2

A First Approach to Classical Mechanics

2.1 Motion in $\mathbb{R}^1$

2.1.1 Newton's law

We begin by considering the motion of a single particle in $\mathbb{R}^1$, which may be thought of as a particle sliding along a wire, or a particle with motion that just happens to lie in a line. We let $x(t)$ denote the particle's position as a function of time. The particle's velocity is then

$$v(t) := \dot{x}(t),$$

where we use a dot over a symbol to denote the derivative of that quantity with respect to the time t.

The particle's acceleration is then

$$a(t) = \dot{v}(t) = \ddot{x}(t),$$

where $\ddot{x}$ denotes the second derivative of x with respect to t. We assume that there is a force acting on the particle and we assume at first that the force F is a function of the particle's position only. (Later, we will look at the case of forces that depend also on velocity.)

Under these assumptions, Newton's second law ($F = ma$) takes the form

$$F(x(t)) = ma = m\ddot{x}(t), \tag{2.1}$$

where m is the mass of the particle, which is assumed to be positive. We will henceforth abbreviate Newton's second law as simply "Newton's law," since

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we will use the second law much more frequently than the others. Since (2.1) is of second order, the appropriate initial conditions (needed to get a unique solution) are the position and velocity at some initial time t_0. So we look for solutions of (2.1) subject to

$$x(t_0) = x_0$$
$$\dot{x}(t_0) = v_0.$$

Assuming that F is a smooth function, standard results from the elementary theory of differential equations tell us that there exists a unique *local* solution to (2.1) for each pair of initial conditions. (A local solution is one defined for t in a neighborhood of the initial time t_0.) Since (2.1) is in general a nonlinear equation, one cannot expect that, for a general force function F, the solutions will exist for all t. If, for example, $F(x) = x^2$, then any solution with positive initial position and positive initial velocity will escape to infinity in finite time. (Apply Exercise 4 with $V(x) = -x^3/3$.) For a proof existence and uniqueness, see Example 8.2 and Theorem 8.13 in [28].

Definition 2.1 *A solution $x(t)$ to Newton's law is called a* ***trajectory****.*

Example 2.2 (Harmonic Oscillator) *If the force is given by Hooke's law, $F(x) = -kx$, where k is a positive constant, then Newton's law can be written as $m\ddot{x} + kx = 0$. The general solution of this equation is*

$$x(t) = a\cos(\omega t) + b\sin(\omega t),$$

where $\omega := \sqrt{k/m}$ is the frequency of oscillation.

The system in Example 2.2 is referred to as a (classical) *harmonic oscillator*. This system can describe a mass on a spring, where the force is proportional to the distance x that the spring is stretched from its equilibrium position. The minus sign in $-kx$ indicates that the force pulls the oscillator back toward equilibrium. Here and elsewhere in the book, we use the "angular" notion of frequency, which is the rate of change of the argument of a sine or cosine function. If ω is the angular frequency, then the "ordinary" frequency—i.e., the number of cycles per unit of time—is $\omega/2\pi$. Saying that x has (angular) frequency ω means that x is periodic with period $2\pi/\omega$.

2.1.2 Conservation of Energy

We return now to the case of a general force function $F(x)$. We define the *kinetic energy* of the system to be $\frac{1}{2}mv^2$. We also define the *potential energy* of the system as the function

$$V(x) = -\int F(x)\, dx, \tag{2.2}$$

so that $F(x) = -dV/dx$. (The potential energy is defined only up to adding a constant.) The *total energy* E of the system is then

$$E(x, v) = \frac{1}{2}mv^2 + V(x). \tag{2.3}$$

The chief significance of the energy function is that it is *conserved*, meaning that its value along any trajectory is constant.

Theorem 2.3 *Suppose a particle satisfies Newton's law in the form* $m\ddot{x} = F(x)$. *Let* V *and* E *be as in (2.2) and (2.3). Then the energy* E *is conserved, meaning that for each solution* $x(t)$ *of Newton's law,* $E(x(t), \dot{x}(t))$ *is independent of* t.

Proof. We verify this by differentiation, using the chain rule:

$$\begin{aligned}\frac{d}{dt}E(x(t), \dot{x}(t)) &= \frac{d}{dt}\left(\frac{1}{2}m(\dot{x}(t))^2 + V(x(t))\right) \\ &= m\dot{x}(t)\ddot{x}(t) + \frac{dV}{dx}\dot{x}(t) \\ &= \dot{x}(t)[m\ddot{x}(t) - F(x(t))].\end{aligned}$$

This last expression is zero by Newton's law. Thus, the time-derivative of the energy along any trajectory is zero, so $E(x(t), \dot{x}(t))$ is independent of t, as claimed. ■

We may call the energy a *conserved quantity* (or *constant of motion*), since the particle neither gains nor loses energy as the particle moves according to Newton's law.

Let us see how conservation of energy helps us understand the solution to Newton's law. We may reduce the second-order equation $m\ddot{x} = F(x)$ to a pair of first-order equations, simply by introducing the velocity v as a new variable. That is, we look for pairs of functions $(x(t), v(t))$ that satisfy the following system of equations

$$\begin{aligned}\frac{dx}{dt} &= v(t) \\ \frac{dv}{dt} &= \frac{1}{m}F(x(t)).\end{aligned} \tag{2.4}$$

If $(x(t), v(t))$ is a solution to this system, then we can immediately see that $x(t)$ satisfies Newton's law, just by substituting dx/dt for v in the second equation. We refer to the set of possible pairs of the form (x, v) (i.e., $\mathbb{R}^2$) as the *phase space* of the particle in $\mathbb{R}^1$. The appropriate initial conditions for this first-order system are $x(0) = x_0$ and $v(0) = v_0$.

Once we are working in phase space, we can use the conservation of energy to help us. Conservation of energy means that each solution to

the system (2.4) must lie entirely on a single "level curve" of the energy function, that is, the set

$$\left\{ (x, v) \in \mathbb{R}^2 \middle| \, E(x, v) = E(x_0, v_0) \right\}. \tag{2.5}$$

If F—and therefore also V—is smooth, then E is a smooth function of x and v. Then as long as (2.5) contains no critical points of E, this set will be a smooth curve in $\mathbb{R}^2$, by the implicit function theorem. If the level set (2.5) is also a simple closed curve, then the solutions of (2.5) will simply wind around and around this curve. Thus, the set that the solutions to (2.5) trace out in phase space can be determined simply from the conservation of energy. The only thing not apparent at the moment is how this curve is parameterized as a function of time.

In mechanics, a conserved quantity—such as the energy in the one-dimensional version of Newton's law—is often referred to as an "integral of motion." The reason for this is that although Newton's second law is a second-order equation in x, the energy depends only on x and $\dot{x}$ and not on $\ddot{x}$. Thus, the equation

$$\frac{m}{2}(\dot{x}(t))^2 + V(x(t)) = E_0,$$

where E_0 is the value of the energy at time t_0, is actually a *first-order* differential equation. We can solve for $\dot{x}$ to put this equation into a more standard form:

$$\dot{x}(t) = \pm\sqrt{\frac{2(E_0 - V(x(t)))}{m}}. \tag{2.6}$$

What this means is that by using conservation of energy we have turned the original second-order equation into a first-order equation. We have therefore "integrated" the original equation once, that is, changed an equation of the form $\ddot{x}(t) = \cdots$ into an equation of the form $\dot{x}(t) = \cdots$. The first-order equation (2.6) is separable and can be solved more-or-less explicitly (Exercise 1).

2.1.3 Systems with Damping

Up to now, we have considered forces that depend only on position. It is common, however, to consider forces that depend on the velocity as well as the position. In the case of a damped harmonic oscillator, for example, one typically assumes that there is, in addition to the force of the spring, a damping force (friction, say) that is proportional to the velocity. Thus, $F = -kx - \gamma\dot{x}$, where k is, as before, the spring constant and where $\gamma > 0$ is the damping constant. The minus sign in front of $\gamma\dot{x}$ reflects that the damping force operates in the opposite direction to the velocity, causing the particle to slow down. The equation of motion for such a system is then

$$m\ddot{x} + \gamma\dot{x} + kx = 0.$$

If γ is small, the solutions to this equation display decaying oscillation, meaning sines and cosines multiplied by a decaying exponential; if γ is large, the solutions are pure decaying exponentials (Exercise 5).

In the case of the damped harmonic oscillator, there is no longer a conserved energy. Specifically, there is no nonconstant continuous function E on $\mathbb{R}^2$ such that $E(x(t), \dot{x}(t))$ is independent of t for all solutions of Newton's law. To see this, we simply observe that for $\gamma > 0$, all solutions $x(t)$ have the property that $(x(t), \dot{x}(t))$ tends to the origin in the plane as t tends to infinity. Thus, if E is continuous and constant along each trajectory, the value of E at the starting point has to be the same as the value at the origin.

We now consider a general system with damping.

Proposition 2.4 *Suppose a particle moves in the presence of a force law given by $F(x, \dot{x}) = F_1(x) - \gamma\dot{x}$, with $\gamma > 0$. Define the energy E of the system by*

$$E(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + V(x),$$

where $dV/dx = -F_1(x)$. Then along any trajectory $x(t)$, we have

$$\frac{d}{dt}E(x(t), \dot{x}(t)) = -\gamma\dot{x}(t)^2 \leq 0.$$

Thus, although the energy is not conserved, it is decreasing with time, which gives us some information about the behavior of the system.

Proof. We differentiate as in the proof of Theorem 2.3, except that now $dV/dx = -F_1(x)$:

$$\frac{d}{dt}E(x(t), \dot{x}(t)) = \dot{x}(t)[m\ddot{x}(t) - F_1(x(t))].$$

Since F_1 is not the full force function, the quantity in square brackets equals not zero but $-\gamma\dot{x}$. Thus, $dE/dt = -\gamma\dot{x}^2$. ■

We can interpret Proposition 2.4 as saying that in the presence of friction, the system we are studying gives up some of its energy to heat energy in the environment, so that the energy of our system decreases with time. We will see that in higher dimensions, it is possible to have conservation of energy in the presence of velocity-dependent forces, provided that these forces act perpendicularly to the velocity.

