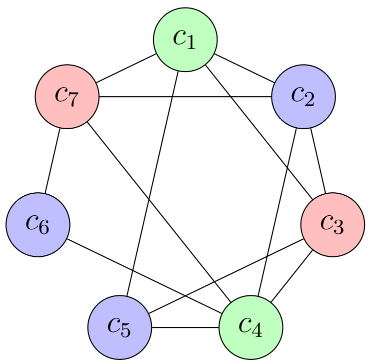
1.1: Graphs and Graph Models

1. What is a logical question to ask in Example 1.1? Answer the question.

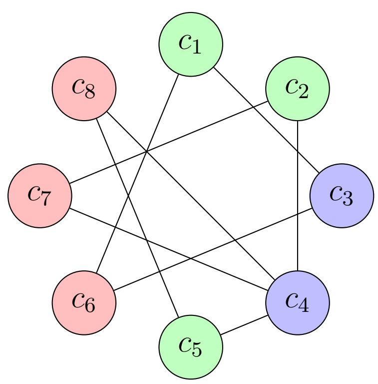
   Given ten editors on seven committees, determine how to schedule committee sessions into three time slots such that any two committees that meet during the same time slot do not have any common members. The graph for this problem represents the committees with nodes. If any two committees have a common member, then their corresponding nodes are adjacent. Thus, the solution to the problem is to partition the nodes into three independent sets. This is an example of a node coloring problem. To solve this problem, use a greedy coloring algorithm:



slot	committees
1	c_1, c_4
2	c_2, c_5, c_6
3	c_3, c_7

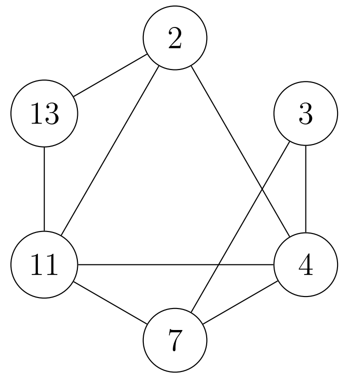
2. Create an example of your own similar to Example 1.1 with nine editors and eight committees and then draw the corresponding graph.

committee	members
1	$1, 2, 7, 8$
2	$3, 4$
3	$1, 7$
4	$3, 4, 5$
5	$5, 6$
6	$2, 7, 8$
7	$4, 9$
8	$5, 6$

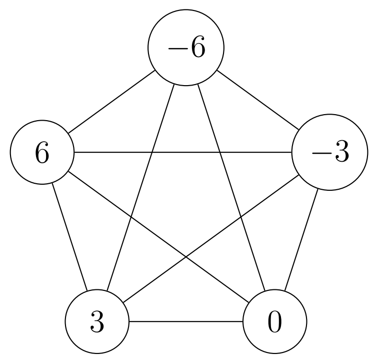


slot	committees
1	c_1, c_2, c_5
2	c_3, c_4
3	c_6, c_7, c_8

3. Let $S = \{2, 3, 4, 7, 11, 13\}$. Draw the graph G whose vertex set is S and such that $ij \in E(G)$ for $i, j \in S$ if $i + j \in S$ or $|i - j| \in S$.

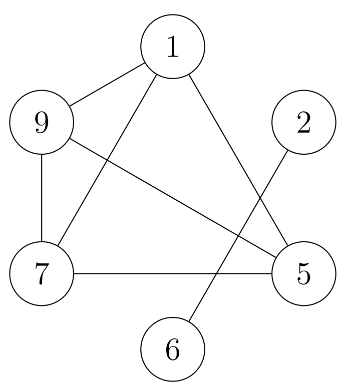


4. Let $S = \{-6, -3, 0, 3, 6\}$. Draw the graph G whose vertex set is S and such that $ij \in E(G)$ for $i, j \in S$ if $i + j \in S$ or $|i - j| \in S$.



