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Math 229
Exam 2

1). Let $J_{a,b}$ be the $a \times b$ matrix with all entries equal to 1 and I_c be the $c \times c$ identity matrix. Consider the matrix:

$$D = \begin{bmatrix} 2(J_{n,n} - I_n) & J_{n,m} \\ J_{m,n} & 2(J_{m,m} - I_m) \end{bmatrix}$$

where $n \geq m \geq 2$.

a). Compute $\operatorname{Sp}(A)$ where $D = A + J_{m+n,m+n}$

Let B_n be the $n \times n$ matrix with -1 on the diagonal and 1 everywhere else. Note that:

$$B_n = J_{n,n} - 2I_n$$

Also note that $J_{n,n}$ is a rank-one matrix that can be generated by $J_{n,1}J_{1,n}$, and so:

$$\operatorname{Sp}(J_{n,n}) = \{0^{(n-1)}, J_{1,n}J_{n,1}\} = \{0^{(n-1)}, n\}$$

and so:

$$\operatorname{Sp}(B_n) = \{(-2)^{(n-1)}, n-2\}$$

Now, since:

$$A = D - J_{m+n,m+n} = \begin{bmatrix} B_n & 0 \\ 0 & B_m \end{bmatrix} - 2I_{n+m}$$

and using the principle to find eigenvalues for block matrices:

$$\operatorname{Sp}(A) = \{(-2)^{n+m-2}, m-2, n-2\}$$

Some examples using MATLAB:

m	n	$\operatorname{Sp}(A)$
2	2	$\{-2, -2, 0, 0\}$
2	3	$\{-2, -2, -2, 0, 1\}$
2	4	$\{-2, -2, -2, -2, 0, 2\}$
3	3	$\{-2, -2, -2, -2, 1, 1\}$
3	4	$\{-2, -2, -2, -2, -2, 1, 2\}$

b). Use MATLAB to complete the table for $\operatorname{Sp}(D)$.

m	n	$\operatorname{Sp}(D)$
2	2	$\{-2, -2, 0, 4\}$
2	3	$\{-2, -2, -2, 0.3542, 5.6458\}$
2	4	$\{-2, -2, -2, -2, 0.5359, 7.4641\}$
3	3	$\{-2, -2, -2, -2, 1, 7\}$
3	4	$\{-2, -2, -2, -2, -2, 1.3944, 8.6056\}$

c). Guess the number of negative eigenvalues of D in general.

For $D_{m,n}$, the number of negative eigenvalues is the same as the number of negative eigenvalues for $A_{m,n}$ which equals $m+n-2$.

d). Prove the guess in (c).

Since D and A are Hermitian and have real eigenvalues, and since $J_{n+m,n+m}$ is rank-one, we can apply the rank-one interlacing theorem to $D = A + J_{m+n,m+n}$ to show that $\lambda_1(D), \ldots, \lambda_{n+m-3}(D) = -2$ and $\lambda_{n+m-1}(D), \lambda_{n+m}(D) \geq 0$. Thus, it remains to show that $\lambda_{n+m-2}(D) < 0$.

Let $\vec{u}_k$ be an eigenvector for A, $\vec{v}_k$ be an eigenvector for B, and $\vec{w}_k$ be an eigenvector for D, and let:

$S_A = \text{span}\{\vec{u}_2, \ldots, \vec{u}_{n+m-2}\}$
$S_B = \text{span}\{\vec{v}_2, \ldots, \vec{v}_{n+m-2}\}$
$S_D = \text{span}\{\vec{w}_{n+m-2}, \vec{w}_{n+m-1}, \vec{w}_{n+m}\}$

$$\begin{aligned} \dim(S_A \cap S_B \cap S_D) &\geq \dim(S_A) + \dim(S_B) + \dim(S_D) - 2(n+m) \\ &= (n+m-1) + (n+m-1) + 3 - 2(n+m) \\ &= 1 \end{aligned}$$

Thus, the intersection of the three spaces is non-empty.

