

Lemma

Let X and Y be topological spaces and let $f : X \rightarrow Y$. For all $B \subset Y$:

$$X - f^{-1}(B) = f^{-1}(Y - B)$$

Proof. Assume $A \subset Y$.

$$\begin{aligned} x \in X - f^{-1}(B) &\iff x \notin f^{-1}(B) \\ &\iff f(x) \notin B \\ &\iff f(x) \in Y - B \\ &\iff x \in f^{-1}(Y - B) \end{aligned}$$

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Theorem: 7.1

Let X and Y be topological spaces and let $f : X \rightarrow Y$. TFAE:

1. f is continuous.
2. For every closed set $K \subset Y$, $f^{-1}(K)$ is closed in X .
3. For all $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.
4. For every $x \in X$ and $V \in \mathcal{N}_{f(x)}$ there exists $U \in \mathcal{N}_x$ such that $f(U) \subset V$.

Proof.

1 \implies 2 Assume that f is continuous.

Assume that $K \subset Y$ is closed, and so $Y - K \in \mathcal{T}_Y$. Since f is continuous, $f^{-1}(Y - K) \in \mathcal{T}_X$. Now, applying the lemma, $f^{-1}(Y - K) = X - f^{-1}(K) \in \mathcal{T}_X$. Therefore $f^{-1}(K)$ is closed.

2 \implies 3 Assume that for every closed set $K \subset Y$, $f^{-1}(K)$ is closed in X .

Assume $A \subset X$. Since $\overline{f(A)}$ is closed, by the assumption, $f^{-1}(\overline{f(A)})$ is closed. Furthermore, since $f(A) \subset \overline{f(A)}$, it must be the case that $f^{-1}(f(A)) = A \subset f^{-1}(\overline{f(A)})$. But \bar{A} is the smallest closed set containing A , and so $\bar{A} \subset f^{-1}(\overline{f(A)})$. Therefore $f(\bar{A}) \subset f(f^{-1}(\overline{f(A)})) \subset \overline{f(A)}$.

3 \implies 4 Assume that for all $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.

Assume $x \in X$ and $V \in \mathcal{N}_{f(x)}$. Note that $Y - V$ is closed. Now, let $U = f^{-1}(V)$ and so $x \in U$ and $f(U) = f(f^{-1}(V)) \subset V$.

WTS: U open.

ABC that U is closed, and hence $X - U$ is open. So assume that $p \in \overline{X - U}$ but $p \notin X - U$. By the assumption (and the lemma):

$$f(p) \in f(\overline{X - U}) \subset \overline{f(X - U)} = \overline{f(X - f^{-1}(V))} = \overline{f(f^{-1}(Y - V))} \subset \overline{Y - V} = Y - V$$

This means that $p \in f^{-1}(Y - V) = X - f^{-1}(V) = X - U$, contradicting the assumption that $p \notin X - U$. Thus $X - U$ contains all of its limit points and is closed. Therefore U is open.

4 \implies 1 Assume that for every $x \in X$ and $V \in \mathcal{N}_{f(x)}$ there exists $U \in \mathcal{N}_x$ such that $f(U) \subset V$.

Assume $V \in \mathcal{T}_Y$ and assume $p \in f^{-1}(V)$. Thus $f(p) \in V \in \mathcal{N}_{f(p)}$. Now, by the assumption, there exists $U \in \mathcal{N}_p$ such that $f(U) \subset V$, and hence $f^{-1}(f(U)) = U \subset f^{-1}(V)$. This means that p is an interior point of $f^{-1}(V)$ and hence $f^{-1}(V)$ is open. Therefore f is continuous.

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Theorem: 7.2

Let X and Y be topological spaces and let $y_0 \in Y$. The constant map $f : X \rightarrow Y$ defined by $f(x) = y_0$ is continuous.

Proof. Assume that $x \in X$ and $V \in \mathcal{N}_{y_0}$. Now, assume that $U \in \mathcal{N}_x$:

$$f(U) = \{y_0\} \subset V$$

Therefore f is continuous.

