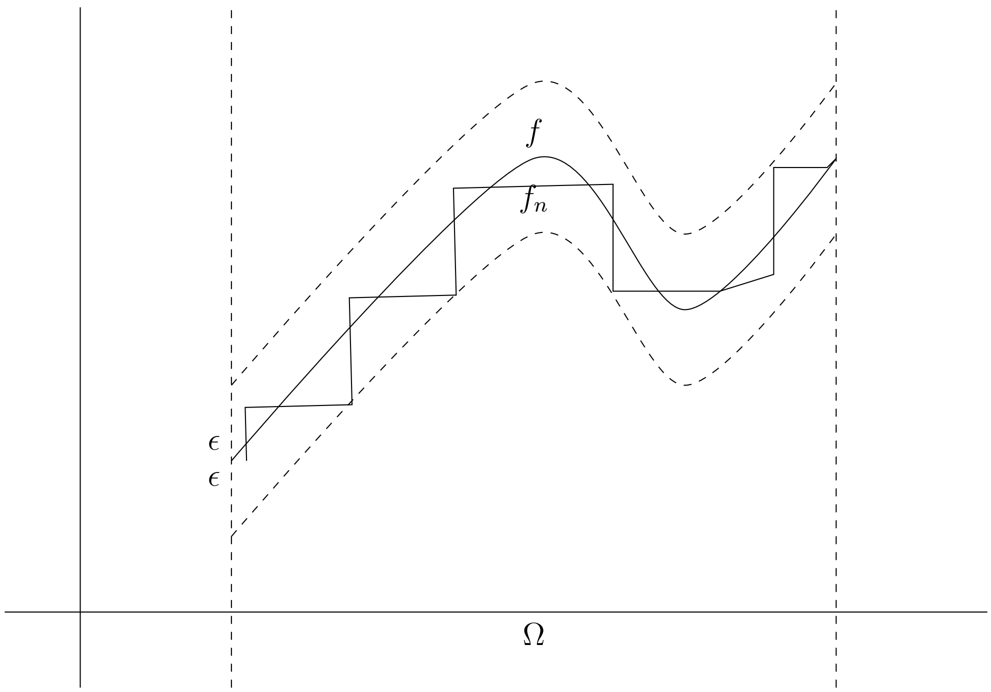
Uniform Convergence

Definition: Uniform Convergence

Let f_n be a sequence of functions. To say that f_n *converges uniformly* to f over an interval Ω, denoted $f_n \rightrightarrows f$, means:

$$\forall\, \epsilon > 0, \exists\, N(\epsilon) > 0, \forall\, x \in \Omega, n > N \implies |f_n(x) - f(x)| < \epsilon$$



Theorem: Uniform Convergence Norm

Let E be a function space and (f_n) be a sequence of functions in E.

$$f_n \rightrightarrows f \iff \|f_n - f\|_\infty \to 0$$

Proof

$$\begin{aligned}
f_n \rightrightarrows f \quad &\iff \quad \forall\, \epsilon > 0, \exists\, N(\epsilon) > 0, \forall\, x, n > N \implies |f_n(x) - f(x)| < \epsilon \\
&\iff \quad \forall\, \epsilon > 0, \exists\, N(\epsilon) > 0, \forall\, x, n > N \implies \max\{|f_n(x) - f(x)|\} < \epsilon \\
&\iff \quad \forall\, \epsilon > 0, \exists\, N(\epsilon) > 0, \forall\, x, n > N \implies \|f_n - f\|_\infty < \epsilon \\
&\iff \quad \|f_n - f\|_\infty \to 0
\end{aligned}$$

But is there such a norm for pointwise convergence?

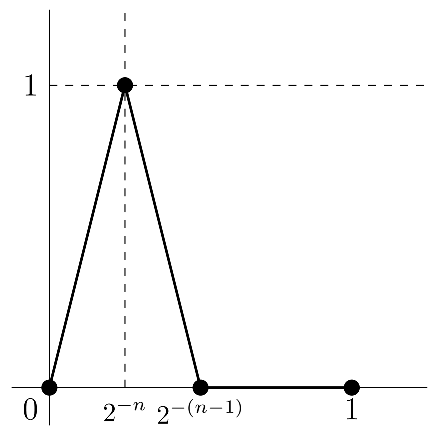
Definition: Pointwise Convergence

Let f_n be a sequence of functions. To say that f_n *converges pointwise* to f over an interval Ω, denoted $f_n \to f$, means:

$$\forall\, x \in \Omega, \forall\, \epsilon > 0, \exists\, N(\epsilon, x) > 0, n > N \implies |f_n(x) - f(x)| < \epsilon$$

Consider the following counterexample. Let $g_n(t) \in \mathcal{C}[0,1]$ be defined by:

$$g_n(t) = \begin{cases} 2^n t, & 0 \le t \le 2^{-n} \\ 2 - 2^n t, & 2^{-n} \le t \le 2^{-(n-1)} \\ 0, & otherwise \end{cases}$$



Assume $\|\cdot\|$ is a norm on $\mathcal{C}[0,1]$. By the properties of the norm, g_n is not the zero function and thus $\|g_n\| \neq 0$. Now, let:

$$f_n = \frac{g_n}{\|g_n\|}$$

By the properties of the norm:

$$\|f_n\| = \left\| \frac{g_n}{\|g_n\|} \right\| = \frac{\|g_n\|}{\|g_n\|} = 1$$

Assume $t \in [0,1]$.
Assume $\epsilon > 0$.
Note that as $t \to 0$, $f_n(t) \to 0$.
So AWLOG $t \neq 0$.
There exists $N > 0$ such that the non-zero part of g_n is pushed to the left of t and thus $f_n(t) \to 0$.
Assume $n > N$:

$$|f_n(t)| = \frac{g(t)}{\|g(t)\|} = \frac{0}{\|g(t)\|} = 0 < \epsilon$$

Thus, $f_n \to 0$ pointwise, but $\|f_n\| \to 1$.

Therefore, there is no suitable norm for pointwise convergence.