Algebraic Numbers

Definition: Algebraic Number

To say that $\alpha \in \mathbb{C}$ is an *algebraic number* means that it is the zero of some monic polynomial with rational coefficients:

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

where $a_k \in \mathbb{Q}$ and $f(\alpha) = 0$.

Otherwise, α is called a *transcendental number*.

Theorem

Every algebraic number has a unique minimal monic polynomial, which is a polynomial of minimal degree that divides all other polynomials with rational coefficients that have α as a zero.

Example

α	$f(x)$
$r \in \mathbb{Q}$	$x - r$
i	$x^2 + 1$
ω	$x^2 + x + 1$
$\sqrt[3]{2}$	$x^3 - 2$
$\frac{1}{\sqrt[3]{2}}$	$x^3 - \frac{1}{2}$

Transcendental: π, e, e^π

Theorem

$\mathbb{Q}[x]$ is a PID.

Theorem

Let $\alpha \in \mathbb{C}$. The set of polynomials for which α is a zero is an ideal in $\mathbb{Q}[x]$.

Theorem

Let $\overline{\mathbb{Q}}$ be the set of algebraic numbers:

$\overline{\mathbb{Q}}$ is a field.

Thus, sums and products of algebraic numbers are also algebraic.