Math-42 Sections 01, 02, 05

Homework #4 Solutions

Problem

One of the most important definitions in mathematics (calculus in particular) is that of the limit of a sequence. Consider the infinite sequence: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. Such a sequence can be represented as follows:

$$a_n = \frac{1}{n}, n \in \mathbb{N}$$

Note that the elements of the sequence get arbitrarily close to 0 as $n \to \infty$. We call such a point the *limit* of the sequence.

The formal definition for "L is the limit of a sequence a_n" is as follows:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, (n > N \to |a_n - L| < \epsilon)$$

Negate this proposition to obtain the definition of "L is NOT the limit of a sequence a_n."

The following a step-by-step application of DeMorgan, but you could jump directly to the solution:

$$\begin{aligned}
&\overline{\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, (n > N \to |a_n - L| < \epsilon}) \equiv \\
&\exists \epsilon > 0, \overline{\exists N \in \mathbb{N}, \forall n \in \mathbb{N}, (n > N \to |a_n - L| < \epsilon}) \equiv \\
&\exists \epsilon > 0, \forall N \in \mathbb{N}, \overline{\forall n \in \mathbb{N}, (n > N \to |a_n - L| < \epsilon}) \equiv \\
&\exists \epsilon > 0, \forall N \in \mathbb{N}, \exists n \in \mathbb{N}, \overline{n > N \to |a_n - L| < \epsilon} \equiv \\
&\exists \epsilon > 0, \forall N \in \mathbb{N}, \exists n \in \mathbb{N}, (n > N \text{ and } |a_n - L| \geq \epsilon)
\end{aligned}$$

Remember that $\overline{p \to q} \equiv p \wedge \bar{q}$. Also note that technically the parentheses are required since quantifiers have the highest precedence.