2.2 Motion in $\mathbb{R}^n$

We now consider a particle moving in $\mathbb{R}^n$. The position $\mathbf{x} = (x_1, \ldots, x_n)$ of a particle is now a vector in $\mathbb{R}^n$, as is the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$. We let

$$\dot{\mathbf{x}} = (\dot{x}_1, \ldots, \dot{x}_n)$$

denote the derivative of $\mathbf{x}$ with respect to t and we let $\ddot{\mathbf{x}}$ denote the second derivative of $\mathbf{x}$ with respect to t. Newton's law now takes the form

$$m\ddot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \dot{\mathbf{x}}(t)), \tag{2.7}$$

where $\mathbf{F} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is some force law, which in general may depend on both the position and velocity of the particle.

We begin by considering forces that are independent of velocity, and we look for a conserved energy function in this setting.

Proposition 2.5 *Consider Newton's law (2.7) in the case of a velocity-independent force: $m\ddot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t))$. Then an energy function of the form*

$$E(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2} m \left|\dot{\mathbf{x}}\right|^2 + V(\mathbf{x})$$

is conserved if and only if V satisfies

$$-\nabla V = \mathbf{F},$$

where ∇V is the gradient of V.

Saying that E is "conserved" means that $E(\mathbf{x}(t), \dot{\mathbf{x}}(t))$ is independent of t for each solution $\mathbf{x}(t)$ of Newton's law. The function V is the *potential energy* of the system.

Proof. Differentiating gives

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2} m \left|\dot{\mathbf{x}}(t)\right|^2 + V(\mathbf{x}(t))\right) &= m \sum_{j=i}^{n} \dot{x}_j(t)\ddot{x}_j(t) + \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}\dot{x}_j(t) \\ &= \dot{\mathbf{x}}(t) \cdot [m\ddot{\mathbf{x}}(t) + \nabla V] \\ &= \dot{\mathbf{x}}(t) \cdot [\mathbf{F}(\mathbf{x}) + \nabla V(\mathbf{x})] \end{aligned}$$

Thus, dE/dt will always be equal to zero if and only if we have

$$-\nabla V(\mathbf{x}) = \mathbf{F}(\mathbf{x})$$

for all $\mathbf{x}$. ■

We now encounter something that did not occur in the one-dimensional case. In $\mathbb{R}^1$, any smooth function can be expressed as the derivative of some other function. In $\mathbb{R}^n$, however, not every vector-valued function $\mathbf{F}(\mathbf{x})$ can be expressed as the (negative of) the gradient of some scalar-valued function V.

Definition 2.6 *Suppose $\mathbf{F}$ is a smooth, $\mathbb{R}^n$-valued function on a domain $U \subset \mathbb{R}^n$. Then $\mathbf{F}$ is called* ***conservative*** *if there exists a smooth, real-valued function V on U such that $\mathbf{F} = -\nabla V$.*

If the domain U is simply connected, then there is a simple local condition that characterizes conservative functions.

Proposition 2.7 *Suppose U is a simply connected domain in $\mathbb{R}^n$ and $\mathbf{F}$ is a smooth, $\mathbb{R}^n$-valued function on U. Then $\mathbf{F}$ is conservative if and only if $\mathbf{F}$ satisfies*

$$\frac{\partial F_j}{\partial x_k} - \frac{\partial F_k}{\partial x_j} = 0 \tag{2.8}$$

at each point in U.

When $n = 3$, it is easy to check that the condition (2.8) is equivalent to the *curl* $\nabla \times \mathbf{F}$ of $\mathbf{F}$ being zero on U. The hypothesis that U be simply connected cannot be omitted; see Exercise 7.

Proof. If $\mathbf{F}$ is conservative, then

$$\frac{\partial F_j}{\partial x_k} = -\frac{\partial^2 V}{\partial x_k \partial x_j} = -\frac{\partial^2 V}{\partial x_j \partial x_k} = \frac{\partial F_k}{\partial x_j}$$

at every point in U. In the other direction, if $\mathbf{F}$ satisfies (2.8), V can be obtained by integrating $\mathbf{F}$ along paths and using the Stokes theorem to establish independence of choice of path. See, for example, Theorem 4.3 on p. 549 of [44] for a proof in the $n = 3$ case. The proof in higher dimensions is the same, provided one knows the general version of the Stokes theorem. ■

We may also consider velocity-dependent forces. If, for example, $\mathbf{F}(\mathbf{x}, \mathbf{v}) = -\gamma \mathbf{v} + \mathbf{F}_1(\mathbf{x})$, where γ is a positive constant, then we will again have energy that is decreasing with time. There is another new phenomenon, however, in dimension greater than 1, namely the possibility of having a conserved energy even when the force depends on velocity.

Proposition 2.8 *Suppose a particle in $\mathbb{R}^n$ moves in the presence of a force $\mathbf{F}$ of the form*

$$\mathbf{F}(\mathbf{x}, \mathbf{v}) = -\nabla V(\mathbf{x}) + \mathbf{F}_2(\mathbf{x}, \mathbf{v}),$$

where V is a smooth function and where $\mathbf{F}_2$ satisfies

$$\mathbf{v} \cdot \mathbf{F}_2(\mathbf{x}, \mathbf{v}) = 0 \tag{2.9}$$

for all $\mathbf{x}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Then the energy function $E(\mathbf{x}, \mathbf{v}) = \frac{1}{2} m |\mathbf{v}|^2 + V(\mathbf{x})$ is constant along each trajectory.

If, for example, $\mathbf{F}_2$ is the force exerted on a charged particle in $\mathbb{R}^3$ by a magnetic field $\mathbf{B}(\mathbf{x})$, then

$$\mathbf{F}_2(\mathbf{x}, \mathbf{v}) = q\mathbf{v} \times \mathbf{B}(\mathbf{x}),$$

where q is the charge of the particle, which clearly satisfies (2.9).

Proof. See Exercise 8. ■

2.3 Systems of Particles

If we have a system if N particles, each moving in $\mathbb{R}^n$, then we denote the position of the jth particle by

$$\mathbf{x}^j = (x_1^j, \ldots, x_n^j).$$

Thus, in the expression x_k^j, the superscript j indicates the jth particle, while the subscript k indicates the kth component. Newton's law then takes the form

$$m_j \ddot{\mathbf{x}}^j = \mathbf{F}^j(\mathbf{x}^1, \ldots, \mathbf{x}^N, \dot{\mathbf{x}}^1, \ldots, \dot{\mathbf{x}}^N), \quad j = 1, 2, \ldots, N,$$

where m_j is the mass of the jth particle. Here, $\mathbf{F}^j$ is the force on the jth particle, which in general will depend on the position and velocity not only of that particle, but also on the position and velocity of the other particles.

2.3.1 Conservation of Energy

In a system of particles, we cannot expect that the energy of each individual particle will be conserved, because as the particles interact, they can exchange energy. Rather, we should expect that, under suitable assumptions on the forces $\mathbf{F}^j$, we can define a conserved energy function for the whole system (the *total* energy of the system).

Let us consider forces depending only on the position of the particles, and let us assume that the energy function will be of the form

$$E(\mathbf{x}^1, \ldots, \mathbf{x}^N, \mathbf{v}^1, \ldots, \mathbf{v}^N) = \sum_{j=1}^{N} \frac{1}{2} m_j \left|\mathbf{v}^j\right|^2 + V(\mathbf{x}^1, \ldots, \mathbf{x}^N). \tag{2.10}$$

We will now try to see what form for V (if any) will allow E to be constant along each trajectory.

Proposition 2.9 *An energy function of the form (2.10) is constant along each trajectory if*

$$\nabla^j V = -\mathbf{F}^j \tag{2.11}$$

for each j, where ∇^j is the gradient with respect to the variable $\mathbf{x}^j$.

Proof. We compute that

$$\begin{aligned} \frac{dE}{dt} &= \sum_{j=1}^{N} \left[m_j \dot{\mathbf{x}}^j \cdot \ddot{\mathbf{x}}^j + \nabla^j V \cdot \dot{\mathbf{x}}^j \right] \\ &= \sum_{j=1}^{N} \dot{\mathbf{x}}^j \cdot \left[m_j \ddot{\mathbf{x}}^j + \nabla^j V \right] \\ &= \sum_{j=1}^{N} \dot{\mathbf{x}}^j \cdot \left[\mathbf{F}^j + \nabla^j V \right]. \end{aligned}$$

If $\nabla^j V = -\mathbf{F}^j$, then E will be conserved. ■

As in the one-particle case, there is a simple condition for the existence of a potential function V satisfying (2.11).

Proposition 2.10 *Suppose a force function* $\mathbf{F} = (\mathbf{F}^1, \ldots, \mathbf{F}^N)$ *is defined on a simply connected domain* U *in* $\mathbb{R}^{nN}$. *Then there exists a smooth function* V *on* U *satisfying*

$$\nabla^j V = -\mathbf{F}^j$$

for all j *if and only if we have*

$$\frac{\partial F_k^j}{\partial x_m^l} = \frac{\partial F_m^l}{\partial x_k^j} \tag{2.12}$$

for all j, k, l, *and* m.

Proof. Apply Proposition 2.7 with n replaced by nN and with j and k replaced by the pairs (j,k) and (l,m). ■

2.3.2 Conservation of Momentum

We now introduce the notion of the momentum of a particle.

Definition 2.11 *In an* N*-particle system, the* ***momentum*** *of the* j*th particle, denoted* $\mathbf{p}^j$, *is the product of the mass and the velocity of that particle:*

$$\mathbf{p}^j = m_j \dot{\mathbf{x}}^j.$$

The ***total momentum*** *of the system, denoted* $\mathbf{p}$, *is defined as*

$$\mathbf{p} = \sum_{j=1}^{N} \mathbf{p}^j.$$

Observe that

$$\frac{d\mathbf{p}^j}{dt} = m_j \ddot{\mathbf{x}}^j = \mathbf{F}^j.$$

Thus, Newton's law may be reformulated as saying, "The force is the rate of change of the momentum." This is how Newton originally formulated his second law.

Newton's third law says, "For every action, there is an equal and opposite reaction." This law will apply if all forces are of the "two-particle" variety and satisfy a natural symmetry property. Having two-particle forces means that the force $\mathbf{F}^j$ on the jth particle is a sum of terms $\mathbf{F}^{j,k}$, $j \neq k$, where $\mathbf{F}^{j,k}$ depends only $\mathbf{x}^j$ and $\mathbf{x}^k$. The relevant symmetry property is that $\mathbf{F}^{j,k}(\mathbf{x}^j, \mathbf{x}^k) = -\mathbf{F}^{k,j}(\mathbf{x}^k, \mathbf{x}^j)$; that is, the force exerted by the jth particle on the kth particle is the negative (i.e., "equal and opposite") of the force

exerted by the kth particle on the jth particle. If the forces are assumed also to be conservative, then the potential energy of the system will be of the form

$$V(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N) = \sum_{j<k} V^{j,k}(\mathbf{x}^j - \mathbf{x}^k). \tag{2.13}$$

One important consequence of Newton's third law is conservation of the total momentum of the system.

