Compact Sets

Definition: Compact

Let E be a normed space and let $K \subseteq E$. To say that K is *compact* means every sequence $(\vec{x}_n)$ in K has a convergent subsequence $\vec{x}_{n_k} \to \vec{x}$ such that $\vec{x} \in K$.

Examples

Let $E = \mathbb{R}^N$ or $\mathbb{C}^N$:

1). Closed balls: $\overline{B}(x, r) = \{y \in E \mid \|\vec{x} - \vec{y}\| \leq r\}$

2). Closed cubes: $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$

Definition: Bounded

Let E be a normed space and let $S \subseteq E$. To say that S is *bounded* means $\exists\, r > 0$ such that $S \subseteq B(\vec{0}, r)$.

Theorem

Let E be a normed space and let $K \subseteq E$:

$$K \text{ compact} \implies K \text{ is closed and bounded.}$$

Proof

Assume K is compact.

Assume $(\vec{x}_n)$ is a sequence in K such that $\vec{x}_n \to \vec{x} \in E$.
WTS: $\vec{x} \in K$.
But K is compact, so $(\vec{x}_n)$ contains a subsequence $(\vec{x}_{n_k})$ such that $\vec{x}_{n_k} \to \vec{y} \in K$.
But convergent subsequences of a convergent sequence must converge to the same value.
And so $\vec{x} = \vec{y} \in K$.

Therefore, K is closed.

Now, ABC: K is not bounded.
Thus, $\forall\, r > 0, K \not\subseteq B(\vec{0}, r)$.
And so, $\forall\, r > 0, \exists\, \vec{x} \in K, \|\vec{x}\| > r$.
Construct the sequence $(\vec{x}_n)$ in K such that $\|\vec{x}_n\| > n$.
For every subsequence $(\vec{x}_{n_k})$, it is the case that $\|\vec{x}_{n_k}\| > n_k \to \infty$.
Thus, $(\vec{x}_n)$ does not have a convergent subsequence in K.
CONTRADICTION! (of the compactness of K)

Therefore, K is bounded.

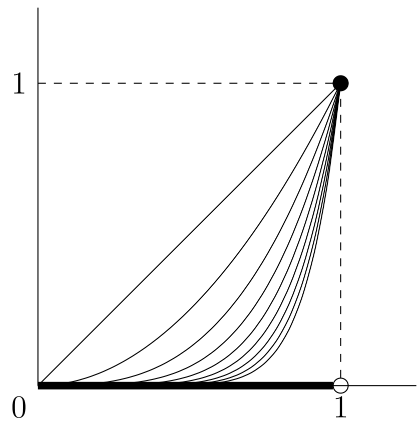
Note that by Heine-Borel, the converse is true as well for finite-dimensional spaces; however, not necessarity for infinite-dimensional spaces.

Example

$E = \mathcal{C}[0,1]$ equipped with the sup norm.

$K = \overline{B}(0,1) = \{f \in \mathcal{C}(0,1) \mid \|f\| \leq 1\} \subset E$

$f_n(t) = t^n \in \mathcal{C}[0,1]$ since $\|f_n\| = \max\limits_{t\in[0,1]} |f_n(t)| = 1 \leq K$.



$$f_n \to f = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & t = 1 \end{cases}$$

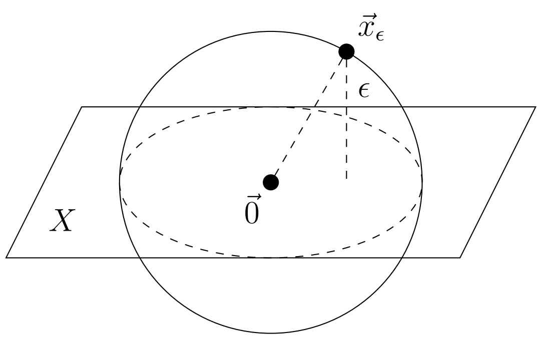
But f is discontinuous and thus $f \notin \mathcal{C}[0,1]$.

Therefore, there exists a sequence in K with a non-converging subsequence, and thus K is not compact.

Lemma: Riesz

Let E be a normed space and let X be a proper, closed subspace of E:

$$\forall\, \epsilon \in (0,1), \exists\, \vec{x}_\epsilon \in E, \|\vec{x}_\epsilon\| = 1 \text{ and } \forall\, \vec{x} \in X, \|\vec{x}_\epsilon - \vec{x}\| \geq \epsilon$$



Proof

Since X is a proper subset of E, $E \setminus X \neq \emptyset$.
So, $\exists\, \vec{y} \in E \setminus X$.

