Invertible Operators

Definition: Inverse

Let E be a vector space and let A be an operator on some subspace of A. To say that an operator B on $\mathcal{R}(A)$ is an *inverse* of A means:

- $\forall \vec{x} \in \mathcal{D}(A), (BA)\vec{x} = x$
- $\forall \vec{y} \in \mathcal{R}(A), (AB)\vec{y} = y$

If B exists then A is said to be *invertible* and B can be denoted by $B = A^{-1}$.

Properties

Let E be a vector space and let A and B be linear operators on some subspace of E.

1). A is invertible $\implies$ A^{-1} is unique.

2). A is invertible $\implies$ A^{-1} linear.

3). A is invertible $\iff$ $\ker(A) = \{\vec{0}\}$.

4). A is invertible and $\{\vec{x}_1, \ldots, \vec{x}_n\}$ is a linearly independent set $\implies$ $\{A\vec{x}_1, \ldots, A\vec{x}_n\}$ is a linearly independent set.

5). A and B invertible $\implies$ AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof

1). Assume A is invertible.

Assume S and T are inverses of A.

$S = SI = S(AT) = (SA)T = IT = I$

2). Assume A is invertible.

Assume $\vec{y}_1, \vec{y}_2 \in \mathcal{R}(A)$.

$\exists\, \vec{x}_1, \vec{x}_2 \in \mathcal{D}(A)$ such that $\vec{y}_1 = A\vec{x}_1$ and $\vec{y}_2 = A\vec{x}_2$.

$A^{-1}\vec{y}_1 = \vec{x}_1$ and $A^{-1}\vec{y}_2 = \vec{x}_2$.

Assume $\alpha, \beta \in \mathbb{F}$:

$$\begin{aligned}
A^{-1}(\alpha\vec{y}_1 + \beta\vec{y}_2) &= A^{-1}(\alpha A\vec{x}_1 + \beta A\vec{x}_2) \\
&= A^{-1}A(\alpha\vec{x}_1 + \beta\vec{x}_2) \\
&= \alpha\vec{x}_1 + \beta\vec{x}_2 \\
&= (\alpha A^{-1}\vec{y}_1 + \beta A^{-1}\vec{y}_2)
\end{aligned}$$

Therefore A^{-1} is linear.

3). $\implies$ Assume A is invertible.

$\vec{x} \in \ker(A) \iff A\vec{x} = \vec{0} \iff A^{-1}A\vec{x} = A^{-1}\vec{0} \iff I\vec{x} = \vec{0} \iff \vec{x} = \vec{0}$

$\Longleftarrow$ Assume $\ker(A) = \{\vec{0}\}$.

A is one-to-one.
But A is also onto $\mathcal{R}(A)$.

Therefore A is bijective and thus invertible.

4). Assume A is invertible.

Assume $\sum_{k=1}^{n} \alpha_k A\vec{x}_k = \vec{0}$:

$$\begin{aligned} A\sum_{k=1}^{n} \alpha_k \vec{x} &= \vec{0} \\ A^{-1}A\sum_{k=1}^{n} \alpha_k \vec{x} &= A^{-1}\vec{0} \\ \sum_{k=1}^{n} \alpha_k \vec{x} &= \vec{0} \end{aligned}$$

But $\{\vec{x}_1, \ldots, \vec{x}_n\}$ is a linearly independent set.

Therefore $\alpha_k = 0$ and thus $\{A\vec{x}_1, \ldots, A\vec{x}_n\}$ is a linearly independent set.

5). Assume A and B are invertible.

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = BB^{-1} = I$

Therefore, since the inverse is unique, AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Corollary

Let E be a finite dimensional vector space and let A be a linear operator on E. If A is invertible then A is onto.

Proof

Assume A is invertible.
Assume $\dim(E) = n$.
Assume $\{\vec{x}_1, \ldots, \vec{x}_n\}$ is a basis for E.
$\{\vec{x}_1, \ldots, \vec{x}_n\}$ is a linearly independent set.
So $\{A\vec{x}_1, \ldots, A\vec{x}_n\}$ is a linearly independent set.
Thus $\{A\vec{x}_1, \ldots, A\vec{x}_n\}$ is also a basis for E.
Now assume $\vec{y} \in E$.

