Math-71 Sections 9, 11, 12

Homework #5 Solutions

Problem

You are a leader for your local Girl Scout troop and it has fallen upon you to plan next year's cookie sales. It has become common to adjust the price of a box of cookies to maximize profits based on the affluence of the community. From past years, you know that when the price was $5.00 per box about 10,000 boxes were sold. When the price was raised to $6.00 per box, only 7500 boxes were sold. Assume that the demand function $n(p)$ is linear. The factory that makes the cookies reports that the fixed costs are $10,000 and the variable costs are $2.00 per box.

1. At what sales price will your troop maximize its profits?

   We start by recognizing that we what a profit function in terms of price: $P(p)$. Once we have this, we can calculate $P'(p)$ and find critical points. First of all, we know that:

   $$P(p) = R(p) - C(p)$$

   where $R(p)$ is a revenue function and $C(p)$ is a cost function.

   Let's start with $R(p)$. We also know that revenue is quantity (n)times price (p):

   $$R(p) = np$$

   However, n is actually a function p via the demand function:

   $$R(p) = n(p) \cdot$$

   So we need the construct the demand function.

   We are told to assume that the demand function is linear, and we are given two points:

   $$n(\$5) = 10\,000 \text{ boxes}$$
   $$n(\$6) = 7500 \text{ boxes}$$

   This is sufficient information to construct a line:

   $$m = \frac{10000 - 7500}{5 - 6} = -2500 \text{ boxes}/\$$$
   $$n - 10000 = -2500(p - 5)$$
   $$n(p) = -2500p + 22500$$

   We can now construct the revenue function:

   $$R(p) = p \cdot n(p) = p(-2500p + 22500) = -2500p^2 + 22500p$$

Now onto the cost function. We are given the fixed and variable costs, so:

$$\begin{aligned} C(p) &= 10000 + 2n(p) \\ &= 10000 + 2(-2500p + 22500) \\ &= 10000 - 5000p + 45000 \\ C(p) &= 55000 - 5000p \end{aligned}$$

And so, the final profit function is:

$$\begin{aligned} P(p) &= R(p) - C(p) \\ &= (-2500p^2 + 22500p) - (55000 - 5000p) \\ P(p) &= -2500p^2 + 27500p - 55000 \end{aligned}$$

We can no differentiate with respect to p:

$$P'(p) = -5000p + 27500$$

To find all critical points we set this to 0:

$$\begin{aligned} &0 = -5000p + 27500 \\ &5000p = 27500 \\ &p = \$5.50 \end{aligned}$$

Therefore, to maximize profits, boxes should be sold at $5.50 per box.

2. At that price, how many boxes is your troop projected to sell?

$$n(\$5.50) = -2500(5.50) + 22500 = 8750 \text{ boxes}$$

3. What is the expected profit?

$$P(\$5.50) = -2500(5.50)^2 + 27500(5.50) - 55000 = \$20625$$