Weak Convergence

Definition: Weak Convergence

Let E be an inner product space and let $(\vec{x}_n)$ be a sequence of vectors in E. To say that $(\vec{x}_n)$ converges to $\vec{x} \in E$ ***weakly***, denoted $\vec{x}_n \xrightarrow{w} \vec{x}$, means $\forall\, \vec{y} \in E$:

$$\langle \vec{x}_n, y \rangle \to \langle \vec{x}, y \rangle$$

Properties

Let H be a Hilbert space over a field $\mathbb{F}$. Let $\vec{x}_n \xrightarrow{w} \vec{x}$ and $\vec{y}_n \xrightarrow{w} \vec{y}$ in H and $\alpha_n \to \alpha$ in $\mathbb{F}$:

1). The weak limit is unique - i.e., if $\forall\, n \in \mathbb{N}, \vec{x}_n = \vec{y}_n$ then $\vec{x} = \vec{y}$.

2). $\vec{x}_n + \vec{y}_n \xrightarrow{w} \vec{x} + \vec{y}$

3). $\alpha_n \vec{x}_n \xrightarrow{w} \alpha \vec{x}$

Proof

1). Assume $\vec{z} \in H$ such that $\vec{z} \neq \vec{0}$ and $\vec{z} \not\perp (\vec{x} - \vec{y})$.

$$\begin{aligned} \langle \vec{x} - \vec{y}, \vec{z} \rangle &= \langle (\vec{x} - \vec{x}_n) + (\vec{x}_n - \vec{y}), \vec{z} \rangle \\ &= \langle \vec{x} - \vec{x}_n, \vec{z} \rangle + \langle \vec{x}_n - \vec{y}, \vec{z} \rangle \\ &\to 0 + 0 \\ &= 0 \end{aligned}$$

And so $\vec{x} - \vec{y} = \vec{0}$.

$\therefore \vec{x} = \vec{y}$

2). $\langle \vec{x}_n + \vec{y}_n, \vec{z} \rangle = \langle \vec{x}_n, \vec{z} \rangle + \langle \vec{y}_n, \vec{z} \rangle \to \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle = \langle \vec{x} + \vec{y}, \vec{z} \rangle$

$\therefore \vec{x}_n + \vec{y}_n \xrightarrow{w} \vec{x} + \vec{y}$

3). $\langle \alpha_n \vec{x}_n, \vec{z} \rangle = \alpha_n \langle \vec{x}_n, \vec{z} \rangle \to \alpha \langle \vec{x}, \vec{z} \rangle = \langle \alpha \vec{x}, \vec{z} \rangle$

$\therefore \alpha_n \vec{x}_n \xrightarrow{w} \alpha \vec{x}$

Theorem

Let E be an inner product space and let $(\vec{x}_n)$ be a sequence of vectors in E:

$$\vec{x}_n \to \vec{x} \implies \vec{x}_n \xrightarrow{w} \vec{x}$$

Proof

Assume $\vec{x}_n \to \vec{x}$.

$\|\vec{x}_n - \vec{x}\| \to 0$

Assume $\vec{y} \in E$.

$|\langle \vec{x}_n, \vec{y}\rangle - \langle \vec{x}, \vec{y}\rangle| = |\langle \vec{x}_n - \vec{x}, \vec{y}\rangle| \leq \|\vec{x}_n - \vec{x}\| \, \|\vec{y}\| \to 0$

$\therefore \vec{x}_n \xrightarrow{w} \vec{x}$

Theorem

Let E be an inner product space and let $(\vec{x}_n)$ be a sequence of vectors in E:

$$\vec{x}_n \xrightarrow{w} \vec{x} \text{ and } \|\vec{x}_n\| \to \|\vec{x}\| \implies \vec{x}_n \to \vec{x}$$

Proof

Assume $\vec{x}_n \xrightarrow{w} \vec{x}$ and $\|\vec{x}_n\| \to \|\vec{x}\|$.

$$\begin{array}{rcl} \|\vec{x}_n - \vec{x}\|^2 & = & \langle \vec{x}_n - \vec{x}, \vec{x}_n - \vec{x}\rangle \\ & = & \langle \vec{x}_n, \vec{x}_n\rangle - \langle \vec{x}_n, \vec{x}\rangle - \langle \vec{x}, \vec{x}_n\rangle + \langle \vec{x}, \vec{x}\rangle \\ & = & \|\vec{x}_n\|^2 - \langle \vec{x}_n, \vec{x}\rangle - \langle \vec{x}, \vec{x}_n\rangle + \|\vec{x}\|^2 \\ & \to & \|\vec{x}\|^2 - \langle \vec{x}, \vec{x}\rangle - \langle \vec{x}, \vec{x}\rangle + \|\vec{x}\|^2 \\ & = & 2\|\vec{x}\|^2 - 2\|\vec{x}\|^2 \\ & = & 0 \end{array}$$