Assume $\vec{x} \in S_A \cap S_B \cap S_D$:

$\lambda_{n+m-2}(D) \leq \frac{\vec{x}^*(A+B)\vec{x}}{\vec{x}^*\vec{x}}$ because $\vec{x} \in S_D$

$\frac{\vec{x}^*A\vec{x}}{\vec{x}^*\vec{x}} \leq \lambda_{n+m-2}(A)$ because $\vec{x} \in S_A$

$\frac{\vec{x}^*B\vec{x}}{\vec{x}^*\vec{x}} \leq \lambda_{n+m-2}(B)$ because $\vec{x} \in S_B$

$\lambda_{n+m-2}(D) \leq \frac{\vec{x}^*(A+B)\vec{x}}{\vec{x}^*\vec{x}} \leq \frac{\vec{x}^*A\vec{x}}{\vec{x}^*\vec{x}} + \frac{\vec{x}^*B\vec{x}}{\vec{x}^*\vec{x}} \leq \lambda_{n+m-2}(A) + \lambda_{n+m-2}(B)$

We already know that $\lambda_{n+m-2}(A) = -2$ and $\lambda_{n+m-2}(B) = 0$, and so:

$\lambda_{n+m-2}(D) \leq -2 + 0 = -2 < 0$

Therefore, D has $n+m-2$ negative eigenvalues.

2). Let $|||\cdot|||_1$ be the ℓ_1 norm induced matrix norm (i.e., the maximum column sum) and $|||\cdot|||_2$ be the ℓ_2 norm induced matrix norm (i.e., max singular value).

a). Define a new matrix norm $|||A|||_\infty = |||A^*|||_1$, called the maximum row sum norm. Verify all 5 conditions of a matrix norm are satisfied.

By definition:

$$|||A^*|||_1 = \max_{\|\vec{x}\|_1=1}\{\|A^*\vec{x}\|_1\}$$

i. Nonnegativity

Assume $A \in M_n$
Assume $\vec{x} \in \mathbb{C}^n$ such that $\|\vec{x}\|_1 = 1$
By nonnegativity of the vector norm, $\|A^*\vec{x}\|_1 \geq 0$
So $\max_{\|\vec{x}\|_1=1}\{\|A^*\vec{x}\|_1\} \geq 0$

$\therefore |||A^*|||_1 \geq 0$

ii. Positivity

$\implies$ Assume $A \neq 0$

$A^* \neq 0$
$\exists\, \vec{y} \in \mathbb{C}^n, \|\vec{y}\|_1 = 1$ and $A^*\vec{y} \neq 0$
$|||A^*|||_1 = \max_{\|\vec{x}\|_1=1}\{\|A^*\vec{x}\|_1\} \geq \|A^*\vec{y}\|_1 > 0$

$\therefore |||A^*|||_1 \neq 0$

$\impliedby$ Assume $A = 0$

$A^* = 0$
Assume $\vec{x} \in \mathbb{C}^n, \|\vec{x}\|_1 = 1$
$A^*\vec{x} = \vec{0}$
$\|A^*\vec{x}\|_1 = \|\vec{0}\|_1 = 0$
$\max_{\|\vec{x}\|_1=1}\{\|A^*\vec{x}\|_1\} = 0$

$\therefore |||A^*|||_1 = 0$

iii. Homogeneity

Assume $c \in \mathbb{C}$:

$$\begin{aligned} |||cA^*|||_1 &= \max_{\|\vec{x}\|_1=1}\{\|cA^*\vec{x}\|_1\} \\ &= \max_{\|\vec{x}\|_1=1}\{|c|\,\|A^*\vec{x}\|_1\} \\ &= |c| \max_{\|\vec{x}\|=1}\{\|A^*\vec{x}\|_1\} \\ &= |c|\,|||A^*|||_1 \end{aligned}$$

iv. Subadditivity

$$\begin{aligned}
|||A+B|||_\infty &= |||(A+B)^*|||_1 \\
&= \max_{\|\vec{x}\|_1=1}\{\|(A+B)^*\vec{x}\|_1\} \\
&= \max_{\|\vec{x}\|_1=1}\{\|A^*\vec{x}+B^*\vec{x}\|_1\} \\
&\leq \max_{\|\vec{x}\|_1=1}\{\|A\vec{x}\|_1+\|B\vec{x}\|_1\} \\
&\leq \max_{\|\vec{x}\|_1=1}\{\|A^*\vec{x}\|_1\}+\max_{\|\vec{x}\|_1=1}\{\|B^*\vec{x}\|_1\} \\
&= |||A^*|||_1+|||B^*|||_1 \\
&= |||A^*|||_\infty+|||B^*|||_\infty
\end{aligned}$$

v. Submultiplicativity

Assume $A, B \in M_n$
Trivial if $A = 0$ or $B = 0$, so AWLOG: $A, B \neq 0$
Assume $\vec{x} \in \mathbb{C}^n$, $\|\vec{x}\|_1 = 1$
$\frac{A^*\vec{x}}{\|A^*\vec{x}\|_1}$ is a unit vector

$$\begin{aligned}
|||B|||_\infty &= |||B^*|||_1 \\
&= \max_{\|\vec{x}\|=1}\{\|B^*\vec{x}\|_1\} \\
&\geq \|B^*\frac{A^*\vec{x}}{\|A^*\vec{x}\|_1}\|_1 \\
|||B|||_\infty\|A^*\vec{x}\|_1 &\geq \|B^*(A^*\vec{x})\|_1 \\
|||B|||_\infty\|A^*\vec{x}\|_1 &\geq \|(AB)^*\vec{x})\|_1
\end{aligned}$$

And now:

$$\begin{aligned}
|||AB|||_\infty &= |||(AB)^*|||_1 \\
&= \max_{\|\vec{x}\|=1}\{\|(AB)^*\vec{x}\|\} \\
&\leq \max_{\|\vec{x}\|=1}\{|||B|||_\infty\|A^*\vec{x}\|_1\} \\
&= |||B|||_\infty\max_{\|\vec{x}\|=1}\{\|A^*\vec{x}\|_1\} \\
&= |||B|||_\infty|||A^*|||_1 \\
&= |||B|||_\infty|||A|||_\infty \\
|||AB|||_\infty &\leq |||A|||_\infty|||B|||_\infty
\end{aligned}$$

b). Prove that $|||A|||_\infty \leq n\,|||A|||_1$ and find a matrix B where equality holds.

It has already been proved $|||A|||_1 = \max_{1\leq j\leq n}\sum_{i=1}^n |a_{ij}|$, which is the maximum column sum for A, so clearly, $|||A|||_\infty = |||A^*|||_1$ is the maximum row sum for A.

By definition:

$$|||A|||_1 = \max_{\|\vec{x}\|_1=1} \|A\vec{x}\|_1$$

Note that $\|\frac{1}{n}J_{n,1}\|_1 = 1$ and so:

$$|||A|||_1 \geq \|A\left(\frac{1}{n}J_{n,1}\right)\|_1 = \frac{1}{n}\|AJ_{n,1}\|_1$$

But $AJ_{n,1}$ is a vector consisting of the row sums of A, and so its norm is greater than or equal to the maximum row sum, and so:

$$n\,|||A|||_1 \geq \|AJ_{n,1}\|_1 \geq |||A^*|||_1 = |||A|||_\infty$$

$\therefore |||A|||_\infty \leq n\,|||A|||_1$

Let B be the rank-one matrix consisting of all 1's in the first row and 0's everywhere else:

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$|||B|||_\infty = n$ and $|||B|||_1 = 1$ and so equality holds.

c). Prove that $|||A|||_2 \leq \sqrt{n}\,|||A|||_1$ and find a matrix B where equality holds.

Assume $\vec{x} \in \mathbb{C}^n$. Using Cauchy-Schwarz:

$$\|\vec{x}\|_1 = \sum_{k=1}^{n} |x_k| \leq \left(\sum_{k=1}^{n} 1^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |x_k|^2\right)^{\frac{1}{2}} = \sqrt{n}\|\vec{x}\|_2$$

Now:

$$\begin{aligned} |||A|||_2 &= \max_{\|\vec{x}\|_2=1} \|A\vec{x}\|_2 \\ &= \max_{\vec{x}\neq 0} \|A\frac{\vec{x}}{\|\vec{x}\|_2}\|_2 \\ &\leq \max_{\vec{x}\neq 0} \|A\frac{\vec{x}}{\frac{1}{\sqrt{n}}\|\vec{x}\|_1}\|_1 \\ &= \sqrt{n}\max_{\vec{x}\neq 0} \|A\frac{\vec{x}}{\|\vec{x}\|_1}\|_1 \\ &= \sqrt{n}\max_{\|\vec{x}\|_1=1} \|A\vec{x}\|_1 \\ &= \sqrt{n}\,|||A|||_1 \end{aligned}$$

3). Let $A \in M_n$ be positive definite.

a). Prove that A has a positive definite square root.

