Integers

Definition

The set of *integers* includes the positive and negative whole numbers and zero and is given by:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Thus, by the trichotomy principle, $\forall\, n \in \mathbb{Z}$, exactly one of the following is true:

1). n is positive

2). n is zero

3). n is negative

Definition

Let a and b be two integer values, regardless of representation. To say that a equals b, denoted $a = b$, means that a and b represent the same element in $\mathbb{Z}$.

Axiom: Substitution Principle

Let a and b be two integer values, regardless of representation. If $a = b$ then a and b can syntactically replace each other in a given context without altering the context.

Properties: Equality

Let a and b be two integer values:

1). Reflexivity

$$a = a$$

2). Symmetry

$$a = b \implies b = a$$

3). Transitivity

$$a = b \text{ and } b = c \implies a = c$$

Properties

The set of integers is a commutative ring with unity under the binary operations of addition and multiplication, and as such, the following axioms hold:

$\forall\, a, b, c, d \in \mathbb{Z}$:

1). Well-defined

- $a + b = c$ and $a + b = d \implies c = d$
- $ab = c$ and $ab = d \implies c = d$

2). Closure

- $a + b \in \mathbb{Z}$
- $ab \in \mathbb{Z}$

3). Cummutativity

- $a + b = b + a$
- $ab = ba$

4). Associativity

- $(a + b) + c = a + (b + c)$
- $(ab)c = a(bc)$

5). Distributivity

- $a(b + c) = ab + ac$

6). Identity

- $a + 0 = a$
- $a1 = a$

7). Additive Inverse

- $\exists\, (-a) \in \mathbb{Z}, a + (-a) = 0$

Since the set of integers is a commutative ring with unity, the following properties also hold:

Properties: Zero

$\forall\, a, b \in \mathbb{Z}$:

1). $-0 = 0$

2). $a0 = 0$

3). $ab = 0 \implies a = 0$ or $b = 0$

Properties: Negatives

$\forall\, a, b \in \mathbb{Z}$:

1). $(-1)a = -a$

2). $-(-a) = a$

3). $(-a)b = -(ab) = a(-b)$

4). $(-a)(-b) = ab$

5). $-(a+b) = (-a) + (-b)$

Notation

Integer subtraction is nothing more than a syntactic convenience:

$$\forall\, a, b \in \mathbb{Z}, a - b = a + (-b)$$

Properties: Cancellation

$\forall\, a, b, c \in \mathbb{Z}$:

1). $a = b \iff a + c = b + c$

2). $a = b \implies ac = bc$

3). $ac = bc$ and $c \neq 0 \implies a = b$

The last cancellation law cannot be proven in the common manner due to the lack of multiplication inverses. Instead:

Proof

Assume $a, b, c \in \mathbb{Z}$
Assume $ac = bc$ and $c \neq 0$
$ac - bc = 0$
$(a - b)c = 0$
Since $c \neq 0$, $a - b = 0$
$\therefore a = b$