Gram Matrix

Theorem

Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space V with basis $\{\vec{f_1}, \dots, \vec{f_2}\}$ and let $G = \left[\left\langle \vec{f_j}, \vec{f_i} \right\rangle\right]$, which is called the *Gram matrix*:

G is positive definite.

Proof

Assume $\vec{x} \in \mathbb{C}^n$:

$$\begin{aligned}
\vec{x}^* G \vec{x} &= \left(\sum_{i=1}^{n} x_i \vec{e}_i\right)^* G \left(\sum_{j=1}^{n} x_j \vec{e}_j\right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{x_i} (\vec{e}_i^* G \vec{e}_j) x_j \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{x_i} x_j a_{ij} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{x_i} x_j \left\langle \vec{f_j}, \vec{f_i} \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \left\langle x_j \vec{f_j}, x_i \vec{f_i} \right\rangle \\
&= \left\langle \sum_{j=1}^{n} x_j \vec{f_j}, \sum_{i=1}^{n} x_i \vec{f_i} \right\rangle
\end{aligned}$$

Let $\vec{u} = \sum_{i=1}^{n} x_i \vec{f_i} \in V$:

$$\vec{x}^* G \vec{x} = \langle \vec{u}, \vec{u} \rangle \geq 0$$

Equality only holds when $\vec{u} = 0$, which means that all the $x_i = 0$ because the $\vec{f_i}$ are linearly independent, which means $\vec{x} = 0$

$\therefore G$ is positive definite.

Example

1). $V = \mathbb{C}^n$ and $\langle x, y \rangle = \sum_{k=1}^{n} \overline{y_i} x_i = y^* x$ and $\vec{f_i} = \vec{e}_i$

$$\left\langle \vec{f_i}, \vec{f_j} \right\rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

So $G = I_n$, which is positive definite.

2). Same as above, except use basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$G = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

3). V consists of polynomials of degree at most n

$\langle p(x), q(x) \rangle = \int_0^1 p(x)\overline{q(x)}dx$

Use basis $\{1, x, x^2, \ldots, x^n\}$

$\langle x^{j-1}, x^{i-1} \rangle = \int_0^1 x^{i+j-2}dx = \frac{1}{i+j-1} x^{i+j-1} \Big|_0^1 = \frac{1}{i+j+1}$

$$G = \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{bmatrix}$$