$\therefore \vec{x}_n \to \vec{x}$

Theorem

Let H be a finite dimensional Hilbert space over a field $\mathbb{F}$. For all sequences $(\vec{x}_n)$ in H:

$$\vec{x}_n \to \vec{x} \iff \vec{x}_n \xrightarrow{w} \vec{x}$$

Thus, weak convergence implies strong convergence in a finite dimensional Hilber space.

Proof

Assume $(\vec{x}_n)$ is a sequence in H.

$\implies$ Assume $\vec{x}_n \to \vec{x}$.

But strong convergence implies weak convergence in any inner product space.

$\therefore \vec{x}_n \xrightarrow{w} \vec{x}$

$\impliedby$ Assume $\vec{x}_n \xrightarrow{w} \vec{x}$.

Assume $B = \{\vec{b}_1, \ldots, \vec{b}_N\}$ is a basis for H.
AWLOG: B is an orthonormal basis (otherwise, use Graham-Schmitt).

$\forall\, n \in \mathbb{N}, \exists\, x_{n,k} \in \mathbb{F}$ such that $\vec{x}_n = \displaystyle\sum_{k=1}^{n} x_{n,k}\vec{b}_k$.

$\exists\, x_k \in \mathbb{F}$ such that $\vec{x} = \displaystyle\sum_{k=1}^{N} x_k\vec{b}_k$.

Assume $\vec{z} \in H$.

$\exists\, z_k \in \mathbb{F}$ such that $\vec{z} = \displaystyle\sum_{k=1}^{N} z_k \vec{b}_k$.

Since $\vec{x}_n \xrightarrow{w} \vec{x}$:

$$\begin{aligned}
\langle \vec{x}_n - \vec{x}, \vec{z} \rangle &= \left\langle \sum_{k=1}^{N} x_{n,k}\vec{b}_k - \sum_{k=1}^{N} x_k \vec{b}_k, \sum_{k=1}^{N} z_k \vec{b}_k \right\rangle \\
&= \left\langle \sum_{k=1}^{N} (x_{n,k} - x_k)\vec{b}_k, \sum_{k=1}^{N} z_k \vec{b}_k \right\rangle \\
&= \sum_{k=1}^{N} (x_{n,k} - x_k)\overline{z_k} \\
&\to 0
\end{aligned}$$

But this is for any $\vec{z} \in E$, and so it must be the case that $x_{n,k} - x_n \to 0$.

So $\vec{x}_n$ converges to $\vec{x}$ component-wise.
It has already been shown that $\|\cdot\|_\infty$ with respect to a particular basis is a proper norm.
Now, since all norms are equivalent in a finite dimensional vector space:

$$\|\vec{x}_n - \vec{x}\|_\infty = \sup_{k \in \mathbb{N}} |x_{n,k} - x_k| \to 0$$

$\therefore \vec{x}_n \to \vec{x}$

Theorem

Let E be an inner product space and let $S \subseteq E$ such that $\operatorname{span}(S)$ is dense in E. Let $(\vec{x}_n)$ be a bounded sequence in E.

$$\vec{x}_n \xrightarrow{w} \vec{x} \iff \forall\, \vec{y} \in S, \langle \vec{x}_n, \vec{y} \rangle \to \langle \vec{x}, \vec{y} \rangle$$

Proof

$\implies$ Assume $\vec{x}_n \xrightarrow{w} \vec{x}$.

$\forall\, \vec{y} \in E, \langle \vec{x}_n, \vec{y} \rangle \to \langle \vec{x}, \vec{y} \rangle$
But $X \subseteq E$.

