Postive Operators

Lemma

Let H be a Hilbert space and $T \in \mathcal{B}(H)$:

$$T = T^* \iff \forall\, \vec{x} \in H, \langle T\vec{x}, \vec{x}\rangle \in \mathbb{R}$$

Proof

Assume $\vec{x} \in H$.

$\implies$ Assume $T = T^*$.

$\langle T\vec{x}, \vec{x}\rangle = \langle \vec{x}, T\vec{x}\rangle = \overline{\langle T\vec{x}, \vec{x}\rangle}$

$\therefore \langle T\vec{x}.\vec{x}\rangle \in \mathbb{R}$

$\impliedby$ Assume $\forall\, \vec{x} \in H, \langle T\vec{x}, \vec{x}\rangle \in \mathbb{R}$.

$\exists A, B \in \mathcal{B}(H)$ such that $T = A + iB$ and A, B are self-adjoint.

$\langle T\vec{x}, \vec{x}\rangle = \langle (A+iB)\vec{x}, \vec{x}\rangle = \langle A\vec{x}, \vec{x}\rangle + i\,\langle B\vec{x}, \vec{x}\rangle$

But by lemma, $\langle A\vec{x}, \vec{x}\rangle, \langle B\vec{x}, \vec{x}\rangle \in \mathbb{R}$.
Thus, for $\langle T\vec{x}, \vec{x}\rangle \in \mathbb{R}$ it must be the case that $\langle B\vec{x}, \vec{x}\rangle = 0$.

Therefore $T = A$ and thus T is self-adjoint.

Definition

Let H be a Hilbert space and $A \in \mathcal{B}(H)$. To say that A is a *positive* operator, denoted $A \geq 0$, means $\forall\, \vec{x} \in H$:

$$\Phi(x) = \langle A\vec{x}, \vec{x}\rangle \geq 0$$

Note that by previous lemma, A is also self-adjoint.

Examples

1). $H = L^2[a,b]$ and $(Tf)(s) = \displaystyle\int_a^b K(s,t)f(t)dt$, where $K(s,t) = K(t,s) \geq 0$

AWLOG: f is simple: $f = \displaystyle\sum_{k=1}^{n} \alpha_k \chi_{E_k}$ where $m(E_k) < \infty$.

$$\begin{aligned}
\langle Tf, f\rangle &= \int_a^b (Tf)(s)\overline{f(s)}ds \\
&= \int_a^b \left(\int_a^b K(s,t)f(t)dt\right)\overline{f(s)}ds \\
&= \int_a^b \left[\int_a^b K(s,t)\left(\sum_{j=1}^{n} \alpha_j \chi_{E_j}\right)dt\right]\overline{\left(\sum_{k=1}^{n} \alpha_k \chi_{E_k}\right)}ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_a^b \left[\int_a^b K(s,t)\left(\sum_{j=1}^n \alpha_j \chi_{E_j}\right) dt\right]\left(\sum_{k=1}^n \overline{\alpha_k}\chi_{E_k}\right) ds \\
&= \sum_{j=1}^n \sum_{k=1}^n \left(\int_{E_j}\int_{E_k} K(s,t)dsdt\right)\alpha_j\alpha_k \\
&= \sum_{j=1}^n \sum_{k=1}^n K_{jk}\alpha_j\alpha_k \\
&= \langle K\vec{x}, \vec{x}\rangle
\end{aligned}
$$

where $K_{jk} = \displaystyle\int_{E_j}\int_{E_k} K(s,t)dsdt$ and $\vec{x} = [\alpha_k]$.

But $K = [K_{jk}]$ is a real, symmetric matrix and $K \geq 0$.

And so $K = U^* D U$ where $D = [\lambda_n]$ and $\lambda_n \geq 0$.

Let $U\vec{x} = \vec{y}$:

$$
\begin{aligned}
\langle Tf, f\rangle &= \langle K\vec{x}, = vx\rangle \\
&= \langle U^* DU\vec{x}, \vec{x}\rangle \\
&= \langle DU\vec{x}, U\vec{x}\rangle \\
&= \langle D\vec{y}, \vec{y}\rangle \\
&= \sum_{k=1}^n \lambda_k y_k \overline{y_k} \\
&= \sum_{k=1}^n \lambda_k |y_k|^2 \\
&\geq 0
\end{aligned}
$$

2). $H = L^2[a,b]$ and fix $\varphi \in \mathcal{C}[a,b]$.

