Vector Spaces

Definition: Vector Space

A *vector space* $V(V, +, \cdot, \mathbb{F})$ is a set of objects V called *vectors* and a field $\mathbb{F}$ of *scalars* with the well-defined operations of *vector addition* and *scalar multiplication* such that the following ten axioms hold: $\forall, \vec{u}, \vec{v} \in V$ and $\forall\, a, b \in \mathbb{F}$:

1). $\vec{u} + \vec{v} \in V$

2). $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

3). $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

4). $\exists\, \vec{0} \in V, \vec{u} + \vec{0} = \vec{u}$

5). $\exists\, (-\vec{u}) \in V, \vec{u} + (-\vec{u}) = \vec{0}$

6). $a\vec{u} \in V$

7). $1\vec{u} = \vec{u}$

8). $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

9). $(a + b)\vec{u} = a\vec{u} + b\vec{u}$

10). $(ab)\vec{u} = a(b\vec{u})$

Example

The set $\mathbb{F}^n$ of column vectors form a vector space under the operations of component-wise addition and scalar multiplication:

$$\mathbb{F}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mid x_k \in \mathbb{F} \right\}$$

Definition

The vector space $\{\vec{0}\}$ is called the *zero vector space*.

Note that vector spaces are never empty because they must contain at least the zero vector.

Subspaces

Definition: Subspace

Let V be a vector space and $S \subseteq V$. To say that S is a *subspace* of V means that S is also a vector space using the same scalar field and the same operations as V.

$\{\vec{0}\}$ and V are called the *trivial* subspaces of V. All other subspaces of V are called *non-trivial*.

To say that S is a *proper* subspace of V means $S \neq V$.

Theorem: Subspace Test

Let V be a vector space. $S \subseteq V$ is a subspace of V iff:

1). $\vec{0} \in S$

2). S is closed under vector addition.

3). S is closed under scalar multiplication.

Theorem

Let V be a vector space and $\{S_i \mid i \in I\}$ be a family of subspaces of V:

$$S = \bigcap_{i \in I} S_i$$

is a subspace of V.

Theorem

Let V be a vector space and let U and W be two subspaces of V:

$$U + W = \{\vec{u} + \vec{w} \mid \vec{u} \in U \text{ and } \vec{w} \in W\}$$

is a subspace of V.

Span

Definition: Span

Let V be a vector space and let $S = \{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$. The *span* of S, denotes $\operatorname{span}(S)$, is the intersection of all subspaces of V containing S.

Thus, $\operatorname{span}(S)$ is itself a subspace of V.

Note that since $\{\vec{0}\}$ is a subset of every subspace of V, by definition:

$$\operatorname{span}(\{\}) = \{\vec{0}\}$$

To say that S *spans* V means:

$$\operatorname{span}(S) = V$$

Definition: Linear Combination

Let V be a vector space over a field $\mathbb{F}$ and let $S = \{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$ and $c_1, \ldots, c_n \in \mathbb{F}$:

$$\sum_{k=1}^{n} c_k \vec{v}_k \in V$$

is called a *linear combination* of S in V.

If all $c_k = 0$ then the linear combination is called *trivial*; otherwise, it is called *non-trivial*.

Note that linear combinations are always finite sums.

Theorem

Let V be a vector space over a field $\mathbb{F}$ and let $S = \{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$. The span of S is the set of all possible linear combinations of S in V:

$$\operatorname{span}(S) = \left\{ \sum_{k=1}^{n} c_k \vec{v}_k \mid c_k \in \mathbb{F} \right\}$$

Theorem

Let V be a vector space and U and W be subspaces of V:

$$U + W = \operatorname{span}(U \cup V)$$

Linear Independence

Definition: Linear Independence

Let V be a vector space over a field F and $S \subseteq V$. To say that S is *linearly independent* means for all $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq S$:

$$\sum_{k=0}^{n} c_k \vec{v}_k = \vec{0} \implies \forall\, c_k = 0$$

In other words, for any subset of vectors from S, only the trivial combination results in the zero vector.

