A COMPLETE ORTHONORMAL SEQUENCE FOR $L^2[-\pi,\pi]$

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Abstract. The goal of this paper is to prove that the sequence (φ_n), where $\varphi_n(x) = \dfrac{1}{\sqrt{2\pi}}e^{inx}$, is a bi-infinite complete orthonormal sequence in $L^2[-\pi,\pi]$. The fact that (φ_n) is orthonormal is fairly straightforward and is proved in the normal way using inner products. The fact that (φ_n) is complete is a bit more complicated. The method used here is to first use Hölder's inequality to show that if $f \in L^2[-\pi,\pi]$ then $f \in L^1[-\pi,\pi]$. Next, the Fejér summability kernel is used to show that for all $f \in L^1[-\pi,\pi]$, if $\hat{f}(n) = \langle f, \varphi_n \rangle_{L_2} = 0$ for all $n \in \mathbb{Z}$ then $f \equiv 0$ a.e., which is sufficient to conclude that (φ_n) is complete in $L^2[-\pi,\pi]$.

1. Introduction

The rise of the relatively new fields of electrical engineering and signal analysis during the late 19^{th} and early 20^{th} centuries motivated the mathematicians of the time to develop new theoretical frameworks for the growing fields. One such leader was the Jewish-Hungarian mathematician Lipót Fejér (1880–1959). Fejér made major contributions to the theory of harmonic analysis, as well as being advisor and mentor to many future giants in the field, including: John Von Neumann, Paul Erdös, and George Pólya.

Since many physical phenomena can be expressed as harmonic functions in $L^2[-\pi,\pi]$, it is important to have a theoretically sound orthonormal basis for the space. The existence of such a basis facilitates the solutions to both theoretical and practical problems, especially in the areas of waveform analysis. A candidate for such a basis is the Dirichlet sequence: (φ_n), where $\varphi_n(t) = \dfrac{1}{\sqrt{2\pi}}e^{int}$. It is fairly easy to prove the orthonormality of this sequence; however, its completeness is a bit more complicated and requires the use of so-called summability kernels. The summability kernel that will be used in this proof is attributed to Fejér . Indeed, the use of summability kernels is an important tool in analysis.

2. Orthonormality

The first order of business is to establish the orthonormal nature of the sequence (φ_n) in $L^2[-\pi,\pi]$, where $\varphi_n(x) = \dfrac{1}{\sqrt{2\pi}}e^{inx}$.

The following lemma is helpful both here and later:

Lemma 2.1. $\displaystyle\int_{-\pi}^{\pi} e^{inx}dx = \begin{cases} 2\pi, & n = 0 \\ 0, & n \in \mathbb{Z} - \{0\} \end{cases}$

Proof. Assume $n \in \mathbb{Z}$.

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Case 1: $n = 0$

$$\int_{-\pi}^{\pi} e^{i0x} dx \quad = \quad \int_{-\pi}^{\pi} dx \quad = \quad 2\pi$$

Case 2: $n \neq 0$

$$\begin{aligned} \int_{-\pi}^{\pi} e^{inx} dx &= \frac{1}{in} e^{inx} \Big|_{-\pi}^{\pi} \\ &= \frac{1}{in}(e^{in\pi} - e^{-in\pi}) \\ &= \frac{2}{n}\left(\frac{e^{in\pi} - e^{-in\pi}}{2i}\right) \\ &= \frac{2}{n}\sin(n\pi) \\ &= 0 \end{aligned}$$

□

The following corollary follows directly from the above lemma:

Corollary 2.2. $\int_{-\pi}^{\pi} \varphi_n(x) dx = \begin{cases} \sqrt{2\pi}, & n = 0 \\ 0, & n \in \mathbb{Z} - \{0\} \end{cases}$

Proof. Assume $n \in \mathbb{Z}$.

