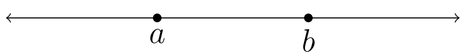
Inequalities

Definition

An inequality is two expressions separated by one of the four inequality signs:

$e1ope2$ where op is one of $<, \leq, \geq, >$

Recall that we have two definitions of "less than". Definition 1 is graphical: $a < b$ means that a occurs to the left of b on the real number line:



This leads us to all the graphical and interval-style notation that we have already seen, which you can review on pp 126-7 in your textbook.

The second definition is more analytical. We start by defining the set of positive real numbers:

Definition

$$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$$

Note that this set is closed under addition and multiplication.

Definition

To say that $a < b$ means $b - a \in \mathbb{R}^+$.

As long as we are working on the *same side* of an inequality then we can use all of our previous rules regarding the manipulation of expressions. But when we need to "do something to both sides", the rules are a little different.

Recall the properties of equality:

Properties

$\forall\, a, b, c \in \mathbb{R}$:

1). Reflexive: $a = a$

2). Symmetric: $a = b \implies b = a$

3). Transitive: $a = b$ and $b = c \implies a = c$

Properties

$\forall\, a, b, c, d \in \mathbb{R}$:

1). Not reflexive

Inequalities are not reflexive unless the “or equals to” part is included:

$a \nless a$
$a \leq a$

2). Not symmetric

Inequalities are not symmetric:

$a < b \;\not\Longrightarrow\; b < a$
$a \leq b \;\not\Longrightarrow\; b \leq a$, unless $a = b$

3). Transitive

$a < b$ and $b < c \implies a < c$

This seems to make sense using the graphical definition. How can we show this using the analytical definition:

Assume $a < b$ and $b < c$
$b - a \in \mathbb{R}^+$ and $c - b \in \mathbb{R}^+$ (definition)
$(b - a) + (c - b) \in \mathbb{R}^+$ (closure)
$c - a \in \mathbb{R}^+$ (axioms)
$a < c$ (definition)

The rest of the properties can be proved in this way.

4). Addition of a constant

$a < b \implies a + c < b + c$

Thus, like equality, we can add the same thing to both sides.

Using the graphical approach, this seems reasonable: if we translate a and b by the same amount then their relative positioning does not change.

5). Addition of inequalities

$a < b$ and $c < d \implies a + c < b + d$

Again, this sounds reasonable: if we translate a by some value but translate b by even more, then the gap widens.

Danger: This does not work with subtraction!

$-5 < 0$ and $1 < 2$ but $1 - (-5) = 6 \nless 1 - 2 = -1$

6). Multiplication by a constant

- $a < b$ and $c > 0 \implies ac < bc$
- $a < b$ and $c < 0 \implies ac > bc$

As long as you multiply both sides by a positive number, the inequality stays the same; however, if you multiply both sides by a negative number then the inequality flips the other direction!

$1 < 2$
$1(2) = 2 < 4 = 2(2)$
$1(-2) = -2 > -4 = 2(-2)$

Linear Inequalities

Remember: the answer is going to be a subset of the real number line, not individual numbers!

Example

$5x - 1 < 2x + 3$
$3x < 4$
$x < \frac{4}{3}$
$(-\infty, \frac{4}{3})$

Example

$1 - 5x \leq 2x + 3$
$-7x \leq 2$
$x \geq -\frac{7}{2}$
$[\frac{7}{2}, \infty)$

(

Polynomial Inequalities)

What do we do with something like:

$$x^2 - 3x - 4 > 0$$

1). Put in a form that compares against 0.

2). Factor.

$$(x-4)(x+1) > 0$$

3). Identify the 0 points (like an equality) and graph them and mark them as either included or excluded (depending on equality allowed):

$$x = -1, 4$$

4). Note that these expressions can only change sign by passing through zero, so make a sign table with test points to see how the sign changes.

test	$x-4$	$x+1$	sign
-2	$-$	$-$	$+$
0	$-$	$+$	$-$
5	$+$	$+$	$+$

5). Choose the intervals with the proper sign, in this case $+$:

$$(-\infty, -1) \cup (4, \infty)$$

Example

$2(x+1)(2-x)(x+3) \geq 0$

- Beware of turned-around factors:

$$-2(x+1)(x-2)(x+3) \geq 0$$

- Divide out leading factors, especially negative ones! If negative then remember to turn the sign around.

$$(x+1)(x-2)(x+3) \leq 0$$

- Solve:

$$(-\infty, -3] \cup [-1, 2]$$

- Beware of special conditions (p 141)
- Note that only odd factors will change sign:

$$(x+4)(x-1)^2(x+2)^3 \leq 0$$

$$[-4, -2]$$

Rational Inequalities

Zeros vs poles/discontinuities.

- Can also change sign across a discontinuity.

$$\frac{x^2 - 4x - 5}{x^2 - 7x + 10} \geq 0$$

- Watch for holes caused by cancelled factors. Answer: $(-\infty, -1] \cup [2, 5) \cup (5, \infty)$

- Cannot cross multiply across an inequality!
- Discontinuity points are never included; however zeros will be if equality is allowed.

$$\frac{x+6}{x+1} \leq 2$$

$$(-\infty, -1) \cup [4, \infty]$$