Hermitian Matrices

Definition: Hermitian

To say that a matrix $A \in M_n$ is *Hermitian (self-adjoint)* means:

$$A^* = A$$

To say that A is *skew-Hermitian* means:

$$A^* = -A$$

Properties: Hermitian

1). A, B Hermitian $\implies \forall\, a, b \in \mathbb{R}, aA + bB$ Hermitian

2). A, B skew-Hermitian $\implies \forall\, a, b \in \mathbb{R}, aA + bB$ Hermitian

3). A Hermitian $\implies iA$ skew-Hermitian

4). A skew-Hermitian $\implies iA$ Hermitian

5). $A = [a_{ij}]$ Hermitian $\implies a_{ii} \in \mathbb{R}$

6). $A = [a_{ij}]$ skew-Hermitian $\implies a_{ii} \in i\mathbb{R}$

Theorem

Let $A, B \in M_n$ be Hermitian:

$$AB \text{ Hermitian} \iff AB = BA$$

Proof

$A = A^*$ and $B = B^*$
AB Hermitian $\iff AB = (AB)^* \iff AB = B^*A^* \iff AB = BA$

Theorem

$A \in M_n$ Hermitian $\iff \forall\, \vec{x} \in \mathbb{C}^n, \vec{x}^*A\vec{x} \in \mathbb{R}$

Proof

$\implies$ Assume $A \in M_n$ is Hermitian

Note that $\vec{x}^*A\vec{x} \in M_1$:

$$\overline{\vec{x}^*A\vec{x}} = (\vec{x}^*A\vec{x})^* = \vec{x}^*A^*\vec{x} = \vec{x}^*A\vec{x}$$

$\therefore \vec{x}^*A\vec{x} \in \mathbb{R}$

$\impliedby$ Assume $\forall\, \vec{x} \in \mathbb{C}^n, \vec{x}^*A\vec{x} \in \mathbb{R}$

Let $A = [a_{ij}]$ and $A^* = [\overline{a_{ji}}]$

Case 1: $i = j$

Let $\vec{x} = \vec{e}_i$
$e_i^* A e_i = a_{ii} \in \mathbb{R}$

$\therefore a_{ii} = \overline{a_{ii}}$

Case 2: $i \neq j$

Let $\vec{x} = \vec{e}_i + \vec{e}_j$
$(\vec{e}_i + \vec{e}_j)^* A(\vec{e}_i + \vec{e}_j) = \vec{e}_i^* A\vec{e}_i + \vec{e}_i^* A\vec{e}_j + \vec{e}_j^* A\vec{e}_i + \vec{e}_j^* A\vec{e}_j = a_{ii} + a_{ij} + a_{ji} + a_{jj} \in \mathbb{R}$
But $a_{ii}, a_{jj} \in \mathbb{R}$, so $a_{ij} + a_{ji} \in \mathbb{R}$

$\therefore \operatorname{Im}(a_{ij}) = -\operatorname{Im}(a_{ji})$

Now let $\vec{x} = i\vec{e}_i + \vec{e}_j$ $(i\vec{e}_i + \vec{e}_j)^* A(i\vec{e}_i + \vec{e}_j) = \vec{e}_i^* A\vec{e}_i - i\vec{e}_i^* A\vec{e}_j + i\vec{e}_j^* A\vec{e}_i + \vec{e}_j^* A\vec{e}_j = a_{ii} + i(-a_{ij} + a_{ji}) + a_{jj} \in \mathbb{R}$
But $a_{ii}, a_{jj} \in \mathbb{R}$, so $i(-a_{ij} + a_{ji}) \in \mathbb{R}$
$-a_{ij} + a_{ji} \in i\mathbb{R}$

$\therefore \operatorname{Re}(a_{ij}) = \operatorname{Re}(a_{ji})$

Thus, $a_{ij} = \overline{a_{ji}}$ and so $A = A^*$

Therefore, A is Hermitian.

Theorem

Let $A \in M_n$:

- A Hermitian $\implies \sigma(A) \subseteq \mathbb{R}$
- A skew-Hermitian $\implies \sigma(A) \subseteq i\mathbb{R}$

Proof

Assume A is Hermitian
Assume $\lambda \in \sigma(A)$
$\exists\, \vec{x} \neq \vec{x}, A\vec{x} = \lambda\vec{x}$
$\lambda = \vec{x}^* A\vec{x} \in \mathbb{R}$

$\therefore \sigma(A) \subseteq \mathbb{R}$

Assume A is skew-Hermitian
iA is Hermitian
$\sigma(iA) \subseteq \mathbb{R}$
$i\sigma(A) \subseteq \mathbb{R}$
$-\sigma(A) \subseteq i\mathbb{R}$

$\therefore \sigma(A) \subseteq i\mathbb{R}$

Theorem: Spectral Theorem for Hermitian Matrices

Let $A \in M_n$ be Hermitian. A is unitary diagonalizable with diagonal entries $\{\lambda_1, \dots \lambda_n\} = \operatorname{Sp}(A) \subseteq \mathbb{R}$.

Proof

A is Hermitian $\implies$ A is normal $\implies$ A is unitary diagonalizable with $\sigma(A) \subseteq \mathbb{R}$

Indeed, since $A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*$

$AU = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*$

$A\vec{u}_i = \lambda_i \vec{u}_i$, so the $\vec{u}_i$ are the eigenvectors of A.