# Eigenvalues of Compact, Self-adjoint Operators

## Theorem

Let $A$ be a compact, self-adjoint operator on a Hilbert space $H$:

> There exists and eigenvalue $\lambda$ of $A$ such that $|\lambda| = \|A\|$.

Proof

From a previous theorem, there exists a bounded sequence $(\vec{x}_n)$ in $H$ such that:

- $\|\vec{x}\| = 1$
- $\exists \lambda \in \sigma(A)$ such that $A\vec{x}_n - \lambda\vec{x}_n \to 0$
- $|\lambda| = \|A\|$

Since $A$ is compact, $(A\vec{x}_n)$ has a convergent subsequence $(A\vec{x}n_k)$.
Let $(A\vec{x}_{n_k}) \to \vec{y} \in H$.
And so $A\vec{x}_{n_k} - \lambda\vec{x}_{n_k} \to 0 \implies \vec{y} - \lambda\vec{x}_{n_k} \to 0 \implies \vec{x}_{n_k} \to \dfrac{1}{\lambda}\vec{x}_y$.
Thus $\vec{x}_n$ contains a convergent subsequence $\vec{x}_{n_k} \to \dfrac{1}{\lambda}\vec{x}_y = \vec{x} \in H$.
And so $A\vec{x} - \lambda\vec{x} = 0$ or $A\vec{x} = \lambda\vec{x}$ where $\vec{x} \neq 0$.

Therefore $\lambda$ is an eigenvalue of $A$ and $|\lambda| = \|A\|$.

## Corollary

Let $A$ be a compact, self-adjoint operator on a Hilbert space $H$. $\exists\, \vec{x}_0 \in H$ such that $\|\vec{x}_0\| = 1$ and:

$$|\langle A\vec{x}_0, \vec{x}_0\rangle| = \sup_{\|\vec{x}\|=1} |\langle A\vec{x}, \vec{x}\rangle|$$

Proof

By previous theorem, there exists an eigenvector $\lambda$ of $A$ such that $|\lambda| = \|A\|$.
Also by previous theorem, $\sup\limits_{\|\vec{x}\|=1} |\langle A\vec{x}, \vec{x}\rangle| = \|A\|$.

Assume $\vec{x}_0$ is an eigenvalue of $A$ such that $\|\vec{x}_0\| = 1$:

$$|\langle A\vec{x}_0, \vec{x}_0\rangle| = |\langle \lambda\vec{x}_0, \vec{x}_0\rangle| = |\lambda\,\langle \vec{x}_0, \vec{x}_0\rangle| = |\lambda|\,\|\vec{x}_0\|^2 = |\lambda| = \|A\|$$

$\therefore |\langle A\vec{x}_0, \vec{x}_0\rangle| = \sup\limits_{\|\vec{x}\|=1} |\langle A\vec{x}, \vec{x}\rangle|$

## Theorem

Let $A$ be a compact operator on a Hilbert space $H$. For all eigenvalues $\lambda$ of $A$:

$$\lambda \neq 0 \implies \dim E_\lambda < \infty$$

In other words, eigenspaces for non-zero eigenvalues are finite-dimensional.

Proof

Assume $\lambda$ is an eigenvalue of $A$ such that $\lambda \neq 0$.
ABC: $E_\lambda$ is infinite-dimensional.
Since $E_\lambda = \ker(A - \lambda I)$, $E_\lambda$ is a closed subspace of $H$ and is thus also Hilbert (and separable).
So there exists a complete orthonormal sequence $(\vec{x}_n)$ in $E_\lambda$, where $\vec{x}_n \xrightarrow{w} 0$.
Thus, since $A$ is compact, $A\vec{x}_n \to 0$.
And so $A\vec{x}_n = \lambda\vec{x}_n \to 0$.
But $\lambda \neq 0$ and so $\vec{x}_n \to 0$.
CONTRADICTION!

Therefore $E_\lambda$ is finite-dimensional.

**Definition**

Let $H$ be a Hilbert space and let $A$ be an operator on $H$:

$$\Lambda = \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } A\}$$

**Theorem**

Let $A$ be a compact, self-adjoint operator on a Hilbert space $H$. One of the following is true:

1). $\Lambda$ is finite.

2). $L$ is infinitely-countable such that $\lambda_n \to 0$.

Proof

First, Assume $A = 0$.
Thus, $\lambda = 0$ is the only eigenvalue for $A$ and therefore $\Lambda$ is finite.

So, assume $\Lambda$ is infinite.
Since $A$ is self-adjoint and compact, eigenvectors for distinct eigenvalues are orthogonal.
And so there exists an orthonormal sequence $(\vec{x}_n)$ where $\vec{x}_n$ is an eigenvector of $\lambda_n$.
But $H$ is separable and so $(\vec{x}_n)$ is countable.

Therefore $\Lambda$ is countable.

Now, since $(\vec{x}_n)$ is orthonormal, $\vec{x}_n \xrightarrow{w} 0$.
But $A$ is compact and so $A\vec{x} \to 0$.

$$\|A\vec{x}\| = \|\lambda\vec{x}\| = |\lambda| \, \|\vec{x}\| = |\lambda| \to 0$$