Ring Homomorphisms

Definition: Ring Homomorphism

Let R and S be rings. A *ring homomorphism* from R to S is a function $\phi : R \to S$ such that $\forall\, x, y \in R$:

$$\begin{aligned} \phi(x+y) &= \phi(x) + \phi(y) \\ \phi(xy) &= \phi(x)\phi(y) \end{aligned}$$

In other words, ϕ is a group homomorphism that preserves multiplication.

Definition: Kernel

Let $\phi : R \to S$ be homomorphism of rings. The *kernel* of ϕ, denoted $\ker(\phi)$, is given by:

$$\ker(\phi) = \{r \in R \mid \phi(r) = 0_S\}$$

Theorem

Let $\phi : R \to S$ be homomorphism of rings:

$$\ker(R) \leq R$$

Proof

By group theory we know $\phi(0_R) = 0_S$
Thus, $0 \in \ker(R)$ and $\ker(R) \neq \emptyset$

Assume $x, y \in \ker(R)$

$\phi(x - y) = \phi(x) - \phi(y) = 0 - 0 = 0$
$x - y \in \ker(R)$

$\phi(xy) = \phi(x)\phi(y) = 0 \cdot 0 = 0$
$xy \in \ker(R)$

Therefore, by the subring test, $\ker(R) \leq R$.

Definition: Kernel

Let $\phi : R \to S$ be homomorphism of rings. The *image* of ϕ, denoted $\mathrm{im}(\phi)$ or $\phi[R]$, is given by:

$$\mathrm{im}(\phi) = \{\phi(r) \mid r \in R\}$$

Theorem

Let $\phi : R \to S$ be homomorphism of rings:

$$\mathrm{im}(R) \leq S$$

<u>Proof</u>

By group theory we know $\phi(0_R) = 0_S$
Thus, $0_S \in \mathrm{im}(R)$ and $\mathrm{im}(R) \neq \emptyset$

Assume $u, v \in \mathrm{im}(R)$ $\exists\, x, y \in R$ such that $\phi(x) = u$ and $\phi(y) = v$

$u - v = \phi(x) - \phi(y) = \phi(x - y) \in S$
But by closure, $x - y \in R$

$\therefore u - v \in \mathrm{im}(R)$

$uv = \phi(x)\phi(y) = \phi(xy) \in S$
But by closure, $xy \in R$

$\therefore uv \in \mathrm{im}(R)$

Therefore, by the subring test, $\mathrm{im}(R) \leq S$.