Subspaces

Definition: Subspace

Let V be a vector space over a field $\mathbb{F}$ and let $S \subseteq V$. To say that S is a *subspace* of V means that S is also a vector space over the same field $\mathbb{F}$ using the same operations of vector addition and scalar multiplication as V.

$\{\vec{0}\}$ is called the *zero subspace* and is a subspace of all vector spaces.

The zero subspace and V are called the *trivial* subspaces of V. All other subspaces of V are called *non-trivial*.

To say that S is a *proper* subspace of V means that S is a subspace of V but $S \neq V$.

Theorem: Subspace Test

Let V be a vector space over a field $\mathbb{F}$.

$S \subseteq V$ is a subspace of V iff:

1). $\vec{0} \in S$

2). S is closed under vector addition.

3). S is closed under scalar multiplication.

Proof

$\implies$ Assume S is a subspace of V

By definition, the closure properties hold and there exists additive identity $\vec{0'} \in S$
Assume $\vec{x} \in S$
$\vec{x} + \vec{0'} = \vec{x}$
But $\vec{x} \in V$ as well, so $\vec{x} + \vec{0} = \vec{x}$
$\vec{x} + \vec{0'} = \vec{x} + \vec{0}$
Thus, $\vec{0'} = \vec{0}$, and so $\vec{0} \in S$

Therefore, the three properties hold.

$\impliedby$ Assume the three properties hold

Note that S inherits all of the vector space axioms from V with the exceptions of closure, additive identity, and additive inverses. Closure and identity are supplied by the assumed properties.

For inverses, assume $x \in S$
By closure, $(-1)x = (-x) \in S$

Therefore, all ten axioms hold, and S is a subspace of V.

Example

1). Let $\mathbb{F}_n[x]$ denote all polynomials of degree $\leq n$ using scalars and coefficients from $\mathbb{F}$. This is a subspace of $\mathbb{F}[x]$.

2). Let $C(\mathbb{R})$ denote all continuous functions $f : \mathbb{R} \to \mathbb{R}$. This is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

But note that polynomials of degree exactly n is not a subspace since it does not include the zero polynomial and closure fails. For example: $(x^3 + x^2)$ and $(-x^3 + x^2)$ are both degree 3 polynomials; however:

$$(x^3 + x^2) + (-x^3 + x^2) = 2x^2$$

which is only degree 2.

Theorem

Let V be a vector space over a field $\mathbb{F}$ and let $\{S_i \mid i \in I\}$ be a family of subspaces of V:

$$S = \bigcap_{i \in I} S_i$$

is a subspace of V.

Proof

Assume $i \in I$

S_i is a subspace of V, so $\vec{0} \in S_i$

$\therefore \vec{0} \in S$

Assume $\vec{x}, \vec{y} \in S$
$\vec{x}, \vec{y} \in S_i$
So by closure, $\vec{x} + \vec{y} \in S_i$
Thus $\vec{x} + \vec{y} \in S$

Therefore S is closed under vector addition.

Assume $c \in \mathbb{F}$
By closure, $c\vec{x} \in S_i$
Thus $c\vec{x} \in S$

Therefore S is closed under scalar multiplication.

Therefore, by the subspace test, S is a subspace of V.

Theorem

Let V be a vector space and let U and W be two subspaces of V:

$$U + W = \{\vec{u} + \vec{w} \mid \vec{u} \in U \text{ and } \vec{w} \in W\}$$

is a subspace of V.

Proof

Since U and W are subspaces of V, $\vec{0} \in U$ and $\vec{0} \in W$

Therefore, $\vec{0} = \vec{0} + \vec{0} \in U + W$

Assume $\vec{x}, \vec{y} \in U + V$
There exists $\vec{u}_x \in U$ and $\vec{w}_x \in W$ such that $\vec{x} = \vec{u}_x + \vec{w}_x$
Also, there exists $\vec{u}_y \in U$ and $\vec{w}_y \in W$ such that $\vec{y} = \vec{u}_y + \vec{w}_y$
$\vec{x} + \vec{y} = (\vec{u}_x + \vec{w}_x) + (\vec{u}_y + \vec{w}_y) = (\vec{u}_x + \vec{u}_y) + (\vec{w}_x + \vec{w}_y)$
But by closure $\vec{u}_x + \vec{u}_y \in U$ and $\vec{w}_x + \vec{w}_y \in W$
Thus, $\vec{x} + \vec{y} \in U + W$

Therefore $U + W$ is closed under vector addition.

Assume $c \in \mathbb{F}$
$c\vec{x} = c(\vec{u}_x + \vec{w}_x) = c\vec{u}_x + c\vec{w}_x$
But by closure, $c\vec{u}_x \in U$ and $c\vec{w}_x \in W$
Thus, $c\vec{x} \in U + W$

Therefore $U + W$ is closed under scalar multiplication.

Therefore, by the subspace test, $U + W$ is a subspace of V.