Conditional Probability & Independence

– Math 161a, Spring 2019, San José State University

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Section 2.4 Conditional probability

Section 2.5 Independence

Introduction

Consider the following two problems.

Ex 0.1 (Toss two fair dice)**.** Let $B = \{\text{Sum}{=}10\}$. Find $P(B)$.

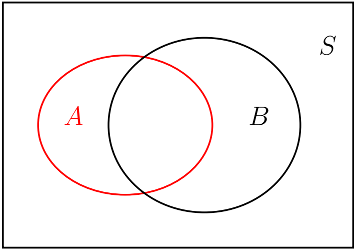
Ex 0.2 (cont'd)**.** What if we know that the two numbers are identical (event A)?

Conditional probability

When extra information (about the result of a random phenomenon) is available, this effectively reduces the sample space.

Def 0.1. Suppose $A, B \subseteq S$ and $P(A) > 0$. The **conditional probability** of B given A (which has occurred) is defined as

$$P(B \mid \underbrace{A}_{\text{given}}) = \frac{P(A \cap B)}{P(A)}$$



Ex 0.3. Consider the experiment of tossing two fair dice. Let $A = \{\text{Both even}\}, B = \{\text{Sum=8}\}$. Find $P(A \mid B)$ and $P(B \mid A)$. Are they equal to each other?

Multiplication rule

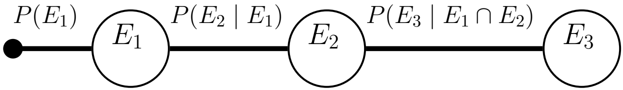
Rewriting the equation in the definition of conditional probability leads to a rule for computing the probability of two events occuring together.

Theorem 0.1. *For any two events $A, B \subseteq S$ with $P(A) > 0$,*

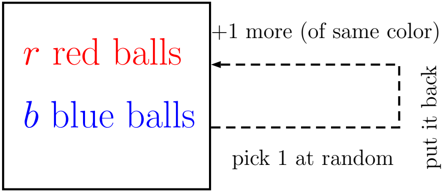
$$P(A \cap B) = P(B \mid A) \cdot P(A).$$

More generally,

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P(E_2 \mid E_1) \cdot P(E_3 \mid E_1 \cap E_2).$$



Ex 0.4 (Polya's urn scheme)**.** Suppose an urn initially has r red balls and b blue balls. A ball is drawn at random and its color noted. The it together with an extra ball of the same color (as the drawn ball) is put back into the urn. Now select a second ball at random. What is the probability that the two drawn balls are both red?



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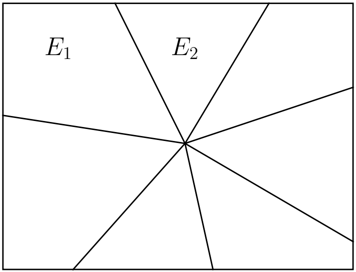
Ex 0.5. Three cards are dealt from the top of a well-shuffled deck of 52 cards. What is the probability that they are all hearts?

Partition of a sample space

Def 0.2. A sequence of nonempty sets $\{E_i\}$ are said to form a **partition** of the sample space S if they are



- both **mutually exclusive**:

$$E_i \cap E_j = \emptyset \quad \text{for all } i \neq j,$$

- and **exhaustive**:

$$\cup E_i = S.$$

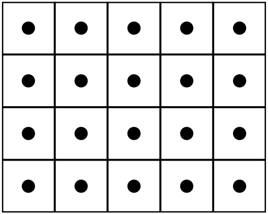
Ex 0.6 (Toss a single die). The sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

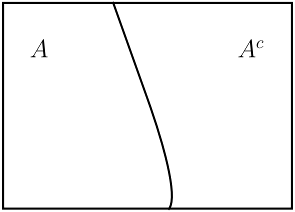
Which of the following are partitions of the sample space?

