Generating Sets

Theorem

Let G be a group and $\{H_i, i \in I\}$ such that $\forall\, i \in I, H_i \leq G$:

$$\bigcap_{i \in I} H_i \leq G$$

Proof

Assume $h \in \bigcap_{i \in I} H_i$
$\forall\, i \in I, h \in H_i$
But $H_i \leq G$
So $h \in G$
$\therefore \bigcap_{i \in I} H_i \subseteq G$

Assume $h_1, h_2 \in \bigcap_{i \in I} H_i$
$\forall\, i \in I, h_1, h_2 \in H_i$
So $\forall\, i \in I, h_1 h_2 \in H_i$
$\therefore \bigcap_{i \in I} H_i$ is closed under the operation

$\forall\, i \in I, e \in H_i$
$\therefore e \in \bigcap_{i \in I} H_i$

Assume $h \in \bigcap_{i \in I} H_i$
$\forall\, i \in I, h \in H_i$
So $\forall\, i \in I, h^{-1} \in H_i$
$h^{-1} \in \bigcap_{i \in I} H_i$
$\therefore \bigcap_{i \in I} H_i$ is closed under inverses

$\therefore \bigcap_{i \in I} H_i \leq G$

Definition

Let G be a group and $S \subseteq G, S \neq \emptyset$. The subgroup of G generated by S is given by:

$$\langle S \rangle = \bigcap_{\substack{H \leq G \\ S \subseteq H}} H$$

To say that G is *generated* by S means $\langle S \rangle = G$; the elements of S are called *generators* of G.

To say that G is *finitely generated* means S is finite.

Corollary

Let G be a group and $S \subseteq G, S \neq \emptyset$:

1). $\langle S \rangle \leq G$

2). $\langle S \rangle$ is the smallest subgroup of G containing S

Proof

Let all of the H be represented by $\{H_i, i \in I\}$
$\therefore \langle S \rangle = \bigcap_{i \in I} H_i \leq G$

Assume $K \leq G$ such that $S \subseteq K$
$\exists\, k \in I, K = H_k$
$\therefore \langle S \rangle = \bigcap_{i \in I} H_i \leq H_k = K$

Definition

Let G be a group and $S \subseteq G, S \neq \emptyset$. A *word* of S is a product:

$$\prod s_i^{n_i}$$

where $s_i \in S$ and $n_i \in \mathbb{Z}$.

Note that the s_i can be repeated, and cannot be grouped unless G is abelian.

The elements of $\langle S \rangle$ are:

1). Elements of S

2). Powers of elements of S

3). Words consisting of powers of elements of S

Theorem

Let G be a group and $S \subseteq G, S \neq \emptyset$. $\langle S \rangle$ is exactly the set of all finite words formed by the elements of S.

Proof

Let K be the set of all finite words formed by the elements of S
Clearly, $K \subseteq \langle S \rangle$

Assume $k_1, k_2 \in K$
k_1 and k_2 are words formed by the elements of S
So $k_1 k_2$ is also a word formed by the elements of S
$\therefore K$ is closed under the operation

Assume $k \in K$
$k^0 = e$
$\therefore e \in K$

Assume $k \in K$
Let $k = \prod_{i=1}^{n} s_i^{j_i}$
$k^{-1} = (\prod_{i=1}^{n} s_i^{j_i})^{-1} = \prod_{i=1}^{n} (s_{n-i})^{-j_{n-i}}$
So k^{-1} is a word formed by the elements of S

$k^{-1} \in K$
$\therefore K$ is closed under inverses

$\therefore K \leq \langle S \rangle$

But $\forall\, s \in S, s^1 = s \in K$
So $K \leq G$ containing S
But $\langle S \rangle$ is the smallest such subgroup
So $\langle S \rangle \leq K$

$\therefore \langle S \rangle = K$