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Math 229
Homework #4

5.1.9

Let $\|\cdot\|$ be a norm on V that is derived from an inner product. Let $\vec{x}, \vec{y} \in V$ and $\vec{y} \neq 0$.

a) Show that the scalar α_0 that minimizes the value of $\|\vec{x} - \alpha\vec{y}\|$ is $\alpha_0 = \frac{\vec{x},\vec{y}}{\|\vec{y}\|^2}$.

   It is a bit easier to minimize $\|\vec{x} - \alpha\vec{y}\|^2$:

$$\begin{array}{rcl} \|\vec{x} - \alpha\vec{y}\|^2 & = & \langle \vec{x} - \alpha\vec{y}, \vec{x} - \alpha\vec{y} \rangle \\ & = & \langle \vec{x}, \vec{x} \rangle - \langle \vec{x}, \alpha\vec{y} \rangle - \langle \alpha\vec{y}, \vec{x} \rangle + \langle \alpha\vec{y}, \alpha\vec{y} \rangle \\ & = & \langle \vec{x}, \vec{x} \rangle - \overline{\alpha} \langle \vec{x}, \vec{y} \rangle - \alpha \langle \vec{y}, \vec{x} \rangle + |\alpha|^2 \langle \vec{y}, \vec{y} \rangle \end{array}$$

   From complex analysis, we know that:

$$\frac{d\overline{\alpha}}{d\alpha} = 0$$

   And so, by the product rule:

$$\frac{d}{d\alpha} |\alpha|^2 = \frac{d}{d\alpha} a\overline{\alpha} = \frac{d\alpha}{d\alpha}\overline{\alpha} + \alpha\frac{d\overline{\alpha}}{d\alpha} = \overline{\alpha} + 0 = \overline{\alpha}$$

   We can also show that:

$$\overline{\|\vec{y}\|^2} = \overline{\langle \vec{y}, \vec{y} \rangle} = \langle \vec{y}, \vec{y} \rangle = \|\vec{y}\|^2$$

   Now minimize:

$$\begin{array}{rcl} 0 - 0 - \langle \vec{y}, \vec{x} \rangle + \overline{\alpha_0} \langle \vec{y}, \vec{y} \rangle & = & 0 \\ \overline{\alpha_0} \langle \vec{y}, \vec{y} \rangle & = & \langle \vec{y}, \vec{x} \rangle \\ \overline{\alpha_0} \|\vec{y}\|^2 & = & \overline{\langle \vec{x}, \vec{y} \rangle} \end{array}$$

$$\alpha_0 \|\vec{y}\|^2 = \langle \vec{x}, \vec{y} \rangle$$

$$\therefore \alpha_0 = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2}$$

b) Show $\|\vec{x} - \alpha_0\vec{y}\|^2 = \|\vec{x}\|^2 - \frac{|\langle \vec{x},\vec{y} \rangle|^2}{\|\vec{y}\|^2}$

From part(a) we have:

$$\begin{aligned}
\|\vec{x}-\alpha_0\vec{y}\|^2 &= \langle\vec{x},\vec{x}\rangle - \overline{\alpha_0}\langle\vec{x},\vec{y}\rangle - \alpha_0\langle\vec{y},\vec{x}\rangle + |\alpha_0|^2\langle\vec{y},\vec{y}\rangle \\
&= \langle\vec{x},\vec{x}\rangle - \overline{\left(\frac{\langle\vec{x},\vec{y}\rangle}{\|\vec{y}\|^2}\right)}\langle\vec{x},\vec{y}\rangle - \frac{\langle\vec{x},\vec{y}\rangle}{\|\vec{y}\|^2}\langle\vec{y},\vec{x}\rangle + \left|\frac{\langle\vec{x},\vec{y}\rangle}{\|\vec{y}\|^2}\right|^2\langle\vec{y},\vec{y}\rangle \\
&= \|\vec{x}\|^2 - \frac{|\langle\vec{x},\vec{y}\rangle|^2}{\|\vec{y}\|^2} - \frac{|\langle\vec{x},\vec{y}\rangle|^2}{\|\vec{y}\|^2} + \frac{|\langle\vec{x},\vec{y}\rangle|^2}{\|\vec{y}\|^4}\|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 - 2\frac{|\langle\vec{x},\vec{y}\rangle|^2}{\|\vec{y}\|^2} + \frac{|\langle\vec{x},\vec{y}\rangle|^2}{\|\vec{y}\|^2} \\
&= \|\vec{x}\|^2 - \frac{|\langle\vec{x},\vec{y}\rangle|^2}{\|\vec{y}\|^2}
\end{aligned}$$