- $E_1 = \{1\}, \ldots, E_6 = \{6\}$
- $A = \{1, 3, 5\},\ A^c = \{2, 4, 6\}$
- $A = \{1, 2, 3\},\ B = \{4, 5\},\ C = \{6\}$
- $A = \{1, 3, 5\},\ B = \{2, 4\}$
- $A = \{1, 3, 5\},\ B = \{2, 4\},\ C = \{5, 6\}$

Remark. For any sample space S, the following are partitions:

- All the simple events: $S = \cup_{s \in S} \{s\}$.
- Any nonempty event $A \subset S$ and its complement: $S = A \cup A^c$.

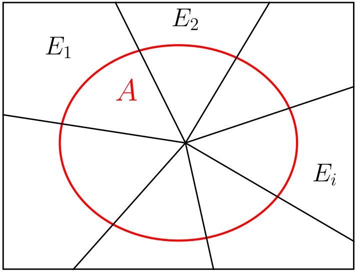




Law of total probability (LTP)

Theorem 0.2. *Assume a partition of a sample space* $S = E_1 \cup E_2 \cup \cdots$. *For any event* $A \subseteq S$, *we have*

$$P(A) = \sum_i P(A \cap E_i)$$
$$= \sum_i P(A \mid E_i) P(E_i).$$

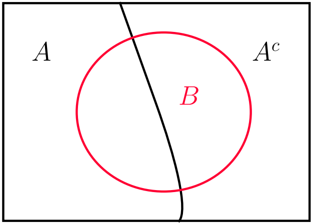


Proof. This is a direct consequence of **additivity of probability function** and the **multiplication rule**. □

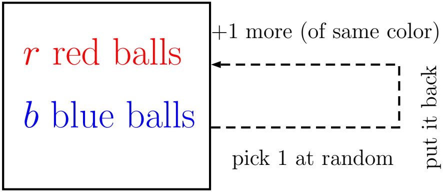
For a partition of size two: $S = A \cup A^c$, the LTP reduces to the following.

Corollary 0.3. *Let $A \subset S$, with $P(A) > 0$. Then for any event $B \subseteq S$,*

$$\begin{aligned} P(B) &= P(B \cap A) + P(B \cap A^c) \\ &= P(B \mid A)P(A) + P(B \mid A^c)P(A^c). \end{aligned}$$



Ex 0.7 (Polya's urn scheme)**.** Suppose an urn initially has r red balls and b blue balls. A ball is drawn and its color noted. The it together with an extra ball of the same color (as the drawn ball) is added to the urn. Now select a second ball at random. What is the probability that the second drawn ball is red?



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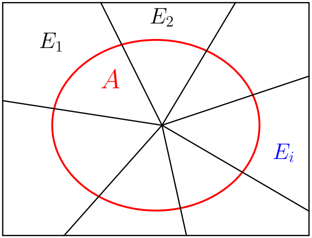
Bayes' Rule

... is a formula for computing the "posterior probabilities" $P(E_i \mid A)$.

Theorem 0.4. *Suppose that the events $E_1, E_2, \ldots$ form a partition of S. Let $A \subseteq S$ be any event with $P(A) > 0$. Then, for any i,*

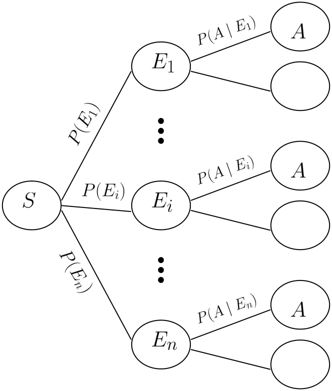
$$P(E_i \mid A) \overset{def}{=} \frac{P(A \cap E_i)}{P(A)} = \frac{P(A \mid E_i)P(E_i)}{\sum_j P(A|E_j)P(E_j)}$$



Interpretation:

