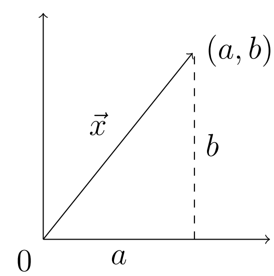
Vectors

Definition

A *vector* $\vec{x}$ in 2D (or 3D) space is a quantity with both *magnitude* and *direction*. Vectors with the same magnitude and direction are considered equal, regardless of position.

The zero vector, denoted $\vec{0}$, is the vector with zero magnitude and no direction.

Graphically, a non-zero vector is represented by a line terminated by an arrow. The length of the line indicates the magnitude and the arrow indicates the direction. When a vector is positioned with its tail at the origin, its head coincides with a *terminal point*:



$$\vec{x} = (a, b)$$

Analytically, a vector is represented by the coordinates of its terminal point (when originating from the origin), regardless of its actual position.

The zero vector has all zero components: $\vec{0} = (0, 0)$.

Notation

The magnitude of a non-zero vector $\vec{x} = (a, b)$, denoted $\|\vec{x}\|$, can be determined using the Pythagorean Theorem:

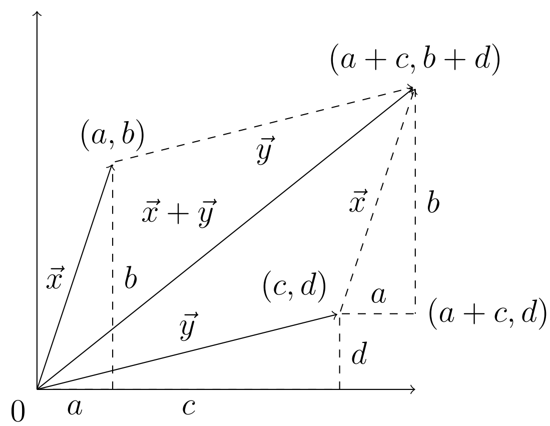
$$\|\vec{x}\| = \sqrt{a^2 + b^2}$$

The zero vector has zero magnitude:

$$\|\vec{0}\| = 0$$

Vector Addition

Graphically, vectors are added by placing them head-to-tail or by using the parallelogram method:



Analytically, vectors are added using component-wise addition of their terminal points:

$$\begin{aligned}\vec{x} &= (a, b)\\ \vec{y} &= (c, d)\\ \vec{x}+\vec{y} &= (a+c, b+d)\end{aligned}$$

Theorem

The zero vector acts as an additive identity:

$$\vec{x}+\vec{0}=\vec{0}+\vec{x}=\vec{x}$$

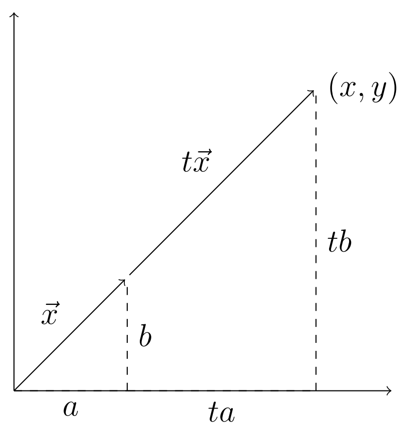
Proof

Let $\vec{x}=(a, b)$
$\vec{x}+\vec{0}=(a, b)+(0,0)=(a+0, b+0)=(a, b)=\vec{x}$
$\vec{0}+\vec{x}=(0,0)+(a, b)=(0+a, 0+b)=(a, b)=\vec{x}$

Scalar Multiplication

A vector can be multiplied by a real number, called a scalar, to change its magnitude and/or direction:

$t > 0$: Increases/decreases the magnitude by a factor of t
$t = 0$: Results in the zero vector
$t < 0$: Increases/decreases the magnitude by a factor of $|t|$ and reverses the direction



$\vec{x} = (a, b)$

$\frac{\|\vec{x}\|}{t\|\vec{x}\|} = \frac{a}{x} \implies x = ta$

$\frac{\|\vec{x}\|}{t\|\vec{x}\|} = \frac{b}{y} \implies y = tb$

$t\vec{x} = (ta, tb)$

Definition

To say that two non-zero vectors $\vec{x}$ and $\vec{y}$ are *parallel* means there exists some non-zero scalar t such that $\vec{y} = t\vec{x}$. If $t > 0$ then the vectors are in the same direction. If $t < 0$ then the vectors are in reverse directions.

Notation

The vector $(-1)\vec{x}$, denoted $-\vec{x}$, is a vector parallel to $\vec{x}$ with the same magnitude as $\vec{x}$, but with the opposite direction of $\vec{x}$.

