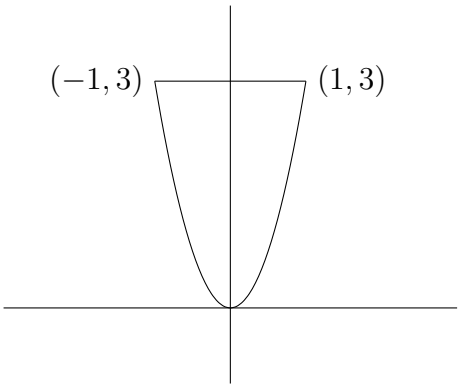
Math-19 Homework #11 Solutions

Problems

1). You are designing a new spotlight with a parabolic reflector. The diameter of the reflector is 2 feet and the height is 3 feet. How high above the bottom of the dish should the lightbulb be placed.

Let's place a sketch of the spotlight on a coordinate system with the vertex at the origin:



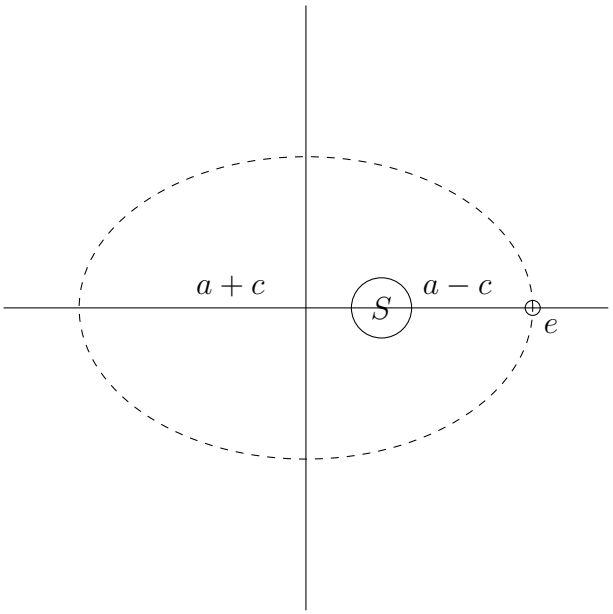
Based on the reflective nature of the parabola, the lightbulb should be placed at the focus: $(0, p)$, so we need to find p:

$x^2 = 4py$
$(1)^2 = 4p(3)$
$1 = 12p$
$p = \dfrac{1}{12}$

And so the lightbulb should be placed at $\dfrac{1}{12}$ of a foot, or at the 1 inch mark.

2). The earth's orbit around the sun is elliptical with the sun at one of the foci. The closest the earth gets to the sun is about 91 million miles. The eccentricity of the earth's orbit is about 0.0167. What is the farthest distance between the earth and the sun?

let's place the orbit of the earth on a coordinate system with the center at the origin and the sun at one focus:



Place the sun at the focus $(c, 0)$. When the earth is at its perihelion (closest point to sun) it is at the vertex $(a, 0)$. Thus, the distance from the sun to the earth is $a - c = 91$ million miles. We want the distance at aphelion (farthest point from sun), which is $a + c$. Since we know that the eccentricity $e = \dfrac{c}{a} = 0.0167$, we have two equations in two unknowns:

$$\begin{aligned} a - c &= 91 \\ c &= 0.0167a \end{aligned}$$

Using substitution to solve for a:

$$\begin{aligned} a - 0.0167a &= 91 \\ 0.9833a &= 91 \\ a &= \frac{91}{0.9833} \end{aligned}$$

Don't jump right to the calculator yet; that introduces needless error. Instead, keep things rational until the end. Now, solve for c:

$$c = 0.0167a = \frac{0.0167}{0.9833} \cdot 91$$

And finally, calculate $a + c$:

$$a + c = \frac{91}{0.9833} + \frac{0.0167}{0.9833} \cdot 91 = \frac{91}{0.9833}(1 + 0.0167) = \frac{1.0167}{0.9833} \cdot 91 \approx 94$$

So, the earth is at about 94 million miles away from the sun at its aphelion position.

3). Consider the following parabola:

$$y^2 - 6y - 12x + 33 = 0$$

a). What is the vertex?

$$\begin{aligned} y^2 - 6y &= 12x - 33 \\ y^2 - 6y + 9 &= 12x - 33 + 9 \\ (y-3)^2 &= 12x - 24 \\ (y-3)^2 &= 12(x-2) \end{aligned}$$

$V(2, 3)$

b). What is the axis of symmetry?

