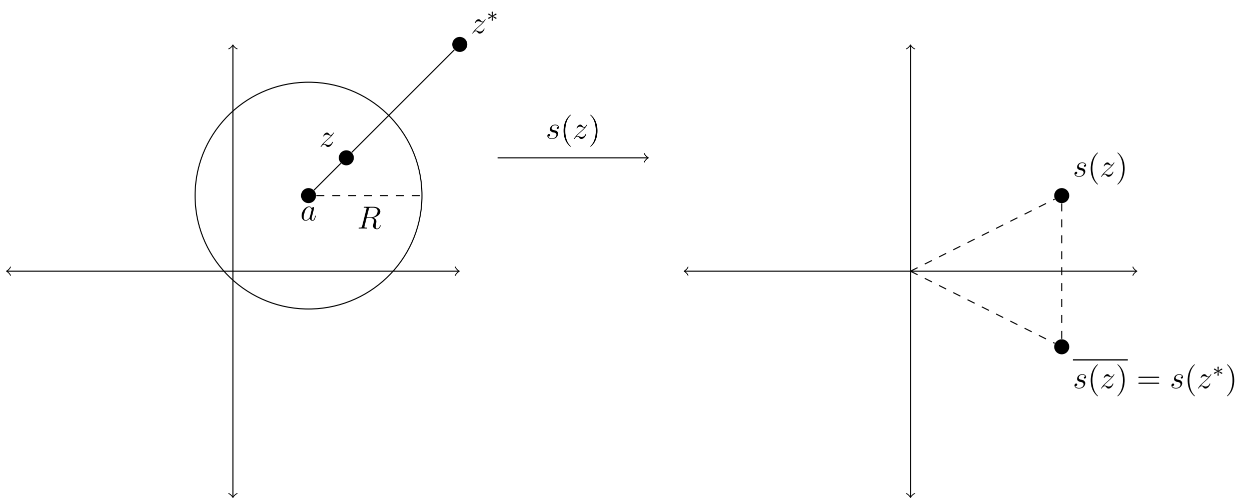
Symmetry

Definition

To say z and z^* are symmetric with respect to a circle C means there exists a LFT $= s(z)$ that maps C onto the real number line and:

$$\overline{s(z)} = s(z^*)$$



Theorem

Let z and z^* be symmetric with respect to circle $|z-a| = R$:

$$z^* = \frac{R^2}{\overline{z-a}} + a$$

$$(z^* - a)(\overline{z} - \overline{a}) = R^2$$

Note that when $a = 0$, we get the familiar $z^* = \frac{R^2}{\overline{z}}$.

Lemma

$$\overline{(z_1, z_2, z_3, z_4)} = (\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4})$$

Proof

$$\begin{aligned}
\overline{(z_1, z_2, z_3, z_4)} &= \overline{\left[\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}\right]} \\
&= \frac{\overline{(z_1 - z_3)(z_2 - z_4)}}{\overline{(z_1 - z_4)(z_2 - z_3)}} \\
&= \frac{\overline{(z_1 - z_3)}\,\overline{(z_2 - z_4)}}{\overline{(z_1 - z_4)}\,\overline{(z_2 - z_3)}} \\
&= \frac{(\overline{z_1} - \overline{z_3})(\overline{z_2} - \overline{z_4})}{(\overline{z_1} - \overline{z_4})(\overline{z_2} - \overline{z_3})} \\
&= (\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4})
\end{aligned}$$

Theorem

Let z_1, z_2, z_3 be on a circle C:
z and z^* are symmetric wrt C iff

$$\overline{(z, z_1, z_2, z_3)} = (z^*, z_1, z_2, z_3)$$

Proof

$\Longrightarrow$ Assume z and z^* are symmetric wrt C

Let $s(z)$ be a LFT from C onto the real number line

$$\begin{aligned}
\overline{(z, z_1, z_2, z_3)} &= \overline{(s(z), s(z_1), s(z_2), s(z_3))} \\
&= (\overline{s(z)}, \overline{s(z_1)}, \overline{s(z_2)}, \overline{s(z_3)}) \\
&= (s(z^*), s(z_1), s(z_2), s(z_3))
\end{aligned}$$

Now, apply the inverse relation $s^{-1}(z)$:

$$\overline{(z, z_1, z_2, z_3)} = (z^*, z_1, z_2, z_3)$$

$\Longleftarrow$ Assume $\overline{(z, z_1, z_2, z_3)} = (z^*, z_1, z_2, z_3)$

Theorem

Let z and z^* be symmetric wrt a circle C and let $s \in \mathcal{S}$:
$s(z)$ and $s(z^*)$ are symmetric wrt some circle Γ:

$$\overline{(s(z), s(z_1), s(z_2), s(z_3))} = (s(z^*), s(z_1), s(z_2), s(z_3))$$

Thus, a LFT preserves symmetry.

Proof

Assume z and z^* are symmetric wrt circle C

$$\begin{aligned}
\overline{(s(z), s(z_1), s(z_2), s(z_3))} &= \overline{(z, z_1, z_2, z_3)} \\
&= (z^*, z_1, z_2, z_3) \\
&= (s(z^*), s(z_1), s(z_2), s(z_3))
\end{aligned}$$

Example

Use symmetry to construct a conformal mapping from $|z| < 1$ to $\mathrm{Im}(w) > 0$.

Find a suitable LFT:

$$\begin{aligned}
0 &\to i \\
\infty &\to -i \\
1 &\to 2 \\
z &\to w
\end{aligned}$$

$$\begin{aligned}
(0, \infty, 1, z) &= (i, -i, 2, w) \\
\frac{(0-1)(\infty - z)}{(0-z)(\infty - 1)} &= \frac{(i-2)(-i-w)}{(i-w)(-i-2)} \\
\frac{1}{z} &= \frac{(i-2)(i+w)}{(i+2)(i-w)} \\
z(i-2)(i+w) &= (i+2)(i-w) \\
-z + izw - i2z - 2zw &= -1 - iw + i2 - 2w \\
w(iz - 2z + i + 2) &= z + i2z - 1 + i2 \\
w((-2+i)z + (i+2)) &= (1+2i)z + (-1+2i) \\
w &= \frac{(1+2i)z + (-1+2i)}{(-2+i)z + (i+2)}
\end{aligned}$$