Cauchy Sequences

Definition: Cauchy

Let E be a normed space and $(\vec{x}_n)$ be a sequence in E. To say that $(\vec{x}_n)$ is *Cauchy* means:

$$\forall\, \epsilon > 0, \exists\, N > 0, m, n > N \implies \|\vec{x}_n - \vec{x}_m\| < \epsilon$$

In other words, $\|\vec{x}_n - \vec{x}_m\| \to 0$.

Theorem

Let E be a normed space and let $(\vec{x}_n)$ be a sequence in E. TFAE:

1). $(\vec{x}_n)$ is Cauchy sequence.

2). For every pair of increasing sequences (p_n) and (q_n) in $\mathbb{N}$, $\|\vec{x}_{p_n} - \vec{x}_{q_n}\| \to 0$.

3). For every increasing sequence (p_n) in $\mathbb{N}$, $\left\|\vec{x}_{p_{n+1}} - \vec{x}_{p_n}\right\| \to 0$.

Proof

$1 \implies 2$: Assume $(\vec{x}_n)$ is Cauchy.

Assume $\epsilon > 0$.
$\exists\, N > 0, m, n > N \implies \|\vec{x}_n - \vec{x}_m\| < \epsilon$

Assume (p_n) and (q_n) are increasing sequences in $\mathbb{N}$.
Assume $p_n, q_n > N$.
$\|\vec{x}_{p_n} - \vec{x}_{q_n}\| < \epsilon$

$\therefore \|\vec{x}_{p_n} - \vec{x}_{q_n}\| \to 0$

$2 \implies 3$: Assume for every pair of increasing sequences (p_n) and (q_n) in $\mathbb{N}$, $\|\vec{x}_{p_n} - \vec{x}_{q_n}\| \to 0$.

Let $p_n = n + 1$ and $q_n = n$.

$\|\vec{x}_{p_n} - \vec{x}_{q_n}\| = \|\vec{x}_{n+1} - \vec{x}_n\| \to 0$.

$3 \implies 1$: Assume $(\vec{x}_n)$ is not Cauchy.

$\exists\, \epsilon > 0, \forall\, N > 0, \exists\, n, m > N, \|\vec{x}_n - \vec{x}_m\| \geq \epsilon$
Let ϵ be such an ϵ.
Assume $N > 0$.
$\exists\, n > N, \|\vec{x}_{n+1} - \vec{x}_n\| \geq \epsilon$

WTS: $\exists\, (p_n), \exists\, \epsilon > 0, \forall\, N > 0, \exists\, p_n > N, \left\|\vec{x}_{p_{n+1}} - \vec{x}_{p_n}\right\| \geq \epsilon$

Let $p_k = k$.

$\left\|\vec{x}_{p_{n+1}} - \vec{x}_{p_n}\right\| = \|\vec{x}_{n+1} - \vec{x}_n\| \geq \epsilon$

Examples

1). $E = \mathbb{R}$ and $\|x\| = |x|$

$(x_n) = \{3, 3.1, 3.14, 3.141, \ldots\}$

(x_n) is a Cauchy sequence in $\mathbb{Q}$; however, it converges to $\pi \notin \mathbb{Q}$.

2). $E = \mathcal{P}[0,1]$ and the sup norm.

$$f_n = \sum_{k=1}^{n} \frac{t^k}{k!}$$

f_n is a Cauchy sequence in $\mathcal{P}[0,1]$; however, it converges to $e^t \notin \mathcal{P}[0,1]$.

Theorem

Let E be a normed space and let $(\vec{x}_n)$ be a sequence in E:

$$\vec{x}_n \to \vec{x} \in E \implies (\vec{x}_n) \text{ is Cauchy}$$

Proof

Assume $\vec{x}_n \to \vec{x} \in E$.
$\|\vec{x}_n - \vec{x}_m\| = \|(\vec{x}_n - \vec{x}) + (\vec{x} - \vec{x}_m)\| \leq \|\vec{x}_n - \vec{x}\| + \|\vec{x}_m - \vec{x}\| \to 0 + 0 = 0$

Therefore $(\vec{x}_n)$ is Cauchy.

Note that the converse holds in finite-dimensional spaces, but does not always hold in infinite=dimensional spaces.

Theorem

Let E be a normed space and let $(\vec{x}_n)$ be a sequence in E:

$$(\vec{x}_n) \text{ Cauchy} \implies (\|\vec{x}_n\|) \text{ converges}$$

Proof

Assume $(\vec{x}_n)$ is Cauchy.

$|\|\vec{x}_n\| - \|\vec{x}_m\|| \leq \|\vec{x}_n - \vec{x}_m\| \to 0$

$\therefore$ $(\|\vec{x}_n\|)$ converges.

Corollary

Let E be a normed space and let $(\vec{x}_n)$ be a sequence in E:

$$(\vec{x}_n) \text{ Cauchy} \implies (\vec{x}_n) \text{ bounded}$$

Proof

Assume $(\vec{x}_n)$ is Cauchy.

Thus $(\|\vec{x}_n\|)$ converges in $\mathbb{R}$, and therefore is bounded.

Theorem

Let E be a normed space and let $(\vec{x}_n)$ be Cauchy in E:

$$(\vec{x}_n) \text{ has a convergent subsequence} \implies (\vec{x}_n) \text{ converges to the same limit.}$$

Proof

Assume $(\vec{x}_n)$ has a convergent subsequence $(\vec{x}_{n_k}) \to \vec{x} \in E$.

$\|\vec{x}_n - \vec{x}\| = \|(\vec{x}_n - \vec{x}_{n_k}) + (\vec{x}_{n_k} - \vec{x})\| \leq \|\vec{x}_n - \vec{x}_{n_k}\| + \|\vec{x}_{n_k} - \vec{x}\|$

But $(\vec{x}_n)$ is Cauchy and so $\|\vec{x}_n - \vec{x}_{n_k}\| \to 0$.
Furthermore, $\vec{x}_{n_k} \to \vec{x}$ and so $\|\vec{x}_{n_k} - \vec{x}\| \to 0$.
And so $\|\vec{x}_n - \vec{x}\| \to 0$.

Therefore $(\vec{x}_n)$ converges to the same limit.