Characteristic Polynomial

Definition: Characteristic Polynomial

Let $A \in M_n(\mathbb{C})$. The *characteristic polynomial* of A is given by:

$$p_A(t) = \det(tI_n - A)$$

Thus, the eigenvalues are the zeros of the characteristic polynomial.

Properties

1). $p_A(t)$ is monic

2). For $\mathbb{C}$, $1 \leq |\sigma(A)| \leq n$

Definition: Algebraic Multiplicity

The *algebraic multiplicity* of an eigenvalue λ for a matrix A, denoted $a_A(\lambda)$ is the multiplicity of λ as a zero of the characteristic polynomial for A.

Thus:

$$\sum_{\lambda \in \sigma(A)} a_A(\lambda) = n$$

Definition: Spectrum

The *spectrum* of a matrix A, denoted $\mathrm{Sp}(A)$, is the collection of eigenvalues for A with each eigenvalue repeated according to its algebraic multiplicity.

Thus, finding $\mathrm{Sp}(A)$ requires finding all of the zeros for the characteristic polynomial, which is usually very hard and requires numerical methods.

Recall that to find the characteristic equation:

$$a_{n-k} = (-1)^k S_k(\lambda_1, \ldots, \lambda_n)$$

where S_k is the symmetric function given by:

$$S_k(\lambda_1, \ldots, \lambda_n) = \sum_{\mathcal{P}_k[n]} \prod_{i=1}^{k} \lambda_i$$

Theorem

Let $A \in UT(n)$:

$$\mathrm{Sp}(A) = \{A_{kk} \mid 1 \leq k \leq n\}$$

In other words, the diagonal entries.

Proof

Note that $tI_n - A$ is also in $UT(n)$ where $(tI_n - A)_{kk} = t - A_{kk}$, so:
$p_A(t) = \det(tI_n - A) = \prod_{k=1}^{n}(t - A_{kk})$
$\therefore \mathrm{Sp}(A) = \{A_{kk} \mid 1 \le k \le n\}$.

Definition

Let $A \in M_n(\mathbb{C})$. A *principle minor* of A is a $k \times k$ matrix ($1 \le k \le n$) where the same row and column numbers are selected.

Thus, there are $\binom{n}{k}$ principle minors for each k.