Equivalence Relations and Partitions

Definition

An *equivalence relation* on a set A is a relation on A that satisfies the following three properties:

1). **Reflexive**

$$\forall\, a \in A, a \sim a$$

2). **Symmetric**

$$\forall\, a, b \in A, a \sim b \implies b \sim a$$

3). **Transitive**

$$\forall\, a, b, c \in A, a \sim b \text{ and } b \sim c \implies a \sim c$$

Definition

To say that two sets A and B are *disjoint* means:

$$A \cap B = \emptyset$$

To say that a family of indexed sets $\{A_i \mid i \in I\}$ is *mutually disjoint* means:

$$\forall\, i, j \in I, i \neq j \implies A_i \cap A_j = \emptyset$$

Definition

Let A be a non-empty set. To say that an indexed family of sets $\{A_i \mid i \in I\}$ *partitions* A means:

1). All of the A_i are non-empty:

$$\forall\, i \in I, A_i \neq \emptyset$$

2). The A_i are mutually disjoint:

$$\forall\, i, j \in I, i \neq j \implies A_i \cap A_j = \emptyset$$

3). The union of the A_i equals A:

$$A = \bigcup_{i \in I} A_i$$

Each A_i is called a *cell* of the partition.

Definition

An *equivalence class* of an equivalence relation on a set A is given by:

$$\bar{a} = \{b \in A \mid a \sim b\}$$

Theorem

An equivalence relation on a set A defines a partition of A, where the equivalence classes of the equivalence relation are the cells of the partition.

Proof

1). Assume $\bar{a}$ is an equivalence class.
$a \in \bar{a}$
$\bar{a} \neq \emptyset$
Therefore, the equivalence classes are non-empty.

2). Assume $\bar{a} \cap \bar{b} \neq \emptyset$
$\exists\, y \in A, y \in \bar{a} \cap \bar{b}$
$y \in \bar{a}$ and $y \in \bar{b}$
$y \sim a$ and $y \sim b$

Assume $x \in \bar{a}$
$x \sim a$
$a \sim y$
$x \sim y$
$x \sim b$
$x \in \bar{b}$

Assume $x \in \bar{b}$
$x \sim b$
$b \sim y$
$x \sim y$
$x \sim a$
$x \in \bar{a}$

$\bar{a} \cap \bar{b} \neq \emptyset \implies \bar{a} = \bar{b}$
$\bar{a} \neq \bar{b} \implies \bar{a} \cap \bar{b} = \emptyset$
Therefore, the equivalence classes are mutually disjoint.

3). Assume $x \in \bigcap_{a \in A} \bar{a}$
$\exists\, a \in A, x \in \bar{a}$
$x \in A$

Assume $x \in A$
$x \sim x$
$x \in \bar{x}$
Let $a = x$
$\exists\, a \in A, x \in \bar{a}$
$x \in \bigcap_{a \in A} \bar{a}$

Therefore, $\bigcap_{a \in A} \bar{a} = A$

Theorem

A partition on a set A defines an equivalence relation on A, where the cells of the partition are the equivalence classes for the equivalence relation.

Proof

Assume $\{A_i \mid i \in I\}$ is a partition of A.

1). Assume $a \in A$
$a \in \bigcap_{i \in I} A_i$
$\exists\, i \in I, a \in A_i$
$a \in A_i$ and $a \in A_i$
$a \sim a$
Therefore, the relation is reflexive.

2). Assume $a \sim b$
$\exists\, i \in I, a \in A_i$ and $b \in A_i$
$b \in A_i$ and $a \in A_i$
$b \sim a$
Therefore, the relation is symmetric.

3). Assume $a \sim b$ and $b \sim c$
$\exists\, i \in I, a \in A_i$ and $b \in A_i$
$\exists\, j \in I, b \in A_j$ and $c \in A_j$
$b \in A_i$ and $b \in A_j$
$A_i \cap A_j \neq \emptyset$
$A_i = A_j$
$a \in A_i$ and $c \in A_i$
$a \sim c$
Therefore, the relation is transitive.