Math-42 Worksheet #7

Introduction to Proofs

1. Write the formal definition for $x \in \mathbb{Q}$.

2. Prove that Q is closed under addition (hint: see text).

3. Prove that Q is closed under multiplication.

4. In class, you probably saw a proof by contradiction that $\sqrt{2}$ is irrational. Here is a direct proof:

   Let $x = \sqrt{2}$. This means that $x^2 = 2$ and $x^2 - 2 = 0$. So, by the rational roots theorem, any rational root of $x^2 - 2 = 0$ must be of the form $\frac{p}{q}$ where p is a factor of -2 and q is a factor of 1. Thus, the only possible rational roots are ± 1 and ± 2. But none of these four values is a solution to $x^2 - 2 = 0$. Therefore x is not rational, and is thus irrational.

   Repeat this proof for $\sqrt{7}$. Make sure that you understand each step.

5. Write the formal definition for $n \in \mathbb{Z}$ is an even number.

6. Write the formal definition for $n \in \mathbb{Z}$ is an odd number.

7. Study the definitions and convince yourself they do not preclude a value from being both even and odd. Forget about what you think you know, just consider the content of the definitions. Using the definitions, write down what it would mean for an integer to be both odd and even. Be sure to use a different variable in each part.

8. We are used to assuming that “not even” means “odd”; however, the definitions of even and odd do not support this immediately—it must be proved. This exercise provides an outline of

this proof.

(a) First, prove that 0 is even (based on the definition of even).

(b) Next, prove that if n is even then $n+1$ and $n-1$ are odd. Note that you will need to make use of the closure principle.

(c) Next, prove that if n is odd then $n+1$ and $n-1$ are even. Note that you will need to make use of the closure principle.

(d) These first three steps prove that each integer can be labeled as even or odd. Why did we need to start with showing that 0 is even?

(e) Next, prove that if n is even then it is not odd (by contradiction), using your equations from the previous problem.

(f) Finally, prove that if n is odd then it is not even. This should be a one line proof! Do not repeat the proof from the previous step.