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Math 229
Homework #5

4.1.3

Let $A, B \in M_n$ be Hermitian. Show that A and B are similar iff A and B are unitary similar.

$\Longrightarrow$ Assume A and B are similar.

Since A and B are similar we have $\mathrm{Sp}(A) = \mathrm{Sp}(B)$. Now, since A and B are Hermitian, they are unitary diagonalizable. Let:

Let $A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*$ and $B = V \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} V^*$ for unitary matrices U and V.

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = V^*BV$$

$A = U(V^*BV)U^* = (UV^*)B(VU^*) = (UV^*)B(UV^*)^*$

But the product of unitary matrices is unitary, so UV^* is unitary.

Therefore, A and B are unitary similar.

$\Longleftarrow$ Assume A and B are unitary similar.

Let $A = UBU^*$ for unitary matrix U.
$UU^* = U^*U = I$, so unitary matrices are invertible with $U^{-1} = U^*$.
Thus, $A = UBU^{-1}$

Therefore, A and B are similar.

4.1.11

Let $A, B \in M_n$ be Hermitian. Explain why $AB - BA$ is skew-Hermitian and deduce from (4.1.P10) that $\mathrm{tr}(AB)^2 \leq \mathrm{tr}(A^2B^2)$ with equality iff $AB = BA$.

$(AB - BA)^* = (AB)^* - (BA)^* = B^*A^* - A^*B^* = BA - AB = -(AB - BA)$

Therefore $AB - BA$ is skew-Hermitian.

4.1.P10 shows that if $C \in M_n$ is skew-Hermitian then the eigenvalues of C are pure imaginary and the eigenvalues of B^2 are real and non-positive, and all zero iff $B = 0$. with equality iff $B = 0$.

$$\begin{aligned}
\operatorname{tr}(AB - BA)^2 &= \operatorname{tr}(ABAB - ABBA - BAAB + BABA) \\
&= \operatorname{tr}(ABAB) + \operatorname{tr}(BABA) - \operatorname{tr}(ABBA) - \operatorname{tr}(BAAB) \\
&= \operatorname{tr}(ABAB) + \operatorname{tr}((BAB)A) - \operatorname{tr}((ABB)A) - \operatorname{tr}(B(AAB)) \\
&= \operatorname{tr}(ABAB) + \operatorname{tr}(A(BAB)) - \operatorname{tr}(A(ABB)) - \operatorname{tr}((AAB)B) \\
&= \operatorname{tr}(ABAB) + \operatorname{tr}(ABAB) - \operatorname{tr}(AABB) - \operatorname{tr}(AABB) \\
&= 2\operatorname{tr}(ABAB) - 2\operatorname{tr}(AABB) \\
&= 2\operatorname{tr}(AB)^2 - 2\operatorname{tr}(A^2B^2)
\end{aligned}$$

But $AB - BA$ is skew Hermitian, so the eigenvalues of $(AB - BA)^2$ are real and non-positive, so:

$$\begin{aligned}
2\operatorname{tr}(AB)^2 - 2\operatorname{tr}(A^2B^2) &\leq 0 \\
\operatorname{tr}(AB)^2 - \operatorname{tr}(A^2B^2) &\leq 0 \\
\therefore \operatorname{tr}(AB)^2 &\leq \operatorname{tr}(A^2B^2)
\end{aligned}$$

Furthermore, the eigenvalues of $AB - BA$ are all zero iff $AB - BA = 0$. Therefore, $\operatorname{tr}(AB)^2 \leq \operatorname{tr}(A^2B^2)$ iff $AB - BA = 0$, or $AB = BA$.

4.2.3

Let $A \in M_n$ be Hermitian with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Use the key lemma to show that $\lambda_1 \leq a_{ii} \leq \lambda_n$ for all $1 \leq i \leq n$ with equality in one of the inequalities for some i only if $a_{ij} = a_{ji} = 0$ for all $j \neq i$. Consider $A = \operatorname{Diag}(1, 2, 3)$ and explain why the condition $a_{ij} = a_{ji} = 0$ for all $j \neq i$ does not imply that $a_{ii} = \lambda_1$ or $a_{ii} = \lambda_n$.

