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Math 275A
Homework #4

**Theorem: 3.1**

Let $(X, \mathscr{T})$ be a topological space and let $\mathcal{B} \subset \mathscr{T}$. $\mathcal{B}$ is a basis for $\mathscr{T}$ iff:

$$\forall U \in \mathscr{T}, \forall p \in U, \exists V \in \mathcal{B}, p \in V \subset U$$

*Proof.*

$\Longrightarrow$ Assume that $\mathcal{B}$ is a basis for $\mathscr{T}$.

Assume $U \in \mathscr{T}$. This means that there exists $\mathcal{B}_U \subset \mathcal{B}$ such that $U = \bigcup \mathcal{B}_U$. Now, assume that $p \in U$. Thus, $p \in \bigcup \mathcal{B}_U$ and therefore there exists some $V \in \mathcal{B}_U$ such that $p \in V \subset U$.

$\Longleftarrow$ Assume $\forall U \in \mathscr{T}, \forall p \in U, \exists V \in \mathcal{B}, p \in V \subset U$

Assume that $U \in \mathscr{T}$. For each $p \in U$, choose a set $V_p \in \mathcal{B}$ such that $p \in V_p \subset U$. Thus $U = \bigcup_{p \in U} V_p$ and so every $U \in \mathscr{T}$ is generated by $\mathcal{B}$. Therefore $\mathcal{B}$ is a basis for $\mathscr{T}$.

$\blacksquare$

**Example: Exercise 3.2**

Let $\mathscr{T}$ be the standard topology on $\mathbb{R}$. Show that the following are bases for $\mathscr{T}$:

1. $\mathcal{B}_1 = \{(a, b) \subset \mathbb{R} \mid a, b \in \mathbb{Q}\}$

   All the $(a, b)$ are open and hence $\mathcal{B}_1 \subset \mathscr{T}$. So assume $U \in \mathscr{T}$ and assume $p \in U$. Since $U$ is open, there exists an open ball $B(p, \epsilon) \subset U$. Since there exists an infinite number of rationals between any two reals, select two rationals $a \in (p - \epsilon, p)$ and $b \in (p, p + \epsilon)$. Thus $(a, b) \in \mathcal{B}_1$ and $p \in (a, b) \subset U$. Therefore, by the previous theorem, $\mathcal{B}_1$ is a basis for $\mathscr{T}$.

2. $\mathcal{B}_2 = \{(a, b) \cup (c, d) \subset \mathbb{R} \mid a, b, c, d \in \mathbb{R} - \mathbb{Q}$ and $a < b < c < d\}$

   All the $(a, b) \cup (c, d)$ are unions of open sets, so they are open as well and hence $\mathcal{B}_2 \subset \mathscr{T}$. So assume $U \in \mathscr{T}$ and assume $p \in U$. Since $U$ is open, there exists an open ball $B(p, \epsilon) \in U$. Since there exists an infinite number of irrationals between any two real numbers, select four irrationals as follows:

   $$\begin{aligned}
   &a \in (p - \epsilon, p) \\
   &b \in (p, p + \epsilon) \\
   &c \in (b, p + \epsilon) \\
   &d \in (c, p + \epsilon)
   \end{aligned}$$

   Thus $a < b < c < d$ and so $(a, b) \cup (c, d) \in \mathcal{B}_2$. Furthermore, $p \in (a, b) \cup (c, d) \subset U$. Therefore, by the previous theorem, $\mathcal{B}_2$ is a basis for $\mathscr{T}$.

**Theorem: 3.3**

Let $X$ be a set and let $\mathcal{B} \subset 2^X$. $\mathcal{B}$ is a basis for some topology $\mathscr{T}$ on $X$ iff:

1. $\forall\, p \in X, \exists\, V_p \in \mathcal{B}, p \in V_p$
2. $\forall\, U, V \in \mathcal{B}, \forall\, p \in U \cap V, \exists\, W \in \mathcal{B}, p \in W \subset U \cap V$

*Proof.*

$\implies$ Assume $\mathcal{B}$ is a basis for some topology $\mathscr{T}$ on $X$.

Since $X \in \mathscr{T}$, (1) must hold by the previous theorem. So assume $U, V \in \mathcal{B}$. If $U \cap V = \emptyset$ then (2) is vacuously true, so assume that $U \cap V \neq \emptyset$ and assume $p \in U \cap V$. Since $U, V \in \mathcal{B} \subset \mathscr{T}$, both $U$ and $V$ are open and hence $U \cap V$ is open. So there exists some $U_p \in \mathscr{T}$ such that $p \in U_p \subset U \cap V$. But $U_p$ is generated by $\mathcal{B}$, and so there must exist some $W \in \mathcal{B}$ such that $p \in W \subset U \cap V$. Therefore (2) holds as well.

$\impliedby$ Assume that properties (1) and (2) hold.