Proposition 2.12 *Suppose that for each j, the force on the jth particle is of the form*

$$\mathbf{F}^j(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N) = \sum_{k \neq j} \mathbf{F}^{j,k}(\mathbf{x}^j, \mathbf{x}^k),$$

for certain functions $\mathbf{F}^{j,k}$. Suppose also that we have the "equal and opposite" condition

$$\mathbf{F}^{j,k}(\mathbf{x}^j, \mathbf{x}^k) = -\mathbf{F}^{k,j}(\mathbf{x}^j, \mathbf{x}^k).$$

Then the total momentum of the system is conserved.

Note that since the rate of change of $\mathbf{p}^j$ is $\mathbf{F}^j$, the force on the jth particle, the momentum of each individual particle is not constant in time, except in the trivial case of a noninteracting system (one in which all forces are zero).

Proof. Differentiating gives

$$\frac{d\mathbf{p}}{dt} = \sum_{j=1}^{N} \frac{d\mathbf{p}^j}{dt} = \sum_{j=1}^{N} \mathbf{F}^j = \sum_j \sum_{k \neq j} \mathbf{F}^{j,k}(\mathbf{x}^j, \mathbf{x}^k).$$

By the equal and opposite condition, $\mathbf{F}^{j,k}(\mathbf{x}^j, \mathbf{x}^k)$ cancels with $\mathbf{F}^{k,j}(\mathbf{x}^j, \mathbf{x}^k)$, so $d\mathbf{p}/dt = 0$. ■

Let us consider, now, a more general situation in which we have conservative forces, but not necessarily of the "two-particle" form. It is still possible to have conservation of momentum, as the following result shows.

Proposition 2.13 *If a multiparticle system has a force law coming from a potential V, then the total momentum of the system is conserved if and only if*

$$V(\mathbf{x}^1 + \mathbf{a}, \mathbf{x}^2 + \mathbf{a}, \ldots, \mathbf{x}^N + \mathbf{a}) = V(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N) \tag{2.14}$$

for all $\mathbf{a} \in \mathbb{R}^n$.

Proof. Apply (2.14) with $\mathbf{a} = t\mathbf{e}_k$, where $\mathbf{e}_k$ is the vector with a 1 in the kth spot and zeros elsewhere. Differentiating with respect to t at $t = 0$ gives

$$0 = \sum_{j=1}^{N} \frac{\partial V}{\partial x_k^j} = -\sum_{j=1}^{N} F_k^j = -\sum_{j=1}^{N} \frac{dp_k^j}{dt} = -\frac{dp_k}{dt},$$

where p_k is the kth component of the total momentum $\mathbf{p}$. Thus, if (2.14) holds, $\mathbf{p}$ is constant in time.

Conversely, if the momentum is conserved, then the sum of the forces is zero at every point, and so

$$\begin{aligned}
&\frac{d}{dt}V(\mathbf{x}^1 + t\mathbf{a}, \mathbf{x}^2 + t\mathbf{a}, \ldots, \mathbf{x}^N + t\mathbf{a}) \\
&= \sum_{j=1}^{N} \nabla^j V(\mathbf{x}^1 + t\mathbf{a}, \mathbf{x}^2 + t\mathbf{a}, \ldots, \mathbf{x}^N + t\mathbf{a}) \cdot \mathbf{a} \\
&= -\left(\sum_{j=1}^{N} \mathbf{F}^j(\mathbf{x}^1 + t\mathbf{a}, \mathbf{x}^2 + t\mathbf{a}, \ldots, \mathbf{x}^N + t\mathbf{a}) \right) \cdot \mathbf{a} \\
&= 0
\end{aligned}$$

for all t. Thus, the value of the quantity being differentiated is the same at $t = 0$ as at $t = 1$, which establishes (2.14). ■

The moral of the story is that conservation of momentum is a consequence of translation-invariance of the system, where "translation invariance " means invariance under simultaneous translations of *every* particle by the *same* amount. (See Exercise 11 for a more general version of this result.) If the potential is of the "two-particle" form (2.13), then it is evident that the condition (2.14) is satisfied.

2.3.3 Center of Mass

We now consider an important application of momentum conservation.

Definition 2.14 *For a system of N particles moving in $\mathbb{R}^n$, the **center of mass** of the system at a fixed time is the vector $\mathbf{c} \in \mathbb{R}^n$ given by*

$$\mathbf{c} = \sum_{j=1}^{N} \frac{m_j}{M} \mathbf{x}^j,$$

where $M = \sum_{j=1}^{N} m_j$ is the total mass of the system.

The center of mass is a weighted average of the positions of the various particles. Differentiating $\mathbf{c}(t)$ with respect to t gives

$$\frac{d\mathbf{c}}{dt} = \frac{1}{M} \sum_{j=1}^{N} m_j \dot{\mathbf{x}}^j = \frac{\mathbf{p}}{M}, \tag{2.15}$$

where $\mathbf{p}$ is the total momentum.

Proposition 2.15 *Suppose the total momentum* $\mathbf{p}$ *of a system is conserved. Then the center of mass moves in a straight line at constant speed. Specifically,*

$$\mathbf{c}(t) = \mathbf{c}(t_0) + (t - t_0)\frac{\mathbf{p}}{M},$$

where $\mathbf{c}(t_0)$ *is the center of mass at some initial time* t_0.

Proof. The result follows easily from (2.15). ■

The notion of center of mass is particularly useful in a system of two particles in which momentum is conserved. For a system of two particles, if the potential energy $V(\mathbf{x}^1, \mathbf{x}^2)$ is invariant under simultaneous translations of $\mathbf{x}^1$ and $\mathbf{x}^2$, then it is of the form

$$V(\mathbf{x}^1, \mathbf{x}^2) = \tilde{V}(\mathbf{x}^1 - \mathbf{x}^2),$$

where $\tilde{V}(\mathbf{a}) = V(\mathbf{a}, 0)$.

Now, the positions $\mathbf{x}^1, \mathbf{x}^2$ of the particles can be recovered from knowledge of the center of mass and the *relative position*

$$\mathbf{y} := \mathbf{x}^1 - \mathbf{x}^2$$

as follows:

$$\mathbf{x}^1 = \frac{\mathbf{c} + m_2\mathbf{y}}{m_1 + m_2}$$
$$\mathbf{x}^2 = \frac{\mathbf{c} - m_1\mathbf{y}}{m_1 + m_2}.$$

Meanwhile, we may compute that

$$\ddot{\mathbf{y}}(t) = \ddot{\mathbf{x}}^1 - \ddot{\mathbf{x}}^2 = -\frac{1}{m_1}\nabla\tilde{V}(\mathbf{x}^1 - \mathbf{x}^2) - \frac{1}{m_2}\nabla\tilde{V}(\mathbf{x}^1 - \mathbf{x}^2).$$

This calculation gives the following result.

Proposition 2.16 *For a two-particle system with potential energy of the form* $V(\mathbf{x}^1, \mathbf{x}^2) = \tilde{V}(\mathbf{x}^1 - \mathbf{x}^2)$, *the relative position* $\mathbf{y} := \mathbf{x}^1 - \mathbf{x}^2$ *satisfies the differential equation*

$$\mu\ddot{\mathbf{y}} = -\nabla\tilde{V}(\mathbf{y}),$$

where μ *is the* ***reduced mass*** *given by*

$$\mu = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}} = \frac{m_1 m_2}{m_1 + m_2}.$$

Thus, when the total momentum of a two-particle system is conserved, the relative position evolves as a one-particle system with "effective" mass μ, while the center of mass moves "trivially," as described in Proposition 2.15.

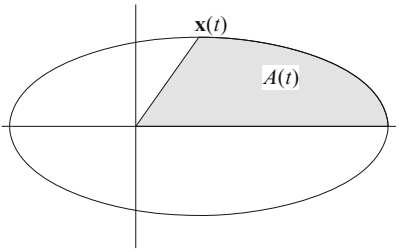

FIGURE 2.1. $A(t)$ is the area of the shaded region.

2.4 Angular Momentum

We start by considering angular momentum in the simplest nontrivial case, motion in $\mathbb{R}^2$.

Definition 2.17 *Consider a particle moving in* $\mathbb{R}^2$, *having position* $\mathbf{x}$, *velocity* $\mathbf{v}$, *and momentum* $\mathbf{p} = m\mathbf{v}$. *Then the* ***angular momentum*** *of the particle, denoted* J, *is given by*

$$J = x_1 p_2 - x_2 p_1. \tag{2.16}$$

In more geometric terms, $J = |\mathbf{x}|\,|\mathbf{p}| \sin\phi$, where ϕ is the angle (measured counterclockwise) between $\mathbf{x}$ and $\mathbf{p}$. We can look at J in yet another way as follows. If θ is the usual angle in polar coordinates on $\mathbb{R}^2$, then an elementary calculation (Exercise 9) shows that

$$J = mr^2 \frac{d\theta}{dt}. \tag{2.17}$$

It then follows that

$$J = 2m \frac{dA}{dt}, \tag{2.18}$$

where $A = (1/2)\int r^2\, d\theta$ is the area being swept out by the curve $\mathbf{x}(t)$. See Fig. 2.1.

One significant property of the angular momentum is that it (like the energy) is *conserved* in certain situations.

Proposition 2.18 *Suppose a particle of mass* m *is moving in* $\mathbb{R}^2$ *under the influence of a conservative force with the potential function* $V(\mathbf{x})$. *If* V *is invariant under rotations in* $\mathbb{R}^2$, *then the angular momentum* $J = x_1 p_2 - x_2 p_1$ *is independent of time along any solution of Newton's equation. Conversely, if* J *is independent of time along every solution of Newton's equation, then* V *is invariant under rotations.*

Proof. Differentiating (2.16) along a solution of Newton's law gives

$$\begin{aligned}\frac{dJ}{dt} &= \frac{dx_1}{dt}p_2 + x_1\frac{dp_2}{dt} - \frac{dx_2}{dt}p_1 - x_2\frac{dp_1}{dt} \\ &= \frac{1}{m}p_1p_2 - x_1\frac{\partial V}{\partial x_2} - \frac{1}{m}p_2p_1 + x_2\frac{\partial V}{\partial x_1} \\ &= x_2\frac{\partial V}{\partial x_1} - x_1\frac{\partial V}{\partial x_2}.\end{aligned}$$

On the other hand, consider rotations R_θ in $\mathbb{R}^2$ given by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

If we differentiate V along this family of rotations, we obtain

$$\frac{d}{d\theta}V\left(R_\theta \mathbf{x}\right)\Big|_{\theta=0} = \frac{\partial V}{\partial x}\frac{dx}{d\theta} + \frac{\partial V}{\partial y}\frac{dy}{d\theta} = -x_2\frac{\partial V}{\partial x_1} + x_1\frac{\partial V}{\partial x_2} = -\frac{dJ}{dt}(\mathbf{x}).$$

Thus, the angular derivative of V is zero if and only if J is constant. ■

Conservation of J [together with the relation (2.18)] gives the following result.

