Decomposition

Definition: Decomposition

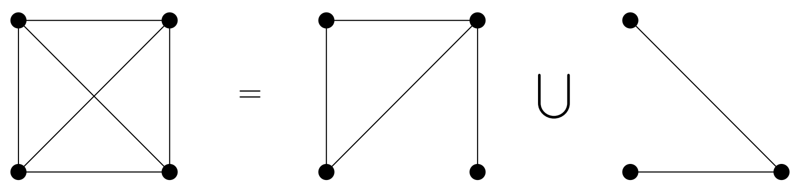
Let G be a simple graph. A *decomposition* of G is a collection of simple graphs $G_1, G_2, \dots G_k$ such that:

1). $E(G) = \bigcup_{k=1}^{n} E(G_k)$

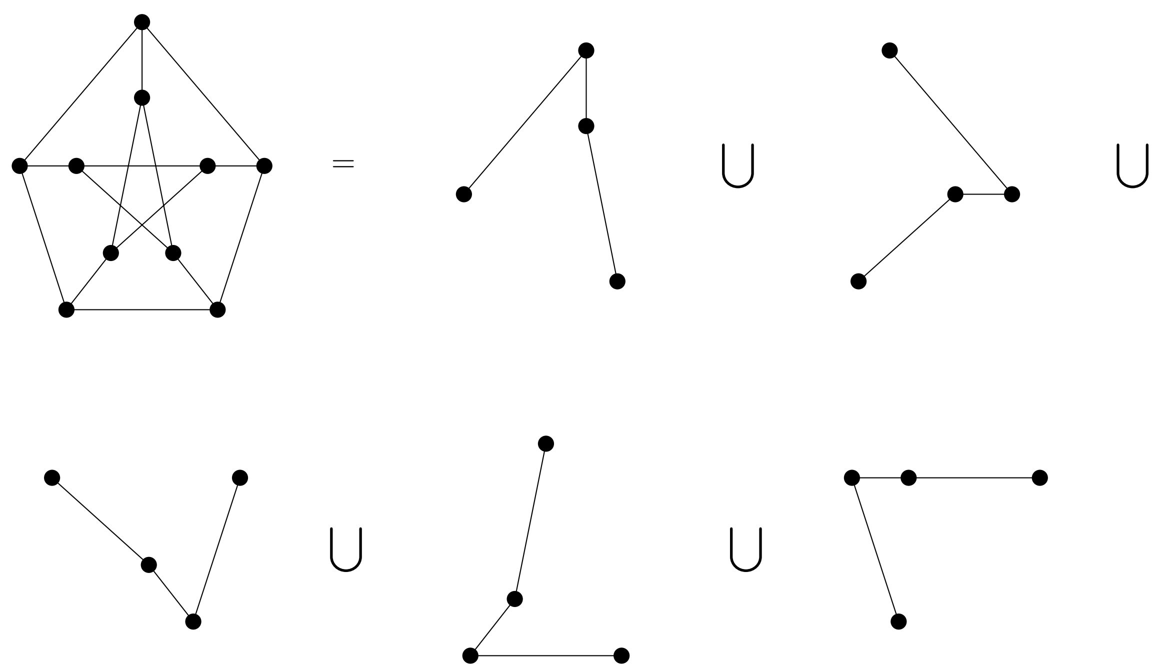
2). $\forall\, i, j, E(G_i) \cap E(G_j) = \emptyset$

Examples

1). K_4



2). $P = 5P_4$



Definition: Path Number

Let G be a simple graph. The *path number* for G, denoted $p(G)$, is the minimum number of paths in any decomposition of G.

Conjecture: Gallai

Let G be a simple graph of order n. G can be decomposed into at most $\left\lceil \frac{n}{2} \right\rceil$ paths. In other words:

$$p(G) \leq \left\lceil \frac{n}{2} \right\rceil$$

Example

Since $n(P) = 10$, so $p(P) \leq \left\lceil \frac{10}{2} \right\rceil = 5$, meaning the Petersen graph can be decomposed into at most 5 paths — indeed: $P = 5P_4$ as shown above and $p(P) = 5$, but we need a way to find a lower bound.

