Isometry Group of a Vector Norm

Definition: Isometry Group

Given a vector norm $\|\cdot\|$ on $\mathbb{C}^n$, the *isometry* group associated with the norm, denoted $G_{\|\cdot\|}$, is given by:

$$G_{\|\cdot\|} = \{A \in M_n \mid \forall\, \vec{x} \in \mathbb{C}^n, \|A\vec{x}\| = \|\vec{x}\|\}$$

Note that $G_{\|\cdot\|} \neq \emptyset$ because $0 \in G_{\|\cdot\|}$.

Lemma

Let $A \in G_{\|\cdot\|}$ and $\lambda \in \sigma(A)$:

$$|\lambda| = 1$$

Proof

$\|A\vec{x}\| = \|\lambda\vec{x}\| = |\lambda|\,\|\vec{x}\|$

$\therefore |\lambda| = 1$

Theorem

$G_{\|\cdot\|} \leq GL(n)$

Proof

Assume $A \in G_{\|\cdot\|}$
$\lambda \in \sigma(A) \implies |\lambda| = 1$
So all $\lambda \neq 0$
Thus A is invertible
$A \in GL(n)$

$\therefore G_{\|\cdot\|} \subseteq GL(n)$

Assume $B \in G_{\|\cdot\|}$
Assume $x \in \mathbb{C}^n$
$B\vec{x} \in \mathbb{C}^n$
$\|AB\vec{x}\| = \|A(B\vec{x})\| = \|B\vec{x}\| = \|\vec{x}\|$

Therefore $G_{\|\cdot\|}$ is closed under the operation (composition).

Since A is invertible, A^{-1} exists
$A^{-1}\vec{x} \in \mathbb{C}^n$
$\|A^{-1}\vec{x}\| = \|A(A^{-1}\vec{x})\| = \|(AA^{-1})\vec{x}\| = \|I_n\vec{x}\| = \|\vec{x}\|$
$A^{-1} \in G_{\|\cdot\|}$

Therefore $G_{\|\cdot\|}$ is closed under inverses.

$\therefore G_{\|\cdot\|} \leq GL(n)$.

Theorem

Let $G_{\|\cdot\|}$ be the unitary group. There exists $\alpha \in \mathbb{R}$ such that $\alpha > 0$ and:

$$\forall \vec{x} \in \mathbb{C}^n, \|\vec{x}\| = \alpha \|\vec{x}\|_2$$

In other words, for all $\vec{x} \in \mathbb{C}^n$, the norm is a positive scalar multiple of the ℓ_2 norm.

Proof

Assume $\vec{x} \in \mathbb{C}^n$
Consider the unit vector $\hat{y}_1 = \frac{\vec{x}}{\|\vec{x}\|_2}$
Using G-S, construct $n-1$ additional orthogonal unit vectors $\{\hat{y}_2, \ldots, \hat{y}_n\}$
Form the matrix:

$$U = \begin{bmatrix} \hat{y}_1 & \hat{y}_2 & \ldots & \hat{y}_n \end{bmatrix}$$

Since the columns of U are orthonormal, U is a unitary matrix
Now, since the norm is unitary invariant:

$$\|\vec{e}_1\| = \|U\vec{e}_1\| = \|\hat{y}_1\| = \left\| \frac{\vec{x}}{\|\vec{x}\|_2} \right\| = \frac{1}{\|\vec{x}\|_2} \|\vec{x}\|$$

Thus, $\|\vec{x}\| = \|\vec{x}\|_2 \|\vec{e}_1\|$
But $\|\vec{e}_1\| > 0$, so let $\alpha = \|\vec{e}_1\|$

$\therefore \|\vec{x}\| = \alpha \|\vec{x}\|_2$