Equations

Definition

An equation is a syntactic construct of the form $e_1 = e_2$, where e_1 and e_2 are expressions.

The usual case is that one or both of the expressions contain one or more variables. Almost all of our equations will contain a single variable, usually, but not necessarily, called x.

Example

Note that the expressions can be as complicated as desired:

$$\frac{x^2+1}{2x} = x^3 - x^2 + 5x - 1$$

Definition

An equation that is true for all possible values of the variables is called an *identity*.

Example

$\forall\, x \in \mathbb{R}, 2x + 3x = 5x$

But more often, an equation is only true for a limited (possible 0) number of values. The goal is to *solve* the equation to determine those values. Remember, expressions are *evaluated* and equations are *solved*.

Example

$x^2 = -1$ has no solutions in $\mathbb{R}$.

$3x + 1 = 4$ is only true for $x = 1$

$(x+1)(x-2) = 0$ is only true for $x = -1$ and $x = 2$

$\sqrt{x} > 1$ is true for $x \in (1, \infty)$

Our toolbox for solving equations contains only the rules from Chapter 0:

1). Arithmetic

2). Properties of equality

3). The substitution principle

4). Well-defined operators (do the same thing to both sides)

5). Closure and the 10 axioms (expand, factor, simplify)

6). Properties of zero

7). Properties of negatives

8). Properties of fractions

9). Exponent rules

Do not make up your own rules!

Linear Equations

$$\begin{aligned} 3x - 1 &= 5 \\ 3x &= 6 \\ x &= 2 \end{aligned} \qquad\qquad \begin{aligned} Ax + B &= 0 \\ Ax &= -B \\ x &= -\frac{B}{A} \end{aligned}$$

Note that the result is a sequence of reversible steps, each implying the other (TFAE). In the final step, we can plug the found solution in to make sure that it is a solution and complete the implication cycle.

$$3(2) - 1 = 6 - 1 = 5$$

Goal: Isolate the variable so that a simplified for of $x =?$ is achieved.

Example: More Complex

$$\begin{aligned} 3(x+1) - 5(2-x) &= 2x - 9 \\ 3x + 3 - 10 + 5x &= 2x - 9 \\ 8x - 7 &= 2x - 9 \\ 6x + 2 = 0 \\ 6x = -2 \\ x = -\frac{1}{3} \end{aligned}$$

$$\begin{aligned} 3\left(-\frac{1}{3} + 1\right) - 5\left(2 + \frac{1}{3}\right) &= 2\left(-\frac{1}{3}\right) - 9 \\ 3\left(\frac{2}{3}\right) - 5\left(\frac{7}{3}\right) &= -\frac{2}{3} - 9 \\ 2 - \frac{35}{3} &= -\frac{29}{3} \\ -\frac{29}{3} &= -\frac{29}{3} \end{aligned}$$

Example: No solution

$$\begin{aligned} 2(x+1) - 2x &= 1 \\ 2x + 2 - 2x &= 1 \\ 2 &\neq 1 \end{aligned}$$

Example: Identity

$$\begin{aligned} 2(x+1) - 2x &= 2 \\ 2x + 2 - 2x &= 2 \\ 2 &= 2 \end{aligned}$$

Rational Equations

Recall:

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc$$

Why does $\frac{1}{2} = \frac{2}{4}$?
Because $1 \cdot 4 = 2 \cdot 2$

Do not confuse this with the addition rule for fractions! That is for expressions, this is for equations.

Example

$$\begin{aligned} \frac{x-1}{x+2} &= \frac{x+3}{x-4} \\ (x-1)(x-4) &= (x+2)(x+3) \\ x^2 - 5x + 4 &= x^2 + 5x + 6 \\ 10x &= -2 \\ x &= \frac{1}{5} \end{aligned}$$

Sometimes, several fraction rules come into play:

Example

$$\begin{aligned}
\frac{1}{x} - \frac{1}{x-1} &= \frac{1}{x-4} \\
\frac{(x-1)-x}{x(x-1)} &= \frac{1}{x-4} \\
\frac{-1}{x(x-1)} &= \frac{1}{x-4} \\
-(x-4) &= x(x-1) \\
-x+4 &= x^2 - x \\
x^2 &= 4 \\
x &= \pm 2
\end{aligned}$$

As an alternative method, multiply both sides by the common denominator. This is not a problem because none of the factors could have been 0:

$$\begin{aligned}
\frac{1}{x} - \frac{1}{x-1} &= \frac{1}{x-4} \\
x(x-1)(x-4)\left(\frac{1}{x} - \frac{1}{x-1}\right) &= x(x-1)(x-4)\left(\frac{1}{x-4}\right) \\
(x-1)(x-4) - x(x-4) &= x(x-1) \\
x^2 - 5x + 4 - x^2 + 4x &= x^2 - x \\
x^2 &= 4 \\
x &= \pm 2
\end{aligned}$$

Pitfalls

1). Multiplying both sides by a variable (0 is an annihilator):

$$\begin{aligned}
1 &= 2 \\
1x &= 2x \\
x &= 0
\end{aligned}$$

2). Multiplying by the recipricol of a variable Solve $x^2 = x$

Incorrect:
$\frac{1}{x}(x^2) = \frac{1}{x}(x)$
$x = 1$

Only one solution? But $x = 0$ is also a solution!

Correct:
$x^2 - x = 0$
$x(x-1) = 0$
$x = 0$ or $x = 1$

Attempting to multiply both sides by x^{-1} is not correct because x can be 0 and 0 has no multiplicative inverse.

3). Extraneous solutions (non-reversible steps)

$$\begin{aligned}
\sqrt{2-x} &= x-2 \\
2-x &= (x-2)^2 \\
2-x &= x^2-4x+4 \\
x^2-3x+2 &= 0 \\
(x-1)(x-2) &= 0 \\
x &= 1,2
\end{aligned}$$

$x = 2$ works; however, $x = 1$ is extraneous.

4). Solutions not in the domain

$$\begin{aligned}
\frac{\frac{1}{x}+\frac{1}{x-1}}{\frac{1}{x}-\frac{1}{x-1}} &= -x \\
\frac{(x-1)+x}{(x-1)-x} &= -x \\
\frac{2x-1}{-1} &= -x \\
2x-1 &= x \\
x &= 1
\end{aligned}$$