The Heine-Borel Theorem

Theorem

For all $a, b \in \mathbb{R}$ such that $a \leq b$, the subspace $[a, b]$ is compact.

Proof. Assume that $a, b \in \mathbb{R}$ such that $a \leq b$ and assume that $\mathcal{U} = \{U_\alpha : \alpha \in \lambda\}$ is an open cover for $[a, b]$. If $a = b$ then $[a, a] = \{a\}$ and there exists $U_a \in \mathcal{U}$ such that U_a is a finite subcover for $\{a\}$. So assume that $a < b$.

Note that for all $x \in [a, b]$ it is the case that $[a, x] \subset \bigcup \mathcal{U}$ as well. So construct the set:

$$C = \{x \in [a, b] \mid [a, x] \text{ has a finite subcover in } \mathcal{U}\} \subset [a, b]$$

Since $[a, a] \in C$, C is not empty. Furthermore, C is bounded by b. Thus, there exists $c = \sup C$. But since $c \in [a, b]$, there exists some $U_c \in \mathcal{U}$ such that $c \in U_c$.

Now, since $U_c \in \mathscr{T}$, there exists $(r, s) \subset U_c$ such that $c \in (r, s)$. Furthermore, there must exist some $x \in C$ such that $r < x \leq c$, otherwise, $x < r < c$, violating the fact that $c = \sup C$. So $x \in [x, a]$, which has a finite subcover $\mathcal{U}_x \subset \mathcal{U}$, and $x, c \in U_0$. Therefore $\mathcal{U}_x \cup \{U_0\}$ is a finite subcover for $[a, c]$ and thus $c \in C$.

Finally, ABC that $c < b$. But $(c, b) \cap U_0 \neq \emptyset$, so assume $x \in (c, b) \cap U_0$. So $[a, c]$ has a finite subcover and $c, x \in U_0$ such that $x < c$. Thus $[a, x]$ has a finite subcover and so $x \in C$, violating the fact that $c = \sup C$. Therefore $c = b$ and $[a, b]$ has a finite subcover in $\mathcal{U}$.

Therefore $[a, b]$ is compact. ∎

Theorem: Heine-Borel

For all $A \subset \mathbb{R}$, A is compact $\iff$ A is closed and bounded.

Proof. Assume that $A \subset \mathbb{R}$.

$\implies$ Assume that A is compact.

Since $\mathbb{R}$ is Hausdorff and A is compact, therefore A is closed.

Now, let $\mathcal{U} = \{(a-1, a+1) : a \in A\}$ be an open cover for A. Since A is compact, there exists $S = \{a_1, \ldots a_n\} \subset A$ such that $\mathcal{U}' = \{(a-1, a+1) : a \in S\} \subset \mathcal{U}$ is a finite subcover for A. So let $M = \max\limits_{a \in S} |a|$. Therefore $A \subset [-M, M]$ and hence A is bounded.

$\impliedby$ Assume that A is closed and bounded.

Since A is bounded, there exists $M \in \mathbb{R}$ such that $A \subset [-M, M]$. But, by the previous theorem, $[-M, M]$ is compact, and so A is a closed subset of a compact set. Therefore A is compact.

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Theorem: The Tube Lemma

Let $X \times Y$ be a product space with Y compact. If $U \in \mathscr{T}_{X\times Y}$ and $\{x_0\} \times Y \subset U$ then there exists some $W \in \mathscr{T}_X$ such that $x_0 \in W$ and $W \times Y \subset U$.

Proof. Assume $U \in \mathscr{T}_{X\times Y}$ and $\{x_0\} \times Y \subset U$. Note that for each $y \in Y$, $(x_0, y) \in \{x_0\} \times Y \subset U$. Thus there exists sets $U_y \in \mathscr{T}_X$ and $V_y \in \mathscr{T}_Y$ such that $\{x_0\} \times Y \subset \bigcup\limits_{y\in Y}(U_y \times V_y) \subset U$, where the V_y are an open cover of Y. But Y is compact, so there exists some finite subcover $\{V_{y_1}, \ldots, V_{y_n}\}$ of Y. So select up to n open subsets of U_y and let $W = \bigcap\limits_{k=1}^{n} U_{y_k}$. Note that $W \in \mathscr{T}_X$ because it is a finite intersection of open sets.

Claim: $W \times Y \subset U$

Assume that $(x, y) \in W \times Y$. This means that for some k, $x \in U_{y_k}$ and $y \in V_{y_k}$. And so $(x, y) \in U_{y_k} \times V_{y_k} \subset U$. ∎

Theorem

If X and Y are compact spaces then $X \times Y$ is compact.

Proof. Assume that X and Y are compact and let $\mathcal{U}$ be an open cover of $X \times Y$. Since Y is compact, for all $x \in X$, $\{x\} \times Y$ has a finite subcover $\mathcal{U}_x = \{U_{x_k} : 1 \leq k \leq n\} \subset \mathcal{U}$. Furthermore, for each $x \in X$, by the previous theorem, there exists a tube $W_x \times Y \subset \mathcal{U}_x$. But $\{W_x : x \in X\}$ is an open cover of X using the tubes, and since X is compact, there exists a finite subcover of tubes $\{W_{x_1}, \ldots, W_{x_n}\}$ such that $W_{x_k} \times Y \subset U_{x_k}$. And so:

$$X \times Y = \bigcup_{k=1}^{n}(W_{x_k} \times Y) \subset \bigcup_{k=1}^{n} U_{x_k}$$

which is a finite subcover of $X \times Y$. Therefore $X \times Y$ is compact. ∎

Theorem: Heine-Borel

For all $A \subset \mathbb{R}^n$, A is compact $\iff$ A is closed and bounded.

Proof. Assume $A \subset \mathbb{R}^n$.

$\implies$ Assume that A is compact.

Since R^n is Hausdorff and $A \subset \mathbb{R}^n$ is compact, A is closed. Now, let $\{(-k, k)^n : k \in N\}$ be an open cover for A. But since A is compact, there exists a finite subcover $\{(-k_i, k_i)^n : 1 \leq i \leq n\}$. Furthermore:

$$\bigcup_{i=1}^{n}(-k_i, k_i)^n = (-k_{max}, k_{max})^n \supset A$$

Therefore A is bounded.

$\Longleftarrow$ Assume that A is closed and bounded.

Since A is bounded, there exists $M > 0$ such that $A \subset [-M, M]^n$. But $[-M, M]$ is compact, and so by repeated application of the previous theorem, $[-M, M]^n$ is compact. Therefore, since A is a closed subset of a compact set, A is also compact.

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Theorem: Tychonoff

Any product of compact spaces is compact.

Recall that for $\{X_\alpha : \alpha \in \lambda\}$, $X = \prod_{\alpha\in\lambda} X_\alpha$ is defined by letting $x \in X$ be a function:

$$x : \alpha \to \bigcup_{\alpha\in\lambda} X_\alpha$$

such that $x(\alpha) \in X_\alpha$.

Also recall that the basis elements $\prod_{\alpha\in\lambda} U_\alpha$ each differ from X by a finite number of components. Otherwise, the topology is the so-called *box* topology, in which Tychonoff does not hold.