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Math 221b
Homework #2

1). Let $R = M_2(\mathbb{Z})$ and let:

$$I = \left\{ \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \mid a, c \in \mathbb{Z} \right\}$$

Show that I is a left ideal in R but not a right ideal in R.

It is known that R is a ring
Clearly, I is a non-empty subset of R
Assume $A, B \in I$
Let $A = \begin{bmatrix} a_1 & 0 \\ c_1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & 0 \\ c_2 & 0 \end{bmatrix}$, $a_1, c_1, a_2, c_2 \in \mathbb{Z}$
$A - B = \begin{bmatrix} a_1 & 0 \\ c_1 & 0 \end{bmatrix} - \begin{bmatrix} a_2 & 0 \\ c_2 & 0 \end{bmatrix} = \begin{bmatrix} a_1 - a_2 & 0 \\ c_1 - c_2 & 0 \end{bmatrix}$
But by closure, $a_1 - a_2 \in \mathbb{Z}$ and $c_1 - c_2 \in \mathbb{Z}$, so $A - B \in I$

Therefore, by the subgroup test, I is an additive subgroup of R.

Furthermore, matrix addition is commutative, so I is an additive abelian subgroup of R.

Assume $C \in R$
Let $C = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}$
$a_3, b_3, c_3, d_3 \in \mathbb{Z}$
$CA = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ c_1 & 0 \end{bmatrix} = \begin{bmatrix} a_1 a_3 + b_3 c_1 & 0 \\ a_1 c_3 + c_1 d_3 & 0 \end{bmatrix}$
But by closure, $a_1 a_3 + b_3 c_1 \in \mathbb{Z}$ and $a_1 c_3 + c_1 d_3 \in \mathbb{Z}$, so $CA \in I$

Therefore, I is a left ideal in R.

$AC = \begin{bmatrix} a_1 & 0 \\ c_1 & 0 \end{bmatrix} \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_1 a_3 & a_1 b_3 \\ a_3 c_1 & b_3 c_1 \end{bmatrix} \notin I$, unless a_1, b_3, or $c_1 = 0$

Therefore, I is not a right ideal in R.

2). Let R be a ring with $1 \neq 0$ and $I \trianglelefteq R$. Prove: $I = R \iff \exists\, r \in I, r$ is a unit in R.

$\implies$ Assume $I = R$

$1 \in R$ and $I = R$, so $1 \in I$
$1 \cdot 1 = 1$
So 1 is a unit in R
Let $r = 1$

$\therefore \exists\, r \in I, r$ is a unit in R.

$\impliedby$ Assume $\exists\, r \in I, r$ is a unit in R

$\exists\, s \in R, rs = rs = 1$

$\implies$ Assume $i \in I$

Since $I \trianglelefteq R$, $I \leq R$ and thus $I \subseteq R$

$\therefore i \in R$

$\impliedby$ Assume $a \in R$

Assume $b \in R$
R is a ring, and thus multiplication is associative
$ab = (ab)(1) = (ab)(sr) = (abs)r$
But, by closure, $abs \in R$ and I is an ideal, so $(abs)r \in I$
Similarly, $ba = (1)(ba) = (rs)(ba) = r(sba) \in I$

$\therefore a \in I$

$\therefore I = R$

3). Prove: $M_2(\mathbb{R})$ is a simple ring.

Assume $I \trianglelefteq M_2(\mathbb{R})$ such that $I \neq \{0\}$
$\exists\, A \in I, A \neq 0$
Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $a_{11}, a_{12}, a_{13}, a_{14} \in \mathbb{R}$
AWLOG: $a_{ij} \neq 0$ (since $A \neq 0$)
Consider the standard basis for $M_2(\mathbb{R})$: $\{E_{11}, E_{12}, E_{21}, E_{22}\}$
Note that left multiply by E_{ij} selects the i^{th} row and left multiply by E_{ij} selects the j^{th} column, so $\left(\frac{1}{a_{ij}} E_{ij}\right) A E_{ij} = E_{ij}$
But $I \trianglelefteq M_2(\mathbb{R})$, so $E_{ij} \in I$
Let $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
By using left multiply by T to switch rows and right multiply by T to switch columns, all four basis matrices can be generated from E_{ij}
But since $E_{ij} \in I$ and $T \in M_2(\mathbb{R})$, all four basis matrices are in I
Assume $B \in M_2(\mathbb{R})$
Let $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, $b_{11}, b_{12}, b_{21}, b_{22} \in \mathbb{R}$
Let $B_{k\ell} \in M_2(\mathbb{R})$ such that the $k\ell^{th}$ entry is $b_{k\ell}$ and 0 everywhere else
$B_{k\ell} E_{k\ell} = B_{k\ell}$
But since $E_{k\ell} \in I$, $B_{k\ell} \in I$
Moreover, $B = B_{11} + B_{12} + B_{21} + B_{22}$
But I is a ring, and thus an additive group, and so by closure, $B \in I$
Thus, $I = M_2(\mathbb{R})$ and $M_2(\mathbb{R})$ has no proper, non-trivial ideals

Therefore $M_2(\mathbb{R})$ is simple.

4). Let R be a commutative ring with $1 \neq 0$ and let $P \trianglelefteq R$. Prove: P is a prime ideal in R iff R/P is an integral domain.

Since P is an ideal in R and R is commutative, R/P is a commutative ring with additive identity $0 + P = P$
It is also true that $a + P = P \iff a \in P$

$\implies$ Assume P is a prime ideal in R

Assume $a, b \in R$ such that $a, b \notin P$
Since P is prime, $ab \notin P$
$a + P \neq P$ and $b + P \neq P$
$(a + P)(b + P) = ab + P \neq P$
Therefore, R/P has no zero-divisors and is thus an integral domain.

$\impliedby$ Assume R/P is an integral domain

Assume $a, b \in R$ such that $ab \in P$
$ab + P = (a + P)(b + P) \in P$
If $a \in P$ then done, so AWLOG: $a \notin P$
$a + P \notin P$
But R/P is an integral domain, so $b + P = P$
Thus $b \in P$

Therefore, P is a prime ideal in R.