Operator Norm

Definition: Operator Norm

Let $\|\cdot\|$ be a vector norm on C^n and $A \in M_n$. The vector-induced matrix norm with respect to $\|\cdot\|$ is given by:

$$|||A||| = \max_{\|\vec{x}\|=1}\{\|A\vec{x}\|\}$$

This norm is often called the *operator norm*.

Check the five properties to make sure that the operator norm is in fact a proper matrix norm:

1). Nonnegativity

Assume $A \in M_n$
Assume $\vec{x} \in \mathbb{C}^n$ such that $\|\vec{x}\| = 1$
By nonnegativity of the vector norm, $\|A\vec{x}\| \geq 0$
So $\max_{\|\vec{x}\|=1}\{\|A\vec{x}\|\} \geq 0$

$\therefore |||A||| \geq 0$

2). Positivity

$\implies$ Assume $A \neq 0$

$\exists\, \vec{y} \in \mathbb{C}^n, \|\vec{y}\| = 1$ and $A\vec{y} \neq 0$
$|||A||| = \max_{\|\vec{x}\|=1}\{\|A\vec{x}\|\} \geq \|A\vec{y}\| > 0$

$\therefore |||A||| \neq 0$

$\impliedby$ Assume $A = 0$

Assume $\vec{x} \in \mathbb{C}^n, \|\vec{x}\| = 0$
$A\vec{x} = \vec{0}$
$\|A\vec{x}\| = \left\|\vec{0}\right\| = 0$
$\max_{\|\vec{x}\|=1}\{\|A\vec{x}\|\} = 0$

$\therefore |||A||| = 0$

3). Homogeneity Assume $c \in \mathbb{C}$:

$$\begin{aligned} |||cA||| &= \max_{\|\vec{x}\|=1}\{\|cA\vec{x}\|\} \\ &= \max_{\|\vec{x}\|=1}\{|c|\,\|A\vec{x}\|\} \\ &= |c| \max_{\|\vec{x}\|=1}\{\|A\vec{x}\|\} \\ &= |c|\,|||A||| \end{aligned}$$

4). Subadditivity

$$\begin{aligned} |||A+B||| &= \max_{\|\vec{x}\|=1}\{\|(A+B)\vec{x}\|\} \\ &= \max_{\|\vec{x}\|=1}\{\|A\vec{x}+B\vec{x}\|\} \\ &\leq \max_{\|\vec{x}\|=1}\{\|A\vec{x}\|+\|B\vec{x}\|\} \\ &\leq \max_{\|\vec{x}\|=1}\{\|A\vec{x}\|\}+\max_{\|\vec{x}\|=1}\{\|B\vec{x}\|\} \\ &= |||A|||+|||B||| \end{aligned}$$

5). Submultiplicativity

Assume $A, B \in M_n$
Assume $\vec{x} \in \mathbb{C}^n, \|\vec{x}\| = 1$
Assume $B\vec{x} \in \mathbb{C}^n$

$$\begin{aligned} \left\|A\frac{B\vec{x}}{\|B\vec{x}\|}\right\| &\leq |||A||| \\ \|A(B\vec{x})\| &\leq |||A|||\,\|B\vec{x}\| \\ \|(AB)\vec{x}\| &\leq |||A|||\,\|B\vec{x}\| \end{aligned}$$

$$\begin{aligned} |||AB||| &= \max_{\|\vec{x}\|=1}\{\|(AB)\vec{x}\|\} \\ &\leq \max_{\|\vec{x}\|=1}\{|||A|||\,\|B\vec{x}\|\} \\ &= |||A|||\max_{\|\vec{x}\|=1}\{\|B\vec{x}\|\} \\ &= |||A|||\,|||B||| \end{aligned}$$