Binomial Theorem

Theorem

$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$

Proof

1). Analytical

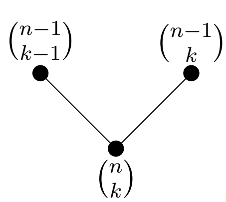
$$
\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)![(n-1)-(k-1)]!} \\
&= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-1)!(n-k)}{k!(n-k)!} + \frac{(n-1)!k}{k!(n-k)!} \\
&= \frac{(n-1)![(n-k)+k]}{k!(n-k)!} \\
&= \frac{(n-1)!n}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}
\end{aligned}
$$

2). Combinatorial

Consider the selection of a committee of size k from a pool of n people. The RHS is the number of ways to select such a committee: $\binom{n}{k}$ ways. For the LHS, select a particular person A and partition the selection process into committees without A and committees with A. For a committee without A, all k members must be selected from the remaining $n-1$ members, or $\binom{n-1}{k}$ ways. For a committee with A, the remaining $k-1$ members must be selected from the remaining $n-1$ members, or $\binom{n-1}{k-1}$ ways. Therefore:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

3). Block-walking



Binomial Theorem

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Proof

By induction:

Base: n=1

$$(a+b)^1 = a+b$$

$$\begin{aligned}
\sum_{k=0}^{1} \binom{n}{k} a^{n-k} b^k &= \binom{1}{0} a^{1-0} b^0 + \binom{1}{1} a^{1-1} b^1 \\
&= 1a^1 b^0 + 1a^0 b^1 \\
&= a+b
\end{aligned}$$

Inductive Assumption

Assume $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b) \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n-(k-1)} b^{(k-1)+1} \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n+1-k} b^k \\
&= a^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] a^{n+1-k} b^k + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^k + b^{n+1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{(n+1)-k} b^k
\end{aligned}$$