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Math 231b
Homework #1 Rewrite

1.7.45

Show that $L(f)(x) = \int_0^x f(t)dt$ defines a continuous linear mapping from $\mathcal{C}[0,1]$ into itself.

Claim: L is linear.

Assume $f, g \in \mathcal{C}[0,1]$ and $\alpha, \beta \in \mathbb{F}$:

$$L(\alpha f + \beta g) = \int_0^x (\alpha f + \beta g) = \alpha \int_0^x f + \beta \int_0^x g = \alpha Lf + \beta Lg$$

Therefore, L is linear.

Furthermore, by the FTC, since $f \in \mathcal{C}[0,1]$, it must be the case that $Lf \in \mathcal{C}[0,1]$.

Claim: L is continuous.

Let the norm be the sup norm: $\|\cdot\|_\infty$.

Assume (f_n) is a sequence in $\mathcal{C}[0,1]$ such that $f_n \to f$ in the norm.

Thus $\|f_n(x) - f(x)\| \to 0$.

Check for pointwise converge and take the $\sup$ later:

$$\begin{array}{rcl}
|(Lf_n)(x) - (Lf)(x)| & = & |L(f_n(x) - f(x))| \\
& = & \left| \int_0^x (f_n(t) - f(t))dt \right| \\
& \leq & \int_0^x |f_n(t) - f(t)|\, dt \\
& \leq & \int_0^x \max_{t \in [0,x]} |f_n(t) - f(t)|\, dt \\
& = & \int_0^x \|f_n(t) - f(t)\|\, dt \\
& = & \|f_n(x) - f(x)\| \\
& \to & 0
\end{array}$$

So we have $|(Lf_n)(x) - (Lf)(x)| \to 0$ and thus:

$$\max_{x \in [0,1]} |(Lf_n)(x) - (Lf)(x)| = \|(Lf_n)(x) - (Lf)(x)\| \to 0$$

Therefore, L is continuous.