Case 1: $n = 0$

$$\int_{-\pi}^{\pi} \varphi_0(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{i0x} dx = \frac{1}{\sqrt{2\pi}} 2\pi = \sqrt{2\pi}$$

Case 2: $n \neq 0$

$$\int_{-\pi}^{\pi} \varphi_n(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{inx} dx = \frac{1}{\sqrt{2\pi}} \cdot 0 = 0$$

□

And so orthonormality in $L^2[-\pi, \pi]$ follows easily:

Theorem 2.3. *(φ_n) is an orthonormal sequence in $L^2[-\pi, \pi]$.*

Proof. Assume $n, m \in \mathbb{Z}$

Case 1: $m = n$

$$\begin{aligned}
\langle \varphi_n, \varphi_n \rangle &= \int_{-\pi}^{\pi} \varphi_n(x)\overline{\varphi_n(x)}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\overline{e^{inx}}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} dx \\
&= \frac{1}{2\pi}2\pi \\
&= 1
\end{aligned}$$

Case 2: $m \neq n$

$$\begin{aligned}
\langle \varphi_m, \varphi_n \rangle &= \int_{-\pi}^{\pi} \varphi_m(x)\overline{\varphi_n(x)}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx}\overline{e^{inx}}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx}e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x}dx \\
&= \frac{1}{2\pi}\cdot 0 \\
&= 0
\end{aligned}$$

□

3. Convolution

The proof will require the services of the *convolution* binary operator on $L^1[-\pi,\pi]$. In order to get a feel for this operator, it is helpful to observe its behavior in $L^1(\mathbb{R})$:

Definition 3.1 (Convolution)**.** *Let $f, g \in L^1(\mathbb{R})$. The convolution of f and g, denoted $f \star g$, is given by:*

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

The first thing to note about this definition is that since $f \in L^1(\mathbb{R})$, reflecting and translating f does not affect its integrability over $\mathbb{R}$, and thus $f \star g \in L^1(\mathbb{R})$ as well.

Next, consider what is happening from a function transformation standpoint:

(1) Express f and g in terms of a dummy variable t.
(2) Reflect $f(t)$ to get $f(-t)$.
(3) As x varies from $-\infty$ to ∞, the reflected $f(t)$ sweeps across $g(t)$.

Therefore, for a given x, $(f \star g)(x)$ provides a weighted summation of the intersection of $f(x-t)$ and $g(t)$.

For example, let $f(x) = g(x)$ be the unit rectangular pulse:

$$f(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & otherwise \end{cases}$$

Note that for $x \notin (0,2)$ there is no overlap and thus $(f \star f)(x) = 0$ (see Figure 1).

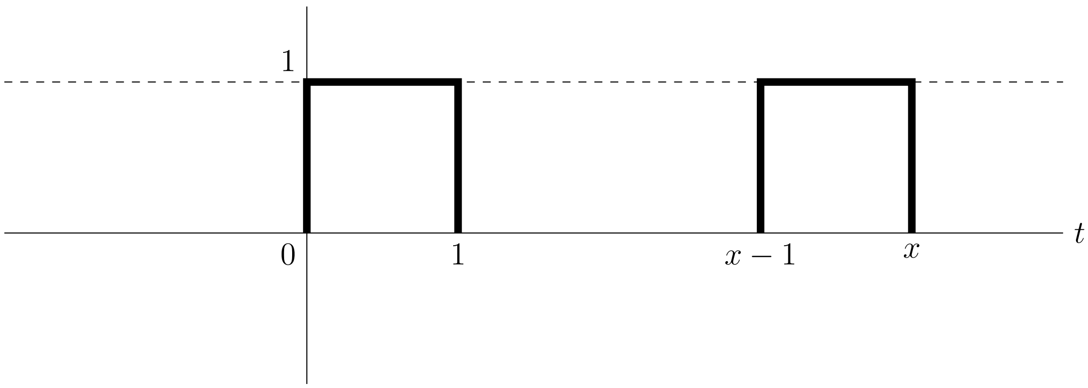

FIGURE 1. Convolution with No Overlap

However, for $x \in (0,2)$ there is indeed overlap (see Figure 2).

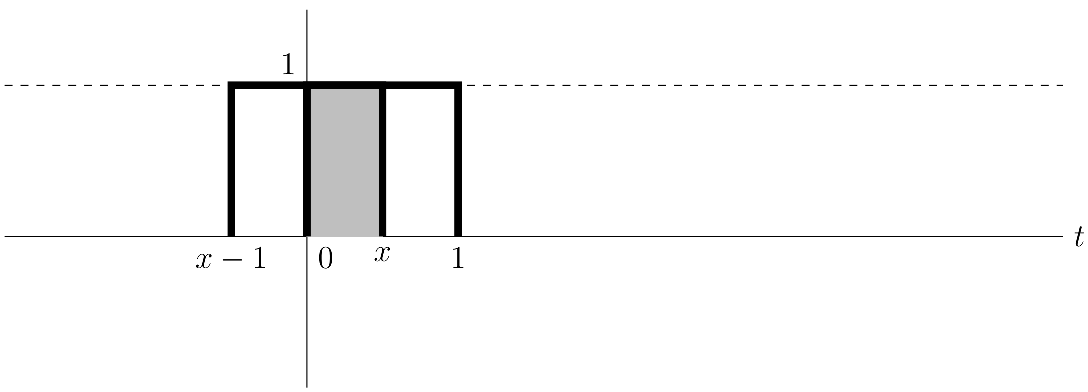

FIGURE 2. Convolution with Overlap

Indeed, for this example, the area of overlap increases linearly for $x \in [0,1]$, reaching its peak at $x = 1$, and then decreases linearly for $x \in [1,2]$. Therefore, the result is a triangular pulse of width 2 (see Figure 3).

Now, turning specifically to the interval $[-\pi, \pi]$, there is a one-to-one correspondence between functions in $L^1[-\pi,\pi]$ and 2π-periodic functions in $L^1(\mathbb{R})$. Thus, it will be convenient to use a slightly modified version of the convolution operator:

Definition 3.2 (Convolution on the Circle)**.** *Let $f, g \in L^1(\mathbb{R})$ be 2π-periodic. Convolution on the circle is given by:*

$$(f \star g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)g(t)dt$$

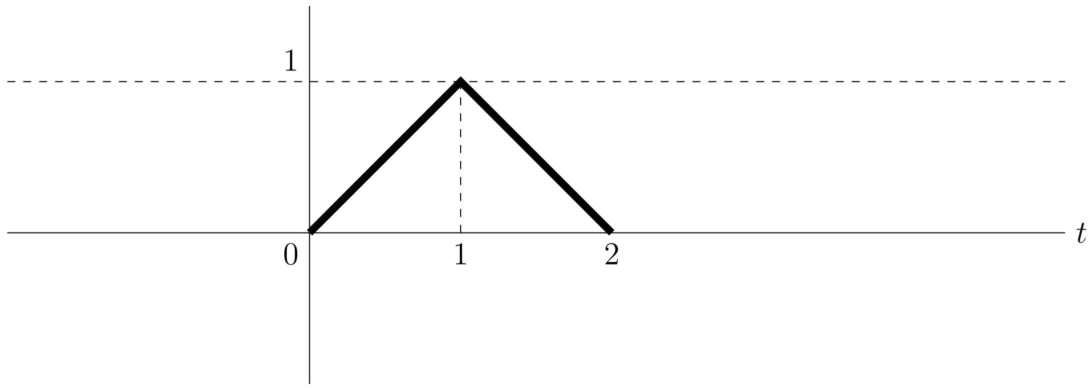

FIGURE 3. Convolution of Two Unit Rectangular Pulses

Note that this adjusted definition provides an average weighted value over the circle. Regardless of which definition is used, convolution is commutative.

Theorem 3.3. *Let $f, g \in L^1(\mathbb{R})$. Then:*

$$(f \star g)(x) = (g \star f)(x)$$

Proof. Using the substitution $u = x - t$:

$$\begin{aligned}
(f \star g)(x) &= \int_{-\infty}^{\infty} f(x-t)g(t)dt \\
&= \int_{\infty}^{-\infty} f(u)g(x-u)(-du) \\
&= \int_{-\infty}^{\infty} f(u)g(x-u)du \\
&= \int_{-\infty}^{\infty} g(x-u)f(u)du \\
&= (g \star f)(x)
\end{aligned}$$

∎

Theorem 3.4. *Let $f, g \in L^1(\mathbb{R})$ such that f and g are 2π-periodic. Then:*

$$(f \star g)(x) = (g \star f)(x)$$

where convolution is on the circle.

Proof. Using the substitution $u = x - t$:

$$\begin{aligned}(f \star g)(x) &= \int_{-\pi}^{\pi} f(x-t)g(t)dt \\ &= \int_{x+\pi}^{x-\pi} f(u)g(x-u)(-du) \\ &= \int_{x-\pi}^{x+\pi} f(u)g(x-u)du \\ &= \int_{-\pi}^{\pi} g(x-u)f(u)du \\ &= (g \star f)(x)\end{aligned}$$

□

Of course, commutativity makes sense because the "sweeping" of either function is relative.

