$\mathbf{L^2}$ Space

Definition

$$L^2(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{C} \mid \text{ f is measurable and } \int_{\mathbb{R}^d} |f|^2 < \infty\}$$

$$\|f\|_{L^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f|^2\right)^{\frac{1}{2}}$$

$$\langle f, g\rangle = \int_{\mathbb{R}^d} f\bar{g}$$

Hence: $\langle f, f\rangle = \int_{\mathbb{R}^d} f\bar{f} = \int_{\mathbb{R}^d} |f|^2 = \|f\|^2$

Theorem

L^2 is a vector space.

Proof

Assume $f, g \in L^2$
$|f+g| \leq |f| + |g| \leq 2 \cdot max\{|f|, |g|\}$
$|f+g|^2 \leq 4 \cdot max\{|f|, |g|\}^2 = 4 \cdot max\{|f|^2, |g|^2\} \leq 4(|f|^2 + |g|^2)$
$\int |f+g|^2 \leq 4 \int (|f|^2 + |g|^2) = 4 \int |f|^2 + 4 \int |g|^2 < \infty$
$\therefore f + g \in L^2$

Assume $\alpha \in \mathbb{C}$
$|\alpha f|^2 = |\alpha|^2 |f|^2$
$\int |\alpha f|^2 = |\alpha|^2 \int |f|^2 < \infty$
$\therefore \alpha f \in L^2$

$\therefore L^2$ is a vector space

Lemma

$\forall f, g \in L^2, |f\bar{g}| \leq \frac{1}{2}\left(|f|^2 + |g|^2\right)$

Proof

Assume $f, g \in L^2$
$(|f| - |g|)^2 \geq 0$
$|f|^2 + |g|^2 - 2|f||g| \geq 0$
$|f||g| = |f||\bar{g}| = |f\bar{g}|$
$\therefore |f\bar{g}| \leq \frac{1}{2}\left(|f|^2 + |g|^2\right)$

Properties

$\forall f, g \in L^2$:

1). $\bar{f} \in L^2$

2). $f\bar{g} \in L^1$

3). $\|f\| = 0 \iff f = 0 \;\; a.e.$

4). $\langle f, g\rangle$ is linear in f and conjugate-linear in g

Proof

1). Assume $f \in L^2$

$\int |\bar{f}|^2 = \int |f|^2 < \infty$
$\therefore \bar{f} \in L^2$

2). Assume $f, g \in L^2$

$\int |f\bar{g}| \leq \frac{1}{2}\left(|f|^2 + |g|^2\right) = \frac{1}{2}\left(\int |f|^2 + \int |g|^2\right) < \infty$
$\therefore f\bar{g} \in L^1$

3). Assume $f \in L^2$

$\|f\| = 0 \iff \left(\int |f|^2\right)^{\frac{1}{2}} = 0 \iff \int |f|^2 = 0 \iff |f|^2 = 0 \;\; a.e. \iff f = 0 \;\; a.e.$

4). Assume $f, g \in L^2$ and $\alpha \in \mathbb{C}$

$\langle \alpha f, g\rangle = \int (\alpha f)\bar{g} = \alpha \int f\bar{g} = \alpha\langle f, g\rangle$

$\langle f, \alpha g\rangle = \int f\overline{\alpha g} = \int f\bar{\alpha}\bar{g} = \bar{\alpha}\int f\bar{g} = \bar{\alpha}\langle f, g\rangle$