Rouche's Theorem

Theorem: Cauchy

Let $f(z) = \sum_{k=0}^{n} a_k z^k$ where $a_k \in \mathbb{R}$. All of the zeros of $f(z)$ are enclosed in the circle:

$$|z| = 1 + \max\{|a_k| \mid 0 \leq k \leq n\}$$

Example

Let $f(z) = z^4 - z^2 - 2z + 2 = (z-1)^2(z+1 \pm i)$

$|z| = 1 + \max\{1, 2\} = 1 + 2 = 3$

$|1| = 1 < 3$
$|-1 \pm i| = \sqrt{2} < 3$

But, we can do better with Rouche's Theorem:

Theorem

Let $f(z)$ and $g(z)$ be analytic on $\overline{D}$ with boundary γ such that $|g(z)| < |f(z)|$ on γ. The number of zeros of $(f+g)(z)$ in γ equals the number of zeros of $f(z)$ in γ.

Proof

Let $F(z) = \frac{g(z)}{f(z)}$
On γ: $|F(z)| = \frac{|g(z)|}{|f(z)|} < 1$
$g(z) = f(z)F(z)$

Let $N_1 =$ the number of zeros of (f+g) inside γ
Let $N_2 =$ the number of zeros of f inside γ

$$\begin{aligned}
N_1 - N_2 &= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)+g'(z)}{f(z)+g(z)}dz - \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz \\
&= \frac{1}{2\pi i}\int_\gamma \left[\frac{f'+g'}{f+g} - \frac{f'}{f}\right] dz \\
&= \frac{1}{2\pi i}\int_\gamma \left[\frac{f'+(fF)'}{f+fF} - \frac{f'}{f}\right] dz \\
&= \frac{1}{2\pi i}\int_\gamma \left[\frac{f'+f'F+fF'}{f+fF} - \frac{f'}{f}\right] dz \\
&= \frac{1}{2\pi i}\int_\gamma \left[\frac{f'(1+F)+fF'}{f(1+F)} - \frac{f'}{f}\right] dz \\
&= \frac{1}{2\pi i}\int_\gamma \left[\frac{f'}{f} + \frac{F'}{1+F} - \frac{f'}{f}\right] dz
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_\gamma \left[\frac{F'}{1+F}\right] dz \\
&= \frac{1}{2\pi i}\int_\gamma \left[F(z)'\sum_{n=0}^{\infty}(-1)^n[F(z)]^n\right] dz \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(-1)^n\int_\gamma [F(z)]^n F'(z)dz \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(-1)^n\frac{[F(z)]^{n+1}}{n+1}\ |_\gamma
\end{aligned}
$$

But γ is closed, so $N_1 - N_2 = 0$
$\therefore N_1 = N_2$

Example

Let $a > e$ and $h(z) = e^z - az^n$
Show that $h(z)$ has n zeros inside $|z| = 1$

Let $f(z) = -az^n$ and $g(z) = e^z$
On $|z| = 1$:
$|f(z)| = |-az^n| = a\,|z|^n = a(1^n) = a$
$|g(z)| = |e^z| = e^x\,|e^{iy}| = e^x(1) = e^x \leq e^1 = e$
So $|g(z)| < |f(z)|$ on $|z| = 1$

$f(z) = -az^n$ has n repeated zeros at $z = 0$, which is inside $|z| = 1$
$\therefore h(z) = (f+g)(z)$ has n zeros inside $|z| = 1$

Example

Let $h(z) = z^7 - 5z^3 + 12$
Show that $h(z)$ has 7 zeros between $|z| = 1$ and $|z| = 2$

First, let $f(z) = z^7$ and $g(z) = 12 - 5z^3$

On $|z| = 2$:

$$\frac{|g(z)|}{|f(z)|} = \frac{|12-5z^3|}{|z^7|} \leq \frac{12+5\,|z|^3}{|z|^7} = \frac{12+5(8)}{128} = \frac{52}{128} < 1$$

So $|g(x)| < |f(x)|$ on $|z| = 2$
But $f(z) = x^7$ has 7 repeated zeros at $z = 0$, which is inside $|z| = 2$
$\therefore h(x) = (f+g)(x)$ has 7 zeros inside $|z| = 2$

Now, let $f(z) = 12$ and $g(z) = z^7 - 5z^3$

On $|z| = 1$:

$$\frac{|g(z)|}{|f(z)|} = \frac{|z^7 - 5z^3|}{|12|} \leq \frac{|z|^7 + 5\,|z|^3}{12} = \frac{1+5}{12} = \frac{1}{2} < 1$$

So $|g(x)| < |f(x)|$ on $|z| = 1$
But $f(z) = 12$ has no zeros inside $|z| = 1$
$\therefore h(x) = (f+g)(x)$ has no zeros inside $|z| = 1$

$\therefore h(z)$ has 7 zeros between $|z| = 1$ and $|z| = 2$

<u>Theorem: Enestome</u>

Let $p(z) = \sum_{k=0}^{n} a_k z^k$ such that $a_k \in \mathbb{R}$ and $0 < a_{k-1} < a_k < a_n$.
All of the zeros of $p(z)$ are inside $|z| = 1$.