Theorem

For every vector $\vec{x}$, the vector $-\vec{x}$ acts as an additive inverse:

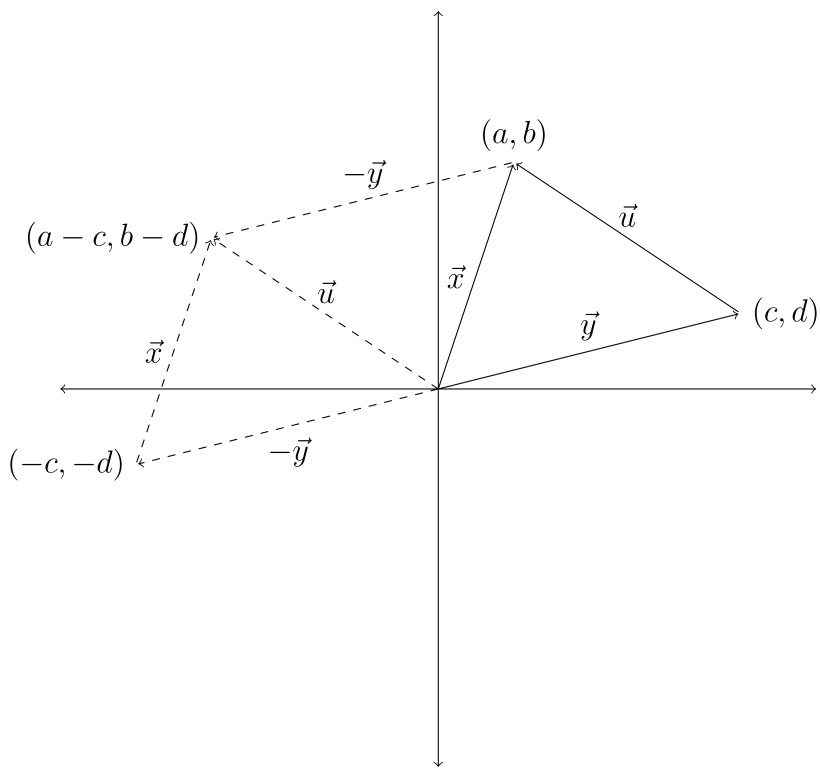
$$\vec{x} + (-\vec{x}) = (-\vec{x}) + \vec{x} = \vec{0}$$

Proof

Let $\vec{x} = (a, b)$
$-\vec{x} = (-1)(a, b) = (-a, -b)$
$\vec{x} + (-\vec{x}) = (a, b) + (-a, -b) = (a - a, b - b) = (0, 0) = \vec{0}$
$(-\vec{x}) + \vec{x} = (-a, -b) + (a, b) = (-a + a, -b + b) = (0, 0) = \vec{0}$

Vector Subtraction

Using vector addition, inverses and identity, we can develop a notion of vector subtraction:

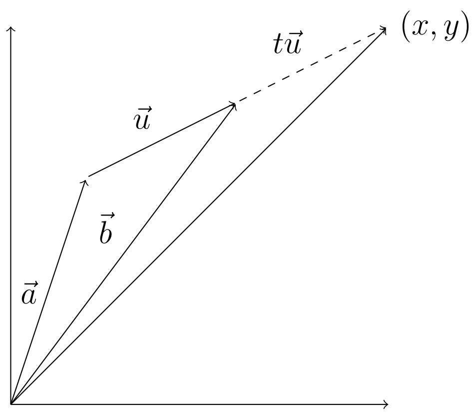


$$\begin{aligned} \vec{y}+\vec{u} &= \vec{x} \\ \vec{u} &= \vec{x}-\vec{y} \\ \vec{u}=(a,b)-(c,d) &= (a-c, b-d) \end{aligned}$$

Geometry

Lines

Problem: Find the vector equation for the line that passes through the terminal points of two vectors $\vec{a}$ and $\vec{b}$.

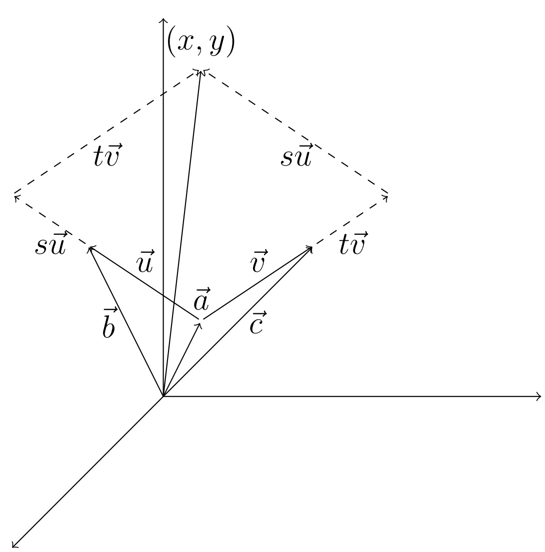


$$\vec{u} = \vec{b} - \vec{a}$$

$$(x, y) = \vec{a} + t\vec{u} = \vec{a} + t(\vec{b} - \vec{a})$$

Planes

Problem: Find the vector equation for the plane that passes through the terminal points of three non-colinear vectors $\vec{a}, \vec{b}$, and $\vec{c}$.



$$\vec{u} = \vec{b} - \vec{a}$$
$$\vec{v} = \vec{c} - \vec{a}$$

$$(x, y) = \vec{a} + s\vec{u} + t\vec{v} = \vec{a} + s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{a})$$