MATH 231B, FALL 2017
HOMEWORK 2 SOLUTIONS

1. (Sec. 3.8, ex. 4) (a) Let $f(x) = 1$, for all $a \leq x \leq b$. Then $\|f\|^2 = \langle f, f\rangle = 0$ even though $f \neq \mathbf{0}$, so $\langle \cdot, \cdot \rangle$ is not an inner product.

(b) It is easy to see that $\langle \cdot, \cdot \rangle$ satisfies axioms (a)-(c) for inner products. To show that (d) is also satisfied, assume $\langle f, f\rangle = 0$. Then

$$\int_a^b \left|f'(x)\right|^2 \, dx = 0,$$

so since f' is continuous, it follows that $f' = 0$. Thus f is constant and since $f(a) = 0$, it follows that $f(x) = 0$, for all x. Therefore, (d) holds and $\langle \cdot, \cdot \rangle$ is an inner product.

However, the space is not Hilbert. Denote by F the space of all functions in $C^1[-1, 1]$ such that $f(0) = 0$ and endow F with the inner product as above. (It doesn't make a difference that we are stipulating that $f(0) = 0$ instead of $f(-1) = 0$.) We will show that there exists a non-convergent Cauchy sequence in F. The idea is to construct a sequence (f_n) in F such that f_n' converges in the L^2 sense to a discontinuous bounded function (such as the sign function) whose integral is then not differentiable at every point. We will therefore begin by constructing a sequence of derivatives of f_n's, which we denote by g_n.

Define $g_n : [-1, 1] \to \mathbb{R}$ by

$$g_n(t) = \begin{cases} -1 & \text{if } -1 \leq t \leq -\dfrac{1}{n} \\ nt & \text{if } -\dfrac{1}{n} \leq t \leq \dfrac{1}{n} \\ 1 & \text{if } \dfrac{1}{n} \leq t \leq 1. \end{cases}$$

Clearly, each g_n is continuous and if we denote by g the sign function on $[-1, 1]$, then it is not hard to verify that

$$\|g_n - g\|_{L^2}^2 = \frac{2}{3n} \to 0,$$

as $n \to \infty$, so $g_n \to g$ in the L^2 sense.

Now define $f_n : [-1, 1] \to \mathbb{R}$ by

$$f_n(x) = \int_0^x g_n(t) \, dt.$$

Then $f_n(0) = 0$ and $f_n' = g_n$, so $f_n \in F$, for all n.

Set also

$$f(x) = \int_0^x g(t) \, dt.$$

It is not hard to check that $f(x) = |x|$ and that $f'(x) = g(x)$, for all $x \neq 0$. We claim that (f_n) is Cauchy in F (i.e., with respect to the norm in F). Indeed,

$$\begin{aligned} \|f_n - f_m\|_F &= \|f_n' - f_m'\|_{L^2} \\ &= \|g_n - g_m\|_{L^2} \\ &\leq \|g_n - g\|_{L^2} + \|g_m - g\|_{L^2} \\ &\to 0, \end{aligned}$$

as $m, n \to \infty$. This proves that (f_n) is Cauchy.

However, it is not hard to check that $f_n(x) \to f(x)$, as $n \to \infty$, for all $-1 \le x \le 1$ and clearly $f \notin F$. This proves that F is not complete, hence not Hilbert. $\square$

2. (Sec. 3.8, ex. 10) We have:
$$\begin{aligned} \|x+y\|^2 - \|x-y\|^2 &= \|x\|^2 + 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2 - (\|x\|^2 - 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2) \\ &= 4\,\mathrm{Re}\langle x,y\rangle. \end{aligned}$$
Similarly:
$$\begin{aligned} \|x+iy\|^2 - \|x-iy\|^2 &= \|x\|^2 + 2\,\mathrm{Re}\langle x,iy\rangle + \|y\|^2 - (\|x\|^2 - 2\,\mathrm{Re}\langle x,iy\rangle + \|y\|^2) \\ &= 4\,\mathrm{Re}\langle x,iy\rangle \\ &= -4\,\mathrm{Re}(i\langle x,y\rangle) \\ &= 4\,\mathrm{Im}\langle x,y\rangle, \end{aligned}$$
where we used the fact that $\mathrm{Re}(iz) = -\mathrm{Im}(z)$, for all complex numbers z. Therefore,
$$\|x+y\|^2 - \|x-y\|^2 + i(\|x+iy\|^2 - \|x-iy\|^2) = 4\mathrm{Re}\langle x,y\rangle + 4i\langle x,y\rangle = 4\langle x,y\rangle. \quad \square$$

3. (Sec. 3.8, ex. 11) We will use the following fact: if f is a real-valued function on a set S satisfying $f(x) \le M$, for all $x \in S$ and $f(x_0) = M$, for some $x_0 \in S$, then $\sup_S f = M$. (In fact, $\max_S f = M$.)

Observe that the statement of the exercise is trivially true for $x = 0$. So assume $x \neq 0$. Then by the Cauchy-Schwarz inequality, we have
$$|\langle x,y\rangle| \le \|x\|\|y\| = \|x\|,$$
for all $y \in E$ with $y = 1$. Taking $y = x/\|x\|$, we obtain
$$\begin{aligned} |\langle x,y\rangle| &= \left|\langle x, \frac{x}{\|x\|}\rangle\right| \\ &= \left|\frac{\langle x,x\rangle}{\|x\|}\right| \\ &= \frac{\|x\|^2}{\|x\|} \\ &= \|x\|, \end{aligned}$$
which by the observation above implies that
$$\sup_{\|y\|=1} |\langle x,y\rangle| = \|x\|. \quad \square$$

Remark. This exercise is saying that if f_x is the linear functional on E defined by $f_x(y) = \langle x,y\rangle$, then $\|f_x\| = \|x\|$.