Inner Product Induced Norm

Definition: Inner Product Induced Norm

Let V be a vector space equipped with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. To say that the norm is an *inner product-induced norm* means:

$$\|\vec{x}\| = \langle\vec{x},\vec{x}\rangle^{\frac{1}{2}}$$

Example

The ℓ_2 norm is an inner product-induced norm because:

$$\|\vec{x}\|_2 = \left(\sum_{k=1}^{n} |x_k|^2\right)^{\frac{1}{2}} = (\vec{x}^*\vec{x})^{\frac{1}{2}} = \langle\vec{x},\vec{x}\rangle^{\frac{1}{2}}$$

Theorem

A norm is inner product-induced iff it satisfies the parallelogram identity:

$$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2(\|\vec{x}\|^2 + \|\vec{y}\|^2)$$

Proof

Assume $\|\cdot\|$ is inner product-induced

$$\begin{aligned}
\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\| &= \langle\vec{x}+\vec{y},\vec{x}+\vec{y}\rangle + \langle\vec{x}-\vec{y},\vec{x}-\vec{y}\rangle \\
&= \langle\vec{x},\vec{x}\rangle + \langle\vec{x},\vec{y}\rangle + \langle\vec{y},\vec{x}\rangle + \langle\vec{y},\vec{y}\rangle + \langle\vec{x},\vec{x}\rangle - \langle\vec{x},\vec{y}\rangle - \langle\vec{y},\vec{x}\rangle + \langle\vec{y},\vec{y}\rangle \\
&= 2\langle\vec{x},\vec{x}\rangle + 2\langle\vec{y},\vec{y}\rangle \\
&= 2(\langle\vec{x},\vec{x}\rangle + \langle\vec{y},\vec{y}\rangle) \\
&= 2(\|\vec{x}\|^2 + \|\vec{y}\|^2)
\end{aligned}$$

Note that the ℓ_1 normal fails the parallelogram identity and thus is not inner product-induced:

Let $\vec{x} = \vec{e}_1$ and $\vec{y} = \vec{e}_2$:

$$\|\vec{e}_1+\vec{e}_2\|_1 = \left\|\begin{bmatrix}1\\1\\0\\\vdots\\0\end{bmatrix}\right\|^2 + \left\|\begin{bmatrix}1\\-1\\0\\\vdots\\0\end{bmatrix}\right\|^2 = 2^2 + 2^2 = 4 + 4 = 8$$

$$2(\|\vec{e}_1\|_1^2 + \|\vec{e}_2\|_1^2) = 2(1^2 + 1^2) = 2(1+1) = 2(2) = 4$$