Second Ring Isomorphism Theorem

Given a ring R and $I \trianglelefteq R$, we want to classify the ideals of R/I.

Theorem

Let R be a ring with $1 \neq 0$ and $I \trianglelefteq R$:

$$I = R \iff \exists\, r \in I, r \text{ is a unit in } R$$

Proof

$\implies$ Assume $I = R$

$1 \in R$ and $I = R$, so $1 \in I$
$1 \cdot 1 = 1$
So 1 is a unit in R
Let $r = 1$

$\therefore \exists\, r \in I, r$ is a unit in R.

$\impliedby$ Assume $\exists\, r \in I, r$ is a unit in R

$\implies$ By definition, $I \subseteq R$

$\impliedby$ Assume $a \in R$

$\exists\, s \in R, rs = rs = 1 \in I$
Assume $a \in R$
$1a = a1 = a \in I$

$\therefore R \subseteq I$

$\therefore I = R$

Thus, if I contains a unit in R then $R/I = R/R = R$ and all ideals of I are just ideals of R.

Theorem

Let R be a ring and $I, J \trianglelefteq R$ such that $I \subseteq J$:

$$J/I \trianglelefteq R/I$$

Proof

From group theory, we already know that $J/I \leq R/I$

Assume $j + I \in J/I$ and $r + I \in R/I$
$(r + I)(j + I) = rj + I = j + I \in I/J$
$(j + I)(r + I) = jr + I = j + I \in I/J$

$\therefore J/I \trianglelefteq R/I$

Theorem

Let R be a ring and $I \trianglelefteq R$:

$$S \trianglelefteq R/I \implies S = J/I$$

where $J \trianglelefteq R$ and $I \subseteq J$.

Proof

Consider the map $\phi : \{\text{ideals in } R \text{ containing } I\} \to \{ \text{ ideals in } R/I\}$ where $J \mapsto J/I$

From group theory we know that the map from subgroups in R containing I to subgroups of R/I is a bijection, and ϕ is simply a restriction of this, so ϕ is at least one-to-one.

Now, assume $S \trianglelefteq R/I$
From group theory, we know that $S = J/I$ for some (normal) subgroup J in R containing I
Assume $j \in J$ and $r \in R$
$(j + I)(r + I) = jr + I \in J/I$
So $\exists\, j' \in J, jr + I = j' + I$
$jr - j' = i \in I \subseteq J$
Thus, by closure, $jr = j' + i \in J$, and so J is a right ideal in R

Similarly, $(r + I)(j + I) = jr + I \in J/I$
So $\exists\, j' \in J, rj + I = j' + I$
$rj - j' = i \in I \subseteq J$
Thus, by closure, $rj = j' + i \in J$, and so J is a left ideal in R

So $J \trianglelefteq R$ and $\phi(J) = S$, so ϕ is onto, and thus a bijection

Therefore $S \trianglelefteq R/I \implies S = J/I$.

Theorem

Let R be a ring, $I \trianglelefteq R$, and $I \subseteq J \trianglelefteq R$:

$$(R/I)/(J/I) \simeq R/J$$

Proof

Consider $\phi : R/I \to R/J$ where $r + I \mapsto r + J$
This is clearly a homomorphism of rings
$\ker(\phi) = J/I$ because $r + J = J \iff r \in J$

Therefore, by the first homomorphism theorem: $(R/I)/(J/I) \simeq R/J$.

Example

Find all of the ideals of $Z/12Z$

Mantra: To contain is to divide.

$12Z \subset 6Z \subset 3Z \subset Z$
$12Z \subset 4Z \subset 2Z \subset Z$

So the ideals of $Z/12Z$ are $\{12Z/Z, 12Z/6Z, 12Z/4Z, 12Z/3Z, 12Z/2Z, 12Z/Z\}$.