Real Numbers

Definition

- The set of *rational* numbers is given by:

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

- A number that is not rational is called *irrational*.
- The set of *real* numbers, denoted $\mathbb{R}$, is the union of the sets of rational and irrational numbers.

The set of irrational numbers is often denoted: $\mathbb{R} - \mathbb{Q}$.

Note that $\mathbb{Z} \subset \mathbb{Q}$ because $\forall\, n \in \mathbb{Z}, n = \frac{n}{1}$.

Definition

- To say that $\alpha \in \mathbb{R}$ is *algebraic* means that it is a zero of some polynomial with integer coefficients:

$$\exists\, a_k, \sum_{k=0}^{n} a_k \alpha^k = 0$$

- A number that is not algebraic is called *transcendental*.

Theorem

$\mathbb{Q}$ is closed under addition and multiplication.

Proof

Assume $a, b \in \mathbb{Q}$
Let $a = \frac{p}{q}$ and $b = \frac{r}{s}$ where $p, q, r, s \in \mathbb{Z}$ and $p, q \neq 0$

$a + b = \frac{p}{q} + \frac{r}{s} = \frac{ps+qr}{qs}$
$ps + qr \in \mathbb{Z}$
$qs \in \mathbb{Z}$ and $qs \neq 0$
$\therefore a + b \in \mathbb{Q}$

$ab = \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$
$pr \in \mathbb{Z}$
$qs \in \mathbb{Z}$ and $qs \neq 0$
$\therefore ab \in \mathbb{Q}$

Example

Prove that $\sqrt{2}$ is irrational.

Proof 1

ABC: $\sqrt{2}$ is rational

$\exists\, a, b \in \mathbb{Z}^+, \frac{a}{b} = \sqrt{2}$ and either a or b not even
$\frac{a^2}{b^2} = 2$
$a^2 = 2b^2$
a^2 is even, so a is even, so b must be odd
Let $a = 2k, k \in \mathbb{Z}$
$(2k)^2 = 4k^2 = 2(2k^2) = b^2$
b^2 is even, so b is even
CONTRADICTION!
$\therefore \sqrt{2}$ is irrational.

Proof 2

ABC: $\sqrt{2}$ is rational

$\exists\, a, b \in \mathbb{Z}^+, \frac{a}{b} = \sqrt{2}$
Let $S = \{k\sqrt{2} \mid k, k\sqrt{2} \in \mathbb{Z}^+\}$
Note that $S \neq \emptyset$ because $a = b\sqrt{2}$
By the well-ordering principle, S has a minimum
Let $s = \min S \in \mathbb{Z}^+$
Let $s = t\sqrt{2}, t \in \mathbb{Z}^+$
$s\sqrt{2} - s = s\sqrt{2} - t\sqrt{2} = (s-t)\sqrt{2}$
$s\sqrt{2} = (t\sqrt{2})\sqrt{2} = 2t \in \mathbb{Z}^+$
Since $\sqrt{2} > 1$, $s\sqrt{2} > s$ and $s\sqrt{2} - s > 0$
$(s-t)\sqrt{2} > 0$
$s - t > 0$
So $(s-t)\sqrt{2} \in S$
But $s\sqrt{2} - s = s(\sqrt{2} - 1) < s$
CONTRADICTION (of the minimality of s)!
$\therefore \sqrt{2}$ is irrational.

Proof 3

Let $x = \sqrt{2}$
$x^2 = 2$
$x^2 - 2 = 0$
$\sqrt{2}$ is algebraic
The only possible rational zeros are ± 2
$(\pm 2)^2 - 2 = 4 - 2 = 2 \neq 0$
So there are no rational zeros
$\therefore \sqrt{2}$ is irrational.