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Math 229
Homework #2

2.1.2

Let $U \in M_n$ be unitary and let λ be a given eigenvalue of U.

a) Show: $|\lambda| = 1$

   There exists eigenvector $\vec{x} \neq 0$ associated with λ such that:

   $$U\vec{x} = \lambda\vec{x}$$

   Since U preserves length and because $\|\vec{x}\| \neq 0$:

   $$\begin{aligned} \|U\vec{x}\| &= \|\lambda\vec{x}\| \\ \|\vec{x}\| &= |\lambda| \, \|\vec{x}\| \\ |\lambda| &= 1 \end{aligned}$$

b) Prove: $\vec{x}$ is a right eigenvector of U associated with λ iff $\vec{x}$ is a left eigenvector of U associated with λ.

   $$\begin{aligned} U\vec{x} = \lambda\vec{x} \quad &\iff \quad \vec{x} = U^*\lambda\vec{x} \\ &\iff \quad \overline{\lambda}\vec{x} = |\lambda|^2 U^*\vec{x} = U^*\vec{x} \\ &\iff \quad \vec{x}^*U = \lambda\vec{x}^* \end{aligned}$$

2.3.6

Let $A, B \in M_n$ be given and suppose A and B are simultaneously similar to upper triangular matrices - there exists nonsingular $S \in M_n$ such that:

$$SAS^{-1} = T_1 \in UT(n)$$

$$SBS^{-1} = T_2 \in UT(n)$$

Show that every eigenvalue of $AB - BA$ must be 0.

Lemma

Let $A, B \in UT(n)$:

1). $AB \in UT(n)$

2). $(AB)_{ii} = A_{ii}B_{ii}$

Proof

$(AB)_{ij} = \sum_{k=0}^{n} A_{ik}B_{kj}$

Assume $i > j$
if $k < i$ then $A_{ik} = 0$
if $k > i$ then $k > j$ and $B_{kj} = 0$
Therefore, $(AB)_{ij} = 0$ and $AB \in UT(n)$

Now, assume $i = j$
$(AB)_{ii} = \sum_{k=0}^{n} A_{ik}B_{ki}$
if $k < i$ then $A_{ik} = 0$
if $k > i$ then $B_{ki} = 0$
Therefore, $(AB)_{ii} = A_{ii}B_{ii}$

Now back to original proof:

$A = S^{-1}T_1S$ and $B = S^{-1}T_2S$
$AB = (S^{-1}T_1S)(S^{-1}T_2S) = S^{-1}T_1T_2S$
$BA = (S^{-1}T_2S)(S^{-1}T_1S) = S^{-1}T_2T_1S$
$AB - BA = S^{-1}T_1T_2S - S^{-1}T_2T_1S = S^{-1}(T_1T_2 - T_2T_1)S$

But $T_1T_2 - T_2T_1 \in UT(n)$, and is thus a Schur triangularization of $AB - BA$. Furthermore:

$$(T_1T_2 - T_2T_1)_{ii} = (T_1)_{ii}(T_2)_{ii} - (T_2)_{ii}(T_1)_{ii} = 0$$

Thus, the Schur triangularization of $AB - BA$ has all zeros on its diagonal

Therefore, all of the eigenvalues of $AB - BA$ are 0.

2.4.13

Let $A \in M_n$ and $B \in M_m$. Prove: $\forall C \in M_{n,m}$ there exists a unique solution $X \in M_{n,m}$ to the equation $AX - XB = C$ iff $\sigma(A) \cap \sigma(B) = \emptyset$. Moreover, if $C = 0$ then $X = 0$.

Consider the linear transformations $T_1, T_2 : M_{n,m} \to M_{n,m}$ defined by:

$$\begin{aligned} T_1(X) &= AX \\ T_2(X) &= XB \end{aligned}$$

Let $T = T_1 - T_2$ be the linear transformation corresponding to $AX - XB$.

$\implies$ Assume $AX - XB = C$, and hence $T(X) = C$, has a unique solution for every $C \in M_{n,m}$

Thus T is both one-to-one (unique solution) and onto (all $C \in M_{n,m}$), and so T is a bijection. This means that T is invertible and by the IMT, $0 \notin \sigma(T)$.

Let $\vec{x}$ be an eigenvector of A (T_1) with respect to eigenvalue λ and let $\vec{y}$ be a left eigenvector of B (T_2) with respect to eigenvalue μ. Also, let $X = xy^*$:

$$\begin{aligned}
T(X) &= T(xy^*) \\
&= (T_1 - T_2)(xy^*) \\
&= T_1(xy^*) - T_2(xy^*) \\
&= Axy^* - xy^*B \\
&= \lambda xy^* - x\mu y^* \\
&= \lambda xy^* - \mu xy^* \\
&= (\lambda - \mu)xy^* \\
&= (\lambda - \mu)X
\end{aligned}$$

And so all of the eigenvalues of T are differences of eigenvalues of T_1 and T_2. But $0 \notin \sigma(T) \implies \lambda \neq \mu$.

$\therefore \sigma(A) \cap \sigma(B) = \emptyset$.

$\Longleftarrow$ Assume $\sigma(A) \cap \sigma(B) = \emptyset$

Assume $X \in M_{n,m}$
$(T_1T_2)(X) = T_1(T_2(X)) = T_1(XB) = AXB$
$(T_2T_1)(X) = T_2(T_1(X)) = T_2(AX) = AXB$
Thus, T_1 and T_2 commute, and so $\sigma(T) \subseteq \sigma(T_1) - \sigma(T_2)$.
In other words all eigenvalues of T can be computed as differences of the eigenvalues of T_1 and T_2.

Now, $\lambda \in \sigma(T_1)$ iff there exists $X \in M_{m,n}$ such that $X \neq 0$ and $T_1(X) = \lambda X$. But this is true iff $AX = \lambda X$, which means that for every non-zero column of X, $\vec{x}_i \in \mathrm{Eig}_A(\lambda)$. Thus, $\mathrm{Sp}(A) = \mathrm{Sp}(T_1)$, and by similar argument, $\mathrm{Sp}(B) = \mathrm{Sp}(T_2)$.

Since A and B, and hence T_1 and T_2, have no eigenvalues in common, $0 \notin \sigma(T)$ and thus, by the IMT, T is invertible, and thus a bijection - both one-to-one and onto.

Therefore, $T(X) = C$, and hence the equation $AX - XB = C$, has a unique solution (one-to-one) for every $C \in M_{n,m}$ (onto). Moreover, since T is one-to-one the null space is trivial and therefore $AX - XB = 0 \implies X = 0$.

2.5.6

Let $A \in M_n$. Prove: A is normal iff A commutes with some normal matrix with distinct eigenvalues.

$\Longrightarrow$ Assume A is normal

A is unitary diagonalizable, so let $A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*$ for some unitary U.

Let $B = U \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & n \end{bmatrix} U^*$

Note that B is diagonalizable, and hence normal, and has distinct eigenvalues $\{1, \ldots, n\}$.

$$\begin{aligned}
AB &= U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*U \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & n \end{bmatrix} U^* \\
&= U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & n \end{bmatrix} U^* \\
&= U \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^* \\
&= U \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & n \end{bmatrix} U^*U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^* \\
&= BA
\end{aligned}$$

$\Longleftarrow$ Assume A commutes with some normal matrix with distinct eigenvalues.

Lemma

Let $A, B \in UT(n)$ such that B is diagonal with distinct eigenvalues:

$$AB = BA \implies A \text{ is diagonal}$$

Proof

Assume $AB = BA$

Proof by induction on n:

Base Case: $n = 1$

Nothing to prove.

Assume $A \in UT(n-1)$ is diagonal.

Consider $A \in UT(n)$

Let $A = \left[\begin{array}{c|c} S & \vec{x} \\ \hline 0 & a \end{array}\right]$, where $S \in UT(n-1)$, $\vec{x} \in \mathbb{C}^{n-1}$ and $a \in \mathbb{C}$.

Let $B = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & \lambda_n \end{array}\right]$, where $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n-1} \end{bmatrix}$ and the λ_k are distinct.

$AB = \left[\begin{array}{c|c} SD & \lambda_n \vec{x} \\ \hline 0 & \lambda_n a \end{array}\right]$ and $BA = \left[\begin{array}{c|c} DS & \lambda_1 \vec{x} \\ \hline 0 & \lambda_n a \end{array}\right]$

The upper-left quadrant tells us that $SD = DS$, so by the inductive assumption, we can conclude that S is diagonal.

Moreover, the upper-right quadrant tells us that $\lambda_1 \vec{x} = \lambda_n \vec{x}$, and since the $\lambda_1 \neq \lambda_n$, it must be the case that $\vec{x} = 0$.

Therefore, A is diagonal.

Now, back to the original question. Let B be the normal matrix with distinct eigenvalues with which A commutes. Since A and B commute, they are simultaneously triangularizable, so let:

$A = UTU^*$ and $B = UDU^*$ for $T, D \in UT(n)$ and D diagonal.

$$\begin{aligned} AB &= BA \\ UTU^*UDU^* &= UDU^*UTU^* \\ UTDU^* &= UDTU^* \\ TD = DT & \end{aligned}$$

And so by the lemma, T is also diagonal, and so A is unitary diagonalizable.

Therefore A is normal.

2.6.15

Let $A = [a_{ij}] \in M_n$ have eigenvalues $\lambda_1, \ldots, \lambda_n$ ordered so that $|\lambda_1| \geq \cdots \geq |\lambda_n|$ and singular values $\sigma_1, \ldots, \sigma_n$ ordered so that $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$.

a) Prove:

$$\sum_{i,j=1}^{n} |a_{ij}|^2 = \operatorname{tr}(A^*A) = \sum_{k=1}^{n} \sigma_k^2$$

$A = [a_{ij}]$
$A^* = [\overline{a_{ji}}]$
$(A^*A)_{ij} = \sum_{k=1}^n (A^*)_{ik} A_{kj} = \sum_{k=1}^n \overline{a_{ki}} a_{kj}$
$(A^*A)_{ii} = \sum_{k=1}^n \overline{a_{ki}} a_{ki} = \sum_{k=1}^n |a_{ki}|^2$
$\therefore \operatorname{tr}(A^*A) = \sum_{i=1}^n \sum_{k=1}^n |a_{ki}|^2 = \sum_{i,j=1}^n |a_{ij}|^2$

Let the SVD for A be:

$$A = U \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} V$$

for some unitary matrices U and V

$$\begin{aligned}
\operatorname{tr}(A^*A) &= \operatorname{tr}\left(\left(U \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} V\right)^* \left(U \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} V\right)\right) \\
&= \operatorname{tr}\left(V^* \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}^* U^*U \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} V\right) \\
&= \operatorname{tr}\left(V^* \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} V\right) \\
&= \operatorname{tr}\left(V^* \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} V\right) \\
&= \operatorname{tr}\left(VV^* \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix}\right) \\
&= \operatorname{tr}\left(\begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix}\right) \\
&= \sum_{k=1}^{n} \sigma_k^2
\end{aligned}$$

b) Prove: $\sum_{k=1}^{n} |\lambda_1|^2 \leq \sum_{k=1}^{n} \sigma_k^2$ with equality iff A is normal.

By Schur triangularization, there existslet $A = U \begin{bmatrix} \lambda_1 & & t_{ij} \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*$ for some unitary U, and so:

$$\operatorname{tr}(A^*A) = \sum_{1 \leq i,j \leq n} |a_{ij}|^2 = \sum_{k=1}^{n} |\lambda_k|^k + \sum_{i<j} |t_{ij}|^2$$

But from the last problem:

$$tr(A^*A) = \sum_{k=1}^{n} \sigma_k^2$$

and so:

$$\sum_{k=1}^{n} |\lambda_k|^k + \sum_{i<j} |t_{ij}|^2 = \sum_{k=1}^{n} \sigma_k^2$$

But $\sum_{i<j} |t_{ij}|^2 \geq 0$, with equality only when A is normal and thus unitary diagonalizable

Therefore $\sum_{k=1}^{n} |\lambda_k|^k \leq \sum_{k=1}^{n} \sigma_k^2$ with equality only when A is normal.

c) Prove: $\sigma_k = |\lambda_k| \iff A$ is normal.

$\implies$ Assume $\sigma_k = |\lambda_k|$

$$A = U \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} V \text{ for some unitary } U \text{ and } V:$$

$$\begin{aligned}
A^*A &= \left(U \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} V \right)^* U \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} V \\
&= V^* \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}^* U^*U \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} V \\
&= V^* \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} V \\
&= V^* \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} V \\
&= V^* \begin{bmatrix} |\lambda_1|^2 & & 0 \\ & \ddots & \\ 0 & & |\lambda_n|^2 \end{bmatrix} V \\
&= V^* \begin{bmatrix} \overline{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} V \\
&= V^* \begin{bmatrix} \overline{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda_n} \end{bmatrix} VV^* \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} V \\
&= \left(V \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} V \right)^* V^* \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} V
\end{aligned}$$

And so $A = V^* \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} V$ and hence is unitary diagonalizable.

Therefore A is normal.

$\Longleftarrow$ Assume A is normal.

$A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ for some unitary U.

$A^*A = U \begin{bmatrix} |\lambda_1|^2 & & 0 \\ & \ddots & \\ 0 & & |\lambda_n|^2 \end{bmatrix} U^*$

But also, $A = V \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} W$ for some unitary V and W.

$A^*A = W^* \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} W^*$

But diagonalizations are unique up to permutation, and since the λ_k and σ_k are properly ordered, it must be the case that $U = W^*$ and $|\lambda_k|^2 = \sigma_k^2$.

$\therefore \sigma_k = |\lambda_k|$

d) Prove: $|a_{ii}| = \sigma_i \implies A$ is diagonal.

Assume $|a_{ii}| = \sigma_i$

$$tr(A^*A) = \sum_{1 \le i,j \le n} |a_{ij}|^2 = \sum_{i=1}^{n} |a_{ii}|^2 + \sum_{i \ne j} |a_{ij}|^2$$

But also:

$$tr(A^*A) = \sum_{i=1}^{n} \sigma_i^2$$

And so:

$$\begin{aligned} \sum_{i=1}^{n} |a_{ii}|^2 + \sum_{i \ne j} |a_{ij}|^2 &= \sum_{i=1}^{n} \sigma_i^2 \\ \sum_{i=1}^{n} \sigma_i^2 + \sum_{i \ne j} |a_{ij}|^2 &= \sum_{i=1}^{n} \sigma_i^2 \end{aligned}$$

$$\sum_{i \neq j} |a_{ij}|^2 \quad = \quad 0$$

And thus $a_{ij} = 0$ for $i \neq j$.

Therefore, A is diagonal.

e) Prove: A is normal and $|a_{ii}| = |\lambda_i| \implies A$ is diagonal.

Assume A is normal and $|a_{ii}| = |\lambda_i|$
Since A is normal, $|\lambda_i| = \sigma_k$ and so $|a_{ii}| = \sigma_i$

Therefore A is diagonal.