4. Summability Kernel

The proof uses the notion of a *summability* kernel to help show convergence in a norm.

Definition 4.1 (Summability Kernel)**.** *To say that a sequence (κ_n) of 2π-periodic continuous fuctions is a summability kernel means that κ_n satisfies the following properties:*

(1) $\displaystyle\int_{-\pi}^{\pi} \kappa_n(t)dt = 2\pi$

(2) $\displaystyle\int_{-\pi}^{\pi} |\kappa_n(t)|\, dt \le M$ *for some* $M > 0$ *and all* $n \in \mathbb{N}$

(3) $\displaystyle\int_{\delta \le |t| \le \pi} |\kappa_n(t)|\, dt \to 0$ *for all* $\delta \in (0, \pi)$

Note that the third property indicates that given a $\delta > 0$, for all $\epsilon > 0$ there exists an n sufficiently large such:

$$2\pi(1-\epsilon) < \int_{-\delta}^{\delta} \kappa_n(t)dt \le 2\pi$$

The importance of summability kernels and convolution is embodied by the following key theorem:

Theorem 4.2. *Let (κ_n) be a summability kernel and let $f \in L^1[-\pi,\pi]$:*

$$\|(k_n \star f) - f\|_1 \to 0$$

In other words, $k_n \star f$ converges to f in the $L^1[-\pi,\pi]$ norm.

Proof. From the first property:

$$\begin{aligned}
\int_{-\pi}^{\pi} \kappa_n(t)dt &= 2\pi \\
\frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_n(t)dt &= 1 \\
f(x)\frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_n(t)dt &= f(x) \\
\frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_n(t)f(x)dt &= f(x)
\end{aligned}$$

And so:

$$\begin{aligned}
(\kappa_n \star f)(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_n(t)f(x-t)dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_n(t)f(x)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_n(t)[f(x-t)-f(x)]dt
\end{aligned}$$

Now construct the $L^1[-\pi,\pi]$ norm:

$$\int_{-\pi}^{\pi} |(\kappa_n \star f)(x) - f(x)|\, dx = \int_{-\pi}^{\pi} \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_n(t)[f(x-t)-f(x)]dt\right| dx$$

Now, assume $\delta \in (0,\pi)$:

$$\begin{aligned}
\|(\kappa_n \star f) - f\|_1 &= \int_{-\pi}^{\pi} \left|\frac{1}{2\pi}\left\{\int_{-\delta}^{\delta} \kappa_n(t)[f(x-t)-f(x)]dt + \int_{\delta\le|t|\le\pi} \kappa_n(t)[f(x-t)-f(x)]dt\right\}\right| dx \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\{\left|\int_{-\delta}^{\delta} \kappa_n(t)[f(x-t)-f(x)]dt\right| + \left|\int_{\delta\le|t|\le\pi} \kappa_n(t)[f(x-t)-f(x)]dt\right|\right\} dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\int_{-\delta}^{\delta} \kappa_n(t)[f(x-t)-f(x)]dt\right| dx \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\int_{\delta\le|t|\le\pi} \kappa_n(t)[f(x-t)-f(x)]dt\right| dx \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\delta}^{\delta} |\kappa_n(t)[f(x-t)-f(x)]|\, dtdx \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\delta\le|t|\le\pi} |\kappa_n(t)[f(x-t)-f(x)]|\, dtdx
\end{aligned}$$

Assume $\epsilon > 0$.

Focusing on the first term in the last sum:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\delta}^{\delta}|\kappa_n(t)[f(x-t)-f(x)]|\,dtdx &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\delta}^{\delta}|\kappa_n(t)|\,|f(x-t)-f(x)|\,dtdx \\
&\leq \frac{1}{2\pi}\left[\max_{|t|\leq\delta}\int_{-\pi}^{\pi}|f(x-t)-f(x)|\,dx\right]\int_{-\delta}^{\delta}|\kappa_n(t)|\,dt \\
&\leq \frac{1}{2\pi}\left[\max_{|t|\leq\delta}\int_{-\pi}^{\pi}|f(x-t)-f(x)|\,dx\right]\int_{-\pi}^{\pi}|\kappa_n(t)|\,dt
\end{aligned}$$

As $t\to 0$, the translated function $f(x-t)\to f(x)$ and $\displaystyle\int_{-\pi}^{\pi}|f(x-t)-f(x)|\,dx\to 0$. Furthermore, $\displaystyle\int_{-\pi}^{\pi}|\kappa_n(t)|\,dt$ is bounded by the second property. Thus, it is possible to select δ small enough such that:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\delta}^{\delta}|\kappa_n(t)[f(x-t)-f(x)]|\,dtdx < \frac{\epsilon}{2}$$

for all $n\in\mathbb{N}$.

