Unitary Similarity

Definition: Unitary Similarity

Let $A, B \in M_n$. To say that A is *unitary similar* to B means there exists a unitary matrix U such that:

$$B = UAU^*$$

Theorem

Let $A, B \in M_n$ be unitary similar:

$$\mathrm{tr}(AA^*) = \mathrm{tr}(BB^*)$$

Proof

There exists unitary matrix U such that $B = UAU^*$
$BB^* = (UAU^*)(UAU^*)^* = UAU^*UA^*U^* = (UA)(A^*U^*)$
$\mathrm{tr}(BB^*) = \mathrm{tr}((UA)(A^*U^*)) = \mathrm{tr}((A^*U^*)(UA)) = \mathrm{tr}(A^*A) = \mathrm{tr}(AA^*)$

Thus, we can rule out unitary similarity if $\mathrm{tr}(AA^*) \neq \mathrm{tr}(BB^*)$.

Theorem: Frobenius Norm

Let $A \in M_n$:

$$\mathrm{tr}(AA^*) = \sum_{1 \le i,j \le n} |A_{ij}|^2$$

Proof

$A = [a_{ij}]$
$A^* = [\overline{a_{ji}}]$
$(A^*A)_{ij} = \sum_{k=1}^{n} (A^*)_{ik} A_{kj} = \sum_{k=1}^{n} \overline{a_{ki}} a_{kj}$
$(A^*A)_{ii} = \sum_{k=1}^{n} \overline{a_{ki}} a_{ki} = \sum_{k=1}^{n} |a_{ki}|^2$
$\therefore \mathrm{tr}(A^*A) = \sum_{i=1}^{n} \sum_{k=1}^{n} |a_{ki}|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2$

Example

Let $A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

These matrices are similar; however:

$\mathrm{tr}(AA^*) = 3^2 + 1^2 + (-2)^2 + 0^2 = 14$

$\mathrm{tr}(BB^*) = 1^2 + 1^2 + 0^2 + 2^2 = 6$

Thus, A and B are not unitary similar.