Argument Principle

<u>Theorem</u>

Let $f(z)$ be analytic on $\overline{D}$ with boundary γ. The number of zeros of $f(z)$ in D is given by:

$$N = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$$

<u>Proof</u>

$\overline{D}$ is a compact set, so the number of zeros is finite
Let $f(z) = \left[\prod_{k=1}^{N}(z - z_k)\right] g(z)$, where $g(z) \neq 0$ in D
$\log f(z) = \sum_{k=1}^{N} \log{(z - z_k)} + \log g(z)$
Now differentiate wrt z:
$\frac{f'(z)}{f(z)} = \sum_{k=1}^{N} \frac{1}{z-z_k} + \frac{g'(z)}{g(z)}$
Now, multiply by $\frac{1}{2\pi i}$ and integrate on γ:
$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{k=1}^{N} \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_k}dz + \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}dz$
But $\frac{g'(z)}{g(z)}$ is analytic in D:
$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{k=1}^{N}(1) + 0$
$\therefore N = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$

<u>Theorem: Newton</u>

Let $f(z) = \sum_{k=0}^{n} a_k z^k$:

$$\sum_{k=0}^{n} a_k = -\frac{a_{n-1}}{a_n}$$

<u>Theorem</u>

Let $f(z) = \sum_{k=0}^{n} a_k z^k$ and let γ enclose all of the zeros of $f(z)$:

$$\frac{1}{2\pi i}\int_\gamma z\frac{f'(z)}{f(z)} = -\frac{a_{n-1}}{a_n}$$

<u>Proof</u>

$f(z) = a_n \prod_{k=1}^{n}(z - z_k)$
$\log f(z) = \log a_n + \sum_{k=0}^{n} \log{(z - z_k)}$
Now differentiate wrt z:
$\frac{f'(z)}{f(z)} = 0 + \sum_{k=0}^{n} \frac{1}{z-z_k}$
Now, multiply by $\frac{1}{2\pi i}z$ and integrate on γ:
$\frac{1}{2\pi i}\int_\gamma z\frac{f'(z)}{f(z)}dz = \sum_{k=0}^{n} \frac{1}{2\pi i}\int_\gamma \frac{z}{z-z_k}dz = \sum_{k=0}^{n} z_k = -\frac{a_{n-1}}{a_n}$