Fundamental Groups of Topological Spaces

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Motivation

To say that two toplogical spaces X and Y are *homeomorphic* means that there exists a continuous bijection $f : X \to Y$ such that f^{-1} is also continuous. Analogous to isomorphic groups, homeomorphic spaces have the same topological structure. But as with isomorphic groups, finding a suitable mapping often proves difficult. A slightly easier task is to show that two spaces are not homeomorphic by demonstrating that they differ in some topological property. One such property is compactness; however, there are some fairly simple spaces that although compact are not homeomorphic. Some examples are shown in Figure 1.

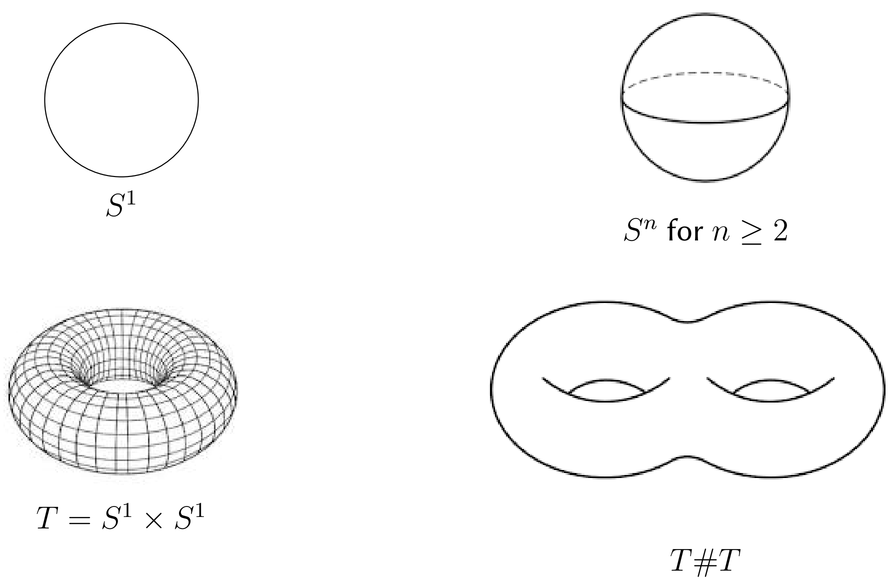


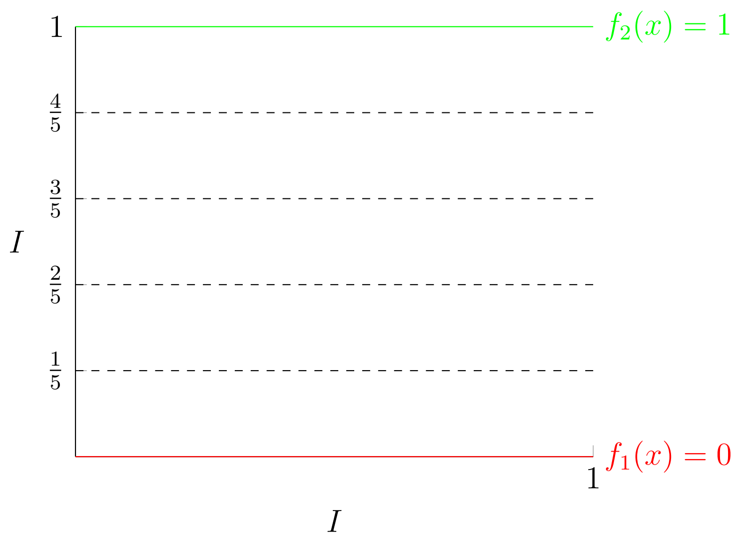
Figure 1: Compact but non-homeomorphic spaces.

The circle, closed ball, torus, and double torus are all clearly compact; however, they are not homeomorphic. In fact, these spaces share most of the standard topological properties, so something different is needed to prove whether or not they are homeomorphic. Such a property is the *fundamental group* of a space. This paper provides an overview of the fundamental group property of a space and compares the fundamental groups of the circle, closed ball, torus, and double torus to prove that these spaces are not homeomorphic.

Homotopy

The development of the concept of the fundamental group of a space begins with the concept of homotopy. First, let $I = [0,1] \subset \mathbb{R}$, imbued with the subspace topology. Next, let X and Y be topological spaces and let $f_1, f_2 : X \to Y$ be continuous functions. To say that f_1 is *homotopic* to f_2, denoted by $f_1 \simeq f_2$, means that there exists a continuous function $F : X \times I \to Y$ such that $F(x,0) = f_1(x)$ and $F(x,1) = f_2(x)$. In particular, if f_2 is a constant function then f_1 is said to be *nulhomotopic*.

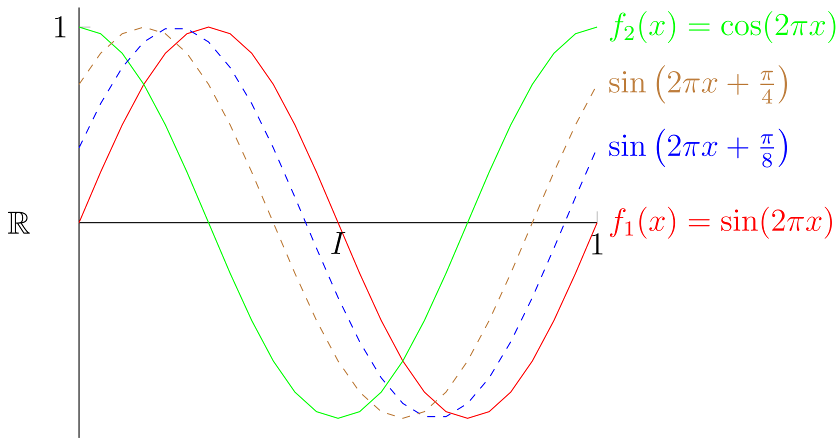
A homotopy can be viewed as a continuous deformation of f_1 into f_2 via a parameterized family of continuous functions. The homotopy example shown in Figure 2 translates a constant function vertically. Note that by definition, such a constant function is nulhomotopic.



$$F(x,t) = \pi_I(x,t) = t$$

Figure 2: A vertical translation homotopy.

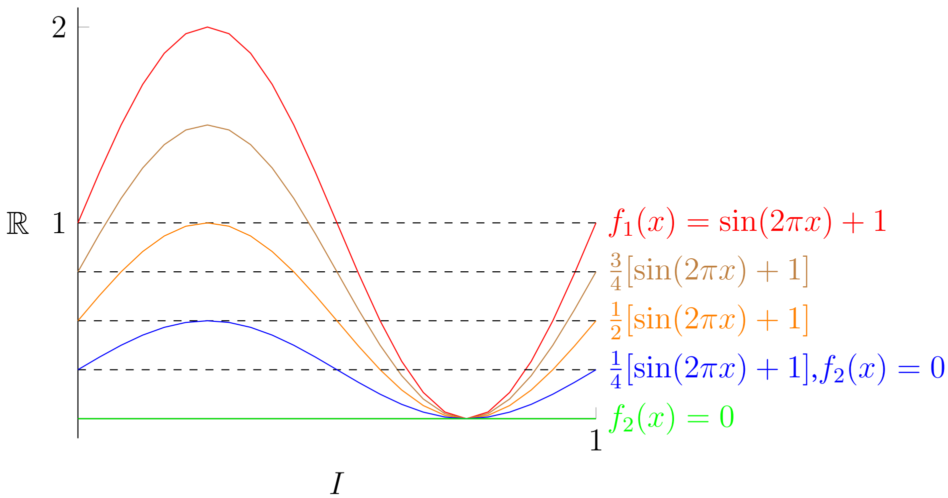
The homotopy example shown in Figure 3 is a horizontal translation.



$$F(x,t) = \sin\left(2\pi x + t\frac{\pi}{2}\right)$$

Figure 3: A horizontal translation homotopy.

