Cross Ratio

Definition

Let $z_1, z_2, z_3, z_4 \in \mathbb{C} \cup \{\infty\}$. The *cross ratio* of z_1, z_2, z_3, z_4 is given by:

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

Example

$(1,2,3,4) = \frac{(1-3)(2-4)}{(1-4)(2-3)} = \frac{(-2)(-2)}{(-3)(-1)} = \frac{4}{3}$

$(4,3,2,1) = \frac{(4-2)(3-1)}{(4-1)(3-2)} = \frac{(2)(2)}{(3)(1)} = \frac{4}{3}$

$(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}) = \frac{(1-\frac{1}{3})(\frac{1}{2}-\frac{1}{4})}{(1-\frac{1}{4})(\frac{1}{2}-\frac{1}{3})} = \frac{\frac{2}{3}\cdot\frac{1}{4}}{\frac{3}{4}\cdot\frac{1}{6}} = \frac{2}{12}\cdot\frac{24}{3} = \frac{1}{6}\cdot 8 = \frac{4}{3}$

Example

$(0,1,1+i,i) = \frac{(0-(1+i))(1-i)}{(0-i)(1-(1+i)} = \frac{-(1+i)(1-i)}{(-i)(-i)} = \frac{-(1+i)(1-i)}{-1} = (1+i)(1-i) = 2$

$(0,1,i,1+i) = \frac{(0-i)(1-(1+i))}{(0-(1+i))(1-i)} = \frac{(-i)(-i)}{-(1+i)(1-i)} = \frac{-1}{-(1+i)(1-i)} = \frac{1}{(1+i)(1-i)} = \frac{1}{2}$

Lemma

Let $s = \frac{az+b}{cz+d}$ and let $\Delta = ad - bc$:

$$s(z_1) - s(z_2) = \frac{\Delta(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)}$$

Proof

$$\begin{aligned}
s(z_1) - s(z_2) &= \frac{az_1 + b}{cz_1 + d} - \frac{az_2 + b}{cz_2 + d} \\
&= \frac{(az_1 + b)(cz_2 + d) - (az_2 + b)(cz_1 + d)}{(cz_1 + d)(cz_2 + d)} \\
&= \frac{acz_1z_2 + adz_1 + bcz_2 + bd - acz_1z_2 - adz_2 - bcz_1 - bd}{(cz_1 + d)(cz_2 + d)} \\
&= \frac{adz_1 + bcz_2 - adz_2 - bcz_1}{(cz_1 + d)(cz_2 + d)} \\
&= \frac{ad(z_1 - z_2) - bc(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)} \\
&= \frac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)} \\
&= \frac{\Delta(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)}
\end{aligned}$$

Theorem

Let $s \in \mathcal{S}$ such that $\Delta \neq 0$ and let $w = s(z)$:

$$(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4)$$

Thus, the cross ratio is invariant under LFT.

Proof

$$\begin{aligned}
(w_1, w_2, w_3, w_4) &= \frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)} \\
&= \frac{\left[\frac{\Delta(z_1 - z_3)}{(cz_1 + d)(cz_3 + d)}\right]\left[\frac{\Delta(z_2 - z_4)}{(cz_2 + d)(cz_4 + d)}\right]}{\left[\frac{\Delta(z_1 - z_4)}{(cz_1 + d)(cz_4 + d)}\right]\left[\frac{\Delta(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)}\right]} \\
&= \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \\
&= (z_1, z_2, z_3, z_4)
\end{aligned}$$

Example

Show that there exists an LFT such that:

$$\begin{aligned}
z &\to w \\
z_1 &\to 0 \\
z_2 &\to \infty \\
z_3 &\to 1
\end{aligned}$$

$$\begin{aligned}
(z_1, z_2, z_3, z) &= (0, \infty, 1, w) \\
\frac{(z_1 - z_3)(z_2 - z)}{(z_1 - z)(z_2 - z_3)} &= \frac{(0 - 1)(\infty - w)}{(0 - w)(\infty - 1)} \\
\frac{1}{w} &= \frac{(z_1 - z_3)(z_2 - z)}{(z_1 - z)(z_2 - z_3)} \\
w &= \frac{(z_1 - z)(z_2 - z_3)}{(z_1 - z_3)(z_2 - z)} \\
&= \frac{(z - z_1)(z_2 - z_3)}{(z - z_2)(z_1 - z_3)} \\
w(z) &= \frac{(z_2 - z_3)z - z_1(z_2 - z_3)}{(z_1 - z_3)z - z_2(z_1 - z_3)}
\end{aligned}$$

Theorem

Let $s \in \mathcal{S}$ such that $z \neq -\frac{d}{c}$.
$s(z)$ is conformal.

