Baire Space

Definition: Nowhere Dense

Let E be a normed space and let $X \subset E$. To say that X is *nowhere dense* in E means that the interior of its closure is empty. In other words, $\overline{X}$ contains no open subsets.

Examples

1). $\mathbb{Z}$ is nowhere dense in R because Z is its own closure, which has no interior points.

2). The Cantor set $\mathcal{C}$.

Definition: Categories

Let E be a normed space and let $X \subset E$ be a countable union of sets:

$$X = \bigcup_{n=1}^{\infty} U_n$$

To say that X is of the *first category (meager)* means $\forall\, n \in \mathbb{N}$, U_n is nowhere dense in E.

Otherwise X is of the *second category (nonmeagre)* - i.e., $\exists\, n \in \mathbb{N}$ such that $\overline{U_n}$ has a non-empty interior.

Definition: Baire Space

Let E be a normed space. To say that E is a *Baire* space means every non-empty open subset X of E is of the second category in E.

The following restatements are equivalent:

- Every countable union of nowhere dense sets in E is nowhere dense.
- Every countable intersection of dense sets in E is dense.

Theorem: Baire Category Theorem

Every complete normed (Banach) space E is a Baire Space.

Proof

Assume $X = \bigcap_{n=1}^{\infty} U_n$ is an intersection of dense sets in E.
Assume W is an open subset in E.
Since U_1 is dense in E, $\exists\, \vec{x}_1 \in U_1$ such that $\vec{x}_1 \in W \cap U_1$.
So $\exists\, r_1 \in (0, 1)$ such that $\overline{B}(\vec{x}_1, r_1) \subset W \cap U_1$. Recursively construct a sequence (x_n) in E such that:

$$\overline{B}(\vec{x}_{n+1}, r_{n+1}) \subset B(\vec{x}_n, r_n) \cap U_n$$

where $0 < r_n < \dfrac{1}{n}$.
Thus, (x_n) is Cauchy and so, by completeness, $\vec{x}_n \to \vec{x} \in E$.
But $\forall\, n \in \mathbb{N}$, a tail part of $(\vec{x}_n)$ is in $\overline{B}(\vec{x}_n, r_n)$.
So, by closedness, $\vec{x} \in \overline{B}(\vec{x}_n, r_n) \subset U_n$.
Thus $\forall\, n \in \mathbb{N}, \vec{x} \in U_n$, and so $\vec{x} \in X$.
But $\vec{x} \in W$ also, and so $\vec{x} \in W \cap X$.

Therefore X is dense, and so E is a Baire Space.