Milne-Thomson

Example

Let $f(z)$ be entire and $u(x,y) = x^4 - 6x^2y^2 + y^4$. Find $v(x,y)$.

$u_x = 4x^3 - 12xy^2 = v_y$
$v(x,y) = 4x^3y - 4xy^3 + g(x)$
$v_x = 12x^2y - 4y^3 + g'(x) = -u_y$
$u_y = -12x^2y + 4y^3$
$12x^2 - 4y^3 + g'(x) = 12x^2y - 4y^3$
$g'(x) = 0$
$g(x) = C$

$v(x,y) = 4x^3y - 4xy^3 + C$

To find $f(z)$ from $u(x,y)$ and $v(x,y)$, use:

$$x = \frac{z + \bar{z}}{2}$$

$$y = \frac{z - \bar{z}}{2i}$$

This can be very tedious. For the above example, the answer would be:

$$f(x) = z^4 + iC$$

As an alternative, use the Milne-Thompson formula:

Theorem

Let $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain D such that $z_0 = x_0 + iy_0 \in D$:

$$f(z) = 2u\left(\frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i}\right) - u(x_0, y_0) + iC, \qquad C \in \mathbb{R}$$

Proof

$u(x,y) = \frac{1}{2}\left[f(z) + \overline{f(z)}\right] = \frac{1}{2}\left[f(z) + \bar{f}(\bar{z})\right] = \frac{1}{2}\left[f(x+iy) + \bar{f}(x-iy)\right]$

Let $x = \frac{z+\overline{z_0}}{2}$ and $y = \frac{z-\overline{z_0}}{2i}$:

$$\begin{aligned} u\left(\frac{z+\overline{z_0}}{2}, \frac{z-\overline{z_0}}{2i}\right) &= \frac{1}{2}\left[f(\frac{z+\overline{z_0}}{2} + i\frac{z-\overline{z_0}}{2i}) + \bar{f}(\frac{z+\overline{z_0}}{2} - i\frac{z-\overline{z_0}}{2i})\right] \\ &= \frac{1}{2}\left[f(z) + \bar{f}(\overline{z_0})\right] \end{aligned}$$

$$
\begin{aligned}
2u\left(\frac{z+\overline{z_0}}{2}, \frac{z-\overline{z_0}}{2i}\right) &= f(z) + \bar{f}\left(\overline{z_0}\right) \\
&= f(z) + \overline{f(z_0)} \\
&= f(z) + u(x_0, y_0) - iv(x_0, y_0)
\end{aligned}
$$

Let $v(x_0, y_0) = C \in \mathbb{R}$:

$$
f(z) = 2u\left(\frac{z+\overline{z_0}}{2}, \frac{z-\overline{z_0}}{2i}\right) - u(x_0, y_0) + iC
$$

Example

Continuing with the above example with $z_0 = 0$:

$$
\begin{aligned}
f(z) &= 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + iC \\
&= 2\left[\left(\frac{z}{2}\right)^4 - 6\left(\frac{z}{2}\right)^2\left(\frac{z}{2i}\right)^2 + \left(\frac{z}{2i}\right)^4\right] - 0 + iC \\
&= 2\left(\frac{z^4}{16} + \frac{6z^4}{16} + \frac{z^4}{16}\right) + iC \\
&= 2\left(\frac{8z^4}{16}\right) + iC \\
&= z^4 + iC
\end{aligned}
$$

Theorem

Let $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain D such that $z_0 = x_0 + iy_0 \in D$:

$$
f(z) = 2iv\left(\frac{z+\bar{z}_0}{2}, \frac{z-\bar{z}_0}{2i}\right) - iv(x_0, y_0) + C, \qquad C \in \mathbb{R}
$$

Proof

Let $f(z) = u + iv$:

$$
\begin{aligned}
if(z) &= -v + iu \\
if(z) &= -2v\left(\frac{z+\overline{z_0}}{2}, \frac{z-\overline{z_0}}{2i}\right) + v(x_0, y_0) + iC \\
f(z) &= 2iv\left(\frac{z+\overline{z_0}}{2}, \frac{z-\overline{z_0}}{2i}\right) - iv(x_0, y_0) + C
\end{aligned}
$$

If $u - v$ is known then:

$$(1+i)(u+iv) = u + iv + iu - v = (u-v) + i(u+v)$$

So, perform the following steps:

1). Let $F(z) = (1+i)f(z) = (u-v) + i(u+v)$

2). Perform M-T on $F(z)$

3). Divide the result by $1+i$

Theorem

Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ be analytic in D such that $0 \in D$:

$$f(z) = u(z,0) + i(v,0)$$

Proof

$u(x,y) = \frac{1}{2}[f(z) + \overline{f(z)}] = \frac{1}{2}[f(z) + \bar{f}(\bar{z})] = \frac{1}{2}[f(x+iy) + \bar{f}(x-iy)]$
$v(x,y) = \frac{1}{2i}[f(z) - \overline{f(z)}] = \frac{1}{2i}[f(z) - \bar{f}(\bar{z})] = \frac{1}{2i}[f(x+iy) - \bar{f}(x-iy)]$

Let $x = z$ and $y = 0$

$u(z,0) = \frac{1}{2}[f(z) + \bar{f}(z)]$
$v(z,0) = \frac{1}{2i}[f(z) - \bar{f}(z)]$

$\therefore f(z) = u(z,0) + iv(z,0)$

Example

$f(z) = (x^2 - y^2) + i2xy$

$u(z,0) = z^2$
$v(z,0) = 0$

$f(z) = z^2 + i0 = z^2$

Theorem

Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ be analytic in D such that $0 \in D$:

$$f(z) = \int [u_x(z,0) - iu_y(z,0)]\,dz + C$$

Proof

$f'(z) = f_x = u_x + iv_x = u_x - iu_y$

Let $x = z$ and $y = 0$

$f'(z) = u_x(z,0) - iu_y(z,0)$
$\therefore f(z) = \int [u_x(z,0) - iu_y(z,0)]\,dz + C$

Example

$f(z) = (x^2 - y^2) + i2xy$

$u_x = 2x$
$u_y = -2y$

$u_x(z, 0) = 2z$
$u_y(z, 0) = 0$

$f(z) = \int 2z dz = z^2 + C$