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Math 231b
Homework #6

4.12.20

Let $(\vec{e}_n)$ be a complete orthonormal sequence in a Hilbert space H. Show that a bounded operator A on H is unitary if and only if $(A\vec{e}_n)$ is a complete orthonormal sequence in H.

$\Longrightarrow$ Assume A is unitary.

Since $A \in \mathcal{B}(H)$ it is also the case that $A^* \in \mathcal{B}(H)$.

A is isometric and thus preserves the norm and inner product.

$\langle \vec{e}_i, \vec{e}_j \rangle = \langle A\vec{e}_i, A\vec{e}_j \rangle$
But $\vec{e}_i \perp \vec{e}_j$ and so $\langle \vec{e}_i, \vec{e}_j \rangle = \langle A\vec{e}_i, A\vec{e}_j \rangle = 0$
$\therefore A\vec{e}_i \perp A\vec{e}_j$

Also $\|A\vec{x}\| = \|\vec{x}\| = 1$.

Therefore $(A\vec{e}_n)$ is an orthonormal sequence.

Now, assume $\vec{y} \in H$.
Since A is onto, $\exists\, \vec{x} \in H$ such that $\vec{y} = A\vec{x}$.
Since $(\vec{e}_n)$ is complete orthonormal and since A is linear and isometric:

$$\vec{y} = A\vec{x} = A\sum_{k=1}^{\infty} \langle \vec{x}, \vec{e}_k \rangle\, \vec{e}_k = \sum_{k=1}^{\infty} \langle A\vec{x}, A\vec{e}_k \rangle\, A\vec{e}_k$$

Therefore $(A\vec{e}_n)$ is complete.

$\Longleftarrow$ Assume $(A\vec{e}_n)$ is a complete orthonormal sequence.

Assume $\vec{x} \in H$.

$$\vec{x} = \sum_{k=1}^{\infty} \langle \vec{x}, A\vec{e}_k \rangle\, A\vec{e}_k = A\sum_{k=1}^{\infty} \langle A^*\vec{x}, \vec{e}_k \rangle\, \vec{e}_k$$

But $(\vec{e}_n)$ is complete orthonormal, and so:

$\vec{x} = AA^*\vec{x}$ for all $\vec{x} \in H$ and thus $AA^* = I$

Now, note that:

$$A^*A\vec{x} = \sum_{k=1}^{\infty} \langle A^*A\vec{x}, \vec{e}_k \rangle\, \vec{e}_k = \sum_{k=1}^{\infty} \langle A\vec{x}, A\vec{e}_k \rangle\, \vec{e}_k$$

Let $\vec{x} = \vec{e}_j$:

$$A^*A\vec{e}_j = \sum_{k=1}^{\infty} \langle A\vec{e}_j, A\vec{e}_k \rangle\, \vec{e}_k = \|A\vec{e}_j\|^2\, \vec{e}_j = 1 \cdot \vec{e}_j = \vec{e}_j$$

Thus $(A^*A\vec{e}_n)$ is also a complete orthonormal sequence.

Assume $\vec{x} \in H$:

$$\vec{x} = \sum_{k=1}^{\infty} \langle \vec{x}, A^*A\vec{e}_k \rangle A^*A\vec{e}_k = A^* \sum_{k=1}^{\infty} \langle A\vec{x}, A\vec{e}_k \rangle A\vec{e}_k = A^*A\vec{x}$$

And so $\vec{x} = A^*A\vec{x}$ for all $\vec{x} \in H$ and thus $A^*A = I$

Therefore $AA^* = A^*A = I$ and thus A is unitary.

4.12.23

Let A be a bounded operator on a Hilbert space. Define the exponential operator:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

where $A^0 = I$.

Show that e^A is a well-defined operator.

$$\sum_{k=1}^{\infty} \left\| \frac{A^n}{n!} \right\| = \sum_{k=1}^{\infty} \frac{\|A^n\|}{n!} \leq \sum_{k=1}^{\infty} \frac{\|A\|^n}{n!}$$

which converges to $e^{\|A\|}$ for all $\|A\| \in \mathbb{R}$.

So e^A converges absolutely. But H is complete, so e^A converges.

Therefore e^A is well-defined.

Prove the following:

(a) $(e^A)^n = e^{nA}$

Proof by induction on n:

Base case: $n = 1$

Trivial.

Assume $(e^A)^n = e^{nA}$

Consider $(e^A)^{n+1}$.

$$(e^A)^{n+1} = \left[\sum_{k=0}^{\infty} \frac{A^k}{k!}\right]^{n+1} = \left[\sum_{k=0}^{\infty} \frac{A^k}{k!}\right]^{n} \left[\sum_{k=0}^{\infty} \frac{A^k}{k!}\right] = (e^A)^n e^A = e^{nA}e^A$$

Note that nA and A commute, so applying part (d):

$$(e^A)^{n+1} = e^{nA}e^A = e^{nA+A} = e^{(n+1)A}$$

(b) $e^0 = I$

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \frac{A^0}{0!} + \sum_{n=1}^{\infty} \frac{A^n}{n!} = I + \sum_{n=1}^{\infty} \frac{A^n}{n!}$$

Now let $A = 0$:

$$e^0 = I + \sum_{n=1}^{\infty} \frac{0^n}{n!} = I + 0 = I$$

(c) e^A is invertible (even if A is not) and its inverse is e^{-A}.

Note that A and $-A$ commute, so applying part (d):

$e^A e^{-A} = e^{A-A} = e^0 = I$

$e^{-A} e^A = e^{-A+A} = e^0 = I$

Therefore e^A is invertible with inverse e^{-A}.

