Convergence

Definition: Convergence

Let $(E, \|\cdot\|)$ be a normed space. To say that a sequence $(\vec{x}_n)$ of elements in E converges to some $\vec{x} \in E$ means:

$$\forall \epsilon > 0, \exists\, N(\epsilon) > 0, n > N \implies \|\vec{x}_n - \vec{x}\| < \epsilon$$

Notation

Any of the following can be used to indicate convergence:

- $\lim \vec{x}_n = \vec{x}$
- $\|\vec{x}_n - \vec{x}\| \to 0$
- $\vec{x}_n \to \vec{x}$
- $d(\vec{x}_n, \vec{x}) \to 0$

Theorem: Properties

Let $(E, \|\cdot\|)$ be a normed space:

1). $(\vec{x}_n)$ converges $\implies$ the limit is unique.

2). $\vec{x}_n \to \vec{x}$ and $\lambda_n \to \lambda \implies \lambda_n \vec{x}_n \to \lambda \vec{x}$.

3). $\vec{x}_n \to \vec{x}$ and $\vec{y}_n \to \vec{y} \implies (\vec{x}_n + \vec{y}_n) \to (\vec{x} + \vec{y})$.

Proof

1). Assume $\vec{x}_n \to \vec{x}$ and $\vec{x}_n \to \vec{y}$.

$\forall\, \epsilon_1 > 0, \exists\, N_1 > 0, n > N_1 \implies \|\vec{x}_n - \vec{x}\| < \epsilon_1$
$\forall\, \epsilon_2 > 0, \exists\, N_2 > 0, n > N_2 \implies \|\vec{x}_n - \vec{y}\| < \epsilon_2$
Assume $\epsilon > 0$.
Let $\epsilon_1, \epsilon_2 = \dfrac{\epsilon}{2}$.
Let $N = \max\{N_1, N_2\}$.
Assume $n > N$.

$$\|\vec{x} - \vec{y}\| = \|\vec{x} - \vec{x}_n + \vec{x}_n + \vec{y}\| \leq \|\vec{x} - \vec{x}_n\| + \|\vec{y} - \vec{x}_n\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore \vec{x} = \vec{y}$

2). Assume $\vec{x}_n \to \vec{x}$ and $\lambda_n \to \lambda$.

$\forall\, \epsilon_1 > 0, \exists\, N_1 > 0, n > N_1 \implies \|\vec{x}_n - \vec{x}\| < \epsilon_1$
$\forall\, \epsilon_2 > 0, \exists\, N_2 > 0, n > N_2 \implies |\lambda_n - \lambda| < \epsilon_2$
So $\exists\, N_3 > 0, n > N_3 \implies |\lambda_n - \lambda| < 1$.

Assume $\epsilon > 0$.
Let $N = \max\{N_1, N_2, N_3\}$.
Assume $n > N$.
$|\lambda_n| = |\lambda_n - \lambda + \lambda| \leq |\lambda_n - \lambda| + |\lambda| < 1 + |\lambda|$
Let $\epsilon_1 = \dfrac{\epsilon}{2(1+|\lambda|)}$ and $\epsilon_2 = \dfrac{\epsilon}{2\|\vec{x}\|}$.

$$\begin{aligned}
\|\lambda_n \vec{x}_n - \lambda \vec{x}\| &= \|\lambda_n \vec{x}_n - \lambda_n \vec{x} + \lambda_n \vec{x} - \lambda \vec{x}\| \\
&\leq \|\lambda_n \vec{x}_n - \lambda_n \vec{x}\| + \|\lambda_n \vec{x} - \lambda \vec{x}\| \\
&= \|\lambda_n(\vec{x}_n - \vec{x})\| + \|(\lambda_n - \lambda)\vec{x}\| \\
&= |\lambda_n| \, \|\vec{x}_n - \vec{x}\| + |\lambda_n - \lambda| \, \|\vec{x}\| \\
&< (1+|\lambda|)\frac{\epsilon}{2(1+|\lambda|)} + \frac{\epsilon}{2\|\vec{x}\|}\|x\| \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

$\therefore \lambda_n \vec{x}_n \to \lambda \vec{x}$

3). Assume $\vec{x}_n \to \vec{x}$ and $\vec{y}_n \to \vec{y}$

$\forall\, \epsilon_1 > 0, \exists\, N_1 > 0, n > N_1 \implies \|\vec{x}_n - \vec{x}\| < \epsilon_1$
$\forall\, \epsilon_2 > 0, \exists\, N_2 > 0, n > N_2 \implies \|\vec{y}_n - \vec{y}\| < \epsilon_2$
Assume $\epsilon > 0$.
Let $\epsilon_1, \epsilon_2 = \dfrac{\epsilon}{2}$.
Let $N = \max\{N_1, N_2\}$.
Assume $n > N$.

$$\begin{aligned}
\|(\vec{x}_n + \vec{y}_n) - (\vec{x} + \vec{y})\| &= \|(\vec{x}_n - \vec{x}) + (\vec{y}_n + \vec{y})\| \\
&\leq \|\vec{x}_n - \vec{x}\| + \|\vec{y}_n + \vec{y}\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

$\therefore (\vec{x}_n + \vec{y}_n) \to (\vec{x} + \vec{y})$