Let $d = d(\vec{x}, \vec{y}) = \inf_{\vec{x} \in X} \|\vec{y} - \vec{x}\|$.
Since X is closed and $\vec{y} \notin X$, $d(\vec{x}, \vec{y}) > 0$.
Assume $\epsilon \in (0, 1)$, as so $d < \dfrac{d}{\epsilon}$.
$\exists\, \vec{x}_0 \in X$ such that $d \leq \|\vec{y} - \vec{x}_0\| \leq \dfrac{d}{\epsilon}$.
Let $\vec{x}_\epsilon = \dfrac{\vec{y} - \vec{x}_0}{\|\vec{y} - \vec{x}_0\|}$.
Assume $\vec{x} \in X$:

$$\begin{aligned}
\|\vec{x}_\epsilon - \vec{x}\| &= \left\| \frac{\vec{y} - \vec{x}_0}{\|\vec{y} - \vec{x}_0\|} - \vec{x} \right\| \\
&= \frac{1}{\|\vec{y} - \vec{x}_0\|} \|\vec{y} - \vec{x}_0 - \|\vec{y} - \vec{x}_0\| \, \vec{x}\| \\
&= \frac{1}{\|\vec{y} - \vec{x}_0\|} \|\vec{y} - (\vec{x}_0 + \|\vec{y} - \vec{x}_0\|) \, \vec{x}\|
\end{aligned}$$

But, by closure, $(\vec{x}_0 + \|\vec{y} - \vec{x}_0\|) \, \vec{x} = \vec{x}_1 \in X$, and so:

$$\|\vec{x}_\epsilon - \vec{x}\| = \frac{1}{\|\vec{y} - \vec{x}_0\|} \|\vec{y} - \vec{x}_1\| \geq \frac{\epsilon}{d} d = \epsilon$$

<u>Theorem</u>

Let E be a normed space. E is finite-dimensional iff $\overline{B}(0, 1)$ is compact.

<u>Proof</u>

$\Longrightarrow$ Assume E is finite-dimensional.

Since E is finite-dimensional, all norms are equivalent, so AWLOG the Euclidean norm.
Thus $\overline{B}(0, 1)$ is closed and bounded.

Therefore, by Heine-Borel, $\overline{B}(0, 1)$ is compact.

$\Longleftarrow$ Assume E is infinite-dimensional.

Construct (x_n) in E by induction using Riesz's Lemma.
Start by selecting any $\vec{x}_1 \in E$ such that $\|\vec{x}_1\| = 1$.
Let $X_1 = \mathrm{Span}\{\vec{x}_1\}$.
By Riesz's Lemma, $\exists\, \vec{x}_2 \in E \setminus X_1$ such that $\|\vec{x}_2\| = 1$ and $\|\vec{x}_2 - \vec{x}_1\| \geq \dfrac{1}{2}$.
Assume $\vec{x}_1, \ldots, \vec{x}_n$ have been selected in this fashion and let $X_n = \mathrm{Span}\{\vec{x}_1, \ldots, \vec{x}_n\}$.
$\exists\, \vec{x}_{n+1} \in E \setminus X_n$ such that $\|\vec{x}_{n+1}\| = 1$ and $\forall\, k \leq n$, $\|\vec{x}_{n+1} - v_k\| \geq \dfrac{1}{2}$.
Thus, (x_n) is a sequence in $\overline{B}[0, 1]$.

ABC: $\overline{B}[0, 1]$ is compact.

Thus, $(\vec{x}_n)$ contains a convergent subsequence $(\vec{x}_{n_k})$ such that $\vec{x}_{n_k} \to \vec{x} \in \overline{B}[0, 1]$.

$$\frac{1}{2} \leq \left\|\vec{x}_{n_{k+1}} - \vec{x}_{n_k}\right\| = \left\|\vec{x}_{n_{k+1}} - \vec{x} + \vec{x} - \vec{x}_{n_k}\right\| \leq \left\|\vec{x}_{n_{k+1}} - \vec{x}\right\| + \|\vec{x}_{n_k} - \vec{x}\| \to 0 + 0 = 0$$

CONTRADICTION!

Therefore, $\overline{B}[0, 1]$ is not compact.