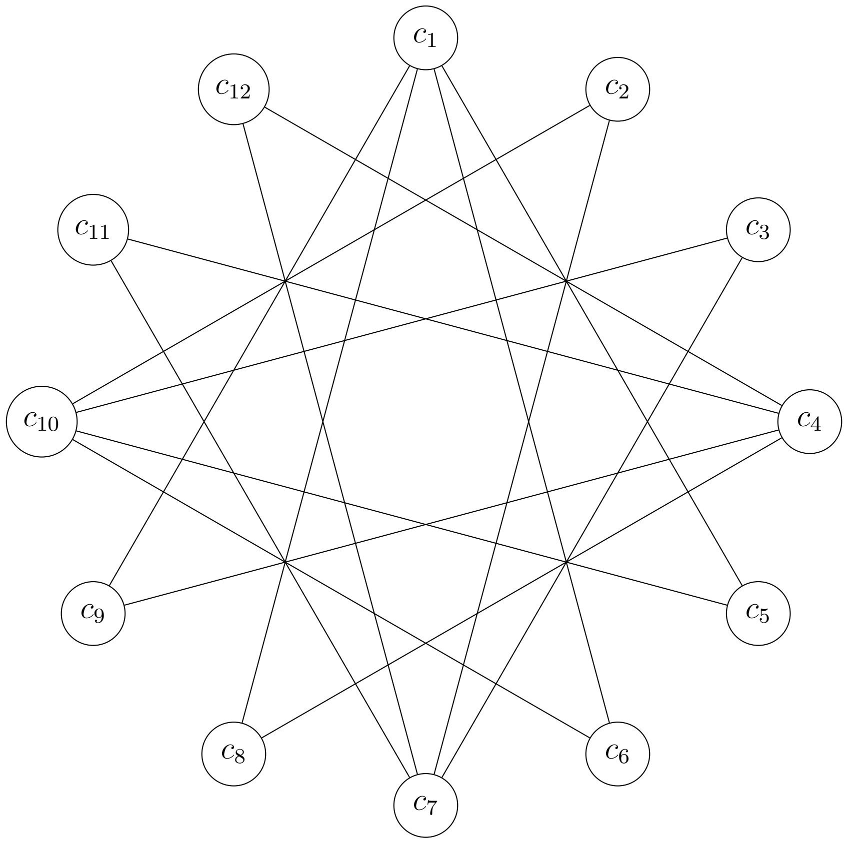
5. Create your own set S of integers and draw the graph G whose vertex set is S and such that $ij \in E(G)$ if i and j are related by some rule imposed on i and j.

   Let $S = \{1, 2, 5, 6, 7, 9\}$ and let $ij \in E(G)$ if $i + j$ is an even number.



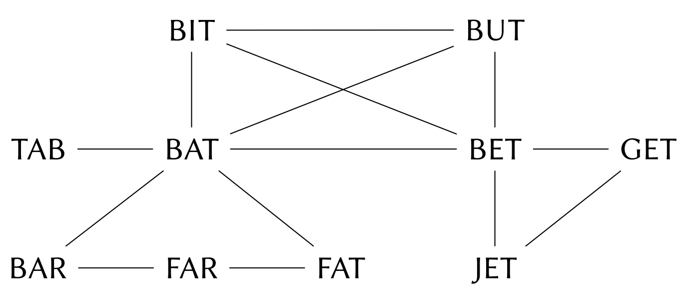
6. Consider the twelve configurations $c_1, c_2, \ldots, c_{12}$ in Figure 1.4. For every two configurations c_i and c_j, where $1 \leq i, j \leq 12, i \neq j$, it may be possible to obtain c_j from c_i by first shifting one of the coins in c_i horizontally or vertically *and* then interchanging the two

coins. Model this by a graph F such that $V(F) = \{c_1, c_2, \ldots, c_{12}\}$ and c_ic_j is an edge of F if c_i and c_j can be transformed into each other by this two step process.



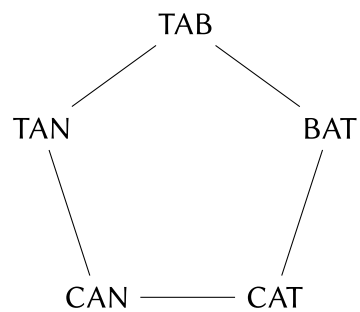
7. Following Example 1.4,

   (a) give an example of ten 3-letter words, none of which are mentioned in Example 1.4 and whose corresponding word graph has at least six edges. Draw this graph.



   (b) give a set of five 3-letter words whose word graph is shown in Figure 1.11 (with the vertices appropriately labeled).

(c) give a set of five 3-letter words whose word graph is shown in Figure 1.12 (with the vertices appropriately labeled).



8. Let S be a finite set of 3-letter and/or 4-letter words. In this case, the word graph $G(S)$ of S is that graph whose vertex set is S and such that two vertices (words) w_1 and w_2 are adjacent if either (1) or (2) below occurs:

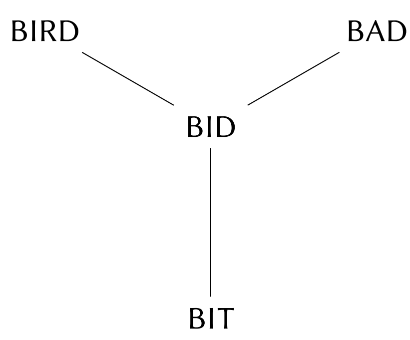
(1) one of the words can be obtained from the other by replacing one letter by another letter.

