Open and Closed Sets

Definition: Ball

Let E be a normed space, $\vec{x} \in E$, and $r > 0$:

1). $B(\vec{x}, r) = \{\vec{y} \in E \mid \|\vec{x} - \vec{y}\| < r$ is called an *open ball.*

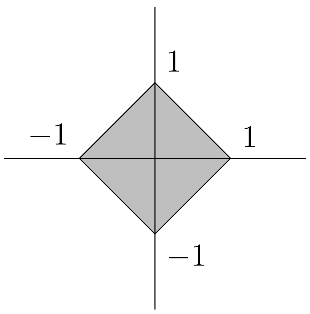
2). $\overline{B}(\vec{x}, r) = \{\vec{y} \in E \mid \|\vec{x} - \vec{y}\| \leq r$ is called a *closed ball.*

3). $S(\vec{x}, r) = \{\vec{y} \in E \mid \|\vec{x} - \vec{y}\| = r$ is called a *sphere.*

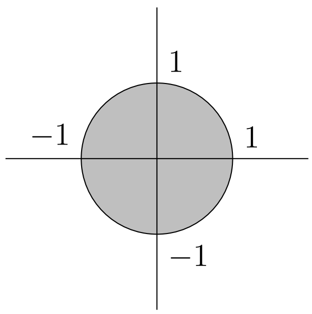
In all cases, $\vec{x}$ is called the *center* and r is called the *radius.*

Example

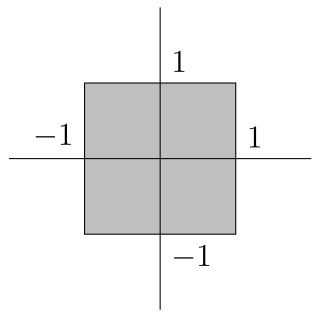
Let $E = R^2$ and let $B_k(0, 1)$ be the unit ball for $\|\cdot\|_k$ for $1 \leq k \leq \infty$:



B_1



B_2



B_∞

Definition: Open

Let E be a normed vector space and $S \subset E$. To say that S is *open* means:

$$\forall\, \vec{x} \in S, \exists\, \epsilon > 0, B(\vec{x}, \epsilon) \subset S$$

To say that S is *closed* means $E \setminus S$ is open.

To say that S is *clopen* means it is both open and closed.

Theorem

Let E be a non-trivial normed vector space:

1). The union of any collection of open subsets of E is open.

2). The empty set and E are clopen.

3). The intersection of a finite number of open subsets of E is open.

4). The union of a finite number of closed subsets of E is closed.

5). The intersection of any collection of closed subsets of E is closed.

Proof

1). Assume $\{U_i \mid i \in I\}$ is a collection of open subsets of E and let $U = \bigcup_{i \in I} U_i$.

   Assume $\vec{x} \in U$.
   $\exists\, i \in I, \vec{x} \in U_i$
   But U_i is open, so $\exists\, \epsilon > 0, B(\vec{x}, \epsilon) \subset U_i \subset U$.
   Thus $B(\vec{x}, \epsilon) \subset U$.

   Therefore, U is open.

2). By definition, the empty set is vacuously open, and since $E \setminus \emptyset = E$, E is closed by definition. Conversely, since E is the union of all possible open subsets of E, E must also be open. And since $\emptyset = E \setminus E$, $\emptyset$ must also be closed. Therefore, $\emptyset$ and E are clopen.

3). Proof by induction on the number of open sets n.

   Base case: $n = 2$

   Assume $U, V \subset E$ are open sets and let $W = U \cap V$.
   If $W = \emptyset$ then W is open, so AWLOG that $W \neq \emptyset$.
   Assume $\vec{x} \in W$.
   $\vec{x} \in U$ and $\vec{x} \in V$.
   But U and V are open, so $\exists\, \epsilon_u, \epsilon_v > 0$ such that $B(\vec{x}, \epsilon_u) \subset U$ and $B(\vec{x}, \epsilon_v) \subset V$.
   Let $\epsilon = \min\{\epsilon_u, \epsilon_b\}$.
   $B(\vec{x}, \epsilon) \subseteq B(\vec{x}, \epsilon_u) \subset U$ and $B(\vec{x}, \epsilon) \subseteq B(\vec{x}, \epsilon_v) \subset V$.
   And so $B(\vec{x}, \epsilon) \subset U \cap V = W$.

