Unitary Operators

Definition: Unitary

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. To say that T is a *unitary* operator means:

$$T^*T = TT^* = I$$

Theorem

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$:

T is unitary $\iff$ T is invertible and $T^{-1} = T^*$.

Proof

T is unitary $\iff$ $T^*T = TT^* = I$ $\iff$ T is invertible and $T^{-1} = T^*$.

Properties

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ such that T is unitary:

1). $\mathcal{R}(T) = H$

2). T is isometric.

3). T is a Hilbert space isomorphism on H.

In fact, T unitary $\implies$ T isometric; however, T isometric and onto $\implies$ T unitary.

Theorem

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ such that T is unitary:

T^{-1} and T^* are unitary.

Proof

Since $T^{-1} = T^*$, it is sufficient to show that T^* is unitary.

$(T^*)^*T^* = TT^* = I$ and $T^*(T^*)^* = T^*T = I$

Therefore T^*, and hence T^{-1}, are unitary.

Examples

1). $H = \ell^2(\mathbb{Z})$ (bi-infinite)

$$\ell^2(Z) = \left\{ (z_n)_{n\in\mathbb{Z}} \middle| \sum_{-\infty}^{\infty} |z_n| < \infty \right\}$$

$$\langle x, y \rangle = \sum_{-\infty}^{\infty} x_n \overline{y_n}$$

Let S be shift right by one position: $S(x_n) = (x_{n-1})$

$$\langle Sx, y\rangle = \sum_{-\infty}^{\infty} x_{n-1}\overline{y_n} = \sum_{-\infty}^{\infty} x_n\overline{y_{n+1}} = \langle x, S^*y\rangle$$

So $S^*(y_n) = (y_{n+1})$, or a left shift by one position.

$SS^* = S^*S = I$ and therefore S is unitary.

2). $H = L^2[0,1]$ and $(Tf)(t) = f(1-t)$

$$\begin{aligned}
\|Tf\|^2 &= \int_0^1 |(Tf)(t)|^2\,dt \\
&= \int_0^1 |f(1-t)|^2\,dt \\
&= \int_1^0 |f(s)|^2\,(-ds) \\
&= \int_0^1 |f(s)|\,ds \\
&= \|f\|
\end{aligned}$$

Therefore T is isometric.

Now, assume $g \in H$.
Let $f \in H$ such that $f(t) = g(1-t)$.
$(Tf)(t) = (Tg)(1-t) = g(1-(1-t)) = g(t)$

Therefore T is onto.

Therefore T is unitary.