By the key lemma we have $\forall\, \vec{x} \in \mathbb{C}^n$ such that $\vec{x} \neq \vec{0}$:

$$\lambda_1 \leq \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} \leq \lambda_n$$

Let $\vec{x} = \vec{e}_i$:

$$\frac{\vec{e}_i^* A \vec{e}_i}{\vec{e}_i^* \vec{e}_i} = \frac{a_{ii}}{1} = a_{ii}$$

Therefore:

$$\lambda_1 \leq a_{ii} \leq \lambda_n$$

Claim: $\forall \vec{x} \in \mathbb{C}^n$ such that $\vec{x} \neq \vec{0}$:

$$\lambda_i = \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} \iff \vec{x} \in \mathrm{Eig}_A(\lambda_i)$$

$\implies$ Assume $\lambda_i = \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}}$

$$\begin{aligned} \vec{x}^* A \vec{x} &= \lambda_i \vec{x}^* \vec{x} \\ \vec{x}^* A \vec{x} &= \vec{x}^* \lambda_i \vec{x} \\ A\vec{x} &= \lambda_i \vec{x} \end{aligned}$$

Therefore, since $\vec{x} \neq \vec{0}$, $\vec{x} \in \mathrm{Eig}_A(\lambda_i)$

$\impliedby$ Assume $\vec{x} \in \mathrm{Eig}_A(\lambda_i)$

$$\frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} = \frac{\vec{x}^* \lambda_i \vec{x}}{\vec{x}^* \vec{x}} = \lambda_i \frac{\vec{x}^* \vec{x}}{\vec{x}^* \vec{x}} = \lambda_i$$

Now assume that $a_{ii} = \lambda_1$ or $a_{ii} = \lambda_n$ for some i. Since:

$$a_{ii} = \frac{\vec{e}_i^* A \vec{e}_i}{\vec{e}_i^* \vec{e}_i}$$

It must be the case that $\vec{e}_i \in \mathrm{Eig}_A(a_{ii})$, and so:

$$\begin{aligned} A\vec{e}_i &= a_{ii}\vec{e}_i \\ \vec{a}_i &= a_{ii}\vec{e}_i \\ a_{ij} &= \begin{cases} a_{ii}, & i = j \\ 0, & i \neq j \end{cases} \end{aligned}$$

But A is Hermitian, so $a_{ji} = 0$ as well.

Now, consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ We have $\lambda_1 = 1$ and $\lambda_n = 3$. Consider $i = 2$. $a_{ii} = 2$, which is neither λ_1 nor λ_n.

4.3.1

Let $A, B \in M_n$ be Hermitian. Show that:

$$\lambda_1(B) \leq \lambda_i(A + B) - \lambda_i(A) \leq \lambda_n(B)$$

Conclude that $|\lambda_i(A + B) - \lambda_i(A)| \leq \rho(B)$.

Start with Weyl's inequalities:

$$\lambda_{i+j-n}(A + B) \leq \lambda_i(A) + \lambda_j(B) \leq \lambda_{i+j-1}(A + B)$$

First, let j=1:

$$\lambda_{i+1-1}(A+B) = \lambda_i(A+B) \geq \lambda_i(A) + \lambda_1(B)$$

Now, for the same i, let $j = n$:

$$\lambda_{i+n-n}(A+B) = \lambda_i(A+B) \leq \lambda_i(A) + \lambda_n(B)$$

Putting these two together we have:

$$\lambda_i(A) + \lambda_1(B) \leq \lambda_i(A+B) \leq \lambda_i(A) + \lambda_n(B)$$

and finally:

$$\lambda_1(B) \leq \lambda_i(A+B) - \lambda_i(A) \leq \lambda_n(B)$$

Note that due to the assumed ordering for the eigenvalues of B:

$$\lambda_1(B) \leq \cdots \leq \lambda_n(B)$$

It is the case that:

$$\rho(B) = \max\{|\lambda_1(B)|, |\lambda_n(B)|\}$$

Case 1: $\rho(B) = |\lambda_1(B)|$

It must be the case that $\lambda_1(B) \leq 0$ and thus:

$$-|\lambda_1(B)| \leq \lambda_i(A+B) - \lambda_i(A) \leq |\lambda_n(B)| \leq |\lambda_1(B)|$$

and thus:

$$|\lambda_i(A+B) - \lambda_i(A)| \leq |\lambda_1(B)| = \rho(B)$$

Case 2: $\rho(B) = |\lambda_n(B)|$

It must be the case that $\lambda_n(B) \geq 0$ and thus:

$$-|\lambda_n(B)| \leq |\lambda_1(B)| \leq \lambda_i(A+B) - \lambda_i(A) \leq |\lambda_n(B)|$$

and thus:

$$|\lambda_i(A+B) - \lambda_i(A)| \leq |\lambda_n(B)| = \rho(B)$$

$\therefore |\lambda_i(A+B) - \lambda_i(A)| \leq \rho(B)$

4.3.3

Let $A, B \in M_n$ be Hermitian. Explain why:

$$\lambda_i(A+B) \leq \min_{j+k=i+n} \{\lambda_j(A) + \lambda_k(B)\}$$

Starting with the first part of Weyl's inequalities:

$$\lambda_{j+k-n}(A+B) \leq \lambda_j(A) + \lambda_k(B)$$

For $j+k=i+n$ we have:

$$\lambda_i(A+B) \leq \lambda_j(A) + \lambda_k(B)$$

and therefore:

$$\lambda_i(A+B) \leq \min_{j+k=i+n} \{\lambda_j(A) + \lambda_k(B)\}$$