Proof

$$\begin{aligned} s(z) &= \frac{az+b}{cz+d} \\ s'(z) &= \frac{a(cz+d)-c(az+b)}{(cz+d)^2} \end{aligned}$$

But $z \neq -\frac{d}{c}$, so $(cz+d)^2 \neq 0$
Thus $s(z)$ is analytic and $s'(z) \neq 0$

$\therefore s(z)$ is conformal.

Note that a line is a circle with infinite radius.

Theorem

$(z_1, z_2, z_3, z_4) \in \mathbb{R} \iff z_1, z_2, z_2, z_4$ lie on a circle.

Proof

$\Longrightarrow$ Assume $(z_1, z_2, z_3, z_4) \in \mathbb{R}$

Let $s = (z_1, z_2, z_3, z_4) = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$

Case 1: $s = 0$

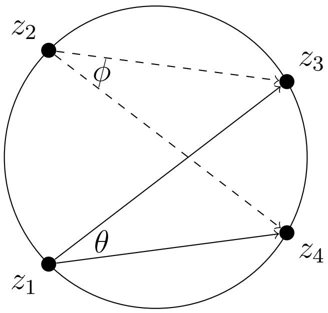
$z_1 = z_3$ or $z_2 = z_4$ Thus there are only 2 or 3 distinct points
But 3 points define a circle
$\therefore z_1, z_2, z_2, z_4$ lie on a circle.

Case 2: $s > 0$

$$\begin{aligned} \arg s &= \arg(z_1, z_2, z_3, z_4) \\ &= \arg\left[\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}\right] \\ &= \arg\left[\frac{z_1-z_3}{z_1-z_4}\right] - \arg\left[\frac{z_2-z_3}{z_2-z_4}\right] \\ &= \arg\left[\frac{z_3-z_1}{z_4-z_1}\right] - \arg\left[\frac{z_3-z_2}{z_4-z_2}\right] \end{aligned}$$

But $\arg s = 0$, so

$$\arg\left[\frac{z_3-z_1}{z_4-z_1}\right] = \arg\left[\frac{z_3-z_2}{z_4-z_2}\right]$$



But $\theta = \phi$

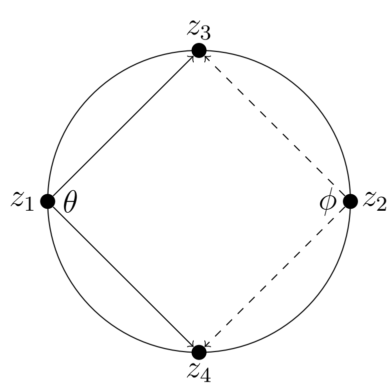
$\therefore z_1, z_2, z_2, z_4$ lie on a circle.

Case 3: $s < 0$

$$\begin{aligned}
\arg s &= \arg(z_1, z_2, z_3, z_4) \\
&= \arg\left[\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}\right] \\
&= \arg\left[\frac{z_1 - z_3}{z_1 - z_4}\right] + \arg\left[\frac{z_2 - z_4}{z_2 - z_3}\right] \\
&= \arg\left[\frac{z_3 - z_1}{z_4 - z_1}\right] + \arg\left[\frac{z_4 - z_2}{z_3 - z_2}\right]
\end{aligned}$$

But $\arg s = \pi$, so

$$\arg\left[\frac{z_3 - z_1}{z_4 - z_1}\right] + \arg\left[\frac{z_4 - z_2}{z_3 - z_2}\right] = \pi$$



But $\theta + \phi = \pi$

$\therefore z_1, z_2, z_2, z_4$ lie on a circle.

$\Longleftarrow$ Assume z_1, z_2, z_2, z_4 lie on a circle

Corollary

A LFT maps a circle (or line) onto a circle (or line).

Proof

Let $w = s(z)$ be a LFT
Assume z_1, z_2, z_3, z_4 lie on a circle
$(z_1, z_2, z_3, z_4) \in \mathbb{R}$
But $(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4)$
So $(w_1, w_2, w_3, w_4) \in \mathbb{R}$
$\therefore w_1, w_2, w_3, w_4$ lie on a circle.