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Math 231b
Homework #1

## 1.7.9

Prove: $1 \leq p < q \implies \ell^p$ is a proper subspace of $\ell^q$.

Assume $1 \leq p < q$.
It was previously shown that $\ell^p$ and $\ell^q$ are both vector spaces, so it sufficies to show that $\ell^p \subset \ell^q$.

Assume $(a_n) \in \ell^p$.
By definition: $\sum_{n=1}^{\infty} |a_n|^p < \infty$.
By the convergence theorem, it must be the case that $|a_n|^p \to 0$, and thus $|a_n| \to 0$.
So $\exists\, N > 0$ such that $n > N \implies |a_n| < 1$.
Assume $n > N$.
Since $|a_n| < 1$ and $1 \leq p < 1$, we have $|a_n|^q < |a_n|^p$.
And so by the comparison theorem: $\sum_{n>N} |a_n|^q < \infty$.
And then by the tail convergence theorem: $\sum_{k=1}^{\infty} |a_n|^q < \infty$.

$\therefore\, (a_n) \in \ell^q$ and $\ell^p \subseteq \ell^q$.

Now, consider $a_n = \left(\frac{1}{n}\right)^{\frac{1}{p}}$.
$\sum_{n=1}^{\infty} \left[\left(\frac{1}{n}\right)^{\frac{1}{p}}\right]^p = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges (harmonic series).
$\sum_{n=1}^{\infty} \left[\left(\frac{1}{n}\right)^{\frac{1}{p}}\right]^q = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\frac{q}{p}}$, which converges since $\frac{q}{p} > 1$.
Thus, $(a_n) \in \ell^q$ but $(a_n) \notin \ell^p$.

Therefore, $\ell^p$ is a proper subset of $\ell^q$.

## 1.7.14

Prove: $\mathcal{C}(\Omega)$, $\mathcal{C}^k(\mathbb{R}^N)$, and $\mathcal{C}^\infty(\mathbb{R}^N)$ are infinite dimensional, where $\Omega$ is a open subset of $\mathbb{R}^N$.

Consider $\mathcal{P}(\Omega)$, the vector space of all polynomials in $N$ variables over domain $\Omega$.
Since polynomials are infinitely continuously differentiable, $\mathcal{P}(\Omega)$ is a subspace of $\mathcal{C}(\Omega)$, and $\mathcal{P}(\Omega)$ with $\Omega = \mathbb{R}$ is a subspace of both $\mathcal{C}^k(\mathbb{R}^N)$ and $\mathcal{C}^\infty(\mathbb{R}^N)$.
Claim: $\mathcal{P}(\Omega)$ is infinite dimensional, and thus so are the rest.

ABC: $\mathcal{P}(\Omega)$ is finite dimensional and let $\dim \mathcal{P}(\Omega) = n$.
Then every linearly independent set of $n$ elements is a basis for $\mathcal{P}(\Omega)$.
Let $\{1, x, x^2, \ldots, x^n\}$ be such a basis (i.e., the powers of the other $n-1$ variables are $0$).
Consider $x^{n+1} \in \mathcal{P}(\Omega)$.
But $x^{n+1} \notin \mathrm{Span}\{1, x, x^2, \ldots, x^n\} \implies$ CONTRADICTION!

Therefore, $\mathcal{P}(\Omega)$ is infinite dimensional.

### 1.7.15

Denote by $\ell_0$ the space of all infinite sequences of complex numbers $(z_n)$ such that $z_n = 0$ for all but a finite number of indices $n$. Find a basis for $\ell_0$.

Let $e_n$ be the sequence in $\ell_0$ such that the $n^{th}$ element is $1$ and all other elements are $0$.

Claim: $E = \{e_n \mid n \in \mathbb{N}\}$ is a basis for $\ell_0$.

Assume $S = \{e_{n_1}, e_{n_2}, \ldots, e_{n_r}\}$ is a finite subset of $E$.
Assume $\sum_{k=1}^{r} \alpha_k e_{n_k} = (0)$.
Consider the $j^{th}$ element in the sequence:

$$z_j = a_j \cdot 1 + \sum_{j \neq k} (a_k \cdot 0) = 0$$

And so $a_j = 0$. This means that all of the $a_k = 0$ and so $S$ is a linearly independent set.
Thus, every finite subset of $E$ is a linearly independent set.

Therefore, $E$ is a linearly independent set.

Now, assume $(z_n) \in \ell_0$ such that the last non-zero element occurs in the $r^{th}$ position.
Consider $S = \{e_n \mid 1 \leq n \leq r\} \subset E$.

$$\begin{aligned} (z_n) &= (z_1, z_2, \ldots, z_r, 0, \ldots) \\ &= (z_1, 0, \ldots) + (0, z_2, 0, \ldots) + \cdots + (0, \ldots, 0, z_r, 0, \ldots) \\ &= \sum_{k=1}^{r} z_k e_k \end{aligned}$$

And so $(z_n) \in \operatorname{Span} S$.
Thus, every element in $\ell_0$ is in the span of some finite subset of $\ell_0$.

Therefore, $E$ spans $\ell_0$.

Therefore, $E$ is a basis for $\ell_0$.

