Cavallaro, Jeffery
Math 231b
Homework #3

3.8.19

Assume $\vec{x}_n \xrightarrow{w} \vec{x}$ and $\vec{y}_n \xrightarrow{w} \vec{y}$ as $n \to \infty$ in a Hilbert space, and $\alpha_n \to \alpha$ in $\mathbb{C}$. Prove or give a counterexample:

(a) $\vec{x}_n + \vec{y}_n \xrightarrow{w} \vec{x} + \vec{y}$

TRUE

$\langle \vec{x}_n + \vec{y}_n, \vec{z} \rangle = \langle \vec{x}_n, \vec{z} \rangle + \langle \vec{y}_n, \vec{z} \rangle \to \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle = \langle \vec{x} + \vec{y}, \vec{z} \rangle$

$\therefore \vec{x}_n + \vec{y}_n \xrightarrow{w} \vec{x} + \vec{y}$

(b) $\alpha_n \vec{x}_n \xrightarrow{w} \alpha \vec{x}$

TRUE

$\langle \alpha_n \vec{x}_n, \vec{z} \rangle = \alpha_n \langle \vec{x}_n, \vec{z} \rangle \to \alpha \langle \vec{x}, \vec{z} \rangle = \langle \alpha \vec{x}, \vec{z} \rangle$

$\therefore \alpha_n \vec{x}_n \xrightarrow{w} \alpha \vec{x}$

(c) $\langle \vec{x}_n, \vec{y}_n \rangle \to \langle \vec{x}, \vec{y} \rangle$

FALSE

If this were the case, then:

$$\langle \vec{x}_n, \vec{x}_n \rangle \to \langle \vec{x}, \vec{x} \rangle$$

which is the same (sans squared) as:

$$\|\vec{x}_n\| \to \|\vec{x}\|$$

However, see (d).

(d) $\|\vec{x}_n\| \to \|\vec{x}\|$

FALSE

Consider (e_n) in ℓ^2.

Claim $e_n \xrightarrow{w} 0$

Assume $y \in \ell^2$.
Assume $\epsilon > 0$.
$\exists\, N > 0, n > N \implies |y_n| < \epsilon$

Assume $n > N$:

$$\begin{aligned} |\langle e_n, y\rangle - \langle 0, y\rangle| &= |\langle e_n, y\rangle| \\ &= \left|\sum_{k=1}^{\infty} e_{n,k}\overline{k_k}\right| \\ &= |\overline{y_k}| \\ &= |y_k| \\ &\leq \epsilon \end{aligned}$$

$\therefore (e_n) \xrightarrow{w} 0$

But $\|e_n\| = 1$ and $\|0\| = 0$.

$\therefore \|e_n\| \neq \|0\|$

(e) $(\forall\, n \in \mathbb{N}, \vec{x}_n = \vec{y}_n) \implies \vec{x} = \vec{y}$

Thus, is the weak limit unique.

TRUE

Assume $\vec{z} \in H$ such that $\vec{z} \neq \vec{0}$ and $\vec{z} \not\perp (\vec{x} - \vec{y})$.

$$\begin{aligned} \langle \vec{x} - \vec{y}, \vec{z}\rangle &= \langle (\vec{x} - \vec{x}_n) + (\vec{x}_n - \vec{y}), \vec{z}\rangle \\ &= \langle \vec{x} - \vec{x}_n, \vec{z}\rangle + \langle \vec{x}_n - \vec{y}, \vec{z}\rangle \\ &\to 0 + 0 \\ &= 0 \end{aligned}$$

And so $\vec{x} - \vec{y} = \vec{0}$.

$\therefore \vec{x} = \vec{y}$

3.8.20

Show that in a finite dimensional Hilbert space, weak convergence implies strong convergence.

