Cavallaro, Jeffery
Math 275A
Homework #9

Theorem: 6.18

Let $X \times Y$ be a product space with Y compact. If $U \in \mathscr{T}_{X\times Y}$ and $\{x_0\} \times Y \subset U$ then there exists some $W \in \mathscr{T}_X$ such that $x_0 \in W$ and $W \times Y \subset U$.

Proof. Assume $U \in \mathscr{T}_{X\times Y}$ and $\{x_0\}\times Y \subset U$. Note that for each $y \in Y$, $(x_0, y) \in \{x_0\}\times Y \subset U$. Thus there exists sets $U_y \in \mathscr{T}_X$ and $V_y \in \mathscr{T}_Y$ such that $\{x_0\}\times Y \subset \bigcup\limits_{y\in Y}(U_y \times V_y) \subset U$, where the V_y are an open cover of Y. But Y is compact, so there exists some finite subcover $\{V_{y_1}, \ldots, V_{y_n}\}$ of Y. So select up to n open subsets of U_y and let $W = \bigcap\limits_{k=1}^{n} U_{y_k}$. Note that $W \in \mathscr{T}_X$ because it is a finite intersection of open sets.

Claim: $W \times Y \subset U$

Assume that $(x, y) \in W \times Y$. This means that for some k, $x \in U_{y_k}$ and $y \in V_{y_k}$. And so $(x, y) \in U_{y_k} \times V_{y_k} \subset U$. ∎