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Math 275A
Homework #6

Example: Exercise 4.13

$SPACE$	T_1	T_2	REGULAR	NORMAL
R_{std}	✓	✓	✓	✓
R^n_{std}	✓	✓	✓	✓
indiscrete	$\times$	$\times$	$\times$	$\times$
discrete	✓	✓	✓	✓
cofinite	✓	X finite: ✓ X infinite: $\times$	X finite: ✓ X infinite: $\times$	X finite: ✓ X infinite: $\times$
cocountable	✓	X finite: ✓ X infinite: $\times$	X finite: ✓ X infinite: $\times$	X finite: ✓ X infinite: $\times$
R_{LL}	✓	✓	✓	✓
R_{+00}	✓	$\times$	$\times$	$\times$
LOS	✓	✓	✓	✓

R and $\mathbb{R}^n$

Since there is a finite distance between points and closed sets (not containing those points), there is always room for enclosing disjoint balls.

indiscrete

Since the only non-empty set is the entire space, there is no separation.

discrete

Since all disjoint subsets are both open and closed, they are self-enclosed.

cofinite

All finite sets are closed. Thus, single points can be viewed as closed sets. Given any two closed sets A and B, the sets $X - A$ and $X - B$ with $A \notin X - A$, $A \in X - B$, $B \notin X - B$, and $B \in X - A$. Thus, cofinite is T_1. If X is finite then all disjoint subsets are both open and closed and hence self-enclosing. Otherwise, enclosing open sets will always have some overlap.

cocountable

Analagous to cofinite.

$\underline{\mathbb{R}_{LL}}$

Since R_{LL} is finer than $\mathbb{R}$, it has the same separation properties.

$\underline{\mathbb{R}_{+00}}$

Any two points can be T_1 separated using the basis elements; however, if one point or closed set contains $0'$ and the other point or closed set contains $0''$ then there is always overlap between the two containing basis elements.

Lexigraphically Ordered Square

Use the alternate definitions. For any point $p \in X$, there exists some containing open set (strip), and it is always possible to use a smaller strip whose closure is contained in the original strip. For any closed set $A \in X$, $X - A$ is an enclosing open set, and likewise, a smaller open set with contained closure is possible.

Theorem: 4.16

X, Y are $T_2 \implies X \times Y$ is T_2.

Proof. Assume that X and Y are T_2 and assume $p_1, p_2 \in X \times Y$ where $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$. Since X is T_2, there exists $U_1, U_2 \in \mathscr{T}_X$ such that $x_1 \in U_1$ and $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. Likewise, since Y is T_2, there exists $V_1, V_2 \in \mathscr{T}_Y$ such that $y_1 \in V_1$ and $y_2 \in V_2$ and $V_1 \cap V_2 = \emptyset$. So $p_1 \in U_1 \times V_1$ and $p_2 \in U_2 \times V_2$. Furthermore, $U_1 \times V_1, U_2 \times V_2 \in \mathscr{T}_{X \times Y}$ and

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) = \emptyset$$

Therefore $X \times Y$ is T_2. ∎

Lemma

Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$:

$$\overline{A \times B} = \bar{A} \times \bar{B}$$

Proof. Assume that $p \in \overline{A \times B}$. This means that for all $U \in \mathscr{T}_{X \times B}$ such that $p \in U$:

$$U \cap (A \times B) \neq \emptyset$$

Now assume $U_1 \in \mathscr{T}_X$ and $U_2 \in T_Y$ such that $p \in U_1 \times U_2 \in \mathscr{T}_{A \times B}$. Then it must be the case that $(U_1 \times U_2) \cap (A \times B) = (U_1 \cap A) \times (U_2 \cap B) \neq \emptyset$. This is only possible if $U_1 \cap A \neq \emptyset$ and $U_2 \cap B \neq \emptyset$.

Therefore $p \in \bar{A} \times \bar{B}$.

Assume that $p \in \bar{A} \times \bar{B}$. This means that for all $U_1 \in \mathscr{T}_X$ and $U_2 \in \mathscr{T}_Y$ such that $p \in U_1 \times U_2$:

$$(U_1 \cap A) \times (U_2 \cap B) \neq \emptyset$$

Now assume $U \in \mathscr{T}_{A\times B}$ such that $p \in U \in \mathscr{T}_{A\times B}$. Then there exists $U_1 \in \mathscr{T}_X$ and $U_2 \in T_Y$ such that $p \in U_1 \times U_2 = U$. So it must be the case that:

$$U \cap (A \times B) = (U_1 \times U_2) \cap (A \times B) = (U_1 \cap A) \times (U_2 \cap B) \neq \emptyset$$

Therefore $p \in \overline{A \times B}$. ■

Theorem: 4.17

X, Y are regular $\implies$ $X \times Y$ is regular.

Proof. Assume that X and Y are regular and assume $p \in X \times Y$ and $U \in \mathcal{U}_p$. Then there exists $U_1 \in \mathscr{T}_X$ and $U_2 \in \mathscr{T}_Y$ such that $p \in U_1 \times U_2 \subset U$. Now, since X and Y are regular, there exists $V_1 \in \mathscr{T}_X$ and $V_2 \in \mathscr{T}_y$ such that $p \in V_1 \times V_2$, $V_1 \subset \overline{V_1} \subset U_1$, and $V_2 \subset \overline{V_2} \subset U_2$. Furthermore, since $\overline{V_1}$ is closed in X and $\overline{V_2}$ is closed in Y, $\overline{V_1} \times \overline{V_2}$ (and hence $\overline{V_1 \times V_2}$) is closed in $X \times Y$. And so:

$$p \in V_1 \times V_2 \subset \overline{V_1 \times V_2} = \overline{V_1} \times \overline{V_2} \subset U_1 \times U_2$$

Therefore $X \times Y$ is regular. ■

Theorem: 4.19

Every T_2 space is hereditarily T_2.

Proof. Assume that X is a T_2 topological space and assume that $Y \subset X$. Now assume that $a, b \in Y$. Thus $a, b \in X$ and, since X is T_2, there exists $U, V \in \mathscr{T}_X$ such that $a \in U$, $b \in V$, and $U \cap V = \emptyset$. Furthermore, $a \in U \cap Y \in \mathscr{T}_Y$ and $b \in V \cap Y \in \mathscr{T}_Y$. And so:

$$(Y \cap U) \cap (Y \cap V) = Y \cap (U \cap V) = Y \cap \emptyset = \emptyset$$

Therefore Y is also T_2. ■

Theorem: 4.20

Every regular space is hereditarily regular.

Proof. Assume that X is a regular topological space and assume that $Y \subset X$. Assume $p \in Y$. There exists $U_X \in \mathscr{T}_X$ such that $p \in U_X$ and so $p \in U_X \cap Y = U_Y \in \mathscr{T}_Y$. Now, since X is regular, there exists $V_X \in \mathscr{T}_X$ such that $p \in V_X \subset \overline{V_X} \subset U_X$, and hence $p \in V_X \cap Y = V_Y \in \mathscr{T}_Y$. Furthermore, since $\overline{V_X}$ is closed in X, $\overline{V_X} \cap Y = W_Y$ is closed in Y. Finally, since $\overline{V_Y}$ is the smallest closed set in Y containing V_Y:

$$p \in V_Y \subset \overline{V_Y} \subset W_Y \subset U_Y$$

Therefore Y is regular. ■

Theorem: 4.23

Let X be a normal topological space and let $Y \subset X$ such that Y is closed in X. Y is normal when given the relative topology.

Proof. Assume $A, B \subset Y$ such that A and B are closed in Y and $A \cap B = \emptyset$. Since A is closed in Y, $Y - A \in \mathscr{T}_Y$. This means that there exists $W \in \mathscr{T}_X$ such that $W \cap Y = Y - A$. Furthermore, $X - W$ is closed in X. Now:

$$(X - W) \cap Y = (X \cap Y) - (W \cap Y) = Y - (Y - A) = A$$

But $X - W$ and Y are closed in X and hence A is also closed in X. By similar argument, B is also closed in X. And, since X is normal, there exists $U, V \in \mathscr{T}_X$ such that $A \in U$, $B \in V$, and $U \cap V = \emptyset$. Finally, since $A \subset (U \cap Y) \in \mathscr{T}_Y$ and $B \subset (V \cap Y) \in \mathscr{T}_Y$:

$$(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \emptyset \cap Y = \emptyset$$

Therefore Y is normal. ■