$y = 3$

c). What is the focus?

$4p = 12$
$p = 3$

This is a horizontal/open right parabola, so the focus is at:

$(2 + p, 3) = (2 + 3, 3) = (5, 3)$

$F(5, 3)$

d). What is the directrix?

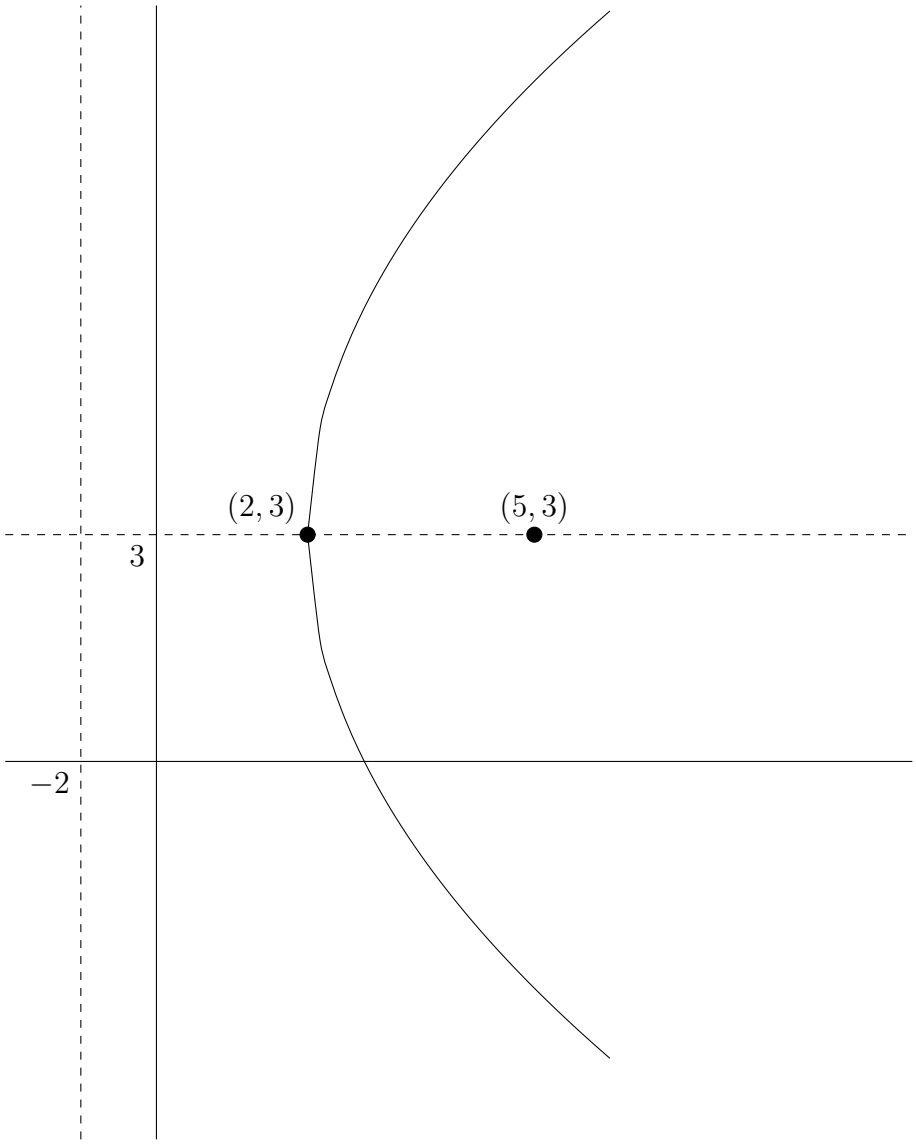
$x = 2 - p = 2 - 3 = -1$

$x = -1$

e). What is the focal diameter?

$|4p| = 12$

f). Sketch the parabola, labeling all of the above items.



4). Consider the following ellipse:

$$4x^2 + 25y^2 - 50y - 75 = 0$$

a). What is the center?

$$\begin{aligned} 4x^2 + 25(y^2 - 2y) &= 75 \\ 4x^2 + 25(y^2 - 2y + 1) &= 75 + 25 \\ 4x^2 + 25(y-1)^2 &= 100 \\ \frac{x^2}{25} + \frac{(y-1)^2}{4} &= 1 \end{aligned}$$

$C(0, 1)$

b). What is the length of the major axis?

