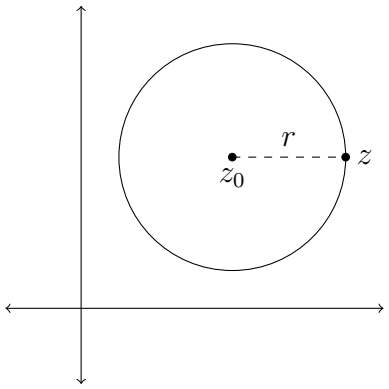
Geometry

1). Circle



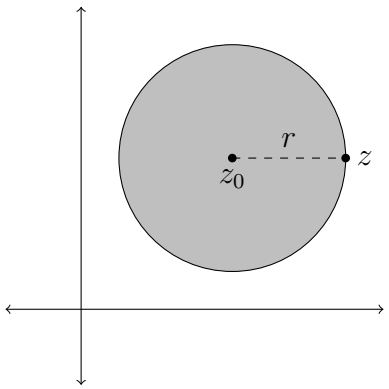
$$|z - z_0| = r$$

Example

Consider the unit circle: $x^2 + y^2 = 1$

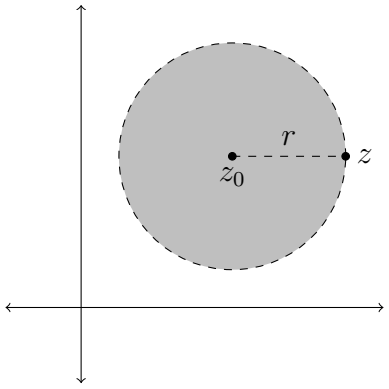
$$\begin{aligned}
x^2 + y^2 &= 1 \\
Re(z)^2 + Im(z)^2 &= 1 \\
\left(\frac{z+\bar{z}}{2}\right)^2 + \left(\frac{z-\bar{z}}{2i}\right)^2 &= 1 \\
(z+\bar{z})^2 - (z-\bar{z})^2 &= 4 \\
(z^2 + \bar{z}^2 + 2z\bar{z}) - (z^2 + \bar{z}^2 - 2z\bar{z}) &= 4 \\
4z\bar{z} &= 4 \\
|z|^2 &= 1 \\
|z| &= 1
\end{aligned}$$

2). Closed Disk



$$0 \leq |z - z_0| \leq r$$

3). Open Disk



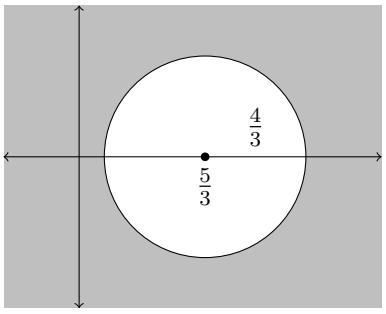
$$0 \leq |z - z_0| < r$$

Example

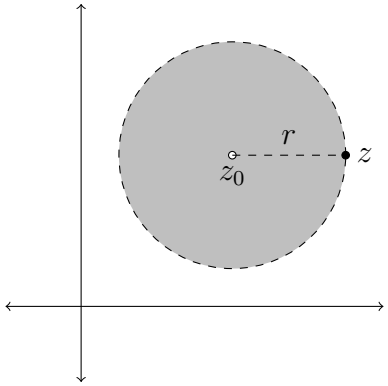
Describe: $|z-1| \leq 2|z+1|$

$$\begin{aligned}
|z-1|^2 &\leq 4|z+1|^2 \\
|z|^2 + 1 - 2Re(z) &\leq 4(|z|^2 + 1 + 2Re(z)) \\
3|z|^2 + 3 + 10Re(z) &\geq 0 \\
|z|^2 + 1 + \frac{10}{3}Re(z) &\geq 0 \\
|z|^2 + \frac{10}{3}Re(z) &\geq -1 \\
|z|^2 + \frac{25}{9} + \frac{10}{3}Re(z) &\geq -1 + \frac{25}{9} \\
\left|z + \frac{5}{3}\right|^2 &\geq \frac{16}{9} \\
\left|z + \frac{5}{3}\right| &\geq \frac{4}{3}
\end{aligned}$$

This is the exterior of the open disk with center $z_0 = \frac{5}{3}$ and radius $r = \frac{4}{3}$.

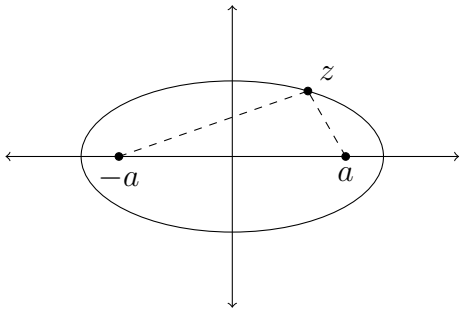


4). Punctured Disk



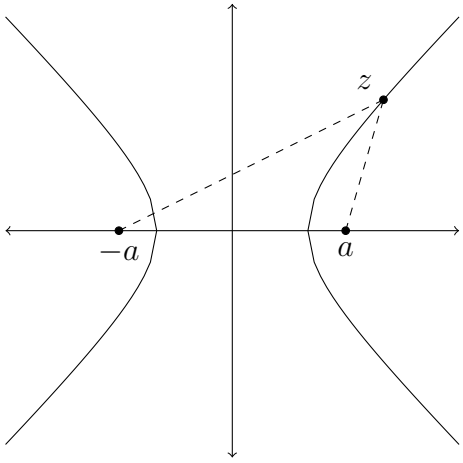
$$0 < |z - z_0| < r$$

5). Ellipse



$$|z + a| + |z - a| = \ell$$

6). Hyperbola



$$||z + a| - |z - a|| = \ell$$

Example

Consider the hyperbola: $x^2 - y^2 = 1$

$$\begin{aligned} x^2 - y^2 &= 1 \\ Re(z)^2 - Im(z)^2 &= 1 \\ \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 &= 1 \\ (z+\bar{z})^2 + (z-\bar{z})^2 &= 4 \\ (z^2 + \bar{z}^2 + 2z\bar{z}) + (z^2 + \bar{z}^2 - 2z\bar{z}) &= 4 \\ 2z^2 + 2\bar{z}^2 &= 4 \\ z^2 + \bar{z}^2 &= 2 \end{aligned}$$