Focusing on the second term:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\delta\leq|t|\leq\pi}|\kappa_n(t)[f(x-t)-f(x)]|\,dtdx &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\delta\leq|t|\leq\pi}|\kappa_n(t)|\,|f(x-t)-f(x)|\,dtdx \\
&\leq \frac{1}{2\pi}\left[\max_{\delta\leq|t|\leq\pi}\int_{-\pi}^{\pi}|f(x-t)-f(x)|\,dx\right]\int_{\delta\leq|t|\leq\pi}|\kappa_n(t)|\,dt
\end{aligned}$$

But note that:

$$\begin{aligned}
\int_{-\pi}^{\pi}|f(x-t)-f(x)|\,dx &\leq \int_{-\pi}^{\pi}[|f(x-t)|+|f(x)|]dx \\
&= \int_{-\pi}^{\pi}|f(x-t)|\,dx+\int_{-\pi}^{\pi}|f(x)|\,dx \\
&= 2\int_{-\pi}^{\pi}|f(x)|\,dx
\end{aligned}$$

when f is viewed as 2π-periodic in $L^1(\mathbb{R})$, and so:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\delta\leq|t|\leq\pi}|\kappa_n(t)[f(x-t)-f(x)]|\,dtdx \;\leq\; \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|\,dx\int_{\delta\leq|t|\leq\pi}|\kappa_n(t)|\,dt$$

But $f\in L^1[-\pi,\pi]$ and so $\displaystyle\int_{-\pi}^{\pi}|f(x)|\,dx<\infty$. Furthermore, $\displaystyle\int_{\delta\leq|t|\leq\pi}|\kappa_n(t)|\,dt\to 0$, and so sufficiently small δ can be selected so that for sufficiently large n:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\delta\leq|t|\leq\pi}|\kappa_n(t)[f(x-t)-f(x)]|\,dtdx \;\leq\; \frac{\epsilon}{2}$$

Therefore $\|(\kappa_n\star f)-f\|<\epsilon$. $\square$

5. The Fejér Kernel

The summability kernel of particular importance to this proof is the Fejér kernel. The Fejér kernel is actually the Cesáro sum of the Dirichlet sequence.

Definition 5.1 (Dirichlet Sequence)**.** *The Dirichlet sequence (D_n) is given by:*

$$D_n(x) = \sum_{k=-n}^{n} e^{inx}$$

Definition 5.2 (Fejér Kernel)**.** *The Fejér kernel (F_N) is given by:*

$$F_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(x)$$

Depending on need, the Fejér kernel can be expressed in a couple of different forms:

Theorem 5.3. *The following are equivalent forms of the Fejér kernel:*

(1) $F_n(x) = \dfrac{1}{n+1} \displaystyle\sum_{j=0}^{n} \sum_{k=-j}^{j} e^{ikx}$

(2) $F_n(x) = \displaystyle\sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikx}$

(3) $F_n(x) = \left(\dfrac{1}{N+1}\right) \dfrac{\sin^2\left[(n+1)\frac{x}{2}\right]}{\sin^2\left(\frac{x}{2}\right)}$

Proof. The first form is just a restatement of the definition.

Starting with the first form, note that the double sum guarantees that each term will be present $(n+1) - |k|$ times, and so:

$$\frac{1}{n+1} \sum_{j=0}^{n} \sum_{k=-j}^{j} e^{ikx} = \frac{1}{n+1} \sum_{k=-n}^{n} [(n+1) - |k|] e^{ikx} = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikx}$$

