MATH 231B, FALL 2017
HOMEWORK 4 SOLUTIONS

1. (Sec. 3.8, ex. 47) Let E, O denote the subspaces of $L^2(\mathbb{R})$ consisting of even and odd functions, respectively. If $f \in E$ and $g \in O$, then
$$\langle f, g\rangle = \int_{\mathbb{R}} f\overline{g} = 0,$$
since $f\overline{g}$ is an odd function and the integral of an odd function over any symmetric interval of real numbers equals zero. Therefore, $E \perp O$, so $O \subset E^{\perp}$. To prove that O equals $E^{\perp}$, we recall that every function f can be uniquely written as $g + h$, where g is even and h is odd. Indeed,
$$g(x) = \frac{f(x) + f(-x)}{2}, \qquad h(x) = \frac{f(x) - f(-x)}{2}.$$
It is clear that if $f \in L^2(\mathbb{R})$, then $g \in E$ and $h \in O$.
Now assume $f \in E^{\perp}$. Then $f = g + h$, where $g \in E$ and $h \in O$. So $g = f - h$ is in $E^{\perp}$ since $h \in O \subset E^{\perp}$ and $E^{\perp}$ is a subspace. But g is also in E and $E \cap E^{\perp} = \{0\}$. Thus $g = 0$ and therefore $f = h \in O$. This proves $E^{\perp} \subset O$ and completes the proof. □

2. (Sec. 3.8, ex. 50) First observe that if $A \subset B$, then clearly $B^{\perp} \subset A^{\perp}$. Therefore, since $S \subset \operatorname{span}(S)$, it follows that $\operatorname{span}(S)^{\perp} \subset S^{\perp}$. To prove the opposite inclusion, assume $x \in S^{\perp}$. Let $y \in \operatorname{span}(S)$ be arbitrary. Then $y = \alpha_1 x_1 + \cdots + \alpha_n x_n$, for some scalars α_j and vectors $x_j \in S$. Then:
$$\langle x, y\rangle = \sum_{j=1}^{n} \overline{\alpha}_j \langle x, x_j\rangle = 0,$$
since $x \perp S$. This shows $x \in \operatorname{span}(S)^{\perp}$, completing the proof. □

3. (Sec. 3.8, ex. 52) Let $S = \{(x, y) \in \mathbb{R}^2 : -1 \le x \le 1, y = 0\}$ and $p = (0, 1)$. Consider an arbitrary point $q = (t, 0) \in S$. Since $|t| \le 1$, we have
$$\|p - q\| = \max\{|t|, 1\} = 1.$$
Thus $\operatorname{dist}(p, S) = 1$ and *every* point $q \in S$ is the closest point to p. □

4. (Sec. 3.8, ex. 53) We claim that
$$y = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n.$$
To prove this, let us fist show that $x - y \perp e_k$, for all k. Indeed,
$$\begin{aligned}\langle x - y, e_k\rangle &= \langle x, e_k\rangle - \langle y, e_k\rangle \\ &= \langle x, e_k\rangle - \sum_{n=1}^{\infty} \langle x, e_n\rangle\langle e_n, e_k\rangle \\ &= 0,\end{aligned}$$
since $\langle e_n, e_k\rangle = \delta_{nk}$ (the Kronecker delta). Since (e_n) is a complete orthonormal sequence in S, it follows that $x - y \perp S$.

Let us show that for every $z \in S$, $\|x - y\| \leq \|x - z\|$. Since $x - y \perp y - z$, by the Pythagorean formula we have:

$$\begin{aligned} \|x - z\|^2 &= \|(x - y) + (y - z)\|^2 \\ &= \|x - y\|^2 + \|y - z\|^2 \\ &\geq \|x - y\|^2. \end{aligned}$$

This proves that $\|x - y\| = \operatorname{dist}(x, S)$. $\square$