# Joint distributions

**– Math 161a, Spring 2019, San Jose State University**

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# Outline

# Introduction

So far we have considered the distribution of only a single random variable, discrete or continuous.

When two or more random variables are defined on the same sample space, we can talk about their **joint distribution**.

**Ex 0.1** (Toss two fair dice)**.** Let $X$ denote the sum and $Y$ the absolute value of their difference. These are two discrete random variables, and we can find their individual distributions easily:

| $x$ | 2 | 3 | $\cdots$ | 12 |
|---|---|---|---|---|
| $P(X=x)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\cdots$ | $\frac{1}{36}$ |

| $y$ | 0 | 1 | $\cdots$ | 5 |
|---|---|---|---|---|
| $P(Y=y)$ | $\frac{6}{36}$ | $\frac{10}{36}$ | $\cdots$ | $\frac{2}{36}$ |

Now consider $X, Y$ together as a pair $(X, Y)$, or a vectored-valued function.

**Questions**:

- Can $(X, Y)$ attain all the $66 = 11 \times 6$ pairs?

$$\{(x, y) \mid 2 \leq x \leq 12,\ 0 \leq y \leq 5\}$$

  If not all, identify the subset of feasible pairs.

- What are the corresponding probabilities for $(X, Y)$ to take those (feasible) pairs as values?

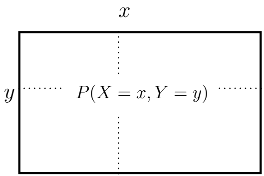
Answering the above two questions together is equivalent to specifying the **joint probability distribution of** $(X, Y)$ in terms of range and frequency.

# Joint pmf

**Def 0.1.** Let $X, Y$ be two discrete random variables associated to the same sample space. We define the joint pmf $f : \mathbb{R}^2 \to \mathbb{R}$ for $X, Y$ as

$$f(x,y) = \begin{cases} P(X = x, Y = y), & \text{for all feasible } (x,y) \\ 0, & \text{otherwise} \end{cases}$$

**Remark**. $f(x, y)$ can be conveniently displayed as a table.



**Ex 0.2.** Find the joint pmf of $X, Y$ in the previous example.

| $y$ \ $x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | | | |
| 1 | | | | | | | | | | | |
| 2 | | | | | | | | | | | |
| 3 | | | | | | | | | | | |
| 4 | | | | | | | | | | | |
| 5 | | | | | | | | | | | |

| $y \backslash x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ |
| 1 | | | | | | | | | | | |
| 2 | | | | | | | | | | | |
| 3 | | | | | | | | | | | |
| 4 | | | | | | | | | | | |
| 5 | | | | | | | | | | | |

# Joint distributions

| $y \backslash x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ |
| 1 | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{2}{36}$ | |
| 2 | | | | | | | | | | | |
| 3 | | | | | | | | | | | |
| 4 | | | | | | | | | | | |
| 5 | | | | | | | | | | | |

| $y \backslash x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ |
| 1 | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{2}{36}$ | |
| 2 | | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | |
| 3 | | | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | | |
| 4 | | | | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | | | |
| 5 | | | | | | $\frac{2}{36}$ | | | | | |

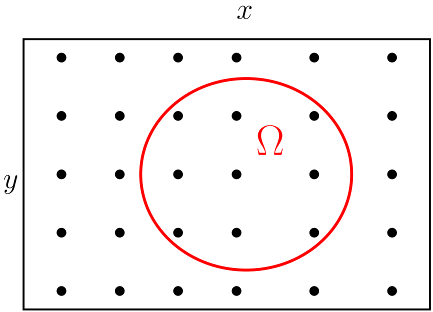
## Properties of the joint pmf

Any joint pmf $f(x,y) : \mathbb{R}^2 \mapsto \mathbb{R}$ must satisfy (and vice versa)

- $f(x,y) \geq 0$ for all $x, y \in \mathbb{R}$
- $f(x,y) > 0$ for finitely or countably many pairs $(x,y)$;
- $\sum_x \sum_y f(x,y) = 1$.