Corollary 2.19 (Kepler's Second Law) *Suppose a particle is moving in $\mathbb{R}^2$ in the presence of a force associated with a rotationally invariant potential. If $\mathbf{x}(t)$ is the trajectory of the particle, then the area swept out by $\mathbf{x}(t)$ between times $t = a$ and $t = b$ is $(b-a)J/(2m)$, where J is the constant value of the angular momentum along the trajectory. Since the area swept out depends only on $b - a$, we may say that "equal areas are swept out in equal times."*

Kepler, of course, was interested in the motion of planets in $\mathbb{R}^3$, not in $\mathbb{R}^2$. The motion of a planet moving in the "inverse square" force of a sun will, however, always lie in a plane. (This claim follows from the three-dimensional version of conservation of angular momentum, as explained in Sect. 2.6.1.)

In $\mathbb{R}^3$, the angular momentum of the particle is a vector, given by

$$\mathbf{J} = \mathbf{x} \times \mathbf{p}, \tag{2.19}$$

where $\times$ denotes the cross product (or vector product). Thus, for example,

$$J_3 = x_1p_2 - x_2p_1. \tag{2.20}$$

If, then, we have a particle in $\mathbb{R}^3$ that just happens to be moving in $\mathbb{R}^2$ (i.e., $x_3 = 0$ and $p_3 = 0$), then the angular momentum will be in the z-direction with z-component given by the quantity J defined in Definition 2.17.

The representation of the angular momentum of a particle in $\mathbb{R}^3$ as a vector is a low-dimensional peculiarity. For a particle in $\mathbb{R}^n$, the angular momentum is a skew-symmetric matrix given by

$$J_{jk} = x_j p_k - x_k p_j. \tag{2.21}$$

In the $\mathbb{R}^3$ case, the entries of the 3×3 angular momentum *matrix* are made up by the three components of the angular momentum *vector* together with their negatives, with zeros along the diagonal. [Compare, e.g., (2.20) and (2.21).]

Definition 2.20 *For a system of N particles moving in $\mathbb{R}^n$, the **total angular momentum** of the system is the skew-symmetric matrix $\mathbf{J}$ given by*

$$J_{jk} = \sum_{l=1}^{N} \left(x_j^l p_k^l - x_k^l p_j^l \right). \tag{2.22}$$

Theorem 2.21 *Suppose a system of N particles in $\mathbb{R}^n$ is moving under the influence of conservative forces with potential function V. If V satisfies*

$$V(R\mathbf{x}^1, R\mathbf{x}^2, \ldots, R\mathbf{x}^N) = V(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N) \tag{2.23}$$

for every rotation matrix R, then the total angular momentum of the system is conserved (constant along each trajectory). Conversely, if the total angular momentum is constant along each trajectory, then V satisfies (2.23).

The proof of this result is similar to that of Proposition 2.18 and is left as an exercise (Exercise 10). We will re-examine the concept of angular momentum in the next section using the language of Poisson brackets and Hamiltonian flows.

2.5 Poisson Brackets and Hamiltonian Mechanics

We consider now the Hamiltonian approach to classical mechanics. (There is also the Lagrangian approach, but that approach is not as relevant for our purposes.) The Hamiltonian approach, and in particular the Poisson bracket, will help us to understand the general phenomenon of conserved quantities. The Poisson bracket is also an important source of motivation for the use of commutators in quantum mechanics.

In the Hamiltonian approach to mechanics, we think of the energy function as a function of position and momentum, rather than position and velocity, and we refer to it as the "Hamiltonian." If a particle in $\mathbb{R}^n$ has the usual sort of energy function (kinetic energy plus potential energy), we have

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2m} \sum_{j=1}^{n} p_j^2 + V(\mathbf{x}). \tag{2.24}$$

Here, as usual, $p_j = m_j \dot{x}_j$. We now observe that Newton's law can be expressed in the following form:

$$\begin{aligned} \frac{dx_j}{dt} &= \frac{\partial H}{\partial p_j} \\ \frac{dp_j}{dt} &= -\frac{\partial H}{\partial x_j}. \end{aligned} \tag{2.25}$$

After all, with H of the indicated form, these equations read $dx_j/dt = p_j/m$, which is just the definition of p_j, and $dp_j/dt = -\partial V/\partial x_j = F_j$, which is just Newton's law, in the form originally given by Newton. We refer to Newton's law, in the form (2.25) as *Hamilton's equations*.

Although it is not obvious at the moment that we have gained anything by writing Newton's law in the form (2.25), let us proceed on a bit further and see. Our next step is to introduce the *Poisson bracket*.

Definition 2.22 *Let f and g be two smooth functions on $\mathbb{R}^{2n}$, where an element of $\mathbb{R}^{2n}$ is thought of as a pair $(\mathbf{x}, \mathbf{p})$, with $\mathbf{x} \in \mathbb{R}^n$ representing the position of a particle and $\mathbf{p} \in \mathbb{R}^n$ representing the momentum of a particle. Then the* ***Poisson bracket*** *of f and g, denoted $\{f, g\}$, is the function on $\mathbb{R}^{2n}$ given by*

$$\{f, g\}(\mathbf{x}, \mathbf{p}) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j} \right).$$

The Poisson bracket has the following properties.

Proposition 2.23 *For all smooth functions f, g, and h on $\mathbb{R}^{2n}$ we have the following:*

1. $\{f, g + ch\} = \{f, g\} + c\{f, h\}$ *for all* $c \in \mathbb{R}$
2. $\{g, f\} = -\{f, g\}$
3. $\{f, gh\} = \{f, g\}h + g\{f, h\}$
4. $\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$

Properties 1 and 2 of Proposition 2.23 say that the Poisson bracket is bilinear and skew-symmetric. Property 3 says that the operation of "bracket with f" satisfies the derivation property (similar to the product rule for derivatives) with respect to pointwise multiplication of functions, while Property 4 says that "bracket with f" satisfies the derivation property with respect to the Poisson bracket itself. Property 4 is equivalent to the *Jacobi identity*:

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0, \tag{2.26}$$

as may easily be seen using the skew-symmetry of the Poisson bracket. The Jacobi identity, along with bilinearity and skew-symmetry, means that the space of C^∞ functions on $\mathbb{R}^{2n}$ forms a *Lie algebra* under the operation of a Poisson bracket. (See Chap. 16.)

Proof. The first two properties of the Poisson bracket are obvious and the third is an easy consequence of the product rule. Let us think about what goes into proving Property 4 by direct computation. (An alternative proof is given in Exercise 15.) We compute that

$$\{f,\{g,h\}\} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial}{\partial p_j}\left(\frac{\partial g}{\partial x_j}\frac{\partial h}{\partial p_j} - \frac{\partial g}{\partial p_j}\frac{\partial h}{\partial x_j}\right) - \sum_{j=1}^{n} \frac{\partial f}{\partial p_j}\frac{\partial}{\partial x_j}\left(\frac{\partial g}{\partial x_j}\frac{\partial h}{\partial p_j} - \frac{\partial g}{\partial p_j}\frac{\partial h}{\partial x_j}\right).$$

Just the first term in the expression for $\{f,\{g,h\}\}$ generates the following four terms (all summed over j) after we use the product rule:

$$\frac{\partial f}{\partial x_j}\frac{\partial^2 g}{\partial x_j \partial p_j}\frac{\partial h}{\partial p_j} + \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_j}\frac{\partial^2 h}{\partial p_j^2} - \frac{\partial f}{\partial x_j}\frac{\partial^2 g}{\partial p_j^2}\frac{\partial h}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial x_j \partial p_j}. \quad (2.27)$$

We see, then, that the left-hand side of (2.26) will have a total of 24 terms, each summed over j. Each term will have a single derivative on two of the three functions, and two derivatives on the third function. There are three possibilities for which function gets two derivatives. Once that function is chosen, there are four possibilities for which derivatives go on the other two functions, with the function that gets two derivatives getting whatever derivatives remain (for a total of two x-derivatives and two p-derivatives). That makes 12 possible terms. It is a tedious but straightforward exercise to check that each of these 12 possible terms occurs twice in the left-hand side of (2.26), with opposite signs. To check just one case explicitly, in computing $\{h,\{f,g\}\}$, we will get a term like the second term in (2.27), but with (f,g,h) replaced by (h,f,g):

$$\frac{\partial h}{\partial x_j}\frac{\partial f}{\partial x_j}\frac{\partial^2 g}{\partial p_j^2}.$$

This term (in the computation of $\{h,\{f,g\}\}$) cancels with the third term in (2.27) (in the computation of $\{f,\{g,h\}\}$). ∎

The following elementary result will provide a helpful analogy to the "canonical commutation relations" in quantum mechanics.

Proposition 2.24 *The position and momentum functions satisfy the following Poisson bracket relations:*

$$\{x_j, x_k\} = 0$$
$$\{p_j, p_k\} = 0$$
$$\{x_j, p_k\} = \delta_{jk}.$$

Proof. Direct calculation. ■

One of the main reasons for considering the Poisson bracket is the following simple result.

Proposition 2.25 *If* $(\mathbf{x}(t), \mathbf{p}(t))$ *is a solution to Hamilton's equation (2.25), then for any smooth function* f *on* $\mathbb{R}^{2n}$ *we have*

$$\frac{d}{dt} f(\mathbf{x}(t), \mathbf{p}(t)) = \{f, H\}(\mathbf{x}(t), \mathbf{p}(t)).$$

We generally write Proposition 2.25 in a more concise form as

$$\frac{df}{dt} = \{f, H\},$$

where the time derivative is understood as being along some trajectory.