And finally, the example shown in Figure 4 compresses a function into a constant function. Thus, the former is nulhomotopic.



$$F(x,t) = (1-t)[\sin(2\pi x) + 1].$$

Figure 4: A compression homotopy.

An important property of homotopies is that homotopic is an equivalence relation. Thus, for topological spaces X and Y and all homotopies between them, $[f]$ denotes the equivalence class of all functions that are homotopic to f. A special case occurs when Y is a convex subset of $\mathbb{R}^n$ in which any two continuous functions are homotopic via the so-called *straight-line homotopy*:

$$F(x,t) = (1-t)f_1(x) + tf_2(x)$$

This is demonstrated in Figure 5; corresponding points $f_1(x)$ and $f_2(x)$ are connected by a straight line that is completely contained in Y.

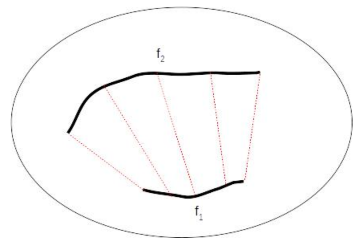


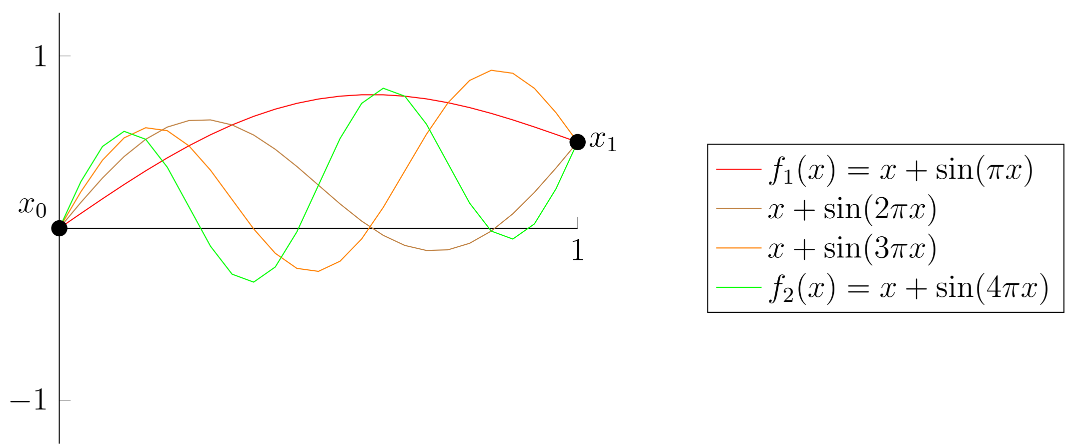
Figure 5: A straight-line homotopy example

Path Homotopy

Path homotopies are, as the name applies, homotopies between paths. Given a topological space X, a path in X from an initial point x_0 to a final point x_1 is a continuous function $f : I \to X$ where $f(0) = x_0$ and $f(1) = x_1$. Thus, a path homotopy between two paths $f_1, f_2 : I \to X$ with the same initial and final points is a continuous function $F : X \times I \to X$ such that:

- $F(x, 0) = f_1(x)$ and $F(x, 1) = f_2(x)$
- $F(0, t) = x_0$ and $F(1, t) = x_1$

An example of a path homotopy is shown in Figure 6.



$$F(s, t) = s + \sin[(3t + 1)\pi s]$$

Figure 6: A path homotopy example.

The equivalence classes of the homotopic paths within a topological space will serve as elements for the fundamental group. The next thing that is needed is a binary operator.

The Product Operator

Let X be a topological space. Let f_1 be a path in X between initial point x_0 and final point x_1, and let f_2 be a path in X between initial point x_1 and final point x_2. The *product* of f_1 and f_2, denoted by $f_1 * f_2$, is the path from x_0 to x_2 defined by:

$$f_1 * f_2 = \begin{cases} f_1(2t), & t \in \left[0, \frac{1}{2}\right] \\ f_2(2t-1), & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Note that $f_1 * f_2$ is continuous by the pasting lemma.

