Rational Numbers

Definition

The set of *Rational Numbers* is given by:

$$\mathbb{Q} = \left\{ \frac{p}{q} \middle| p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

If a number is not rational then it is called *irrational*.

Properties

A rational number can be written as:

1). A ratio of integers

2). A finite decimal

3). A repeating decimal

Example

Convert $0.1\overline{23}$ into a ratio of integers.

Let $x = 0.1\overline{23}$.
$1000x = 123.\overline{23}$
$10x = 1.\overline{23}$
$990x = 122$
$x = \frac{122}{990} = \frac{61}{495}$

Theorem

$\forall r, s \in \mathbb{Q}, r + s \in \mathbb{Q}$

Proof

Assume $r, s \in \mathbb{Q}$.
$\exists a, b \in \mathbb{Z}, r = \frac{a}{b}, b \neq 0$
$\exists c, d \in \mathbb{Z}, s = \frac{c}{d}, d \neq 0$
$r + s = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$
$ab + bc \in \mathbb{Z}$
$bd \in \mathbb{Z}$ and $bd \neq 0$
$\therefore r + s \in \mathbb{Q}$

Lemma

$r \in \mathbb{Q} \iff -r \in \mathbb{Q}$

Proof

$$\begin{aligned} r \in \mathbb{Q} \quad &\iff \quad \exists p, q \in \mathbb{Z}, r = \frac{p}{q}, q \neq 0 \\ &\iff \quad \frac{-p}{q} \in \mathbb{Q} \\ &\iff \quad -r \in \mathbb{Q} \end{aligned}$$

Theorem

$\forall r \in \mathbb{Q}$ and $s \notin \mathbb{Q}, r + s \notin \mathbb{Q}$

Proof

Assume $r \in \mathbb{Q}$ and $s \notin \mathbb{Q}$.
ABC: $r + s \in \mathbb{Q}$
Let $t = r + s$
$-r \in \mathbb{Q}$
$s = t - r$
But $t - r \in \mathbb{Q}$.
Thus $s \in \mathbb{Q}$.
Contradiction.
$\therefore r + s \notin \mathbb{Q}$

Theorem

$\forall r, s \in \mathbb{Q}, rs \in \mathbb{Q}$

Proof

Assume $r, s \in \mathbb{Q}$.
$\exists a, b \in \mathbb{Z}, r = \frac{a}{b}, b \neq 0$
$\exists c, d \in \mathbb{Z}, s = \frac{c}{d}, d \neq 0$
$rs = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
$ac \in \mathbb{Z}$
$bd \in \mathbb{Z}$ and $bd \neq 0$
$\therefore rs \in \mathbb{Q}$

Lemma

$r \in \mathbb{Q} - \{0\} \iff \frac{1}{r} \in \mathbb{Q}$

Proof

$$
\begin{aligned}
r \in \mathbb{Q} - \{0\} &\iff \exists p, q \in \mathbb{Z} - \{0\}, r = \frac{p}{q} \\
&\iff \frac{q}{p} \in \mathbb{Q} \\
&\iff \frac{1}{r} \in \mathbb{Q}
\end{aligned}
$$

Theorem

$\forall r \in \mathbb{Q}$ and $s \notin \mathbb{Q}, rs \notin \mathbb{Q}$

Proof

Assume $r \in \mathbb{Q}$ and $s \notin \mathbb{Q}$.
ABC: $rs \in \mathbb{Q}$
Let $t = rs$
$\frac{1}{r} \in \mathbb{Q}$
$s = t \cdot \frac{1}{r}$
But $t \cdot \frac{1}{r} \in \mathbb{Q}$.
Thus $s \in \mathbb{Q}$.
Contradiction.
$\therefore rs \notin \mathbb{Q}$