- E_i: different **causes or hypotheses**
- A: **new evidence**
- $P(E_i)$: **prior** probabilities (*before* seeing any evidence)
- $P(A \mid E_i)$: **forward** probabilities
- $P(E_i \mid A)$: **posterior** probabilities (*after* seeing the evidence)

Law of Total Probability(LTP)

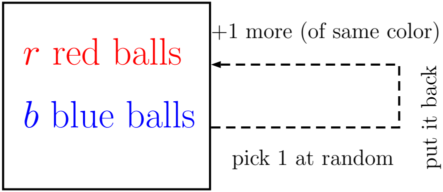


$$P(A) = P(E_1)P(A \mid E_1) + \cdots + P(E_i)P(A \mid E_i) + \cdots + P(E_n)P(A \mid E_n)$$

Bayes' Rule

$$\underbrace{P(E_i \mid A)}_{\text{Posterior Probability}} = \frac{\overbrace{P(A|E_i)}^{\text{Forward Probability}} \cdot \overbrace{P(E_i)}^{\text{Prior Probability}}}{\underbrace{P(A)}_{\text{Total Probability}}}$$

Ex 0.8 (Polya's urn scheme)**.** Suppose an urn has r red balls and b blue balls. A ball is drawn and its color noted. The it together with an extra ball of the same color as the drawn ball is added to the urn. Find the probability that the first drawn ball was red given that the second ball drawn is red.



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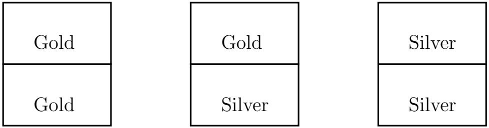
Ex 0.9. Suppose that 65% of the defendants are truly guilty. Suppose also that juries vote a guilty person innocent with probability 0.2 whereas the probability that a jury votes an innocent person guilty is 0.1. Find the probability that a defendant is convicted. What percentage of convicted defendants are truly guilty?

Ex 0.10. Suppose there are three chests each having two drawers. One chest has a gold coin in each drawer, one chest has a gold coin in one drawer and a silver coin in the other drawer, and the third chest has a silver coin in each drawer. A chest is first picked at random and then a random drawer is opened.

(a) What is the probability that the opened drawer contains a gold coin?

(b) If the drawer contains a gold coin, what is the probability that the other drawer also contains a gold coin?



Ex 0.11 (The Monte Hall problem)**.** First watch a YouTube video at `https://www.youtube.com/watch?v=mhlc7peGlGg` and then use a tree diagram to verify the probabilities.

Ex 0.12 (Three prisoners)**.** Three prisoners A, B and C are on death row. The governor decides to pardon one of the three and chooses at random the prisoner to pardon. He informs the warden to of his choice but requests that the name be kept secret for a few days.

The next day, A tries to get the warden to tell him who had been pardoned. The warden refuses. A then asks which of B or C will be executed. The warden thinks for a while and then tells A that B is to be executed.

Warden's reasoning: Each prisoner has a $1/3$ chance of being pardoned. Clearly, either B or C must be executed, so I have given A no information about whether A will be pardoned.

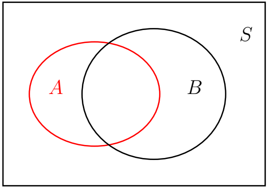
A's reasoning: Given that B will be executed, then either A or C will be pardoned. My chance of being pardoned has risen to $1/2$.

Whose reasoning is correct?

Independence

Def 0.3. Two events $A, B \subseteq S$ with $P(A) > 0$ are said to be (statistically) *independent* if

$$P(B \mid A) = P(B).$$



Ex 0.13. A card is selected at random from an ordinary deck of 52. Let A denote the event that the selected card is an ace, and B a spade. Are A, B independent?

Ex 0.14. Determine if the following events are independent.

