Integer Ordering

Definition

The set of *positive integers*, a subset of $\mathbb{Z}$, is given by:

$$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$$

Note that $\mathbb{Z}^+$ inherits from $\mathbb{Z}$ all those properties that do not involve inverses. In particular, $\mathbb{Z}^+$ is closed under addition and multiplication.

Definition

$\forall a, b \in \mathbb{Z}$:

- To say that a is less than b, denoted $a < b$, means $b - a$ is a positive value. This can also be stated as b is greater than a, denoted $b > a$.
- To say that a is less than or equal to b, denoted $a \leq b$, means $a < b$ or $a = b$: $b - a$ is either positive or zero. This can also be stated as b is greater than or equal to a, denoted $b \geq a$

Theorem

$\forall\, a \in \mathbb{Z}$:

- a is positive $\iff a > 0$
- a is negative $\iff a < 0$

Proof

Assume $a \in \mathbb{Z}$

$$\begin{aligned} a \text{ is positive} &\iff a \in \mathbb{Z}^+ \\ &\iff a - 0 \in \mathbb{Z}^+ \\ &\iff 0 < a \\ &\iff a > 0 \end{aligned} \qquad\qquad \begin{aligned} a \text{ is negative} &\iff -a \in \mathbb{Z}^+ \\ &\iff -a + 0 \in \mathbb{Z}^+ \\ &\iff 0 - a \in \mathbb{Z}^+ \\ &\iff a < 0 \end{aligned}$$

Thus, the trichotomy principle can be rewritten as follows:

$\forall\, n \in \mathbb{Z}$, exactly one of the following is true:

1). $n > 0$

2). $n = 0$

3). $n < 0$

Properties

$\forall a, b, c \in \mathbb{Z}$:

1). $a < b$ and $b < c \implies a < c$

2). $a < b$ and $c < d \implies a + c < b + d$

3). $a < b \iff a + c < b + c$

4). $c > 0 \implies (a < b \iff ac < bc)$

5). $c < 0 \implies (a < b \iff ac > bc)$

Note that all of the above properties hold if '$<$' is replaced with '$\leq$'.

Proof

Assume $a, b, c \in \mathbb{Z}$

1). Assume $a < b$ and $b < c$

$b - a \in \mathbb{Z}^+$ and $c - b \in \mathbb{Z}^+$
By closure $(b - a) + (c - b) \in \mathbb{Z}^+$
$(b - a) + (c - b) > 0$
$c - a > 0$
$\therefore a < c$

2). Assume $a < b$ and $c < d$

$b - a \in \mathbb{Z}^+$ and $d - c \in \mathbb{Z}^+$
By closure $(b - a) + (d - c) \in \mathbb{Z}^+$
$(b - a) + (d - c) > 0$
$(b + d) - (a + c) > 0$
$\therefore a + c < b + d$

3).

$$\begin{aligned} a < b &\iff b - a > 0 \\ &\iff b - a + 0 > 0 \\ &\iff b - a + c - c > 0 \\ &\iff (b + c) - (a + c) > 0 \\ &\iff a + c < b + c \end{aligned}$$

4). Assume $c > 0$

$c \in \mathbb{Z}^+$

$$\begin{aligned} a < b &\iff b - a \in \mathbb{Z}^+ \\ &\iff c(b-a) \in \mathbb{Z}^+ \\ &\iff c(b-a) > 0 \\ &\iff bc - ac > 0 \\ &\iff ac < bc \end{aligned}$$

5). Assume $c < 0$

$0 - c = -c \in \mathbb{Z}^+$

$$\begin{aligned} a < b &\iff b - a \in \mathbb{Z}^+ \\ &\iff (-c)(b-a) \in \mathbb{Z}^+ \\ &\iff (-c)(b-a) > 0 \\ &\iff ac - bc > 0 \\ &\iff bc < ac \\ &\iff ac > bc \end{aligned}$$

Theorem

$\forall\, a, k \in \mathbb{Z}^+, ka \geq a$

Proof

Assume $k, a \in \mathbb{Z}^+$
ABC: $ka < a$
$ka - a = a(k-1) < 0$

Case 1: $k = 1$

$a(1-1) = a0 = 0$ CONTRADICTION!

Case 2: $k > 1$

$k - 1 > 0$
$k - 1 \in \mathbb{Z}^+$
$a(k-1) \in \mathbb{Z}^+$
$a(k-1) > 0$
CONTRADICTION!

$\therefore ka \geq a$

Definition

Let $S \subseteq \mathbb{Z}$:

- To say that S has a *minimum* element means:

$$\exists\, m \in S, \forall\, n \in S, m \leq n$$

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$$\exists\, m \in S, \forall\, n \in S, n \leq m$$

Axiom: Well-ordering Principle

Every non-empty subset of Z^+ has a minimum value.

In fact, any ordered set that has this property is said to be *well-ordered*.