Open and Closed Functions

Definition: Open and Closed Functions

Let X and Y be topological spaces. To say that $f : X \to Y$ is open means that for all $A \in \mathscr{T}_X$, $f(A) \in \mathscr{T}_Y$. Likewise, to say that $f : X \to Y$ is closed means that for all closed $A \subset X$, $f(A) \subset Y$ is closed.

Example

1. An open function that is not continuous.

   Consider $f : \mathbb{R}_{\text{cof}} \to \mathbb{R}_{\text{coc}}$ defined by $f(x) = x$. Since every open set in the cofinite topology is open in the cocountable topology, f is open. However, $\mathbb{R} - \mathbb{Q}$ is open in the cocountable topology but not in cofinite topology and so f is not continuous.

2. A closed function that is not continuous.

   Consider $f : \mathbb{R}_{\text{cof}} \to \mathbb{R}_{\text{coc}}$ defined by $f(x) = x$. Since every closed set in the cofinite topology is closed in the cocountable topology, f is closed. However, $\mathbb{Q}$ is closed in the cocountable topology but not in cofinite topology and so f is not continuous.

3. A continuous function that is neither open nor closed.

   Consider $f : \mathbb{R}_{\text{dis}} \to \mathbb{R}_{\text{ind}}$ defined by $f(x) = x$. Since the only open (and closed) sets in the indiscrete topology are $\emptyset$ and R, and these sets are also open in the indiscrete topology, f is continuous. However, $[0, 1]$ is open and closed in the discrete topology, but neither in the indiscrete topology, so f is neither open nor closed.

4. A continuous function that is open but not closed.

   Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$, which is continuous. Since $f((a, b)) = (e^a, e^b)$, open sets will always map to open sets. However, $\mathbb{R}$ is closed in $\mathbb{R}$ and $f(\mathbb{R}) = (0, \infty)$, which is not closed in R. Thus, f is open but not closed.

5. A continuous function that is closed but not open.

   Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) \to y_0$. This was already shown to be continuous. Note that $\{y\}$ is closed in $\mathbb{R}$ so closed sets will always map to closed sets; however, open sets will also map to the closed set and thus f is closed but not open.

Lemma

Let X and Y be topological spaces and let $f : X \to Y$ be bijective. For all $A \subset X$:

$$f(A) = Y - f(X - A)$$

Proof. Assume $A \subset X$.

$(\subset)$ Assume $y \in f(A)$.

Since f is injective, there exists one and only one $x \in X$ such that $y = f(x)$ and that $x \in A$. Thus, $x \notin X - A$ and so $y = f(x) \notin f(X - A)$. Therefore $y \in Y - f(X - A)$.

($\supset$) Assume $y \in Y - f(X - A)$.

Thus, $y \notin f(X - A)$ and so there is no $x \in X - A$ such that $y = f(x)$. But f is surjective, and so there is such an $x \in X$ and that $x \in A$. Therefore $y = f(x) \in f(A)$.

■

Lemma

Let X and Y be topological spaces and let $f : X \to Y$ be bijective. f is open iff f is closed.

Proof.

$\Longrightarrow$ Assume f is open.

Assume that $A \subset X$ is closed. Then $X - A$ is open, and since f is open, $f(X - A)$ is open and $Y - f(X - A)$ is closed. But f is bijective, so $Y - f(X - A) = f(A)$. Therefore $f(A)$ is closed.

$\Longleftarrow$ Assume f is closed.

Assume that $A \subset X$ is open. Then $X - A$ is closed, and since f is closed, $f(X - A)$ is closed and $Y - f(X - A)$ is open. But f is bijective, so $Y - f(X - A) = f(A)$. Therefore $f(A)$ is open.

■

Lemma

Let X and Y be topological spaces and let $f : X \to Y$ be continuous and closed. For all $A \subset X$, $f(\bar{A}) = \overline{f(A)}$.

Proof. Assume that $A \subset X$. Since f is continuous, $f(\bar{A}) \subset \overline{f(A)}$. Now, since $A \subset \bar{A}$, $f(A) \subset f(\bar{A})$. Furthermore, $\bar{A}$ is closed and f is closed, so $f(\bar{A})$ is closed. But $\overline{f(A)}$ is the smallest closed set containing $f(A)$, and so $f(A) \subset \overline{f(A)} \subset f(\bar{A})$. Therefore $f(\bar{A}) = \overline{f(A)}$. ■

Theorem

Let X and Y be topological spaces. If X is normal and $f : X \to Y$ is continuous, surjective, and closed then Y is normal.

Proof. Assume that X is normal and $f : X \to Y$ is continuous, surjective, and closed. Assume that B is closed in Y and assume that $V \in \mathscr{T}_Y$ such that $B \subset V$. Since f is continuous, $f^{-1}(B)$ is closed in X and $f^{-1}(V) \in \mathscr{T}_X$ with $f^{-1}(B) \subset f^{-1}(V)$. Now, since X is normal, there exists $U \in \mathscr{T}_X$ such that $f^{-1}(B) \subset U$ and $\bar{U} \subset f^{-1}(V)$. Now, since $U \in \mathscr{T}_X$, $X - U$ is closed in X. Since f is closed, $f(X - U)$ is closed in Y and thus $Y - f(X - U) \subset f(U) \in \mathscr{T}_Y$.

Claim: $B \subset Y - f(X - U)$

Assume $y \in B$. Since f is surjective, y is mapped and all such $x \in f^{-1}(B) \subset U$. Thus, $x \notin X - U$, so $y = f(x) \notin f(X - U)$, and hence $y \in Y - f(X - U)$.

Claim: $\overline{Y - f(X - U)} \subset V$

Since $Y - f(X - U) \subset f(U)$, $\overline{Y - f(X - U)} \subset \overline{f(U)}$. Now, since $\bar{U} \subset f^{-1}(V)$, $f(\bar{U}) \subset f(f^{-1}(V)) \subset V$. But f is continuous, so $f(\bar{U}) = \overline{f(U)}$, and so $\overline{Y - f(X - U)} \subset \overline{f(U)} \subset V$.

Therefore Y is normal. ■

Theorem

Let X and Y be topological spaces such that X is compact and Y is Hausdorff. For all $f : X \to Y$, if f is continuous then f is closed.

Proof. Assume that f is continuous and assume that $A \subset X$ is closed in X. Since X is compact, A is also compact. Now, consider $f(A)$ as a subspace of Y. Since $f|_A$ is surjective, $f(A)$ is compact. Finally, since Y is Hausdorff, $f(A)$ is closed. Therefore f is closed. ■