Sums of Powers

Theorem

1). $\sum_{k=1}^{n} 1 = n$

2). $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

3). $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

4). $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

Proof

By derivation:

1). $\sum_{k=1}^{n} 1 = n$

trivial

2). $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$$\begin{aligned}\sum_{k=1}^{n}[(k+1)^2 - k^2] &= \sum_{k=1}^{n}[k^2 + 2k + 1 - k^2] \\ &= \sum_{k=1}^{n}[2k+1] \\ &= 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\ &= 2\sum_{k=1}^{n} k + n\end{aligned}$$

But note that the LHS is a telescoping sum, so we have:

$$\begin{aligned}\sum_{k=1}^{n}[(k+1)^2 - k^2] &= (n+1)^2 - 1^2 \\ &= n^2 + 2n + 1 - 1 \\ &= n^2 + 2n\end{aligned}$$

Equating the two, we have:

$$\begin{aligned}
n^2 + 2n &= 2\sum_{k=1}^{n} k + n \\
2\sum_{k=1}^{n} k &= n^2 + n \\
2\sum_{k=1}^{n} k &= n(n+1) \\
\sum_{k=1}^{n} k &= \frac{n(n+1)}{2}
\end{aligned}$$

3). $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$$\begin{aligned}
\sum_{k=1}^{n}[(k+1)^3 - k^3] &= \sum_{k=1}^{n}[k^3 + 3k^2 + 3k + 1 - k^3] \\
&= \sum_{k=1}^{n}[3k^2 + 3k + 1] \\
&= 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
&= 3\sum_{k=1}^{n} k^2 + 3\frac{n(n+1)}{2} + n \\
&= 3\sum_{k=1}^{n} k^2 + \frac{3n^2 + 3n}{2} + n \\
&= 3\sum_{k=1}^{n} k^2 + \frac{3n^2 + 3n + 2n}{2} \\
&= 3\sum_{k=1}^{n} k^2 + \frac{3n^2 + 5n}{2}
\end{aligned}$$

But note that the LHS is a telescoping sum, so we have:

$$\begin{aligned}
\sum_{k=1}^{n}[(k+1)^3 - k^3] &= (n+1)^3 - 1^3 \\
&= n^3 + 3n^2 + 3n + 1 - 1 \\
&= n^3 + 3n^2 + 3n
\end{aligned}$$

Equating the two, we have:

$$
\begin{aligned}
n^3+3n^2+3n &= 3\sum_{k=1}^{n}k^2+\frac{3n^2+5n}{2}\\
3\sum_{k=1}^{n}k^2 &= n^3+3n^2+3n-\frac{3n^2+5n}{2}\\
&= \frac{2n^3+6n^2+6n-3n^2-5n}{2}\\
&= \frac{2n^3+3n^2+n}{2}\\
&= \frac{n(2n^2+3n+1)}{2}\\
&= \frac{n(n+1)(2n+1)}{2}\\
\sum_{k=1}^{n}k^2 &= \frac{n(n+1)(2n+1)}{6}
\end{aligned}
$$

4). $\sum_{k=1}^{n}k^3=\frac{n^2(n+1)^2}{4}$

$$
\begin{aligned}
\sum_{k=1}^{n}[(k+1)^4-k^4] &= \sum_{k=1}^{n}[k^4+4k^3+6k^2+4k+1-k^4]\\
&= \sum_{k=1}^{n}[4k^3+6k^2+4k+1]\\
&= 4\sum_{k=1}^{n}k^3+6\sum_{k=1}^{n}k^2+4\sum_{k=1}^{n}k+\sum_{k=1}^{n}1\\
&= 4\sum_{k=1}^{n}k^3+6\frac{n(n+1)(2n+1)}{6}+4\frac{n(n+1)}{2}+n\\
&= 4\sum_{k=1}^{n}k^3+n(n+1)(2n+1)+2n(n+1)+n\\
&= 4\sum_{k=1}^{n}k^3+(n^2+n)(2n+1)+2n^2+2n+n\\
&= 4\sum_{k=1}^{n}k^3+2n^3+3n^2+n+2n^2+3n\\
&= 4\sum_{k=1}^{n}k^3+2n^3+5n^2+4n
\end{aligned}
$$

But note that the LHS is a telescoping sum, so we have:

$$\begin{aligned}\sum_{k=1}^{n}[(k+1)^4 - k^4] &= (n+1)^4 - 1^4 \\ &= n^4 + 4n^3 + 6n^2 + 4n + 1 - 1 \\ &= n^4 + 4n^3 + 6n^2 + 4n\end{aligned}$$

Equating the two, we have:

$$\begin{aligned}n^4 + 4n^3 + 6n^2 + 4n &= 4\sum_{k=1}^{n} k^3 + 2n^3 + 5n^2 + 4n \\ 4\sum_{k=1}^{n} k^3 &= n^4 + 2n^3 + n^2 \\ &= n^2(n^2 + 2n + 1) \\ &= n^2(n+1)^2 \\ \sum_{k=1}^{n} k^3 &= \frac{n^2(n+1)^2}{4}\end{aligned}$$

<u>Proof</u>

By induction:

1). $\sum_{k=1}^{n} 1 = n$

Base: $n = 1$

$$\sum_{k=1}^{1} 1 = 1$$

Inductive Assumption:

Assume $\sum_{k=1}^{n} 1 = n$

$$\sum_{k=1}^{n+1} 1 = 1 + \sum_{k=1}^{n} 1 = 1 + n = n + 1$$

2). $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Base: $n = 1$

$$\sum_{k=1}^{1} k = 1$$

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1$$

Inductive Assumption:

Assume $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$$\begin{aligned}
\sum_{k=1}^{n+1} k &= (n+1) + \sum_{k=1}^{n} k \\
&= n + 1 + \frac{n(n+1)}{2} \\
&= \frac{2(n+1) + n(n+1)}{2} \\
&= \frac{2n + 2 + n^2 + n}{2} \\
&= \frac{n^2 + 3n + 2}{2} \\
&= \frac{(n+1)(n+2)}{2} \\
&= \frac{(n+1)[(n+1)+1]}{2}
\end{aligned}$$

3). $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Base: $n = 1$

$$\sum_{k=1}^{1} k^2 = 1^2 = 1$$

$$\frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = \frac{6}{6} = 1$$

Inductive Assumption:

Assume $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= (n+1)^2 + \sum_{k=1}^{n} k^2 \\
&= (n+1)^2 + \frac{n(n+1)(2n+1)}{6} \\
&= \frac{(n+1)[6(n+1) + n(2n+1)]}{6} \\
&= \frac{(n+1)(6n+6+2n^2+n)}{6} \\
&= \frac{(n+1)(2n^2+7n+6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)[(n+1)+1][2(n+1)+1]}{6}
\end{aligned}$$

4). $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

Base: $n = 1$

$$\sum_{k=1}^{1} k^3 = 1$$

$$\frac{1^2(1+1)^2}{4} = \frac{2^2}{4} = \frac{4}{4} = 1$$

Inductive Assumption:

Assume $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

$$\begin{aligned}
\sum_{k=1}^{n+1} k^3 &= (n+1)^3 + \sum_{k=1}^{n} k^3 \\
&= (n+1)^3 + \frac{n^2(n+1)^2}{4} \\
&= \frac{(n+1)^2[4(n+1)+n^2]}{4} \\
&= \frac{(n+1)^2(n^2+4n+4)}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= \frac{(n+1)^2[(n+1)+1]^2}{4}
\end{aligned}$$