Inner Product

Definition: Inner Product

Let E be a vector space over a field $\mathbb{F}$. An *inner product* on E is a mapping $\langle\cdot,\cdot\rangle : E \times E \to \mathbb{C}$ such that the following axioms are satisfied $\forall\, \vec{x}, \vec{y}, \vec{z} \in E$ and $\forall\, \alpha, \beta \in \mathbb{F}$:

1). $\langle \vec{x}, \vec{x} \rangle \geq 0$

2). $\langle \vec{x}, \vec{x} \rangle = 0 \iff \vec{x} = \vec{0}$

3). $\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle}$

4). $\langle \alpha\vec{x} + \beta\vec{y}, \vec{z} \rangle = \alpha \langle \vec{x}, \vec{z} \rangle + \beta \langle \vec{y}, \vec{z} \rangle$

A vector space equipped with such an inner product is called an *inner product space* or a *pre-Hilbert space*.

Properties

Let E be a vector space over a field $\mathbb{F}$. $\forall\, \vec{x}, \vec{y}, \vec{z} \in E$ and $\forall\, \alpha, \beta \in \mathbb{F}$:

1). $\langle \vec{x}, \vec{x} \rangle \in \mathbb{R}$

2). $\left\langle \vec{0}, \vec{y} \right\rangle = \left\langle \vec{x}, \vec{0} \right\rangle = 0$

3). $\langle \vec{x}, \alpha\vec{y} + \beta\vec{z} \rangle = \overline{\alpha} \langle \vec{x}, \vec{y} \rangle + \overline{\beta} \langle \vec{x}, \vec{z} \rangle$

Proof

Assume $\vec{x}, \vec{y}, \vec{z} \in E$ and $\alpha, \beta \in \mathbb{F}$.

1). $\langle \vec{x}, \vec{x} \rangle = \overline{\langle \vec{x}, \vec{x} \rangle}$.

$\therefore \langle \vec{x}, \vec{x} \rangle \in \mathbb{R}$

2). $\left\langle \vec{0}, \vec{y} \right\rangle = \left\langle 0 \cdot \vec{0}, \vec{y} \right\rangle = 0 \left\langle \vec{0}, \vec{y} \right\rangle = 0$

$\left\langle \vec{x}, \vec{0} \right\rangle = \left\langle \vec{x}, 0 \cdot \vec{0} \right\rangle = \overline{0} \left\langle \vec{x}, \vec{0} \right\rangle = 0 \left\langle \vec{x}, \vec{0} \right\rangle = 0$

3).

$$\begin{aligned}
\langle \vec{x}, \alpha\vec{y} + \beta\vec{z} \rangle &= \overline{\langle \alpha\vec{y} + \beta\vec{z}, \vec{x} \rangle} \\
&= \overline{\alpha \langle \vec{y}, \vec{x} \rangle + \beta \langle \vec{z}, \vec{x} \rangle} \\
&= \overline{\alpha \langle \vec{y}, \vec{x} \rangle} + \overline{\beta \langle \vec{z}, \vec{x} \rangle} \\
&= \overline{\alpha}\, \overline{\langle \vec{y}, \vec{x} \rangle} + \overline{\beta}\, \overline{\langle \vec{z}, \vec{x} \rangle} \\
&= \overline{\alpha} \langle \vec{x}, \vec{y} \rangle + \overline{\beta} \langle \vec{x}, \vec{z} \rangle
\end{aligned}$$

$\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle}$ is referred to as *Hermitian* symmetry.

The inner product is *sesquilinear*, as opposed to *bilinear*.

Examples

1). $E = C^N$ and $\langle x, y\rangle = \sum_{k=1}^{N} x_k \overline{y_k}$

$$\langle x, x\rangle = \sum_{k=1}^{N} x_k \overline{x_k} = \sum_{k=1}^{N} |x_k|^2 \geq 0$$

with equality iff $x = 0$.

$$\langle x, y\rangle = \sum_{k=1}^{N} x_k \overline{y_k} = \overline{\sum_{k=1}^{N} \overline{x_k} y_k} = \overline{\sum_{k=1}^{N} y_k \overline{x_k}} = \overline{\langle y, x\rangle}$$

$$\langle \alpha x + \beta y, z\rangle = \sum_{k=1}^{N} (\alpha x + \beta y)\overline{z} = \alpha \sum_{k=1}^{N} x\overline{z} + \beta \sum_{k=1}^{N} y\overline{z} = \alpha \langle x, z\rangle + \beta \langle y, z\rangle$$

2). $E = \ell^2$ and $\langle (x_n), (y_n)\rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}$

Recall: $\ell^2 = \left\{ (x_n) \middle| \sum_{k=1}^{n} |x_n|^2 < \infty \right\}$.