Proof. Using the chain rule and Hamilton's equations, we have

$$\begin{aligned} \frac{df}{dt} &= \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} \frac{dx_j}{dt} + \frac{\partial f}{\partial p_j} \frac{dp_j}{dt} \right) \\ &= \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} \frac{\partial H}{\partial p_j} + \frac{\partial f}{\partial p_j} \left(-\frac{\partial H}{\partial x_j} \right) \right) \\ &= \{f, H\}, \end{aligned}$$

as claimed. ■

Observe that Proposition 2.25 includes Hamilton's equations themselves as special cases, by taking $f(x, p) = x_j$ and by taking $f(x, p) = p_j$. Thus, this proposition gives a more coordinate-independent way of expressing the time-evolution.

Corollary 2.26 *Call a smooth function* f *on* $\mathbb{R}^{2n}$ *a conserved quantity if* $f(\mathbf{x}(t), \mathbf{p}(t))$ *is independent of* t *for each solution* $(\mathbf{x}(t), \mathbf{p}(t))$ *of Hamilton's equations. Then* f *is a conserved quantity if and only if*

$$\{f, H\} = 0.$$

In particular, the Hamiltonian H *is a conserved quantity.*

Conserved quantities are also called *constants of motion*. See Conclusion 2.31 for another perspective on this result. Conserved quantities (when one can find them) are useful in that we know that trajectories must lie in the level surfaces of any conserved quantity. Suppose, for example, that we have a particle moving in $\mathbb{R}^2$ and that the Hamiltonian H and one other independent function f (such as, say, the angular momentum) are conserved quantities. Then, rather than looking for trajectories in the four-dimensional phase space, we look for them inside the joint level sets of H

and f (sets of the form $H(x,p) = a$, $f(x,p) = b$, for some constants a and b). These joint level sets are (generically) two-dimensional instead of four-dimensional, so using the constants of motion greatly simplifies the problem—from an equation in four variables to one in only two variables.

Solving Hamilton's equations on $\mathbb{R}^{2n}$ gives rise to a *flow* on $\mathbb{R}^{2n}$, that is, a family Φ_t of diffeomorphisms of $\mathbb{R}^{2n}$, where $\Phi_t(\mathbf{x}, \mathbf{p})$ is equal to the solution at time t of Hamilton's equations with initial condition $(\mathbf{x}, \mathbf{p})$. Since it is possible (depending on the choice of potential function V) that a particle can escape to infinity in finite time, the maps Φ_t are not necessarily defined on all of $\mathbb{R}^{2n}$, but only on some open subset thereof. If Φ_t *does* happen to be defined on all of $\mathbb{R}^{2n}$ (for all t), then we say that the flow is *complete*.

Theorem 2.27 (Liouville's Theorem) *The flow associated with Hamilton's equations, for an arbitrary Hamiltonian function H, preserves the $(2n)$-dimensional volume measure*

$$dx_1 dx_2 \cdots dx_n dp_1 dp_2 \cdots dp_n.$$

What this means, more precisely, is that if a measurable set E is contained in the domain of Φ_t for some $t \in \mathbb{R}$, then the volume of $\Phi_t(E)$ is equal to the volume of E.

Proof. Hamilton's equations may be written as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_n \\ p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_n} \\ -\frac{\partial H}{\partial x_1} \\ \vdots \\ -\frac{\partial H}{\partial x_n} \end{bmatrix}. \tag{2.28}$$

This means that Hamilton's Equations describe the flow along the vector field on $\mathbb{R}^{2n}$ appearing on the right-hand side of (2.28). By a standard result from vector calculus (see, e.g., Proposition 16.33 in [29]), this flow will be volume-preserving if and only if the *divergence* of the vector field is zero. We compute this divergence as

$$\frac{\partial}{\partial x_1}\frac{\partial H}{\partial p_1} + \cdots + \frac{\partial}{\partial x_n}\frac{\partial H}{\partial p_n} - \frac{\partial}{\partial p_1}\frac{\partial H}{\partial x_1} - \cdots - \frac{\partial}{\partial p_n}\frac{\partial H}{\partial x_n}. \tag{2.29}$$

Since

$$\frac{\partial^2 H}{\partial x_j \partial p_j} = \frac{\partial^2 H}{\partial p_j \partial x_j},$$

the divergence is zero. ■

The existence of an invariant volume has important consequences for the dynamics of a system. For example, for "confined" systems, an invariant volume implies that the system exhibits "recurrence," which means

(roughly) that for most initial conditions, the particle will eventually come back arbitrarily close to its initial state in phase space. We will not, however, delve into this aspect of the theory.

Note that the divergence of X_H, computed in (2.29), vanishes in a very particular way, namely the sum of the jth and $(n+j)$th terms vanishes for all $1 \leq j \leq n$. This stronger condition turns out to be equivalent to the condition that the Hamiltonian flow Φ_t associated with an arbitrary smooth function on $\mathbb{R}^{2n}$ preserves the symplectic form ω, defined by

$$\omega((\mathbf{x}, \mathbf{p}), (\mathbf{x}', \mathbf{p}')) = \mathbf{x} \cdot \mathbf{p}' - \mathbf{p} \cdot \mathbf{x}'.$$

What this means, more precisely, is that for any $t \in \mathbb{R}$ and any $(\mathbf{x}, \mathbf{p}) \in \mathbb{R}^{2n}$, the matrix of partial derivatives of Φ_t at the point $(\mathbf{x}, \mathbf{p})$—thought of as a linear map of $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$—preserves ω. This property of Φ_t, as it turns out, is equivalent to the property that Φ_t preserves Poisson brackets, meaning that

$$\{f \circ \Phi_t, g \circ \Phi_t\} = \{f, g\} \circ \Phi_t$$

for all $f, g \in C^\infty(\mathbb{R}^n)$. A map $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ that preserves ω is called a *symplectomorphism* (in mathematics notation) or a *canonical transformation* (in physics notation). We defer the proofs of these claims until Chap. 21, where we can consider them in a more general setting.

Definition 2.28 *For any smooth function f on $\mathbb{R}^{2n}$, the **Hamiltonian flow** generated by f is the flow obtained by solving Hamilton's equation (2.25) with the Hamiltonian H replaced by f. The function f is called the **Hamiltonian generator** of the associated flow.*

Although *any* smooth function on $\mathbb{R}^{2n}$ can be inserted into Hamilton's equations to produce a flow, physically one should think that there is a distinguished function, the Hamiltonian H of the system, such that the flow generated by H is the time-evolution of the system. For any other function f, the Hamiltonian flow generated by f should not be thought of as time-evolution, but as some other flow, which might, for example, represent some family of symmetries of our system.

Proposition 2.29 *The Hamiltonian flow generated by the function*

$$f_{\mathbf{a}}(\mathbf{x}, \mathbf{p}) := \mathbf{a} \cdot \mathbf{p} \tag{2.30}$$

is given by

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}_0 + t\mathbf{a} \\ \mathbf{p}(t) &= \mathbf{p}_0, \end{aligned} \tag{2.31}$$

and the Hamiltonian flow generated by the function

$$g_{\mathbf{b}}(\mathbf{x}, \mathbf{p}) := \mathbf{b} \cdot \mathbf{x} \tag{2.32}$$

is given by

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{x}_0\\ \mathbf{p}(t) &= \mathbf{p}_0 - t\mathbf{b}.\end{aligned}$$

Proof. Direct calculation. ■

What this means is that the Hamiltonian flow generated by a linear combination of the *momentum* functions consists of translations in *position* of the particle. That is to say, in the flow (2.31) generated by the function $f_{\mathbf{a}}$ in (2.30), the particle's initial position $\mathbf{x}_0$ is translated by $t\mathbf{a}$ while the particle's momentum is independent of t. Similarly, the Hamiltonian flow generated by a linear combination of the *position* functions [the function $g_{\mathbf{b}}$ in (2.32)] consists of translations in the particle's *momentum*.

Proposition 2.30 *For a particle moving in* $\mathbb{R}^2$, *the Hamiltonian flow generated by the angular momentum function*

$$J(\mathbf{x}, \mathbf{p}) = x_1 p_2 - x_2 p_1$$

consists of simultaneous rotations of $\mathbf{x}$ *and* $\mathbf{p}$. *That is to say,*

$$\begin{aligned}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}\\ \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} &= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix}.\end{aligned} \tag{2.33}$$

Proof. If we plug the angular momentum function J into Hamilton's equations in place of H, we obtain

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{\partial J}{\partial p_1} = -x_2; \qquad & \frac{dp_1}{dt} &= -\frac{\partial J}{\partial x_1} = -p_2\\ \frac{dx_2}{dt} &= \frac{\partial J}{\partial p_2} = x_1; \qquad & \frac{dp_2}{dt} &= -\frac{\partial J}{\partial x_2} = p_1\end{aligned}.$$

The solution to this system is given by the expression in the proposition, as is easily verified by differentiation of (2.33). ■

Note that since the Hamiltonian flow generated by J does not have the interpretation of the time-evolution of the particle, the parameter t in (2.33) should not be interpreted as the physical time; it is just the parameter in a one-parameter group of diffeomorphisms. In this case, t is the angle of rotation. Thus, one answer to the question, "What *is* the angular momentum?" is that J is the *Hamiltonian generator of rotations*.

If f is any smooth function, then by the proof of Proposition 2.25, the time derivative of any other function g along the Hamiltonian flow generated by f is given by $dg/dt = \{g, f\}$. In particular, the derivative of the Hamiltonian H along the flow generated by f is $\{H, f\}$. Thus, f is constant

along the flow generated by H if and only if $\{f, H\} = 0$, which holds if and only if $\{f, H\} = 0$, which holds if and only if H is constant along the flow generated by f. This line of reasoning leads to the following result.

Conclusion 2.31 *A function f is a conserved quantity for solutions of Hamilton's equation (2.25) if and only if H is invariant under the Hamiltonian flow generated by f. In particular, the angular momentum J is conserved if and only if H is invariant under simultaneous rotations of $\mathbf{x}$ and $\mathbf{p}$.*

We will return to this way of thinking about conserved quantities in Chap. 21. Compare Exercise 12.

The Hamiltonian framework can be extended in a straightforward way to systems of particles.

