

Exponential Distribution

Definition: Exponential PDF

To say that a continuous random variable X has an *exponential distribution* with parameter $\lambda > 0$, denoted $X \sim \text{Exp}(\lambda)$, means that it has the following pdf:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \lambda \int_0^{\infty} e^{-\lambda x} dx \\ &= \lambda \left(-\frac{1}{\lambda} \right) e^{-\lambda x} \Big|_0^{\infty} \\ &= -e^{-\lambda x} \Big|_0^{\infty} \\ &= e^{-\lambda x} \Big|_{\infty}^0 \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Theorem: Exponential CDF

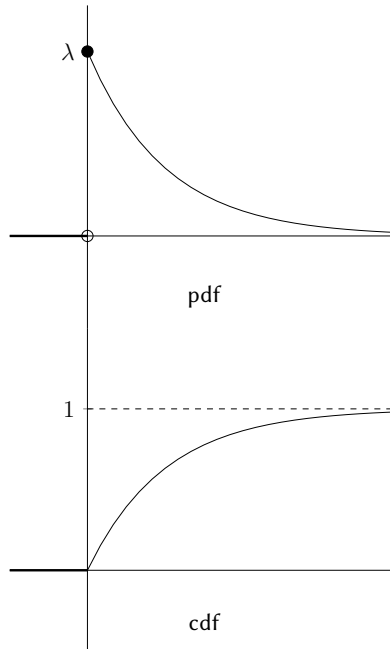
Let X be a continuous random variable such that $X \sim \text{Exp}(\lambda)$. The cdf of X is given by:

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof.

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(x) dx \\ &= \lambda \int_0^x e^{-\lambda x} dx \\ &= \lambda \left(-\frac{1}{\lambda} \right) e^{-\lambda x} \Big|_0^x \\ &= -e^{-\lambda x} \Big|_0^x \\ &= e^{-\lambda x} \Big|_x^0 \\ &= 1 - e^{-\lambda x} \end{aligned}$$

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Corollary

Let X be a continuous random variable such that $X \sim \text{Exp}(\lambda)$:

$$P(a \leq X \leq b) = e^{-\lambda a} - e^{-\lambda b}$$

Corollary

Let X be a continuous random variable such that $X \sim \text{Exp}(\lambda)$:

$$\bar{F}(x) = e^{-\lambda x}$$

Theorem: Exponential Mean and Variance

Let X be a continuous random variable such that $X \sim \text{Exp}(\lambda)$:

- $E(X) = \frac{1}{\lambda}$
- $V(X) = \frac{1}{\lambda^2}$

Proof.

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \lambda \int_0^{\infty} x e^{-\lambda x} dx$$

$$\begin{aligned} u &= x & dv &= e^{-\lambda x} dx \\ du &= dx & v &= -\frac{1}{\lambda} e^{-\lambda x} \end{aligned}$$

$$E(X) = \lambda \left(-\frac{1}{\lambda} x e^{-\lambda x} \Big|_0^{\infty} + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda x} dx \right)$$

$$\begin{aligned}
&= \lambda \left(0 - \frac{1}{\lambda^2} e^{-\lambda x} \Big|_0^\infty \right) \\
&= - \frac{1}{\lambda} e^{-\lambda x} \Big|_0^\infty \\
&= \frac{1}{\lambda} e^{-\lambda x} \Big|_\infty^0 \\
&= \frac{1}{\lambda} (1 - 0) \\
&= \frac{1}{\lambda}
\end{aligned}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \lambda \int_0^{\infty} x^2 e^{-\lambda x} dx$$

$$\begin{array}{rcl}
x^2 & e^{-\lambda x} \\
2x & -\frac{1}{\lambda} e^{-\lambda x} \\
2 & \frac{1}{\lambda^2} e^{-\lambda x} \\
& -\frac{1}{\lambda^3} e^{-\lambda x}
\end{array}$$

$$\begin{aligned}
E(X^2) &= \lambda \left(-\frac{1}{\lambda} x^2 e^{-\lambda x} - \frac{2}{\lambda^2} x e^{-\lambda x} - \frac{2}{\lambda^3} e^{-\lambda x} \right) \Big|_0^\infty \\
&= \left(-x^2 e^{-\lambda x} - \frac{2}{\lambda} x e^{-\lambda x} - \frac{2}{\lambda^2} e^{-\lambda x} \right) \Big|_0^\infty \\
&= \left(x^2 e^{-\lambda x} + \frac{2}{\lambda} x e^{-\lambda x} + \frac{2}{\lambda^2} e^{-\lambda x} \right) \Big|_\infty^0 \\
&= \left(0 + 0 + \frac{2}{\lambda^2} \right) - (0 + 0 + 0) \\
&= \frac{2}{\lambda^2}
\end{aligned}$$

$$V(X) = E(X^2) - E(X)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

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Example

The lifetime of a certain brand of lightbulbs is exponentially distributed with an average of 1000 hours. What is the probability that a new lightbulb will still be working after this amount of time? Between 1000 and 2000 hours?

$$\frac{1}{\lambda} = 1000 \implies \lambda = \frac{1}{1000}$$

$$P(1000 < X) = 1 - P(X \leq 1000) = 1 - F(1000) = 1 - \left(1 - e^{-\frac{1000}{1000}} \right) = e^{-1} \approx 0.3679$$

$$P(1000 \leq X \leq 2000) = e^{-\frac{1000}{1000}} - e^{-\frac{2000}{1000}} = e^{-1} - e^{-2} \approx 0.2325$$

Theorem: The Memoryless Property

Let X be a continuous random variable such that $X \sim \text{Exp}(\lambda)$:

$$\forall t_0, t \geq 0, P(X \geq t_0 + t | X \geq t_0) = P(X \geq t)$$

Proof. Assume $t_0, t \geq 0$:

$$\begin{aligned} P(X \geq t_0 + t | X \geq t_0) &= \frac{P((X \geq t_0 + t) \cap (X \geq t_0))}{P(X \geq t_0)} \\ &= \frac{P(X \geq t_0 + t)}{P(X \geq t_0)} \\ &= \frac{\bar{F}(t_0 + t)}{\bar{F}(t_0)} \\ &= \frac{e^{-\lambda(t_0 + t)}}{e^{-\lambda t_0}} \\ &= e^{\lambda t} \\ &= \bar{F}(t) \\ &= P(X \geq t) \end{aligned}$$

Note that the event $X \geq t_0$ must have occurred if the event $X \geq t_0 + t$ has occurred. ■

The exponential distribution is the only distribution that has the memoryless property.

Example

Jones figures that the total number of thousands of miles an auto can be driven before it would need to be junked is an exponential random variable with parameter $\lambda = \frac{1}{20}$. Smith has a used car that he claims has been driven only 10,000 miles. If Jones purchases the car, what is the probability that he would get at least 20,000 additional miles out of it? Repeat under the assumption that the lifetime mileage of the car is not exponentially distributed but rather is (in thousands of miles) uniformly distributed over $[0, 40]$.

Exponential:

$$P(X \geq 30 | X \geq 10) = P(X \geq 20) = \bar{F}(20) = e^{-\frac{20}{20}} = e^{-1} = 0.3679$$

Uniform:

$$P(X \geq 30 | X \geq 10) = \frac{P((X \geq 30) \cap (X \geq 10))}{P(X \geq 10)} = \frac{P(X \geq 30)}{P(X \geq 10)} = \frac{\bar{F}(30)}{\bar{F}(10)} = \frac{(1 - \frac{30}{40})}{(1 - \frac{10}{40})} = \frac{1}{3}$$