WTS: $\mathscr{T} = \left\{ U \subset 2^X \,\middle|\, \exists\, \mathcal{B}_U \subset \mathcal{B}, U = \bigcup \mathcal{B}_U \right\}$ is a topology on $X$.

First, consider $\emptyset$. Since $\emptyset \subset \mathcal{B}$ and $\emptyset$ is the union of no sets, $\emptyset \in \mathscr{T}$.

Next, consider $X$. By (1), $X$ is generated by $\mathcal{B}$ and hence $X \in \mathscr{T}$.

Next, assume $U, V \in \mathscr{T}$ and let $U$ be generated by $\mathcal{B}_U \subset \mathcal{B}$ and let $V$ be generated by $\mathcal{B}_V \subset \mathcal{B}$, specifically:

$$U = \bigcup \mathcal{B}_U = \bigcup_{\alpha \in A} B_\alpha$$
$$V = \bigcup \mathcal{B}_V = \bigcup_{\lambda \in \Lambda} B_\lambda$$

Then:

$$U \cap V = \bigcup \mathcal{B}_U \cap \bigcup \mathcal{B}_V = \bigcup_{\alpha \in A} B_\alpha \cap \bigcup_{\lambda \in \Lambda} B_\lambda = \bigcup_{\alpha \in A, \lambda \in \Lambda} (B_\alpha \cap B_\lambda)$$

But by (2), each of the $B_\alpha \cap B_\lambda$ is generated by some subset of $\mathcal{B}$, and hence $U \cap V$ is generated by some subset of $\mathcal{B}$. Therefore $U \cap V \in \mathscr{T}$.

Finally, assume that $\{U_\alpha : \alpha \in A\} \subset \mathscr{T}$ and let $U = \bigcup_{\alpha \in A} U_\alpha$. But each $U_\alpha$ is generated by some subset of $\mathcal{B}$ and hence $U$ is generated by some subset of $\mathcal{B}$. Therefore $U \in \mathscr{T}$.

Therefore $\mathscr{T}$ is a topology on $X$.

■

**Example: Exercise 3.6**

Give an example of two topologies on $R$ such that neither is finer than the other, that is, the two topologies are not comparable.

Consider the standard and cocountable topologies on $R$. $(0, 1)$ is open in the standard topology; however, $\mathbb{R} - (0, 1)$ is uncountable and hence $(0, 1)$ is not in the cocountable topology. Likewise, $\mathbb{R} - \mathbb{Q}$ is open in the cocountable topology (since $\mathbb{Q}$ is countable); however, since there are an infinite number of rationals between any two irrationals, it is impossible to draw an open ball around any irrational that is a subset of the irrationals and so $\mathbb{R} - \mathbb{Q}$ is not in the standard topology. Therefore the two topologies are not comparable.

**Theorem: Exercise 3.8**

Let $\mathscr{T}$ be the double-headed snake topology on $\mathbb{R}_{00+}$:

1. Every point in $\mathbb{R}_{00+}$ is a closed set.
2. It is impossible for find disjoint open sets $U$ and $V$ such that $0' \in U$ and $0'' \in V$.

*Proof.*

1. Assume $x \in \mathbb{R}_{00+}$ and consider $\{x\}$. Let $A = \mathbb{R}_{00+} - \{x\}$:

   **Case 1:** $x = 0'$

   $$A = (\{0''\} \cup (0, 2)) \cup \bigcup_{1<b\in\mathbb{R}^+}(1, b) \in \mathscr{T}$$

   **Case 2:** $x = 0''$

   $$A = (\{0'\} \cup (0, 2)) \cup \bigcup_{1<b\in\mathbb{R}^+}(1, b) \in \mathscr{T}$$

   **Case 3:** $x \in \mathbb{R}^+$

   $$A = (\{0'\} \cup (0, x)) \cup (\{0''\} \cup (0, x)) \cup \bigcup_{x<b\in\mathbb{R}^+}(x, b) \in \mathscr{T}$$

   Therefore $A$ is open and thus $\{x\}$ is closed.

2. WTS: $\forall U, V \in \mathscr{T}, (0' \in U \text{ and } 0'' \in V \implies U \cap V \neq \emptyset)$

   Assume $U, V \in \mathscr{T}$ and assume $0' \in U$ and $0'' \in V$. This means that $\{0'\} \cup (0, c) \subset U$ and $\{0''\} \cup (0, d) \subset V$ for some $c, d \in \mathbb{R}+$. Since $U$ and $V$ are generated by these and possibly other basis elements, it must be the case that:

   $$(\{0'\} \cup (0, c)) \cap (\{0''\} \cup (0, d)) = (0, \min\{c, d\}) \subset U \cap V$$

   But $(0, \min\{c, d\}) \neq \emptyset$.

   Therefore $U \cap V \neq \emptyset$.

