A Complete Orthonormal Sequence for $L^2[-\pi, \pi]$

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Introduction

Question: Is (φ_n), where $\varphi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, a bi-infinite complete orthonormal sequence in $L^2[-\pi, \pi]$?

- Orthonormality
- Convolution
- Summability Kernels
- The Fejér Kernel
- $\|K_n \star f - f\|_1 \to 0$
- $f \in L^1[-\pi, \pi]$ and $\langle f, \varphi_n \rangle_{L_2} = 0 \implies f \equiv 0$
- $f \in L^2[-\pi, \pi] \implies f \in L^1[-\pi, \pi]$ (Hölder)
- (φ_n) is complete

Orthonormality

- $\int_{-\pi}^{\pi} e^{inx} dx = \begin{cases} 2\pi, & n = 0 \\ 0, & n \in \mathbb{Z} - \{0\} \end{cases}$
- $\int_{-\pi}^{\pi} \varphi_n(x) dx = \begin{cases} \sqrt{2\pi}, & n = 0 \\ 0, & n \in \mathbb{Z} - \{0\} \end{cases}$
- $\langle \varphi_m, \varphi_n \rangle = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$

Convolution

Definition (Convolution)

Let $f, g \in L^1(\mathbb{R})$. The convolution of f and g, denoted $f \star g$, is given by:

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

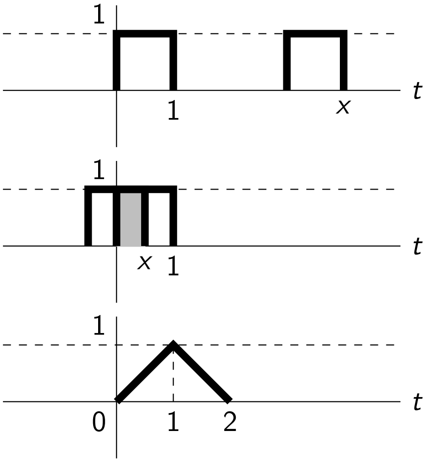
Definition (Convolution on the Circle)

Let $f, g \in L^1(\mathbb{R})$ be 2π-periodic. Convolution on the circle is given by:

$$(f \star g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)g(t)dt$$

Convolution is commutative: $f \star g = g \star f$

Convolution Example



Summability Kernel

Definition (Summability Kernel)

To say that a sequence (κ_n) of 2π-periodic continuous fuctions is a summability kernel means that κ_n satisfies the following properties:

1. $\int_{-\pi}^{\pi} \kappa_n(t)dt = 2\pi$
2. $\int_{-\pi}^{\pi} |\kappa_n(t)| \, dt \leq M$ for some $M > 0$ and all $n \in \mathbb{N}$
3. $\int_{\delta \leq |t| \leq \pi} |\kappa_n(t)| \, dt \to 0$ for all $\delta \in (0, \pi)$

Note that the third property indicates that given a $\delta > 0$, For all $\epsilon > 0$ there exists an n sufficiently large such:

$$2\pi(1 - \epsilon) < \int_{-\delta}^{\delta} \kappa_n(t)dt \leq 2\pi$$

The Fejér Kernel

Definition (Dirichlet Sequence)

The Dirichlet sequence (D_n) is given by:

$$D_n(x) = \sum_{k=-n}^{n} e^{inx}$$

Definition (Fejér Kernel)

The Fejér kernel (F_N) is given by:

$$F_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(x)$$

Fejér Kernel Forms

Theorem 1

The following are equivalent forms of the Fejér kernel:

1. $F_n(x) = \frac{1}{n+1} \sum_{j=0}^{n} \sum_{k=-j}^{j} e^{ikx}$
2. $F_n(x) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikx}$
3. $F_n(x) = \left(\frac{1}{N+1}\right) \frac{\sin^2\left[(n+1)\frac{x}{2}\right]}{\sin^2\left(\frac{x}{2}\right)}$

F_0 through F_5