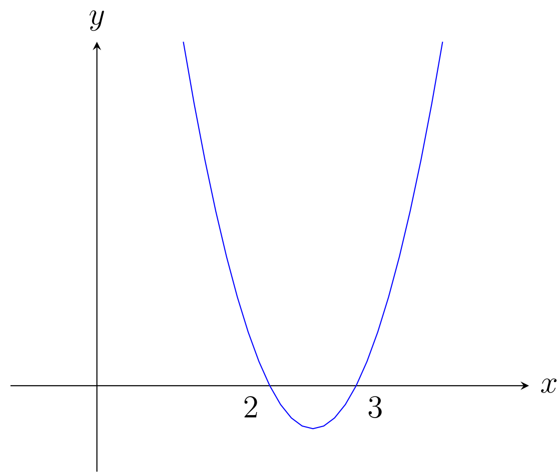
Limits

Example

Consider the quadratic function $f(x) = x^2 - 5x + 6$:



What happens to $f(x)$ as $x \to 2$, but $x \neq 2$?

x	$f(x)$
2.1	-0.09
2.01	-0.0099
2.001	-0.000999
2	
1.999	0.001001
1.99	0.0101
1.9	0.11

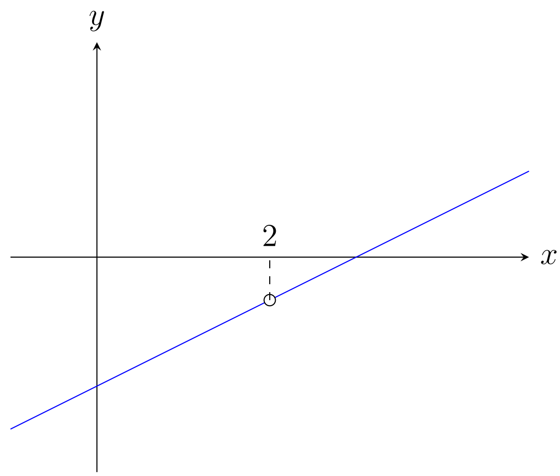
It appears that $f(x) \to 0$ as $x \to 2$ (from either direction).

In the previous example, it turns out that $f(x)$ is actually defined at $x = 2$ and furthermore, $f(2) = 0$. This special case will be used later as a formal definition of *continuity*. However, as previously stated, we don't actually care about the function value at $x = 2$. In fact, the function might not even be defined at the x value in question.

Example

Consider the rational function:

$$f(x) = \frac{x^2 - 5x + 6}{x - 2}$$

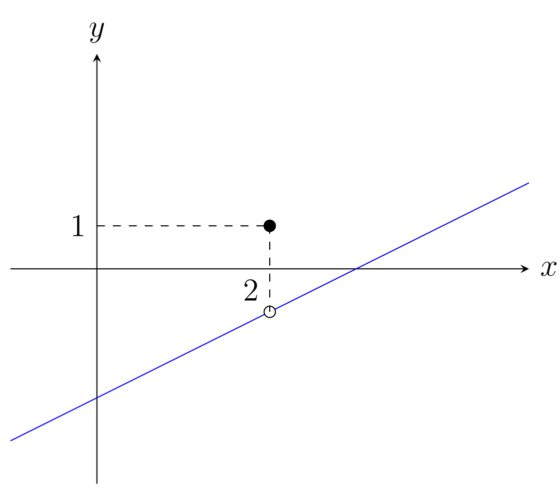


Now, as $x \to 2$:

x	$f(x)$
2.1	-0.9
2.01	-0.99
2.001	-0.999
2	
1.999	-1.001
1.99	-1.01
1.9	-1.1

It appears that $f(x) \to -1$ as $x \to 2$ (from either direction), even though $f(2)$ is not defined. To reiterate, we do not care what actually happens at $x = 2$. In fact, let's define $f(2) = 1$:

$$f(x) = \begin{cases} \frac{x^2-5x+6}{x-2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

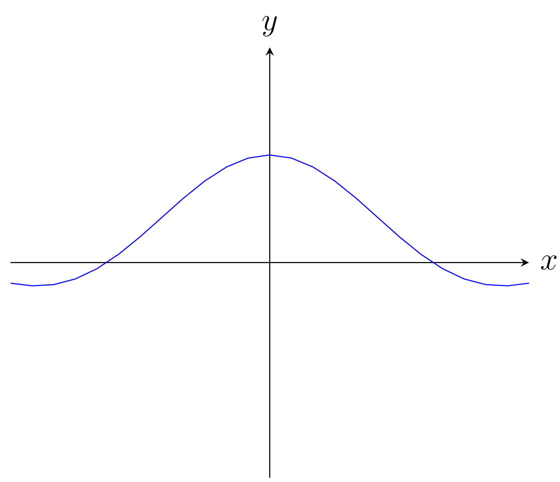


Still, $f(x) \to -1$ as $x \to 2$, regardless of the fact that $f(2) = 1$. Once again, we do not care about the function at $x = 2$; we only care what happens near $x = 2$.

