Norms

Definition: Norm

Let E be a vector space over a scalar field $\mathbb{F}$. A *norm* on E is a function $\|\cdot\| : E \to \mathbb{R}$ such that the following axioms are satisfied $\forall\, \vec{x} \in E$ and $\forall \lambda \in \mathbb{F}$:

1). Definiteness: $\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$

2). Homogeneity: $\|\lambda\vec{x}\| = |\lambda|\, \|\vec{x}\|$

3). Subadditive: $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \qquad$ (triangle inequality)

A vector space equipped with such a norm is called a *normed* space.

Theorem

Let E be a vector space over a scalar field $\mathbb{F}$ and let $\|\cdot\|$ be a norm on E. $\forall, \vec{x} \in E$:

$$\|\vec{x}\| = \|-\vec{x}\|$$

Proof

Assume $\vec{x} \in E$.

$$\|-\vec{x}\| = \|(-1)\vec{x}\| = |-1|\, \|\vec{x}\| = 1 \cdot \|\vec{x}\| = \|\vec{x}\|$$

Theorem

Let E be a vector space over a scalar field $\mathbb{F}$ and let $\|\cdot\|$ be a norm on E. $\forall, \vec{x} \in E$:

$$\|\vec{x}\| \geq 0$$

Thus, the norm is positive-definite.

Proof

Assume $\vec{x} \in E$.

$$0 = \|\vec{0}\| = \|\vec{x} + (-\vec{x})\| \leq \|\vec{x}\| + \|-\vec{x}\| = \|\vec{x}\| + \|\vec{x}\| = 2\|\vec{x}\|$$

$\therefore \|\vec{x}\| \geq 0$

Theorem

Let E be a vector space over a scalar field $\mathbb{F}$ and let $\|\cdot\|$ be a norm on E. $\forall, \vec{x}, \vec{y} \in E$:

$$|\|\vec{x}\| - \|\vec{y}\|| \leq \|\vec{x} - \vec{y}\|$$

Proof

Assume $\vec{x}, \vec{y} \in E$.

$\|\vec{x}\| = \|(\vec{x} - \vec{y}) + \vec{y}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y}\|$
$\|\vec{x}\| - \|\vec{y}\| \leq \|\vec{x} - \vec{y}\|$

$\|\vec{y}\| = \|(\vec{y} - \vec{x}) + \vec{x}\| \leq \|\vec{y} - \vec{x}\| + \|\vec{x}\|$
$\|\vec{y}\| - \|\vec{x}\| \leq \|\vec{y} - \vec{x}\|$
$-(\|\vec{x}\| - \|\vec{y}\|) \leq \|\vec{x} - \vec{y}\|$

$\therefore |\|\vec{x}\| - \|\vec{y}\|| \leq \|\vec{x} - \vec{y}\|$

Examples

1). $E = \mathbb{R}^n$ or $\mathbb{C}^n$:

$$\|\vec{z}\| = \sqrt{\sum_{k=1}^{n} |z_n|^2}$$

is called the *Euclidean* norm.

2). $E = \mathbb{R}^n$ or $\mathbb{C}^n$:

$$\|\vec{z}\| = \max_{1 \leq k \leq n} |z_n|$$

is called the *sup* norm.

3). Let Ω be a closed, bounded subset of $\mathbb{R}^n$ and $E = C(\Omega)$:

$$\|f\|_p = \left(\int_\Omega |f(t)|^p \, dt \right)^{\frac{1}{p}}$$

is a norm for $1 \leq p < \infty$. The integral form of Minkowski is needed for this:

$$\left(\int_\Omega |f + g|^p \right)^{\frac{1}{p}} \leq \left(\int_\Omega |f|^p \right)^{\frac{1}{p}} + \left(\int_\Omega |g|^p \right)^{\frac{1}{p}}$$

4). Let Ω be a closed, bounded subset of $\mathbb{R}^n$ and $E = C(\Omega)$:

$$\|f\|_\infty = \max_{t \in \Omega} |f(t)|$$

is a sup norm for E.

5). Let $E = \ell^p$:

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

is a norm for $1 \leq p < \infty$. In fact, Minkowski serves as the triangle inequality for this norm.

6). Let $E = \ell^p$:

$$\|x\|_\infty = \sup |x_n|$$

is a sup norm for ℓ^p.

Definition: Metric

Let E be a vector space and define function $d : E \times E \to \mathbb{R}$ such that d satisfies the following axioms $\forall \vec{x}, \vec{y}, \vec{z} \in E$:

1). $d(\vec{x}, \vec{y}) = 0 \iff \vec{x} = \vec{y}$

2). $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$

3). $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$

The function d is called a *metric* for E.

A vector space equipped with a metric is called a *metric* space.

Theorem

Let E be a normed space. E is a metric space with metric $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$.

Proof

Assume $\vec{x}, \vec{y}, \vec{z} \in E$

$d(\vec{x}, \vec{y}) = 0 \iff \|\vec{x} - \vec{y}\| = 0 \iff \vec{x} - \vec{y} = \vec{0} \iff \vec{x} = \vec{y}$.

$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| = \|\vec{y} - \vec{x}\| = d(\vec{y}, \vec{x})$

$d(\vec{x}, \vec{z}) = \|\vec{x} - \vec{z}\| = \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| = d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$