Joint Distributions

Definition: Joint Distribution

Let X and Y be two discrete random variables associated with the same sample space. The *joint* pmf is a function $F : \mathbb{R}^2 \to \mathbb{R}$, denoted $f(X, Y)$ where:

$$f(x,y) = \begin{cases} P(X = x, Y = y) & \text{for all feasible } (x,y) \\ 0 & \text{otherwise} \end{cases}$$

that satisfies the following:

1) $p(x, y) \geq 0$
2) $p(x, y) > 0$ for a countable number of (x, y) pairs.
3) $\displaystyle\sum_x \sum_y p(x, y) = 1$
4) The probability that (x, y) occurs in a set A is given by:

$$P[(X,Y) \in A] = \sum_{(x,y) \in A} \sum p(x,y)$$

A joint discrete distribution is easily represented by a table with the possible values of X and Y across the top and left, respectively, and the *marginal* pdfs $f_X(x)$ and $f_Y(y)$ across the bottom and right, respectively, and the joint pmf $f(x, y)$ in the middle.

Example

Toss two fair dice. Let $X =$ their sum and let $Y =$ the absolute value of their difference.

y \ x	2	3	4	5	6	7	8	9	10	11	12	$f_Y(y)$
0	$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$	$\frac{6}{36}$
1		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{10}{36}$
2			$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$			$\frac{8}{36}$
3				$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$				$\frac{6}{36}$
4					$\frac{2}{36}$		$\frac{2}{36}$					$\frac{4}{36}$
5						$\frac{2}{36}$						$\frac{2}{36}$
$f_X(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	

- $P(X \leq 4, Y \leq 2) = \frac{6}{36}$
- $P(X \leq 5) = \frac{10}{36}$
- $P(X \geq 11, Y \geq 2) = 0$
- $P(Y \leq 1) = \frac{16}{36}$

Definition: Conditional Probability

Let X and Y be two discrete random variables with joint pmf $f(x, y)$. The *conditional* pmf of Y given $X = x$ (with $f_X(x) \neq 0$) is given by:

$$f(y \mid x) = \frac{f(x, y)}{f_X(x)}$$

for all feasible y.

Example

From the previous example, the conditional pmfs of Y given $X = x$:

y \ x	2	3	4	5	6	7	8	9	10	11	12
0	1		$\frac{1}{3}$		$\frac{1}{5}$		$\frac{1}{5}$		$\frac{1}{3}$		1
1		1		$\frac{1}{2}$		$\frac{1}{3}$		$\frac{1}{2}$		1	
2			$\frac{2}{3}$		$\frac{2}{5}$		$\frac{2}{5}$		$\frac{2}{3}$		
3				$\frac{1}{2}$		$\frac{1}{3}$		$\frac{1}{2}$			
4					$\frac{2}{5}$		$\frac{2}{5}$				
5						$\frac{1}{3}$					

And the conditional pmfs of X given $Y = y$:

y \ x	2	3	4	5	6	7	8	9	10	11	12
0	$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$
1		1		$\frac{1}{5}$		$\frac{1}{5}$		$\frac{1}{5}$		$\frac{1}{5}$	
2			$\frac{1}{4}$		$\frac{1}{4}$		$\frac{1}{4}$		$\frac{1}{4}$		
3				$\frac{1}{3}$		$\frac{1}{3}$		$\frac{1}{3}$			
4					$\frac{1}{2}$		$\frac{1}{2}$				
5						1					

- Y given $X = 6$

y	0	2	4
$f(y \mid x = 6)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

- Y given $X = 4$

y	0	2
$f(y \mid x = 4)$	$\frac{1}{3}$	$\frac{2}{3}$

- X given $Y = 3$

x	5	7	9
$f(x \mid y = 3)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

- X given $Y = 0$

x	2	4	6	8	10	12
$f(x \mid y = 0)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Definition

Independence Let X and Y be two discrete random variables. To say that X and Y are *independent* means that:

$$\forall\, x, y, f(x, y) = f_X(x) f_Y(y)$$

Example

In the previous example, X and Y are not independent because:

$$f_X(2) f_Y(0) = \frac{1}{36} \cdot \frac{6}{36} = \frac{1}{216} \neq \frac{1}{36} = f(2, 0)$$

In fact, if the joint pmf has 0's then the variables are never independent.

Example

The following joint and marginal pmf's demonstrate independent variables:

y \ x	0	1	2	$p_Y(y)$
-1	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{3}$
1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
$p_X(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	

Theorem

Let X and Y be two discrete random variables. If for all x and y:

$$f\left(y \mid x\right) = f_Y(y) \text{ or } f\left(x \mid y\right) = f_X(x)$$

then X and Y are independent.