# Self-adjoint Operators

## Definition: Self-adjoint

Let $H$ be a Hilbert space and $A \in \mathcal{B}(H)$. To say that $A$ is *self-adjoint* means:

$$A^* = A$$

Thus, $\langle A\vec{x}, \vec{y}\rangle = \langle \vec{x}, A\vec{y}\rangle$.

## Examples

1). $H = C^N$ and $A \in \mathcal{B}(H)$.

$[A]_e = [a_{ij}]$ and $[A^*]_e = [\overline{a_{ji}}]$.
So for $A = A^*$, $a_{ij} = \overline{a_{ji}}$.
Such a matrix is called *Hermitian*.

2). Let $H = L^2[a,b]$ and let $f_0 \in H$ such that $f_0$ is continuuous and real-valued.

Let $T \in \mathcal{B}(H)$ where $Tf = f_0 f$

Claim: $T$ is self-adjoint.

$$\langle Tf, g\rangle = \int_a^b (f_0 f)\overline{g} = \int_a^b f\overline{f_0 g} = \langle f, Tg\rangle$$

3). Let $H = L^2[a,b]$ and let $T \in \mathcal{B}(H)$ where:

$$(Tf)(s) = \int_a^b K(s,t)f(t)dt$$

where $K \in L^2(Q)$ for $Q = [a,b] \times [a,b]$.

Claim: $T$ is self-adjoint iff $K(s,t) = \overline{K(t,s)}$

Assume $f, g \in L^2[a,b]$:

$$\begin{aligned}
\langle Tf, g\rangle &= \int_a^b (Tf)(s)\overline{g(s)}ds \\
&= \int_a^b \left[\int_a^b K(s,t)f(t)dt\right]\overline{g(s)}ds \\
&= \int_a^b \left[\int_a^b K(s,t)f(t)\overline{g(s)}dt\right]ds \\
&= \int_a^b \left[\int_a^b K(s,t)f(t)\overline{g(s)}ds\right]dt \\
&= \int_a^b f(t)\left[\int_a^b K(s,t)\overline{g(s)}ds\right]dt \\
&= \int_a^b f(t)\overline{\left[\int_a^b \overline{K(s,t)}g(s)ds\right]}dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_a^b f(s)\overline{\left[\int_a^b \overline{K(t,s)}g(t)dt\right]}ds \\
&= \langle f, T^* g\rangle
\end{aligned}
$$

where $(T^*g)(s) = \displaystyle\int_a^b \overline{K(t,s)}g(t)dt$

Therefore $T = T^*$ iff $K(s,t) = \overline{K(t,s)}$

**<u>Theorem</u>**

Let $H$ be a Hilbert space and let $A \in \mathcal{B}(H)$:

1). $A + A^*$ is self-adjoint.

2). $A^*A$ is self-adjoint.

<u>Proof</u>

Assume $\vec{x}, \vec{y} \in H$.

1). $\langle (A+A^*)\vec{x}, \vec{y}\rangle = \langle \vec{x}, (A+A^*)^*\vec{y}\rangle = \langle \vec{x}, (A^* + (A^*)^*)\vec{y}\rangle = \langle \vec{x}, (A^*+A)\vec{y}\rangle = \langle \vec{x}, (A+A^*)\vec{y}\rangle$

2). $\langle (A^*A)\vec{x}, \vec{y}\rangle = \langle \vec{x}, (A^*A)^*\vec{y}\rangle = \langle \vec{x}, (A^*(A^*)^*)\vec{y}\rangle = \langle \vec{x}, (A^*A)\vec{y}\rangle$

**<u>Theorem</u>**

Let $H$ be a Hilbert space and let $T \in \mathcal{B}(H)$. There exists unique self-adjoint $A, B \in \mathcal{B}(H)$ such that:

- $T = A + iB$
- $T^* = A - iB$

<u>Proof</u>

Solving for $A$ and $B$:

$$A = \frac{1}{2}(T + T^*)$$

$$B = \frac{1}{2i}(T - T^*)$$

and this solution is unique. Furthermore:

$$A^* = \left[\frac{1}{2}(T+T^*)\right]^* = \frac{1}{2}(T+T^*)^* = \frac{1}{2}(T+T^*) = A$$

$$B^* = \left[\frac{1}{2i}(T-T^*)\right]^* = -\frac{1}{2i}(T^*-T) = \frac{1}{2i}(T-T^*) = B$$

**Theorem**

Let $H$ be a Hilbert space and let $A, B \in \mathcal{B}(H)$ be self-adjoint:

$AB$ is self-adjoint iff $A$ and $B$ commute.

Proof

$\implies$ Assume $AB$ is self-adjoint.

$(AB)^* = B^*A^* = BA$

$\impliedby$ Assume $A$ and $B$ commute.

$AB = (AB)^* = BA$

**Corollary**

Let $p(z) = \sum_{k=1}^{n} \alpha_k z^n$ be a polynomial such that $\alpha_k \in \mathbb{R}$. Let $H$ be a Hilbert space and let $A \in \mathcal{B}(H)$ be self-adjoint:

$p(A)$ is self-adjoint.

Proof

$$p(A)^* = \left[\sum_{k=1}^{n} \alpha_k A^k\right]^* = \sum_{k=1}^{n} \overline{\alpha_k}(A^*)^k = \sum_{k=1}^{n} \alpha_k A^k = p(A)$$

**Theorem**

Let $H$ be a Hilbert space and let $A \in \mathcal{B}(H)$ such that $A$ is self-adjoint:

$$\|A\| = \sup_{\|\vec{x}\|=1} |\langle A\vec{x}, \vec{x}\rangle|$$

Proof

Let $M = \sup_{\|\vec{x}\|=1} |\langle A\vec{x}, \vec{x}\rangle|$.

Assume $\|\vec{x}\| = 1$.

$|\langle A\vec{x}, \vec{x}\rangle| \leq \|A\vec{x}\| \, \|\vec{x}\| = \|A\vec{x}\| \leq \|A\| \, \|\vec{x}\| = \|A\|$

$\therefore M \leq \|A\|$.