Unitary Matrices

Definition: Unitary

Let $U \in M_n$. To say that U is *unitary* means:

$$UU^* = U^*U = I_n$$

Example: Unitary Matrices

1). Diagonal matrices of the form:

$$\begin{bmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{bmatrix}$$

2). Diagonal signed matrices where $\theta = 0, \pi$:

$$\begin{bmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1 \end{bmatrix}$$

3). Permutation matrices

4). Products of unitary matrices

Theorem

Let U_n be the set of all $n \times n$ unitary matrices:

U_n is a group.

Proof

Clearly, $I_n \in U_n$ and thus $U_n \neq \emptyset$
Assume $U \in U_n$
By definition, U is invertible with $U^{-1} = U^*$ and so $U_n \subset GL(n)$
Assume $V \in U_n$
$(UV^*)(UV^*)^* = UV^*VU^* = UI_nU^* = UU^* = I_n$
Thus, $UV^* \in U_n$

Therefore, by the subgroup test, $U_n \leq GL(n)$.

Theorem

Let $U \in U_n$:

$$\det(U) = \pm 1$$

Proof

$UU^* = I_n$
$\det(UU^*) = \det(I_n) = 1$
$\det(U)\det(U^*) = 1$
But $\det(U^*) = \overline{\det(U^T)} = \overline{\det(U)}$
$\det(U)\det(U^*) = \det(U)\overline{\det(U)} = |\det(U)|^2 = 1$

$\therefore \det(U) = \pm 1$

Theorem

Let $U \in M_n$. TFAE:

1). U is unitary

2). The columns of U form an orthonormal basis for U_n

3). U preserves inner product:

$$\langle \vec{x}, \vec{y} \rangle = \langle U\vec{x}, U\vec{y} \rangle$$

4). U preserves length:

$$\|U\vec{x}\| = \|x\|$$

Proof

$1 \to 3$: Assume U is unitary

$\langle U\vec{x}, U\vec{y} \rangle = (U\vec{y})^*(U\vec{x}) = \vec{y}^* U^* U\vec{x} = \vec{y}^* I_n \vec{x} = \vec{y}^* \vec{x} = \langle \vec{x}, \vec{y} \rangle$

$3 \to 4$: Assume U preserves inner product

$\|U\vec{x}\|^2 = \langle U\vec{x}, U\vec{x} \rangle = \langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2$
$\therefore \|Ux\| = \|x\|$

$4 \to 2$: Assume U preserves length

WTS: $\langle \vec{u}_i, \vec{u}_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

Consider $\vec{e}_i$, the columns of I_n
The $\vec{e}_i$ are an orthonormal basis for $\mathbb{C}^n$
$\|\vec{e}_i\| = \|U\vec{e}_i\| = \|\vec{u}_i\| = 1$

$\langle \vec{u}_i, \vec{u}_i \rangle = \|\vec{u}_i\|^2 = 1^2 = 1$

Now consider $\vec{e}_i$ and $\vec{e}_j$ where $i \neq j$

$\|ve_i + \vec{e}_j\|^2 = \|U(ve_i + \vec{e}_j)\|^2 = \|U\vec{e}_i + U\vec{e}_j\|^2 = \|\vec{u}_i + \vec{u}_j\|^2$
$\langle \vec{e}_i + \vec{e}_j, \vec{e}_i + \vec{e}_j \rangle = \langle \vec{u}_i + \vec{u}_j, \vec{u}_i + \vec{u}_j \rangle$

$$\langle \vec{e}_i, \vec{e}_i \rangle + \langle \vec{e}_i, \vec{e}_j \rangle + \langle \vec{e}_j, \vec{e}_i \rangle + \langle \vec{e}_j, \vec{e}_j \rangle = \langle \vec{u}_i, \vec{u}_i \rangle + \langle \vec{u}_i, \vec{u}_j \rangle + \langle \vec{u}_j, \vec{u}_i \rangle + \langle \vec{u}_j, \vec{u}_j \rangle$$
$$1 + 0 + 0 + 1 = 1 + \langle \vec{u}_i, \vec{u}_j \rangle + \langle \vec{u}_j, \vec{u}_i \rangle + 1$$

$$\langle \vec{u}_i, \vec{u}_j \rangle + \langle \vec{u}_j, \vec{u}_i \rangle = 0$$

$$\|ve_i + i\vec{e}_j\|^2 = \|U(ve_i + i\vec{e}_j)\|^2 = \|U\vec{e}_i + iU\vec{e}_j\|^2 = \|\vec{u}_i + i\vec{u}_j\|^2$$
$$\langle \vec{e}_i + i\vec{e}_j, \vec{e}_i + i\vec{e}_j \rangle = \langle \vec{u}_i + i\vec{u}_j, \vec{u}_i + i\vec{u}_j \rangle$$
$$\langle \vec{e}_i, \vec{e}_i \rangle + \langle \vec{e}_i, i\vec{e}_j \rangle + \langle i\vec{e}_j, \vec{e}_i \rangle + \langle i\vec{e}_j, i\vec{e}_j \rangle = \langle \vec{u}_i, \vec{u}_i \rangle + \langle \vec{u}_i, i\vec{u}_j \rangle + \langle i\vec{u}_j, \vec{u}_i \rangle + \langle i\vec{u}_j, i\vec{u}_j \rangle$$
$$\langle \vec{e}_i, \vec{e}_i \rangle - i \langle \vec{e}_i, \vec{e}_j \rangle + i \langle \vec{e}_j, \vec{e}_i \rangle + |i|^2 \langle i\vec{e}_j, i\vec{e}_j \rangle = \langle \vec{u}_i, \vec{u}_i \rangle - i \langle \vec{u}_i, \vec{u}_j \rangle + i \langle \vec{u}_j, \vec{u}_i \rangle + |i|^2 \langle \vec{u}_j, \vec{u}_j \rangle$$
$$1 - 0 + 0 + 1 = 1 - i \langle \vec{u}_i, \vec{u}_j \rangle + i \langle \vec{u}_j, \vec{u}_i \rangle + 1$$
$$-i \langle \vec{u}_i, \vec{u}_j \rangle + i \langle \vec{u}_j, \vec{u}_i \rangle = 0$$

$$- \langle \vec{u}_i, \vec{u}_j \rangle + \langle \vec{u}_j, \vec{u}_i \rangle = 0$$

Sum and difference $= 0$

$$\therefore \langle \vec{u}_i, \vec{u}_j \rangle = \langle \vec{u}_j, \vec{u}_i \rangle = 0$$

$2 \to 1$: Assume the columns of U form an orthonormal basis for $\mathbb{C}^n$

$$(U^*U)_{ij} = \langle \vec{u}_i, \vec{u}_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$U^*U = I_n$$

$$(U^*U)^* = I_n^* = I_n$$

$$U^*U = I_n$$

Therefore U is unitary.