$a = 5$

Length of major axis $= 2a = 2(5) = 10$

c). What is the length of the minor axis?

$b = 2$

Length of minor axis $= 2b = 2(2) = 4$

d). What are the four vertices?

$(0 + a, 1) = (0 + 5, 1) = (5, 1)$
$(0 - a, 1) = (0 - 5, 1) = (-5, 1)$
$(0, 1 + b) = (0, 1 + 2) = (0, 3)$
$(0, 1 - b) = (0, 1 - 2) = (0, -1)$

e). What are the two foci?

$c = \sqrt{a^2 - b^2} = \sqrt{25 - 4} = \sqrt{21}$

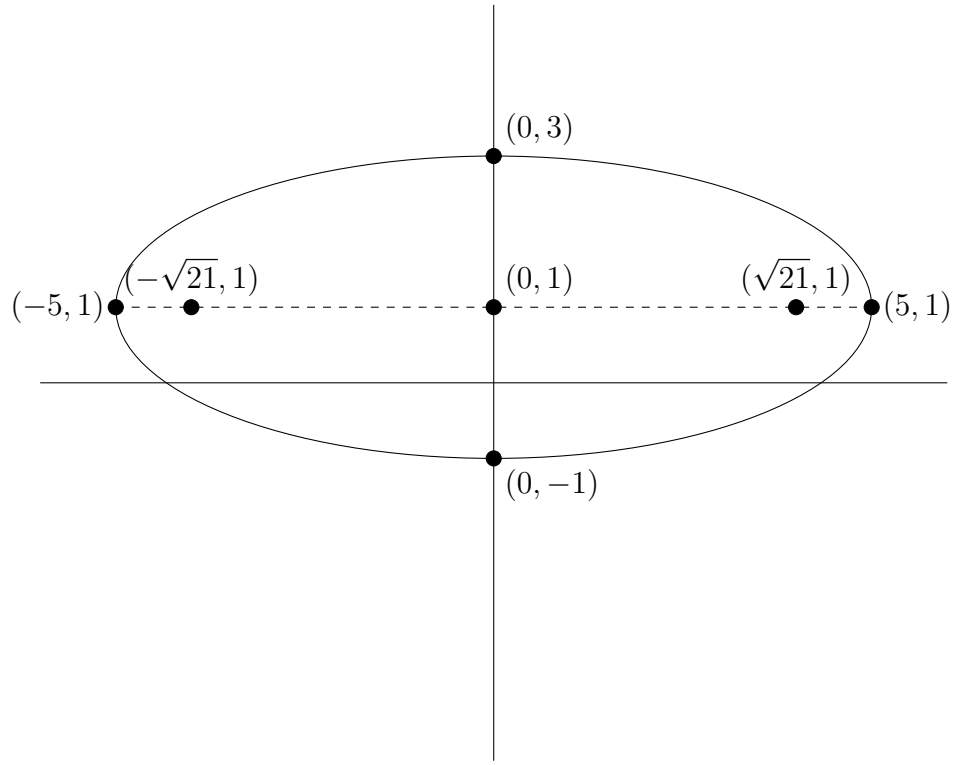
The major axis is parallel to the x-axis, so the foci are also on the x-axis:

$(0 + c, 1) = (0 + \sqrt{21}, 1) = (\sqrt{21}, 1)$
$(0 - c, 1) = (0 - \sqrt{21}, 1) = (-\sqrt{21}, 1)$

f). What is the eccentricity?

$e = \dfrac{c}{a} = \dfrac{\sqrt{21}}{5} \approx 0.92$

g). Sketch the ellipse, labeling all of the above items.



5). Consider the following hyperbola:

$$36x^2 + 72x - 4y^2 + 32y + 116 = 0$$

a). What is the center?

$$\begin{aligned} 36(x^2 + 2x) - 4(y^2 - 8y) &= -116 \\ 36(x^2 + 2x + 1) - 4(y^2 - 8y + 16) &= -116 + 36 - 64 \\ 36(x+1)^2 - 4(y-4)^2 &= -144 \\ \frac{(y-4)^2}{36} - \frac{(x+1)^2}{4} &= 1 \end{aligned}$$

$C(-1, 4)$

b). What is the length of the horizontal axis?

$b = 2$

Length of horizontal axis $= 2b = 2(2) = 4$

c). What is the length of the vertical axis?

