Propositions

The objects in the Boolean logic system are *propositions*:

Definition: Proposition

A *proposition* is a declarative statement (a sentence that states a fact) that is objectively and unambiguously either *true* or *false*. Thus, the *value* of a proposition is either *true*, denoted by T or 1, or *false*, denoted by F or 0.

Examples

The following are all valid propositions:

1. Sacramento is the current capital of California. (T)
2. SJSU is part of the UC system. (F)
3. $1 + 2 = 3$ (T)
4. $2 + 3 = 4$ (F)
5. Every even integer can be expressed as the sum of two odd integers. (T)
6. $\sqrt{2}$ is a rational number. (F)

The following are not propositions:

1. What time is it? (interrogative)
2. Do your homework! (imperative)
3. 100 is a big number. (subjective)
4. I am lying. (paradoxical)
5. $x + 2 = 5$ (inconclusive)

Propositions are represented by variables: $p, q, r, s, \ldots$.

Examples

$p \coloneqq 1 + 2 = 3$ (T)

$q \coloneqq$ Every integer is either odd or even. (T)

$r \coloneqq 10$ is a prime number. (F)

Note of the use of $\coloneqq$, which means "is defined as," as opposed to $=$, which assigns a value to a variable.

Definition: Simple and Compound

Propositions that are not expressed in terms of other propositions are called *simple* or *atomic* propositions. Propositions that are constructed from other propositions using *logical operators* are called *compound* propositions. The basic logical operators are *not*, *and*, and *or*.

Not

Definition: Negation

Let p be a proposition. The *negation* of p, also called "*not* p" and denoted by $\neg p$ or $\bar{p}$ or $\sim p$, is the proposition represented by the statement: "It is not the case that p," which is true when p is false and false when p is true.

p	$\neg p$
F	T
T	F

When stating negations, always look for the most compact form.

Examples

Let:

$$
\begin{aligned}
p &:= \text{There are more than 30 students taking this class.} \quad (T) \\
q &:= 2 < 3 \quad (T) \\
r &:= 10 \text{ is an odd number.} \quad (F)
\end{aligned}
$$

$$
\begin{aligned}
\neg p &= \text{It is not the case that there are more than 30 students taking this class.} \\
&= \text{There are not more than 30 students taking this class.} \\
&= \text{There are at most 30 students taking this class.} \quad (F)
\end{aligned}
$$

$$
\begin{aligned}
\neg q &= \text{It is not the case than } 2 < 3 \\
&= 2 \not< 3 \\
&= 2 \geq 3 \quad (F)
\end{aligned}
$$

$$
\begin{aligned}
\neg r &= \text{It is not the case that 10 is an odd number.} \\
&= 10 \text{ is not an odd number.} \\
&= 10 \text{ is an even number.} \quad (T)
\end{aligned}
$$

Conjunction

Definition: Conjunction

Let p and q be propositions. The *conjunction* of p and q, also called "p *and* q" and denoted by $p \wedge q$ or simply pq, is the proposition represented by the statement: "p and q," which is true when p and q are both true and false otherwise.

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Examples

Let:

$$
\begin{aligned}
p &\coloneqq 1 < 2 \quad (T) \\
q &\coloneqq 2 < 3 \quad (T) \\
r &\coloneqq \text{There are more than 30 students in this class.} \quad (T) \\
s &\coloneqq \text{All of the students in this class are Freshmen.} \quad (F) \\
t &\coloneqq 10 \text{ is an odd number.} \quad (F)
\end{aligned}
$$

$$p \wedge q = 1 < 2 \wedge 2 < 3 = 1 < 2 < 3 \quad (TT = T)$$

$$
\begin{aligned}
r \wedge s &= \text{There are more than 30 students in this class and all of the students in this class are freshmen.} \\
&= \text{There are more than 30 students in this class and they are all freshmen.} \quad (TF = F)
\end{aligned}
$$

$$t \wedge p = 10 \text{ is an odd number and } 1 < 2 \quad (FT = F)$$

$$
\begin{aligned}
s \wedge t = {} & \text{All of the students in this class are freshmen and } 10 \text{ is an odd number.} \\
& (FF = F)
\end{aligned}
$$

Disjunction

Care must be taken to distinguish between the common English use of the word *or*, as in: "Do you want soup or salad?" and the more precise logical definition. The English usage typically presents two mutually exclusive choices, whereas the logical usage does not.

Definition: Disjunction

Let p and q be propositions. The *disjunction* of p and q, also called "p *or* q" or "p *inclusive-or* q" and denoted by $p \vee q$ or $p + q$, is the proposition represented by the statement: "p or q," which is false when p and q are both false and true otherwise.

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Examples

Assume that p, q, r, s, and t are defined as above.

$p \vee q = 1 < 2 \vee 2 < 3 \quad (T + T = T)$

$r \vee s =$ There are more than 30 students in this class or all of the students in this class are freshmen. $(T + F = T)$

$t \vee p = 10$ is an odd number or $1 < 2 \quad (F + T = T)$

$s \vee t =$ All of the students in this class are freshmen or 10 is an odd number. $(F + F = F)$

Exclusivity is obtained using the exclusive-OR (XOR) operator.

Definition: Exclusive-OR

Let p and q be propositions. The *exclusive-or* of p and q, denoted by $p \oplus q$, is the proposition represented by the statement: "either p or q," which is true when p and q have different truth values and false otherwise.

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

Examples

Assume that p, q, r, s, and t are defined as above.

$$p \oplus q = 1 < 2 \oplus 2 < 3 \quad (T \oplus T = F)$$

$r \oplus s =$ Either there are more than 30 students in this class or all of the students in this class are freshmen. $(T \oplus F = T)$

$t \oplus p =$ Either 10 is an odd number or $1 < 2$ $\quad (F \oplus T = T)$

$s \oplus t =$ Either all of the students in this class are freshmen or 10 is an odd number. $(F \oplus F = F)$

Compound Propositions

The logical operators can be used to construct more complex propositions. Order of evaluation is left to right with precedence: not, and, or, xor. Use parentheses to override normal precedence or for clarity.

Example

Let p, q, and r be propositions. Construct a truth table for:

$$s = (p \vee \neg q) \wedge (\neg p \vee r) \wedge \neg(q \vee r)$$

p	q	r	$\neg p$	$\neg q$	$\neg r$	$p \vee \neg q$	$\neg p \vee r$	$q \oplus r$	$\neg(q \oplus r)$	s
F	F	F	T	T	T	T	T	F	T	T
F	F	T	T	T	F	T	T	T	F	F
F	T	F	T	F	T	F	T	T	F	F
F	T	T	T	F	F	F	T	F	T	F
T	F	F	F	T	T	T	F	F	T	F
T	F	T	F	T	F	T	T	T	F	F
T	T	F	F	F	T	T	F	T	F	F
T	T	T	F	F	F	T	T	F	T	T

We can also go backwards, from the final column to the so-called *canonical* form, where each true row contributes a conjunctive term containing each variable with false valued variables negated. variables.

Example

In the previous example, s is true in only two cases:

$$s = (\neg p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r) = \bar{p}\bar{q}\bar{r} + pqr$$

Example

Consider the truth table for $p \oplus q$:

$$p \oplus q = \neg p \wedge q \vee p \wedge \neg q = \bar{p}q + p\bar{q}$$

Example

Consider the truth table:

p	q	r	s
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	F
T	F	F	T
T	F	T	T
T	T	F	F
T	T	T	T

$$s = (\neg p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r) = \bar{p}\bar{q}r + p\bar{q}\bar{r} + p\bar{q}r + pqr$$