Triangle Inequality

For ℓ_1:

$$\begin{aligned} \|\vec{x}+\vec{y}\|_1 &= \sum_{k=1}^{n} |x_i + y_i| \\ &\leq \sum_{k=1}^{n} (|x_i| + |y_i|) \\ &= \sum_{k=1}^{n} |x_i| + \sum_{k=1}^{n} |y_i|) \\ &= \|\vec{x}\|_1 + \|\vec{y}\|_1 \end{aligned}$$

For ℓ_∞:

$$\begin{aligned} \|\vec{x}+\vec{y}\|_\infty &= \max_{1\leq k\leq n} |x_i + y_i| \\ &\leq \max_{1\leq k\leq n} (|x_i| + |y_i|) \\ &\leq \max_{1\leq k\leq n} |x_i| + \max_{1\leq k\leq n} |y_i|) \\ &= \|\vec{x}\|_\infty + \|\vec{y}\|_\infty \end{aligned}$$

For ℓ_2:

Lemma

Let $a, b \in \mathbb{C}$:

$$2\,|a|\,|b| \leq |a|^2 + |b|^2$$

Proof

$(|a| - |b|)^2 \geq 0$
$|a|^2 - 2\,|a|\,|b| + |b|^2 \geq 0$

$\therefore 2\,|a|\,|b| \leq |a|^2 + |b|^2$

Theorem: Cauchy-Schwarz

Let $\vec{x}, \vec{y} \in \mathbb{C}^n$:

$$\sum_{k=1}^{n} x_k y_k \leq \left|\sum_{k=1}^{n} x_k y_k\right| \leq \sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |y_k|^2\right)^{\frac{1}{2}}$$

Proof

$$\begin{aligned}
\left(\sum_{k=1}^{n}|x_k|\,|y_k|\right)^2 &= \sum_{k=1}^{n}|x_k|^2\,|y_k|^2 + 2\sum_{i<j}|x_i|\,|y_i|\,|x_j|\,|y_j| \\
&= \sum_{k=1}^{n}|x_k|^2\,|y_k|^2 + 2\sum_{i<j}(|x_i|\,|y_j|)(|y_i|\,|x_j|) \\
&\leq \sum_{k=1}^{n}|x_k|^2\,|y_k|^2 + \sum_{i<j}(|x_i|^2\,|y_j|^2 + |y_i|^2\,|x_j|^2) \\
&= \sum_{1\leq i,j\leq n}|x_i|^2\,|y_j|^2 \\
&= \left(\sum_{k=1}^{n}|x_k|^2\right)\left(\sum_{k=1}^{n}|y_k|^2\right) \\
\therefore \sum_{k=1}^{n}|x_k y_k| &\leq \left(\sum_{k=1}^{n}|x_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}|y_k|^2\right)^{\frac{1}{2}}
\end{aligned}$$

Theorem

Let $\vec{x}, \vec{y} \in \mathbb{C}^n$:

$$\|\vec{x}+\vec{y}\|_2 \leq \|\vec{x}\|_2 + \|\vec{y}\|_2$$

Proof

$$\begin{aligned}
\sum_{k=1}^{n}|x_k+y_k|^2 &= \sum_{k=1}^{n}|x_k^2 + 2x_k y_k + y_k^2| \\
&\leq \sum_{k=1}^{n}(|x_k|^2 + |2x_k y_k| + |y_k|^2) \\
&= \sum_{k=1}^{n}|x_k|^2 + 2\sum_{k=1}^{n}|x_k y_k| + \sum_{k=1}^{n}|y_k|^2 \\
&\leq \sum_{k=1}^{n}|x_k|^2 + 2\left(\sum_{k=1}^{n}|x_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}|y_k|^2\right)^{\frac{1}{2}} + \sum_{k=1}^{n}|y_k|^2 \\
&= \left[\left(\sum_{k=1}^{n}|x_k|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{n}|y_k|^2\right)^{\frac{1}{2}}\right]^2 \\
\left(\sum_{k=1}^{n}|x_k+y_k|^2\right)^{\frac{1}{2}} &= \left(\sum_{k=1}^{n}|x_k|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{n}|y_k|^2\right)^{\frac{1}{2}} \\
\therefore \|\vec{x}+\vec{y}\|_2 &\leq \|\vec{x}\|_2 + \|\vec{y}\|_2
\end{aligned}$$