Proposition 2.32 *Consider the phase space for a system of N particles moving in $\mathbb{R}^n$, namely $\mathbb{R}^{2nN}$, thought of as the set of $(2N)$-tuples of the form*

$$(\mathbf{x}^1, \ldots, \mathbf{x}^N, \mathbf{p}^1, \ldots, \mathbf{p}^N)$$

with $\mathbf{x}^j$ and $\mathbf{p}^j$ belonging to $\mathbb{R}^n$. Define the Poisson bracket of two smooth functions f and g on the phase space by

$$\{f, g\} = \sum_{j=1}^{N} \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k^j} \frac{\partial g}{\partial p_k^j} - \frac{\partial f}{\partial p_k^j} \frac{\partial g}{\partial x_k^j} \right)$$

and consider a Hamiltonian function of the form

$$H(\mathbf{x}^1, \ldots, \mathbf{x}^N, \mathbf{p}^1, \ldots, \mathbf{p}^N) = \sum_{j=1}^{N} \frac{1}{2m_j} \left| \mathbf{p}^j \right|^2 + V(\mathbf{x}^1, \ldots, \mathbf{x}^N).$$

Then Newton's law in the form $m_j \ddot{\mathbf{x}}^j = -\nabla^j V$ is equivalent to Hamilton's equations in the form

$$\begin{aligned} \frac{dx_k^j}{dt} &= \frac{\partial H}{\partial p_k^j} \\ \frac{dp_k^j}{dt} &= -\frac{\partial H}{\partial x_k^j}. \end{aligned} \tag{2.34}$$

For any smooth function f, the derivative of f along a solution of Hamilton's equations is given by

$$\frac{df}{dt} = \{f, H\}.$$

The proof of these results is entirely similar to the one-particle case and is omitted.

2.6 The Kepler Problem and the Runge–Lenz Vector

2.6.1 The Kepler Problem

We consider now the classical Kepler problem, that of finding the trajectories of a planet orbiting the sun. Since the sun is very much more massive than any of the planets, we may consider the position of the sun to be fixed at the origin of our coordinate system. The sun exerts a force on a planet given by

$$\mathbf{F} = -k\frac{\mathbf{x}}{|\mathbf{x}|^3}. \tag{2.35}$$

Here $k = GmM$, where m is the mass of the planet, M is the mass of the sun, and G is the universal gravitational constant. Note that the magnitude of $\mathbf{F}$ is proportional to the reciprocal of the square of the distance from the origin; thus, the force follows an *inverse square law*. Since k contains a factor of the mass m of the planet, this quantity drops out of the equation of motion, $m\ddot{\mathbf{x}} = \mathbf{F}$. The potential associated with the force (2.35) is easily seen to be

$$V(\mathbf{x}) = -\frac{k}{|\mathbf{x}|}. \tag{2.36}$$

Since our potential V is invariant under rotations, the angular momentum vector $\mathbf{J} = \mathbf{x} \times \mathbf{p}$ is a conserved quantity (Theorem 2.21 with $N = 1$ and $n = 3$). If $\mathbf{J} = 0$, the particle is moving along a ray through the origin. In that case, either the particle will pass through the origin at some point in the future (if the initial momentum points toward the origin), or else the particle must have passed through the origin at some point in the past (if the initial momentum points away from the origin). Trajectories of this sort are called *collision trajectories*, and we will regard such trajectories as pathological.

We will, from now on, consider only trajectories along which the angular momentum vector is nonzero. Fixing the energy and angular momentum of the particle guarantees that the particle stays a certain minimum distance from the origin (Exercise 20). Meanwhile, since $\mathbf{J} = \mathbf{x} \times \mathbf{p}$, the position $\mathbf{x}(t)$ of the particle will always be perpendicular to the constant value of $\mathbf{J}$. We will therefore refer to the plane (through the origin) perpendicular to $\mathbf{J}$ as the "plane of motion."

2.6.2 Conservation of the Runge–Lenz Vector

We are going to obtain a description of the classical trajectories in an indirect way, using something called the Runge–Lenz vector.

Definition 2.33 *The* ***Runge–Lenz vector*** *is the vector-valued function on* $\mathbb{R}^3\backslash\{0\} \times \mathbb{R}^3$ *given by*

$$\mathbf{A}(\mathbf{x}, \mathbf{p}) = \frac{1}{mk}\mathbf{p} \times \mathbf{J} - \frac{\mathbf{x}}{|\mathbf{x}|}.$$

Here $\mathbf{x}$ *represents the position of a classical particle and* $\mathbf{p}$ *its momentum.*

The significance of this vector is that it is a *conserved quantity* for the Kepler problem. Of course, whenever the potential energy is radial (a function of the distance from the origin), the angular momentum vector is a conserved quantity. What is special about the $1/r$ potential of the Kepler problem is that there is another conserved vector-valued quantity.

Proposition 2.34 *The Runge–Lenz vector is conserved quantity for Newton's law with force given by (2.35).*

Proof. Since $\mathbf{J}$ is conserved, we compute that

$$\begin{aligned}
\dot{\mathbf{A}}(t) &= \frac{1}{mk}\mathbf{F} \times \mathbf{J} - \frac{1}{|\mathbf{x}|}\frac{\mathbf{p}}{m} + \frac{\mathbf{x}}{|\mathbf{x}|^2}\sum_{j=1}^{3}\frac{\partial |\mathbf{x}|}{\partial x_j}\frac{dx_j}{dt} \\
&= -\frac{1}{m}\frac{1}{|\mathbf{x}|^3}\mathbf{x} \times (\mathbf{x} \times \mathbf{p}) - \frac{1}{|\mathbf{x}|}\frac{\mathbf{p}}{m} + \frac{\mathbf{x}}{|\mathbf{x}|^2}\sum_{j=1}^{3}\frac{x_j}{|\mathbf{x}|}\frac{p_j}{m} \\
&= \frac{1}{m}\left(-\frac{1}{|\mathbf{x}|^3}\mathbf{x}(\mathbf{x}\cdot\mathbf{p}) + \frac{1}{|\mathbf{x}|^3}\mathbf{p}(\mathbf{x}\cdot\mathbf{x}) - \frac{\mathbf{p}}{|\mathbf{x}|} + \frac{\mathbf{x}(\mathbf{x}\cdot\mathbf{p})}{|\mathbf{x}|^3}\right) \\
&= 0.
\end{aligned}$$

Here we have used the identity $\mathbf{b} \times (\mathbf{c} \times \mathbf{d}) = \mathbf{c}(\mathbf{b}\cdot\mathbf{d}) - \mathbf{d}(\mathbf{b}\cdot\mathbf{c})$, which holds for all vectors $\mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$. ■

2.6.3 Ellipses, Hyperbolas, and Parabolas

We now use the Runge–Lenz vector to determine the trajectories for the Kepler problem.

Proposition 2.35 *The magnitude of the Runge–Lenz vector* $\mathbf{A}$ *satisfies*

$$|\mathbf{A}|^2 = 1 + \frac{2|\mathbf{J}|^2}{mk^2}E,$$

where $E = |\mathbf{p}|^2/(2m) - k/|\mathbf{x}|$ *is the energy of the particle. Furthermore, if* $\hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}|$ *is the unit vector in the* $\mathbf{x}$*-direction, we have*

$$\mathbf{A} \cdot \hat{\mathbf{x}} = \frac{|\mathbf{J}|^2}{mk|\mathbf{x}|} - 1 \tag{2.37}$$

for all nonzero $\mathbf{x}$. *It follows from (2.37) that*

$$|\mathbf{x}| = \frac{|\mathbf{J}|^2}{mk(1 + \mathbf{A} \cdot \hat{\mathbf{x}})}.$$

Note that from (2.37), $\mathbf{A} \cdot \hat{\mathbf{x}} > -1$ for all points $(\mathbf{x}, \mathbf{p})$ with $\mathbf{x} \neq 0$.

Proof. Using the identity $\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d}) = \mathbf{d} \cdot (\mathbf{b} \times \mathbf{c})$, we see that

$$\hat{\mathbf{x}} \cdot (\mathbf{p} \times \mathbf{J}) = \mathbf{J} \cdot (\hat{\mathbf{x}} \times \mathbf{p}) = |\mathbf{J}|^2 / |\mathbf{x}|.$$

Since $\mathbf{J}$ and $\mathbf{p}$ are orthogonal, we get

$$\begin{aligned} |\mathbf{A}|^2 &= \frac{1}{m^2k^2} |\mathbf{p}|^2 |\mathbf{J}|^2 + 1 - \frac{2}{mk} \hat{\mathbf{x}} \cdot (\mathbf{p} \times \mathbf{J}) \\ &= 1 + \frac{2|\mathbf{J}|^2}{mk^2} \left(\frac{|\mathbf{p}|^2}{2m} - \frac{k}{|\mathbf{x}|} \right) \\ &= 1 + \frac{2|\mathbf{J}|^2}{mk^2} E. \end{aligned}$$

Using again the identity for $\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})$, we next compute that

$$\begin{aligned} \mathbf{A} \cdot \mathbf{x} &= \frac{1}{mk} \mathbf{J} \cdot (\mathbf{x} \times \mathbf{p}) - \frac{\mathbf{x} \cdot \mathbf{x}}{|\mathbf{x}|} \\ &= \frac{|\mathbf{J}|^2}{mk} - |\mathbf{x}|. \end{aligned}$$

We may now divide by $|\mathbf{x}|$ to obtain the desired expression for $\mathbf{A} \cdot \hat{\mathbf{x}}$. It is then straightforward to solve for $|\mathbf{x}|$. ■

Corollary 2.36 *Choose orthonormal coordinates in the plane of motion so that* $\mathbf{A}$ *lies along the positive* x_1*-axis. If* r *and* θ *are the polar coordinates associated with this coordinate system, then along each trajectory* $(r(t), \theta(t))$, *we have*

$$r(t) = \frac{|\mathbf{J}|^2}{mk} \frac{1}{1 + A \cos \theta(t)}, \tag{2.38}$$

where $A = |\mathbf{A}|$.

If $\mathbf{A} = 0$, any orthonormal coordinates can be used.

Proposition 2.37 *If* $A := |\mathbf{A}| < 1$, *(2.38) is the equation of an ellipse with eccentricity* A *and with the origin being one focus of the ellipse. If* $A > 1$, *(2.38) is the equation of a hyperbola, and if* $A = 1$, *(2.38) is the equation of a parabola.*

The orbit of the particle in the plane of motion is an ellipse if the energy of the particle is negative, a hyperbola if the energy is positive, and a parabola if the energy is zero.

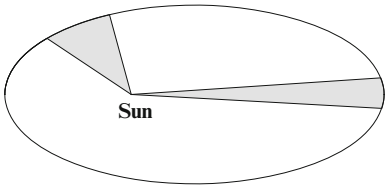

FIGURE 2.2. Elliptical orbit for the Kepler problem, with two equal areas shaded.

