Unit Ball of a Vector Norm

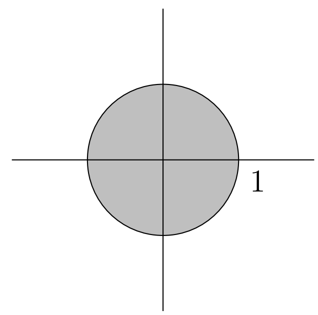
Definition: Unit Ball

Let $\|\cdot\|$ be a vector norm on $\mathbb{C}^n$. The *closed unit ball* of $\|\cdot\|$, denoted $B_{\|\cdot\|}$, is given by:

$$B_{\|\cdot\|} = \{\vec{x} \in \mathbb{C}^n \mid \|\vec{x}\| \leq 1\}$$

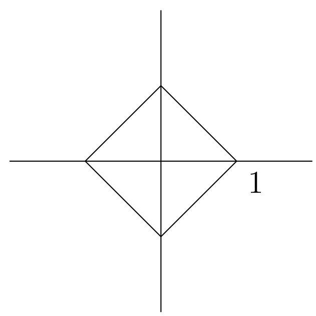
Note that when $n = 1$:

- $\ell_1, \ell_2, \ell_\infty$ are the same.
- $B_{\ell_1} = B_{\ell_2} = B_{\ell_\infty} = \{z \in \mathbb{C} \mid |z| \leq 1\}$

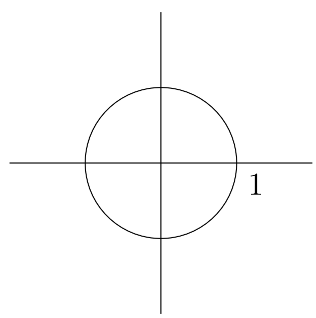


If restricted to $\mathbb{R}^2$ then we have the following:

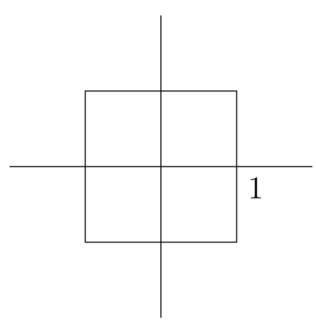
$$B_{\ell_1} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid |x| + |y| \leq 1 \right\}$$



$$B_{\ell_2} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid \sqrt{x^2 + y^2} \leq 1 \right\}$$



$$B_{\ell_\infty} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid \max\{x, y\} \leq 1 \right\}$$



Note that $\forall\,\vec{x}$ we have:

$$\|\vec{x}\|_\infty \le \|\vec{x}\|_2 \le \|\vec{x}\|_1$$

However:

$$B_{\ell_1} \subseteq B_{\ell_2} \subseteq B_{\ell_\infty}$$

<u>Theorem</u>

$\forall \vec{x} \in \mathbb{C}^n$:

$$\|\vec{x}\|_\alpha \le \|\vec{x}\|_\beta \iff B_{\|\cdot\|_\alpha} \supseteq B_{\|\cdot\|_\alpha}$$

The larger the norm, the smaller the unit ball.