Once again start with the first form and this time use geometric series:

$$
\begin{aligned}
\frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=-j}^{j} e^{ikx} &= \frac{1}{n+1}\sum_{j=0}^{n}\frac{e^{-ijx}-e^{i(j+1)x}}{1-e^{ix}} \\
&= \frac{1}{n+1}\left(\frac{1}{1-e^{ix}}\right)\left(\sum_{j=0}^{n} e^{-ijx}-\sum_{j=0}^{n} e^{i(j+1)x}\right) \\
&= \frac{1}{n+1}\left(\frac{1}{1-e^{ix}}\right)\left(\frac{1-e^{-i(n+1)x}}{1-e^{-ix}}-\frac{e^{ix}-e^{i(n+2)x}}{1-e^{ix}}\right) \\
&= \frac{1}{n+1}\left(\frac{1}{1-e^{ix}}\right)\left(\frac{1-e^{-i(n+1)x}}{1-e^{-ix}}-\frac{1-e^{i(n+1)x}}{e^{-ix}-1}\right) \\
&= \frac{1}{n+1}\left(\frac{1}{1-e^{ix}}\right)\left(\frac{1-e^{-i(n+1)x}}{1-e^{-ix}}+\frac{1-e^{i(n+1)x}}{1-e^{-ix}}\right) \\
&= \frac{1}{n+1}\left(\frac{1}{1-e^{ix}}\right)\left(\frac{-e^{i(n+1)x}+2-e^{-i(n+1)x}}{1-e^{-ix}}\right) \\
&= \frac{1}{n+1}\left(\frac{-e^{i(n+1)x}+2-e^{-i(n+1)x}}{-e^{ix}+2-e^{-ix}}\right) \\
&= \frac{1}{n+1}\left(\frac{e^{i(n+1)x}-2+e^{-i(n+1)x}}{e^{ix}-2+e^{-ix}}\right) \\
&= \frac{1}{n+1}\left[\frac{e^{i(n+1)\frac{x}{2}}-e^{-i(n+1)\frac{x}{2}}}{e^{i\frac{x}{2}}-e^{-i\frac{x}{2}}}\right]^2 \\
&= \left(\frac{1}{n+1}\right)\frac{\sin^2\left[(n+1)\frac{x}{2}\right]}{\sin^2\left(\frac{x}{2}\right)}
\end{aligned}
$$

□

Thus, $F_n(x)$ is a real-valued function. $F_0(x)$ through $F_5(x)$ are shown in Figure 4.

Theorem 5.4. *The Fejér kernel is a summability kernel.*

Proof. From Lemma 2.1, since $\int_{-\pi}^{\pi} e^{ikx}dx = 2\pi$ for $k=0$ and 0 otherwise:

$$\int_{-\pi}^{\pi} F_n(x)dx = \int_{-\pi}^{\pi}\sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)e^{ikx}dx = \sum_{k=-n}^{n}\int_{-\pi}^{\pi}\left(1-\frac{|k|}{n+1}\right)e^{ikx}dx = 2\pi$$

Furthermore, as evinced by the third form in Theorem 5.3, $F_n(x) \geq 0$ for all x and thus:

$$\int_{-\pi}^{\pi}|F_n(x)|\,dx = \int_{-\pi}^{\pi} F_n(x)dx = 2\pi$$

Therefore F_n is bounded for all $n \in \mathbb{N}$.