A is postive definite $\implies$ A is Hermitian $\implies$ A is unitary diagonalizable with all $\lambda_k(A) \in (0, \infty)$

$$\begin{aligned} A &= U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^* \\ &= U \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} U^* U \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} U^* \end{aligned}$$

Let $S = U \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} U^*$

$\therefore A = S^2$

Now, note that since $\lambda_k(S) \in (0, \infty)$:

$$\begin{aligned} S^* &= \left(U \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} U^* \right)^* \\ &= (U^*)^* \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix}^* U^* \\ &= U \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} U^* \\ &= S \end{aligned}$$

So S is Hermitian and $\lambda_k(S) \in (0, \infty)$

Therefore S is positive definite.

b). Prove that S is unique.

Assume that S and T are square roots of A. Since S and T are positive definite, and thus normal, they are diagonalizable. Since a diagonalization of S or T must contain the square roots of the eigenvalues of A and since such diagonalizations are permutation similar, AWLOG that: $S = UDU^*$ and $T = VDV^*$ for some unitary U and V.

$S^2 = UD^2U^*$ and $T^2 = VD^2V^*$ and so:

$$\begin{array}{rcl} S^2 &=& T^2 \\ UD^2U^* &=& VD^2V^* \\ D^2 &=& U^*VD^2V^*U \\ D^2 &=& U^*VD^2(U^*V)^* \\ D &=& U^*VD(U^*V)^* \\ D &=& U^*VDV^*U \end{array}$$

And now:

$$S = UDU^* = U(U^*VDV^*U)U^* = VDV^* = T$$

c). Show that the matrix $\begin{bmatrix} 28 & -12 & 12 & 12 \\ -12 & 28 & 12 & -12 \\ -12 & 12 & 28 & -12 \\ 12 & -12 & -12 & 28 \end{bmatrix}$ is positive definite.

Note that the matrix is Hermitian, so it suffices to prove that all of its eigenvalues are nonnegative.

Let $A = \begin{bmatrix} 0 & -1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 28 & -12 & 12 & 12 \\ -12 & 28 & 12 & -12 \\ -12 & 12 & 28 & -12 \\ 12 & -12 & -12 & 28 \end{bmatrix} = 12A + 28I_4$$

So we need the eigenvalues of A. Use the principle minors trick to find the characteristic polynomial of A:

$$E_1 = \mathrm{tr}(A) = 0$$

$$\begin{array}{rcl} E_2 &=& |A_{12}| + |A_{13}| + |A_{14}| + |A_{23}| + |A_{24}| + |A_{34}| \\ &=& \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} \\ &=& (-1) + (-1) + (-1) + (-1) + (-1) + (-1) \\ &=& -6 \end{array}$$

$$\begin{aligned}
E_3 &= |A_{123}| + |A_{124}| + |A_{134}| + |A_{234}| \\
&= \begin{vmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{vmatrix} \\
&= 2+2+2+2 \\
&= 8
\end{aligned}$$

$$E_4 = \begin{vmatrix} 0 & -1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{vmatrix} = -3$$

$p_A(t) = t^4 - E_1 t^3 + E_2 t^2 - E_3 t + E_4 = t^4 - 6t^2 - 8t - 3$

Using the rational root test and long division, this factors as follows:

$p_A(t) = (t+1)^3(t-3)$

And so $\mathrm{Sp}(A) = \{-1, -1, -1, 3\}$ and thus $\mathrm{Sp}(A) = \{16, 16, 16, 64\}$

So the original matrix is Hermitian and has nonnegative eigenvalues and is therefore positive definite.

4). Find the unique positive definite square root of the matrix in part (c).

One could sit and solves SOLEs to find the eigenvectors in order to construct U; however, MATLAB has a *sqrtm* command:

$$S = \begin{bmatrix} 5 & -1 & -1 & 1 \\ -1 & 5 & 1 & -1 \\ -1 & 1 & 5 & -1 \\ 1 & -1 & -1 & 5 \end{bmatrix}$$