Math-19 Homework #10 Solutions

Problems

1). Divide $8x^4 + 4x^3 + 6x^2$ by $2x^2 + 1$. Show *all* work and state your final answer in division algorithm form.

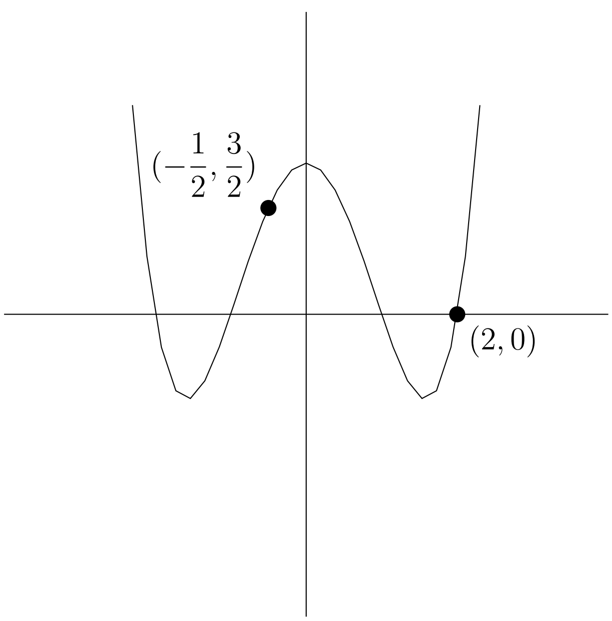
This is a bit hard to typeset properly. The package that does this automatically doesn't put in the fillers and applies the minus sign, but you get the idea.

$$
\begin{array}{rrrrrr}
 & & & 4x^2 & +\,2x & +\,1 \\ \cline{2-6}
2x^2+1 \big) & 8x^4 & +\,4x^3 & +\,6x^2 & & \\
 & -\,8x^4 & & -\,4x^2 & & \\ \cline{2-4}
 & & 4x^3 & +\,2x^2 & & \\
 & & -\,4x^3 & & -\,2x & \\ \cline{3-5}
 & & & 2x^2 & -\,2x & \\
 & & & -\,2x^2 & & -\,1 \\ \cline{4-6}
 & & & & -\,2x & -\,1
\end{array}
$$

So, in DA form:

$8x^4 + 4x^3 + 6x^2 = (2x^2 + 1)(4x^2 + 2x + 1) - 2x - 1 = (2x^2 + 1)(4x^2 + 2x + 1) - (2x + 1)$

2). Consider the following graph of a polynomial:



a). What is the remainder when the polynomial is divided by $(x - 2)$?

By the remainder theorem, $p(2) = 0 = r$.

$r = 0$

b). What is the remainder when the polynomial is divided by $\left(x + \frac{1}{2}\right)$?

By the remainder theorem, $p\left(-\frac{1}{2}\right) = \frac{3}{2} = r$.

$r = \frac{3}{2}$

3). Consider the polynomial function:

$$y = 4x^7 - 4x^6 - 15x^5 + 16x^4 - 4x^3$$

a). Factor completely. You must show *all* work (clearly), including selection of candidates, determining which candidates are roots, and successively dividing out factors. Be sure to clearly state your final factorized form.

First, factor out an x^2 and then work on what is left.

$y = x^3(4x^4 - 4x^3 - 15x^2 + 16x - 4)$

Now, identify candidates for $p_1(x) = 4x^4 - 4x^3 - 15x^2 + 16x - 4$:

$a_0 = -4$ and $a_n = 4$

$p = \pm 1, \pm 2, \pm 4$
$q = \pm 1, \pm 2, \pm 4$

$\frac{p}{q} : \pm 1, \pm 2, \pm 4, \pm\frac{1}{2}, \pm\frac{1}{4}$

Use the remainder theorem to find a zero:

$p_1(1) = 4 - 4 - 15 + 16 - 4 \neq 0$
$p_1(-1) = 4 + 4 - 15 - 16 - 4 \neq 0$
$p_1(2) = 64 - 32 - 60 + 32 - 4 = 0$

