Ordered Fields

Definition

An *ordered field* F is a field that defines an ordering on its elements by the *"less than or equal to"* $(\leq)$ test such that the following five axioms are satisfied:

O1: $\forall a, b \in F, a \leq b$ or $b \leq a$.

O2: $\forall a, b \in F, a \leq b$ and $b \leq a \implies a = b$

O3: $\forall a, b, c \in F, a \leq b$ and $b \leq c \implies a \leq c$

O4: $\forall a, b, c \in F, a \leq b \implies a + c \leq b + c$

O5: $\forall a, b, c \in F, a \leq b$ and $0 \leq c \implies ac \leq bc$

Example

- The set of rational numbers $\mathbb{Q}$
- The set of real numbers $\mathbb{R}$

Notation

$$\begin{aligned} a \geq b &:= b \leq a \\ a < b &:= a \leq b \text{ and } a \neq b \\ a > b &:= b < a \end{aligned}$$

Properties

1). $\forall a, b \in F, a \leq b$ and $b \leq a \iff a = b$

2). $\forall a, b, c \in F, a \leq b \iff a + c \leq b + c$

3). $\forall a, b, c \in F, ac \leq bc$ and $0 \leq c \implies a \leq b$

4). $\forall a, b \in F, a \leq b \iff -b \leq -a$