Rejection Region

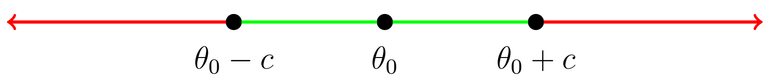
Definition: Decision Rule

A *decision rule* specifies the criteria by which H_0 should be rejected. The criteria is in the form of a so-called *rejection region*: H_0 is rejected if $\hat{\theta}$ falls within the rejection region.

Given a tolerance c, the rejection regions for the three alternate hypothesis types are as follows:

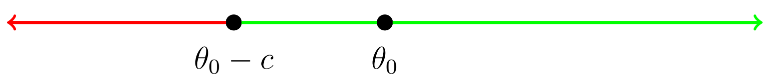
1. $H_a : \theta \neq \theta_0$

   $$|\hat{\theta} - \theta_0| > c$$

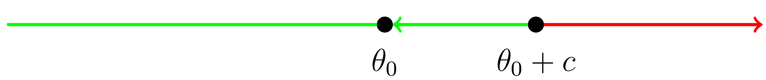
2. $H_a : \theta < \theta_0$

   $$\hat{\theta} < \theta_0 - c$$

3. $H_a : \theta > \theta_0$

   $$\hat{\theta} > \theta_0 + c$$

Definition: Hypothesis Test Errors

Hypothesis tests can have the following results:

1. Retaining a true H_0 and rejecting a false H_0 are both *correct decisions*.
2. Rejecting a true H_0 is called a *type I error*.
3. Retaining a false H_0 is called a *type II error*.

Calculating Type I Error

Note that as c increases, the rejection region gets smaller, making it harder to reject H_0. Thus, the possibility of a type I error is decreased.

Definition: Level

The *level* of a hypothesis test, denoted α, is the probability of making a type I error.

Example

A brown egg farm claims that the average weight of their eggs is $\mu = 65\,\text{g}$. The true standard deviation is known to be $\sigma = 2\,\text{g}$. What is the level of a two-sided test with a carton sample $(n = 12)$ and with $c = 1$ and $c = 2$?

$$H_0 : \mu = 65 \qquad H_a : \mu \neq 65$$

$$X_i \overset{\text{iid}}{\sim} \mathrm{N}(65, 2^2) \qquad \bar{X} \sim \mathrm{N}\left(65, \frac{2^2}{12}\right)$$

$$|\bar{x} - 65| > c$$

$$\begin{aligned}
\alpha &= P\left(\text{reject } H_0 | H_0 \text{ true}\right) \\
&= P\left(|\bar{X} - 65| > 1 \middle| \mu = 65\right) \\
&= P\left(\bar{X} < 64 \text{ or } \bar{X} > 66 \middle| \mu = 65\right) \\
&= P\left(\bar{X} < 64 \middle| \mu = 65\right) + P\left(\bar{X} > 66 \middle| \mu = 65\right) \\
&= P\left(Z < \frac{64 - 65}{\frac{2}{\sqrt{12}}}\right) + P\left(Z > \frac{66 - 65}{\frac{2}{\sqrt{12}}}\right) \\
&= P(Z < -\sqrt{3}) + P(Z > \sqrt{3}) \\
&= 2P(Z < -1.73) \\
&= 2\Phi(-1.73) \\
&= 2(0.0418) \\
&= 0.0836
\end{aligned}$$

$$\begin{aligned}
\alpha &= P\left(|\bar{X} - 65| > 2 \middle| \mu = 65\right) \\
&= P\left(\bar{X} < 63 \text{ or } \bar{X} > 67 \middle| \mu = 65\right) \\
&= P\left(\bar{X} < 63 \middle| \mu = 65\right) + P\left(\bar{X} > 67 \middle| \mu = 65\right) \\
&= P\left(Z < \frac{63 - 65}{\frac{2}{\sqrt{12}}}\right) + P\left(Z > \frac{67 - 65}{\frac{2}{\sqrt{12}}}\right) \\
&= P(Z < -2\sqrt{3}) + P(Z > 2\sqrt{3}) \\
&= 2P(Z < -3.46) \\
&= 2\Phi(-3.46) \\
&= 2(0.0003) \\
&= 0.0006
\end{aligned}$$

Note that the probability of making a type I error decreased as c increased.

Example

Repeat the above example for a one-sided test with $H_a : \mu < 65$.

In each case, only one side of the symmetric distribution is selected, and so the probability is halved:

$$c = 1 : 0.0418$$
$$c = 2 : 0.0003$$

Calculating Type II Error

Although decreasing the size of the rejection region decreases the probability of a type I error, the smaller rejection region makes it harder to reject H_0 if it is false. Thus, the probability for a type II error increases. Since it is unknown whether whether H_0 is true or not, a balance between the possibility of either type of error is needed.

If H_0 is false, then each possible value of θ results in a different probability for type II error.

Definition: Power

Let $\beta(\theta)$ be the probability for a type II error. The *power* of a hypothesis test, given by $1 - \beta(\theta)$, is the probability of correctly rejecting a false H_0.

So the desire is to have a small type I error probability (typically 5%) and large power (typically 80%) and hence a small type II error probability (typically 20%).