<u>Proof</u>

Let $0 < \lambda_k < 1$ such that $\lambda_k a_k > a_{k-1}$
Let $\lambda = \max\{\lambda_k \mid 0 \leq k \leq n\}$
$(\lambda - z)p(z) = (\lambda - z)\sum_{k=0}^{n} a_k z^k = \sum_{k=0}^{n}(\lambda a_k z^k - a_k z^{n+1}) = \lambda a_0 + \sum_{k=1}^{n}(\lambda a_k - a_{k-1})z^k - a_n z^{n+1}$
$(\lambda - z)p(z) + a_k z^{n+1} = \lambda a_0 + \sum_{k=1}^{n}(\lambda a_k - a_{k-1})z^k$

Let $f(z) = -a_n z^{n+1}$ and $g(z) = (\lambda - z)p(z) + a_n z^{n+1}$

On $|z| = 1$:

$$|f(z)| = \left|-a_n z^{n+1}\right| = a_n\,|z|^{n+1} = a_n$$

$$\begin{aligned}
\left|(\lambda - z)p(z) + a_n z^{n+1}\right| &= \left|\lambda a_0 + \sum_{k=1}^{n}(\lambda a_k - a_{k-1})z^k\right| \\
&\leq |\lambda a_0| + \left|\sum_{k=1}^{n}(\lambda a_k - a_{k-1})z^k\right| \\
&\leq \lambda a_0 + \sum_{k=1}^{n}\left|(\lambda a_k - a_{k-1})z^k\right| \\
&= \lambda a_0 + \sum_{k=1}^{n}(\lambda a_k - a_{k-1})\,|z|^k \\
&\leq \lambda a_0 + \sum_{k=1}^{n}(\lambda a_k - a_{k-1}) \\
&= \sum_{k=0}^{n-1}(1-\lambda)a_k + \lambda a_n
\end{aligned}$$

But $(\lambda - 1) < 0$ and $\lambda a_n > 0$, so:

$$\left|(\lambda - z)p(z) + a_k z^{n+1}\right| \leq \lambda a_n < a_n$$

So, $|gz| < |f(z)|$ on $|z| = 1$
But $f(z)$ has $(n+1)$ repeated zeros at $z = 0$, which is inside $|z| = 1$
So, $(f+g)(z) = (\lambda - z)p(z)$ has $(n+1)$ zeros inside $|z| = 1$

Therefore $p(z)$ has n zeros inside $|z| = 1$.

Theorem: Rouche Alternate Form

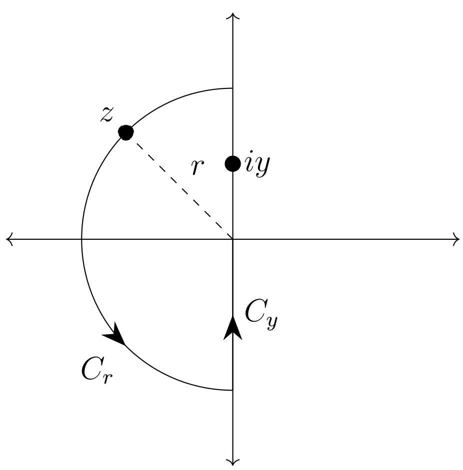
Let $f(z)$ and $g(z)$ be analytic on $\overline{D}$ with boundary γ such that $|f(z) - g(z)| < |f(z)|$ on γ. The number of zeros of $f(z)$ in γ equals the number of zeros of $g(z)$ in γ.

Proof

Let $h(z) = (g - f)(z)$
$|h(z)| = |g(z) - f(z)| = |f(z) - g(z)| < |f(z)|$
$N_f = N_{f+h} = N_{f+(g-f)} = N_g$

Example

Show that $g(z) = z + 3 + 2e^z$
Show that $g(z)$ has exactly one zero in the left-hand plane.



Let $f(z) = z + 3$

On C_y:

$$|f(z)| = |z + 3| = |3 + iy| \geq 3$$

Assume $r > 5$
On C_r:

$$|f(z)| = |z + 3| \geq |z| - |3| > 5 - 3 = 2$$

Therefore, on $C_y \cup C_r$, $|f(z)| > 2$

$|f - g| = |-2e^z| = |-2|\,|e^z| = 2e^x$

On C_y, $x = 0$, so $|f - g| = 2$

On C_r, $x \leq 0$, so $|f - g| < 2$

Therefore, on $C_y \cup C_r$, $|f - g| \leq 2$

So, on $C_y \cup C_r$, $|f(z) - g(z)| < |f(z)|$
But $f(z)$ has only one zero at $x = -3$ inside $C_y \cup C_r$
Therefore, $g(z)$ has only one zero inside $C_y \cup C_r$
Now let $r \to \infty$
$g(z)$ has only one zero in the left-hand plane.