \{1, 1\}$

Not possible because:

$$\begin{array}{rcl} \mathrm{tr}(A) & = & 1 \\ \mathrm{tr}(B) & = & 7 \\ \mathrm{tr}(A+B) & = & 2 \\ 1 + 7 \neq 2 & & \end{array}$$

Lemma

Let $A, B \in M_n$ be Hermitian with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$:

1). $\lambda_1(A+B) \geq \lambda_1(A) + \lambda_1(B)$

2). $\lambda_n(A+B) \leq \lambda_n(A) + \lambda_n(B)$

Proof

$$\begin{array}{rcl} \lambda_1(A+B) & = & \min\limits_{\vec{x} \neq \vec{0}} \dfrac{\vec{x}^*(A+B)\vec{x}}{\vec{x}^*\vec{x}} \\ & = & \min\limits_{\vec{x} \neq \vec{0}} \dfrac{\vec{x}^*A\vec{x} + \vec{x}^*B\vec{x}}{\vec{x}^*\vec{x}} \\ & \geq & \min\limits_{\vec{x} \neq \vec{0}} \dfrac{\vec{x}^*A\vec{x}}{\vec{x}^*\vec{x}} + \min\limits_{\vec{x} \neq \vec{x}} \dfrac{\vec{x}^*B\vec{x}}{\vec{x}^*\vec{x}} \\ & = & \lambda_1(A) + \lambda_1(B) \end{array}$$

$$\begin{array}{rcl} \lambda_n(A+B) & = & \max\limits_{\vec{x} \neq \vec{0}} \dfrac{\vec{x}^*(A+B)\vec{x}}{\vec{x}^*\vec{x}} \\ & = & \max\limits_{\vec{x} \neq \vec{0}} \dfrac{\vec{x}^*A\vec{x} + \vec{x}^*B\vec{x}}{\vec{x}^*\vec{x}} \\ & \leq & \max\limits_{\vec{x} \neq \vec{0}} \dfrac{\vec{x}^*A\vec{x}}{\vec{x}^*\vec{x}} + \max\limits_{\vec{x} \neq \vec{x}} \dfrac{\vec{x}^*B\vec{x}}{\vec{x}^*\vec{x}} \\ & = & \lambda_n(A) + \lambda_n(B) \end{array}$$

Theorem: Weyl's Inequalities

Let $A, B \in M_n$ be Hermitian with eigenvalues arranged such that $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ and $\lambda_1(B) \leq \cdots \leq \lambda_n(B)$:

$$\lambda_{j+k-n}(A+B) \leq \lambda_j(A) + \lambda_k(B) \leq \lambda_{j+k-1}(A+B)$$

Proof

Let $\vec{u}_k$ be an eigenvector of $\lambda_k(A)$
Let $\vec{v}_k$ be an eigenvector of $\lambda_k(B)$
Let $\vec{w}_k$ be an eigenvector of $\lambda_k(A+B)$

Let $S_1 = \mathrm{span}\{\vec{u}_1, \ldots, \vec{u}_j\}$
Let $S_2 = \mathrm{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$
Let $S_3 = \mathrm{span}\{\vec{w}_{j+k-n}, \ldots, \vec{w}_n\}$

$$\begin{aligned} \dim(S_1 \cap S_2 \cap S_3) &\geq \dim(S_1) + \dim(S_2) + \dim(S_3) - 2n \\ &= j + k + [n - (j + k - n) + 1] - 2n \\ &= 1 \end{aligned}$$

Thus, $\exists \vec{x} \in S_1 \cap S_2 \cap S_3$ such that $\vec{x} \neq 0$

$\lambda_{j+k-n}(A+B) \leq \frac{\vec{x}^*(A+B)\vec{x}}{\vec{x}^*\vec{x}}$ because $\vec{x} \in S_3$

$\frac{\vec{x}^*A\vec{x}}{\vec{x}^*\vec{x}} \leq \lambda_j(A)$ because $\vec{x} \in S_1$

$\frac{\vec{x}^*B\vec{x}}{\vec{x}^*\vec{x}} \leq \lambda_k(B)$ because $\vec{x} \in S_2$

$\therefore \lambda_{j+k-n} \leq \frac{\vec{x}^*(A+B)\vec{x}}{\vec{x}^*\vec{x}} = \frac{\vec{x}^*(A)\vec{x}}{\vec{x}^*\vec{x}} + \frac{\vec{x}^*(B)\vec{x}}{\vec{x}^*\vec{x}} \leq \lambda_j(A) + \lambda_k(B)$

Let $S_4 = \mathrm{span}\{\vec{u}_j, \ldots, \vec{u}_n\}$
Let $S_5 = \mathrm{span}\{\vec{v}_k, \ldots, \vec{v}_n\}$
Let $S_6 = \mathrm{span}\{\vec{w}_1, \ldots, \vec{w}_{j+k-1}\}$

$$\begin{aligned} \dim(S_4 \cap S_5 \cap S_6) &\geq \dim(S_4) + \dim(S_5) + \dim(S_6) - 2n \\ &= (n - j + 1) + (n - k + 1) + (j + k - 1) - 2n \\ &= 1 \end{aligned}$$

Thus, $\exists \vec{x} \in S_4 \cap S_5 \cap S_6$ such that $\vec{x} \neq 0$

$\lambda_{j+k-1}(A+B) \geq \frac{\vec{x}^*(A+B)\vec{x}}{\vec{x}^*\vec{x}}$ because $\vec{x} \in S_6$

$\frac{\vec{x}^*A\vec{x}}{\vec{x}^*\vec{x}} ge \lambda_j(A)$ because $\vec{x} \in S_4$

$\frac{\vec{x}^*B\vec{x}}{\vec{x}^*\vec{x}} \geq \lambda_k(B)$ because $\vec{x} \in S_5$

$\therefore \lambda_{j+k-1} \geq \frac{\vec{x}^*(A+B)\vec{x}}{\vec{x}^*\vec{x}} = \frac{\vec{x}^*(A)\vec{x}}{\vec{x}^*\vec{x}} + \frac{\vec{x}^*(B)\vec{x}}{\vec{x}^*\vec{x}} \geq \lambda_j(A) + \lambda_k(B)$

$\therefore \lambda_{j+k-n}(A+B) \leq \lambda_j(A) + \lambda_k(B) \leq \lambda_{j+k-1}(A+B)$

Example

$\mathrm{Sp}(A) = \{0, 1\}$

$\mathrm{Sp}(B) = \{2, 5\}$

$\lambda_{1+2-2}(A + B) = \lambda_1(A + B) \leq \lambda_1(A) + \lambda_2(B) = 0 + 5 = 5$
$\lambda_{2+1-2}(A + B) = \lambda_1(A + B) \leq \lambda_2(A) + \lambda_1(B) = 1 + 2 = 3$
$\lambda_{2+2-2}(A + B) = \lambda_2(A + B) \leq \lambda_2(A) + \lambda_2(B) = 1 + 5 = 6$

$\lambda_{1+1-1}(A + B) = \lambda_1(A + B) \geq \lambda_1(A) + \lambda_1(B) = 0 + 2 = 2$
$\lambda_{1+2-1}(A + B) = \lambda_2(A + B) \geq \lambda_1(A) + \lambda_2(B) = 0 + 5 = 5$
$\lambda_{2+1-1}(A + B) = \lambda_2(A + B) \geq \lambda_2(A) + \lambda_1(B) = 1 + 2 = 3$

$2 \leq \gamma_1 \leq 3$
$5 \leq \gamma_2 \leq 6$
$\gamma_1 + \gamma_2 = 1 + 7 = 8$