Otherwise, S is said to be *linearly dependent*—there exists a non-trivial linear combination of the vectors that equals the zero vector.

By definition, $\{\vec{0}\}$ is linearly dependent; however, $\{\}$ is linearly independent.

Theorem

A set of two vectors is linearly independent iff one is a scalar multiple of the other.

Theorem

A set of vectors is linearly independent iff one can be written as a linear combination of the others.

Basis

Definition

Let V be a vector space and let $S \subseteq V$. To say that S is a basis for V means:

1). $\operatorname{span}(S) = V$

2). S is linearly independent

Theorem

Every basis for a particular vector space V has the same cardinality, called the dimension of the vector space and denoted $\dim V$.

Theorem

Let V be a vector space and $S \subseteq V$. S can be either reduced or extended (with additional vectors from V) to form a basis for V.

Theorem

Let W be a subspace of a vector space V:

$$\dim W \leq \dim V$$

Theorem

Let U and W be subspaces of a vector space V:

$$\dim(U \cap W) = \dim U + \dim W - \dim(U + W)$$

Proof

$U \cap W \subseteq U$ and $U \cap W \subseteq W$
So, $\dim(U \cap W) \leq \dim U$ and $\dim(U \cap W) \leq \dim W$
Let $\{\vec{v}_1, \ldots, \vec{v}_k\}$ be a basis for $U \cap W$
Extend the basis for $U \cap W$ to a basis for $U : \{\vec{v}_1, \ldots, \vec{v}_k, \vec{u}_1, \ldots, \vec{u}_j\}$
Likewise, extend the basis for $U \cap W$ to a basis for $W : \{\vec{v}_1, \ldots, \vec{v}_k, \vec{w}_1, \ldots, \vec{w}_p\}$
Thus, $\dim U = k + j$ and $\dim W = k + p$

Consider the set $S = \{\vec{v}_1, \ldots, \vec{v}_k, \vec{u}_1, \ldots, \vec{u}_k, \vec{w}_1, \ldots, \vec{w}_p\}$

Assume $\vec{v} \in U + W$
$\exists\, \vec{u} \in U$ and $\exists\, \vec{w} \in W$ such that $\vec{v} = \vec{u} + \vec{w}$
So $\exists\, a_i, b_i \in \mathbb{F}, \vec{u} = \sum_{i=1}^{k} a_i\vec{v}_i + \sum_{i=1}^{j} b_i\vec{u}_i$
Likewise, $\exists\, c_i, d_i \in \mathbb{F}, \vec{w} = \sum_{i=1}^{k} c_i\vec{v}_i + \sum_{i=1}^{p} d_i\vec{w}_i$

And so:

$$\begin{aligned}
\vec{v} &= \vec{u} + \vec{w} \\
&= \left(\sum_{i=1}^{k} a_i \vec{v}_i + \sum_{i=1}^{j} b_i \vec{u}_i \right) + \left(\sum_{i=1}^{k} c_i \vec{v}_i + \sum_{i=1}^{p} d_i \vec{w}_i \right) \\
&= \sum_{i=1}^{k} (a_i + c_i) \vec{v}_i + \sum_{i=1}^{j} b_i \vec{u}_i + \sum_{i=1}^{p} d_i \vec{w}_i \\
&\in \operatorname{span}(S)
\end{aligned}$$

Now, assume $\vec{v} \in \operatorname{span}(S)$
So $\exists\, a_i, b_i, c_i \in \mathbb{F}, \vec{v} = \sum_{i=1}^{k} a_i \vec{v}_i + \sum_{i=1}^{j} b_i \vec{u}_i + \sum_{i=1}^{p} d_i \vec{w}_i$
But $\sum_{i=1}^{k} a_i \vec{v}_i + \sum_{i=1}^{j} b_i \vec{u}_i \in U$ and $\sum_{i=1}^{p} d_i \vec{w}_i \in W$
$\therefore \vec{v} \in U + W$