Assume H is a Hilbert space over a field $\mathbb{F}$.
Assume $(\vec{x}_n)$ is a sequence in H such that $\vec{x}_n \xrightarrow{w} \vec{x} \in H$.
Assume $B = \{\vec{b}_1, \ldots, \vec{b}_N\}$ is a basis for H.
AWLOG: B is an orthonormal basis (otherwise, use Graham-Schmitt).
$\forall\, n \in \mathbb{N}, \exists\, x_{n,k} \in \mathbb{F}$ such that $\vec{x}_n = \displaystyle\sum_{k=1}^{n} x_{n,k}\vec{b}_k$.
$\exists\, x_k \in \mathbb{F}$ such that $\vec{x} = \displaystyle\sum_{k=1}^{N} x_k\vec{b}_k$.
Assume $\vec{z} \in H$.

$\exists\, z_k \in \mathbb{F}$ such that $\vec{z} = \sum_{k=1}^{N} z_k \vec{b}_k$.

Since $\vec{x}_n \xrightarrow{w} \vec{x}$:

$$\begin{aligned}
\langle \vec{x}_n - \vec{x}, \vec{z} \rangle &= \left\langle \sum_{k=1}^{N} x_{n,k}\vec{b}_k - \sum_{k=1}^{N} x_k\vec{b}_k, \sum_{k=1}^{N} z_k\vec{b}_k \right\rangle \\
&= \left\langle \sum_{k=1}^{N} (x_{n,k} - x_k)\vec{b}_k, \sum_{k=1}^{N} z_k\vec{b}_k \right\rangle \\
&= \sum_{k=1}^{N} (x_{n,k} - x_k)\overline{z_k} \\
&\to 0
\end{aligned}$$

But this is for any $\vec{z} \in E$, and so it must be the case that $x_{n,k} - x_n \to 0$.

So $\vec{x}_n$ converges to $\vec{x}$ component-wise.
It has already been shown that $\|\cdot\|_\infty$ with respect to a particular basis is a proper norm.
Now, since all norms are equivalent in a finite dimensional vector space:

$$\|\vec{x}_n - \vec{x}\|_\infty = \sup_{k \in \mathbb{N}} |x_{n,k} - x_k| \to 0$$

$\therefore \vec{x}_n \to \vec{x}$

3.8.23

In the inner product space $\mathcal{C}[-\pi, \pi]$, show that the following sequence of functions are orthogonal.

Note: $\forall\, n \in \mathbb{Z}, \sin(n\pi) = 0$.

1). $x_k(t) = \sin kt$ for $k \in \mathbb{N}$

Case 1: $j = k$

$$\begin{aligned}
\langle \sin(kt), \sin(kt) \rangle &= \int_{-\pi}^{\pi} \sin^2(kt)dt \\
&= \frac{1}{2}\int_{-\pi}^{\pi} [1 - \cos(2kt)]dt \\
&= \frac{1}{2}\left[t - \frac{1}{2k}\sin(2kt)\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}[\pi - (-\pi)] \\
&= \frac{1}{2}(2\pi) \\
&= \pi
\end{aligned}$$

Case 2: $j \neq k$

$$\begin{aligned}
\langle \sin(jt), \sin(kt) \rangle &= \int_{-\pi}^{\pi} \sin(jt)\sin(kt)dt \\
&= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(j-k)t - \cos(j+k)t]dt \\
&= \frac{1}{2}\left[\frac{1}{j-k}\sin(j-k)t - \frac{1}{j+k}\sin(j+k)t\right]_{-\pi}^{\pi} \\
&= 0
\end{aligned}$$

$\therefore \sin(jt) \perp \sin(kt)$ for $j \neq k$.

2). $y_n(t) = \cos nt$ for $n \in \mathbb{N}$

Case 1: $n = m$

$$\begin{aligned}
\langle \cos(nt), \cos(nt) \rangle &= \int_{-\pi}^{\pi} \cos^2(nt)dt \\
&= \frac{1}{2}\int_{-\pi}^{\pi} [1 + \cos(2nt)]dt \\
&= \frac{1}{2}\left[t + \frac{1}{2n}\sin(2nt)\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}[\pi - (-\pi)] \\
&= \frac{1}{2}(2\pi) \\
&= \pi
\end{aligned}$$

Case 2: $n \neq m$

$$\begin{aligned}
\langle \cos(nt), \cos(mt) \rangle &= \int_{-\pi}^{\pi} \cos(nt)\cos(mt)dt \\
&= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(n-m)t + \cos(n+m)t]dt \\
&= \frac{1}{2}\left[\frac{1}{n-m}\sin(n-m)t + \frac{1}{n+m}\sin(n+m)t\right]_{-\pi}^{\pi} \\
&= 0
\end{aligned}$$

$\therefore \cos(nt) \perp \cos(mt)$ for $n \neq m$.