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Theorem: 7.3

Let Y be a topological space and let X be a subspace of Y . The inclusion map $i : X \rightarrow Y$ defined by $i(x) = x$ is continuous.

Proof. Assume $V \in \mathcal{T}_Y$. Then $i^{-1}(V) = V \cap X \in \mathcal{T}_X$. Therefore i is continuous.

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Theorem: 7.5

A function $f : \mathbb{R}_{\text{std}} \rightarrow \mathbb{R}_{\text{std}}$ is continuous iff for every $x \in \mathbb{R}$ and $\epsilon > 0$ there exists $\delta > 0$ such that for every $y \in \mathbb{R}$:

$$d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon$$

Proof.

\implies Assume that f is continuous.

Assume $x \in \mathbb{R}$ and $\epsilon > 0$. Let $V = B(f(x), \epsilon) \in \mathcal{N}_{f(x)}$. Since f is continuous, there exists $U \in \mathcal{N}_x$ such that $f(U) \subset V$. But, since U is open, there exists $\delta > 0$ such that $B(x, \delta) \subset U$. Now, assume $y \in \mathbb{R}$ such that $d(x, y) < \delta$. This means $y \in B(x, \delta) \subset U \subset f^{-1}(V)$. Therefore $f(y) \in V$ and thus $d(f(x), f(y)) < \epsilon$.

\Leftarrow Assume for every $x \in \mathbb{R}$ and $\epsilon > 0$ there exists $\delta > 0$ such that for every $y \in \mathbb{R}$:

$$d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon$$

Assume $x \in \mathbb{R}$ and $V \in \mathcal{N}_{f(x)}$. Since $f(x)$ is an interior point of V , there exists $\epsilon > 0$ such that $B(f(x), \epsilon) \subset V$. But by the assumption, this means that there exists $\delta > 0$ such that $U = B(x, \delta) \subset f^{-1}(V)$. Therefore $f(U) \subset V$ and thus f is continuous. ■

Lemma

Let X be a 1st countable topological space, $A \subset X$, and $p \in A$. There exists a sequence (a_n) in A such that $a_n \rightarrow p$ iff $p \in \bar{A}$.

Proof.

\implies Assume that there exists a sequence (a_n) in A such that $a_n \rightarrow p$.

Assume that $U \in \mathcal{N}_p$. This means that there exists some $N \in \mathbb{N}$ such that for all $n > N$, $a_n \in U$. But $a_n \in A$ also, so $U \cap A \neq \emptyset$. Therefore $p \in \bar{A}$.

\Leftarrow Assume that $p \in \bar{A}$.

This means that for all $U \in \mathcal{N}_p$ it must be the case that $U \cap A \neq \emptyset$. Now, since X is 1st countable, assume that $\{B_k : k \in \mathbb{N}\}$ is a countable neighborhood basis for p . Define the collection $\{U_n : n \in \mathbb{N}\}$ such that:

$$U_n = \bigcap_{k=1}^n B_k$$

Note that each U_n is a finite intersection of open sets and so $U_n \in \mathcal{N}_p$. Furthermore, since $p \in U_n$ and $p \in \bar{A}$, it must be the case that $U_n \cap A \neq \emptyset$. So select $a_n \in U_n \cap A$. Therefore (a_n) is in sequence in A and $a_n \rightarrow p$. ■

Theorem: 7.6

Let X and Y be topological spaces such that X is 1st countable. $f : X \rightarrow Y$ is continuous iff for every convergent sequence $x_n \rightarrow x$ in X , $f(x_n) \rightarrow f(x)$ in Y .

Proof.

\implies Assume that f is continuous.

Assume that $f(x_n) \not\rightarrow f(x)$. This means that there exists a $V \in \mathcal{N}_{f(x)}$ such that for all $N \in \mathbb{N}$ there exists an $n > N$ such that $f(x_n) \notin V$. So $f(x_n) \in Y - V$ and hence $x_n \in f^{-1}(Y - V) = X - f^{-1}(V)$. Thus $x_n \notin f^{-1}(V)$. But f is continuous, so $f^{-1}(V) \in \mathcal{N}_x$. Let $U = f^{-1}(V)$. Therefore, there exists $U \in \mathcal{N}_x$ such that for all $N \in \mathbb{N}$ there exists $n > N$ such that $x_n \notin U$, and thus $x_n \not\rightarrow x$.