**Theorem 0.1.** *Let $X, Y$ be two discrete random variables with joint pmf $f(x, y)$. Then for any region $\Omega \subset \mathbb{R}^2$,*

$$P((X, Y) \in \Omega) = \sum_{(x,y)\in\Omega} f(x, y)$$



**Ex 0.3** (Toss 2 fair dice, cont'd). Find the following probabilities:

- $P(X \leq 4, Y \leq 2) =$
- $P(X \leq 5) =$
- $P(X \geq 11, Y \geq 2) =$
- $P(Y \leq 1) =$

## From joint to marginal

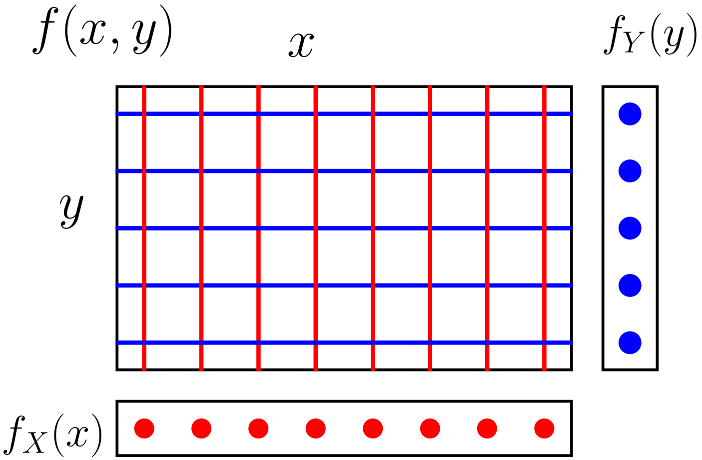
**Def 0.2.** In the joint distribution setting, we call the individual pmfs $f_X(x), f_Y(y)$ the marginal pmfs.

**Proposition 0.2.** *Let $f(x, y)$ be the joint pmf for $X, Y$. Then*

$$f_X(x) = \sum_y f(x, y), \quad \textit{and} \quad f_Y(y) = \sum_x f(x, y).$$

*Proof.* This is just the Law of Total Probability:

$$\underbrace{P(X = x)}_{f_X(x)} = \sum_y \underbrace{P(X = x, Y = y)}_{f(x,y)}.$$


$f(x,y)$
$x$
$f_Y(y)$
$y$
$f_X(x)$

| $y \backslash x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | $f_Y(y$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ | | $\frac{1}{36}$ | $\frac{6}{36}$ |
| 1 | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{10}{36}$ |
| 2 | | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | | $\frac{8}{36}$ |
| 3 | | | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | | | $\frac{6}{36}$ |
| 4 | | | | | $\frac{2}{36}$ | | $\frac{2}{36}$ | | | | | $\frac{4}{36}$ |
| 5 | | | | | | $\frac{2}{36}$ | | | | | | $\frac{2}{36}$ |
| $f_X(x)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | |

## Conditional pmfs

Consider the following question:

**Ex 0.4** (Toss 2 fair dice)**.** Suppose we are told that the sum is $X = 6$. What is the (conditional) distribution of $Y$?

**Answer**:

| $y$ | 0 | 2 | 4 |
|---|---|---|---|
| $P(Y = y \mid X = 6)$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{2}{5}$ |

**Def 0.3.** Let $X, Y$ be two discrete random variables with joint pmf $f(x, y)$. The conditional pmf of $Y$ given $X = x$ (with $f_X(x) \neq 0$)) is defined as

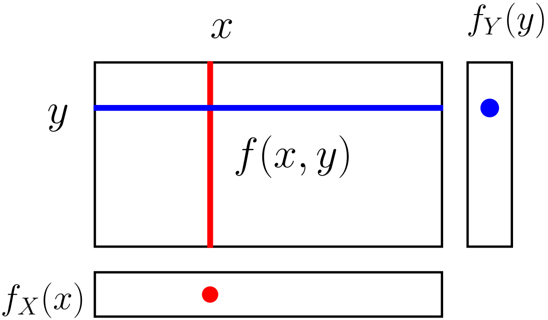
$$f(\underbrace{y}_{\text{variable}} \mid \underbrace{x}_{\text{fixed}}) = \frac{f(x,y)}{f_X(x)}, \quad \text{for all feasible } y$$

**Remarks:**

(1) This definition is just based on the conditional probability of events:

$$P(Y = y \mid X = x) = \frac{P(X = x, Y = y)}{P(X = x)}.$$

(2) **For each fixed value $x$ of $X$, there is a separate distribution for $Y$** (thus $x$ may be regarded as a location parameter).




