Cauchy-Schwarz Inequality

Theorem

$\forall\, a, b \in \mathbb{C}$:

$$\left|\sum a_k b_k\right|^2 \leq \left(\sum |a_k|^2\right)\left(\sum |b_k|^2\right)$$

Proof

If $\forall\, a_k = 0$ or $\forall\, b_k = 0$ then trivial, so
AWLOG: $\exists\, a_k \neq 0$ and $\exists\, b_k \neq 0$
Assume $c \in \mathbb{C}$

$$\begin{aligned}
\sum \left|a_k - c\overline{b_k}\right|^2 &= \sum (a_k - c\overline{b_k})(\overline{a_k - c\overline{b_k}}) \\
&= \sum (a_k - c\overline{b_k})(\overline{a_k} - \overline{c} b_k) \\
&= \sum \left(a_k\overline{a_k} + c\overline{b_k}\overline{c} b_k - a_k \overline{c} b_k - c\overline{b_k}\overline{a_k}\right) \\
&= \sum \left[|a_k|^2 + |c|^2 |b_k|^2 - (a_k b_k \overline{c} + \overline{a_k b_c \overline{c}})\right] \\
&= \sum \left[|a_k|^2 + |c|^2 |b_k|^2 - 2Re(a_k b_k \overline{c})\right] \\
&= \sum |a_k|^2 + |c|^2 \sum |b_k|^2 - 2Re\left(\overline{c} \sum a_k b_k\right) \\
&\geq 0
\end{aligned}$$

Let $c = \frac{\sum a_k b_k}{\sum |b_k|^2}$

$$\sum |a_k|^2 + |c|^2 \sum |b_k|^2 - 2Re\left(\overline{c}\sum a_k b_k\right) \geq 0$$

$$\sum |a_k|^2 + \left|\frac{\sum a_k b_k}{\sum |b_k|^2}\right|^2 \sum |b_k|^2 - 2Re\left(\frac{\overline{\sum a_k b_k}}{\sum |b_k|^2}\sum a_k b_k\right) \geq 0$$

$$\sum |a_k|^2 + \frac{|\sum a_k b_k|^2}{\sum |b_k|^2} - 2Re\left(\frac{|\sum a_k b_k|^2}{\sum |b_k|^2}\right) \geq 0$$

$$\sum |a_k|^2 + \frac{|\sum a_k b_k|^2}{\sum |b_k|^2} - 2\frac{|\sum a_k b_k|^2}{\sum |b_k|^2} \geq 0$$

$$\sum |a_k|^2 - \frac{|\sum a_k b_k|^2}{\sum |b_k|^2} \geq 0$$

$$\left(\sum |a_k|^2\right)\left(\sum |b_k|^2\right) - \left|\sum a_k b_k\right|^2 \geq 0$$

$$\left|\sum a_k b_k\right|^2 \leq \left(\sum |a_k|^2\right)\left(\sum |b_k|^2\right)$$