\impliedby Assume for all sequences (x_n) in A , $x_n \rightarrow x$ implies $f(x_n) \rightarrow f(x)$.

Assume $A \subset X$ and $x \in \bar{A}$, and hence $f(x) \in f(\bar{A})$. By the lemma, there exists a sequence (x_n) in A such that $x_n \rightarrow x$. Furthermore, by the assumption, $f(x_n) \rightarrow f(x)$. But $f(x_n) \in f(A)$ and so $f(x) \in \overline{f(A)}$. Therefore $f(\bar{A}) \subset \overline{f(A)}$ and thus f is continuous. ■

Theorem: 7.7

Let X and Y be topological spaces such that $D \subset X$ is dense and Y is Hausdorff. Let $f : X \rightarrow Y$ and $g : X \rightarrow Y$ be continuous such that $\forall d \in D, f(d) = g(d)$. Then $\forall x \in X, f(x) = g(x)$.

Proof. ABC that there exists $x \in X$ such that $f(x) \neq g(x)$. Now, since Y is Hausdorff, there exists $U \in \mathcal{N}_{f(x)}$ and $V \in \mathcal{N}_{g(x)}$ such that $U \cap V = \emptyset$. Furthermore, since f and g are continuous, $f^{-1}(U) \in \mathcal{N}_x$ and $f^{-1}(V) \in \mathcal{N}_x$. Since $x \in f^{-1}(U)$ and $x \in f^{-1}(V)$, this means that $f^{-1}(U) \cap f^{-1}(V) \neq \emptyset$, and so, since D is dense in X , there must exist $d \in D$ such that $d \in f^{-1}(U) \cap f^{-1}(V)$. But this means that $f(d) \in U \cap V$, contradicting the assumption that U and V are disjoint. Therefore $\forall x \in X, f(x) = g(x)$. ■

Lemma

Let X, Y, Z be topological spaces. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous then for all $W \subset Z$:

$$(g \circ f)^{-1}(W) = (f^{-1} \circ g^{-1})(W)$$

Proof. Assume $W \subset Z$.

$$\begin{aligned} x \in (g \circ f)^{-1}(W) &\iff (g \circ f)(x) \in W \\ &\iff g(f(x)) \in W \\ &\iff f(x) \in g^{-1}(W) \\ &\iff x \in f^{-1}(g^{-1}(W)) \\ &\iff x \in (f^{-1} \circ g^{-1})(W) \end{aligned}$$

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Theorem: 7.9

Let X, Y, Z be topological spaces. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous then their composition $g \circ f : X \rightarrow Z$ is continuous.

Proof. Assume that f and g are continuous and $W \in \mathcal{T}_Z$. Since g is continuous, $g^{-1}(W) \in \mathcal{T}_Y$. And, since f is continuous, $f^{-1}(g^{-1}(W)) = (f^{-1} \circ g^{-1})(W) = (g \circ f)^{-1}(W) \in \mathcal{T}_X$. Therefore $g \circ f$ is continuous. ■

Theorem: 7.10

Let X and Y be topological spaces such that $A \cup B = X$ for A, B closed in X and $f, g : A \rightarrow Y$ continuous functions that agree on $A \cap B$. The function $h : A \cup B \rightarrow Y$ defined by $h = f$ on A and $h = g$ on B is continuous.

Proof. Assume $K \subset Y$ is closed in Y . Since f and g are continuous, $f^{-1}(K)$ and $g^{-1}(K)$ are closed in X . Now, since f and g agree on $A \cap B$:

$$h^{-1}(K) = (h^{-1}(K) \cap A) \cup (h^{-1}(K) \cap B) = f^{-1}(K) \cup g^{-1}(K)$$

which is closed in X . Therefore h is continuous. ■