(2) w_1 is a 3-letter word and w_2 is a 4-letter word and w_2 can be obtained from w_1 by the insertion of a single letter (anywhere, including the beginning or the end) into w_1.

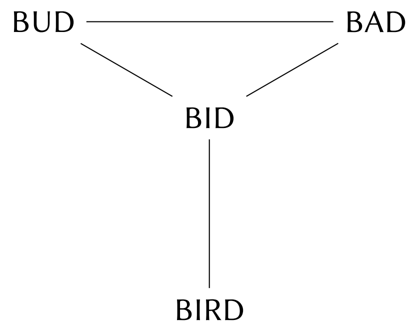
(a) Find six sets $S_1, S_2, \ldots, S_6$ of 3-letter and/or 4-letter words so that for each integer i $(1 \leq i \leq 6)$ the graph G_i of Figure 1.13 is the word graph of S_i.

ART —— CART —— CARD —— HARD —— ART

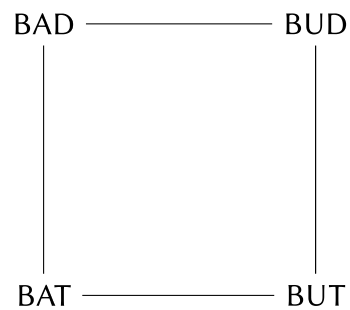
G_1



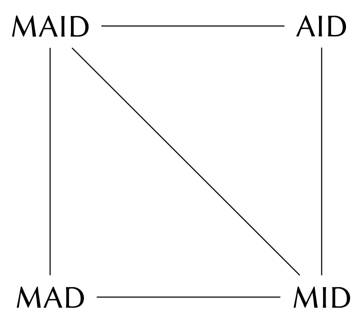
G_2


BUD
BAD
BID
BIRD

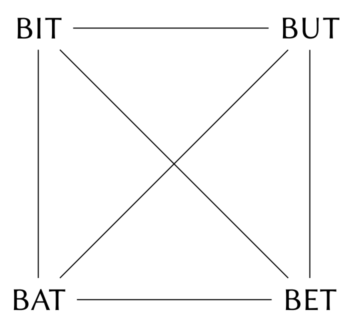
G_3


BAD
BUD
BAT
BUT

G_4


MAID
AID
MAD
MID

G_5


BIT
BUT
BAT
BET

G_6

(b) For another graph H (of your choice), determine whether H is a word graph of some sort.

$$\begin{tikzcd}[arrows=dash]
\text{BAR} \arrow[r] & \text{BOAR} \arrow[r] \arrow[d] & \text{OAR} \\
\text{GOAT} \arrow[r] & \text{BOAT} \arrow[r] & \text{BOLT}
\end{tikzcd}$$

9. Define a word graph differently from the word graphs defined in Example 1.4 and Exercise 1.8 and illustrate your definition.

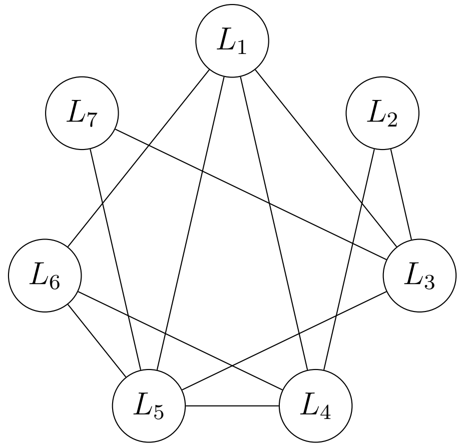
Let S be a finite set of 3-letter words. The word graph $G(S)$ of S is that graph whose vertex set is S and such that two vertices (words) w_1 and w_2 are adjacent if either (1) or (2) below occurs:

(1) one can be obtained from the other by replacing one letter by another letter.

(2) one is an anagram of the other.

$$\begin{tikzcd}[arrows=dash]
\text{ART} \arrow[r] \arrow[d] & \text{TAR} \arrow[d] \\
\text{ARC} \arrow[r] & \text{CAR}
\end{tikzcd}$$

10. Figure 1.14 illustrates the traffic lanes at the intersection of two streets. When a vehicle approaches this intersection, it could be in one of the seven lanes: $L_1, L_2, \ldots, L_7$. Draw a graph G that models this situation, where $V(G) = \{L_1, L_2, \ldots, L_7\}$ and where two vertices are joined by an edge if vehicles in these two lanes cannot safely enter this intersection at the same time.


L_1
L_2
L_3
L_4
L_5
L_6
L_7