Example

Consider the function:

$$f(x) = \frac{\sin x}{x}$$



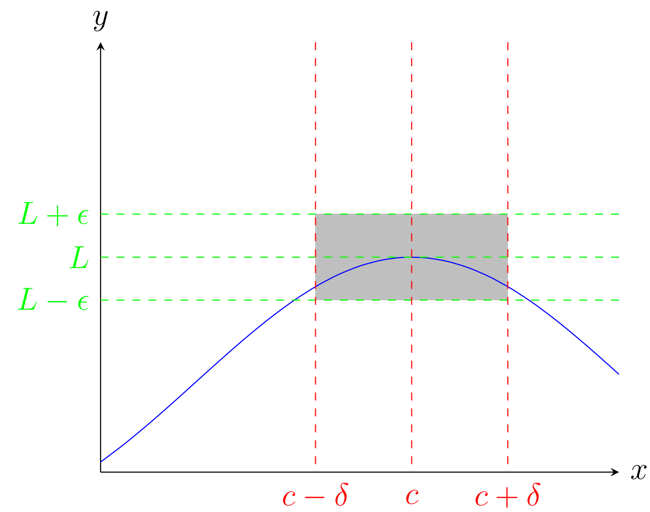
As $x \to 0$:

x	$f(x)$
1	0.841471
0.1	0.998334
0.01	0.999983
0	
-0.01	0.999983
-0.1	0.998334
-1	0.841471

It appears that $f(x) \to 1$ as $x \to 0$. Note that at $x = 0$, $f(x) = \frac{0}{0}$, which is a so-called *indeterminate form*; we cannot tell if the function is actually defined at $x = 0$ or not. In this case it is and $f(0) = 1$.

Definition: Limit of a Function at a Point

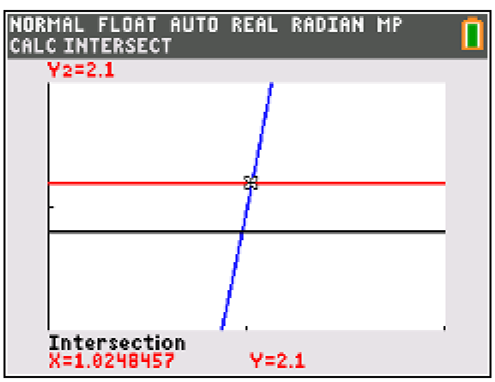
Let $f(x)$ be a function on $\mathbb{R}$. To say that the *limit* of $f(x)$ at $x = c$ is L, denoted by $\lim_{x \to c} f(x) = L$, means that $f(x) \to L$ as $x \to c$ but $x \neq c$. In other words, for all $\epsilon > 0$ there exists some $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$.

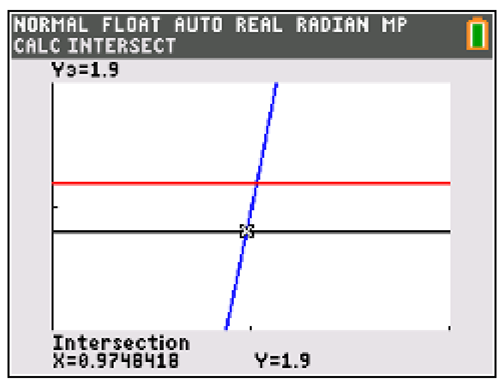


Select an $\epsilon > 0$ and then find a $\delta > 0$ such that $f(x)$ is contained in the bounding box. As $\epsilon \to 0$, this forces $\delta \to 0$ and the bounding box converges to the point (c, L). This does not imply that $f(c) = L$. In fact since $|x - c| > 0$, $x \neq c$ so we don't care what actually happens at $x = c$.

Example

Consider the function $f(x) = x^2 + 2x - 1$ and note that $\lim\limits_{x \to 1} = 2$. Find a suitable δ to two decimal places for $\epsilon = 0.1$.



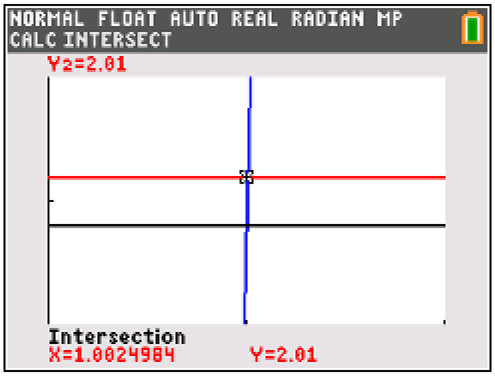


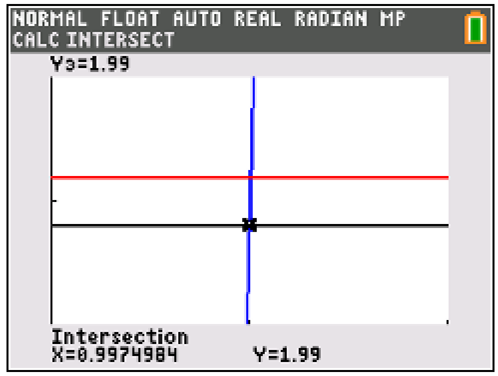
$$\delta_1 = 1.0248457 - 1 = 0.0248457$$
$$\delta_2 = 1 - 0.9748418 = 0.0251582$$

$$\delta = \min\{\delta_1, \delta_2\} = 0.0248457$$

Be sure to round down: $\delta = 0.24$.

Find a suitable δ to four decimal places for $\epsilon = 0.01$.





$$\delta_1 = 1.0024984 - 1 = 0.0024984$$
$$\delta_2 = 1 - 0.9974984 = 0.0025016$$

$$\delta = \min\{\delta_1, \delta_2\} = 0.0024984$$

Be sure to round down: $\delta = 0.0024$.