Boundaries

Notation

Let $(X, \mathscr{T})$ be a topological space and let $A \subset X$:

$$\mathcal{U}_A = \{U \in \mathscr{T} \,|\, U \subset A\}$$

Definition: Interior

Let $(X, \mathscr{T})$ be a topological space and let $A \subset X$. The *interior* of A, denoted by $\mathrm{Int}(A)$, is given by:

$$\mathrm{Int}(A) = \bigcup \mathcal{U}_A$$

To say that a $p \in X$ is an *interior point* of A means that $p \in \mathrm{Int}(A)$.

Theorem

Let $(X, \mathscr{T})$ be a topological space, $A \subset X$, and $p \in X$. p is an interior point of A iff there exists $U \in \mathscr{T}$ such that $p \in U \subset A$.

Proof. $p \in \mathrm{Int}(A) \iff p \in \bigcup \mathcal{U}_A \iff \exists U \in \mathcal{U}_A, p \in U \subset A$ ∎

Theorem

Let $(X, \mathscr{T})$ be a topological space and $U \subset \mathscr{T}$:

$$U \in \mathscr{T} \iff \forall p \in U, p \in \mathrm{Int}(U)$$

Proof. $\mathcal{U} \in T \iff \forall p \in U, \exists U_p \in \mathscr{T}, p \in U_p \subset U \iff \forall p \in U, p \in \mathrm{Int}(U)$ ∎

Definition: Boundary

Let $(X, \mathscr{T})$ be a topological space and let $A \subset X$. The *boundary* or A, denoted by $\mathrm{Bd}(A)$, is given by:

$$\mathrm{Bd}(A) = \bar{A} \cap \overline{X - A}$$

Theorem

Let $(X, \mathscr{T})$ be a topological space and let $A \subset X$. $\mathrm{Int}(A)$, $\mathrm{Bd}(A)$, and $\mathrm{Int}(X - A)$ are disjoint sets whose union is X.

Proof. Assume that $p \in \mathrm{Int}(A)$. This means that there exists $U \in \mathcal{U}_A$ such that $p \in U \subset A$. Now ABC that $p \in \mathrm{Bd}(A)$. This means that $p \in \overline{X - A}$ and so for all $U \in \mathcal{U}_p, U \cap (X - A) \neq \emptyset$. This contradicts the fact that there exists a $U \in \mathcal{U}_p$ that is a subset of A.

Therefore $\mathrm{Int}(A) \cap \mathrm{Bd}(A) = \emptyset$.

Similarly, assume that $p \in \mathrm{Int}(X - A)$. This means that there exists $U \in \mathcal{U}_{X-A}$ such that $p \in U \subset (X - A)$. Now ABC that $p \in \mathrm{Bd}(A)$. This means that $p \in \bar{A}$ and so for all $U \in \mathcal{U}_p, U \cap A \neq \emptyset$. This contradicts the fact that there exists a $U \in \mathcal{U}_p$ that is a subset of $X - A$.

Therefore $\mathrm{Int}(X - A) \cap \mathrm{Bd}(A) = \emptyset$.

Finally, note that for all $U \in \mathcal{U}_p$, U cannot be a subset of both A and $X - A$.

Therefore $\mathrm{Int}(A) \cap \mathrm{Int}(X - A) = \emptyset$.

Clearly, $\mathrm{Int}(A) \cup \mathrm{Int}(X - A) \cup \mathrm{Bd}(A) \subset X$. Assume that $p \in X$. If $p \in \mathrm{Int}(A)$ or $p \in \mathrm{Int}(X - A)$ then done, so assume that X is in neither. This means that for all $U \in \mathcal{U}_p$, $U \cap A \neq \emptyset$ and $U \cap (X - A) \neq \emptyset$, and thus $p \in \bar{A}$ and $p \in \overline{X - A}$.

Therefore, $p \in \mathrm{Bd}(A)$. ■

Example

Pick several different subsets A of $\mathbb{R}$ and for each one find its interior and boundary using:

1. The discrete topology.

   Since $A = \bar{A}$ and $\mathbb{R} - A = \overline{\mathbb{R} - A}$, $\mathrm{Bd}(A) = A \cap (\mathbb{R} - A) = \emptyset$. Therefore $\mathrm{Int}(A) = A$.

2. The indiscrete topology.

$$\mathrm{Int}(A) = \begin{cases} \emptyset, & A \neq \mathbb{R} \\ \mathbb{R}, & A = \mathbb{R} \end{cases}$$

$$\bar{A} = \begin{cases} \emptyset, & A = \emptyset \\ \mathbb{R}, & A \neq \emptyset \end{cases}$$

$$\overline{\mathbb{R} - A} = \begin{cases} \emptyset, & A = \mathbb{R} \\ \mathbb{R}, & A \neq \mathbb{R} \end{cases}$$

$$\mathrm{Bd}(A) = \begin{cases} \emptyset \cap \mathbb{R} = \emptyset, & A = \emptyset \\ \mathbb{R} \cap \mathbb{R} = \mathbb{R}, & A \neq \emptyset, \mathbb{R} \\ \mathbb{R} \cap \emptyset = \emptyset, & A = \mathbb{R} \end{cases}$$

3. The cofinite topology.

Assume that A is finite (closed):

$$\text{Int}(A) = \emptyset$$
$$\bar{A} = A$$
$$\text{Int}(\mathbb{R} - A) = \mathbb{R} - A$$
$$\overline{\mathbb{R} - A} = \mathbb{R}$$
$$\text{Bd}(A) = A \cap \mathbb{R} = A$$

Assume that $A = \mathbb{R} - F$ where F is finite (thus A is open and F is closed):

$$\text{Int}(A) = A$$
$$\bar{A} = \mathbb{R}$$
$$\text{Int}(\mathbb{R} - A) = \emptyset$$
$$\overline{\mathbb{R} - A} = F$$
$$\text{Bd}(A) = \mathbb{R} \cap F = F$$

Assume that $A = \mathbb{Z}$:

$$\text{Int}(\mathbb{Z}) = \emptyset$$
$$\bar{\mathbb{Z}} = \mathbb{R}$$
$$\text{Int}(\mathbb{R} - \mathbb{Z}) = \emptyset$$
$$\overline{\mathbb{R} - \mathbb{Z}} = \mathbb{R}$$
$$\text{Bd}(Z) = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$$

4. The standard topology.

Assume that $A = (a, b)$:

$$\text{Int}(A) = A$$
$$\bar{A} = [a, b]$$
$$\text{Int}(\mathbb{R} - A) = (-\infty, a) \cup (b, \infty)$$
$$\bar{(}\mathbb{R} - A) = (-\infty, a] \cup [b, \infty)$$
$$\text{Bd}(A) = [a, b] \cap (-\infty, a] \cup [b, \infty) = \{a, b\}$$

Assume that $A = [a, b]$:

$$\text{Int}(A) = (a, b)$$
$$\bar{A} = A$$
$$\text{Int}(\mathbb{R} - A) = (-\infty, a) \cup (b, \infty)$$
$$\bar{(}\mathbb{R} - A) = (-\infty, a] \cup [b, \infty)$$
$$\text{Bd}(A) = [a, b] \cap (-\infty, a] \cup [b, \infty) = \{a, b\}$$

Assume that $A = \mathbb{Z}$:

$$\begin{aligned}
&\mathrm{Int}(\mathbb{Z}) = \emptyset \\
&\bar{\mathbb{Z}} = \mathbb{Z} \\
&\mathrm{Int}(\mathbb{R} - \mathbb{Z}) = \mathbb{R} - \mathbb{Z} \\
&\overline{\mathbb{R} - \mathbb{Z}} = \mathbb{R} \\
&\mathrm{Bd}(Z) = \mathbb{R} \cap \mathbb{Z} = \mathbb{Z}
\end{aligned}$$

Assume that $A = \mathbb{Q}$:

$$\begin{aligned}
&\mathrm{Int}(\mathbb{Q}) = \emptyset \\
&\bar{\mathbb{Q}} = \mathbb{R} \\
&\mathrm{Int}(\mathbb{R} - \mathbb{Q}) = \mathbb{R} - \mathbb{Q} \\
&\overline{\mathbb{R} - \mathbb{Q}} = \mathbb{R} \\
&\mathrm{Bd}(\mathbb{Q}) = \mathbb{R} \cap \mathbb{R} = \mathbb{R}
\end{aligned}$$