# Linear Independence

## Definition: Linear Combination

Let $E$ be a vector space over a field $\mathbb{F}$ and let $X = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ be a finite, nonempty subset of $E$. A *linear combination* of $X$ is given by:

$$\vec{x} = \sum_{k=1}^{n} \lambda_k \vec{x}_k$$

where $\lambda_k \in \mathbb{F}$ and $\vec{x} \in E$ (by closure).

## Definition: Trivial

Let $E$ be a vector space over a field $\mathbb{F}$ an let $X = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ be a finite, nonempty subset of $E$. The linear combination $\displaystyle\sum_{k=1}^{n} 0\vec{x}_k = \vec{0}$ is called the *trivial* linear combination of $X$.

Otherwise, a linear combination is called *non-trivial*.

## Definition: Linearly Independent

Let $E$ be a vector space over a field $\mathbb{F}$ and let $X = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ be a finite, non-empty subset of $E$. To say that $X$ is a linearly independent set means:

$$\sum_{k=1}^{n} \lambda_k \vec{x}_k = \vec{0} \implies \forall\, \lambda_k = 0$$

Otherwise, $X$ is said to be *linearly dependent*.

Thus, $X$ is linearly independent means only the trivial linear combination results in the zero vector. If a non-trivial linear combination that equals the zero vector exists then $X$ is a linearly dependent set.

This definition can be extended to allow for infinite subsets:

## Definition: Linearly Independent (general)

Let $E$ be a vector space over a field $\mathbb{F}$ and let $X$ be a non-empty subset of $E$. To say that $X$ is a linearly independent set means that any finite subset of $X$ is linearly independent.

Otherwise, $X$ is said to be linearly dependent.

## Examples

1). $E = \ell^p$ and $e_n = (\delta_{kn})$

$$\sum_{k=1}^{\infty} (\delta_{kn})^p = 1 < \infty \text{ and so } e_n \in \ell^p.$$

2). $E = C(\Omega)$ and $f_n(t) = t^n$

**Theorem**

Let $E$ be a vector space over a field $\mathbb{F}$ and let $X$ be a linearly independent subset of $E$. $\vec{0} \notin X$.

Proof

ABC: $\vec{0} \in X$.

Assume $\{\vec{0}, \vec{x}_2, \ldots, \vec{x}_n\} \subset X$.

Assume $\alpha_1\vec{0} + \sum_{k=2}^{n} \alpha_k\vec{x}_k = \vec{0}$.
Let $\alpha_1 = 1$ and the remaining $\alpha_k = 0$.
Assume $1\vec{0} + \sum_{k=2}^{n} 0\vec{x}_k = \vec{0}$.
So a non-trivial solution exists and thus $X$ is a linearly dependent set.
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$\therefore \vec{0} \notin X$.

**Theorem**

Let $E$ be a vector space over a field $\mathbb{F}$ and let $X = \{\vec{x}_1, \ldots, \vec{x}_n\}$ be a non-empty subset of $E$. Define a new set $X' = \{\lambda_1\vec{x}_1, \ldots, \lambda_n\vec{x}_n\}$ for some $\lambda_k \in \mathbb{F}$ and $\lambda_k \neq 0$:

> $X$ is linearly independent iff $X'$ is linearly independent.

Proof

$\implies$ Assume $X$ is a linearly independent set.

$$\sum_{k=0}^{n} \alpha_k\vec{x}_k = \vec{0} \implies \alpha_k = 0$$

Assume $\sum_{k=0}^{n} \beta(\lambda_k\vec{x}_k) = \vec{0}$.

$$\sum_{k=0}^{n} (\beta\lambda_k)\vec{x}_k = \vec{0}$$

But $X$ is linearly independent, so $\beta_k\lambda_k = 0$.
By assumption, $\lambda_k \neq 0$, and thus $\beta_k = 0$.

Therefore $X'$ is a linearly independent set.

$\impliedby$ Assume $X'$ is a linearly independent set.

$$\sum_{k=0}^{n} \alpha_k(\lambda_k\vec{x}_k) = \vec{0} \implies \alpha_k = 0$$

Assume $\sum_{k=0}^{n} \beta_k \vec{x}_k = \vec{0}$.

Since $\lambda_k \neq 0$ (by assumption), $\beta_k = \frac{\beta_k}{\lambda_k}\lambda_k$.

$$\sum_{k=0}^{n} \left(\frac{\beta_k}{\lambda_k}\lambda_k\right)\vec{x}_k = \vec{0}$$

$$\sum_{k=0}^{n} \frac{\beta_k}{\lambda_k}(\lambda_k \vec{x}_k) = \vec{0}$$

But $X'$ is linearly independent, so $\frac{\beta_k}{\lambda_k} = 0$.

Therefore $\beta_k = 0$ and thus $X$ is a linearly independent set.