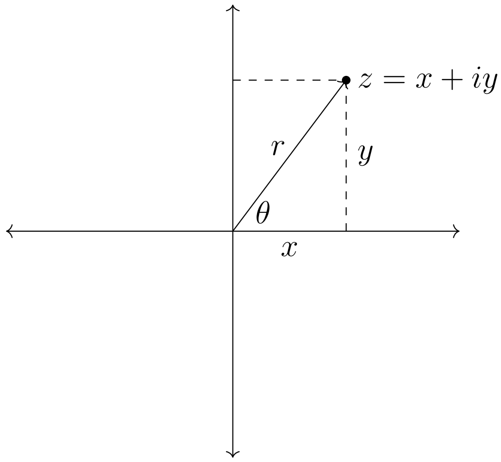
Exponential Form



$$r = \sqrt{x^2 + y^2} = |z| \neq 0$$

$$x = r\cos\theta = |z|\cos\theta$$

$$y = r\sin\theta = |z|\sin\theta$$

$$\tan\theta = \frac{y}{x}$$

Note that the calculation for θ depends on quadrant.

Definition

Let $z \in \mathbb{C}$. The polar and exponential forms for z are given by:

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + isin\theta) = |z|\,(\cos\theta + isin\theta) = |z|\,e^{i\theta}$$

Definition

Let $z = |z|\,e^{i\theta}$. θ is called the *argument* of z. The set of all coterminal angles of θ, denoted $\arg z$, is given by:

$$\arg z = \{\theta + 2\pi n \mid n \in \mathbb{Z}\}$$

For convenience, this can be shortened to:

$$\arg z = \theta + 2\pi n$$

with the understanding that $\arg z$ is actually a set and $n \in \mathbb{Z}$.

The *principle value* of $\arg z$, denoted $\operatorname{Arg} z$, is the value $\Theta \in \arg z$ such that:

$$\Theta \in (-\pi, \pi]$$

Thus:

$$\arg z = \operatorname{Arg} z + 2\pi n$$

Example

Let $z = -1 - i\sqrt{3}$.

$$|z| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\theta = \arctan\left(\frac{-\sqrt{3}}{-1}\right) = \arctan\sqrt{3} = \frac{4\pi}{3}$$

$$\Theta = \frac{4\pi}{3} - 2\pi = -\frac{2\pi}{3}$$

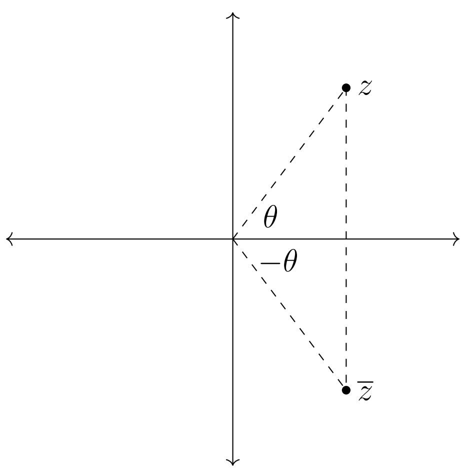
$$\arg z = \left\{-\frac{2\pi}{3} + 2\pi n \mid n \in \mathbb{Z}\right\}$$

$$z = e^{-\frac{2\pi}{3}+2\pi n} = e^{\frac{2\pi}{3}(3n-1)}$$

<u>Theorem</u>

$$\arg\overline{z} = -\arg z = \arg\frac{1}{z}$$

This first part is obvious from the following diagram:



<u>Proof</u>

Let $z = |z|\, e^{i\theta}$
$\overline{z} = |\overline{z}|\, e^{-i\theta} = |z|\, e^{-i\theta}$
$\operatorname{Arg} z = -\operatorname{Arg}\overline{z}$
$\operatorname{Arg}\overline{z} + 2\pi n = -\operatorname{Arg} z + 2\pi n$
$\therefore \arg\overline{z} = -\arg z$

$\frac{1}{z} = \frac{1}{|z|e^{i\theta}} = \frac{1}{|z|}e^{-i\theta}$
$\operatorname{Arg}\frac{1}{z} = -\operatorname{Arg} z = \operatorname{Arg}\overline{z}$
$\operatorname{Arg}\frac{1}{z} + 2\pi n = \operatorname{Arg}\overline{z} + 2\pi n$
$\therefore \arg\frac{1}{z} = \arg\overline{z}$

Theorem

Let $z_1 = |z_1| e^{i\theta_1}$ and $z_2 = |z_2| e^{i\theta_2}$:

1). $\arg(z_1 z_2) = \theta_1 + \theta_2 + 2\pi n$

2). $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 + 2\pi n$

Proof

1). $\arg(z_1 z_2) = \arg\left(|z_1| e^{i\theta_1} |z_2| e^{i\theta_2}\right) = \arg\left(|z_1| |z_2| e^{i(\theta_1+\theta_2)}\right) = \theta_1 + \theta_2 + 2\pi n$

2). $\arg\left(\frac{z_1}{z_2}\right) = \arg\left(\frac{|z_1| e^{i\theta_1}}{|z_2| e^{i\theta_2}}\right) = \arg\left(\frac{|z_1|}{|z_2|} e^{i(\theta_1-\theta_2)}\right) = \theta_1 - \theta_2 + 2\pi n$

Example

Let $z_1 = i$ and $z_2 = -1 + i$

$z_1 = e^{i\frac{\pi}{2}}$ and $z_2 = \sqrt{2} e^{i\frac{3\pi}{4}}$

$\theta_1 + \theta_2 = \frac{\pi}{2} + \frac{3\pi}{4} = \frac{5\pi}{4}$

$\mathrm{Arg}(z_1 z_2) = \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$

$\arg(z_1 z_2) = -\frac{3\pi}{4} + 2\pi n$

Theorem

Let $z = e^{i\theta}$:

$$z^k + \frac{1}{z^k} = 2\cos k\theta$$

Proof

$$z^k + \frac{1}{z^k} = e^{ik\theta} + e^{-ik\theta} = 2\cos k\theta$$

Note that $z = z_0 + Re^{i\theta}$ is the circle with center z_0 and radius R: