Arbitrarily Close

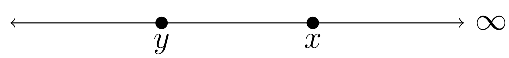
A new concept is needed to solve problems that algebra alone cannot solve: arbitrarily close.

Arbitrarily Large

Infinity (∞) is not an actual number, but instead is indicative of a process:

1. Select a positive number.
2. Now select a next number that is larger than the previous number.
3. Go to 2.

This is possible because the real numbers are unbounded: for every $y \in \mathbb{R}$ there exists some $x \in \mathbb{R}$ such that $x > y$.

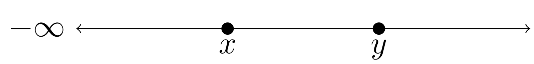


Note that as y increases, x is pushed to the right.

Definition: Arbitrarily Large

To say that a value $x \in \mathbb{R}$ is *arbitrarily large*, denoted by $x \to \infty$, means that for every $y \in \mathbb{R}$, $x > y$.

This also works in the negative direction. For $x \to -\infty$, select a negative number and then continually select numbers that are less than the previous number. In other words, for every $y \in \mathbb{R}, x < y$.



Note that as y decreases, x is pushed to the left.

Arbitrarily Small

A number can also be said to be arbitrarily small. Like infinity, this is not an actual number, but is indicative of a process:

1. Select a positive number.
2. Now select a next positive number that is smaller than the previous number.
3. Go to 2.

This is possible because between any two real numbers there are an infinite number of real numbers. Thus, for any value $y > 0$ there exists some x such that $0 < x < y$.

Note that as y decreases, x is squeezed between 0 and y.

Definition: Arbitrarily Small

To say that a value $x \in \mathbb{R}^+$ is *arbitrarily small*, denoted by $x \to 0^+$, means that for every $y \in \mathbb{R}^+, 0 < x < y$.

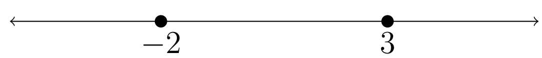
The Greek letters epsilon (ϵ) and delta (δ) are typically used to represent arbitrarily small values.

Distance

The first time students are introduced to the concept of *absolute value*, they are given a formula:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

This is a perfectly good definition; however, it lacks meaning. Instead, consider two points on the real number line:



How does one calculate the *distance* from -2 to 3?

$$3 - (-2) = 5$$

How about from 3 to -2:

$$-2 - 3 = -5$$

But distance is an unsigned quantity (different from displacement). Furthermore, the distance from -2 to 3 should be the same as the distance from 3 to -2. Thus, we use absolute value:

$$|3 - (-2)| = |-2 - 3| = 5$$

Definition: Distance

Let $x, y \in \mathbb{R}$. The *distance* from x to y (and from y to x) is given by:

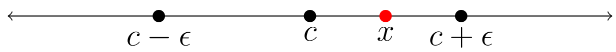
$$d(x, y) = |x - y| = |y - x|$$

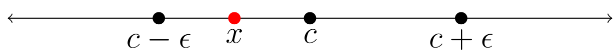
Thus, $|x| = |x - 0|$, which is the distance from x to 0.

Arbitrarily Close

Definition: Arbitrarily Close

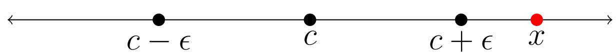
To say that a value $x \in \mathbb{R}$ is *arbitrarily close* to another value $c \in \mathbb{R}$, denoted by $x \to c$, means that for all $\epsilon > 0, |x - c| < \epsilon$. In other words: $c - \epsilon < x < c + \epsilon$.

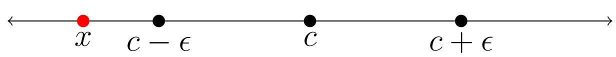




Thus, as ϵ gets arbitrarily small, x gets arbitrarily close to c.

Also important is the negation: there exists an $\epsilon > 0$ such that $|x - c| \geq \epsilon$.





Theorem

Arbitrarily close is equivalent to equality.

Proof.

$\Longrightarrow$ Assume that $x \to c$.

ABC that $x \neq c$. Thus, there exist some $d > 0$ such that $|x - c| \geq d$. So let $\epsilon = d$:

$$|x - c| \geq d = \epsilon$$

This means that there exists an $\epsilon > 0$ such that $|x - c| \geq \epsilon$ and hence $x \not\to c$, contradicting the assumption.

Therefore $x = c$.

$\Longleftarrow$ Assume that $x = c$

Assume that $\epsilon > 0$:

$$|x - c| = 0 < \epsilon$$

Therefore $x \to c$.

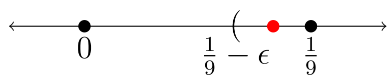
$\blacksquare$

Example

Recall that one of the ways of representing a rational number is by using an infinite sequence of repeated decimal digits. For example:

$$\frac{1}{9} = 0.11111\ldots = 0.\overline{1}$$

It is easy to mark $\frac{1}{9}$ on the number line. But how does $0.\overline{1}$ correspond to this point? As each repeated digit is added, the value $0.\overline{1}$ gets *arbitrarily close* to $\frac{1}{9}$. For every $\epsilon > 0$, enough digits can eventually be added so that the result is within ϵ of $\frac{1}{9}$.



Furthermore:

$$\frac{1}{9} = 0.\overline{1}$$
$$\frac{2}{9} = 0.\overline{2}$$
$$\frac{3}{9} = 0.\overline{3}$$
$$\vdots$$
$$\frac{8}{9} = 0.\overline{8}$$
$$\frac{9}{9} = 0.\overline{9} = 1$$

Therefore $0.\overline{9}$ is arbitrarily close and thus equal to 1.

Example

This works for (and is indeed a definition) for irrational numbers, which are represented by infinite sequences of non-repeating digits. Consider $\pi = 3.1415926\ldots$. Eventually, enough digits can be added in order to get arbitrarily closed to π on the number line: