Field of Fractions

Example

Consider the injective homomorphism (monomorphism) from the integral domain $\mathbb{Z}$ to the field $\mathbb{Q}$ where $n \mapsto \frac{n}{1}$.

Definition

Let R be a commutative ring with $1 \neq 0$. Let $S \subseteq R$ such that S is multiplicatively closed. Define a binary relation on $R \times S$ as follows:

$$(a, b) \sim (c, d) \iff ad - bc = 0$$

This is an equivalence relation with the equivalence class of (a, b) denoted by $\frac{a}{b}$. The set R_S of equivalence classes is a ring that is called the *localization* of R at S.

Definition

Let R be an integral domain and let $S = R \setminus \{0\}$. Define operations:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad - bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Note that:

$$\frac{a}{b} + \frac{0}{1} = \frac{a \cdot 1 + b \cdot 0}{b \cdot 1} = \frac{a}{b}$$

$$\frac{a}{b} \cdot \frac{1}{1} = \frac{a \cdot 1}{b \cdot 1} = \frac{a}{b}$$

Also note that if $(a, b) \neq (0, 1)$ then $\frac{b}{a}$ is a valid equivalence class and:

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = \frac{ab}{ab} = \frac{1}{1}$$

Thus R_S is a field and is called the field of fractions for R.

All other localizations of and integral domain are contained in the domain's field of fractions.