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**Theorem: Exercise 3.14**

Let $\mathscr{T}$ be the standard topology on $\mathbb{R}$ and let:

$$\mathcal{S} = \{(-\infty, b) \mid b \in \mathbb{R}\} \cup \{(a, \infty) \mid a \in \mathbb{R}\}$$

$\mathcal{S}$ is a subbasis for $\mathscr{T}$.

*Proof.* Since all of the sets in $S$ are open, all finite intersections are also open. In particular, for all $a, b \in \mathbb{R}$ such that $a < b$:

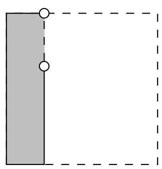
$$(-\infty, b) \cap (a, \infty) = (a, b)$$

But the $(a, b)$ are known to be a basis of $\mathscr{T}$. Furthermore, adding more open sets to a basis just results in a finer basis.

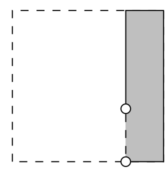
Therefore, $\mathcal{S}$ is a subbasis of $\mathscr{T}$. ■

**Example: Exercise 3.20**

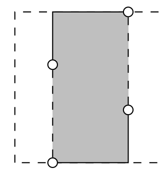
Draw pictures of various open sets in the lexicographically ordered square.



$a_<$



$a_>$



$(a, b)$

**Theorem: 3.25**

Let $(X, \mathscr{T})$ be a topological space and let $Y \subset X$. $\mathscr{T}_Y$ is a topology on $Y$.

*Proof.* $\emptyset \cap Y = \emptyset \in \mathscr{T}_Y$ and $X \cap Y = Y \in \mathscr{T}_Y$.

Assume $U, V \in \mathscr{T}_Y$. Then there exists $U', V' \in \mathscr{T}$ such that $U = U' \cap Y$ and $V = V' \cap Y$. So:

$$U \cap V = (U' \cap Y) \cap (V' \cap Y) = (U' \cap V') \cap Y$$

But $U' \cap V' \in \mathscr{T}$. Therefore $U \cap V \in \mathscr{T}_Y$.

Now, assume that $\{U_\alpha : \alpha \in \lambda\}$ such that $U_\alpha \in \mathscr{T}_Y$. Then for each $U_\alpha$ there exists a $U'_\alpha \in \mathscr{T}$ such that $U_\alpha = U'_\alpha \cap Y$. So:

$$U = \bigcup_{\alpha \in \lambda} U_\alpha = \bigcup_{\alpha \in \lambda} (U'_\alpha \cap Y) = \left( \bigcup_{\alpha \in \lambda} U'_\alpha \right) \cap Y$$

But $\bigcup_{\alpha \in \lambda} U'_\alpha \in \mathscr{T}$. Therefore, $U \in \mathscr{T}_Y$.

Therefore $\mathscr{T}_Y$ is a topology on $Y$. ■

**Example: Exercise 3.26**

Consider $Y = [0, 1)$ as a subspace of $\mathbb{R}_{std}$. In $Y$, is the set $\left[\frac{1}{2}, 1\right)$ open, closed, neither, or both?

There is no open set in $X$ that will result in a closed endpoint at $\frac{1}{2}$ so the set is not open. However, $[0, 1) \cap \left(\frac{1}{2}, 1\right) = \left(\frac{1}{2}, 1\right) \in \mathscr{T}_Y$ and $\frac{1}{2}$ serves as a limit point in $Y$ so $\left[\frac{1}{2}, 1\right)$ is closed in $Y$. Hence it is not neither and not both.

**Theorem: 3.28**

Let $(Y, \mathscr{T}_Y)$ be a subspace of $(X, \mathscr{T})$. $C \subset Y$ is closed in $(Y, \mathscr{T}_Y)$ iff there exists $D \subset X$, closed in $(X, \mathscr{T})$, such that $C = D \cap Y$.

*Proof.*

$\Longrightarrow$ Assume $C \subset Y$ is closed in $(Y, \mathscr{T}_Y)$.

Since $C$ in closed in $Y$, $Y - C$ is open in $Y$. So there exists some $U \in \mathscr{T}$ such that $Y - C = U \cap Y$. Let $D = X - U$, which is closed in $X$:

$$D \cap Y = (X - U) \cap Y = (X \cap Y) - (U \cap Y) = Y - (Y - C) = C$$

Therefore there exists $D \subset X$, closed in $(X, \mathscr{T})$, such that $C = D \cap Y$.

$\Longleftarrow$ Assume there exists $D \subset X$, closed in $(X, \mathscr{T})$, such that $C = D \cap Y$.

Since $D$ is closed in $X$, $X - D$ is open in $X$ and $(X - D) \cap Y$ is open in $Y$:

$$(X - D) \cap Y = (X \cap Y) - (D \cap Y) = Y - C$$

Therefore $C$ is closed in $(Y, \mathscr{T}_Y)$.

■