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Math 229
Homework #1

1.1.5

Let $A \in M_n$ be idempotent. Show that each eigenvalue of A is either 0 or 1. Explain why I_n is the only non-singular idempotent matrix.

Assume λ is an eigenvalue of A
$\exists\, \vec{x} \neq \vec{0}, A\vec{x} = \lambda\vec{x}$
$A(A\vec{x}) = A(\lambda\vec{x})$
$A^2\vec{x} = \lambda(A\vec{x})$
$A^2\vec{x} = \lambda(\lambda\vec{x})$
$A\vec{x} = \lambda^2\vec{x}$
$\lambda^2\vec{x} = \lambda\vec{x}$
$(\lambda^2 - \lambda)\vec{x} = \vec{0}$
$\lambda(\lambda - 1)\vec{x} = \vec{0}$
But $\vec{x}$ is an eigenvector and thus $\vec{x} \neq \vec{0}$, so:
$\lambda(\lambda - 1) = 0$

$\therefore \lambda = 0$ or 1

We know that $(I_n)^2 = I_n$ so I_n is idempotent. Also, $I_n I_n = I_n$, so I_n is its own inverse and is thus non-singular.

Now, assume A is idempotent. If A is singular then done. So AWLOG that A is non-singular (invertible):

$$\begin{aligned}
A^2 &= A \\
A^{-1}A^2 &= A^{-1}A \\
(A^{-1}A)A &= I_n \\
I_n A &= I_n \\
\therefore A &= I_n
\end{aligned}$$

Therefore I_n is the only non-singular idempotent matrix.

1.2.21

Let $A \in M_n$ and non-zero vectors $\vec{x}, \vec{v} \in \mathbb{C}^n$ be given. Suppose that $c \in \mathbb{C}$, $\vec{v}^*\vec{x} = 1$, $A\vec{x} = \lambda\vec{x}$, and $\mathrm{Sp}(A) = \{\lambda, \lambda_2, \ldots, \lambda_n\}$. Define the Google matrix by:

$$A(c) = cA + (1 - c)\lambda\vec{x}\vec{v}^*$$

Show that $\mathrm{Sp}(A(c)) = \{\lambda, c\lambda_2, \ldots, c\lambda_n\}$.

$A\vec{x} = \lambda\vec{x} \iff (cA)\vec{x} = (c\lambda)\vec{x}$
Therefore there is a one-to-one correspondence between the eigenvalues for A and those for cA, and so:

$$\mathrm{Sp}(cA) = \{c\lambda, c\lambda_2, \ldots, c\lambda_n\}$$

In class, we proved:

$$\mathrm{Sp}(A + \vec{x}\vec{y}^T) = \{\lambda + \vec{y}^T\vec{x}, \lambda_2, \ldots, \lambda_n\}$$

So taking $A = cA$, $\vec{x} = (1-c)\lambda\vec{x}$, and $y^T = \vec{v}^*$, we have:

$$\mathrm{Sp}(A(c)) = \mathrm{Sp}(cA + ((1-c)\lambda\vec{x})(\vec{v}^*)) = \{c\lambda + \vec{v}^*(1-c)\lambda\vec{x}, c\lambda_2, \ldots, c\lambda_n\}$$

Now, simplifying the expression for the first eigenvalue:

$$\begin{aligned} c\lambda + \vec{v}^*(1-c)\lambda\vec{x} &= c\lambda + (1-c)\lambda(\vec{v}^*\vec{x}) \\ &= c\lambda + (1-c)\lambda(1) \\ &= c\lambda + \lambda - c\lambda \\ &= \lambda \end{aligned}$$

Therefore:

$$\mathrm{Sp}(A(c)) = \{\lambda, c\lambda_2, \ldots, c\lambda_n\}$$

1.3.7

Show that every diagonalizable $B \in M_n$ has a square root.

Assume that B is diagonalizable
Then B is similar to some diagonal matrix D such that for some invertible matrix S:

$$B = SDS^{-1}$$

Now, let C be a diagonal matrix such that $C_{kk} = \sqrt{D_{kk}}$:

$$C^2 = D$$

And so:

$$B = SC^2S^{-1} = (SCS^{-1})(SCS^{-1})$$

Let $A = SCS^{-1}$

Therefore: $B = A^2$

Show that $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ does not have a square root.

ABC: B does have a square root $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then:

$$A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which results in the four equations:

$$\begin{aligned} a^2 + bc &= 0 \\ (a+d)b &= 1 \\ c(a+d) &= 0 \\ cb + d^2 &= 0 \end{aligned}$$

Assume $c = 0$. Then $a = d = 0$, which contradicts $(a+d)b = 1$
Assume $c \neq 0$. Then $a + d = 0$, which also contradicts $(a+d)b = 1$

Therefore, no such A exists.

1.3.13

Let A and B be diagonalizable matrices. Prove:

$$A \sim B \iff p_A(t) = p_b(t)$$

Since A is diagonalizable, there exists diagonal matrix D_A and invertible matrix S_A such that $D_A = S_A A S_A^{-1}$. Similarly, there exists diagonal matrix D_B and invertible matrix S_B such that $D_B = S_B B S_B^{-1}$.

