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Math 275A
Homework #11

Theorem: 7.15

Let X and Y be topological spaces. If X is compact and $f : X \to Y$ is continuous and surjective then Y is compact.

Proof. Assume that X is compact and $f : X \to Y$ is continuous and surjective. Assume that $\{V_\alpha : \alpha \in \lambda\}$ is an open cover for Y. Since f is continuous, each $f^{-1}(V_\alpha) \in \mathscr{T}_X$. Furthermore, since f is surjective, $f^{-1}(\bigcup\limits_{\alpha \in \lambda} V_\alpha) = \bigcup\limits_{\alpha \in \lambda} f^{-1}(V)$ is an open cover of X. But X is compact, so there exists a finite subcover $\{f^{-1}(V_1), \ldots, f^{-1}(V_n)\}$ of X. And since f is surjective $\{V_1, \ldots, V_n\}$ is a finite subcover for Y. Therefore Y is compact. ■

Theorem: 7.18

Let X and Y be topological spaces. If D is dense in X and $f : X \to Y$ is continuous and surjective then $f(D)$ dense in Y.

Proof. Assume that D is dense in X and $f : X \to Y$ is continuous and surjective. Assume that $V \in \mathscr{T}_Y$ and $V \neq \emptyset$. Since f is continuous, $f^{-1}(V) \in \mathscr{T}_Y$. Furthermore, since f is surjective, $f^{-1}(V) \neq \emptyset$, and since D is dense in X, $f^{-1}(V) \cap D \neq \emptyset$. Therefore $f(U) \cap f(D) \neq \emptyset$ and thus $f(D)$ is dense in Y. ■

Example: Exercise 7.20

1. An open function that is not continuous.

   Consider $f : \mathbb{R}_{\text{cof}} \to \mathbb{R}_{\text{coc}}$ defined by $f(x) = x$. Since every open set in the cofinite topology is open in the cocountable topology, f is open. However, $\mathbb{R} - \mathbb{Q}$ is open in the cocountable topology but not in cofinite topology and so f is not continuous.

2. A closed function that is not continuous.

   Consider $f : \mathbb{R}_{\text{cof}} \to \mathbb{R}_{\text{coc}}$ defined by $f(x) = x$. Since every closed set in the cofinite topology is closed in the cocountable topology, f is closed. However, $\mathbb{Q}$ is closed in the cocountable topology but not in cofinite topology and so f is not continuous.

3. A continuous function that is neither open nor closed.

   Consider $f : \mathbb{R}_{\text{dis}} \to \mathbb{R}_{\text{ind}}$ defined by $f(x) = x$. Since the only open (and closed) sets in the indiscrete topology are $\emptyset$ and R, and these sets are also open in the indiscrete topology, f is continuous. However, $[0, 1]$ is open and closed in the discrete topology, but neither in the indiscrete topology, so f is neither open nor closed.

4. A continuous function that is open but not closed.

   Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$, which is continuous. Since $f((a,b)) = (e^a, e^b)$, open sets will always map to open sets. However, $\mathbb{R}$ is closed in $\mathbb{R}$ and $f(\mathbb{R}) = (0, \infty)$, which is not closed in R. Thus, f is open but not closed.

5. A continuous function that is closed but not open.

   Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) \to y_0$. This was already shown to be continuous. Note that $\{y\}$ is closed in $\mathbb{R}$ so closed sets will always map to closed sets; however, open sets will also map to the closed set and thus f is closed but not open.

Lemma

Let X and Y be topological spaces and let $f : X \to Y$ be continuous and closed. For all $A \subset X$, $f(\bar{A}) = \overline{f(A)}$.

Proof. Assume that $A \subset X$. Since f is continuous, $f(\bar{A}) \subset \overline{f(A)}$. Now, since $A \subset \bar{A}$, $f(A) \subset f(\bar{A})$. Furthermore, $\bar{A}$ is closed and f is closed, so $f(\bar{A})$ is closed. But $\overline{f(A)}$ is the smallest closed set containing $f(A)$, and so $f(A) \subset \overline{f(A)} \subset f(\bar{A})$. Therefore $f(\bar{A}) = \overline{f(A)}$. ∎

Theorem: 7.21

Let X and Y be topological spaces. If X is normal and $f : X \to Y$ is continuous, surjective, and closed then Y is normal.

Proof. Assume that X is normal and $f : X \to Y$ is continuous, surjective, and closed. Assume that B is closed in Y and assume that $V \in \mathscr{T}_Y$ such that $B \subset V$. Since f is continuous, $f^{-1}(B)$ is closed in X and $f^{-1}(V) \in \mathscr{T}_X$ with $f^{-1}(B) \subset f^{-1}(V)$. Now, since X is normal, there exists $U \in \mathscr{T}_X$ such that $f^{-1}(B) \subset U$ and $\bar{U} \subset f^{-1}(V)$. Now, since $U \in \mathscr{T}_X$, $X - U$ is closed in X. Since f is closed, $f(X - U)$ is closed in Y and thus $Y - f(X - U) \subset f(U) \in \mathscr{T}_Y$.

Claim: $B \subset Y - f(X - U)$

Assume $y \in B$. Since f is surjective, y is mapped and all such $x \in f^{-1}(B) \subset U$. Thus, $x \notin X - U$, so $y = f(x) \notin f(X - U)$, and hence $y \in Y - f(X - U)$.

