Extended Complex Plane

Definition

The extended complex plane is $C \cup \{\infty\}$.

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Let C be a circle in the clockwise direction containing all of the singularities of $f(z)$ in the finite complex plane. The residue of $f(z)$ at ∞ is given by:

$$\mathrm{Res}[f, \infty] = \frac{1}{2\pi i} \oint_C f(z) dz = -\frac{1}{2\pi i} \oint_C f(z) dz$$

Example

$f(z) = z$

To determine the behavior at ∞, examine the behavior in the neighborhood of infinity:
$\lim_{z\to\infty} f(z) = \lim_{z\to 0} f(\frac{1}{z}) = \frac{1}{z}$
Since the latter has a simple pole at 0, $f(z)$ has a simple pole at ∞.

Note that $f(z) = z$ is entire in the finite complex place, so:

$$\mathrm{Res}[f, \infty] = -\frac{1}{2\pi i} \int_C z dz = 0$$

Example

$f(z) = \frac{1}{z}$

$lim_{z\to\infty} f(z) = lim_{z\to 0} f(\frac{1}{z}) = z$
Since the latter is analytic at 0, $f(z)$ is analytic at ∞.

$$\mathrm{Res}[f, \infty] = -\frac{1}{2\pi i} \int_C \frac{1}{z} dz = -\frac{1}{2\pi i} [2\pi i(1)] = -1$$

Theorem

$\mathrm{Res}[f, \infty]$ equals the negative of the coefficient of z in the expansion of $f(\frac{1}{z})$.

Theorem

Let $f(z)$ have a finite number of singularities $\{z_k \mid 1 \leq k \leq n\}$, all contained within C:

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}[f, z_k] = -2\pi i \, \mathrm{Res}[f, \infty]$$

Theorem

$Res[f(z), \infty]$ equals the negative of the coefficient of w in the expansion of $f(\frac{1}{w})$.

Example

$G(z) = \frac{b_{n-1}z^{n-1}+\cdots+b_0}{a_nz^n+\cdots+a_0}$

$$
\begin{aligned}
G\left(\frac{1}{w}\right) &= \frac{\frac{b_{n-1}}{w^{n-1}} + \cdots + b_0}{\frac{a_n}{w^n} + \cdots + a_0} \\
&= \frac{b_{n-1}w + \cdots + b_0w^n}{a_n + \cdots + a_0w^n} \\
&= \frac{b_{n-1}}{a_n}w\left[\frac{1 + \frac{b_{n-2}}{b_{n-1}}w + \cdots + \frac{b_0}{b_{n-1}}w^n}{1 + \frac{a_{n-1}}{a_n}w + \cdots + \frac{a_0}{a_n}w^n}\right] \\
&= \frac{b_{n-1}}{a_n}w\left[\frac{1 + \frac{b_{n-2}}{b_{n-1}}w + \cdots + \frac{b_0}{b_{n-1}}w^n}{1+a}\right] \\
&= \frac{b_{n-1}}{a_n}w\left[1 + \frac{b_{n-2}}{b_{n-1}}w + \cdots + \frac{b_0}{b_{n-1}}w^n\right]\left[1 - a + a^2 \cdots\right]
\end{aligned}
$$

So the coefficient of w is $\frac{b_{n-1}}{a_n}$

$Res[G(z), \infty] = -\frac{b_{n-1}}{a_n}$