c) Show that $\vec{x}-\alpha_0\vec{y}$ is orthogonal to $\vec{y}$.

$$\begin{aligned}
\langle\vec{x}-\alpha_0\vec{y},\vec{y}\rangle &= \langle\vec{x},\vec{y}\rangle - \langle\alpha_0\vec{y},\vec{y}\rangle \\
&= \langle\vec{x},\vec{y}\rangle - \alpha_0\langle\vec{y},\vec{y}\rangle \\
&= \langle\vec{x},\vec{y}\rangle - \alpha_0\|\vec{y}\|^2 \\
&= \langle\vec{x},\vec{y}\rangle - \frac{\langle\vec{x},\vec{y}\rangle}{\|\vec{y}\|^2}\|\vec{y}\|^2 \\
&= \langle\vec{x},\vec{y}\rangle - \langle\vec{x},\vec{y}\rangle \\
&= 0
\end{aligned}$$

Therefore, $\vec{x}-\alpha_0\vec{y}$ is orthogonal to $\vec{y}$.

5.2.6

If $\|\cdot\|$ is a unitary invariant norm on C^n, show that $\forall\,\vec{x}\in\mathbb{C}^n$:

$$\|\vec{x}\| = \|\vec{x}\|_2\,\|\vec{e}_1\|$$

Explain why the Euclidean norm is the only unitary invariant norm on $\mathbb{C}^n$ for which $\|\vec{e}_1\| = 1$.

Assume $\vec{x}\in\mathbb{C}^n$ and consider the unit vector $\hat{y}_1 = \frac{\vec{x}}{\|\vec{x}\|_2}$. Using G-S, construct $n-1$ additional orthogonal unit vectors $\{\hat{y}_2,\ldots,\hat{y}_n\}$ and form the matrix $U = \begin{bmatrix}\hat{y}_1 & \hat{y}_2 & \ldots & \hat{y}_n\end{bmatrix}$. Since the columns of U are orthonormal, U is a unitary matrix.

Now, since the norm is unitary invariant:

$$\|\vec{e}_1\| = \|U\vec{e}_1\| = \|\hat{y}_1\| = \left\|\frac{\vec{x}}{\|\vec{x}\|_2}\right\| = \frac{1}{\|\vec{x}\|_2}\|\vec{x}\|$$

Therefore:

$$\|\vec{x}\| = \|\vec{x}\|_2\,\|\vec{e}_1\|$$

So, if $\|\vec{e}_1\| = 1$ then:

$$\|\vec{x}\| = \|\vec{x}\|_2\cdot 1 = \|\vec{x}\|_2$$

5.4.11

Let $\|\cdot\|$ be a norm on F^n where $F = \mathbb{R}$ or $\mathbb{C}$.

a) Show that every isometry for $\|\cdot\|$ is nonsingular.

   Assume $A \in G_{\|\cdot\|}$
   Assume $\lambda \in \sigma(A)$
   $\|A\vec{x}\| = \|\lambda\vec{x}\| = |\lambda|\,\|\vec{x}\| = \|\vec{x}\|$
   But $\vec{x}$ is an eigenvector and thus $\vec{x} \neq \vec{0}$
   Thus $|\lambda| = 1$ and so $\lambda \neq 0$

   Therefore, by the IMT, A is invertible (nonsingular).

b) Prove: $G_{\|\cdot\|} \leq GL(n)$

   Assume $A, B \in G_{\|\cdot\|}$ and $\vec{x} \in F^n$
   A is invertible
   $A \in GL(n)$

   Therefore $G_{\|\cdot\|} \subseteq GL(n)$.

   $B\vec{x} \in F^n$
   $\|(AB)\vec{x}\| = \|A(B\vec{x})\| = \|B\vec{x}\| = \|\vec{x}\|$
   $AB \in G_{\|\cdot\|}$

   Therefore $G_{\|\cdot\|}$ is closed under the operation (composition).

   $\|I_n\vec{x}\| = \|\vec{x}\|$

   Therefore $I_n \in G_{\|\cdot\|}$.

   Since A is invertible we have $A^{-1} \in GL(n)$
   $A^{-1}\vec{x} \in F^n$
   $\|A^{-1}\vec{x}\| = \|A(A^{-1}\vec{x})\| = \|(AA^{-1})\vec{x}\| = \|I_n\vec{x}\| = \|\vec{x}\|$
   $A^{-1} \in G_{\|\cdot\|}$

   Therefore $G_{\|\cdot\|}$ is closed under inverses.

   $\therefore G_{\|\cdot\|} \leq GL(n)$.

c) Show that if $A \in G_{\|\cdot\|}$ and $\lambda \in \sigma(A)$ then $|\lambda| = 1$.

   See part (a).

d) Prove: If $A \in M_n$ is an isometry for $\|\cdot\|$ then $|\det(A)| = 1$.

By Auerbach's Theorem, A is similar to some unitary matrix $U \in M_n$:

$$\begin{aligned}
A &= SUS^{-1} \\
\det(A) &= \det(SUS^{-1}) \\
&= \det(S)\det(U)\det(S^{-1}) \\
&= \det(U) \\
&= \pm 1 \\
\therefore |\det(A)| &= 1
\end{aligned}$$

e) Prove: Any unitary generalized permutation matrix is an isometry for every k-norm and every ℓ_p norm for $1 \leq p \leq \infty$.

Assume that U is a generalized permutation matrix and assume $\vec{x} \in \mathbb{C}^n$. The result of $U\vec{x}$ is to permute the components of $\vec{x}$ and multiply moved components by $e^{i\theta}$ for some $\theta \in \mathbb{R}$. Assume x_k is such a permuted component. Then:

$$\left|x_k e^{i\theta}\right| = |x_k| \left|e^{i\theta}\right| = |x_k| \cdot 1 = |x_k|$$

Thus, the absolute values of the moved components do not change, just their positions.

This does not affect the k-norm because the k-norm adds the k greatest component absolute values regardless of position. It does not affect any of the ℓ_p norms because all of the component absolute values are involved in the calculation regardless of position. Finally, it does not affect the ℓ_∞ norm because the largest component absolute value is selected regardless of position.

5.5.7

If $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are norms on a vector space and if $\|\cdot\|$ is the norm defined by:

$$\|\vec{x}\| = \max\{\|\vec{x}\|_\alpha, \|\vec{x}\|_\beta\}$$

then show that:

$$B_{\|\cdot\|} = B_{\|\cdot\|_\alpha} \cap B_{\|\cdot\|_\beta}$$

First show that $\|\cdot\|$ is a norm by checking the four norm properties:

1). Assume $\vec{x} \in \mathbb{C}^n$

$\|\vec{x}\|_\alpha \geq 0$
$\|\vec{x}\|_\beta \geq 0$
$\max\{\|\vec{x}\|_\alpha, \|\vec{x}\|_\beta\} \geq 0$

