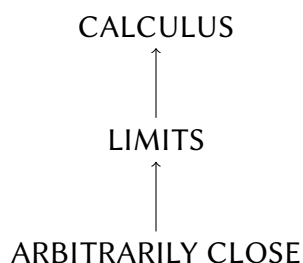


## Arbitrarily Close

- Everything that we can do in algebra is ultimately based on three things:
  1. The substitution principle.
  2. The closed and well-defined nature of addition and multiplication.
  3. The nine real number (field) axioms.
- But there are some problems that algebra cannot solve:
  1. The slope of a tangent line to a non-linear curve.
  2. The area under a non-linear curve.
- A new concept is needed to solve problems that algebra alone cannot solve: arbitrarily close.



Q: What is meant by saying that one thing is *close* to another?

A: The *distance* between them is *small*.

But this is a subjective statement. In math, we want objective facts.

### Definition: Distance

Let  $a, b \in \mathbb{R}$ . The *distance* from  $a$  to  $b$  is given by:

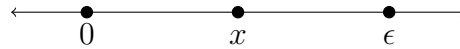
$$d(a, b) = |b - a|$$



### Properties: Distance

1.  $d(a, b) = |b - a| = |a - b| = d(b, a)$
2.  $d(a, 0) = |a - 0| = |a|$

Let  $\epsilon > 0$ . By the density of  $\mathbb{R}$ , there always exists some  $x$  such that  $0 < x < \epsilon$ .



### Example

Consider the following game:

1. Select some  $\epsilon > 0$ .
2. Select some  $x \in (0, \epsilon)$ .
3. Let  $\epsilon = x$ .
4. Go to step 2.

1

$\frac{1}{2}$

$\frac{1}{4}$

$\frac{1}{8}$

0.1

0.0001

0.00005

0.00000000001

$\vdots$

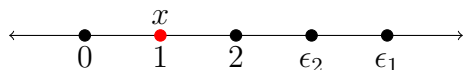
### Definition: Arbitrarily Small

To say that a value  $x \in \mathbb{R}$  is *arbitrarily small* means that for every  $\epsilon > 0$ ,  $0 < x < \epsilon$ .

- This does not imply that  $x$  is assigned a particular value nor does it say that  $x = 0$ .
- It is indicative of an infinite argument: no matter which  $\epsilon$  is selected,  $x$  is smaller (less) than  $\epsilon$  (but not 0).
- Another interpretation is that as  $\epsilon$  approaches 0,  $x$  is squeezed towards 0 as well. This is valid as long as all values of  $\epsilon$  are considered: there are ways to continually decrease  $\epsilon$  such that  $x$  gets no closer to 0.

### Example

Consider  $\epsilon > 2$  and  $x = 1$ . Now, for each step, move  $\epsilon$  to the halfway point between the previous  $\epsilon$  and 2. Although  $\epsilon$  is continually getting smaller,  $x$  is not forced to move.



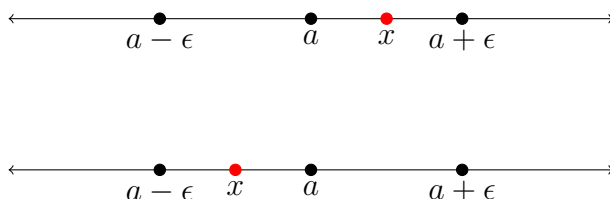
The problem is that values of  $\epsilon \leq 2$  are not considered.

### Definition: Arbitrarily Close

To say that a value  $x \in \mathbb{R}$  is *arbitrarily close* to another value  $a \in \mathbb{R}$ , denoted by  $x \rightarrow a$ , means that the distance between  $x$  and  $a$  becomes arbitrarily small (but not 0):

$$\forall \epsilon > 0, 0 < |x - a| < \epsilon$$

This means that for every  $\epsilon > 0$ ,  $a - \epsilon < x < a + \epsilon$ :



### Definition: Neighborhood

Let  $x, \epsilon \in \mathbb{R}$  such that  $\epsilon > 0$ . The open interval  $(x - \epsilon, x + \epsilon)$  is called an  $\epsilon$ -neighborhood of  $x$ .

