Relations and Functions

Definition

A relation $\Re$ between two sets A and B is a subset of $A \times B$:

$$\Re \subseteq A \times B$$

To say that $a \in A$ is related to $b \in B$, often denoted $a \sim b$, means $(a, b) \in \Re$.

A relation between a set A and itself ($\Re \subseteq A \times A$) is referred to as a relation *on* A.

Definition

A function ϕ is a relation between sets A and B such that:

1). $\forall\, a \in A, \exists\, b \in B, (a, b) \in \phi$

2). $\forall\, a \in A, \forall\, b, c \in B, (a, b) \in \phi$ and $(a, c) \in \phi \implies b = c$

The function is said to map A into B and is denoted $\phi : A \to B$, where A is called the *domain* of ϕ and B is called the *codomain* of ϕ. Also, $(a, b) \in \phi$ is denoted using the more conventional functional notation: $\phi(a) = b$.

A function between a set A and itself is referred to as a function *on* A.

Definition

Let $\phi : A \to B$ with $C \subseteq A$ and $D \subseteq B$:

- The *image* of C under ϕ, denoted $\phi[C]$, is give by:

  $$\phi[C] = \{\phi(c) \mid c \in C\} \subseteq B$$

  $\phi[A]$ is called the *range* of ϕ.

- The *pre-image* of D under ϕ, denoted $\phi^{-1}[D]$, is given by:

  $$\phi^{-1}[D] = \{a \in A \mid \phi(a) \in D\} \subseteq A$$

  Note that $\phi^{-1}[D]$ is a (possibly empty) set and should not be confused with any inverse function $\phi^{-1} : B \to A$.

Definition

Let $\phi : A \to B$ and $\mu : C \to D$. To say that ϕ and μ are equal, denoted $\phi = \mu$, means:

1). $A = C$ (same domain)

2). $B = D$ (same codomain)

3). $\forall\, a \in A, \phi(a) = \mu(a)$

Definition

Let $\phi : A \to B$:

- To say that ϕ is *one-to-one* (*injective*) means:

$$\forall\, a, b \in A, \phi(a) = \phi(b) \implies a = b$$

- To say that ϕ is *onto* (*surjective*) means $\phi[A] = B$, or:

$$\forall\, b \in B, \exists\, a \in A, \phi(a) = b$$

- To say that ϕ is a *one-to-one correspondence* (*bijective*) means ϕ is both one-to-one and onto.

Definition

The *identity* function on a set A, denoted i_A, is given by:

$$\forall\, a \in A, \phi(a) = a$$

Definition

Let $\phi : A \to B$ and $\mu : B \to C$. The composition of ϕ and μ, denoted $\mu \circ \phi$ or simply $\mu\phi$, is a new function $\mu\phi : A \to C$ given by:

$$\forall\, a \in A, (\mu\phi)(a) = \mu[\phi(a)]$$

Theorem

Function composition is associative.

Proof

Let $\phi : A \to B$, $\mu : B \to C$, and $\gamma : C \to D$.
Assume $a \in A$

$$[(\gamma\mu)\phi)](a) = (\gamma\mu)[\phi(a)] = \gamma\{\mu[\phi(a)]\} = \gamma[\mu\phi(a)] = [\gamma(\mu\phi)](a)$$

Definition

Let $\phi : A \to B$. To say that $\phi^{-1} : B \to A$ is an inverse function for ϕ means:

1). $\phi^{-1}\phi = i_A$

2). $\phi\phi^{-1} = i_B$

Note that an inverse must be both a left and a right inverse; there are cases where only one side exists.

Theorem

Let $\phi : A \to B$:

ϕ is invertible iff $\phi(a) = b \iff \phi^{-1}(b) = a$

Proof

$\implies$ Assume ϕ is invertible.

$\implies$ Assume $\phi(a) = b$

$$\begin{aligned} (\phi^{-1}\phi)(a) &= \phi^{-1}(b) \\ \therefore a &= \phi^{-1}(b) \end{aligned}$$

$\impliedby$ Assume $\phi^{-1}(b) = a$

$$\begin{aligned} (\phi\phi^{-1})(b) &= \phi(a) \\ \therefore b &= \phi(a) \end{aligned}$$

$\impliedby$ Assume $\phi(a) = b \iff \phi^{-1}(b) = a$

$$(\phi^{-1}\phi)(a) = \phi^{-1}[\phi(a)] = \phi^{-1}(b) = a = i_A(a)$$

$$(\phi\phi^{-1})(b) = \phi[\phi^{-1}(b)] = \phi(a) = b = i_B(b)$$

Theorem

Let $\phi : A \to B$:

ϕ is invertible $\iff$ ϕ is bijective

Proof

$\implies$ Assume ϕ is invertible

Assume $\phi(a) = \phi(b)$

$$(\phi^{-1}\phi)(a) = (\phi^{-1}\phi)(b)$$

$$a = b$$

$\therefore \phi$ is one-to-one

ϕ^{-1} is a function with domain B
Assume $b \in B$
$\exists\, a \in A, \phi^{-1}(b) = a$

$$(\phi\phi^{-1})(b) = \phi(a)$$

$$b = \phi(a)$$

$\therefore \phi$ is onto

$\therefore \phi$ is bijective

$\Longleftarrow$ Assume ϕ is bijective

Assume $a \in A$
Since ϕ is a function, $\exists\, b \in B, \phi(a) = b$
But, since ϕ is one-to-one, $\exists\, \phi^{-1}, \phi^{-1}(b) = a$.
$(\phi\phi^{-1})(b) = \phi(a) = b$

Assume $b \in B$
But, since ϕ is onto, $\exists\, a \in A, \phi(a) = b$
$(\phi^{-1}\phi)(a) = \phi^{-1}(b) = a$

$\therefore \phi^{-1}$ exists and ϕ is invertible.

Theorem

Inverse functions are unique.

Proof

Assume $\phi : A \to B$ is invertible
Assume $\mu : B \to A$ and $\gamma : B \to A$ are both inverses of ϕ
Assume $b \in B$

$$\begin{aligned}
(\phi\mu)(b) &= b \\
[\gamma(\phi\mu)]\,(b) &= \gamma(b) \\
[(\gamma\phi)\mu)]\,(b) &= \gamma(b) \\
(\gamma\phi)[\mu(b)] &= \gamma(b) \\
\mu(b) &= \gamma(b) \\
\therefore \mu &= \gamma
\end{aligned}$$