Completion of Normed Spaces

Definition: Completion

Let E be a normed space over a field $\mathbb{F}$. To say that a normed space $\tilde{E}$ is a *completion* of E means:

1). There exists an injective mapping $\Phi : E \to \tilde{E}$ such that $\forall\, \vec{x}, \vec{y} \in E$ and $\forall\, \alpha, \beta \in \mathbb{F}$:

$$\Phi(\alpha\vec{x} + \beta\vec{y}) = \alpha\Phi(\vec{x}) + \beta\Phi(\vec{y})$$

2). $\|\Phi(\vec{x})\|_{\sim} = \|\vec{x}\|$

3). $\Phi(E)$ is dense in $\tilde{E}$.

4). $\tilde{E}$ is complete (Banach).

Definition: Equivalence

Let E be a normed space and let $(\vec{x}_n)$ and $(\vec{y}_n)$ be Cauchy in E. To say that $(\vec{x}_n)$ and $(\vec{y}_n)$ are *equivalent* in E means:

$$\|\vec{x}_n - \vec{y}_n\| \to 0$$

Theorem

Every normed space E has a completion $\tilde{E}$.

Proof

Let $[(\vec{x}_n)]$ denote the set of all sequences in E that are equivalent to $(\vec{x}_n)$.
Let $\tilde{E}$ be the set of all equivalence classes of sequences in E.
Equip $\tilde{E}$ with the binary operations:

- $[(\vec{x}_n)] + [(\vec{y}_n)] = [(\vec{x}_n + \vec{y}_n)]$
- $\lambda[(\vec{x}_n)] = [(\lambda\vec{x}_n)]$

Assume $(\vec{x}_n)_1 \sim (\vec{x}_n)_2$ and $(\vec{y}_n)_1 \sim (\vec{y}_n)_2$:

$$\begin{aligned}
\|((\vec{x}_n)_1 + (\vec{y}_n)_1) - ((\vec{x}_n)_2 + (\vec{y}_n)_2)\| &= \|((\vec{x}_n)_1 - (\vec{x}_n)_2) + ((\vec{y}_n)_1 - (\vec{y}_n)_2)\| \\
&\leq \|(\vec{x}_n)_1 - (\vec{x}_n)_2\| + \|(\vec{y}_n)_1 - (\vec{y}_n)_2\| \\
&\to 0 + 0 \\
&= 0
\end{aligned}$$

and:

$$\|\lambda(\vec{x}_n)_1 - \lambda(\vec{x}_n)_2\| = |\lambda|\, \|(\vec{x}_n)_1 - (\vec{x}_n)_2\| \to 0$$

Therefore, the operations are well-defined and $\tilde{E}$ is a vector space.

Now, define $\|[\vec{x}_n]\|_\sim = \|[\vec{x}_n]\|_1 = \lim_{n\to\infty} \|\vec{x}_n\|$.
By previous theorem, this norm always converges for Cauchy $(\vec{x}_n)$.
Furthermore, if $(\vec{x}_n) \sim (\vec{y}_n)$:

$$|\|\vec{x}_n\| - \|\vec{y}_n\|| \leq \|\vec{x}_n - \vec{y}_n\| \to 0$$

And so $\lim_{n\to\infty} \|\vec{x}_n\| = \lim_{n\to\infty} \|\vec{y}_n\|$.

Now, define $\Phi : E \to \tilde{E}$ by $\Phi(\vec{x}) = [(\vec{x}, \vec{x}, \ldots)]$.
Clearly, Φ satisfies the linearity conditions under the binary operations.

Since every $[(x_n)]$ is the result of some sequence in E, $\Phi(E)$ is dense in $\tilde{E}$.

Assume (X_n) be a sequence in $\tilde{E}$.
Since $\Phi(E)$ is dense in $\tilde{E}$, $\exists\, (\vec{x}_n)$ in E such that:

$$\|\Phi(\vec{x}_n) - X_n\| < \frac{1}{n}$$

And so:

$$\begin{array}{rcl}
\|\vec{x}_n - \vec{x}_m\| & = & \|\Phi(\vec{x}_n) - \Phi(\vec{x}_m)\| \\
& = & \|(\Phi(\vec{x}_n) - X_n) + (X_n - X_m) + (X_m - \Phi(\vec{x}_m))\| \\
& \leq & \|\Phi(\vec{x}_n) - X_n\| + \|X_n - X_m\| + \|X_m - \Phi(\vec{x}_m)\| \\
& \leq & \frac{1}{n} + \|X_n - X_m\| + \frac{1}{m} \\
& \to & 0 + 0 + 0 \\
& = & 0
\end{array}$$

Therefore, $(\vec{x}_n)$ is Cauchy in E.

Finally, let $X = [(\vec{x}_n)]$:

$$\begin{array}{rcl}
\|X_n - X\| & = & \|(X_n - \Phi(\vec{x}_n)) + (\Phi(\vec{x}_n) - X)\| \\
& \leq & \|X_n - \Phi(\vec{x}_n)\| + \|\Phi(\vec{x}_n) - X\| \\
& \leq & \frac{1}{n} + \|\Phi(\vec{x}_n) - X\| \\
& \to & 0 + 0 \\
& = & 0
\end{array}$$

Therefore E is complete (Banach).

Therefore there exists $\tilde{E}$ which is a completion of E.