Math-42 Worksheet #4

Predicates and Qualifiers

1. Let $P(x)$ be the predicate "x is a rational number." Determine the truth values of the following propositions:
   (a) $P(0)$
   (b) $P\left(\frac{1}{2}\right)$
   (c) $P(\pi)$
   (d) $P(0.125)$
   (e) $P(-5)$
   (f) $P(123.45\overline{67})$
   (g) $P(\sqrt{2})$

2. Let $P(x, y)$ be the predicate "xy is an even number." Determine the truth values of the following propositions:
   (a) $P(2, 4)$
   (b) $P(3, 1)$
   (c) $P(5, -2)$
   (d) $P(0, \pi)$
   (e) $P\left(\pi, \frac{1}{\pi}\right)$

3. Let $x \in \{-1, 0, 1, 2, 3, 5, 9\}$ and let $P(x) \coloneqq x \leq 5$. Determine the truth values of the following quantified propositions. State a counterexample for false universal quantifiers and an example for true existential quantifiers.
   (a) $\forall\, x P(x)$
   (b) $\neg\forall\, x P(x)$
   (c) $\forall\, x \neg P(x)$
   (d) $\exists\, x P(x)$
   (e) $\neg\exists\, x P(x)$

(f) $\exists\, x \neg P(x)$

4. Recall that $p \rightarrow q$ is true whenever p is false. Thus, constructs like:

$$\forall\, x(P(x) \rightarrow Q(x))$$

let $P(x)$ act like a filter to only select particular values of x. If $P(x)$ is false then the implication is true and the truth value of the universal quantifier is unaffected. So let x and $P(x)$ be the same as the previous problem and let $Q(x) := x > 0$. Determine the truth values of the following quantified propositions. State a counterexample for false universal quantifiers.

   (a) $\forall\, x(P(x) \rightarrow Q(x))$

   (b) $\forall\, x(P(x) \rightarrow \neg Q(x))$

   (c) $\forall\, x(\neg P(x) \rightarrow Q(x))$

   (d) $\forall\, x(P(x) \rightarrow \neg Q(x))$

5. Let $R(x, y) :=$ "x is related to y" and $T(x) :=$ "x is a teenager", where x and y are taken from a particular set of people. Convert each of the following to logic expressions:

   (a) George is related to Mary.

   (b) Everyone is related to Mary.

   (c) Someone is related to Mary.

   (d) Mary is related to someone.

   (e) No one is related to Mary.

   (f) Someone is not related to Mary.

   (g) No one is related to themselves.

   (h) If someone is a teenager then they are related to Mary.

   (i) If someone is related to Mary then they are a teenager.

   (j) Everyone is a teenager and related to Mary.

6. Negate each expression in the previous problem (hint: deMorgan).

7. An important concept that we will use in our first proofs is the concept of *closure*. Closure states that if you perform an operation on one or more operands (e.g., addition or multiplication) then the result is the same type of object as the two operands. For example, if you add any two natural numbers then the result must be a natural number. If $\mathbb{N}$ is the set of natural numbers, then:

$$\forall\, n, m \in \mathbb{N}, n + m \in \mathbb{N}$$

is a statement of the closure of natural numbers under addition.

(a) Convince yourself that the natural numbers are in fact closed under addition.

(b) Construct a similar logical expression for the closure of natural numbers under multiplication. Is it true?

(c) If $\mathbb{Z}$ is the set of integers, write similar logical expressions for the closure of integers under addition and multiplication. Are they true?

(d) If $\mathbb{Q}$ is the set of rational numbers, write similar logical expressions for the closure of rational numbers under addition and multiplication. Are they true?

(e) If $\mathbb{R}$ is the set of real numbers, write similar logical expressions for the closure of real numbers under addition and multiplication. Are they true?

8. The set of irrational numbers is *not* closed under addition or multiplication. Find counterexamples where adding or multiplying two irrational numbers results in a rational number.