Math-42 Sections 01, 02, 05

Homework #9 Solutions

Problems

Prove the following using the telescoping sum method discussed in class:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

First, start with a telescoping sum:

$$\begin{aligned}
\sum_{k=1}^{n} [(k+1)^3 - k^3] &= (n+1)^3 - 1^3 \\
&= n^3 + 3n^2 + 3n + 1 - 1 \\
&= n^3 + 3n^2 + 3n
\end{aligned}$$

Next, evaluate the sum directly:

$$\begin{aligned}
\sum_{k=1}^{n} [(k+1)^3 - k^3] &= \sum_{k=1}^{n} (k^3 + 3k^2 + 3k + 1 - k^3) \\
&= \sum_{k=1}^{n} (3k^2 + 3k + 1) \\
&= 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
&= 3\sum_{k=1}^{n} k^2 + 3\left[\frac{n(n+1)}{2}\right] + n
\end{aligned}$$

Finally, equate the two and do some algebra:

$$
\begin{aligned}
3\sum_{k=1}^{n} k^2 + 3\left[\frac{n(n+1)}{2}\right] + n &= n^3 + 3n^2 + 3n \\
3\sum_{k=1}^{n} k^2 &= n^3 + 3n^2 + 3n - 3\left[\frac{n(n+1)}{2}\right] - n \\
3\sum_{k=1}^{n} k^2 &= n^3 + 3n^2 + 2n - \frac{3}{2}n^2 - \frac{3}{2}n \\
3\sum_{k=1}^{n} k^2 &= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \\
3\sum_{k=1}^{n} k^2 &= \frac{1}{2}n(2n^2 + 3n + 1) \\
3\sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{2} \\
\sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6}
\end{aligned}
$$