Hilbert Spaces

Definition

A *Hilbert* space is a complete inner product space.

Examples

1). Finite dimensional: $\mathbb{C}^N$ where $\langle z, w\rangle = \sum_{k=1}^{N} z\overline{w}$.

2). Infinite dimensional: ℓ^2 where $\langle x, y\rangle = \sum_{k=1}^{\infty} x_k\overline{y_k}$.

3). Infinite dimensional: $L^2(\Omega)$ where $\langle f, g\rangle = \int_\Omega f\overline{g}$.

Non-Hilbert inner product spaces arise from:

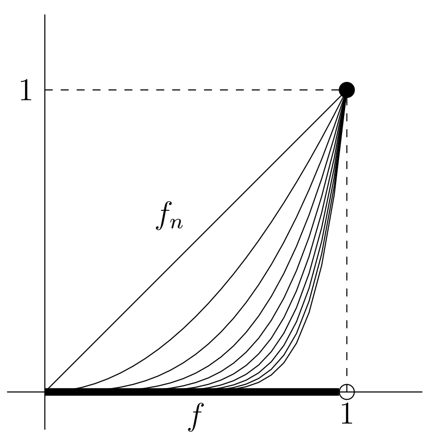
1). The norm is not induced by an inner product.

2). The space is not complete.

Examples

1). $\mathcal{C}[a, b]$ where $\langle f, g\rangle = \int_a^b f\overline{g}$.

Consider the counterexample $f_n = t^n \in \mathcal{C}[0, 1]$:



Claim: f_n is Cauchy:

AWLOG: $n < m$

$$\begin{aligned}
\|f_n - f_m\|^2 &= \int_0^1 |f_n - f_m|^2 \\
&= \int_0^1 (t^n - t^m)^2 dt \\
&= \int_0^1 (t^{2n} - 2t^{n+m} + t^{2m}) dt \\
&= \left[\frac{1}{2n+1} t^{2n+1} - \frac{2}{n+m+1} t^{n+m+1} + \frac{1}{2m+1} t^{2m+1} \right]_0^1 \\
&= \frac{1}{2n+1} - \frac{2}{n+m+1} + \frac{1}{2m+1} \\
&\to 0 - 0 + 0 \\
&= 0
\end{aligned}$$

Thus, f_n is Cauchy in the norm.

Claim: $f_n \to f$ where $f = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$

$$\begin{aligned}
\|f_n - f\|^2 &= \|f_n - 0\|^2 \\
&= \|f_n\|^2 \\
&= \int_0^1 t^{2n} dt \\
&= \left. \frac{1}{2n+1} t^{2n+1} \right|_0^1 \\
&= \frac{1}{2n+1} \\
&\to 0
\end{aligned}$$

Thus, $f_n \to f$ in the norm; however, f is discontinuous and thus $f \notin \mathcal{C}[0,1]$.

Therefore, $\mathcal{C}[0,1]$ is not complete, and thus not Hilbert.

2). Let ℓ_0 be the set of complex sequences x such that only a finite number of the $x_k \neq 0$, with $\langle x, y \rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}$.

Let $x_n = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots\right)$.

Claim: x_n is Cauchy in the norm:

AWLOG: $n < m$

$$
\begin{aligned}
\|x_n - x_m\|^2 &= \sum_{k=1}^{\infty} |x_{n_k} - x_{m_k}|^2 \\
&= \sum_{k=1}^{\infty} x_{n_k}^2 - 2x_{n_k}x_{m_k} + x_{m_k}^2 \\
&= \sum_{k=1}^{\infty} \left(x_{n_k}^2 - 2\sum_{k=1}^{\infty} x_{n_k}x_{m_k} + \sum_{k=1}^{\infty} x_{m_k}^2 \right) \\
&= \sum_{k=1}^{n} \frac{1}{k^2} - 2\sum_{k=1}^{n} \frac{1}{k^2} + \sum_{k=1}^{m} \frac{1}{k^2} \\
&\to \frac{\pi^2}{6} - 2\frac{\pi^2}{6} + \frac{\pi^2}{6} \\
&= 0
\end{aligned}
$$

Thus x_n is Cauchy in the norm.

By letting $m \to \infty$ above, it is clear that $x_n \to x = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$, the harmonic sequence.

However, $x \notin l_0$ and so ℓ_0 is not complete, and therefore ℓ_0 is not Hilbert.