Direct Products

Definition: Direct (Cartesian) Product

Let $E_1, E_2, \dots E_n$ be an ordered collection of vector spaces. The *direct (Cartesian) product* of these spaces is given by:

$$\prod_{k=1}^{n} E_k = E_1 \times E_2 \times \cdots \times E_n = \{(x_k) \mid x_k \in E_k\} = \{(x_1, x_2, \dots, x_n) \mid x_k \in E_k\}$$

This definition can be extended to allow for a countable number of vector spaces:

$$\prod_{k=1}^{\infty} E_k = E_1 \times E_2 \times \cdots = \{(x_k) \mid x_k \in E_k\} = \{(x_1, x_2, \dots) \mid x_k \in E_k\}$$

And this definition can be further extended to allow for an uncountable number of vector spaces via the use of functions. Let $E_j \mid j \in J$ be an indexed family of vector spaces:

$$\prod_{j \in J} E_j = \{x : J \to \bigcup_{j \in J} E_j \mid x(j) \in E_j\}$$

Thus, direct products of vector spaces on the same field $\mathbb{F}$ are vector spaces using the standard function operations:

$$(x + y)(j) = x(j) +_j y(j)$$

$$(\lambda x)(j) = \lambda x(j)$$