Cavallaro, Jeffery
Math 221b
Final Exam

1). Let $K$ be the splitting field of $x^4 - x^2 + 1$ over $\mathbb{Q}$. Compute $\operatorname{Gal} K/\mathbb{Q}$ and find all subfields of $K$.

First, find the roots:

$$x^2 = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2} = e^{\pm i\frac{\pi}{3}}$$

$$x = \pm e^{\pm i\frac{\pi}{6}} = \pm\frac{\sqrt{3} \pm i}{2}$$

Thus $x^4 - x^2 + 1$ is irreducible over $\mathbb{Q}$ with $K = \mathbb{Q}(\sqrt{3}, i)$:

$$\begin{tikzcd}
\mathbb{Q}(\sqrt{3}, i) \arrow[d, dash, "2"] \\
\mathbb{Q}(\sqrt{3}) \arrow[d, dash, "2"] \\
\mathbb{Q}
\end{tikzcd}$$

Thus $[K : \mathbb{Q}] = 4$ and so $\operatorname{Gal}(K/\mathbb{Q})$ is either $V$ or $Z/4Z$. To determine which, consider the resolvant. Since $x^4 - x^2 + 1$ is already depressed, we have $p = -1$, $q = 0$, and $r = 1$ for a resolvant of:

$$h(x) = x^3 + 2x^2 - 3x = x(x^2 + 2x - 3) = x(x-1)(x+3)$$

Thus, $h(x)$ has three rational roots.

$\therefore \operatorname{Gal} K/\mathbb{Q} = V$

The corresponding subfield diagram is as follows:

$$\begin{tikzcd}
 & \mathbb{Q}(\sqrt{3}, i) \arrow[dl, dash, "2"'] \arrow[d, dash, "2"'] \arrow[dr, dash, "2"] & \\
\mathbb{Q}(i) \arrow[dr, dash, "2"'] & \mathbb{Q}(\sqrt{3}) \arrow[d, dash, "2"] & \mathbb{Q}(i\sqrt{3}) \arrow[dl, dash, "2"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

2). Let $K$ be the splitting field of $x^4+5x^2+5$ over $\mathbb{Q}$. Compute $\mathrm{Gal}(K/\mathbb{Q})$ and find all subfields of $K$.

By Eisenstein ($p=5$), $x^4+5x^2+5$ is irreducible over $\mathbb{Q}$. Find the roots:

$$x^2 = \frac{-5 \pm \sqrt{25-20}}{2} = \frac{-5 \pm \sqrt{5}}{2}$$

$$x = \pm\sqrt{\frac{-5 \pm \sqrt{5}}{2}} = \pm i\sqrt{\frac{5 \pm \sqrt{5}}{2}}$$

Let:

$r_1 = i\sqrt{\frac{5+\sqrt{5}}{2}}$
$r_2 = i\sqrt{\frac{5-\sqrt{5}}{2}}$
$r_3 = -i\sqrt{\frac{5+\sqrt{5}}{2}}$
$r_4 = -i\sqrt{\frac{5-\sqrt{5}}{2}}$

Consider $K = \mathbb{Q}(r_1)$:

$$r_1^2 = -\frac{5+\sqrt{5}}{2}$$

So:

$$\sqrt{5} = 2r_1^2 + 5 \in \mathbb{Q}(r_1)$$

Furthermore:

$$r_1 r_2 = -\frac{\sqrt{(5+\sqrt{5})(5-\sqrt{5})}}{2} = -\frac{\sqrt{20}}{2} = -\sqrt{5}$$

And so:

$$r_2 = -\frac{\sqrt{5}}{r_1} \in \mathbb{Q}(r_1)$$

Thus all of the roots are contained in $K = \mathbb{Q}(r_1)$:

$$\begin{array}{c} \mathbb{Q}(r_1) \\ \Big| \, 4 \\ \mathbb{Q} \end{array}$$

So $[K:\mathbb{Q}] = 4$ and may be either $\mathbb{Z}/4\mathbb{Z}$ or $V$. Next, since $x^4+5x^2+5$ is already depressed,consider the resolvant with $p=5$, $q=0$, and $r=5$:

$$h[x] = x^3 - 10x^2 + 5x = x(x^2-10x+5) = x[x-(5+2\sqrt{5})][x-(5-2\sqrt{5})]$$

So:

$\theta_1 = 0 \in \mathbb{Q}$
$\theta_2 = 5 + 2\sqrt{5} \notin \mathbb{Q}$
$\theta_3 = 5 - 2\sqrt{5} \notin \mathbb{Q}$

Thus, $\theta_1$ is fixed.

$\therefore \operatorname{Gal}(K/\mathbb{Q}) = \mathbb{Z}/4\mathbb{Z}$.

The corresponding subfield diagram is as follows:

$$\begin{tikzcd}
\mathbb{Q}(r_1) \arrow[d, dash, "2"] \\
\mathbb{Q}(\sqrt{5}) \arrow[d, dash, "2"] \\
\mathbb{Q}
\end{tikzcd}$$

3). Show that the angle $30°$ is constructable but not trisectable.

An angle $\theta$ is constructable iff $\sin\theta$ is constructable. $\sin(30°) = \frac{1}{2}$. But $Q\left(\frac{1}{2}\right) = \mathbb{Q}$.

Therefore an angle of $30°$ is constructable.

Now, let $\phi = 10°$ and $\theta = 3\phi = 30°$. From Euler's formula we have:

$$e^{i3\phi} = (e^{i\phi})^3 = (\cos\phi + i\sin\phi)^3 = \cos^3\phi + 3i\cos^2\phi\sin\phi - 3\cos\phi\sin^2\phi - i\sin^3\phi$$

Since we want $\sin\phi$, take the imaginary part:

$$\sin\theta = 3\cos^2\phi\sin\phi - \sin^3\phi = 3(1-\sin^2\phi)\sin\phi - \sin^3\phi = -4\sin^3\phi + 3\sin\phi$$

Thus, if $4x^3 - 3x + \sin\theta$ is irreducible over $\mathbb{Q}(\sin\theta)[x]$ then a field extension not a power of $2$ is required and thus $\sin\phi$ would not be constructable.

In this case, $\sin\theta = \frac{1}{2} \in \mathbb{Q}$ so

$$f(x) = 4x^3 - 3x + \frac{1}{2} = 2(8x^3 - 6x + 1)$$

must be factorable over $\mathbb{Q}$ for $\phi$ to be constructable. By the rational root test, the only possible rational roots are: $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}$:

| $x$ | $f(x)$ |
|---|---|
| $1$ | $3$ |
| $-1$ | $-1$ |
| $\frac{1}{2}$ | $-1$ |
| $-\frac{1}{2}$ | $3$ |
| $\frac{1}{4}$ | $-\frac{3}{8}$ |
| $-\frac{1}{4}$ | $\frac{19}{8}$ |
| $\frac{1}{8}$ | $\frac{17}{64}$ |
| $-\frac{1}{8}$ | $\frac{111}{64}$ |

Thus, $f(x)$ has no rational roots and so is irreducible over $\mathbb{Q}$.

Therefore $\phi = 10^\circ$ is not constructable.

4). Explain why $f(x) = x^5 - 2x^3 - 8x + 2$ is not solvable by radicals.

Let $K$ be the splitting field of $f(x)$ over $\mathbb{Q}$. By Eisenstein ($p = 2$), $f(x)$ is irreducible in $\mathbb{Q}$.

Since $f(x)$ has two sign changes, by Decartes, $f(x)$ has $0$ or $2$ positive real roots (not $4$, since complex roots must come in conjugate pairs). But $f(0) = 2$ and $(1) = -7$, thus there must be at least one, and therefore there are $2$.

Now $f(-x) = -x^5 + 2x^3 + 8x + 2$ has $1$ sign change and thus $f(x)$ has $0$ or $1$ negative real roots. There must be exactly $1$ since the complex roots must come in conjugate pairs.

Thus, $f(x)$ has $3$ real and $2$ complex roots, and so $\mathrm{Gal}(K/\mathbb{Q}) = S_5$. But we know that:

$$S_5 \geq S_5' = A_5$$

where $S_5'$ is the commutator group of $S_5$. But $A_5$ is simple and so the commutator chain is locked into $A_5$. Thus, $S_5$ is not solvable.

Therefore $f(x)$ is not solvable.