   Therefore, $W = U \cap V$ is open.

   Assume that an intersection of n open subsets of E is open.

   Assume $U_1, \ldots, U_{n+1}$ are $n+1$ open subsets of E.

   Let $U = \bigcap_{k=1}^{n+1} U_k = \left(\bigcap_{k=1}^{n} U_k \right) \cap U_{n+1}$.

   But by the inductive assumption, $\bigcap_{k=1}^{n} U_k$ is an open set.

   Therefore, by the base case, U is an open set.

4). Assume $V_1, V_2, \ldots, V_n$ is a finite number of closed subsets of E.

   Let $V = \bigcup_{k=1}^{n} V_k$.

   By DeMorgan: $\overline{V} = \overline{\bigcup_{k=1}^{n} V_k} = \bigcap_{k=1}^{n} \overline{V_k}$.
   But V_k is closed, so $\overline{V_k}$ is open.
   And so $\overline{V}$ is an intersection of a finite number of open subsets of E and is thus open.

Therefore $V = \bigcup_{k=1}^{n} V_k$ is closed.

5). Assume $\{V_i \mid i \in I\}$ is a collection of closed subsets of E.

Let $V = \bigcap_{i \in I}^{n} V_i$.

By DeMorgan: $\overline{V} = \overline{\bigcap_{i \in I} V_i} = \bigcup_{i \in I} \overline{V_i}$.

But V_i is closed, so $\overline{V_i}$ is open.

And so $\overline{V}$ is a union of a collection of open subsets of E and is thus open.

Therefore $V = \bigcap_{i \in I} V_i$ is closed.

Examples

Let $U_n = (-\frac{1}{n}, \frac{1}{n})$.

$\bigcap_{n=1}^{\infty} U_n = \{0\}$, which is closed.

Therefore, an intersection of an infinite collection of open sets is not necessarily open.

Let $V_n = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$.

$\bigcup_{n=1}^{\infty} V_n = (-1, 1)$, which is open.

Therefore, a union of an infinite collection of closed sets is not necessarily closed.

Theorem

Let E be a normed space and let $S \subseteq E$:

$$S \text{ is closed } \iff \forall\,(\vec{x}_n) \text{ in } S, \vec{x}_n \to \vec{x} \implies \vec{x} \in S$$

Thus, S contains all of its limit points.

Proof

$\implies$ Assume S is closed.

Assume $(\vec{x}_n)$ is in S and $\vec{x}_n \to \vec{x}$.

ABC: $\vec{x} \notin S$.

Since S is closed, $E \setminus S$ is open.

So, $\exists\, \epsilon > 0$ such that $B(\vec{x}, \epsilon) \in E \setminus S$.

And so $\forall\, n \in \mathbb{N}, \|\vec{x}_n - \vec{x}\| > \epsilon$.

But, by assumption, $\vec{x}_n \to \vec{x}$ and so $\|\vec{x}_n - \vec{x}\| < \epsilon$ for n sufficiently large.

CONTRADICTION!

Therefore, $\vec{x} \in S$.

$\Longleftarrow$ Assume S contains all of its limit points.

ABC: S is open.
Thus, $E \setminus S$ is closed, and so $\exists\, \vec{x} \in E \setminus S$ such that $\forall\, \epsilon > 0, B(\vec{x}, \epsilon) \cap S \neq \emptyset$.
Construct a sequence $(\vec{x}_n) \in S$ such that $\vec{x}_n \in B(\vec{x}, \dfrac{1}{n})$.
Clearly, $\vec{x}_n \to \vec{x} \notin S$, so $\vec{x}$ is a limit point for S that is not in S.
CONTRADICTION!

Therefore, S is closed.