Found one. Now divide by $x - 2$:

$$
\begin{array}{r|l}
 & \quad 4x^3 \quad + 4x^2 \quad - 7x + 2 \\
\hline
x - 2 & 4x^4 - 4x^3 - 15x^2 + 16x - 4 \\
 & \underline{-4x^4 + 8x^3} \\
 & \qquad 4x^3 - 15x^2 \\
 & \qquad \underline{-4x^3 + 8x^2} \\
 & \qquad\qquad -7x^2 + 16x \\
 & \qquad\qquad \underline{7x^2 - 14x} \\
 & \qquad\qquad\qquad 2x - 4 \\
 & \qquad\qquad\qquad \underline{-2x + 4} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}
$$

Now we have:

$p(x) = x^3(x - 2)(4x^3 + 4x^2 - 7x + 2)$

Get new candidates for: $p_2(x) = 4x^3 + 4x^2 - 7x + 2$

$p = \pm 1, \pm 2$
$q = \pm 1, \pm 2, \pm 4$

$\frac{p}{q} : \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{4}$

We already eliminated ± 1 is the previous step; however, we need to retry 2, just in case it is a repeated root:

$p_2(2) = 32 + 16 - 14 + 2 \neq 0$ $p_2(-2) = -32 + 16 + 14 + 2 = 0$

So divide out $x + 2$:

$$
\begin{array}{r|l}
 & 4x^2 - 4x + 1 \\
\hline
x+2 & 4x^3 + 4x^2 - 7x + 2 \\
 & -4x^3 - 8x^2 \\
\hline
 & -4x^2 - 7x \\
 & 4x^2 + 8x \\
\hline
 & x + 2 \\
 & -x - 2 \\
\hline
 & 0
\end{array}
$$

Now:

$p(x) = x^3(x-2)(x+2)(4x^2 - 4x + 1)$

The last quadratic factors easily and we have the final result:

$p(x) = x^3(x-2)(x+2)(2x-1)^2$

b). Sketch the graph. All intercepts must be labeled.

Here are the zeros and their multiplicities:

zero	multiplicity	change sign?
-2	1	yes
0	3	yes
$\frac{1}{2}$	2	no
2	1	yes

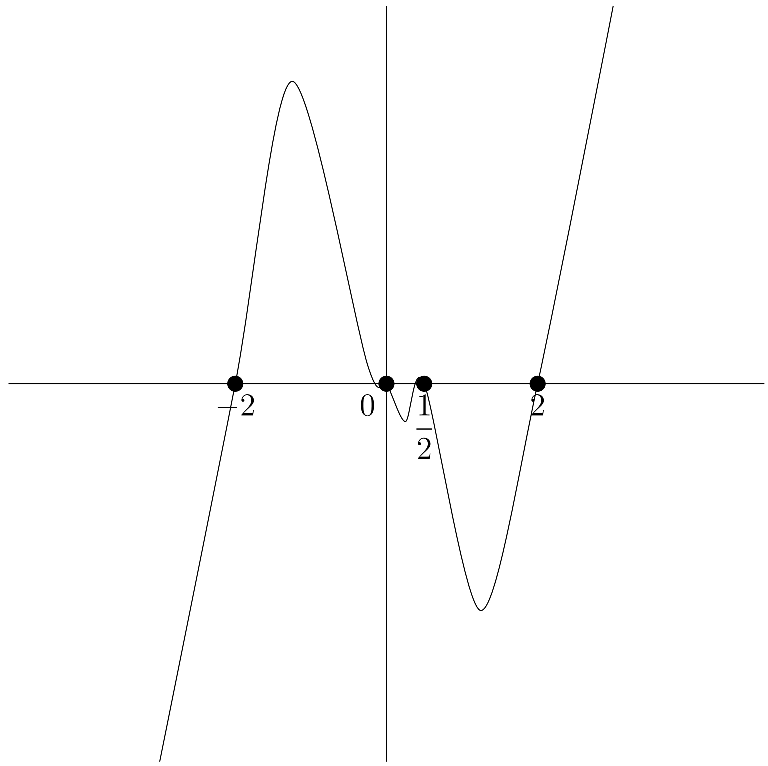
This results in the following x-intercepts:

$(-2, 0), (0, 0), (\frac{1}{2}, 0), (2, 0)$

And y-intercept:

$(0, 0)$

The leading term is $4x^7$: an odd power and a positive coefficient, so the end behavior is like x^3:



Note the shape at $(0,0)$ which is a point of inflection.

