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Math 231b
Homework #4

3.8.47

What is the orthogonal complement in $L^2(\mathbb{R})$ of the set of all even functions?

Every function $f \in L^2(\mathbb{R})$ can be written as a combination of an even function and an odd function: $f = f_e + f_o$. Furthermore, the integral of an even function over all of $\mathbb{R}$ is not necessarily 0, but the integral of an odd function over all of $\mathbb{R}$ is 0.

So assume $f \in L^2(\mathbb{R})$ is an even function and assume $g \in L^2(\mathbb{R})$:

$$\int_{\mathbb{R}} fg = \int_{\mathbb{R}} f(g_e + g_o) = \int_{\mathbb{R}} fg_e + \int_{\mathbb{R}} fg_o = \int_{\mathbb{R}} fg_e$$

because an even function times an odd function is odd. But this will only be 0 for any even f if $g_e = 0$ and thus g must be odd.

Therefore, the orthogonal complement to the set of even functions in $L^2(\mathbb{R})$ is the set of odd functions in $L^2(\mathbb{R})$.

3.8.50

Let S be a subset of an inner product space. Show: $S^\perp = (\operatorname{span} S)^\perp$.

$\subseteq$ Assume $\vec{x} \in S^\perp$.

$\forall\, \vec{y} \in S, \vec{x} \perp \vec{y}$.
Assume $\vec{y} \in \operatorname{span}(S)$.
$\exists\, \{\vec{y}_1, \ldots, \vec{y}_n\} \subseteq S$ and scalars $\alpha_1, \ldots, \alpha_n$ such that $\vec{y} = \sum_{k=1}^{n} \alpha_k \vec{y}_k$.

$\langle \vec{y}, \vec{x} \rangle = \left\langle \sum_{k=1}^{n} \alpha_k \vec{y}_k, \vec{x} \right\rangle = \sum_{k=1}^{n} \alpha_k \langle \vec{y}_k, \vec{x} \rangle = 0$, since $\vec{x} \perp \vec{y}_k$.

Therefore $\vec{x} \perp \vec{y}$ and thus $\vec{x} \in (\operatorname{span} S)^\perp$.

$\supseteq$ Assume $\vec{x} \in (\operatorname{span} S)^\perp$.

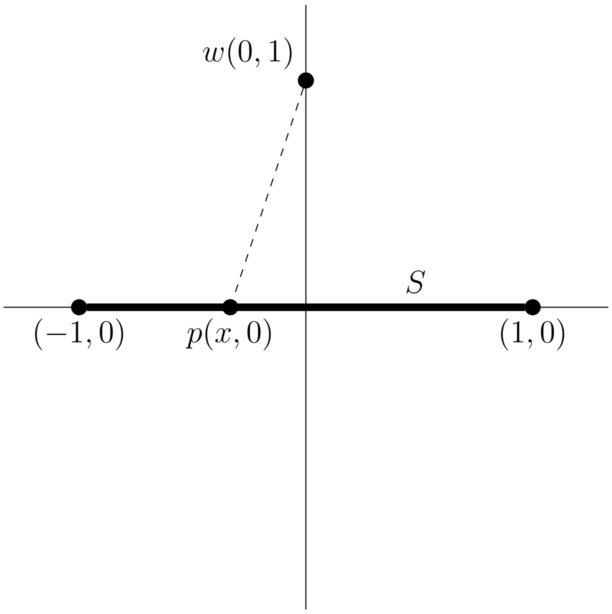
$\forall\, \vec{y} \in \operatorname{span}(S), \vec{x} \perp \vec{y}$.
But $\forall\, \vec{y} \in S, \vec{y} \in \operatorname{span}(S)$.
And so $\forall\, \vec{y} \in S, \vec{x} \perp \vec{y}$.

Therefore $\vec{x} \in S^\perp$.

3.8.52

Let E be the Banach space $\mathbb{R}^2$ with norm $\|(x,y)\| = \max(|x|, |y|)$. Show that E does not have the closest point property.

Let $S = \{(x,0) \mid x \in [-1,1]\}$, which is a closed and convex subset of E. Also, let $w = (0,1) \in E$:



Assume $p = (x,0) \in S$:

$$\begin{aligned} d(w,p) &= \|w-p\| \\ &= \|(0,1)-(x,0)\| \\ &= \|(-x,1)\| \\ &= \max(|-x|, |1|) \\ &= 1 \end{aligned}$$

Since $-1 \leq x \leq 1$.

Thus, $\forall\, p \in S, d(w,p) = 1$ and so the closest point is not unique.

Therefore, E does not have the closest point property with the given norm.

3.8.53

Let S be a closed subspace of a Hilbert space H and let $(\vec{e}_1, \vec{e}_2, \ldots)$ be a complete orthonormal sequence in S. For an arbitrary $\vec{x} \in H$ there exists $\vec{y} \in S$ such that $\|\vec{x}-\vec{y}\| = \inf\limits_{\vec{z}\in S} \|\vec{x}-\vec{z}\|$. Define $\vec{y}$ in terms of $(\vec{e}_1, \vec{e}_2, \ldots)$.

Assume $\vec{x} \in H$.
Thus, $\exists\, \vec{y} \in S, \|\vec{x}-\vec{y}\| = \inf\limits_{\vec{z}\in S} \|\vec{x}-\vec{z}\|$.
By definition, this means that $d(x,S) = \|\vec{x}-\vec{y}\|$.
By Theorem done is class, we can conclude that $\vec{x}-\vec{y} \perp S$.
Thus, $\vec{x}-\vec{y} \perp \vec{e}_n$.

Now, since $(\vec{e}_n)$ is complete: $\vec{y} = \sum\limits_{n=1}^{\infty} \langle \vec{y}, \vec{e}_n \rangle \vec{e}_n$.

$\langle \vec{x}, \vec{e}_n \rangle - \langle \vec{y}, \vec{e}_n \rangle = \langle \vec{x}-\vec{y}, \vec{e}_n \rangle = 0$, since $\vec{x}-\vec{y} \perp \vec{e}_n$, and so $\langle \vec{x}, \vec{e}_n \rangle = \langle \vec{y}, \vec{e}_n \rangle$.

$\therefore \vec{y} = \sum\limits_{n=1}^{\infty} \langle \vec{x}, \vec{e}_n \rangle \vec{e}_n$