Transpositions

Definition

A *transposition* is a cycle of length 2.

Theorem

Every permutation σ of a finite set A with at least 2 elements can be written as a product of transpositions.

Proof

Since each σ of A can be expressed as a product of disjoint cycles, AWLOG that σ contains one cycle
The case $n = 2$ is trivial, so assume $n > 2$

Proof by induction on the length of the cycle n

Base Case: $n = 3$
$(x_1 x_2 x_3) = (x_1 x_2)(x_2 x_3)$

Assume $(x_1 x_2 x_3 \dots x_n) = (x_1 x_2)(x_2 x_3) \dots (x_{n-1} x_n)$

$(x_1 x_2 x_3 \dots x_n x_{n+1}) = (x_1 x_2 x_3 \dots x_{n-1} x_n)(x_n x_{n+1}) = (x_1 x_2)(x_2 x_3) \dots (x_{n-1} x_n)(x_n x_{n+1})$

Example

$(12345678) = (12)(23)(34)(45)(56)(67)(78)$
$(12345678) = (18)(17)(16)(15)(14)(13)(12)$

Corollary

An n-cycle can be represented using $n - 1$ transpositions.

Definition

A permutation σ on a set A that can be expressed as an even number of transpositions is called *even*. Otherwise, it is called *odd*.

Theorem

The evenness or oddness of a permutation is well-defined.

Proof

Assumed $\sigma \in S_n$ is expressed as a composition of transpositions
Associate σ with its corresponding permutation matrix
The determinant of the matrix is either 1 or -1, depending on the either an odd (-1) or even (1) number of transpositions
Therefore the evenness or oddness is well-defined.