Likewise, this can be interpreted as the  $\epsilon$ -neighborhoods getting continually smaller and squeezing  $x$  towards  $a$ . This is OK as long as all values of  $\epsilon$  are considered.

### Notation: One-sided

Note that  $x \rightarrow a$  implies that  $x$  can approach  $a$  from either direction (from the left or from the right). When we are only interested in one direction:

$x \rightarrow a^+$	$x$ approaches from the right of $a$
$x \rightarrow a^-$	$x$ approaches from the left of $a$

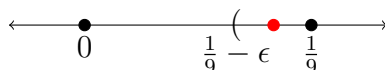
If the direction is understood then the direction indicator can be omitted. For example, if  $x > 0$  then  $x \rightarrow 0^+$  can be written as just  $x \rightarrow 0$ , which can be used to represent the fact that  $x$  gets arbitrarily small.

### Example

Recall that one of the ways of representing a rational number is a terminating infinite repeating sequence of decimal digits. For example:

$$\frac{1}{9} = 0.11111 \dots = 0.\bar{1}$$

So let  $a = \frac{1}{9}$  and let  $x$  represent an estimate of  $a$ . Then  $x$  can be made to be arbitrarily close to  $a$  by adding digits to the approximation.



How many digits are required for  $\epsilon = 0.001$ ?

$$0.\bar{9} - 0.9 = 0.0\bar{9} > 0.001$$

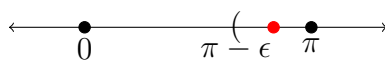
$$0.\bar{9} - 0.99 = 0.00\bar{9} > 0.001$$

$$0.\bar{9} - 0.999 = 0.000\bar{9} = 0.001$$

$$0.\bar{9} - 0.9999 = 0.0000\bar{9} = 0.0001 < 0.001$$

### Example

This works for irrational numbers as well, which are represented by terminating infinite sequences of non-repeating digits. Consider  $\pi = 3.1415926 \dots$ . For every  $\epsilon > 0$ , enough digits can be added so that the result is eventually within  $\epsilon$  of  $\pi$ .



How many digits are required for  $\epsilon = 0.001$ ?

$$3.1415926 \dots - 3 = 0.1415926 \dots > 0.001$$

$$3.1415926 \dots - 3.1 = 0.0415926 \dots > 0.001$$

$$3.1415926 \dots - 3.14 = 0.0015926 \dots > 0.001$$

$$3.1415926 \dots - 3.141 = 0.0005926 \dots < 0.001$$

### Example

Consider the real numbers  $\frac{1}{7}$ ,  $\pi$ , and  $e$ . How many digits in the decimal forms are required such that each value is within 0.005 and then 0.000001 of its corresponding exact value?

$$\frac{1}{7} = 0.14285714 \dots$$

$$\pi = 3.14159265 \dots$$

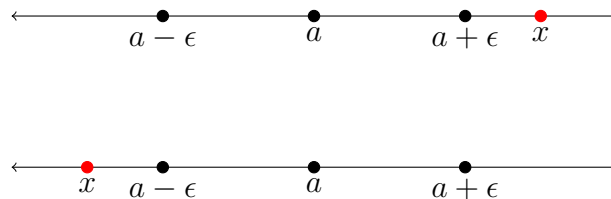
$$e = 2.71828182 \dots$$

$\epsilon$	$\frac{1}{7}$	$\pi$	$e$
0.0005	0.1428	3.1415	2.718
0.000001	0.142857	3.141592	2.718281

Also important is the negation:

### Definition: Not Arbitrarily Close

To say that a value  $x \in \mathbb{R}$  is *not arbitrarily close* to another value  $a \in \mathbb{R}$ , denoted by  $x \not\rightarrow a$ , means that means that there exists an  $\epsilon > 0$  such that  $|x - a| \geq \epsilon$ .



Thus, there is always some finite gap between  $x$  and  $a$ .

### Example

Why isn't  $24.57\bar{9}$  arbitrarily close to 24.6?

Since  $24.57\bar{9} \leq 24.58$ :

$$24.6 - 24.57\bar{9} \geq 24.6 - 24.58 = 0.02$$

So there exists  $\epsilon = 0.02$  such that  $24.6 - 24.57\bar{9} \geq \epsilon$ .