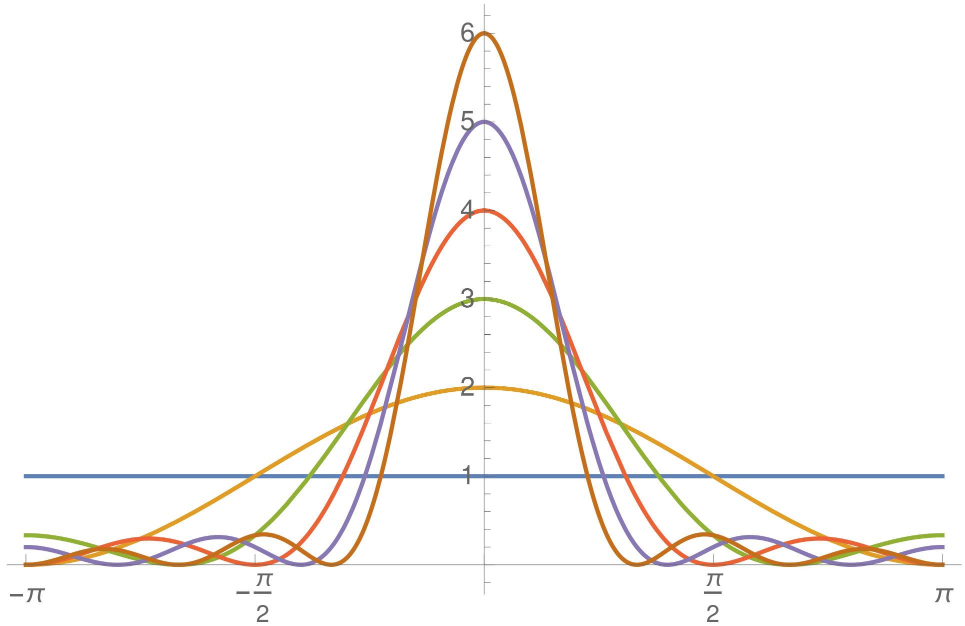

FIGURE 4. The First 6 Terms of the Fejér Kernel

Finally, for some $\delta \in (0,\pi)$:

$$\begin{aligned}
\int_{\delta\le|x|\le\pi} |F_n(x)|\,dx &= \int_{\delta\le|x|\le\pi} F_n(x)dx \\
&= \frac{1}{n+1}\int_{\delta\le|x|\le\pi} \frac{\sin[(n+1)\frac{x}{2}]}{\sin(\frac{x}{2})}dx \\
&\le \frac{1}{n+1}\int_{\delta\le|x|\le\pi} \frac{1}{\sin(\frac{\delta}{2})}dx \\
&\le \frac{1}{(n+1)\sin(\frac{\delta}{2})}\int_{-\pi}^{\pi} dx \\
&= \frac{2\pi}{(n+1)\sin(\frac{\delta}{2})} \\
&\to 0
\end{aligned}$$

$\square$

6. The Final Result

The needed pieces are now in place to prove that (φ_n) is complete in $L^2[-\pi,\pi]$. The final result is actually a corollary to the following theorem:

Theorem 6.1. *If $f \in L^1[-\pi,\pi]$ and $\langle f,\varphi_n\rangle_{L_2} = 0$ for all $n \in \mathbb{Z}$ then $f \equiv 0$ (a.e.).*

Proof. By assumption:

$$\langle f,\varphi_n\rangle = \int_{-\pi}^{\pi} f(t)e^{-int}dt = 0$$

Let:

$$f_n(x) = \sum_{k=-n}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{ik(x-t)} dt$$

Note that:

$$f_n(x) = \sum_{k=-n}^{n} \frac{e^{ikx}}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt = 0$$

Now, taking the Cesáro sum of the f_n:

$$\begin{aligned} \frac{1}{n+1} \sum_{k=0}^{n} f_n(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[\sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1} \right) e^{ik(x-t)} \right] dt \\ &= f \star F_n \\ &= F_n \star f \\ &\to f \end{aligned}$$

in the $L^1[-\pi, \pi]$ norm by Theorem 4.2. But the Cesáro sum of the f_n is also 0.

Therefore, $f \equiv 0$ (a.e.). □

And finally:

Corollary 6.2. *The sequence* (φ_n) *is complete in* $L^2[-\pi, \pi]$.

Proof. Assume $f \in L^2[-\pi, \pi]$. By Hölder's inequality:

$$\int |f| = \int (|f| \cdot 1) \leq \left(\int |f|^2 \right)^{\frac{1}{2}} \left(\int 1 \right)^{\frac{1}{2}} < \infty$$

And so $f \in L^1[-\pi, \pi]$ also. Thus, by Theorem 6.1, the statement:

$$\forall\, n \in \mathbb{Z}, \langle f, \varphi_n \rangle_{L_2} = 0 \implies f \equiv 0$$

is a true statement, which is a sufficient condition for concluding that (φ_n) is complete in $L^2[-\pi, \pi]$. □

7. Final Words

The use of summability kernels and convergence in a norm is an important technique in analysis. When choosing a kernel, the uniform boundedness expressed by the second property is important. For example, the Dirichlet sequence (D_n) is a kernel; however, it is not a summability kernel because it is not bounded. This causes convolution with some (even continuous) functions to diverge in the $L^1[-\pi, \pi]$ norm. On the other hand, the Poisson kernel, denoted by (P_r) and given by:

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

is a summability kernel, which is very helpful in finding solutions to two-dimensional Laplace equations on a sphere with Dirichlet boundary conditions.

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