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Math 229
Homework #0

1). Let A be an $m \times n$ matrix:

a). Prove: $\text{Null}(A^*A) = \text{Null}(A)$

$\implies$ Assume $\vec{x} \in \text{Null}(A^*A)$

$A^*A\vec{x} = \vec{0}$
$\overline{A}^T A\vec{x} = \vec{0}$
$\overline{\vec{x}}^T \overline{A}^T A\vec{x} = \overline{\vec{x}}^T \vec{0} = \vec{0}$
$\left(\overline{A\vec{x}}\right)^T (A\vec{x}) = \vec{0}$
$\|A\vec{x}\|^2 = 0$
$A\vec{x} = \vec{0}$

$\therefore \vec{x} \in \text{Null}(A)$

$\impliedby$ Assume $\vec{x} \in \text{Null}(A)$

$A\vec{x} = \vec{0}$
$A^*A\vec{x} = A^*\vec{0} = \vec{0}$

$\therefore \vec{x} \in \text{Null}(A^*A)$

$\therefore \text{Null}(A^*A) = \text{Null}(A)$

b). Prove: $\text{rank}(A^*A) = \text{rank}(A)$

A is $m \times n$
A^* is $n \times m$
A^*A is $n \times n$

By the dimension theorem:

$$\text{rank}(A^*A) + \text{nullity}(A^*A) = n = \text{rank}(A) + \text{nullity}(A)$$

But from the part (a): $\text{Null}(A^*A) = \text{Null}(A)$

$\therefore \text{rank}(A^*A) = \text{rank}(A)$

c). Prove: $\text{rank}(A^*A) = \text{rank}(AA^*)$

Note that $(A^*)^* = A$, so from the part (b) we get:
$\text{rank}(A^*A) = \text{rank}(A)$ and
$\text{rank}(AA^*) = \text{rank}((A^*)^*A^*) = \text{rank}(A^*)$

Now, consider the null space of A:
$A\vec{x} = 0 \iff (\overline{A})\overline{\vec{x}} = 0$
Thus, there is a one-to-one correspondence between the vectors in $\text{Null}(A)$ and $\text{Null}(\overline{A})$, and so $\text{nullity}(A) = \text{nullity}(\overline{A})$

But A and $\overline{A}$ have the same number of columns, so by the dimension theorem:
$\operatorname{rank}(A) = \operatorname{rank}(\overline{A})$
In class we proved that $\operatorname{rank}(A) = \operatorname{rank}(A^T)$, so:
$\operatorname{rank}(A^*) = \operatorname{rank}(\overline{A}^T) = \operatorname{rank}(\overline{A}) = \operatorname{rank}(A)$

$\therefore$, $\operatorname{rank}(A^*A) = \operatorname{rank}(AA^*)$

2). Let A and B be square ($n \times n$) matrices:

a). Prove: $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

$$\begin{aligned}
tr(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik} \\
&= \sum_{k=1}^{n} (BA)_{kk} \\
&= \operatorname{tr}(BA)
\end{aligned}$$

b). Prove: $\det(AB) = \det(BA)$

$$\begin{aligned}
\det(AB) &= \det(A)\det(B) \\
&= \det(B)\det(A) \\
&= \det(BA)
\end{aligned}$$

c). Is it always true that $\operatorname{rank}(AB) = \operatorname{rank}(BA)$?

No. Counterexample:

$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So $\operatorname{rank}(AB) = 0$

$BA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

So $\operatorname{rank}(BA) = 1$

$\therefore \operatorname{rank}(AB) \neq \operatorname{rank}(BA)$

3). Let A be an $n \times n$ square matrix and let $\vec{x}$ and $\vec{y}$ be two $n \times 1$ column vectors:

a).

$$\begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix} \begin{bmatrix} I_n & 0_{n\times 1} \\ -\vec{y}^T & 1 \end{bmatrix} = \begin{bmatrix} A + \vec{x}\vec{y}^T & -\vec{x} \\ 0 & 1 \end{bmatrix}$$

b). Prove: $\det \begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix} = \det(A + \vec{x}\vec{y}^T)$

$$\det \begin{bmatrix} I_n & 0_{n\times 1} \\ -\vec{y}^T & 1 \end{bmatrix} = \det(I_n)\det(\begin{bmatrix}1\end{bmatrix}) = 1 \cdot 1 = 1$$

$$\det \begin{bmatrix} A + \vec{x}\vec{y}^T & -\vec{x} \\ 0 & 1 \end{bmatrix} = \det(A + \vec{x}\vec{y}^T)\det(\begin{bmatrix}1\end{bmatrix}) = \det(A + \vec{x}\vec{y}^T) \cdot 1 = \det(A + \vec{x}\vec{y}^T)$$

$$\begin{aligned}
\det\left(\begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix} \begin{bmatrix} I_n & 0_{n\times 1} \\ -\vec{y}^T & 1 \end{bmatrix}\right) &= \det \begin{bmatrix} A + \vec{x}\vec{y}^T & -\vec{x} \\ 0 & 1 \end{bmatrix} \\
\left(\det \begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix}\right)\left(\det \begin{bmatrix} I_n & 0_{n\times 1} \\ -\vec{y}^T & 1 \end{bmatrix}\right) &= \det \begin{bmatrix} A + \vec{x}\vec{y}^T & -\vec{x} \\ 0 & 1 \end{bmatrix} \\
\left(\det \begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix}\right)(1) &= \det(A + \vec{x}\vec{y}^T) \\
\det \begin{bmatrix} A & -\vec{x} \\ \vec{y}^T & 1 \end{bmatrix} &= \det(A + \vec{x}\vec{y}^T)
\end{aligned}$$