Math-1003b Homework #1 Solutions

Reading

- Really focus on your lecture notes for the information on rational numbers.
- Review textbook chapters 1, 2, and 5.
- Text book sections 7.1 and 7.2.

Problems

1). Questions about rational numbers. Be sure to explain why a statement is either true or false. Provide and explain counterexamples for all false statements.

a). Describe the three forms of a rational number covered in class and give an example of each.

1. Fractional: $\frac{p}{q}$ where p and q are integers and $q \neq 0$. Example: $\frac{2}{3}$
2. Finite Decimal. Example: 1.23
3. Infinite Repeating Decimal. Example: $1.\overline{23}$

b). Is zero a rational number?

Yes, because it can be written as $\frac{0}{1}$ (different syntax, same semantic), which fulfills the definition of a rational number.

c). Is $\frac{\sqrt{9}}{2}$ a rational number?

Yes, because it is the same as $\frac{3}{2}$ (once again, different syntax, same semantic), which is a rational number.

d). Is every fraction a rational number?

No. Counterexample: $\frac{\pi}{2}$. Since π is not an integer, the fraction does not meet the requirements for a rational number.

e). Based on the class discussion of rational numbers (i.e., the meaning of $\frac{p}{q}$ on the number line), explain why zero can never appear in the denominator.

By the definition of division, $\frac{p}{q} = p\left(\frac{1}{q}\right)$. Thus, we divide the space on the number line between 0 and 1 into q finite partitions, each of length $\frac{1}{q}$, and then take p of them. You cannot divide the space between 0 and 1 into 0 finite partitions.

2). Consider the numbers 120 and 252.

a). State the prime factorization for each.

$120 = 2^3 \cdot 3 \cdot 5$

$252 = 2^2 \cdot 3^2 \cdot 7$

When we take the GCD and LCM of these two numbers, we assume that any missing primes in one of the numbers actually has a power of 0. Thus, it may be helpful to view the prime factorizations as:

$120 = 2^3 \cdot 3^1 \cdot 5^1 \cdot 7^0$

$252 = 2^2 \cdot 3^2 \cdot 5^0 \cdot 7^1$

b). Show how to determine and then state their least common multiple (LCM).

For the LCM we take the highest power of each prime:

$[120, 252] = 2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1 = 2520$

c). Show how to determine and then state their greatest common denominator (GCD).

For the GCD we take the lowest power of each prime:

$(120, 252) = 2^2 \cdot 3^1 \cdot 5^0 \cdot 7^0 = 12$

d). Show how to simplify $\frac{120}{252}$ using the above information.

To reduce to simplest form, divide the numerator and the denominator by the GCD:

$\frac{120}{252} = \frac{\frac{120}{12}}{\frac{252}{12}} = \frac{10}{21}$

Alternatively, use the prime factorizations and the exponent rule:

$\frac{120}{252} = \frac{2^3 \cdot 3 \cdot 5}{2^2 \cdot 3^2 \cdot 7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$

3). Rewrite $4 - 9x^2$ by factoring out $(-3x)$.

Remember that in the reals, we can factor any non-zero value out of any other value. To start, multiply by $\frac{-3x}{-3x} = 1$:

$$4 - 9x^2 = 1(4 - 9x^2) = \frac{-3x}{-3x}(4 - 9x^2)$$

Note that since we multiplied by 1 (the multiplicative identity), we do not change the value of the expression. Now apply the definition of division and the associative rule:

$$4 - 9x^2 = (-3x)\left[\frac{1}{-3x}(4 - 9x^2)\right]$$

We now use the distributive rule:

$$4 - 9x^2 = (-3x)\left(\frac{4}{-3x} + \frac{-9x^2}{-3x}\right) = (-3x)\left(-\frac{4}{3x} + 3x\right)$$

When done, always check by working backwards and seeing if you get the original expression.

4). Operations on rational expressions:

a). Fully factor:

$$\frac{2x^2+5x+2}{x^2-3x} = \frac{(2x+1)(x+2)}{x(x-3)}$$

b). Fully factor:

$$\frac{2x^3+4x^2}{x^2-9} = \frac{2x^2(x+2)}{(x+3)(x-3)}$$

c). Determine and fully simplify:

$$\begin{aligned}
\frac{2x^2+5x+2}{x^2-3x} \cdot \frac{2x^3+4x^2}{x^2-9} &= \frac{(2x+1)(x+2)}{x(x-3)} \cdot \frac{2x^2(x+2)}{(x+3)(x-3)} \\
&= \frac{(2x+1)(x+2)(2x^2)(x+2)}{x(x-3)(x+3)(x-3)} \\
&= \frac{2x^2(x+2)^2(2x+1)}{x(x+3)(x-3)^2} \\
&= \frac{2x(x+2)^2(2x+1)}{(x+3)(x-3)^2}
\end{aligned}$$

d). Determine and fully simplify:

$$\begin{aligned}
\frac{2x^2+5x+2}{x^2-3x} \div \frac{2x^3+4x^2}{x^2-9} &= \frac{2x^2+5x+2}{x^2-3x} \cdot \frac{x^2-9}{2x^3+4x^2} \\
&= \frac{(2x+1)(x+2)}{x(x-3)} \cdot \frac{(x+3)(x-3)}{2x^2(x+2)} \\
&= \frac{(2x+1)(x+2)(x+3)(x-3)}{x(x-3)(2x^2)(x+2)} \\
&= \frac{(2x+1)(x+2)(x+3)(x-3)}{2x^3(x-3)(x+2)} \\
&= \frac{(2x+1)(x+3)}{2x^3}
\end{aligned}$$