$\therefore \forall\, \vec{y} \in S, \langle \vec{x}_n, \vec{y} \rangle \to \langle \vec{x}, \vec{y} \rangle$

$\impliedby$ Assume $\forall\, \vec{y} \in S, \langle \vec{x}_n, \vec{y} \rangle \to \langle \vec{x}, \vec{y} \rangle$

Assume $\epsilon > 0$.
Assume $\vec{z} \in E$.
Since $(\vec{x}_n)$ is bounded, $\exists\, M > 0$ such that $\|\vec{x}_n\|, \|\vec{x}\| \le M$.
Since $\operatorname{span}(S)$ is dense in E, $\exists\, \vec{y}_0 \in \operatorname{span}(S)$ such that:

$$\|\vec{z} - \vec{y}_0\| < \frac{\epsilon}{3M}$$

Furthermore, by assumption, $\exists\, N > 0$ such that $\forall\, n > N$:

$$|\langle \vec{x}_n, \vec{y}_0 \rangle - \langle \vec{x}, \vec{y}_0 \rangle| < \frac{\epsilon}{3}$$

Assume $n > N$:

$$\begin{aligned}
|\langle \vec{x}_n, \vec{z} \rangle - \langle \vec{x}, \vec{z} \rangle| &= |(\langle \vec{x}_n, \vec{z} \rangle - \langle \vec{x}_n, \vec{y}_0 \rangle) + (\langle \vec{x}_n, \vec{y}_0 \rangle - \langle \vec{x}, \vec{y}_0 \rangle) + (\langle \vec{x}, \vec{y}_0 \rangle - \langle \vec{x}, \vec{z} \rangle)| \\
&\leq |\langle \vec{x}_n, \vec{z} \rangle - \langle \vec{x}_n, \vec{y}_0 \rangle| + |\langle \vec{x}_n, \vec{y}_0 \rangle - \langle \vec{x}, \vec{y}_0 \rangle| + |\langle \vec{x}, \vec{y}_0 \rangle - \langle \vec{x}, \vec{z} \rangle| \\
&\leq |\langle \vec{x}_n, \vec{z} - \vec{y}_0 \rangle| + \frac{\epsilon}{3} + |\langle \vec{x}, \vec{y}_0 - \vec{z} \rangle| \\
&\leq \|\vec{x}_n\| \, \|\vec{z} - \vec{y}_0\| + \frac{\epsilon}{3} + \|\vec{x}\| \, \|\vec{y}_0 - \vec{z}\| \\
&\leq M \frac{\epsilon}{3M} + \frac{\epsilon}{3} + M \frac{\epsilon}{3M} \\
&= \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

$\therefore \vec{x}_n \xrightarrow{w} \vec{x}$

<u>Theorem</u>

Let H be a Hilbert space and let $(\vec{x})$ be a sequence in H:

$\vec{x}_n \xrightarrow{w} \vec{x} \implies (\vec{x}_n)$ is bounded.

<u>Proof</u>

Assume $\vec{x}_n \xrightarrow{w} \vec{x}$.
Define $f_n : H \to \mathbb{C}$ by $f_n(\vec{y}) = \langle \vec{y}, \vec{x}_n \rangle, \forall\, n \in \mathbb{N}$.
f_n is clearly linear (due to sesquilinearity of the inner product).

Assume $\vec{y}_n \to \vec{y}$, and so $\|\vec{y}_n - \vec{y}\| \to 0$
Since strong implies weak, $\vec{y}_n \xrightarrow{w} \vec{y}$.
So $\langle \vec{y}_n, \vec{x}_n \rangle \to \langle \vec{y}, \vec{x}_n \rangle$.
$\|f_n(\vec{y}_n) - f_n(\vec{y})\| = |\langle \vec{y}_n, \vec{x}_n \rangle - \langle \vec{y}, \vec{x}_n \rangle| \to 0$
Therefore $f_n(y)$ is continuous, and thus bounded.

Now, $\forall\, \vec{y} \in H$ the sequence $(f_n(\vec{y})) = (\langle \vec{y}, \vec{x}_n \rangle)$ converges to $\langle \vec{y}, \vec{x} \rangle$, and so it is bounded:

$$\forall\, \vec{y} \in H, \exists\, M_y > 0, |f_n(\vec{y})| = \langle \vec{y}, \vec{x}_n \rangle < M_y$$

And so, by the principle of uniform boundedness:

$$\exists\, M > 0, \forall\, n \in \mathbb{N}, \|f_n\| \leq M$$

Finally:

$$\|f_n(\vec{x}_n)\| = |\langle \vec{x}_n, \vec{x}_n \rangle| = \|\vec{x}_n\|^2 \leq \|f_n\| \, \|\vec{x}_n\|$$

Therefore $\|\vec{x}_n\| \leq \|f_n\| \leq M$ and thus $(\vec{x}_n)$ is bounded.