Fields

Definition

To say that R is a *commutative ring* means:

1). R is a ring

2). Multiplication in R is commutative

Note that the binomial theorem holds for any commutative ring R:

$\forall\, a, b \in R, \forall\, n \in \mathbb{N}$:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot a^{n-k} b^k$$

Definition

To say that R is a *division ring (skew field)* means:

1). R is a ring with unity $1 \neq 0$

2). $\langle R - \{0\}, \cdot \rangle$ is a group

In other words, every non-zero element of R is a unit.

Definition

To say that R is a *field* means:

1). R is a ring with unity $1 \neq 0$

2). $\langle R - \{0\}, \cdot \rangle$ is an abelian group

In other words, R is a commutative division ring (skew field).

If R is a non-commutative division ring then it is called a *strictly skew field.*

Example

$\mathbb{Z}$ is not a field, because $2 \in \mathbb{Z}$ but 2 has no multiplicative inverse.

$\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields.