Other Types of Equations

Previously we looked at linear, rational, and quadratic equations. We will now extend that list by looking at:

1). Polynomial equations

2). More on rational equations

3). Quadratic-like equations

4). Absolute value equations

5). Equations involving radicals/rational exponents

Polynomial Equations

General form: $P(x) = 0$, where $P(x)$ is a polynomial.

We have already looked at linear (degree=1) and quadratic (degree=2). For degrees ≥ 2, the goal is to factor them so that we can apply the property of 0 to find one solution for each factor.

Example

$(x-1)(x+2)(x-3)(x+4)^2 = 0$

$x = 1, -2, 3, -4, -4$

Note that if we expanded this, the leading term would be x^4. So the goal is to factor a polynomial equation of degree n into n linear factors, each providing a solution (some repeated). This is not always possible, so the best that we can say is that the actual number of solutions is $\leq n$.

More of this in chapter 3, but for now, we will focus on polynomials that we can easily factor:

1). Factoring out powers of x:

$$\begin{aligned} x^4 - 3x^3 + 2x^2 &= 0 \\ x^2(x^2 - 3x + 2) &= 0 \\ x^2(x-1)(x-2) &= 0 \\ x &= 0, 1, 2 \end{aligned}$$

2). Patterns like difference of squares:

$$\begin{aligned} x^4 - 1 &= 0 \\ (x^2 - 1)(x^2 + 1) &= 0 \\ (x+1)(x-1)(x^2+1) &= 0 \end{aligned}$$

$$x = \pm 1$$

3). Grouping

$$\begin{aligned}
x^3 - x^2 - x + 1 &= 0 \\
x^2(x-1) - (x-1) &= 0 \\
(x-1)(x^2-1) &= 0 \\
(x-1)(x-1)(x+1) &= 0 \\
(x-1)^2(x+1) &= 0 \\
x &= \pm 1
\end{aligned}$$

More on Rational Functions

I want to highlight the trick of multiplying both sides by the common denominator as a short cut for eliminating the fractions. Note that since none of the factors in the common denominator can be 0, we can multiply without fear of 0; however, we need to make sure that all of our found solutions are in the domain.

Example

$$\begin{aligned}
\frac{1}{x^2-9} + \frac{2x}{x+3} - \frac{1}{2} &= \frac{4}{x-3} \\
2 + 4x(x-3) - (x^2-9) &= 8(x+3) \\
2 + 4x^2 - 12x - x^2 + 9 &= 8x + 24 \\
3x^2 - 20x - 13 &= 0 \\
x &= \frac{20 \pm \sqrt{(-20)^2 - 4(3)(-13)}}{2(3)} \\
&= \frac{20 \pm \sqrt{400+156}}{6} \\
&= \frac{20 \pm \sqrt{556}}{6} \\
&= \frac{20 \pm 2\sqrt{139}}{6} \\
&= \frac{10 \pm \sqrt{139}}{3}
\end{aligned}$$

Quadratic-like Equations

$$\begin{aligned}
x^4 - 7x^2 + 12 &= 0 \\
(x^2)^2 - 7(x^2) + 12 &= 0 \\
(x^2 - 3)(x^2 - 4) &= 0 \\
(x - \sqrt{3})(x + \sqrt{3})(x - 2)(x + 2) &= 0 \\
x &= \pm\sqrt{3}, \pm 2
\end{aligned}$$

$$\begin{aligned}
x^4 - 2x^2 - 2 &= 0 \\
(x^2)^2 - 2(x^2) - 2 &= 0 \\
x^2 &= \frac{2 \pm \sqrt{2^2 - 4(1)(-2)}}{2(1)} \\
&= \frac{2 \pm \sqrt{4 + 8}}{2} \\
&= \frac{2 \pm \sqrt{12}}{2} \\
&= \frac{2 \pm 2\sqrt{3}}{2} \\
x^2 &= 1 \pm \sqrt{3} \\
|x| &= [1 \pm \sqrt{3}]^{\frac{1}{2}} \\
|x| &= [1 + \sqrt{3}]^{\frac{1}{2}} \\
x &= \pm[1 + \sqrt{3}]^{\frac{1}{2}} \\
x &= \pm\sqrt{1 + \sqrt{3}}
\end{aligned}$$

Absolute Value Equations

Remember:

$$|a| = c \implies a = \pm c$$

If absolute value equations, always try to isolate the absolute value part on one side before doing plus/minus:

$$\begin{aligned}
2|x + 1| - 1 &= 0 \\
2|x + 1| &= 1
\end{aligned}$$

$$\begin{aligned} |x+1| &= \frac{1}{2} \\ x+1 &= \pm\frac{1}{2} \\ x &= -\frac{3}{2}, -\frac{1}{2} \end{aligned}$$

