Inverses

Definition: Inverses

Let $A \in M_n$. To say that $B \in M_n$ is an *inverse* of A means:

$$AB = BA = I_n$$

B is often denoted by A^{-1}.

Properties

Let $A \in M_n$:

1). A^{-1} is unique (if it exists).

2). $(A^{-1})^{-1} = A$

3). $(A^T)^{-1} = (A^{-1})^T$

4). $(\bar{A})^{-1} = \overline{A^{-1}}$

5). $(A^*)^{-1} = (A^{-1})^*$

6). $(AB)^{-1} = B^{-1}A^{-1}$

Theorem: Sherman-Morrison-Woodbury

Let $A \in M_n$, $X \in M_{n,h}$, $R \in M_{h,k}$, and $Y \in M_{k,n}$:

$$(A + XRY)^{-1} = A^{-1} - A^{-1}X(R^{-1} + YA^{-1}X)^{-1}YA^{-1}$$

Special Cases:

1). $R = \begin{bmatrix}1\end{bmatrix}$, $X \in M_{n,1}$ (a row matrix), and $Y \in M_{1,n}$ (a column matrix):

$$(A + xy^T)^{-1} = A^{-1} - A^{-1}x(1 + y^TA^{-1}x)y^TA^{-1}$$

2). Same conditions, but $A = I$:

$$(I + xy^T)^{-1} = I - x(1 + y^Tx)y^T = I - \frac{xy^T}{1 + y^Tx}$$

3). $A = I_n$, $R = I_m$, and $X, Y \in M_{n,m}$ where $n >> m$:

$$(I + XY^T)^{-1} = I - X(1 + y^Tx)^{-1}y^T$$

This formula results in the inversion of a much smaller $m \times m$ matrix.

Example

$$\begin{aligned}
\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}^{-1} \quad &= \quad \left(I_4 + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \right)^{-1} \\
&= \quad \left(I_4 + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \right)^{-1} \\
&= \quad I_4 - \frac{1}{1+4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\
&= \quad \frac{1}{5} \left(5I_4 - \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \right) \\
&= \quad \frac{1}{5} \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}
\end{aligned}$$