Cavallaro, Jeffery
Math 229
Homework #3

3.1.6

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda + 1 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$

$\sigma(A) = \{0, 2\}$

Start with $\lambda = 0$:

$r_0(0) = \operatorname{rank}(A^0) \operatorname{rank}(I_2) = 2$
$r_1(0) = \operatorname{rank}(A) = 1$
$r_2(0) = n - a(0) = 2 - 1 = 1$ $r_3(0) = r_2(0) = 1$

$b_1(0) = r_0 - 2r_1 + r_2 = 2 - 2(1) + 1 = 1$
So $b_2(0) = 0$

Also, this means:

$b_1(2) = 1$
$b_2(2) = 0$

And therefore:

$J_A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

$B = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\sigma(B) = \{3\}$

$B - 3I = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$r_0(3) = \operatorname{rank}(B - 3I)^0 = \operatorname{rank}(I_3) = 3$
$r_1(3) = \operatorname{rank}(B - 3I)^1 = 1$
$r_2(3) = \operatorname{rank}(B - 3I)^2 = 0$
$r_3(3) = n - a(3) = 3 - 3 = 0$
$r_4(3) = r_3(3) = 0$

$b_1(3) = r_0 - 2r_1 + r_2 = 3 - 2(1) + 0 = 1$
So $b_2(3) = 1$ and $b_3(3) = 0$

$$J_b = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

3.2.7

What are the possible Jordan forms of a matrix $A \in M_n$ such that $A^3 = I$.

Since $A^3 - I = 0$, $t^3 - 1$ is an annihilating polynomial, and thus $q_A(t)$ must divide it. Since:

$$t^3 - 1 = (t-1)(t-\omega)(t-\omega^2)$$

Thus $\sigma(A) \subseteq \{1, \omega, \omega^2\}$ and the actual $q_A(t)$ must have some combination of these linear factors, all with multiplicity of 1. Thus, the maximum Weyr index for any eigenvalue is 1 and J_A is diagonal with any combination of 1, ω, and ω^2 as diagonal values.

3.3.3

Show that every protection (idempotent) matrix is diagonalizable. What is the minimum polynomial of A? What can you say if A is tripotent ($A^3 = A$). What if $A^k = A$?

If $A^2 = A$ then $t(t-1)$ is an annihilating polynomial for A and thus $q_A(t)$ must be some combination of these distinct linear factors with multiplicity 1. Thus, the possibilities for $q_A(t)$ are t, $t-1$ and $t(t-1)$.

If $A^3 = A$ then $t^3 - t = t(t^2-1) = t(t-1)(t+1)$ and thus the possibilities for $q_A(t)$ are any combination of the following linear factors with multiplicity 1: $t, t-1, t+1$.

To generalize, for $A^k = A$, $q_A(t)$ is any combination of linear factors with multiplicity 1 from the set $\{t, t-\alpha \mid \alpha \text{ is a } (k-1)\text{-root of } 1\}$.

3.3.9

If $A \in M_5$ has $p_t(A) = (t-4)^3(t+6)^2$ and $q_A(t) = (t-4)^2(t+6)$, what is J_A?

$\sigma(A) = \{4, -6\}$ with $a_A(4) = 3$ and $a_A(-6) = 2$.

The highest non-zero Weyr index for $\lambda = 4$ is b_2.

The highest non-zero Weyr index for $\lambda = -6$ is b_1.

And so:

$b_2(4) = 1$
$b_1(4) = 1$
$b_1(-6) = 2$

$$J_A = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & -6 \end{bmatrix}$$

3.3.31

Show that there is no real 3×3 matrix whose minimal polynomial is $t^2 + 1$, but that there is a real 2×2 matrix as well as a complex 3×3 matrix with this property.

ABC: $A \in M_3(\mathbb{R})$ is such a matrix.

Since $\deg(p_A(t)) = 3$ and since $q_A(t)$ divides $p_A(t)$, we know that $\pm i \in \sigma(A)$ and that:

$$p_A(t) = (t^2 + 1)(t - \alpha) = t^3 - \alpha t^2 + t - \alpha$$

But we know that α cannot be distinct from $\pm i$, otherwise, all three eigenvalues would have to be present in $p_A(t)$ with linear factors. Thus, $a = \pm i$; however, that would mean that $p_A(t)$ has complex coefficients, which cannot result from a matrix with real components - CONTRADICTION!.

Therefore, no such A exists.

Let $B = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

$B^2 + I = 0$, so $t^2 + 1$ is an annihilator polynomial for B. Furthermore, the only possible linear cases would be $x + i$ and $x - i$, neither of which is an annihilator for B, so $q_B(t) = t^2 + 1$.

Let $C = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$

$C^2 + I = 0$, so $t^2 + 1$ is an annihilator polynomial for C. Furthermore, the only possible linear cases would be $x + i$ and $x - i$, neither of which is an annihilator for C, so $q_C(t) = t^2 + 1$.