$\therefore \|\vec{x}\| \geq 0$

2). Assume $\vec{x} \in \mathbb{C}^n$

$$\begin{aligned}\|\vec{x}\| = \vec{0} &\iff \max\{\|\vec{x}\|_\alpha, \|\vec{x}\|_\beta\} = \vec{0} \\ &\iff \|\vec{x}\|_\alpha = \vec{0} \text{ and } \|\vec{x}\|_\beta = \vec{0} \\ &\iff \vec{x} = \vec{0}\end{aligned}$$

3). Assume $\vec{x} \in \mathbb{C}^n$ and $c \in \mathbb{C}$

$$\begin{aligned}\|c\vec{x}\| &= \max\{\|c\vec{x}\|_\alpha, \|c\vec{x}\|_\beta\} \\ &= \max\{|c|\,\|\vec{x}\|_\alpha, |c|\,\|c\vec{x}\|_\beta\} \\ &= |c| \max\{\|\vec{x}\|_\alpha, \|c\vec{x}\|_\beta\} \\ &= |c|\,\|\vec{x}\|\end{aligned}$$

4). Assume $\vec{x}.\vec{y} \in \mathbb{C}^n$

$$\begin{aligned}\|\vec{x}+\vec{y}\| &= \max\{\|\vec{x}+\vec{y}\|_\alpha, \|\vec{x}+\vec{y}\|_\beta\} \\ &\leq \max\{\|\vec{x}\|_\alpha + \|\vec{y}\|_\alpha, \|\vec{x}\|_\beta + \|\vec{y}\|_\beta\} \\ &\leq \max\{\|\vec{x}\|_\alpha, \|\vec{x}\|_\beta\} + \max\{\|\vec{y}\|_\alpha, \|\vec{y}\|_\beta\} \\ &= \|\vec{x}\| + \|\vec{y}\|\end{aligned}$$

Therefore $\|\cdot\|$ is a proper norm.

$$\begin{aligned}\vec{x} \in B_{\|\cdot\|} &\iff \|\vec{x}\| \leq 1 \\ &\iff \max\{\|\vec{x}\|_\alpha, \|\vec{x}\|_\beta\} \leq 1 \\ &\iff \|\vec{x}\|_\alpha \leq 1 \text{ and } \|\vec{x}\|_\beta \leq 1 \\ &\iff \vec{x} \in B_{\|\cdot\|_\alpha} \text{ and } \vec{x} \in B_{\|\cdot\|_\beta} \\ &\iff \vec{x} \in B_{\|\cdot\|_\alpha} \cap B_{\|\cdot\|_\beta}\end{aligned}$$

$\therefore B_{\|\cdot\|} = B_{\|\cdot\|_\alpha} \cap B_{\|\cdot\|_\beta}$

5.6.24

Let $A \in M_n$. Show:

$$\|A\|_2 \leq (\operatorname{rank} A)^{\frac{1}{2}}\, |||A|||_2$$

Let $r = \operatorname{rank} A$. From a previous homework problem we know:

$$\operatorname{tr}(A^*A) = \sum_{1 \leq i,j} |a_{ij}|^2 = \sum_{k=1}^{n} \sigma_k^2$$

where σ_k are the singular values of A. And we also know from the SVD proof done in class that $\operatorname{rank} A^*A = \operatorname{rank} A =$ the number of non-zero singular values of A, and so:

$$\operatorname{tr}(A^*A) = \sum_{k=1}^{r} \sigma_k^2 \leq \sum_{k=1}^{r} \sigma_1^2 = r\sigma_1^2$$

But:

$$\operatorname{tr}(A^*A) = \sum_{1 \leq i,j \leq n} |a_{ij}|^2 = \|A\|_2^2$$

And:

$$\sigma_1 = |||A|||_2$$

And so:

$$\|A\|_2^2 \leq (\operatorname{rank} A)\, |||A|||_2^2$$

Therefore:

$$\|A\|_2 \leq (\operatorname{rank} A)^{\frac{1}{2}}\, |||A|||_2$$