$\implies$ Assume $A \sim B$

There exists invertible matrix S such that $A = SBS^{-1}$

$$\begin{aligned} p_A(t) &= \det(tI - A) \\ &= \det(tI - SBS^{-1}) \\ &= \det(tSS^{-1}I - SBS^{-1}) \\ &= \det(S(tI)S^{-1} - SBS^{-1}) \\ &= \det(S(tI - B)S^{-1}) \\ &= \det(S)\det(tI - B)\det(S^{-1}) \\ &= \det(S) \cdot \frac{1}{\det(S)} \cdot \det(tI - B) \\ &= \det(tI - B) \\ &= p_B(t) \end{aligned}$$

$\Longleftarrow$ Assume $p_A(t) = p_B(t)$

$$\begin{aligned}
\det(tI - A) &= \det(tI - B) \\
\det(tI - S_A^{-1} D_A S_A) &= \det(tI - S_B^{-1} D_B S_B) \\
\det(tS_A^{-1} S_A I - S_A^{-1} D_A S_A) &= \det(tS_B^{-1} S_B I - S_B^{-1} D_B S_B) \\
\det(S_A^{-1}(tI) S_A - S_A^{-1} D_A S_A) &= \det(S_B^{-1}(tI) S_B - S_B^{-1} D_B S_B) \\
\det(S_A^{-1}(tI - D_A) S_A) &= \det(S_B^{-1}(tI - D_B) S_B) \\
\det(S_A^{-1}) \det(tI - D_A) \det(S_A) &= \det(S_B^{-1}) \det(tI - D_B) \det(S_B) \\
\frac{1}{\det(S_A^{-1})} \cdot \det(S_A) \cdot \det(tI - D_A) &= \frac{1}{\det(S_B^{-1})} \cdot \det(S_B) \det(tI - D_B) \\
\det(tI - D_A) &= \det(tI - D_B)
\end{aligned}$$

Thus, $\mathrm{Sp}(D_A) = \mathrm{Sp}(D_B)$; however, the matching eigenvalues may be in different positions on the diagonals of D_A and D_B. But, using invertible permutation matrices, multiplication on the left (row) and right (column) can be used to reorder the diagonal of one to match the diagonal of the other. Also note that the product of invertible permutation matrices is still an invertible permutation matrix.

So, AWLOG, there exists a product of permutation matrices P such that:

$$D_A = P D_B P^{-1}$$

Substituting back we have:

$$\begin{aligned}
S_A A S_A^{-1} &= P S_B B S_B^{-1} P^{-1} \\
A &= (S_A^{-1} P S_B) B (S_B^{-1} P^{-1} S_A) \\
A &= (S_A^{-1} P S_B) B (S_A^{-1} P S_B)^{-1}
\end{aligned}$$

Finally, let $C = S_A^{-1} P S_B$
$A = CBC^{-1}$

$\therefore A \sim B$

The forward direction does not require diagonalizable; however, the reverse direction does. For a counterexample, we need two matrices with the same characteristic polynomial that are not similar.

Note that if two matrices are similar and if one is diagonalizable then the other must also be diagonalizable. To prove this, assume that $A \sim B$ and AWLOG that A is diagonalizable:

$$\begin{aligned}
A &= SBS^{-1} \\
TDT^{-1} &= SBS^{-1} \\
D &= T^{-1} SBS^{-1} T \\
D &= (T^{-1} S) B (T^{-1} S)^{-1}
\end{aligned}$$

Therefore B must also be diagonalizable.

Thus, we need to find two matrices with the same characteristic polynomial where only one is diagonalizable. Consider:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Note that $\sigma(A) = \sigma(B) = \{1\}$ and $p_A(t) = p_B(t) = (t-1)^2$, so the characteristic polynomials match. Furthermore, since $A = I_2$ and $I_2 I_2 I_2^{-1} = I_2$ it follows that A is diagonalizable.

To find $g_B(1)$, consider the nullity of $(B - I_2)$:

$$B - I_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

So the nullity of $B - I_2$ is 1 and thus $g_B(1) = 1$. But, $a_B(1) = 2$. Thus, $g_B(1) \neq a_B(1)$ and therefore B is not diagonalizable.

1.4.1

Let non-zero vectors $\vec{x}, \vec{y} \in \mathbb{C}^n$ be given, let $A = \vec{x}\vec{y}^*$ and let $\lambda = \vec{y}^*\vec{x}$.

a) Show that $\lambda \in \sigma(A)$

$A = \vec{x}\vec{y}^*$
$A\vec{x} = \vec{x}(\vec{y}^*\vec{x}) = \vec{x}\left[\lambda\right] = \lambda\vec{x}$

Therefore $\lambda \in \sigma(A)$.

b) Show that $\vec{x}$ is a right eigenvector and $\vec{y}$ is a left eigenvalue of A associated with λ.

By part (a), $\vec{x}$ is a right eigenvector of A associated with λ.

$A = \vec{x}\vec{y}^*$
$\vec{y}^*A = (\vec{y}^*\vec{x})\vec{y}^* = \lambda\vec{y}^*$

Therefore $\vec{y}$ is a left eigenvector of A associated with λ.

c) Show that if $\lambda \neq 0$ then it is the only non-zero eigenvalue of A.

Note that by the definition of matrix multiplication, we have:

$$A = xy^* = \left[\overline{y}_1\vec{x} \cdots \overline{y}_n\vec{x}\right]$$

Thus, the columns of A are all scalar multiples of $\vec{x}$ and so A is a rank-one matrix
Next, note that $\operatorname{Null}(A) = \operatorname{Eig}_A(0)$, but by the dimension theorem we have:

$$\operatorname{nullity}(A) = n - \operatorname{rank}(A) = n - 1$$

and $\operatorname{Eig}_A(0)$ has $n-1$ independent vectors, or $g_A(0) = n-1$ Finally, since $g_A(0) \leq a_A(0) \leq n$, it must be the case that: $a_A(0) = n-1$ or $a_A(0) = n$

Therefore, if $\lambda \neq 0$ then $a_A(\lambda) = 1$.