Claim: $\overline{Y - f(X - U)} \subset V$

Since $Y - f(X - U) \subset f(U)$, $\overline{Y - f(X - U)} \subset \overline{f(U)}$. Now, since $\bar{U} \subset f^{-1}(V)$, $f(\bar{U}) \subset f(f^{-1}(V)) \subset V$. But f is continuous, so $f(\bar{U}) = \overline{f(U)}$, and so $\overline{Y - f(X - U)} \subset \overline{f(U)} \subset V$.

Therefore Y is normal. ∎

Theorem: 7.24

Let X and Y be topological spaces such that X is compact and Y is Hausdorff. For all $f : X \to Y$, if f is continuous then f is closed.

Proof. Assume that f is continuous and assume that $A \subset X$ is closed in X. Since X is compact, A is also compact. Now, consider $f(A)$ as a subspace of Y. Since $f|_A$ is surjective, $f(A)$ is compact. Finally, since Y is Hausdorff, $f(A)$ is closed. Therefore f is closed. ■

Theorem: 7.26

Homeomorphic is an equivalence relation.

Proof. Assume that X,Y, and Z are topological spaces.

R: Consider $i_X = i_X^{-1}$, which is continuous. Therefore X is homeomorphic to X.

S: Assume that X is homeomorphic to Y.

Then there exists a homeomorphism $f : X \to Y$. Since f is a homeomorphism, it is invertible and its inverse is continuous. Thus, $f^{-1} : Y \to X$ is a continuous, invertible function and $(f^{-1})^{-1} = f$ is invertible. Therefore Y is homeomorphic to X.

T: Assume that X is homeomorphic to Y and Y is homeomorphic to Z.

Then there exists homeomorphics $f : X \to Y$ and $g : Y \to Z$. So consider $g \circ f : X \to Z$. Since f and g are continuous and invertible, $g \circ f$ is continuous and invertible. Furthermore, since f^{-1} and g^{-1} are continuous, $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$ is continuous. Therefore X is homeomorphic to Z.

■

Lemma

For all $a, b \in \mathbb{R}$ such that $a < b$, (a, b) is homeomorphic to $(0, 1)$.

Proof. Let $f : (0, 1) \to (a, b)$ be defined by $f(t) = a + t(a - b)$. f is linear, and thus continuous and invertible with $f^{-1}(s) = \frac{s-a}{b-a}$ which is also linear and thus continuous. Therefore (a, b) is homeomorphic to $(0, 1)$. ■

Corollary

All open intervals in $\mathbb{R}$ are homeomorphic.

Proof. Assume $(a, b), (c, d) \subset \mathbb{R}$. (a, b) is homeomorphic to $(0, 1)$ and $(0, 1)$ is homeomorphic to (c, d). Therefore, (a, b) is homeomorphic to (c, d). ■

Theorem: 7.27

$(a, b) \subset \mathbb{R}$ is homeomorphic to R.

Proof. (a, b) is homeomorphic to $(-\frac{\pi}{2}, \frac{\pi}{2})$. Now, consider $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ defined by $f(x) = \tan x$. This is a continuous and invertible function whose inverse is also continuous. Thus, $(-\frac{\pi}{2}, \frac{\pi}{2})$ is $\mathbb{R}$. Therefore, (a, b) is homeomorphic to R. ■

Lemma

Let X and Y be topological spaces and let $f : X \to Y$ be bijective. For all $A \subset X$:

$$f(A) = Y - f(X - A)$$

Proof. Assume $A \subset X$.

$(\subset)$ Assume $y \in f(A)$.

Since f is injective, there exists one and only one $x \in X$ such that $y = f(x)$ and that $x \in A$. Thus, $x \notin X - A$ and so $y = f(x) \notin f(X - A)$. Therefore $y \in Y - f(X - A)$.

$(\supset)$ Assume $y \in Y - f(X - A)$.

Thus, $y \notin f(X - A)$ and so there is no $x \in X - A$ such that $y = f(x)$. But f is surjective, and so there is such an $x \in X$ and that $x \in A$. Therefore $y = f(x) \in f(A)$.

■

Theorem: 7.28

Let X and Y be topological spaces and let $f : X \to Y$ be continuous. TFAE:

1. f is a homeomorphism.
2. f is a closed bijection.
3. f is an open bijection.

Proof.

$(1 \implies 2)$ Assume that f is a homeomorphism.

This means that f is a bijection and its inverse is continuous. So assume that $A \subset X$ is closed in X. Since f is bijective, $f(A) = (f^{-1})^{-1}(A)$, and since $(f^{-1})^{-1}$ is continuous, $f(A)$ is also closed. Therefore f is a closed bijection.

$(2 \implies 3)$ Assume that f is a closed bijection.

Assume that $U \in \mathscr{T}_X$. This means that $X - U$ is closed in X, and since f is closed, $f(X - U)$ is closed in Y and so $Y - f(X - U) \in \mathscr{T}_Y$. But f is a bijection and so $Y - f(X - U) = f(U) \in \mathscr{T}_Y$. Therefore, f is an open bijection.

$(3 \implies 1)$ Assume that f is an open bijection.

Assume that $U \in \mathscr{T}_Y$. Since f is continuous, $f^{-1}(U) \in \mathscr{T}_X$. But f is open so $(f^{-1})^{-1}(U) \in \mathscr{T}_Y$. Therefore f^{-1} is continuous and hence f is a homeomorphism.

■

Theorem: 7.29

Let X and Y be topological spaces such that X is compact and Y is Hausdorff and let $f : X \to Y$ be a continuous bijection. f is a homeomorphism.

Proof. Since X is compact, Y is Hausdorff, and f is a bijection, f is closed. Therefore, since f is a continuous closed bijection, f is a homeomorphism. ■