Let $M_\varphi \in H$ such that $M_\varphi f = \varphi f$.

Claim: $\varphi \geq 0 \implies M_\varphi \geq 0$

Assume $\varphi \geq 0$:

$$\langle M_\varphi f, f\rangle = \langle \varphi f, f\rangle = \int_a^b \varphi(t) f(t)\overline{f(t)}dt = \int_a^b \varphi(t)\,|f(t)|^2\, dt \geq 0$$

3). $H = \mathbb{C}^N$ and $A \in H$ such that $A = [a_{ij}]$.

$A \geq 0 \implies A = A^* \implies a_{ij} = \overline{a_{ji}}$

Let $x = e_k$:

$$\langle Ae_k, e_k\rangle = a_{kk}$$

Thus, if a matrix is either not self-adjoint or the diagonal entries are not real, nonnegative then A cannot be positive.

Definition

Let H be a Hilbert space and let $A, B \in \mathcal{B}(H)$. To say that $A \leq B$ (or $B \geq A$) means that $B - A$ is a positive operator.

Note that the above relation is a partial ordering on the space of self-adjoint operators:

1). $A \leq A$

2). $A \leq B$ and $B \leq A \implies A = B$

3). $A \leq B$ and $B \leq C \implies A \leq C$

Properties

Let H be a Hilbert space and then $A, B, C, D \in \mathcal{B}H$ and $\alpha \in \mathbb{R}$:

1). $A \leq B \implies -A \geq -B$

2). $A \leq B$ and $C \leq D \implies A + C \leq B + D$

3). $A \geq 0$ and $\alpha \geq 0 \implies \alpha A \geq 0$

Proof

1). Assume $A \leq B$.

$B - A \geq 0$

$-A - (-B) = B - A \geq 0$

$\therefore -A \geq -B$

2). Assume $A \leq B$ and $C \leq D$.

$B - A \geq 0$ and $D - C \geq 0$

$(B + D) - (A + C) = (B - A) + (D - C) \geq 0$

$\therefore A + C \leq B + D$

3). Assume $A \geq 0$ and $\alpha \geq 0$:

Assume $\vec{x} \in H$.

$\langle \alpha A\vec{x}, \vec{x} \rangle = \alpha \langle A\vec{x}, \vec{x} \rangle \geq 0$

$\therefore \alpha A \geq 0$

Theorem

Let H be a Hilbert space and $A \in \mathcal{B}(H)$:

$A^*A \geq 0$ and $AA^* \geq 0$

Proof

Assume $\vec{x} \in H$.

$\langle A^*A\vec{x}, \vec{x}\rangle = \langle A\vec{x}, A\vec{x}\rangle = \|A\vec{x}\|^2 \geq 0$

$\langle AA^*\vec{x}, \vec{x}\rangle = \langle A^*\vec{x}, A^*\vec{x}\rangle = \|A^*\vec{x}\|^2 \geq 0$

$\therefore A^*A \geq 0$ and $AA^* \geq 0$

Theorem

Let H be a Hilbert space and $A \in \mathcal{B}(H)$:

$A \geq 0$ and A invertible $\implies A^{-1} \geq 0$

Proof

Assume $A \geq 0$ and A invertible.
Thus A is also self-adjoint.
Assume $\vec{y} \in \mathcal{R}(A)$.
$\exists \vec{x} \in \mathcal{D}(A), \vec{y} = A\vec{x}$

$\left\langle A^{-1}\vec{y}, \vec{y}\right\rangle = \left\langle A^{-1}A\vec{x}, A\vec{x}\right\rangle = \langle \vec{x}, A\vec{x}\rangle = \langle A\vec{x}, \vec{x}\rangle \geq 0$

$\therefore A^{-1} \geq 0$

Note that $A, B \geq 0 \not\Longrightarrow AB \geq 0$.

Example

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Note that both A and B are self-adjoint and thus positive.

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Thus, AB is not self-adjoint and therefore AB is not positive.

Theorem

Let H be a Hilbert space and $A, B \in \mathcal{B}(H)$:

$A, B \geq 0$ and $AB = BA \implies AB \geq 0$.