The product operator is demonstrated in Figure 7.

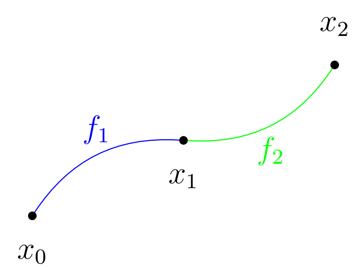


Figure 7: A product operator example.

Note that when the path equivalence classes of a topological space are paired with the product operator, a so-called *groupoid* is formed. The groupoid is not a proper group because the product operator is only a partial function — i.e, it is only well-defined when $f_1(1) = f_2(0)$. In particular, the groupoid has the following group-like properties:

- Associative: $([f] * [g]) * [h]$ is defined if and only if $[f] * ([g] * [h])$ is defined and if defined then they are equal.
- Identity: $[e_{x_0}] * [f] = [f]$ and $[f] * [e_{x_1}] = [f]$.
- Inverse: $[f] * [\bar{f}] = [e_{x_0}]$ and $[\bar{f}] * [f] = [e_{x_1}]$.

where e_x is the *trivial path* that maps I to the constant value x and $\bar{f}(t) = f(1-t)$ is the reverse path from x_1 to x_0.

The Fundamental Group

In order to construct a proper group using the paths in a topological space X and the product operator, a single point $x_0 \in X$ is selected. A path that has x_0 as both its initial and final points is called a *loop* based at x_0. The fundamental group for X is then the homotopic equivalence classes of the loops based at x_0 paired with the product operator. Such a group is denoted by $\pi_1(X, x_0)$. Note that $[e_{x_0}]$ is the group identity element and $\bar{f}$ is the inverse of f.

One might wonder if the fundamental group is dependent on the choice of x_0. It turns out that for two different points $x_0, x_1 \in X$, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$. Thus, although the actual isomorphism may differ, the structure of the fundamental group is invariant based on the choice of x_0 and x_1. Furthermore, if $\pi_1(X, x_0)$ is not isomorphic to $\pi_1(Y, y_0)$ then X and Y are not homeomorphic.

An important point to remember is that the fundamental group only describes the connected component containing x_0. Thus, fundamental groups are usually only discussed for path connected spaces.

Simply Connected Spaces

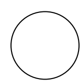
A special case occurs when the fundamental space for a group is trivial, meaning it consists of the identity element $[e_{x_0}]$ only. Such a space is called *simply connected* and is signified as such by the syntax $\pi_1(X, x_0) = 0$.

Note that due to the straight-line homotopy, any two paths in a convex subspace of $\mathbb{R}^n$ are homotopic. Thus, such spaces are simply connected. In particular, all open balls (and their closures) in $\mathbb{R}^n$ are simply connected: $\pi_1(B(p, r), x_0) = 0$.

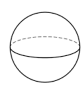
Non-homeomorphic Spaces

The actual analysis used to determine the fundamental groups of the original four spaces mentioned at the beginning of this paper is omitted. However, the results are as follows:

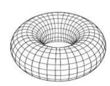
- $\pi_1(S^1, x_0) \sim \mathbb{Z}$



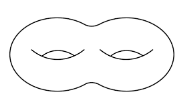
- $\pi_1(S^n, x_0) = 0$ for $n \geq 2$



- $\pi_1(T = S^1 \times S^1, x_0) \sim \mathbb{Z} \times \mathbb{Z}$ (abelian)



- $\pi_1(T \# T)$ is not abelian



Therefore, since these fundamental groups are not isomorphic, the corresponding spaces are non-homeomorphic.

References

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[3] Michael Starbird and Francis Su, *Topology through inquiry*, MAA Press, Providence, RI, 2019.