Limit Failures

Definition

To say that $L \in \mathbb{R}$ is not the limit of a function $f(x)$ at $x = a$ means that $f(x) \not\to L$ as $x \to a$:

$$\exists\,\epsilon > 0, \forall\,\delta > 0, \exists\,x \in \mathbb{R}, 0 < |x - a| < \delta \text{ and } |f(x) - L| \geq 0$$

Find an ϵ such that for every δ, there is at least one x in the δ-neighborhood of a at which the function value is outside the bounding $\epsilon - \delta$ box.

There are three possibilities:

1. Gaps
2. Arbitrarily Large
3. Oscillations

Gaps

Gaps introduced by piecewise functions result in the non-existence of the limit at the gap.

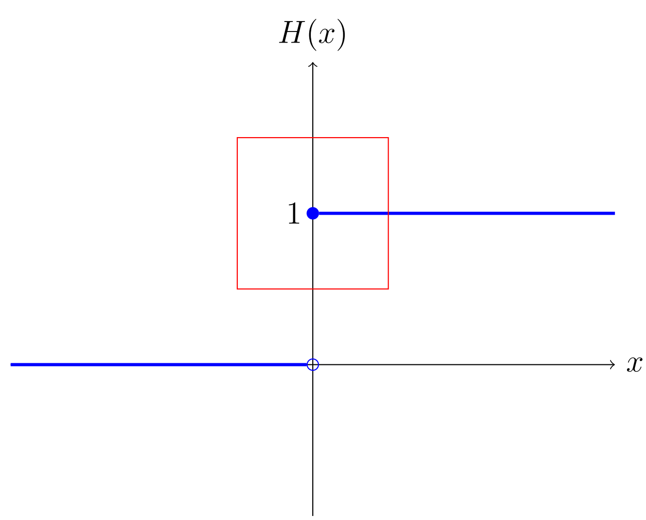
Example: The Heaviside Function

Define $H(x)$ as follows:

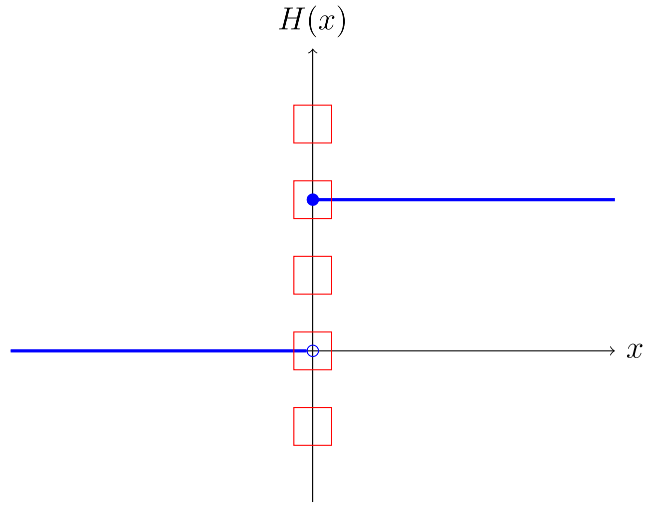
$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Is $\lim_{x \to 0} H(x) = 1$?

Let $\epsilon = \frac{1}{2}$. Note that for any δ, the part of the function for $x < 0$ will always be outside the bounding box.



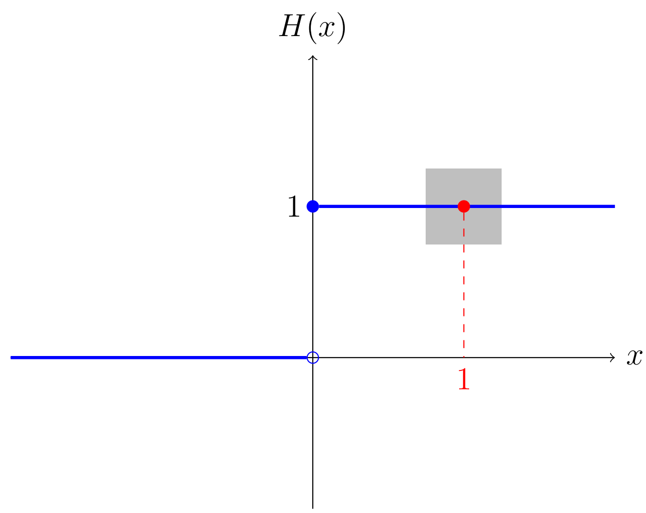
In fact, for any L, an ϵ can be selected such that no suitable bounding box can be drawn.



Thus, $\lim_{x \to 0} H(x)$ does not exist (DNE).