Kepler's first law is the assertion that planets move in elliptical trajectories with the sun at one focus, as shown in Fig. 2.2. The shaded regions indicate two equal areas that are swept out in equal times, in accordance with Kepler's second law (Corollary 2.19).

Recall that the eccentricity of an ellipse is $\sqrt{1-(b/a)^2}$, where a is half the length of the major axis and b is half the length of the minor axis. Thus, when $A = 0$, we have $b = a$, meaning that the ellipse is a circle.

Proof. We continue to work in a coordinate system in which $\mathbf{A}$ is along the positive x_1-axis. Then (2.38) becomes

$$\sqrt{x^2+y^2} = \alpha \frac{1}{1 + A\frac{x}{\sqrt{x^2+y^2}}},$$

where $\alpha = |\mathbf{J}|^2/(mk)$. From this we obtain

$$1 = \frac{1}{\alpha}\left(\sqrt{x^2+y^2} + Ax\right).$$

Now we can solve for $\sqrt{x^2+y^2}$, square both sides of the equation, and simplify. Assuming $A^2 \neq 1$, we obtain

$$\alpha^2\left(\frac{1}{1-A^2}\right) = (1-A^2)\left(x + \frac{A\alpha}{1-A^2}\right)^2 + y^2. \tag{2.39}$$

This is the equation of an ellipse (if $A^2 < 1$) or a hyperbola (if $A^2 > 1$), where the center of the ellipse or hyperbola is the point $(-\alpha/(1-A^2), 0)$. In light of the formula for $A := |\mathbf{A}|$ in Proposition 2.35, we obtain an ellipse if the energy of the particle is negative and a hyperbola if the energy is positive.

In the case $A^2 < 1$, we may readily compute the half-lengths a and b of the major and minor axes as

$$a = \frac{\alpha}{1-A^2}; \quad b = \frac{\alpha}{\sqrt{1-A^2}}.$$

From this, we readily calculate that the eccentricity is A. Now, the distance between the foci of an ellipse is the length of the major axis times the eccentricity, in our case, $2A\alpha/(1-A^2)$. Since the center of the ellipse in (2.39) is at the point $(A\alpha/(1-A^2), 0)$, the origin is one focus of the ellipse.

If $A^2 = 1$, then when we perform the same analysis, x^2 drops out of the equation and we obtain

$$x = \frac{1}{2A\alpha}\left(-y^2 + \alpha^2\right)$$

which is the equation of a parabola opening along the x-axis. This case corresponds to energy zero. ■

Note that Proposition 2.37 does not tell us how the particle moves along the ellipse, hyperbola, or parabola as a function of time. We can, however, determine this, at least in principle, by making use of the angular momentum. After all, applying (2.17) in the plane of motion gives

$$\frac{d\theta}{dt} = \frac{1}{mr^2}\left|\mathbf{J}\right|, \tag{2.40}$$

where θ is the polar angle variable in the plane of motion. Since we have computed r as a function of θ in Corollary 2.36, (2.40) gives us a (first-order, separable) differential equation, from which we can attempt to solve to obtain θ—and thus also r—as a function of t.

2.6.4 Special Properties of the Kepler Problem

As we have said, the existence of another conserved vector-valued function—in addition to the conserved energy and angular momentum—is special to a potential of the form $-k/\left|\mathbf{x}\right|$. For a general radial potential, the energy and the angular momentum will be the only conserved quantities. Assuming $\mathbf{J} \neq 0$, the motion of a particle in any radial potential will always lie in the plane perpendicular to $\mathbf{J}$. Taking this into account, we think of our particle

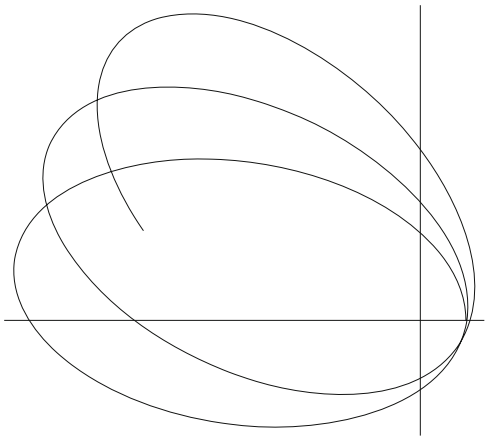

FIGURE 2.3. Trajectory in the plane of motion for a typical radial potential.

as moving in $\mathbb{R}^2$ rather than $\mathbb{R}^3$, and accordingly think of our phase space as being four-dimensional rather than six-dimensional. From this point of view, there are two remaining conserved quantities, the energy E and the scalar angular momentum J in the plane, as given by Definition 2.17. Thus, each trajectory will lie in a set of the form

$$\left\{ (\mathbf{x}, \mathbf{p}) \in \mathbb{R}^2 \times \mathbb{R}^2 \middle| \, E(\mathbf{x}, \mathbf{p}) = a, \;\; J(\mathbf{x}, \mathbf{p}) = b \right\}.$$

We refer to such a set as a *joint level set* of E and J. These sets are two-dimensional surfaces inside our four-dimensional phase space.

For a general radial potential, a trajectory $(\mathbf{x}(t), \mathbf{p}(t))$ in phase space may not be a closed curve, but may fill up a dense subset of the joint level surface on which it lives. In particular, the trajectory $\mathbf{x}(t)$ in position space will typically not be a closed curve. For example, $\mathbf{x}(t)$ may trace out a roughly elliptical region in the plane, but where the axes of the ellipse "precess," that is, vary with time. Such a trajectory is shown in Fig. 2.3, which should be contrasted with Fig. 2.2.

In the Kepler problem, even after restricting attention to the plane of motion, we still have one conserved quantity in addition to E and J, namely the direction of $\mathbf{A}$, which can be expressed in terms of the angle ϕ between $\mathbf{A}$ and the x_1-axis in the plane of motion. (Note that both terms in the definition of $\mathbf{A}$ lie in the plane of motion. Note also that the magnitude of $\mathbf{A}$ is, by Proposition 2.35, computable in terms of E and J.) The trajectories of the Kepler problem, then, lie in the joint level sets of E and J and ϕ, which are one-dimensional. When $E < 0$, the joint level sets of E and J are compact, in which case the joint level sets of E and J and ϕ are compact and one-dimensional, that is, simple closed curves.

Another special property of the Kepler problem is that the period of the closed trajectories (the trajectories with negative energy) is the same for all trajectories with the same energy (Exercise 21). This apparent coincidence can be explained by showing that the Hamiltonian flows (Definition 2.28) generated by $\mathbf{J}$ and $\mathbf{A}$ act transitively on the energy surfaces. These flows commute with the time evolution of the system, because they are all conserved quantities (Conclusion 2.31). Thus, any two points with the same energy are "equivalent" with respect to time evolution. Although we will not go into the details of this analysis, we will gain a better understanding of the flows generated by the components of $\mathbf{A}$ in Sect. 18.4.

2.7 Exercises

1. Consider a particle moving in the real line in the presence of a force coming from a potential function V. Given some value E_0 for the energy of the particle, suppose that $V(x) < E_0$ for all x in some closed interval $[x_0, x_1]$. Then a particle with initial position x_0 and

positive initial velocity will continue to move to the right until it reaches x_1. Using (2.6), show that the time needed to travel from x_0 to x_1 is given by

$$t = \int_{x_0}^{x_1} \sqrt{\frac{m}{2(E_0 - V(y))}}\, dy.$$

Note: This shows that we can solve Newton's equation in $\mathbb{R}^1$ more or less explicitly for time as a function of position, which in principle determines the position as a function of time.

2. In the notation of the previous problem, suppose now that $V(x) < E_0$ for $x_0 \leq x < x_1$, but that $V(x_1) = E_0$.

   (a) Show that if $V'(x_1) \neq 0$, then the particle reaches x_1 in a finite time.

   (b) Show that if $V'(x_1) = 0$, then the time it takes the particle to reach x_1 is infinite; that is, the particle approaches but never actual reaches x_1.

   Note: In Part (b), the point x_1 is an unstable equilibrium for the system, that is, a critical point for V that is not a local minimum.

3. Consider the equation of motion of a pendulum of length L,

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0,$$

   where g is the acceleration of gravity. Here θ is the angle between the pendulum and the negative y-axis in the plane. This system has a stable equilibrium at $\theta = 0$ and an unstable equilibrium at $\theta = \pi$.

   Consider initial conditions of the form $\theta(0) = \pi - \delta$, $\dot{\theta}(0) = 0$, for $0 < \delta < \pi/4$. Fix some angle θ_0 and let $T(\delta)$ denote the time it takes for the pendulum with the given initial conditions to reach the angle θ_0. (Here θ_0 represents an arbitrarily chosen cutoff point at which the pendulum is no longer "close" to $\theta = \pi$.) Show that $T(\delta)$ grows only logarithmically as δ tends to zero.

   Note: Logarithmic growth of T as a function of δ corresponds to exponential decay of δ as a function of T. Thus, if we want T to be large, we must choose δ to be *very* small.

4. Consider a particle moving in the real line in the presence of a "repelling potential," such that there is an A with $V'(x) < 0$ for all $x > A$. Then a particle with initial position $x_0 > A$ and positive initial velocity will have positive velocity for all positive times. Suppose now that $V(x) = -x^a$ for all $x > 1$, for some positive constant

a. Suppose also that the particle is given initial position $x_0 > 1$ and positive initial velocity. Show that for $a > 2$, the particle escapes to infinity in finite time, but that for $a \leq 2$, the position of the particle remains finite for all finite times.

Hint: Use Problem 1.

5. Consider the equation $m\ddot{x} + \gamma\dot{x} + kx = 0$, where γ and k are positive constants (the damping constant and spring constant, respectively). Find the critical value γ_c of γ (for a fixed m and k) such that for $\gamma < \gamma_c$, we get solutions that are sines and cosines times a decaying exponential and for $\gamma > \gamma_c$, we get pure decaying exponentials.

6. Continue with the notation of Exercise 5. Given particular choices for m, γ, and k, let r be the rate of exponential decay of a "generic" solution to the equation of motion. Here, if the solution is of the form $ae^{-rt}\cos(\omega t) + be^{-rt}\sin(\omega t)$, the rate of exponential decay is r. If the solution is of the form $ae^{-r_1 t} + be^{-r_2 t}$, then $r = \min(r_1, r_2)$, since the slower-decaying term will dominate as long as a and b are both nonzero.

