Rational Zeros Theorem

Theorem

Let $\sum_{k=0}^{n} a_k x^k = 0$ be a polynomial equation such that $n \geq 1$ and $a_n, a_0 \neq 0$. If $r \in \mathbb{Q}$ where $r = \frac{p}{q}$ such that $(p, q) = 1$ is a solution to the polynomial equation then $p|a_0$ and $q|a_n$.

Proof

Assume $r = \frac{p}{q}$ is a solution to $\sum_{k=0}^{n} a_k x^k = 0$.
$\sum_{k=0}^{n} a_k \left(\frac{p}{q}\right)^k = 0$ Multiply by q^k.
$\sum_{k=0}^{n} a_k p^k q^{n-k} = 0$

$a_0 q^n + \sum_{k=1}^{n} a_k p^k q^{n-k} = 0$
$a_0 q^n = -\sum_{k=1}^{n} a_k p^k q^{n-k}$
$a_0 q^n = -p \sum_{k=1}^{n} a_k p^{k-1} q^{n-k}$
But $a_k, p^{k-1}, q^{n-k} \in \mathbb{Z}$ for $k \in \{1, ..., n\}$.
so $p|a_0 q^n$, but $(p, q) = 1$.
$\therefore p|a_0$

$\sum_{k=0}^{n-1} a_k p^k q^{n-k} + a_n p^n = 0$
$a_n p^n = -\sum_{k=0}^{n-1} a_k p^k q^{n-k}$
$a_n p^n = -q \sum_{k=0}^{n-1} a_k p^k q^{n-k-1}$
But $a_k, p^k, q^{n-k+1} \in \mathbb{Z}$ for $k \in \{1, ..., n-1\}$.
so $q|a_n p^n$, but $(p, q) = 1$.
$\therefore q|a_n$

Example

Show that $\sqrt{2}$ is irrational.

$\sqrt{2}$ is a solution to $x^2 - 2 = 0$.
But by the RZT, the only possible rational solutions are $\pm 1, \pm 2$.
$2^2 - 2 = 2 \neq 0$
$(-2)^2 - 2 = 2 \neq 0$
$1^2 - 2 = -1 \neq 0$
$(-1)^2 - 2 = -1 \neq 0$
So the polynomial equation has no rational solutions.
$\therefore \sqrt{2} \notin \mathbb{Q}$