$a = 6$

Length of vertical axis $= 2a = 2(6) = 12$

d). What are the two vertices?

This is a vertically-oriented hyperbola, so the vertices (and foci) are on the major axis parallel to the y-axis:

$(-1, 4 + a) = (-1, 4 + 6) = (-1, 10)$
$(-1, 4 - a) = (-1, 4 - 6) = (-1, -2)$

e). What are the two foci?

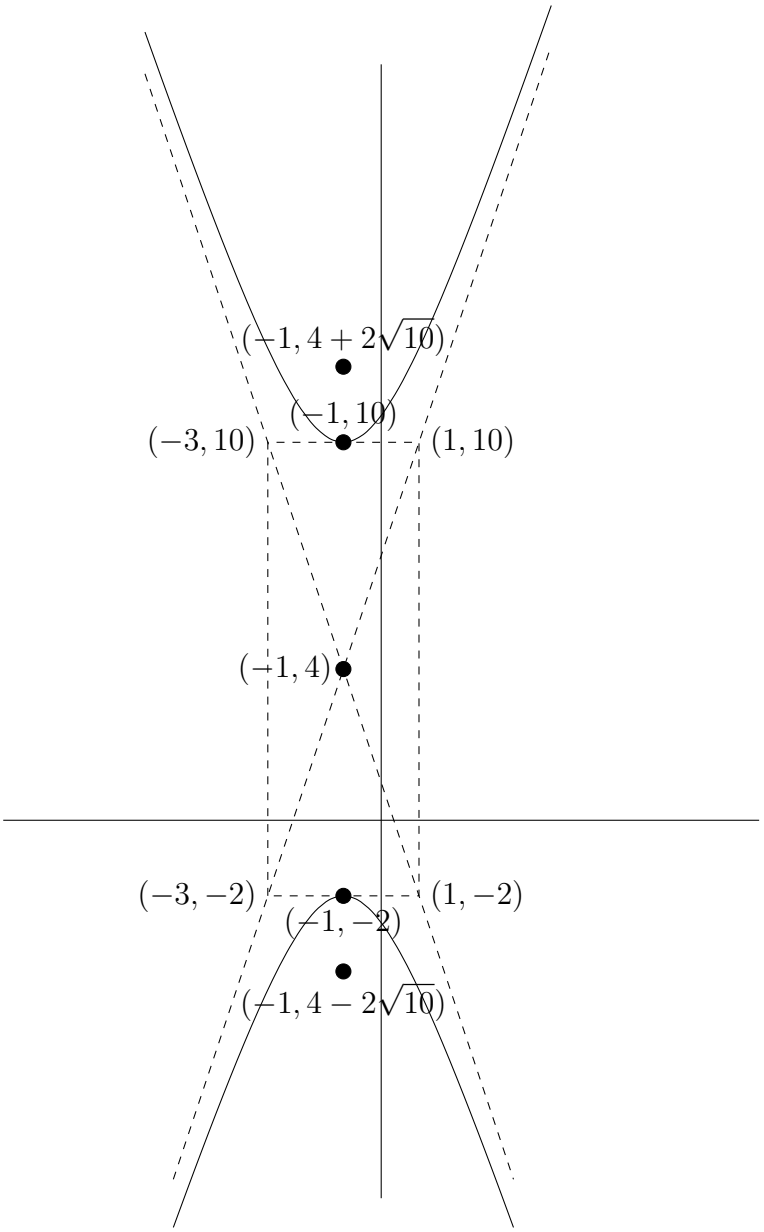
$c = \sqrt{a^2 + b^2} = \sqrt{36 + 4} = \sqrt{40} = 2\sqrt{10} \approx 6.3$

$(-1, 4 + c) = (-1, 4 + 2\sqrt{10}) \approx (-1, 10.3)$
$(-1, 4 - c) = (-1, 4 - 2\sqrt{10}) \approx (-1, -2.3)$

f). What are the two asymptotes?

$$y - 4 = \pm\frac{a}{b}(x+1) = \pm\frac{6}{2}(x+1) = \pm 3(x+1)$$

g). Sketch the hyperbola, labeling all of the above items.


$(-1, 4 + 2\sqrt{10})$
$(-1, 10)$
$(-3, 10)$
$(1, 10)$
$(-1, 4)$
$(-3, -2)$
$(1, -2)$
$(-1, -2)$
$(-1, 4 - 2\sqrt{10})$