Also remember:

$$|a| = |c| \implies a \pm c$$

But when the absolute value is a term in an expression, we need to be a bit more careful and unwind the absolute values one at a time:

$$|x+1| = \frac{|x|-1}{2}$$

$$x+1 = \pm\left(\frac{|x|-1}{2}\right)$$

$$\begin{aligned} x+1 &= \frac{|x|-1}{2} \\ 2x+2 &= |x|-1 \\ |x| &= 2x+3 \\ x &= \pm(2x+3) \end{aligned}$$

$$\begin{aligned} x+1 &= -\left(\frac{|x|-1}{2}\right) \\ 2x+2 &= 1-|x| \\ |x| &= -2x-1 \\ x &= \pm(-2x-1) \end{aligned}$$

$$\begin{aligned} x &= 2x+3 \\ x &= -3 \end{aligned}$$

$$\begin{aligned} x &= -(2x+3) \\ x &= -2x-3 \\ 3x &= -3 \\ x &= -1 \end{aligned}$$

$$\begin{aligned} x &= -2x-1 \\ 3x &= -1 \\ x &= -\frac{1}{3} \end{aligned}$$

$$\begin{aligned} x &= -(-2x-1) \\ x &= 2x+1 \\ x &= -1 \end{aligned}$$

So we have three candidate solutions: $x = -3, -1, -\frac{1}{3}$. Do they all work?

$$\begin{aligned} |-3+1| &\stackrel{?}{=} \frac{|-3|-1}{2} \\ |-2| &\stackrel{?}{=} \frac{3-1}{2} \end{aligned}$$

$$\begin{aligned} 2 &\stackrel{?}{=} \frac{2}{2} \\ 2 &\neq 1 \end{aligned}$$

$$\begin{aligned} |-1+1| &\stackrel{?}{=} \frac{|-1|-1}{2} \\ |0| &\stackrel{?}{=} \frac{1-1}{2} \\ 0 &\stackrel{?}{=} \frac{0}{2} \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} \left|-\frac{1}{3}+1\right| &\stackrel{?}{=} \frac{\left|-\frac{1}{3}\right|-1}{2} \\ \left|\frac{2}{3}\right| &\stackrel{?}{=} \frac{\frac{1}{3}-1}{2} \\ \frac{2}{3} &\stackrel{?}{=} \frac{-\frac{2}{3}}{2} \\ \frac{2}{3} &\neq -\frac{1}{3} \end{aligned}$$

Thus, the only solution is $x = -1$, the others are extraneous.

Radicals and Rational Exponents

When we have something like $(x+1)^{\frac{2}{3}}$ in an equation, we need to peel off the exponent somehow in order to get to the variable.

Remember:

$$(a^n)^{\frac{1}{n}} = \begin{cases} a, & a \text{ n odd} \\ |a|, & a \text{ n even} \end{cases}$$

$$(a^{\frac{1}{n}})^n = \begin{cases} a, & a \text{ n odd} \\ a, & a \text{ n even, since } a \geq 0 \end{cases}$$

But what about $a^{\frac{p}{q}}$?

1). Let's make sure we remember what this means:

$a^{\frac{p}{q}} = (a^{\frac{1}{q}})^p = (a^p)^{\frac{1}{q}} = \sqrt[q]{a^p}$

$8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = 2^2 = 4$
$8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = 64^{\frac{1}{3}} = 4$

2). Remember that we have to accept an equation how it is written.

So something like $x^{\frac{2}{6}}$ cannot be simplified to $x^{\frac{1}{3}}$ until after we have determined domain and start to evaluate.

3). We want to take the appropriate root to both sides of an equation in order to unwrap the variables:

$x^{\frac{2}{3}} = 4$
$(x^{\frac{2}{3}})^{\frac{3}{2}} = 4^{\frac{3}{2}}$
$x = (4^{\frac{1}{2}})^3$
$x = 2^3$
$x = 8$

case 1: p, q odd

No problems, just evaluate:

$(x-1)^{\frac{3}{5}} = 8$
$x - 1 = 8^{\frac{5}{3}}$
$x - 1 = 32$
$x = 33$

case 2: q even

Since we are taking an even root, we can assume that $a \geq 0$:

$(x-1)^{\frac{3}{2}} = 8$
$x - 1 = 8^{\frac{2}{3}}$
$x - 1 = 4$
$x = 5$

Be sure that the candidate solution is in the domain and is not extraneous:

$(x-1)^{\frac{3}{2}} = -8$
$x - 1 = (-8)^{\frac{2}{3}}$
$x - 1 = 4$
$x = 5$

But this solution is extraneous, since the principle root is never < 0.

case 3: p even

This is the absolute value case:

$(x-1)^{\frac{2}{3}} = 4$
$|x-1| = 4^{\frac{3}{2}}$
$x - 1 = \pm 8$
$x = -7, 9$

If you forget to take the absolute value then you lose the -7 solution.

The following has no solution because a square is never negative:

$(x-1)^{\frac{2}{3}} = -4$
$|x-1| = (-4)^{\frac{3}{2}}$

Also seen by not being able to take the square root of a negative value.

And finally, don't forget the problems where we factor out a rational exponent, and be careful of domain considerations:

$$\begin{aligned}
x^{\frac{7}{2}} + 5x^{\frac{5}{2}} - 6x^{\frac{3}{2}} &= 0 \\
x^{\frac{3}{2}}(x^2 + 5x - 6) &= 0 \\
x^{\frac{3}{2}}(x+6)(x-1) &= 0 \\
x &= 0, 1, -6
\end{aligned}$$

But note that $x = -6$ is extraneous, so $x = 0, 1$.