Note that this does not prohibit limits at other values of x. For example:

$$\lim_{x \to 1} H(x) = 1$$



In fact, for any $a > 0$:

$$\lim_{x \to a} H(x) = 1$$

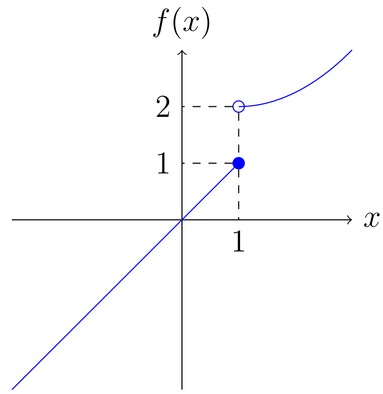
and for any $a < 0$:

$$\lim_{x \to a} H(x) = 0$$

Example

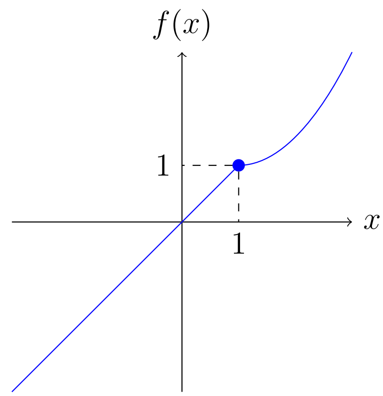
$$f(x) = \begin{cases} x, & x \leq 1 \\ (x-1)^2 + 2, & x > 1 \end{cases}$$

$\lim_{x \to 1} f(x)$ DNE



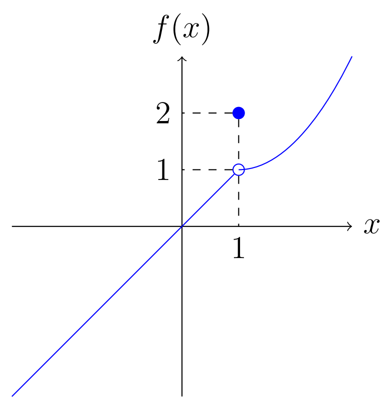
$$f(x) = \begin{cases} x, & x \leq 1 \\ (x-1)^2 + 1, & x > 1 \end{cases}$$

$\lim_{x \to 1} f(x) = 1$



$$f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ (x-1)^2 + 1, & x > 1 \end{cases}$$

$\lim_{x \to 1} f(x) = 1$



Arbitrarily Large

Functions that result in $\frac{b}{0}$ for some constant $b \in \mathbb{R}$ when evaluated at $x = a$ get arbitrarily large:

$$f(x) \to \pm\infty \text{ as } x \to a$$

Definition: Arbitrarily Large

To say that $x \in \mathbb{R}$ is *arbitrarily large*, denoted by $x \to \infty$ or $x \to +\infty$, means that:

$$\forall\, y \in \mathbb{R}, x > y$$

To say that $x \in \mathbb{R}$ is *arbitrarily large negative*, denoted by $x \to -\infty$, means that:

$$\forall\, y \in \mathbb{R}, x < y$$

Similar to arbitrarily small, arbitrarily large is an infinite no-win game: given any $y \in \mathbb{R}$, x is larger (greater) than y.

Definition: Infinite Limit

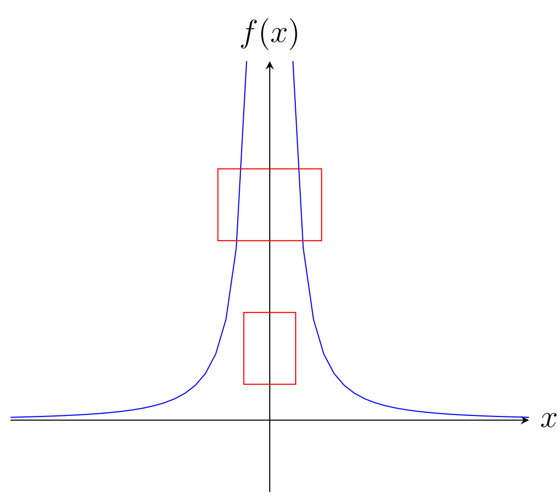
To say that $\lim\limits_{x \to a} f(x) = \infty$ means that for every $M > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) > M$.

To say that $\lim\limits_{x \to a} f(x) = -\infty$ means that for every $M > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) < -M$.

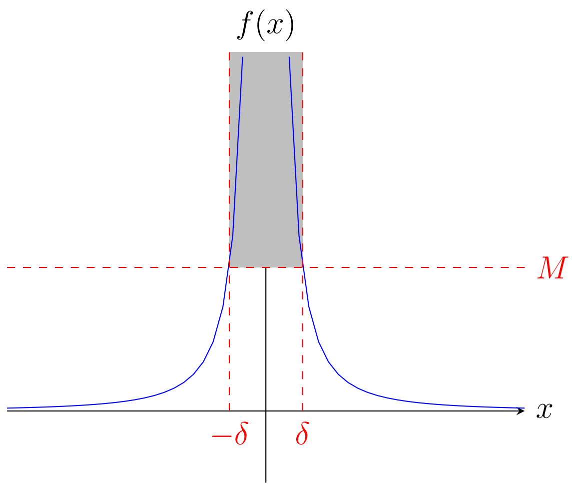
Example

Let $f(x) = \dfrac{1}{x^2}$.

