Positive Definite Matrices

Definition: Positive Definite

To say that $A \in M_n$ is *positive definite* means $\forall\, \vec{x} \in \mathbb{C}^n - \{\vec{0}\}$:

$$\vec{x}^* A \vec{x} > 0$$

Note that A positive definite $\implies$ A Hermitian.

Example

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{aligned}
\begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 2\bar{x}_1 + \bar{x}_2 & \bar{x}_1 + 2\bar{x}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= x_1(2\bar{x}_1 + \bar{x}_2) + x_2(\bar{x}_1 + 2\bar{x}_2) \\
&= 2\left|x_1\right|^2 + x_1\bar{x}_2 + \bar{x}_1 x_2 + 2\left|x_2\right|^2 \\
&= 2\left|x_1\right|^2 + 2\,\mathrm{Re}(x_1\bar{x}_2) + 2\left|x_2\right|^2 \\
&= 2(\left|x_1\right|^2 + \mathrm{Re}(x_1\bar{x}_2) + \left|x_2\right|^2) \\
&= 2(\left|x_1\right|^2 - 2\left|x_1\bar{x}_2\right| + \left|x_2\right|^2 + 2\left|x_1\bar{x}_2\right| + \mathrm{Re}(x_1\bar{x}_2)) \\
&= 2[(\left|x_1\right| - \left|x_2\right|)^2 + 2\left|x_1\bar{x}_2\right| + \mathrm{Re}(x_1\bar{x}_2)] \\
&\geq 0
\end{aligned}$$

With equality only at $x_1 = x_2 = 0$, or $\vec{x} = 0$.

Properties: Positive Definite

1). $A \in M_n$ positive definite $\implies$ $\mathrm{Sp}(A) \subseteq (0, \infty)$

Assume A is positive definite
Assume $\vec{x} \in \mathbb{C}^n$ such that $\vec{x} \neq \vec{0}$
$\vec{x}^* A \vec{x} > 0$
Let $\vec{x} \in \mathrm{Eig}_A(\lambda)$ such that $\vec{x}$ is a unit vector
$\vec{x}^* A \vec{x} = \vec{x}^* \lambda \vec{x} = \lambda \vec{x}^* \vec{x} = \lambda > 0$

2). $A \in M_n$ positive definite $\implies$ $a_{ii} > 0$

Assume A is positive definite
$\vec{e}_i^* A \vec{e}_i = a_{ii} > 0$

3). $A \in M_n$ positive definite $\implies$ $\forall\, S \in GL(n), S^* A S$ positive definite

Assume A is positive definite
Assume $\vec{x} \in \mathbb{C}^n$ such that $\vec{x} \neq \vec{0}$
$\vec{x}^*(S^* A S)\vec{x} = (\vec{x}^* S^*)A(S\vec{x}) = (S\vec{x})^* A (S\vec{x}) = \vec{y}^* A \vec{y} > 0$

$\therefore S^* A S$ is positive definite.

4). $A \in M_n$ positive definite $\implies$ any principle submatrix B of A is positive definite

Assume A is positive definite
AWLOG: B is a leading principle submatrix, otherwise permute and note property (3)
Assume $\vec{x} \in \mathbb{C}^k$ for $1 \leq k \leq n$

$$\begin{bmatrix} \vec{x}^* & 0 \end{bmatrix} \left[\begin{array}{c|c} B & * \\ \hline * & * \end{array}\right] \begin{bmatrix} \vec{x} \\ 0 \end{bmatrix} = \vec{x}^* B \vec{x} > 0$$

$\therefore B$ is positive definite.

Theorem

Let $A \in M_n$. A positive definite $\iff$ A Hermitian and $\mathrm{Sp}(A) \subseteq (0, \infty)$

Proof

$\implies$ Assume A is positive definite

A is also Hermitian
By property (1), $\forall \lambda \in \mathrm{Sp}(A), \lambda > 0$

$\impliedby$ Assume A is Hermitian and $\mathrm{Sp}(A) \subseteq (0, \infty)$

Assume $\lambda \in \mathrm{Sp}(A)$
Let $\vec{x}$ be a unit eigenvector associated with λ
$\vec{x} \neq 0$
$\vec{x}^* A \vec{x} = \vec{x}^* \lambda \vec{x} = \lambda \vec{x}^* \vec{x} = \lambda > 0$

$\therefore A$ is positive definite.

Theorem

Let $A \in M_n$. A positive definite $\iff$ $\exists C \in GL(n), A = C^* C$

Proof

$\implies$ Assume A is positive definite

A is Hermitian and is thus unitary diagonalizable:

$$A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^* = U \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} U^* U \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} U^*$$

Let $C = U \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} U^* = C^*$

Note that since $\lambda_k > 0$, C is invertible

$\therefore A = C^* C$

$\Longleftarrow$ Assume $\exists\, C \in GL(n), A = C^*C$

Assume $\vec{x} \in \mathbb{C}^n$ such that $\vec{x} \neq \vec{0}$
$\vec{x}^* A\vec{x} = \vec{x}^* C^* C\vec{x} = (C\vec{x})^*(C\vec{x}) = \|C\vec{x}\|_2^2 > 0$

Therefore A is positive definite.

Example

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \implies \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \implies \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Use rowops to convert to eschelon form:

1). $-R_1 + R_2$

2). $-R_1 + R_3$

3). $-2R_2 + R_3$

$$E_3E_2E_1 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = (E_3E_2E_1)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = E_1^{-1}E_2^{-1}E_3^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = C^*C$$

$$\therefore C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem

Let $A \in M_n$. A positive definite $\iff$ A Hermitian and $\det A_k > 0$ for all $1 \leq k \leq n$, where A_k is the $k \times k$ leading principle submatrix of A.

Proof

$\implies$ Assume A is positive definite

A is Hermitian
Assume $1 \leq k \leq n$
A_k is positive definite
Assume $\lambda \in \sigma(A_k)$
$\lambda > 0$
$\det A_k = \prod_{i=1}^{k} \lambda_i(A_k) > 0$

$\impliedby$ Assume A is Hermitian and $\det A_k > 0, 1 \leq k \leq n$

Proof by induction on n

Base Case: n=1

$A = \begin{bmatrix} \lambda \end{bmatrix}$ with $\lambda > 0$

Therefore, A is positive definite.

Assume A is positive definite for $A \in M_{n-1}$

Consider $A \in M_n$ and let $A = \left[\begin{array}{c|c} B & * \\ \hline * & * \end{array}\right]$ where $B = A_{n-1}$

Since A is Hermitian, B is also Hermitian
Assume $1 \leq k \leq n-1$
$\det B_k = \det A_k > 0$
So by the inductive assumption, B is positive definite
Thus $\sigma(B) \subseteq (0, \infty)$
But by the interlacing theorem, $\lambda_k(B) \leq \lambda_{k+1}(A)$, so $\lambda_k(A) > 0$ for $2 \leq k \leq n$
But $\det A = \det A_n = \prod_{k=1}^{n} \lambda_k(A) > 0$
Thus, since $\lambda_2, \ldots \lambda_n > 0$ it must be the case that $\lambda_1 > 0$
And so $\sigma(A) \subseteq (0, \infty)$

Therefore A is positive definite.