(d) $e^A e^B = e^{A+B}$ for any commuting operators A and B.

$$\begin{aligned}
e^{A+B} &= \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A^k B^{n-k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{A^k}{k!} \frac{B^{n-k}}{(n-k)!} \\
&= \left[\sum_{n=0}^{\infty} \frac{A^k}{k!} \right] \left[\sum_{n=0}^{\infty} \frac{B^k}{k!} \right] \\
&= e^A e^B
\end{aligned}$$

(by the Cauchy product of two infinite, absolutely converging series).

(e) If A is self-adjoint then e^{iA} is unitary.

Assume A is self-adjoint.
$A = A^*$

$$\begin{aligned}
(e^{iA})^* &= \left[\sum_{n=1}^{\infty} \frac{(iA)^n}{n!}\right]^* \\
&= \lim_{N\to\infty} \left[\sum_{n=1}^{N} \frac{(iA)^n}{n!}\right]^* \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{[(iA)^n]^*}{n!} \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{[(iA)^*]^n}{n!} \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{[-iA^*]^n}{n!} \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{[-iA]^n}{n!} \\
&= e^{-iA}
\end{aligned}$$

Therefore, by part (c), e^{iA} is unitary.

4.12.28

If $T^*T = I$, is it true that $TT^* = I$?

No. Let $H = \ell^2$ and let T be the right-shift operator:

$$T(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, z_3, \ldots)$$

Claim: $\forall\, z \in \ell^2, Tz \in \ell^2$

Assume $z \in \ell^2$.

$$\sum_{n=1}^{\infty} |(Tz)_n|^2 = 0 + \sum_{n=1}^{\infty} |z_n|^2 = \sum_{n=1}^{\infty} |z_n|^2 < \infty$$

Claim: T is linear.

Assume $x, y \in \ell^2$ and $\alpha, \beta \in \mathbb{C}$.

$$\begin{aligned}
T(\alpha x + \beta y) &= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3, \ldots) \\
&= (0, \alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3, \ldots) \\
&= \alpha(0, x_1, x_2, x_3, \ldots) + \beta(0, y_1, y_2, y_3, \ldots) \\
&= \alpha Tx + \beta Ty
\end{aligned}$$

Claim: T is bounded.

$$\|Tz\|^2 = \sum_{n=1}^{\infty} |(Tz)_n|^2 = 0 + \sum_{n=1}^{\infty} |z_n|^2 = \|z\|^2$$

Therefore $\|T\| \leq 1$ and thus T is bounded.

$\therefore T \in \mathcal{B}(\ell^2)$

Now, assume $T^*T = I$.
Thus, T^* is a left-inverse of T and must therefore be the left-shift operator:

$$T^*(z_1, z_2, z_3, z_4 \ldots) = (z_2, z_3, z_4, \ldots)$$

And so:

$$T^*T(z_1, z_2, z_3, \ldots) = T^*(0, z_1, z_2, z_3, \ldots) = (z_1, z_2, z_3, \ldots)$$

However:

$$TT^*(z_1, z_2, z_3, \ldots) = T(z_2, z_3, \ldots) = (0, z_2, z_3, \ldots) \neq (z_1, z_2, z_3, \ldots)$$

And therefore $TT^* \neq I$.

4.12.31

If A and B are positive operators and $A + B = 0$, show that $A = B = 0$.

Assume A and B are positive operators and $A + B = 0$.

$\langle (A+B)\vec{x}, \vec{x} \rangle = \langle A\vec{x} + B\vec{x}, \vec{x} \rangle = \langle A\vec{x}, \vec{x} \rangle + \langle B\vec{x}, \vec{x} \rangle = 0$

But A and B are positive, so $\langle A\vec{x}, \vec{x} \rangle \geq 0$ and $\langle B\vec{x}, \vec{x} \rangle \geq 0$.

And so $\langle A\vec{x}, \vec{x} \rangle = \langle B\vec{x}, \vec{x} \rangle = 0$, for all $\vec{x}$.

$\therefore A = B = 0$

4.12.54

Give an example of a self-adjoint operator that has no eigenvalues.

Let $H = L^2[a, b]$ and let $f_0 \in H$ be a real-valued, continuous, non-constant (and hence non-zero), and bounded function. For example: $f_0(x) = 2 + \sin x$.

Define $Tf = f_0 f$.

Claim: T is linear.

$T(\alpha f + \beta g) = f_0(\alpha f + \beta g) = \alpha f_0 f + \beta f_0 g = \alpha Tf + \beta Tg$

Claim: T is bounded.

$\|Tf\| = \|f_0 f\| \leq \|f_0\| \, \|f\|$

Therefore $\|T\| \leq \|f_0\|$ and thus T is bounded.

$\therefore T \in \mathcal{B}(H)$

Claim: T is self-adjoint.

$$\begin{aligned}
\langle Tf, g \rangle &= \int_a^b ((Tf)(x))\overline{g(x)}dx \\
&= \int_a^b (f_0 f)(x)\overline{g(x)}dx \\
&= \int_a^b f_0(x)f(x)\overline{g(x)}dx \\
&= \int_a^b f(x)\overline{f_0(x)g(x)}dx \\
&= \int_a^b f(x)\overline{(Tg)(x)}dx \\
&= \langle f, Tg \rangle
\end{aligned}$$

Claim: T has no eigenvalues.

If it did, then $(Tf)(x) = (f_0 f)x = f_0(x)f(x) = \lambda f(x)$ for all $x \in [a, b]$. But this is only true for $f(x) \equiv 0$.

Therefore T has no eigenvalues.