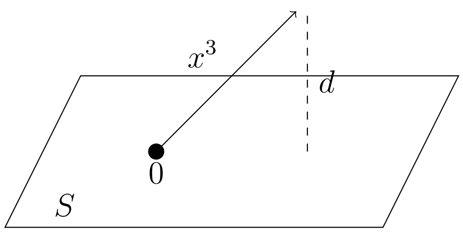
3.8.34

Find $a, b, c \in \mathbb{C}$ which minimizes the value of the integral:

$$\int_{-1}^{1} \left|x^3 - a - bx - cx^2\right|^2 dx$$

This is equivalent to minimizing the distance $\left\|x^3 - (cx^2 + bx + a)\right\|_{L_2}$, which is the distance between x^3 and the space spanned by $\{x^2, x, 1\}$. We need to be careful because this basis is not orthogonal in L_2.

Let $S = \mathrm{Span}\{x^2, x, 1\}$. The problem can be viewed as follows:



In order to minimize the distance d, the vector $x^3 - cx^2 - bx - a$ must be orthogonal to S, and in particular, orthogonal to each of the basis vectors for S:

$$\begin{aligned}
\left\langle x^3 - cx^2 - bx - a, 1\right\rangle &= \left\langle x^3, 1\right\rangle - c\left\langle x^2, 1\right\rangle - b\left\langle x, 1\right\rangle - a\left\langle 1, 1\right\rangle \\
&= \int_{-1}^{1} x^3 dx - c\int_{-1}^{1} x^2 dx - b\int_{-1}^{1} x dx - a\int_{-1}^{1} dx \\
&= 0 - \frac{2}{3}c - 0 - 2a \\
&= -\frac{2}{3}c - 2a \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\left\langle x^3 - cx^2 - bx - a, x\right\rangle &= \left\langle x^3, x\right\rangle - c\left\langle x^2, x\right\rangle - b\left\langle x, x\right\rangle - a\left\langle 1, x\right\rangle \\
&= \int_{-1}^{1} x^4 dx - c\int_{-1}^{1} x^3 dx - b\int_{-1}^{1} x^2 dx - a\int_{-1}^{1} x dx \\
&= \frac{2}{5} - 0 - \frac{2}{3}b - 0 \\
&= \frac{2}{5} - \frac{2}{3}b \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\left\langle x^3 - cx^2 - bx - a, x^2 \right\rangle &= \left\langle x^3, x^2 \right\rangle - c\left\langle x^2, x^2 \right\rangle - b\left\langle x, x^2 \right\rangle - a\left\langle 1, x^2 \right\rangle \\
&= \int_{-1}^{1} x^5 dx - c\int_{-1}^{1} x^4 dx - b\int_{-1}^{1} x^3 dx - a\int_{-1}^{1} x^2 dx \\
&= 0 - \frac{2}{5}c - 0 - \frac{2}{3}a \\
&= -\frac{2}{5}c - \frac{2}{3}a \\
&= 0
\end{aligned}
$$

This results in 3 equations in 3 unknowns:

$$
\begin{aligned}
2a + \frac{2}{3}c &= 0 \\
\frac{2}{3}b - \frac{2}{5} &= 0 \\
\frac{2}{3}a + \frac{2}{5}c &= 0
\end{aligned}
$$

or:

$$
\begin{aligned}
3a + c &= 0 \\
b &= \frac{3}{5} \\
5a + 3c &= 0
\end{aligned}
$$

which has the single solution:

$$
\begin{aligned}
a &= 0 \\
b &= \frac{3}{5} \\
c &= 0
\end{aligned}
$$