Fourier Coefficients

Definition: Fourier Expansion

Let E be an inner product space and let $(\vec{x}_n)$ be an orthonormal sequence in E. $\forall\, \vec{x} \in E$, the expansion:

$$x \sim \sum_{n=1}^{\infty} \langle \vec{x}, \vec{x}_n \rangle \, \vec{x}_n$$

is called the *Fourier expansion* of $\vec{x}$ with respect to $(\vec{x}_n)$ and the $\langle \vec{x}, \vec{x}_n \rangle$ are called the *generalized Fourier coefficients* of the expansion.

Theorem

Let H be a Hilbert space over $\mathbb{C}$ and let $(\vec{x}_n)$ be an orthonormal sequence in H:

$$\sum_{n=1}^{\infty} \alpha_n \vec{x}_n \text{ converges} \iff (\alpha_n) \text{ is a sequence in } \ell^2.$$

Proof

Let $S_n = \displaystyle\sum_{k=1}^{n} \alpha_k \vec{x}_k$ and $s_n = \displaystyle\sum_{k=1}^{n} |\alpha_k|^2$.

AWLOG: $n < m$.

$$\|S_m - S_n\|^2 = \left\| \sum_{k=n+1}^{m} S_k \right\|^2 = \left\| \sum_{k=n+1}^{m} \alpha_k \vec{x}_k \right\|^2 = \sum_{k=n+1}^{m} \|\alpha_k \vec{x}_k\|^2 = \sum_{k=n+1}^{m} |\alpha_k|^2 = |s_m - s_n|$$

Thus, (S_n) is Cauchy iff (s_n) is Cauchy.

$$\begin{aligned}
\sum_{n=1}^{\infty} \alpha_n \vec{x}_n \text{ converges} \quad &\iff \quad (S_n) \text{ converges} \\
&\iff \quad (S_n) \text{ is Cauchy (since } H \text{ is Hilbert)} \\
&\iff \quad (s_n) \text{ is Cauchy} \\
&\iff \quad (s_n) \text{ converges} \\
&\iff \quad (\alpha_n) \text{ is in } \ell^2.
\end{aligned}$$

Corollary

Let H be a Hilbert space and $(\vec{x}_n)$ be an orthonormal sequence in H. $\forall\, \vec{x} \in E$:

$$\sum_{n=1}^{\infty} \langle \vec{x}, \vec{x}_n \rangle \, \vec{x}_n \text{ converges.}$$

Proof

$(\langle \vec{x}, \vec{x}_n \rangle)$ is a sequence in ℓ^2.

Therefore $\displaystyle\sum_{n=1}^{\infty} \langle \vec{x}, \vec{x}_n \rangle \, \vec{x}_n$ converges.

Definition: Complete

Let E be an inner product space and let $(\vec{x}_n)$ be an orthonormal sequence in E. To say that $(\vec{x}_n)$ is complete means $\forall \, \vec{x} \in E$:

$$\vec{x} = \sum_{n=1}^{\infty} \langle \vec{x}, \vec{x}_n \rangle \, \vec{x}_n$$

If H is a Hilbert space and $(\vec{x}_n)$ is an orthonormal sequence in H such $\forall \, \vec{x} \in E$, the Fourier expansion for $\vec{x}$ converges, but not necessarily to $\vec{x}$.

Let $H = L^2[-\pi, \pi]$ and $\langle f, g \rangle = \displaystyle\int_{-\pi}^{\pi} f \overline{g}$.

Let $f_n(t) = \dfrac{1}{\sqrt{\pi}} \sin nt$

$$\begin{aligned}
\langle f_n, f_m \rangle &= \left\langle \frac{1}{\sqrt{\pi}} \sin(nt), \frac{1}{\sqrt{\pi}} \sin(mt) \right\rangle \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nt) \sin(mt) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos(n-m)t - \cos(n+m)t] dt \\
&= \frac{1}{2\pi} \left[\frac{1}{n-m} \sin(n-m)t - \frac{1}{n+m} \sin(n+m)t \right]_{-\pi}^{\pi} \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\langle f_n, f_n \rangle &= \left\langle \frac{1}{\sqrt{\pi}} \sin(nt) \right\rangle \frac{1}{\sqrt{\pi}} \sin(nt) \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(nt) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} [1 - \cos(2nt)] dt \\
&= \frac{1}{2\pi} \left[t - \frac{1}{2n} \sin(2nt) \right]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi} [\pi - (-\pi)] \\
&= \frac{1}{2\pi} (2\pi) \\
&= 1
\end{aligned}
$$

Thus, (f_n) is an orthonormal sequence in $L^2[-\pi, \pi]$.

Now, let $f(t) = \cos t$.

$$
\begin{aligned}
\langle f, f_n \rangle &= \left\langle \cos t, \frac{1}{\sqrt{\pi}} \sin(nt) \right\rangle \\
&= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \cos(t) \sin(nt) dt \\
&= \frac{1}{2\sqrt{\pi}} \int_{-\pi}^{\pi} [\sin(1+n)t - \sin(1-n)t] dt \\
&= \frac{1}{2\sqrt{\pi}} \left[-\frac{1}{1+n} \cos(1+n)t + \frac{1}{1-n} \cos(1-n)t \right]_{-\pi}^{\pi} \\
&= 0
\end{aligned}
$$

Thus, all of the Fourier coefficients, and therefore:

$$
\sum_{n=1}^{\infty} \langle f, f_n \rangle f_n = 0 \neq \cos t
$$