Math-42 Sections 01, 02, 05

Homework #11 Solutions

Problems

Prove (by induction): For all $n \in \mathbb{N}$:

$$21 \,\big|\, \big(4^{n+1} + 5^{2n-1}\big)$$

Make sure that your proof is well structured as specified in class.

(Hint: Example 9)

Proof by induction on n.

Base Case: $n = 1$

$$4^{1+1} + 5^{2(1)-1} = 4^2 + 5^1 = 16 + 5 = 21$$

$$21|21$$

Inductive Hypothesis: Assume that $21 \,|(4^{n+1} + 5^{2n-1})$.

Inductive Step: Consider $n + 1$.

$$\begin{aligned}
4^{(n+1)+1} + 5^{2(n+1)-1} &= 4 \cdot 4^{n+1} + 5^{2n+1} \\
&= 4 \cdot 4^{n+1} + 5^{(2n-1)+2} \\
&= 4 \cdot 4^{n+1} + 5^2 \cdot 5^{2n-1} \\
&= 4 \cdot 4^{n+1} + 25 \cdot 5^{2n-1} \\
&= 4 \cdot 4^{n+1} + 4 \cdot 5^{2n-1} + 21 \cdot 5^{2n-1} \\
&= 4(4^{n+1} + 5^{2n-1}) + 21 \cdot 5^{2n-1}
\end{aligned}$$

But $21|4^{n+1} + 5^{2n-1}$ (inductive hypothesis), so $21|[4(4^{n+1} + 5^{2n-1})]$. Furthermore, $21|21 \cdot 5^{2n-1}$. Therefore $21|[4(4^{n+1} + 5^{2n-1}) + 21 \cdot 5^{2n-1}]$.