### 1.7.44

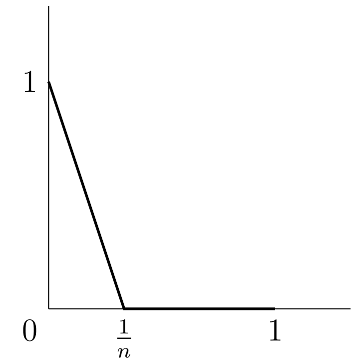
Consider the space $\mathcal{C}[a, b]$ with the norm defined as:

$$\|f\| = \int_a^b |f(t)|\, dt$$

Is this a Banach space?

Consider the following counterexample for $\mathcal{C}[0, 1]$. Define:

$$f_n(t) = \begin{cases} 1 - nt, & 0 \le t \le \frac{1}{n} \\ 0, & \frac{1}{n} \le t \le 1 \end{cases}$$




Clearly $(f_n)$ is a sequence in $\mathcal{C}[0,1]$.

Claim: $(f_n)$ is Cauchy.

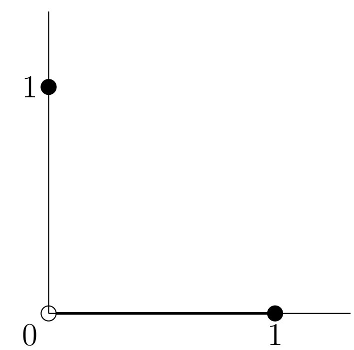
AWLOG: $n < m$. Since $\|f_n\|$ is simply the area under the triangle:

$$\|f_n - f_m\| = \frac{1}{2n} - \frac{1}{2m} \to 0$$

Therefore, $(f_n)$ is Cauchy.

Now, define:

$$f(t) = \begin{cases} 1, & t = 0 \\ 0, & 0 < t \le 1 \end{cases}$$




Claim: $f_n \to f$.

At $t = 0$: $f_n = f = 1$.

For $t \in (0,1]$: $\|f_n - f\| = \|f_n\| = \frac{1}{2n} \to 0$

Therefore, $f_n \to f$.

But clearly, $f \notin \mathcal{C}[0,1]$ and therefore $C[0,1]$ is not complete and thus is not Banach.

## 1.7.45

Show that $L(f)(x) = \int_0^x f(t)dt$ defines a continuous linear mapping from $\mathcal{C}[0,1]$ into itself.

Claim: $L$ is linear.

Assume $f, g \in \mathcal{C}[0,1]$ and $\alpha, \beta \in \mathbb{F}$:

$$L(\alpha f + \beta g) = \int_0^x (\alpha f + \beta g) = \alpha \int_0^x f + \beta \int_0^x g = \alpha L f + \beta L g$$

Therefore, $L$ is linear.

Furthermore, by the FTC, since $f \in \mathcal{C}[a,b]$, it must be the case that $Lf \in \mathcal{C}[a,b]$.

Claim: $L$ is continuous.

Assume $(f_n)$ is a sequence in $\mathcal{C}[a,b]$ such that $f_n \to f$. Note that $\mathcal{C}[a,b]$ need not be Banach.
Thus $\|f_n - f\| \to 0$.
$\|Lf_n - Lf\| = \|L(f_n - f)\| = \left\|\int_0^x (f_n - f)\right\| \leq \int_0^x \|f_n - f\| \to 0$ (since $\|f_n - f\| \to 0$).

Therefore, $L$ is continuous.

## 1.7.46

Give an example of a linear mapping from a normed space into a normed space that is not continuous.

Let $E = \mathcal{C}^\infty[0,\pi]$ and let $L = D$ (the differentiation operator) and let the norm be the uniform convergence norm:

$$\|f(x)\| = \|f(x)\|_\infty = \sup_{x \in [0,\pi]} |f(x)|$$

$D$ is clearly linear:

$$D(\alpha f + \beta g) = \alpha Df + \beta Dg$$

Furthermore, by definition, $Df \in \mathcal{C}^\infty[0,\pi]$.

Claim: $D$ is not continuous.

Consider the counterexample:

$$f_n(x) = \frac{\sin(nx)}{n}$$

Note that $\|f_n(x)\| = \frac{1}{n} \to 0$, which occurs at $x = \frac{\pi}{2n} \in [0,\pi]$.
And thus $f_n$ converges to the zero function.
$\|Df_n - Df\| = \|Df_n\| = \left\|\frac{n\cos(nx)}{n}\right\| = \|\cos(nx)\| = 1$, which occurs at $x = 0 \in [0,\pi]$.

Therefore, $D$ is not continuous.

### 1.7.52

Let $E = \mathcal{C}^\infty[a,b]$ be the space of infinitly differentiable functions on the interval $[a,b]$ with $\|f\| = \max_{x\in[a,b]} |f(x)|$. Is the differential operator $D = \frac{d}{dx}$ a contraction mapping?

Claim: $D$ is not a contraction mapping.

Let $E = \mathcal{C}^\infty[0,\pi]$, $f(x) = \sin x$, and $g(x) = \cos x$.

$\|\sin x - \cos x\| = \left\|\sqrt{2}\sin(x - \frac{\pi}{4})\right\| = \sqrt{2}$, which occurs at $x = \frac{3\pi}{4} \in [0,\pi]$.

$\|D\sin x - D\cos x\| = \|\cos x + \sin x\| = \left\|\sqrt{2}\sin(x + \frac{\pi}{4})\right\| = \sqrt{2}$, which occurs at $x = \frac{\pi}{4} \in [0,\pi]$.

And so there is no $\lambda \in (0,1)$ such that $\|D\sin x - D\cos x\| \leq \lambda \|\sin x - \cos x\|$.

Therefore, $D$ is not a contraction mapping.