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Math 231b
Homework #2 Rewrite

3.8.4

b. Counterexample

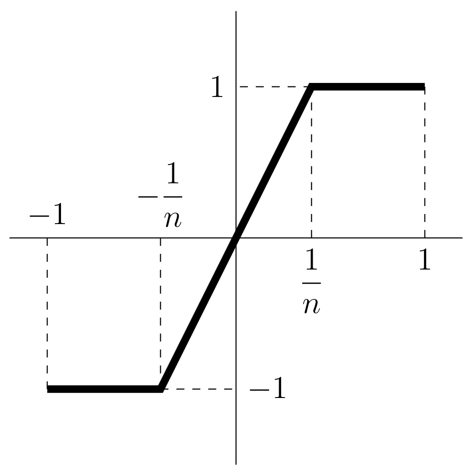
Let $F[a,b] = \{f \in \mathcal{C}^1[a,b] \mid f(a) = 0\}$ with inner product:

$$\langle f, g \rangle = \int_a^b f'(x)\overline{g'(x)}dx$$

It was proven in the original submission that F is an inner product space. The following new (proper) counterexample shows that F is not a Hilbert space.

Consider the interval $[-1,1]$ and let:

$$g_n(t) = \begin{cases} -1, & -1 \le t \le -\frac{1}{n} \\ nt, & -\frac{1}{n} \le t \le \frac{1}{n} \\ 1, & \frac{1}{n} \le t \le 1 \end{cases}$$



Let $f_n(x) = \displaystyle\int_0^x g_n(t)dt$.

Note that $g_n(t) \in \mathcal{C}[-1,1]$, and so, by the FTC:

1). $f_n'(x) = g_n(x)$

2). $f_n(x) \in \mathcal{C}'[-1,1]$

Also, $f_n(0) = 0$, so we can conclude that $f_n(x) \in F[-1,1]$.

Now, consider the standard signum function:

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Claim: $g_n \xrightarrow{L_2} \operatorname{sgn}$

Note that $g_n(0) = \operatorname{sgn}(0) = 0$ and we can ignore this single point (since we are integrating):

$$\begin{aligned}
\|g_n - \operatorname{sgn}\|_{L_2} &= \int_{-1}^{1} |g_n(t) - \operatorname{sgn}(t)|^2\, dt \\
&= \int_{-\frac{1}{n}}^{-\frac{1}{n}} [nt - \operatorname{sgn}(t)]^2 dt \\
&= \int_{-\frac{1}{n}}^{0} (1+nt)^2 dt + \int_{0}^{\frac{1}{n}} (1-nt)^2 dt \\
&= \int_{-\frac{1}{n}}^{0} (1 + 2nt + n^2t^2) dt + \int_{0}^{\frac{1}{n}} (1 - 2nt + n^2t^2) dt \\
&= \left[t + nt^2 + \frac{n^2}{3}t^3\right]_{-\frac{1}{n}}^{0} + \left[t - nt^2 + \frac{n^2}{3}t^3\right]_{0}^{\frac{1}{n}} \\
&= \left[0 - \left(-\frac{1}{n} + \frac{1}{n} - \frac{1}{3n}\right)\right] + \left[\left(\frac{1}{n} - \frac{1}{n} + \frac{1}{3n}\right) - 0\right] \\
&= \frac{2}{3n} \\
&\to 0
\end{aligned}$$

Claim: f_n is Cauchy in $\|\cdot\|_F$.

$$\begin{aligned}
\|f_n - f_m\|_F &= \|f_n' - f_m'\|_{L_2} \\
&= \|g_n - g_m\|_{L_2} \\
&= \|(g_n - \operatorname{sgn}) + (\operatorname{sgn} - g_m)\|_{L_2} \\
&\le \|g_n - \operatorname{sgn}\|_{L_2} + \|\operatorname{sgn} - g_m\|_{L_2} \\
&\to 0 + 0 \\
&= 0
\end{aligned}$$

By geometry, it is clear that:

$$f_n(x) = \begin{cases} -\left(x - \frac{1}{2n}\right), & x \le 0 \\ x - \frac{1}{2n}, & x \ge 0 \end{cases}$$

And so $f_n \to f$, where $f(x) = \begin{cases} -x, & x \le 0 \\ x, & x \ge 0 \end{cases} = |x|$

But $|x| \neq F[-1,1]$ because $|x|$ is not differentiable at 0.

Therefore F is not complete, and thus not Hilbert.