

Category Theory

Definition: Category

A *category* C is a mathematical structure consisting of two *classes*:

- 1). $\text{obj}(C) = \text{class of objects of } C$
- 2). $\text{mor}(C) = \text{class of morphisms of } C$

Objects are typically denoted by A, B, C, \dots

Morphisms are disjoint sets, one per each pair of objects, denoted $\text{mor}(A, B)$ for objects $A, B \in \text{obj}(C)$. An element $f \in \text{mor}(A, B)$, denoted $f : A \rightarrow B$ is called a morphism from A to B and is used to define structure between the objects.

Definition: Composition

Let C be a category and $A, B, C \in \text{obj}(C)$. There exists a function:

$$\text{mor}(B, C) \times \text{mor}(A, B) \rightarrow \text{mor}(A, C)$$

defined by $(g, f) \mapsto g \circ f$, where $g \circ f$ is called the *composite* of f and g and is subject to the following two axioms:

- 1). *Associativity*: $\forall f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D$:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

- 2). *identity*: $\forall B \in \text{obj}(C), \exists \iota_B : B \rightarrow B$ such that:

- $\forall f : A \rightarrow B, \iota_B \circ f = f$
- $\forall g : B \rightarrow C, g \circ \iota_B = g$

Note that $\text{mor}(A, A) \neq \emptyset$ because at least $i_A \in \text{mor}(A, A)$.

Definition: Equivalence

Let C be a category, $A, B \in \text{obj}(C)$, and $f : A \rightarrow B$. To say that f is an *equivalence* means $\exists g : B \rightarrow A$ such that:

- $g \circ f = \iota_A$
- $f \circ g = \iota_B$

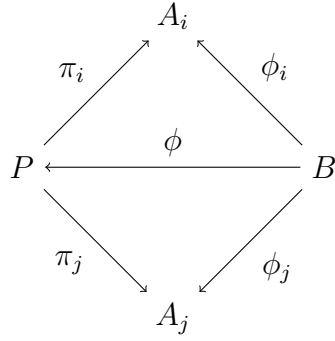
In this case, A and B are said to be *equivalent*.

Example

Let \mathcal{G} = category of groups, where $A = \text{obj}(\mathcal{G})$ is a group, and $\text{mor}(A, B)$ is the (possibly empty) set of homomorphisms from A to B .

Definition: Product

Let C be a category and $\mathcal{S} = \{A_i \mid i \in I\}$ be a family of objects in $\text{obj}(C)$. A *product* of \mathcal{S} , denoted $\prod_{i \in I} A_i$, is an object $P \in \text{obj}(C)$ together with a family of morphisms $\{\pi_i : P \rightarrow A_i \mid i \in I\}$ such that $\forall B \in \text{obj}(C)$ and family of morphisms $\{\phi_i : B \rightarrow A_i \mid i \in I\}$ there exists a unique morphism $\phi : B \rightarrow P$ such that $\forall i \in I, \pi_i \circ \phi = \phi_i$.



For the category of groups, define $\pi_i : G \rightarrow G_i$ by projection.

Theorem

Let $\{A_i \mid i \in I\}$ be a family of objects in a category C . If $\{A_i \mid i \in I\}$ has a product then that product is unique (up to equivalence).

Proof

Assume $\{A_i \mid i \in I\}$ has a product

Let $(P, \{\pi_i\})$ and $(Q, \{\phi_i\})$ be two such products

$\exists \phi : Q \rightarrow P, \pi_i \phi = \phi_i$

$\exists \pi : P \rightarrow Q, \phi_i \pi = \pi_i$

$\phi_i(\pi \phi) = \phi_i$, so $\pi \phi = \iota_Q$

$\pi_i(\phi \pi) = \pi_i$, so $\phi \pi = \iota_P$

$\therefore P$ and Q are equivalent.

Definition: Coproduct

Let C be a category and $\mathcal{S} = \{A_i \mid i \in I\}$ be a family of objects in $\text{obj}(C)$. A *coproduct* of \mathcal{S} is an object $P \in \text{obj}(C)$ together with a family of morphisms $\{\pi_i : P \rightarrow A_i \mid i \in I\}$ such that $\forall B \in \text{obj}(C)$ and family of morphisms $\{\phi_i : B \rightarrow A_i \mid i \in I\}$ there exists a unique morphism $\pi : P \rightarrow B$ such that $\forall i \in I, \phi_i \circ \pi = \pi_i$.

In other words, a cofactor is a factor in the opposite direction.