Math-1005a Homework #6

Exponents: Multiplying/Dividing Common Bases

Problems

1). An exponential expression is an expression of the form a^b where a is called the base and b is called the exponent. We will begin our examination of exponential expressions by assuming that the base and the exponent are positive integers greater than 1. Recall that multiplication is a shorthand for repeated addition. For example:

$$2 \cdot 3 = 2 + 2 + 2 = 6$$

Similarly, exponentiation is a shorthand for repeated multiplication:

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$

Evaluate each of the following:

$2^2 =$	$2^3 =$	$2^4 =$
$3^2 =$	$3^3 =$	$3^4 =$
$4^2 =$	$4^3 =$	$4^4 =$
$5^2 =$	$5^3 =$	$5^4 =$

2). Now let's look at some of the cases where the either the exponent or base are either 0 or 1:

- We will not consider the case where the base and the exponent are both zero: 0^0.
- Any non-zero value a to the zero power is always 1: $a^0 = 1$.
- Any value a to the first power is just a and we omit the 1: $a^1 = a$.
- Zero to any non-zero value a is always zero: $0^a = 0$.
- One to any value a is always one: $1^a = 1$.

Evaluate the following exponential expressions:

$1^0 =$	$1^1 =$	$0^1 =$	0^{100}
$5^1 =$	$0^5 =$	$5^0 =$	1^5
$0^{15} =$	$15^0 =$	$1^{15} =$	15^1

3). When the base in an exponential expression is negative, we still have $a^0 = 1$ and $a^1 = a$, but when the exponent is an integer greater than or equal to 2 then the sign of the result is dependent on whether the exponent is even or odd.

Consider an example with an even power:

$$(2)^4 = (2)(2)(2)(2) = 16$$

Since we are multiplying an even number of negative values the result is positive value.

Now consider an example with an odd power:

$$(2)^5 = (2)(2)(2)(2)(2) = 32$$

Since we are multiplying an odd number of negative values the result is negative value.

The parentheses in the above are important! In general:

$$(-a)^n \neq -a^n$$

This is because the exponent on the RHS is more binding (i.e., has higher precedence) than the minus sign. The expression on the LHS overrides this precedence. For example:

$$(-2)^4 = 16$$

but:

$$-2^4 = -(2^4) = -16$$

Evaluate the following expressions:

a). $(-3)^2 =$

b). $(-3)^3 =$

c). $(-3)^0 =$

d). $(-3)^1 =$

e). $-3^0 =$

f). $-3^1 =$

g). $-3^2 =$

h). $-3^3 =$

4). When we multiply exponential expressions with common bases, we add their exponents:

$$a^n a^m = a^{n+m}$$

You can think of this as:

$$(a \cdot a \cdot a \cdots a)(a \cdot a \cdot a \cdots a)$$

where the first parentheses contain n a's and the second parentheses contain m a's for a total of $n + m$ a's. For example:

$$2^2 \cdot 2^3 = 2^{2+3} = 2^5 = 32$$

Note these special cases:

- $a^n a^0 = a^{n+0} = a^n$
- $a^n a = a^n a^1 = a^{n+1}$

If there is more than two factors with a common base then combine all their exponents. For example:

$$2^2 \cdot 2^5 \cdot 2^3 \cdot 2^2 = 2^{2+5+3+2} = 2^{12}$$

Simplify the following expressions. Leave your answers in exponent form:

a). $3^2 \cdot 3^3 =$

b). $7^3 \cdot 7^5 =$

c). $11 \cdot 11^5 =$

d). $0^2 \cdot 0^3 =$

e). $1^2 \cdot 1^3 =$

f). $5^2 \cdot 5^4 \cdot 5^3 =$

g). $13^{10} \cdot 13^{10} \cdot 13^5 =$

h). $2^3 \cdot 2 \cdot 2^0 \cdot 2^2 =$

5). Sometimes there are other factors between the factors with common bases. For example:

$$2^2 \cdot 3^2 \cdot 2^3$$

But remember, multiplication can be done in any order so we are free to rearrange the factors and then combine exponents:

$$2^2 \cdot 3^2 \cdot 2^3 = 2^2 \cdot 2^3 \cdot 3^2 = 2^{2+3} \cdot 3^2 = 2^5 \cdot 3^2$$

Simplify the following expressions. Leave your answers in exponent form:

a). $5^3 \cdot 7^2 \cdot 5^2 =$

b). $11^2 \cdot 13 \cdot 11^5 =$

c). $3 \cdot 5^2 \cdot 3 =$

d). $7^4 \cdot 17 \cdot 7^3$

e). $2^2 \cdot 3^2 \cdot 2^5 \cdot 3 \cdot 2 \cdot 5^3 =$

6). The next exponent rule is as follows:

$$(a^n)^m = a^{nm}$$

In this case, we multiply the exponents. You can think of this as:

$$a^n \cdot a^n \cdot a^n \cdots a^n$$

a total of n times, where each a^n contains n a's, for a total of nm a's. For example:

$$(2^3)^2 = 2^{3 \cdot 2} = 2^6$$

Make sure that you can distinguish between these two rules:

$$a^n a^m = a^{n+m}$$

$$(a^n)^m = a^{nm}$$

Mixing up these two rules results in lots of algebra errors!

Simplify the following expressions. Leave your answers in exponent form:

a). $(3^4)^2 =$

b). $(5^2)^3 =$

c). $(7^1)^2 =$

d). $(2^0)^1 =$

e). $(11^1)^0 =$

f). $(10^0)^0 =$

g). $(0^2)^3 =$

h). $(1^2)^3 =$

7). Now let's look at the exponent rules for different bases. The first rule is as follows:

$$(ab)^n = a^n b^n$$

In other words, the exponent needs to be applied to *every* factor inside the parens. You can think of this as:

$$(ab)^n = (ab)(ab)(ab) \cdots (ab)$$

n times. Then, since multiplication can be done in any order, we group all n a'a and all n b's:

$$(ab)^n = (a \cdot a \cdot a \cdots a)(b \cdot b \cdot b \cdots b) = a^n b^n$$

For example:

$$(2 \cdot 3)^2 = 2^2 \cdot 3^2$$

and:

$$(2 \cdot 3 \cdot 5)^2 = 2^2 \cdot 3^2 \cdot 5^2$$

Note that $(ab)^2$ is *very* different from $(a+b)^2$. Many student mix up this rule and try to say $(a+b)^2 = a^2 + b^2$; however, this is very wrong — you *cannot* distribute an exponent across addition! But you can distribute it across multiplication.

Simplify the following expressions. Leave your answers in exponent form:

a). $(5 \cdot 7)^2 =$

b). $(5 \cdot (-7))^2 =$

c). $(3 \cdot 11)^5 =$

d). $((-3) \cdot 11)^5 =$

e). $(11 \cdot 13)^0 =$

f). $(2 \cdot 17)^1 =$

g). $(5 \cdot 0)^2 =$

h). $(11 \cdot 19 \cdot 2)^3 =$

i). $(11 \cdot (-19) \cdot 2)^2 =$

j). $(11 \cdot (-19) \cdot 2)^3 =$

8). Sometimes you might have two factors that look like they have the same base, but one is negative and one is positive. If you remember that:

$$(-a) = (-1)a$$

then you can combine the previous rules to simplify. For example:

$$(-2)^2 \cdot 2^3 = ((-1) \cdot 2)^2 \cdot 2^3 = (-1)^2 \cdot 2^2 \cdot 2^3 = 1 \cdot 2^{2+3} = 2^5$$

and:

$$(-2)^3 \cdot 2^3 = ((-1) \cdot 2)^3 \cdot 2^3 = (-1)^3 \cdot 2^2 \cdot 2^3 = (-1) \cdot 2^{2+3} = -2^5$$

Note that the evenness or oddness of the exponent will make a difference in the final sign.

Simplify the following expressions. Leave your answers in exponent form:

a). $((-5) \cdot 7)^5 =$

b). $(5 \cdot (-7))^5 =$

c). $((-5) \cdot 7)^4 =$

d). $(5 \cdot (-7))^4 =$

e). $((-5) \cdot (-7))^3 =$

f). $((-5) \cdot (-7))^2 =$

g). $((-5) \cdot 7)^0 =$

h). $(5 \cdot (-7))^1 =$

9). Many problems require you to apply multiple rules, one at a time. One common pattern that you should know how to handle is something like this:

$$(2 \cdot 3^2)^4$$

Note that one of the factors is an exponential expression itself. So the exponent of 4 needs to be applied to both 2 and 3^2 first:

$$(2 \cdot 3^2)^4 = 2^4 \cdot (3^2)^4 = 2^4 \cdot 3^8$$

Simplify the following expressions. Leave your answers in exponent form:

a). $(2^2 \cdot 3 \cdot 5^3)^2 =$

b). $(2^2 \cdot (-3) \cdot 5^3)^2 =$

c). $(2^2 \cdot (-3) \cdot 5^3)^3 =$

d). $(2^2 \cdot 7^3)^2(5^3 \cdot 7)^2 =$

e). $((2^2 \cdot (-11))^2((-2) \cdot 5)^3)^3 =$

10). The final two rules deal with division. The first rule is for a common base:

$$\frac{a^n}{a^m} = a^{n-m}$$

For now, we will assume that $n \geq m$; we will deal with $n < m$ in the next lesson. You can think of this a n a's in the numerator and m a's in the denominator, so the m a's below will all cancel, leaving $n - m$ a's on top. For example:

$$\frac{a^3}{a^2} = a^{3-1} = a^1 = a$$

Note that the technique of *cancelling* factors in the numerator and denominator are just a consequence of this above rule:

$$\frac{a^n}{a^n} = a^{n-n} = a^0 = 1$$

Simplify the following expressions. Leave your answers in exponent form:

a). $\frac{3^4}{3^2} =$

b). $\frac{5^3}{5^3} =$

c). $\frac{7^1}{5^1} =$

d). $\frac{2^2}{2^0} =$

e). $\frac{3^2}{3} =$

f). $\frac{11}{11} =$

g). $\frac{13}{13^0} =$

h). $\frac{(-3)^4}{3^2} =$

i). $\frac{(-3)^3}{3^2} =$

j). $\frac{5^9}{(-5)^5} =$

11). The next and final rule is for different bases:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Simplify the following expressions. Leave your answers in exponent form:

a). $\left(\frac{3}{2}\right)^3$

b). $\left(\frac{3^2}{2}\right)^3$

c). $\left(\frac{3^2}{2^4}\right)^3$

d). $\left(\frac{(-3)}{2^4}\right)^2$

e). $\left(\frac{(-3)}{2^4}\right)^3$

12). When composite numbers are involved, determine their prime factorizations first — this may result in some unexpected cancelling.

Simplify the following expression by putting everything in prime factored form first. Leave your answers in exponent form:

$$\frac{81 \cdot 12}{2 \cdot 30}$$

13). Simplify the following expression. Leave the answer in exponent form:

$$-4((-5) \cdot 2^6)^2 =$$

14). Simplify the following expression. Leave the answer in exponent form:

$$\frac{2 \cdot (-3)^2(-2)^3 \cdot 5}{2^2 \cdot 3} =$$