Third Ring Isomorphism Theorem

Definition

Let R be a ring and $I, J \trianglelefteq R$:

$$I + J = \{i + j \mid i \in I \text{ and } j \in J\}$$

Theorem

Let R be a ring and $I, J \trianglelefteq R$:

$$I + J \trianglelefteq R$$

In fact, $I + J$ is the smallest ideal in R containing I and J.

Proof

From group theory, we know that $I + J = I \vee J$ (join) when either subgroup is normal in R. But since R is an additive abelian group, all subgroups are normal. Therefore $I + J$ is an additive abelian subgroup of R.

Now, assume $a \in I + J$
By definition, there exists $i \in I$ and $j \in J$ such that $a = i + j$
Assume $b \in R$
$ab = (i + j)b = ib + jb$
But $ib \in I$ and $jb \in J$
Thus, $ab \in I + J$

Likewise, $ba = b(i + j) = bi + bj$
But $bi \in I$ and $bj \in J$
Thus, $ba \in I + J$ Thus $ba \in I + J$

Therefore, by the ideal test, $I + J$ is an ideal in R.

It has already been proven that any intersection of ideals of R is also an ideal of R.

Definition

Let R be a ring and $S \subseteq R$. The ideal:

$$\bigcap_{\substack{I \trianglelefteq R \\ S \subseteq I}} I$$

is called the ideal generated by S and is the smallest ideal of R containing S.

When $S = \{r\}$ for some $r \in R$ then the ideal generated by r, denoted (r), is called a *principal* ideal.

Properties: Principle ideals

Let R be a ring and $r_k \in R$:

1). If R is commutative then $(r) = \{r\alpha \mid \alpha \in R\}$

2). $(r_1, \dots, r_n) = (r_1) + \cdots + (r_n)$

Theorem: Third Ring Isomorphism Theorem

Let R be a ring and $I, J \trianglelefteq R$:

$$(I + J)/I \simeq J/(I \cap J)$$

Proof

From the previous theorem: $I + J \trianglelefteq R$
But $J \trianglelefteq R$ and $j \subseteq I + J$, so $J \trianglelefteq I + J$
Thus $(I + J)/J$ is a factor ring

Now, consider $\phi : I \to (I + J)/J$ defined by $\phi(i) = i + J$.

Assume $i, i' \in I$
$\phi(i + i') = (i + i') + J = (i + J) + (i' + J) = \phi(i) + \phi(i')$
$\phi(ii') = (ii') + J = (i + J)(i' + J) = \phi(i)\phi(i')$

Therefore ϕ is a ring homomorphism.

Now, assume $a \in (I + J)/J$
There exists $b \in (I + J)$ such that $a = b + J$
But, there exists $i \in I$ and $j \in J$ such that $b = i + j$
So, $a = (i + j) + J$
Now, since J is the additive identity for $(I + J)/J$:
$\phi(i) = i + J = (i + j) + J$
And since $j \in J$:
$\phi(i) = (i + J) + (j + J) = (i + j) + J$

Therefore, ϕ is surjective.

Now, consider $i \in I$ such that $\phi(i) = i + J = J$
This means that $i \in J$ as well, so $\ker(\phi) = I \cap J$

Therefore, by the first fundamental ring theorem:

$$I/(I \cap J) \simeq (I + J)/J$$