4). Consider the rational function:

$$y = \frac{3(x-1)^3(x^2-9)}{(2x^3-6x^2)(x^2-4)}$$

a). What are the zeros?

First factor the numerator and denominator:

$$y = \frac{3(x-1)^3(x+3)(x-3)}{2x^2(x-3)(x+2)(x-2)}$$

Cancel the $x-3$, but remember that we may need to make a hole at $x=3$:

$$y = \frac{3(x-1)^3(x+3)}{2x^2(x+2)(x-2)}$$

Now, the zeros are obviously at $x = 1, -3$

b). What are the poles?

$x = 0, \pm 2$

c). What is the y-intercept?

Note that $x = 0$ is a pole, so there is no y-intercept.

d). What is the horizontal asymptote?

Both the numerator and denominator are of degree 4, so the horizontal asymptote is the ratio of the leading coefficients:

$$y = \frac{3}{2}$$

e). Sketch the graph. You must label all intercepts, asymptotes, and any holes that may occur. All zeros must have the proper shape and the end behaviors must be correct.

The end behavior is a bit tricky. We know that it is asymptotic to $y = \dfrac{3}{2}$; however, we don't know if it approaches from above or below. To determine this, we need to expand the numerator and denominator and then do long division:

$$y = \frac{3(x^4 - 6x^2 + 8x - 3)}{2(x^4 - 4x^2)}$$

$$\begin{array}{r|l} & \quad 1 \\ \hline x^4 - 4x^2 & x^4 - 6x^2 + 8x - 3 \\ & -x^4 + 4x^2 \\ \hline & \quad -2x^2 \end{array}$$

$$y = \frac{3}{2}\left(1 + \frac{-2x^2 + 8x - 3}{x^4 - 4x^2}\right) = \frac{3}{2}\left(1 - \frac{2x^2 - 8x + 3}{x^4 - 4x^2}\right)$$

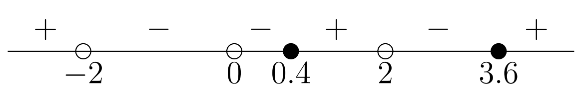
We need to see how $\dfrac{2x^2 - 8x + 3}{x^4 - 4x^2}$ changes sign during the end behavior (i.e., past the last zero/pole). If it is < 0 then the graph is above the asymptote. If it is > 0 then the graph is below the asymptote. The numerator doesn't factor nicely, so we will use the quadratic formula:

$$x = \frac{8 \pm \sqrt{(-8)^2 - 4(2)(3)}}{2(2)} = \frac{8 \pm \sqrt{40}}{4} = \frac{4 \pm \sqrt{10}}{2} \approx 0.4, 3.6$$

So we have:

$$y = \frac{3}{2}\left(1 - \frac{\left(x - \frac{4-\sqrt{10}}{2}\right)\left(x - \frac{4+\sqrt{10}}{2}\right)}{x^2(x+2)(x-2)}\right)$$

Using the techniques from chapter 1:



So before $x = -3$ the correction is always positive, and so the graph approaches the asymptote from below. But past $x = 2$ the correction starts off negative and then goes positive after about $x = 3.6$. Thus, the graph is above the asymptote, crosses at about $x = 3.6$, and then approaches from below.

Also, remember to leave a hole at $x = 3$. The corresponding y value is:

$$y(3) = \frac{3(8)(6)}{2(9)(5)(1)} = \frac{8}{5}$$

Putting this altogether we have: