Events

Definition: Experiment

An *experiment* is any activity or process whose outcome is subject to uncertainty. A *trial* is one execution of an experiment. An *outcome* of an experiment, denoted ω, is one of the possible results from the experiment. The *sample space* of an experiment, denoted $\mathcal{S}$, is the set of all possible outcomes of the experiment.

Examples: Sample Spaces

- Discrete and finite sample spaces:
  - Toss a coin: $\mathcal{S} = \{H, T\}$
  - Roll a die: $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$
  - Draw a card from a deck: $\mathcal{S} = \{sr \mid s \in \{C, D, H, S\}, r \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\}\}$
  - Throw a coin twice: $\mathcal{S} = \{HH, HT, TH, HH\}$
  - Roll two dice: $\mathcal{S} = \{(i, j) \mid i, j \in \{1, 2, 3, 4, 5, 6\}\}$
- Discrete and infinite sample spaces:
  - Throw a coin repeatedly until the first heads: $\mathcal{S} = \{H, TH, TTH, TTTH, \ldots\}$
- Continuous sample spaces:
  - The lifetime of a new lightbulb: $\mathcal{S} = [0, \infty)$

Definition: Event

An *event* is a subset of outcomes from a sample space. To say that an event is *simple* means that it contains exactly one outcome. Otherwise, an event is called *compound*. When an experiment is performed, an event A is said to have *occurred* if the resulting outcome ω is contained in the event ($\omega \in A$). In particular, $\mathcal{S}$ is an event that always occurs and $\emptyset$ is an event that never occurs.

Example: Roll a Die

$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$

- $A = \{1\}$ (simple)
- $B = \{6\}$ (simple)
- $C = \{\text{an even number}\} = \{2, 4, 6\}$ (compound)
- $D = \{\text{an odd number}\} = \{1, 3, 5\}$ (compound)

Although each trial of an experiment has exactly one outcome, multiple events could occur: if $\omega = 1$ then A and D occur, but B and C do not occur.

Example: Roll Two Dice

$\mathcal{S} = \{(i,j) \,|\, i,j \in \{1,2,3,4,5,6\}\}$

- $A = \{\text{sum equals 6}\} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$
- $B = \{\text{both equal}\} = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$
- $C = \{\text{both even}\} = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$

Example: Toss a Coin

Toss a coin until the first heads: $\mathcal{S} = \{H, TH, TTH, TTH, \ldots\}$

- $A = \{\text{at most four tails}\} = \{H, TH, TTH, TTTH, TTTTH\}$

Since events are sets, the various set operations apply. Assuming the following events from rolling two dice:

- Cardinality: $|A|$ = the number of outcomes in A
- Complement: $A^c = \{\omega \,|\, \omega \notin A\}$
- Union: $A \cup B = \{\omega \,|\, \omega \in A \text{ or } \omega \in B\}$
- Intersection: $A \cap B = \{\omega \,|\, \omega \in A \text{ and } \omega \in B\}$
- Difference: $A - B = \{\omega \,|\, \omega \in A \text{ and } \omega \notin B\} = A \cap B^c$
- Distributive:
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- DeMorgan:
  - $(A \cup B)^c = A^c \cap B^c$
  - $(A \cap B)^c = A^c \cup B^c$

Example: Set Operations

Assuming the following events from rolling two dice:

1. $A = \{\text{sum equals 6}\}$
2. $B = \{\text{both equal}\}$
3. $C = \{\text{both even}\}$

$|C| = 9$
$A \cap B = \{(3,3)\}$
$A \cup B = \{(1,1),(1,5),(2,2),(2,4),(3,3),(4,2),(4,4),(5,1),(5,5),(6,6)\}$
$B^c = \{(i,j) \mid i,j \in \{1,2,3,4,5,6\} \text{ and } i \neq j\}$
$A - C = \{(1,5),(3,3),(5,1)\}$

Definition: Disjoint

To say that two events A and B are *disjoint* means that $A \cap B = \emptyset$.

Let $\{E_i, i \in I\}$ be a family of events. To say that the E_i are *pairwise disjoint* or *mutually exclusive* means:

$$\forall i \in I, i \neq j \implies E_i \cap E_j = \emptyset$$

Example: Toss Two Dice

Consider the following events:

$A = \{\text{sum equals 7}\}$

$B = \{\text{both equal}\}$

Since B always results in an even sum, but 7 is odd, it is the case that $A \cap B = \emptyset$ and thus A and B are disjoint.