$\exists\, \alpha_k \in \mathbb{F}$ such that $\vec{y} = \sum_{k=1}^{n} \alpha_k A\vec{x}_k = A\sum_{k=1}^{n} \alpha_k A\vec{x}_k$.

But $\vec{x} = \sum_{k=1}^{n} \alpha_k A\vec{x}_k \in \mathcal{D}(A) = E$ and so $A\vec{x} = \vec{y}$.

Therefore A is onto.

Example

Let $E = \ell^2$ and $A(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, z_3, \ldots)$.

A is linear, one-to-one, and invertible; however, A is not onto.

Note that the inverse of a bounded linear operator need not be bounded:

Example

Consider $T : \ell^2 \to \ell^2$ defined by $T(z_1, z_2, z_3, \ldots) = \left(z_1, \frac{z_2}{2}, \frac{z_3}{3}, \ldots\right)$.

Claim: T is bounded.

Assume $z = (z_n) \in \ell^2$:

$$\|Tz\|^2 = \sum_{k=1}^{\infty} |Tz_k|^2 = \sum_{k=1}^{\infty} \left|\frac{z_k}{k}\right|^2 = \sum_{k=1}^{\infty} \frac{1}{k^2} |z_k|^2 \leq \sum_{k=1}^{\infty} |z_k|^2 = \|z\|^2$$

Thus $\|Tz\| \leq \|z\|$ and so $\|T\| \leq 1$ (bounded).

Claim: T is invertible.

Let $T^{-1}(z_1, z_2, z_3, \ldots) = (z_1, 2z_2, 3z_3, \ldots)$.
$T(T^{-1}z) = T^{-1}(Tz) = z$

Therefore T and T^{-1} are inverses.

Claim: T^{-1} is unbounded.

$\left\|T^{-1}e_n\right\| = \|ne_n\| = n\,\|e_n\| = n \cdot 1 = n$

So $\left\|T^{-1}e_n\right\| \to \infty$ as $n \to \infty$.
But $\|e_n\| = 1$.

Therefore T^{-1} is unbounded.

If E is finite dimensional and A on E is invertible, then both A and A^{-1} are bounded because all operators on finite dimensional vector spaces are bounded.

Theorem

Let H be a Hilbert space and let $A \in \mathcal{B}(H)$ such that $\exists\, m > 0$ such that $\forall\, \vec{x} \in H, \|A\vec{x}\| \geq m\,\|\vec{x}\|$:

> A is invertible and A^{-1} is bounded.

<u>Proof</u>

$A\vec{x} = \vec{0} \iff \vec{x} = \vec{0}$

Therefore A is invertible.

Assume $\vec{y} \in H$.
Since A^{-1} is onto, $\exists\, \vec{x} \in H$ such that $A^{-1}\vec{y} = \vec{x}$.
And so $A\vec{x} = \vec{y}$.
By assumption, $m \|\vec{x}\| \leq \|A\vec{x}\|$.

Therefore $\left\|A^{-1}\vec{y}\right\| \leq \dfrac{1}{m} \|\vec{y}\|$ and thus A^{-1} is bounded.

The number $m(A) = \inf\limits_{\|\vec{x}\|=1} \|A\vec{x}\|$ is called the *conorm* of A. Thus, if $m(A) > 0$ then A is invertible and $\left\|A^{-1}\right\| = \dfrac{1}{m(A)}$

Note that the previous example failed because $\|Te_n\| = \dfrac{1}{n} \to 0$.

<u>Theorem</u>

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be invertible such that $T^{-1} \in \mathcal{B}(H)$:

$$(T^{-1})^* = (T^*)^{-1}$$

<u>Proof</u>

$(T^{-1})^* T^* = (TT^{-1})^* = I^* = I$

$T^*(T^{-1})^* = (T^{-1}T)^* = I^* = I$

Therefore, since inverses are unique, $(T^{-1})^* = (T^*)^{-1}$.

<u>Corollary</u>

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be invertible such that T is self-adjoint and $T^{-1} \in \mathcal{B}(H)$:

T^{-1} is self-adjoint.

<u>Proof</u>

$(T^{-1})^* = (T^*)^{-1} = T^{-1}$