Infinite Cyclic Group Structure

Definition

Let $n \in \mathbb{Z}$:

$$nZ = \langle n \rangle = \{nm \mid m \in \mathbb{Z}\}$$

Theorem

$$\forall\, n \in \mathbb{Z}, n\mathbb{Z} \leq \mathbb{Z}$$

Proof

Assume $n \in Z$
Assume $a, b \in n\mathbb{Z}$
$\exists\, h, k \in \mathbb{Z}, a = nh$ and $b = nk$
$n\mathbb{Z} \subseteq \mathbb{Z}$, so $b = nk \in \mathbb{Z}$
But $\mathbb{Z}$ is a group, so $-b = -nk \in \mathbb{Z}$
$a - b = nh - nk = n(h - k)$
But $h - k \in \mathbb{Z}$
$a - b \in n\mathbb{Z}$
$\therefore$ by the subgroup test, $n\mathbb{Z} \leq \mathbb{Z}$.

Corollary

$\forall\, n \in \mathbb{Z}, n\mathbb{Z}$ is the smallest subgroup of $\mathbb{Z}$ containing n.

Corollary

$n\mathbb{Z}$ are the only subgroups of $\mathbb{Z}$:

$$H \leq G \iff \exists\, n \in \mathbb{Z}, H = n\mathbb{Z}$$

Proof

$\Longrightarrow$ Assume $H \leq G$
H is cyclic
$\exists\, n \in \mathbb{Z}, H = \langle n \rangle$
$\therefore H = n\mathbb{Z}$

$\Longleftarrow$ Assume $\exists\, n \in \mathbb{Z}, H = n\mathbb{Z}$
But $n\mathbb{Z} \leq \mathbb{Z}$
$\therefore H \leq \mathbb{Z}$

Theorem

Let G be infinite cyclic. All subgroups of G must be isomorphic to some $n\mathbb{Z}$:

$$H \leq G \iff H \subseteq G \text{ and } \exists\, n \in \mathbb{Z}, H \simeq n\mathbb{Z}$$

Proof

$\implies$ Assume $H \leq G$

$H \subseteq G$
$G \simeq \mathbb{Z}$
$\exists\, \phi : G \to \mathbb{Z}, \phi$ is an isomorphism
$H \simeq \phi[H] \leq \mathbb{Z}$
$\exists\, n \in \mathbb{Z}, \phi[H] = n\mathbb{Z}$
$\therefore \exists\, n \in \mathbb{Z}, H \simeq n\mathbb{Z}$

$\impliedby$ Assume $H \subseteq G$ and $\exists\, n \in \mathbb{Z}, H \simeq n\mathbb{Z}$

$\mathbb{Z} \simeq G$
$\exists\, \phi : \mathbb{Z} \to G, \phi$ is an isomorphism
$n\mathbb{Z} \leq \mathbb{Z}$
$H = \phi[n\mathbb{Z}] \leq \phi[\mathbb{Z}] = G$
$\therefore H \leq G$