Theorem

Let G be a simple graph. If G can be decomposed into k paths then the number of odd vertices in $G \leq 2k$.

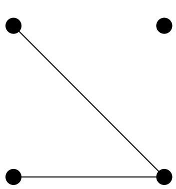
Proof

Assume G can be decomposed into k paths:

$$G = \bigcup_{i=1}^{k} P_{n_i}$$

Isolated vertices are treated as P_1 and are considered even. Each of the other paths consist of 2 odd vertices and $n_i - 2$ even vertices:

Let G_i be the spanning subgraph of G such that $E(G_i) = E(P_{n_i})$, in other words, G_i contains the vertices and edges from P_{n_i} and all of the remaining vertices in G as isolated vertices. For example, if $G = K_4$, one such G_i might be:

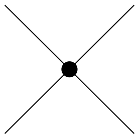


consisting of a P_3 and one isolated vertex.

Let $e(G_i)$ be the number of even (including isolated) vertices in G_i and $o(G_i)$ be the number of odd vertices in G_i:

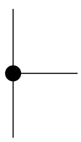
$$\bigcap_{i=1}^{k} e(G_i) \subseteq e(G)$$

The LHS contains all isolated vertices in G and all vertices that are either isolated or in the interior of the path in each G_i. Note that a vertex that is in the interior of multiple paths (never an end vertex of a path) remains even:



However, the LHS is a subset of the RHS, because two end (odd) vertices in different paths can be joined in G to form an even vertex:

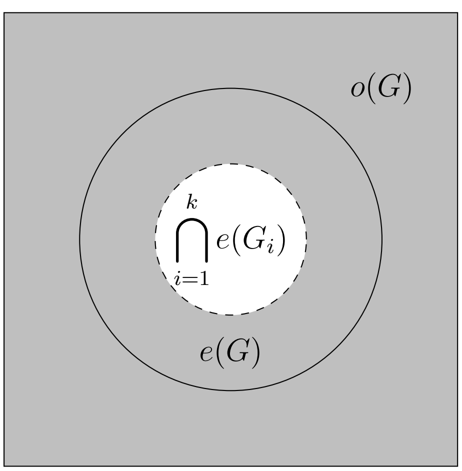
But no even vertex on the LHS can be converted to an odd vertex, because it would have to be an end vertex in at least one path and thus would not be in the LHS:



Now, take the complement of both sides:

$$\overline{\bigcap_{i=1}^{k} e(G_i)} \supseteq \overline{e(G)} = o(G)$$

To see why this is true, consider the following Venn diagram for $V(G)$:



Now, applying DeMorgan:

$$o(G) \subseteq \overline{\bigcap_{i=1}^{k} e(G_i)} = \bigcup_{i=1}^{k} \overline{e(G_i)} = \bigcup_{i=1}^{k} o(G_i)$$

And then the triangle inequality:

$$|o(G)| \leq \left| \sum_{i=1}^{k} o(G_i) \right| \leq \sum_{i=1}^{k} |o(G_i)| = \sum_{i=1}^{k} 2 = 2k$$

$\therefore |o(G)| \leq 2k$

Examples

1). P

By Gallai: $p(P) \leq 5$. By the lemma: $10 \leq 2p(P)$ and so $p(P) \geq 5$. Therefore $p(P) = 5$. A decomposition $P = 5P_4$ is shown above.

2). C_n

$P(C_n) = P_n \cup P_2$ and so $p(C_n) = 2$

3). ST_n

$ST_{2k+1} = \displaystyle\bigcup_{i=1}^{k} P_3$ and so $p(ST_{2k+1}) = k$

$ST_{2k} = \displaystyle\bigcup_{i=1}^{k-1} P_3 \cup P_2$ and so $p(ST_{2k}) = k$

Therefore: $p(ST_n) = \left\lfloor \dfrac{n}{2} \right\rfloor$