$$f(x \mid y) = \frac{f(x,y)}{f_Y(y)}$$

$$f(y \mid x) = \frac{f(x,y)}{f_X(x)}$$

Table 1: Conditional pmfs of $Y$ given $X = x$

| $y$ \ $x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | $\frac{1}{3}$ | | $\frac{1}{5}$ | | $\frac{1}{5}$ | | $\frac{1}{3}$ | | 1 |
| 1 | | 1 | | $\frac{1}{2}$ | | $\frac{1}{3}$ | | $\frac{1}{2}$ | | 1 | |
| 2 | | | $\frac{2}{3}$ | | $\frac{2}{5}$ | | $\frac{2}{5}$ | | $\frac{2}{3}$ | | |
| 3 | | | | $\frac{1}{2}$ | | $\frac{1}{3}$ | | $\frac{1}{2}$ | | | |
| 4 | | | | | $\frac{2}{5}$ | | $\frac{2}{5}$ | | | | |
| 5 | | | | | | $\frac{1}{3}$ | | | | | |

Table 2: Conditional pmfs of $X$ given $Y = y$

| $y$ \ $x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $\frac{1}{6}$ | | $\frac{1}{6}$ | | $\frac{1}{6}$ | | $\frac{1}{6}$ | | $\frac{1}{6}$ | | $\frac{1}{6}$ |
| 1 | | $\frac{1}{5}$ | | $\frac{1}{5}$ | | $\frac{1}{5}$ | | $\frac{1}{5}$ | | $\frac{1}{5}$ | |
| 2 | | | $\frac{1}{4}$ | | $\frac{1}{4}$ | | $\frac{1}{4}$ | | $\frac{1}{4}$ | | |
| 3 | | | | $\frac{1}{3}$ | | $\frac{1}{3}$ | | $\frac{1}{3}$ | | | |
| 4 | | | | | $\frac{1}{2}$ | | $\frac{1}{2}$ | | | | |
| 5 | | | | | | 1 | | | | | |

**Ex 0.5** (Toss two fair dice). Find the following conditional distributions:

- $Y$ given $X = 4$:

| $y$ | 0 | 2 |
|---|---|---|
| $f(y \mid x = 4)$ | $\frac{1}{3}$ | $\frac{2}{3}$ |

- $X$ given $Y = 3$:

| $x$ | 5 | 7 | 9 |
|---|---|---|---|
| $f(x \mid y = 3)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |

- $X$ given $Y = 0$:

| $x$ | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|
| $f(x \mid y = 0)$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

## Independence

**Def 0.4.** Two discrete random variables $X, Y$ are independent if

$$f(x,y) = f_X(x) f_Y(y), \quad \text{for all } x, y$$

**Remark**. For discrete random variables $X, Y$, this is just

$$P(X = x, Y = y) = P(X = x) P(Y = Y).$$

**Ex 0.6** (Toss 2 fair dice)**.** Determine if $X$ (sum) and $Y$ (absolute difference) are independent.

**Proposition 0.3.** *Two discrete random variables $X, Y$ are independent if*

$$f(y \mid x) = f_Y(y), \quad \textit{for all } x, y$$

*(That is, all conditional distributions of $Y$ are identical to its marginal distribution)*

**Ex 0.7.** Are the random variables $X, Y$ independent?

| $y \backslash x$ | 0 | 1 | 2 |
|---|---|---|---|
| $-1$ | $\frac{1}{6}$ | $\frac{1}{12}$ | $\frac{1}{12}$ |
| 1 | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

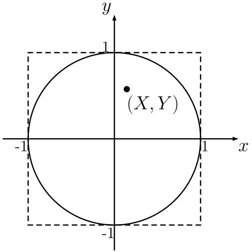
| $y \backslash x$ | 0 | 1 | 2 | $f_Y(y)$ |
|---|---|---|---|---|
| $-1$ | $\frac{1}{6}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{3}$ |
| $1$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |
| $f_X(x)$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | |

| $y \backslash x$ | 0 | 1 | 2 |
|---|---|---|---|
| $-1$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $1$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |

## Two continuous random variables

It is also possible to define joint distributions between continuous random variables.

**Ex 0.8.** Consider the game of throwing a dart toward a unit disk and let $X, Y$ be the coordinates of the landing point (assuming it is always within the disk). Individually, $X, Y$ both range from $-1$ to $1$, but the pair $(X, Y)$ does not attain every point in the square.




The joint pdf of $X, Y$ is a two dimensional function $f(x, y)$. However, probability calculations will involve multiple integration (Math 32). This is left to Math 163.