Inverse Properties

It has already been show that inverses are unique.

Theorem

Let G be a group with identity element e.

$$e^{-1} = e$$

Proof

$ee = e$
But inverses are unique,
$\therefore e^{-1} = e$

Theorem

Let G be a group with identity element e.

$$\forall\, a, b \in G, (ab)^{-1} = b^{-1}a^{-1}$$

Proof

Assume $a, b \in G$
$a^{-1}, b^{-1} \in G$
ab and $b^{-1}a^{-1} \in G$
$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aea^{-1} = aa^{-1} = e$
But inverses are unique,
$\therefore (ab)^{-1} = b^{-1}a^{-1}$

Theorem

Let G be a group with identity element e.

$$\forall\, a \in G, \left(a^{-1}\right)^{-1} = a$$

Proof

Assume $a \in G$
a^{-1} and $\left(a^{-1}\right)^{-1} \in G$
$a^{-1}\left(a^{-1}\right)^{-1} = \left(a^{-1}a\right)^{-1} = e^{-1} = e$
But inverses are unique,
$\therefore \left(a^{-1}\right)^{-1} = a$

Theorem

Let G, H be groups and let $\phi : G \to H$ be an isomorphism.

$$\forall\, a \in G, \phi(a^{-1}) = \phi(a)^{-1}$$

Proof

Assume $a \in G$
$a^{-1} \in G$
$\phi(aa^{-1}) = \phi(a)\phi(a^{-1})$
$\phi(aa^{-1}) = \phi(e)$
$\phi(a)\phi(a^{-1}) = \phi(e)$
But $\phi(e)$ is an identity for H and inverses are unique,
$\therefore \phi(a^{-1}) = \phi(a)^{-1}$