MATH 231B, FALL 2017
HOMEWORK 1 SOLUTIONS

1. (Sec. 1.7, ex. 9) Prove that ℓ^p is a proper vector subspace of ℓ^q whenever $1 \le p < q$.

Proof: First let us show that $\ell^p \subset \ell^q$ for $1 \le p < q < \infty$. Suppose $x = (x_n) \in \ell^p$. Then $\sum_n |x_n|^p < \infty$, so $x_n \to 0$, as $n \to \infty$. Thus there exists N such that $|x_n| \le 1$, for all $n > N$. It follows that $|x_n|^q \le |x_n|^p$, for all $n > N$ and therefore $\sum_n |x_n|^q < \infty$. (Note that finitely many terms of a series have no effect on its convergence.) Thus $x \in \ell_q$ proving that $\ell_p \subset \ell^q$.

Now let us show that $\ell^p \neq \ell^q$. Define a sequence $x = (x_n)$ by

$$x_n = \frac{1}{n^{1/p}}.$$

Then $x_n^q = 1/n^{q/p}$, so $\sum |x_n|^q < \infty$, since $q/p > 1$. However, $\sum |x_n|^p = \sum 1/n = \infty$. Therefore, $x \in \ell^q$, but $x \notin \ell_p$. $\square$

2. (Sec. 1.7, ex. 14) Prove that spaces $C(\Omega), C^k(\mathbb{R}^n), C^\infty(\mathbb{R}^n)$ are infinite dimensional.

Proof: Since

$$C^\infty(\mathbb{R}^n) \subset C^k(\mathbb{R}^n) \subset C(\mathbb{R}^n),$$

it is enough to show that $C^\infty(\mathbb{R}^n)$ is infinite dimensional. To do that, it suffices to show $C^\infty(\mathbb{R}^n)$ contains an infinite linearly independent set.

First, let us assume $n = 1$. Let $f_n(t) = t^n$ and set $S = \{f_n : n \ge 0\}$. It is clear that $S \subset C^\infty(\mathbb{R})$. We claim that S is linearly independent. Indeed, assume that some linear combination f of $f_{n_1}, \dots, f_{n_k}$ is zero (as a function). Since f is a polynomial,

$$f(t) = \alpha_1 t^{n_1} + \cdots + \alpha_k t^{n_k}$$

it can only be identically zero iff $\alpha_j = 0$, for all j. (Recall that a non-trivial polynomial of degree N has exactly N complex zeros.) This proves that S is linearly independent and hence $C^\infty(\mathbb{R})$ is infinite dimensional.

If $n > 1$, then it can be similarly shown that the set of monomials $x_1^{k_1} \cdots x_n^{k_n}$ $(k_1, \dots, k_n \ge 0)$ is a linearly independent (and clearly infinite) set, making $C^\infty(\mathbb{R})^n$ infinite dimensional. $\square$

3. (Sec. 1.7, ex. 15) Denote by ℓ_0 the space of all infinite sequences of complex numbers (z_n) such that $z_n = 0$ for all but a finite number of indices n. Find a basis of ℓ_0.

Solution: Let e_n be the sequence with 1 in the n^{th} place and zeros elsewhere. We claim that $B = \{e_n : n \ge 1\}$ is a basis for ℓ_0. It is clear that B is linearly independent. Let us show that every element of ℓ_0 is a finite linear combination of elements of B. Indeed, let $x = (x_n) \in \ell_0$ be arbitrary. There exists N such that $x_n = 0$, for $n > N$. Then

$$x = \sum_{k=1}^{N} x_k e_k.$$

This completes the proof. $\square$

4. (Sec. 1.7, ex. 44) Consider the space $C([a,b])$ with the norm defined as $\|f\| = \int_a^b |f(t)|\ dt$. Is this a Banach space?

Solution: No. Let $a = 0, b = 1$ and consider the sequence $f_n(t) = t^n$, $n \geq 1$. We claim that (f_n) is Cauchy. Let $m > n$. Then $t^m \leq t^n$ for all $0 \leq t \leq 1$, so

$$\begin{aligned} \|f_m - f_n\| &= \int_0^1 |t^m - t^n|\ dt \\ &= \int_0^1 (t^n - t^m)\ dt \\ &= \frac{1}{n} - \frac{1}{m} \\ &\to 0, \end{aligned}$$

as $n \to \infty$. This proves our claim. However,

$$f(t) := \lim_{n\to\infty} f_n(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t = 1, \end{cases}$$

so (f_n) converges to a function which is not in $C[0,1]$. (Note, by the way, that $\|f_n - f\| \to 0$, as $n \to \infty$, so $f_n \to f$ in the L^1-norm.) Thus the space $C[0,1]$ is not Banach with respect to the given norm. $\square$

5. (Sec. 1.7, ex. 45) Show that $L(f)(x) = \int_0^x f(t)\ dt$ defines a continuous linear mapping from the space $C([0,1])$ into itself.

Proof: Since integration is linear, it is not hard to see that L is linear as well. If $f \in C[0,1]$, then by the Fundamental Theorem of Calculus $L(f)$ is differentiable at every point, hence $L(f) \in C[0,1]$. Thus $L : C[0,1] \to C[0,1]$. To show that L is bounded, observe that

$$\begin{aligned} |L(f)(x)| &\leq \int_0^x |f(t)|\ dt \\ &\leq \int_0^1 \|f\|\ dt \\ &= \|f\|, \end{aligned}$$

where $\|f\| = \max_{[0,1]} |f|$. Thus L is bounded. Since $\|L(1)\| = \|\text{identity}\| = 1$, it follows that $\|L\| = 1$. $\square$

6. (Sec. 1.7, ex. 46) Give an example of a linear mapping from a normed space into a normed space which is not continuous.

Solution: Consider $D : C^1[a,b] \to C[a,b]$ (both spaces equipped with the sup-norm) defined by $Df = f'$. It was shown in one of the worksheets that D is unbounded, hence discontinuous.

Another example is this. Define $L : C[0,1] \to \mathbb{R}$ by $L(f) = f(0)$, where $C[0,1]$ is equipped with the L^1-norm $\|f\| = \int_0^1 |f(t)|\ dt$. It is not hard to see that L is linear. For each $n \geq 1$ consider the function $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(t) = \begin{cases} 2n - 2n^2 t & \text{if } 0 \leq t \leq \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < t \leq 1. \end{cases}$$

Clearly, $f_n \in C[0,1]$. Furthermore, $\|f_n\| = 1$, but $L(f_n) = 2n$, for all n. This proves that L is unbounded.

7. Let $E = C^\infty([a,b])$ be the space of all infinitely differentiable functions on the interval $[a,b]$ with $\|f\| = \max_{[a,b]} |f(x)|$. Is the differential operator $D = \frac{d}{dx}$ a contraction mapping?

Solution: No. First note that if T is a linear map between normed spaces, then

$$\|Tx - Tx\| = \|T(x-y)\| \leq \|T\|\|x-y\|,$$

for all x, y. So T is a contraction iff $\|T\| < 1$.

We showed in a worksheet that D is unbounded. Recall that proof: we set $f_n(x) = \sin nx$, for $n = 1, 2, \cdots$, where we take, e.g., $[a,b] = [-1,1]$. Clearly, $f_n \in C^\infty[-1,1]$. Then $\|f_n\| = 1$, for all $n \geq 2$, but $\|Df_n\| \geq n$, for all n.

Since $\|D\| = \infty \not< 1$, D is not a contraction. $\square$