   For a fixed value of m and k, show that the maximum value for r is achieved by taking $\gamma = \gamma_c$. (This accounts for the terminology "critical damping" for the case in which $\gamma = \gamma_c$.)

7. Consider the $\mathbb{R}^2$-valued function $\mathbf{F}$ on $\mathbb{R}^2 \setminus \{0\}$ given by

$$\mathbf{F}(x_1, x_2) = \left(-\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right).$$

   Show that $\partial F_1/\partial x_2 - \partial F_2/\partial x_1 = 0$ but that there does not exist any smooth function V on $\mathbb{R}^2 \setminus \{0\}$ with $\mathbf{F} = -\nabla V$.

   Hint: If $\mathbf{F}$ were of the form $-\nabla V$, we would have

$$V(\mathbf{x}(b)) - V(\mathbf{x}(a)) = -\int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \frac{d\mathbf{x}}{dt}\, dt$$

   for every smooth path $\mathbf{x}(\cdot) : [a, b] \to \mathbb{R}^2 \setminus \{0\}$, by the fundamental theorem of calculus and the chain rule.

8. Consider a particle moving in $\mathbb{R}^n$ with a velocity-dependent force law given by

$$\mathbf{F}(\mathbf{x}, \mathbf{v}) = -\nabla V(\mathbf{x}) + \mathbf{F}_2(\mathbf{x}, \mathbf{v}),$$

   where the velocity-dependent term $\mathbf{F}_2$ acts perpendicularly to the velocity of the particle. (That is, we assume that $\mathbf{v} \cdot \mathbf{F}_2(\mathbf{x}, \mathbf{v}) = 0$ for all $\mathbf{x}$ and $\mathbf{v}$.) Let E denote the usual energy function $E(\mathbf{x}, \mathbf{v}) = \frac{1}{2}m|\mathbf{v}|^2 + V(\mathbf{x})$, unmodified by the presence of the velocity-dependent term in the force. Show that E is conserved.

9. (a) If r and θ are the usual polar coordinates on $\mathbb{R}^2$, compute $\partial\theta/\partial x_1$ and $\partial\theta/\partial x_2$.

   (b) If $\mathbf{x}(\cdot)$ denotes the trajectory of a particle of mass m moving in $\mathbb{R}^2$, show that

$$\frac{d}{dt}\theta(\mathbf{x}(t)) = \frac{1}{mr^2} J(\mathbf{x}(t), \mathbf{p}(t)).$$

10. Prove Theorem 2.21, by imitating the proof of Proposition 2.18. You may assume that every rotation can be built up as a product of repeated rotations in the various coordinate planes (i.e., rotations in the (x_j, x_k) plane, for various pairs (j, k), where the same plane may be used more than once).

11. Consider Hamilton's equations for N particles moving in $\mathbb{R}^n$, as in Proposition 2.32. Show that the total momentum $\mathbf{p} = \sum_{j=1}^{N} \mathbf{p}^j$ of the system is a conserved quantity if and only if the quantity

$$H(\mathbf{x}^1 + \mathbf{a}, \ldots, \mathbf{x}^N + \mathbf{a}, \mathbf{p}^1 + \mathbf{a}, \ldots, \mathbf{p}^N + \mathbf{a}), \ \mathbf{a} \in \mathbb{R}^n,$$

   is independent of $\mathbf{a}$ for all $\mathbf{x}^1, \ldots, \mathbf{x}^N$ and $\mathbf{p}^1, \ldots, \mathbf{p}^N$ in $\mathbb{R}^n$.

   Hint: Use (the N-particle version of) Conclusion 2.31.

12. Let J denote the angular momentum of a particle moving in $\mathbb{R}^2$. Let R_θ denote a counterclockwise rotation by angle θ in $\mathbb{R}^2$.

   (a) If f is any smooth function on $\mathbb{R}^4$, show that

$$\{f, J\}(\mathbf{x}, \mathbf{p}) = \frac{d}{d\theta} f\left(R_\theta \mathbf{x}, R_\theta \mathbf{p}\right)\Big|_{\theta=0}.$$

   (b) Let H be any smooth function on $\mathbb{R}^4$ and consider Hamilton's equations with this function playing the role of the Hamiltonian. Show that J is conserved (i.e., constant in time along any solution of Hamilton's equations) if and only if

$$H(R_\theta \mathbf{x}, R_\theta \mathbf{p}) = H(\mathbf{x}, \mathbf{p})$$

   for all θ in $\mathbb{R}$ and all $\mathbf{x}$ and $\mathbf{p}$ in $\mathbb{R}^2$. (This argument is a more explicit way to obtain Conclusion 2.31.)

13. Suppose that f and g are smooth functions on $\mathbb{R}^{2n}$ and that at least one of the two functions has compact support. Show that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \{f, g\}(\mathbf{x}, \mathbf{p}) \, d^n\mathbf{x} \, d^n\mathbf{p} = 0.$$

   Hint: Use integration by parts or Liouville's theorem.

14. Let X and Y be "vector fields" on $\mathbb{R}^n$, viewed as first-order differential operators. This means that X and Y are of the form

$$X = \sum_{j=1}^{n} a_j(\mathbf{x}) \frac{\partial}{\partial x_j}; \quad Y = \sum_{j=1}^{n} b_j(\mathbf{x}) \frac{\partial}{\partial x_j}.$$

[If $\tilde{X}(\mathbf{x}) = (a_1(\mathbf{x}), \ldots, a_n(\mathbf{x}))$, then the operator X is the directional derivative in the direction of $\tilde{X}$. It is common to identify the vector-valued function $\tilde{X}$ with the associated first-order differential operator X.]

Show that the commutator $[X, Y]$ of X and Y, defined by

$$[X, Y] = XY - YX$$

is again a vector field (i.e., a *first*-order differential operator).

15. Given a smooth function f on $\mathbb{R}^{2n}$, define an operator X_f, acting on $C^\infty(\mathbb{R}^{2n})$, by the formula

$$X_f(g) = \{f, g\}.$$

That is to say,

$$X_f = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} \frac{\partial}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial}{\partial x_j} \right).$$

The operator X_f is called the *Hamiltonian vector field* associated with the function f. (Here, as in Exercise 14, we identify vector fields with first-order differential operators.)

(a) Show that for all $f, g \in C^\infty(\mathbb{R}^{2n})$, we have

$$X_{\{f,g\}} = [X_f, X_g],$$

where $[X_f, X_g] = X_f X_g - X_g X_f$.
Hint: By Exercise 14, all terms in the computation of $[X_f, X_g](h)$ involving second derivatives of h can be neglected, since they will always cancel out to zero.

(b) Use Part (a) to compute $\{\{f, g\}, h\} = X_{\{f,g\}}(h)$ and thereby obtain another proof of the Jacobi identity for the Poisson bracket.

16. Recall the definition of a Hamiltonian vector field X_f in Exercise 15.

(a) Consider a smooth vector field X on $\mathbb{R}^2$ (viewed as a first-order differential operator as in Exercise 14) of the form

$$X(\mathbf{x}) = g_1(x, p) \frac{\partial}{\partial x} + g_2(x, p) \frac{\partial}{\partial p}.$$

Show that X can be expressed as $X = X_f$, for some $f \in C^\infty(\mathbb{R}^2)$, if and only X is *divergence free*, that is, if and only if

$$\nabla \cdot X := \frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial p} = 0.$$

Hint: As in Proposition 2.7, given a pair of functions h_1 and h_2 on $\mathbb{R}^2$, there exists a function f with $\partial f/\partial x = h_1$ and $\partial f/\partial p = h_2$ if and only if we have $\partial h_1/\partial p = \partial h_2/\partial x$.

(b) Show that there exists a smooth vector field X on $\mathbb{R}^4$ of the form

$$X = \sum_{j=1}^{2} \left(g_j(\mathbf{x}) \frac{\partial}{\partial x_j} + g_{j+2}(\mathbf{x}) \frac{\partial}{\partial p_j} \right)$$

such that

$$\nabla \cdot X := \sum_{j=1}^{2} \left(\frac{\partial g_j}{\partial x_j} + \frac{\partial g_{j+2}}{\partial p_j} \right) = 0$$

but such that there does not exist $f \in C^\infty(\mathbb{R}^4)$ with $X = X_f$.

Hint: You should be able to find a counterexample in which the coefficient functions g_j are linear.

17. Show that the space of homogeneous polynomials of degree 2 on $\mathbb{R}^{2n}$ is closed under the Poisson bracket.

18. Determine the Hamiltonian flow on $\mathbb{R}^2$ generated by the function $f(x, p) = xp$.

19. Let $\mathbf{J}$ denote the angular momentum vector for a particle moving in $\mathbb{R}^3$, namely $\mathbf{J} = \mathbf{x} \times \mathbf{p}$. Show that the components J_1, J_2, and J_3 of $\mathbf{J}$ satisfy the following Poisson bracket relations:

$$\{J_1, J_2\} = J_3; \quad \{J_2, J_3\} = J_1; \quad \{J_3, J_1\} = J_2.$$

20. In the Kepler problem, show that for each real number E and positive number J, there exists $\varepsilon > 0$ such that for all $(\mathbf{x}, \mathbf{p})$ with $E(\mathbf{x}, \mathbf{p}) = E$ and $|\mathbf{J}(\mathbf{x}, \mathbf{p})| = J$, we have $|\mathbf{x}| \geq \varepsilon$.

Hint: Suppose that $(\mathbf{x}_n, \mathbf{p}_n)$ is a sequence with $|\mathbf{J}(\mathbf{x}_n, \mathbf{p}_n)| = J$ and $|\mathbf{x}_n|$ tending to zero. Show that $E(\mathbf{x}_n, \mathbf{p}_n)$ tends to $+\infty$.

21. (a) Determine the area of the ellipse in the plane of motion in Proposition 2.37, in the case $A < 1$.

(b) Show that the time T it takes the particle to travel once around the ellipse is given by

$$\frac{\pi}{\sqrt{2}} GM(-\tilde{E})^{-3/2},$$

where $\tilde{E}$ is the "massless energy" of the particle, given by

$$\tilde{E} = \frac{E}{m} = \frac{1}{2}\,|\dot{\mathbf{x}}| - \frac{GM}{|\mathbf{x}|}.$$

Note in the case where the trajectory in the plane of motion is elliptical, the energy of the particle is negative.

Note: The result of Part (b) is closely related to Kepler's third law.