- Throw a fair die twice.

$$A = \{\text{first number is } 1\}, B = \{\text{second number is } 6\}.$$

- Draw two cards from a deck of 52 (1) with (2) without replacement. $A = \{\text{first card is a heart}\}, B = \{\text{second card is also a heart}\}.$

- Flu test for couples.

$$A = \{\text{husband tests positive}\}, B = \{\text{wife tests negative}\}.$$

Theorem 0.5. *Two events $A, B \subseteq S$ are* independent *if and only if*

$$P(A \cap B) = P(A)P(B).$$

Proof. This can be easily shown by combining the definitions of conditional probability and independence:

$$\begin{aligned} \text{Independence :} \qquad & P(B \mid A) = P(B) \\ \text{Conditional probability :} \qquad & P(B \mid A) = \frac{P(A \cap B)}{P(A)} \end{aligned}$$

Remark. If A, B are independent events, then each of the pairs A^c and B, A and B^c, A^c and B^c is also independent.

Ex 0.15. Suppose we draw two cards from a deck of 52, with replacement. What is the probability that both are diamonds?

Ex 0.16 (Toss 2 fair dice)**.** Let $E = \{\text{Sum=6}\}$ and $F = \{\text{First=4}\}$. Determine if E, F are independent.

A joke on independence

A stats professor plans to travel to a conference by plane. When he passes the security check, they discover a bomb in his carry-on-baggage. Of course, he is hauled off immediately for interrogation.

"I don't understand it!" the interrogating officer exclaims. "You're an accomplished professional, a caring family man, a pillar of your parish - and now you want to destroy that all by blowing up an airplane!"

"Sorry", the professor interrupts him. "I had never intended to blow up the plane."

"So, for what reason else did you try to bring a bomb on board?!"

"Let me explain. Statistics shows that the probability of a bomb being on an airplane is 1/1,000. That's quite high if you think about it - so high that I wouldn't have any peace of mind on a flight."

"And what does this have to do with you bringing a bomb on board of a plane?"

"You see, since the probability of one bomb being on my plane is 1/1,000, the chance that there are two bombs is 1/1,000,000. If I already bring one, the chance of another bomb being around is actually 1/1,000,000, and I am much safer.

i-Clicker activity 2 (extra credit)

Question. Let $A, B \subseteq S$ be two disjoint events with $P(A) > 0, P(B) > 0$. Are they independent?

A. Yes

B. No

C. Maybe

D. Cannot tell (need more information)

Def 0.4. We say that a collection of events $E_1, E_2, \ldots$ are **pairwise independent** if every pair of events E_i, E_j are independent,

$$P(E_i \cap E_j) = P(E_i)P(E_j), \qquad \text{for all} \quad i \neq j$$

Def 0.5. A collection of events $E_1, E_2, \ldots$ are **(mutually) independent** if for any subcollection $E_{i_1}, \ldots, E_{i_k}$ $(1 \leq k \leq n)$, we have

$$P(E_{i_1} \cap \cdots \cap E_{i_k}) = P(E_{i_1}) \cdots P(E_{i_k}).$$

Remark. Pairwise independence is weaker than mutual independence.

Ex 0.17. Consider a sample space $S = \{1, 2, \ldots, 8\}$ whose outcomes are equally likely. Let $A = \{1, 2, 3, 4\}, B = \{1, 4, 5, 6\}, C = \{1, 6, 7, 8\}$. Show that

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

but they are not pairwise independent.

Remark. In your homework you will be asked to show that the other direction is not true either, that is, pairwise independence does not necessarily imply

$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

Ex 0.18 (n coin tosses)**.** Consider the experiment of tossing a coin n times and record results. Let

$$H_i = \{i\text{th toss is a head}\}, \quad i = 1, \ldots, n.$$

Then $H_1, \ldots, H_n$ are (mutually) independent.

Ex 0.19 (Continuation of previous problem)**.** Suppose the probability of the coin landing on heads is p. Find the probability that (1) only the first k tosses are heads (2) at least one toss is a head.