Analytic Functions

Definition

To say that $f(z)$ is *analytic* at a point z_0 means that it is differentiable at every point in some ϵ-neighborhood of z_0:

$$\exists\, \epsilon > 0, \forall\, z \in N_\epsilon(z_0), f \text{ is analytic at } z$$

To say that $f(z)$ is analytic in a domain D means that it is analytic everywhere in D:

$$\forall\, z \in D, f \text{ is analytic at } z$$

To say that $f(z)$ is an *entire* function means that it is analytic everywhere in $\mathbb{C}$:

$$\forall\, z \in \mathbb{C}, f \text{ is analytic at } z$$

To say that z_0 is a *singular* point of $f(z)$ means that f is analytic in some deleted neighborhood of z_0, but not at z_0.

Example

$f(z) = z^2$	$f(z) = \frac{1}{1-z}$	$f(z) = \lvert z\rvert^2$
$f'(z) = 2z$	$f'(z) = \frac{1}{(1-z)^2}, z \neq 1$	$f'(0) = 0$ only (no neighborhood)
$f(z)$ is entire	$z = 1$ is a singular point of f	$f(z)$ is analytic nowhere

The following theorem follows directly from the differentiation laws:

Theorem

Let $f(z)$ and $g(z)$ be analytic in a domain D. The following are also analytic in D:

1). $f(z) \pm g(z)$

2). $f(z)g(z)$

3). $\frac{f(z)}{g(z)}$, wherever $g(z) \neq 0$

4). $(f \circ g)(z)$

Note that by extension, all polynomial and rational functions ($g(z) \neq 0$) are analytic as well

Theorem

Let D be a domain:

$\forall\, z \in D, f'(z) = 0 \implies f(z)$ constant in D

Proof

Assume $\forall\, z \in D, f'(z) = 0$
$f(z) = u + iv$
$f'(z) = u_x + iv_x = 0$
$u_x = v_x = 0$, and CR, $v_x = v_y = 0$
Let $z_0, z \in D$ such that z_0 and z can be connected by a single line segment L
Let s denote the distance from z_0 to z
$\frac{du}{ds} = \nabla u \cdot \hat{u}$
But $\nabla u = u_x \hat{i} + u_y \hat{j} = \hat{0}$
So $\frac{du}{ds} = 0$ along L and thus u is some constant a
Similarly, v is some constant b
$f = a + ib$ along L
But any two points in D can be connected by a finite number of line segments
$\therefore f$ is constant in D

Theorem

Let $f(z) = u + iv$ be analytic in a domain D:

u constant in $D \implies f$ constant in D

Proof

Assume $u = c, c \in \mathbb{C}$ in D
$u_x = 0 = v_y$
$-u_y = 0 = v_x$
So v is constant in D
$\therefore f$ is constant in D

Theorem

Let $f(z)$ be analytic in a domain D:

$\overline{f(z)}$ analytic in $D \iff f(z)$ is constant in D

Proof

$\implies$ Assume $\overline{f(z)}$ is analytic in D

$f(z) = u + iv$
$u_x = v_y$ and $v_x = -u_y$
$\overline{f(z)} = u - iv$
$u_x = -v_y$ and $-v_x = -u_y$, or $v_x = u_y$
$u_x = v_y = -v_y$, so $u_x = v_y = 0$
$v_x = -u_y = u_y$, so $v_x = u_y = 0$
$f'(z) = u_x + iv_x = 0$
$\therefore f(z)$ is constant in D

$\Longleftarrow$ Assume $f(z)$ is constant in D

$\overline{f(z)}$ is constant in D
$\therefore \overline{f(z)}$ is analytic in D

Theorem

Let $f(z)$ be analytic in a domain D:

$f(z)$ constant in $D \iff |f(z)|$ constant in D

Proof

$\Longrightarrow$ Assume $f(z)$ is constant in $D \therefore |f(z)|$ is constant in D

$\Longleftarrow$ Assume $|f(z)|$ is constant in D

case 1: $f(z) = 0$

$\therefore f(z)$ is constant in D

case 2: $f(z) \neq 0$

Let $|f(z)| = c$
$|f(z)|^2 = c^2$
$f(z)\overline{f(z)} = c^2$
$\overline{f(z)} = \frac{c^2}{f(z)}$
So $\overline{f(z)}$ is analytic
$\therefore f(z)$ is constant in D

Proof (alternate)

Assume $|f(z)|$ is constant
$|f(z)|^2$ is constant
Let $f(z) = u + iv$
$|f(z)|^2 = u^2 + v^2$
Let $u^2 + v^2 = c$

case 1: $c = 0$

$u = v = 0$
$f(z) = 0$
$\therefore f(z)$ is constant in D

case 2: $c \neq 0$

$2uu_x + 2vv_x = 0$
$2uu_y + 2vv_y = 0$

Note that if any of $u_x, u_y, v_x, v_y = 0$ then, by above and CR, all must be 0 and $f(z)$ would be constant, so assume none are 0

$2uu_x = -2vv_x$
$\frac{u_x}{v_x} = -\frac{v}{u}$

$2uu_y = -2vv_y$
$\frac{u_y}{v_y} = -\frac{v}{u}$

$\frac{u_x}{v_x} = \frac{u_y}{v_y}$
$u_x v_y = v_x u_y$
$u_x v_y - v_x u_y = 0$

By CR, $u_x^2 + v_x^2 = 0$, so $u_x = v_x = 0$ and $f'(z) = 0$
$\therefore f(z)$ is constant on D

Theorem

Let $f(z) = u(x, y) + iv(x, y)$ be analytic on a domain D and let $f'(z) \neq 0$ at a point $z_0 \in D$, which is the point of intersection of the level curves $u(x, y) = c_1$ and $v(x, y) = c_2$:

u and v are orthogonal at z_0

Proof

$du = u_x dx + u_y dy = 0$
$dv = v_x dx + v_y dy = 0$

$\frac{dy}{dx} = -\frac{u_x}{u_y} = m_1$
$\frac{dy}{dx} = -\frac{v_x}{v_y} = m_2$

$m_1 m_2 = \left(-\frac{u_x}{u_y}\right)\left(-\frac{v_x}{v_y}\right) = \left(\frac{u_x}{u_y}\right)\left(\frac{v_x}{v_y}\right) = \left(-\frac{v_y}{v_x}\right)\left(\frac{v_x}{v_y}\right) = -1$

$\therefore u$ and v are orthogonal at z_0