Important Theorems

Separation

- X is $T_1 \iff \forall\, p \in X, \{p\}$ is closed in X
- $T_2 \implies T_1$
- $T_3 \implies T_2$
- $T_4 \implies T_3$
- X regular $\iff \forall\, p \in X, \forall\, U_p \in \mathscr{T}, \exists\, V_p \in \mathscr{T}, \overline{V_p} \subset U_p$
- X normal $\iff \forall\, A \in X$ closed, $\forall\, U_A \in \mathscr{T}, \exists\, V_A \in \mathscr{T}, \overline{V_A} \subset U_A$
- X, Y $T_2 \implies X \times Y$ T_2
- X, Y regular $\implies X \times Y$ regular
- T_2 is hereditary
- Regular is hereditary
- X normal and $A \subset X$ closed $\implies A$ normal

Separable

- D dense in $X \iff \forall\, U \in \mathscr{T}, U \neq \emptyset \implies U \cap D \neq \emptyset$
- X, Y separable $\implies X \times Y$ separable
- X 2^{nd} countable $\implies X$ separable
- X 2^{nd} countable and $A \subset X$ uncountable $\implies A$ has a limit point.
- 2^{nd} countable is hereditary
- X, Y 2^{nd} countable $\implies X \times Y$ 2^{nd} countable
- 2^{nd} countable $\implies 1^{st}$ countable
- 1^{st} countable is hereditary
- X, Y 1^{st} countable $\implies X \times Y$ 1^{st} countable

Compact

- X finite $\implies$ X compact
- X compact $\implies \forall A \subset X, A$ infinite $\implies$ A has a limit point
- X compact $\iff$ $\forall \mathcal{A} = \{A_\alpha : \alpha \in \lambda\}$ such that the A_α are closed, $\mathcal{A}$ has the finite intersection property $\implies \bigcap \mathcal{A} \neq \emptyset$
- X compact $\iff$ $\forall U \in \mathscr{T}, \forall \mathcal{K} = \{K_\alpha : \alpha \in \lambda\}$ such that the $\mathcal{K}_\alpha$ are closed and $\bigcap \mathcal{K} \subset U$, there exists $\mathcal{K}' \subset \mathcal{K}$ such that $\mathcal{K}'$ finite and $\mathcal{K}' \subset U$
- $\mathcal{A} = \{A_\alpha : \alpha \in \lambda\}$ such that the A_α compact $\implies \bigcup \mathcal{A}$ compact
- X compact and subspace A closed $\implies$ A compact
- X T_2 and subspace A compact $\implies$ A closed
- X compact and T_2 $\implies$ X normal.
- $[a, b]$ compact.
- A, B compact $\implies$ $A \times B$ compact.
- $A \subset \mathbb{R}^n$ compact $\iff$ A is closed and bounded.
- Any product of compact spaces is compact.

Continuity

- $X - f^{-1}(A) = f^{-1}(Y - A)$
- f bijection $\implies$ $f(A) = Y - f(X - A)$
- The constant function is continuous
- The inclusion function is continuous
- A restricted function of a continuous function is continuous
- f continuous $\iff$ $\forall K \subset Y, K$ closed $\implies$ $f^{-1}(K)$ closed
- f continuous $\iff$ $\forall A \subset X, f(\bar{A}) \subset \overline{f(A)}$
- f continuous $\iff$ $\forall x \in X, \forall V \in \mathcal{N}_{f(x)}, \exists U \in \mathcal{N}_x, f(U) \subset V$
- f 1^{st} countable $\implies$ $f : X \to Y$ continuous $\iff$ $\forall (x_n), x_n \to x \implies f(x_n) \to f(x)$
- $f : X \to Y$ and $g : Y \to Z$ continuous $\implies$ $g \circ f : X \to Z$ continuous
- X compact and $f : X \to Y$ continuous and surjective $\implies$ Y compact
- D dense in X and $f : X \to Y$ continuous and surjective $\implies$ $f(D)$ dense in Y
- X normal and $f : X \to Y$ continuous, surjective, and closed $\implies$ Y is normal
- X compact and Y T_2 $\implies$ $f : X \to Y$ continuous $\implies$ f closed

- $f: X \to Y$ bijective $\implies f$ open $\iff f$ closed
- f continuous and closed $\implies f(\bar{A}) = \overline{f(A)}$
- X normal and f continuous, surjective, and closed $\implies Y$ normal
- X compact, Y T_2, f continuous $\implies f$ closed

Homeomorphism

- All $(a, b) \subset \mathbb{R}$ are homeomorphic to each other and are homeomorphic to $\mathbb{R}$
- f continuous, TFAE:
  1. f is a homeomorphism
  2. f is a closed bijection
  3. f is an open bijection
- f X compact, Y T_2, f continuous $\implies f$ a homeomorphism.