$\therefore \operatorname{span}(S) = U + V$

Now, assume $\sum_{i=1}^{k} a_i \vec{v}_i + \sum_{i=1}^{j} b_i \vec{u}_i + \sum_{i=1}^{p} c_i \vec{w}_i = \vec{0}$
$\sum_{i=1}^{k} a_i \vec{v}_i + \sum_{i=1}^{j} b_i \vec{u}_i = -\sum_{i=1}^{p} c_i \vec{w}_i$
Let $\vec{v} = \sum_{i=1}^{k} a_i \vec{v}_i + \sum_{i=1}^{j} b_i \vec{u}_i = -\sum_{i=1}^{p} c_i \vec{w}_i$
But $\vec{v} \in U$ and $\vec{v} \in W$, and so $\vec{v} \in U \cap W$
So, $\exists\, d_i \in \mathbb{F}, \vec{v} = \sum_{i=1}^{k} d_i \vec{v}_i$
$\sum_{i=1}^{k} a_i \vec{v}_i + \sum_{i=1}^{j} b_i \vec{u}_i = \sum_{i=1}^{k} d_i \vec{v}_i$
$\sum_{i=1}^{k} (a_i - di) \vec{v}_i + \sum_{i=1}^{j} b_i \vec{u}_i = \vec{0}$
But $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{u}_1, \ldots, \vec{u}_j\}$ is a basis for U and is thus an independent set
$\therefore \forall\, b_i = 0$

We now have $\sum_{i=1}^{k} a_i \vec{v}_i + \sum_{i=1}^{p} c_i \vec{w}_i = \vec{0}$
But $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{w}_1, \ldots, \vec{w}_p\}$ is a basis for W and is thus an independent set
$\therefore \forall\, a_i = 0$ and $\forall\, c_i = 0$

Thus, S is an independent set that spans $U + W$ and is therefore a basis for $U + W$ and:

$$\dim(U + W) = k + j + p$$

Finally:

$$\begin{aligned}
\dim U + \dim V - \dim(U + V) &= (k + j) + (k + p) - (k + j + p) \\
&= k \\
&= \dim(U \cup W)
\end{aligned}$$

Example

Let:

$$U = \operatorname{span}\left(\left\{\begin{bmatrix}1\\0\\3\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\1\end{bmatrix}\right\}\right) = \operatorname{range}\left(\begin{bmatrix}1 & 0\\0 & 2\\3 & 0\\0 & 1\end{bmatrix}\right)$$

$$W = \operatorname{span}\left(\left\{\begin{bmatrix}1\\2\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\}\right) = \operatorname{range}\left(\begin{bmatrix}1 & 1 & 0\\2 & 0 & 1\\2 & 0 & 1\\1 & 0 & 0\end{bmatrix}\right)$$

$\dim U = 2$
$\dim W = 3$

$$U + W = \operatorname{range}\left(\begin{bmatrix}1 & 0 & 1 & 1 & 0\\0 & 2 & 2 & 0 & 1\\3 & 0 & 2 & 0 & 1\\0 & 1 & 1 & 0 & 0\end{bmatrix}\right)$$

$$\begin{aligned}\begin{bmatrix}1 & 0 & 1 & 1 & 0\\0 & 2 & 2 & 0 & 1\\3 & 0 & 2 & 0 & 1\\0 & 1 & 1 & 0 & 0\end{bmatrix} &\sim \begin{bmatrix}1 & 0 & 1 & 1 & 0\\0 & 2 & 2 & 0 & 1\\0 & 0 & -1 & -3 & 1\\0 & 1 & 1 & 0 & 0\end{bmatrix}\\ &\sim \begin{bmatrix}1 & 0 & 1 & 1 & 0\\0 & 1 & 1 & 0 & 0\\0 & 0 & -1 & -3 & 1\\0 & 2 & 2 & 0 & 1\end{bmatrix}\\ &\sim \begin{bmatrix}1 & 0 & 1 & 1 & 0\\0 & 1 & 1 & 0 & 0\\0 & 0 & -1 & -3 & 1\\0 & 0 & 0 & 0 & 1\end{bmatrix}\end{aligned}$$

$\dim(U + W) = 4$

$\dim(U \cap W) = 2 + 3 - 4 = 1$