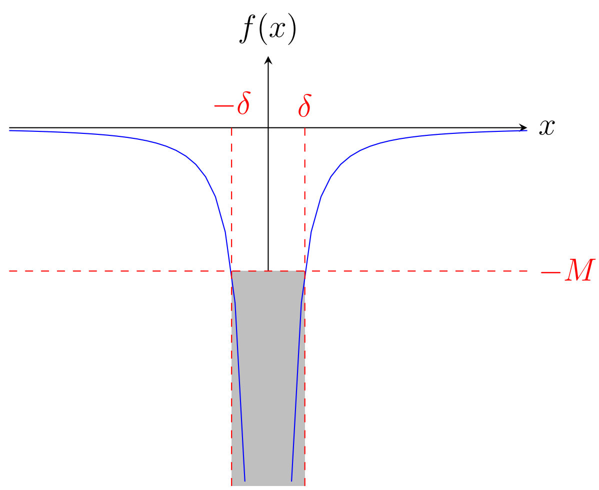
Does $\lim\limits_{x \to 0} f(x)$ exist?



No matter how the bounding box is drawn, there will be a portion of the function outside of the box since $f(x) \to \infty$ as $x \to 0$. In fact, for every $M > 0$, a suitable δ can be found:



Likewise: $\lim_{x \to 0} \left(-\frac{1}{x^2} \right) = -\infty$

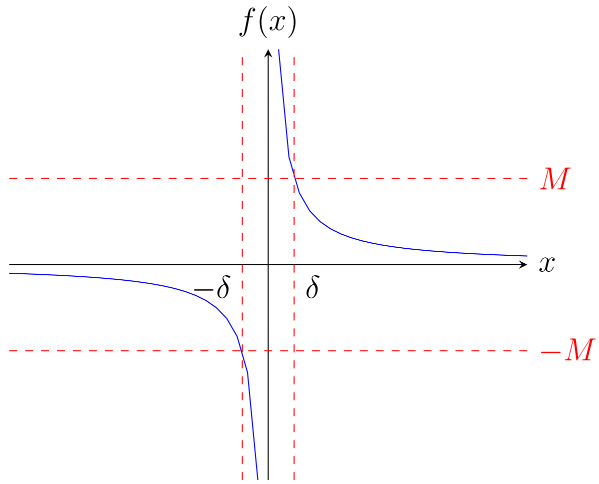


Saying that $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$ is preferred to saying that the limit DNE.

Example

Let $f(x) = \dfrac{1}{x}$.

Does $\lim_{x \to 0} f(x)$ exist?



No matter what the choice of M, part of the graph will always be below (less than) M and part of the graph will be above (greater than) $-M$. Thus the limit DNE.

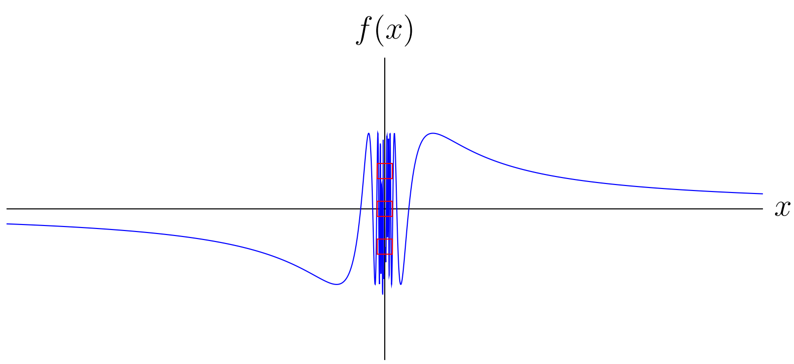
Oscillations

Functions that oscillate infinitely about a point will always escape the bounding box for some ϵ at the point.

Example

Let $f(x) = \sin \frac{1}{x}$.

Does $\lim_{x \to 0} f(x)$ exist?



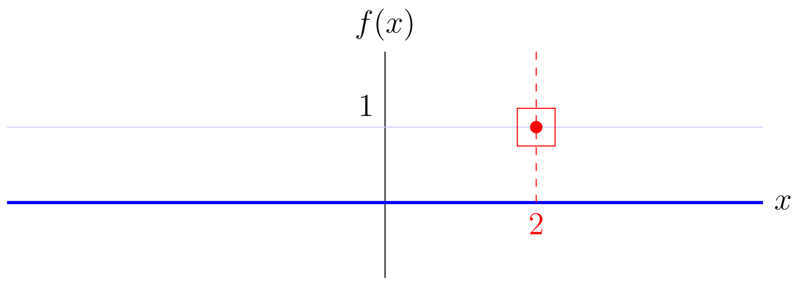
For small ϵ, no matter how small d is, $f(x)$ will oscillate outside the bounding box. Thus, the limit DNE.

Example

Define $f(x)$ as follows:

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} - Q \end{cases}$$

Does $\lim\limits_{x \to 2} f(x)$ exist?



Due to the density of the reals, in any δ-neighborhood there will be irrational numbers that pull $f(x)$ out of the bounding box. Thus, the limit DNE.