Example

For the above example, calculate the probability of a two-sided test type II error when $\mu = 64$ and $c = \frac{1}{2}$, 1, and 2.

$$\beta(\mu) = P\left(\text{fail to reject } H_0 | H_0 \text{ is false}\right)$$
$$\beta(64) = P\left(|\bar{X} - 65| < c \middle| \mu \neq 65\right)$$

$$\bar{X} \sim \mathrm{N}\left(64, \frac{2^2}{12}\right)$$

$$
\begin{aligned}
\beta(64) &= P\left(|\bar{X} - 65| < c\middle|\mu \neq 65\right) \\
&= P\left(64.5 < \bar{X} < 65.5\middle|\mu = 64\right) \\
&= P\left(\frac{64.5 - 64}{\frac{2}{\sqrt{12}}} < Z < \frac{65.5 - 64}{\frac{2}{\sqrt{12}}}\right) \\
&= P(0.5\sqrt{3} < Z < 1.5\sqrt{3}) \\
&= P(2.60) - \Phi(0.87) \\
&= 0.9953 - 0.8078 \\
&= 0.1875
\end{aligned}
$$

$$
\begin{aligned}
\beta(64) &= P\left(|\bar{X} - 65| < c\middle|\mu \neq 65\right) \\
&= P\left(64 < \bar{X} < 66\middle|\mu = 64\right) \\
&= P\left(\frac{64 - 64}{\frac{2}{\sqrt{12}}} < Z < \frac{66 - 64}{\frac{2}{\sqrt{12}}}\right) \\
&= P(0 < Z < 2\sqrt{3}) \\
&= \Phi(3.46) - P(0) \\
&= 0.9997 - 0.500 \\
&= 0.4997 \\
\beta(64) &= P\left(|\bar{X} - 65| < c\middle|\mu \neq 65\right) \\
&= P\left(63 < \bar{X} < 67\middle|\mu = 64\right) \\
&= P\left(\frac{63 - 64}{\frac{2}{\sqrt{12}}} < Z < \frac{67 - 64}{\frac{2}{\sqrt{12}}}\right) \\
&= P(-\sqrt{3} < Z < 3\sqrt{3}) \\
&= \Phi(5.20) - P(-1.73) \\
&= 1 - 0.0418 \\
&= 0.9582
\end{aligned}
$$

The goal is to select appropriate c and n values for a desired a (typically 5%) and b (typically 20%).

Theorem

Let α be the desired level and $1 - \beta(\theta')$ be the desired power for a hypothesis test with claim $H_0 : \theta = \theta_0$ and known standard deviation σ. For a two-sided hypothesis test $Ha : \theta \neq \theta_0$:

$$
\begin{aligned}
c &= z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \\
n &\approx \left(\frac{\sigma\left(z_{\frac{\alpha}{2}} + z_\beta\right)}{\theta - \theta'}\right)^2
\end{aligned}
$$

For a one-sided hypothesis test:

$$c = z_\alpha \frac{\sigma}{\sqrt{n}}$$
$$n \approx \left(\frac{\sigma \left(z_\alpha + z_\beta \right)}{\theta - \theta'} \right)^2$$

Example

For the above example, determine the desired c and n of a two-sided and a one-sided hypothesis test for $\alpha = 5\%$ and $\beta = 20\%$.

$$n \approx \left(\frac{\sigma \left(z_{0.025} + z_{0.20} \right)}{\theta - \theta'} \right)^2 = \left(\frac{2 \left(1.96 + 0.84 \right)}{65 - 64} \right)^2 = 31.36 \approx 32$$
$$c = z_{0.025} \frac{\sigma}{\sqrt{n}} = 1.96 \frac{2}{\sqrt{32}} = 0.6930 \approx 0.7$$

$$n \approx \left(\frac{\sigma \left(z_{0.05} + z_{0.20} \right)}{\theta - \theta'} \right)^2 = \left(\frac{2 \left(1.645 + 0.84 \right)}{65 - 64} \right)^2 = 24.7 \approx 25$$
$$c = z_{0.05} \frac{\sigma}{\sqrt{n}} = 1.645 \frac{2}{\sqrt{25}} = 0.6580 \approx 0.7$$

Test Result

Note that $c = z_\alpha 2 \frac{\sigma}{\sqrt{n}}$ is actually the margin of error for the $1-\alpha$ confidence interval. Therefore:

H_0 is not rejected if $\hat{\theta}$ falls within the $1 - \alpha$ confidence interval.

H_0 is rejected if $\hat{\theta}$ falls outside the $1 - \alpha$ confidence interval.

Example

The heights of a random sample of 400 male high school sophomores in a mid-western state are measured. The sample mean is $\bar{x} = 66.2$ in. Suppose that the heights of all male high school sophomores in that state follow a normal distribution with a standard deviation of $\sigma = 4.1$ in. Conduct the following hypothesis test at the 5% level:

$$H_0 : \mu = 66.8 \qquad H_a : \mu \neq 66.8$$

$$\pm z_{0.025} \frac{\sigma}{\sqrt{n}} = 1.96 \frac{4.1}{\sqrt{400}} \approx 0.4$$

Since $66.2 \notin (66.4, 67.2)$, reject H_0.

Reconduct the test for the one-sided alternative: $H_a : \mu < 66.8$.

$$z_{0.05}\frac{\sigma}{\sqrt{n}} = 1.645\frac{4.1}{\sqrt{400}} \approx 0.3$$

Since $66.2 \notin (66.5, \infty)$, reject H_0.