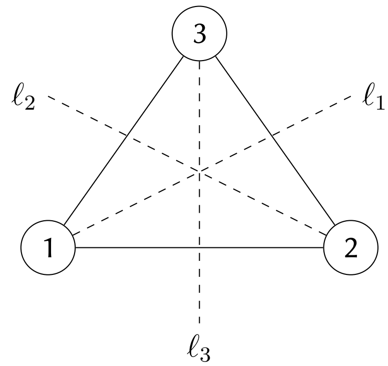
Dihedral Groups

Definition

A dihedral group, denoted D_n, represents the symmetries (rotation and reflection) of a regular polygon with n vertices by permutations of the vertices.

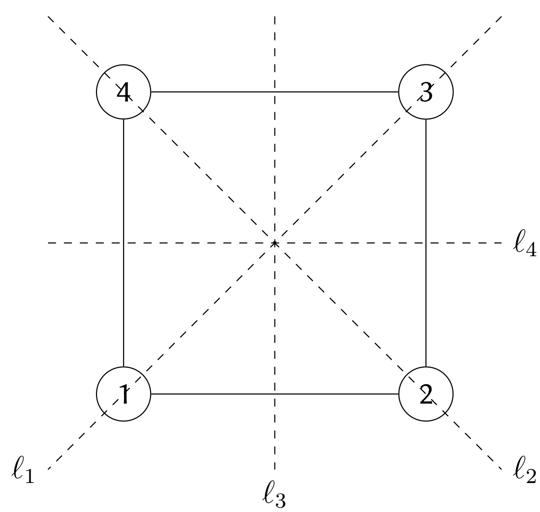
Example: D_3



$e = ()$	no action
$r = (123)$	rotate CCW
$r^{-1} = (132)$	rotate CW
$f_1 = (23)$	flip about ℓ_1
$f_2 = (13)$	flip about ℓ_2
$f_3 = (12)$	flip about ℓ_3

$r^2 = r^{-1}$
$r^3 = (r^{-1})^3 = e$
$f_1 = f_1^{-1}$
$f_2 = f_2^{-1}$
$f_3 = f_3^{-1}$
$f_1^2 = f_2^2 = f_3^2 = e$

Example: D_4



$e = ()$	no action
$r_1 = (1234)$	rotate once CCW
$r_2 = (13)(24)$	rotate twice CCW
$r_3 = (1432)$	rotate thrice CCW
$f_1 = (24)$	flip about ℓ_1
$f_2 = (13)$	flip about ℓ_2
$f_3 = (12)(34)$	flip about ℓ_3
$f_4 = (14)(23)$	flip about ℓ_4

$|D_3| = |S_3| = 6$

$|D_4| = 8$

In general:

$$|D_n| = 2n = (n \text{ rotations}) + (n \text{ reflections})$$

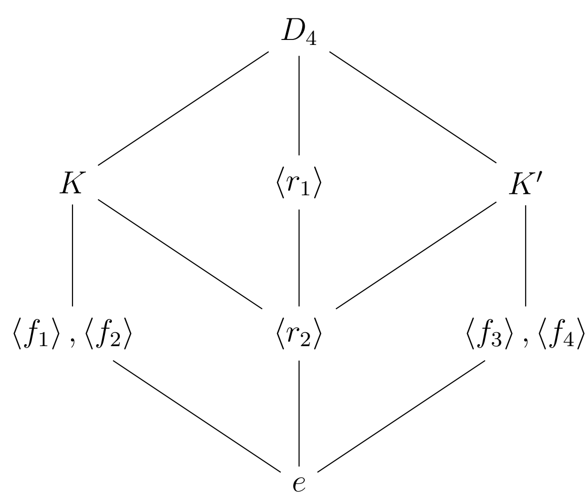
where:

$$n \text{ reflections} = \begin{cases} \frac{n}{2} \text{ opposite vertices} + \frac{n}{2} \text{ opposite sides}, & n \text{ even} \\ n \text{ vertex-opposite side}, & n \text{ odd} \end{cases}$$

Hasse Diagram for D_4:

	e	r_1	r_2	r_3	f_1	f_2	f_3	f_4
e	e	r_1	r_2	r_3	f_1	f_2	f_3	f_4
r_1	r_1	r_2	r_3	e	f_3	f_4	f_2	f_1
r_2	r_2	r_3	e	r_1	f_2	f_1	f_4	f_3
r_3	r_3	e	r_1	r_2	f_4	f_3	f_1	f_2
f_1	f_1	f_4	f_2	f_3	e	r_2	r_3	r_1
f_2	f_2	f_3	f_1	f_4	r_2	e	r_1	r_3
f_3	f_3	f_1	f_4	f_2	r_1	r_3	e	r_2
f_4	f_4	f_2	f_3	f_1	r_3	r_1	r_2	e

$\langle e \rangle = \{e\}$
$\langle r_1 \rangle = \langle r_3 \rangle = \{e, r_1, r_2, r_3\}$
$\langle r_2 \rangle = \{e, r_2\}$
$\langle f_1 \rangle = \{e, f_1\}$
$\langle f_2 \rangle = \{e, f_2\}$
$\langle f_3 \rangle = \{e, f_3\}$
$\langle f_4 \rangle = \{e, f_4\}$
$K = \{e, r_2, f_1, f_2\}$
$K' = \{e, r_2, f_3, f_4\}$
$D_4 = \{e, r_1, r_2, r_3, f_1, f_2, f_3, f_4\}$



	e	r_2	f_1	f_2
e	e	r_2	f_1	f_2
r_2	r_2	e	f_2	f_1
f_1	f_1	f_2	e	r_2
f_2	f_2	f_1	r_2	e

$= K$

	e	r_2	f_3	f_4
e	e	r_2	f_3	f_4
r_2	r_2	e	f_4	f_3
f_3	f_3	f_4	e	r_2
f_4	f_4	f_3	r_2	e

$= K'$