Note that $(y_n) \in \ell^2 \iff (\overline{y_n}) \in \ell^2$:

$$\sum_{k=1}^{\infty} |y|^2 = \sum_{k=1}^{\infty} |\overline{y}|^2 < \infty$$

By Hölder's inequality with $p = q = 2$:

$$\sum_{k=1}^{\infty} |x_k \overline{y_k}| \leq \left(\sum_{k=1}^{\infty} |x_k|^2 \right)^2 \left(\sum_{k=1}^{\infty} |\overline{y_k}|^2 \right)^2 = \left(\sum_{k=1}^{\infty} |x_k|^2 \right)^2 \left(\sum_{k=1}^{\infty} |y_k|^2 \right)^2 < \infty$$

So the inner product sum converges absolutely, and ℓ^2 is Banach, thus the inner product sum converges.

$$\langle (x_n), (x_n)\rangle = \sum_{k=1}^{\infty} x_k \overline{x_k} = \sum_{k=1}^{\infty} |x_k|^2 \geq 0$$

with equality iff $(x_n) = (0)$.

$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_k \overline{y_k} = \overline{\sum_{k=1}^{\infty} \overline{x_k} y_k} = \overline{\sum_{k=1}^{\infty} y_k \overline{x_k}} = \overline{\langle y, x\rangle}$$

is allowed since the inner product sum converges.

$$\langle \alpha x + \beta y, z\rangle = \sum_{k=1}^{\infty} (\alpha x + \beta y)\overline{z} = \alpha \sum_{k=1}^{\infty} x\overline{z} + \beta \sum_{k=1}^{\infty} y\overline{z} = \alpha \langle x, z\rangle + \beta \langle y, z\rangle$$

is allowed since the inner product sum converges.

3). $E = \mathcal{C}[a,b]$ and $\langle f, g\rangle = \displaystyle\int_a^b f\bar{g}$.

Note that the integral converges because f and g (and $\bar{g}$) are continuous over a closed interval, and so $f\bar{g}$, a product of continuous functions, is also continuous over a closed interval.

$$\langle f, f\rangle = \int_a^b f\bar{f} = \int_a^b |f|^2 \geq 0$$

with equality when $f \equiv 0$.

$$\langle f, g\rangle = \int_a^n f\bar{g} = \overline{\int_a^b \bar{f}g} = \overline{\int_a^b g\bar{f}} = \overline{\langle g, f\rangle}$$

$$\langle \alpha f + \beta g, h\rangle = \int_a^b (\alpha f + \beta g)\bar{h} = \alpha \int_a^b f\bar{h} + \beta \int_a^b g\bar{h} = \alpha \langle f, h\rangle + \beta \langle g, h\rangle$$

4). $E = L_2(\Omega)$ and $\langle f, g\rangle = \displaystyle\int_\Omega f\bar{g}$

Note that the integral converges because f and g (and $\bar{g}$) are square integrable, and so $f\bar{g}$, a product of square integrable functions, is also square integrable.

$$\langle f, f\rangle = \int_\Omega f\bar{f} = \int_\Omega |f|^2 \geq 0$$

with equality when $f \equiv 0$ (ae).

$$\langle f, g\rangle = \int_\Omega f\bar{g} = \overline{\int_\Omega \bar{f}g} = \overline{\int_\Omega g\bar{f}} = \overline{\langle g, f\rangle}$$

$$\langle \alpha f + \beta g, h\rangle = \int_\Omega (\alpha f + \beta g)\bar{h} = \alpha \int_\Omega f\bar{h} + \beta \int_\Omega g\bar{h} = \alpha \langle f, h\rangle + \beta \langle g, h\rangle$$