Orthogonal Complement

Definition: Orthogonal Complement

Let E be an inner product space and let S be a non-empty subset of E. To say that an $\vec{x} \in E$ is *orthogonal* to S, denoted $\vec{x} \perp S$, means $\forall\, \vec{y} \in S, \vec{x} \perp \vec{y}$.

The set of all elements of E that are orthogonal to S, denoted $S^{\perp}$, is called the *orthogonal complement* of S:

$$S^{\perp} = \{\vec{x} \in E \mid \vec{x} \perp S\}$$

To say that two non-empty subsets A and B of E are orthogonal, denoted $A \perp B$, means $\forall\, \vec{x} \in A, \forall\, \vec{y} \in B, \vec{x} \perp \vec{y}$.

Examples

Let E be an inner product space:

1). $E^{\perp} = \{\vec{0}\}$

2). $\{\vec{0}\}^{\perp} = E$

3). Let $(\vec{x}_n)$ be a complete orthonormal sequence is E:

$$\{\vec{x}_n \mid n \in \mathbb{N}\}^{\perp} = \{\vec{0}\}$$

4). $E = \mathcal{C}[0,1]$ with $\langle f, g\rangle = \displaystyle\int_0^1 f\overline{g}$.

Let $S = \mathcal{P}[0,1]$, all polynomials.
Assume $f \in S^{\perp}$.
But there exists (p_n) in $\mathcal{P}[0,1]$ such that $p_n \rightrightarrows f$, and thus $p_n \overset{L_2}{\to} f$.
Now, by continuity of the inner product, $\forall\, n \in \mathbb{N}$:

$$\|f\|^2 = \langle f, f\rangle = \left\langle f, \lim_{n\to\infty} p_n \right\rangle = \lim_{n\to\infty} \langle f, p_n\rangle = 0$$

$\therefore S^{\perp} = \{\vec{0}\}$

Theorem

Let E be an inner product space and $S \subseteq E$:

$S^{\perp}$ is a closed subspace of E.

Proof

Assume $\vec{x}, \vec{y} \in S^{\perp}$.
Assume $\alpha, \beta \in \mathbb{C}$.

Assume $\vec{z} \in S$.
$\langle \alpha\vec{x} + \beta\vec{y}, \vec{z} \rangle = \alpha \langle \vec{x}, \vec{z} \rangle + \beta \langle \vec{y}, \vec{z} \rangle = 0$
Thus $\alpha\vec{x} + \beta\vec{y} \perp \vec{z}$ and so $\alpha\vec{x} + \beta\vec{y} \in S^{\perp}$.

Therefore, $S^{\perp}$ is a subspace of E.

Assume $(\vec{x}_n)$ is a sequence in $S^{\perp}$ such that $\vec{x}_n \to \vec{x} \in E$.
Assume $\vec{y} \in S$.
So $\forall\, n \in \mathbb{N}, \vec{y} \perp \vec{x}_n$.
$\langle \vec{x}, \vec{y} \rangle = \left\langle \lim_{n\to\infty} \vec{x}_n, \vec{y} \right\rangle = \lim_{n\to\infty} \langle \vec{x}_n, \vec{y} \rangle = 0$
Thus $\vec{x} \perp \vec{y}$ and so $\vec{x} \in S^{\perp}$.

Therefore $S^{\perp}$ is closed.

Theorem

Let H be a Hilbert space and let S be a convex subset of H. Let $\vec{x} \in H \setminus S$ and let $\vec{y} \in S$ such that $d(\vec{x}, S) = \|\vec{x} - \vec{y}\|$:

$$\vec{x} - \vec{y} \perp S$$

Thus, $\forall\, \vec{z} \in S, \vec{x} - \vec{y} \perp \vec{z}$.

Proof

Assume $\vec{z} \in S$ such that $\vec{z} \neq \vec{0}$.
Consider the perturbation $\vec{y} + \epsilon\vec{z}$ for some $\epsilon \in \mathbb{R}$.
Let $d = d(\vec{x}, S)$.

$$\begin{array}{rcl}
d^2 & \leq & \|\vec{x} - (\vec{y} + \epsilon\vec{z}\|^2 \\
& = & \|(\vec{x} - \vec{y}) + \epsilon\vec{z}\|^2 \\
& = & \langle (\vec{x} - \vec{y}) + \epsilon\vec{z}, (\vec{x} - \vec{y}) + \epsilon\vec{z} \rangle \\
& = & \|\vec{x} - \vec{y}\|^2 - \langle \vec{x} - \vec{y}, \epsilon\vec{z} \rangle - \langle \epsilon\vec{z}, \vec{x} - \vec{y} \rangle + \epsilon^2 \|\vec{z}\|^2 \\
& = & \|\vec{x} - \vec{y}\|^2 - \epsilon[\langle \vec{x} - \vec{y}, \vec{z} \rangle - \overline{\langle \vec{x} - \vec{y}, \vec{z} \rangle}] + \epsilon^2 \|\vec{z}\|^2 \\
& = & \|\vec{x} - \vec{y}\|^2 - 2\epsilon \operatorname{Re}(\langle \vec{x} - \vec{y}, \vec{z} \rangle) + \epsilon^2 \|\vec{z}\|^2 \\
& = & d^2 - 2\epsilon \operatorname{Re}(\langle \vec{x} - \vec{y}, \vec{z} \rangle) + \epsilon^2 \|\vec{z}\|^2 \\
0 & \leq & \epsilon^2 \|\vec{z}\|^2 - 2\epsilon \operatorname{Re}(\langle \vec{x} - \vec{y}, \vec{z} \rangle)
\end{array}$$

Now, let $a = \|\vec{z}\|^2$ and $b = 2\operatorname{Re}(\langle \vec{x} - \vec{y}, \vec{z} \rangle)$:

$$\begin{array}{rcl}
a\epsilon^2 - b\epsilon & \geq & 0 \\
a\epsilon \left(\epsilon - \dfrac{b}{a} \right) & \geq & 0
\end{array}$$

But for this to always be true it must be the case that $b = 0$.

And so $\operatorname{Re}(\langle \vec{x} - \vec{y}, \vec{z} \rangle) = 0$.

By similar argument for $\vec{y} + i\epsilon\vec{z}$, $\mathrm{Im}(\langle \vec{x} - \vec{y}, \vec{z} \rangle) = 0$.

Thus $\langle \vec{x} - \vec{y}, \vec{z} \rangle = 0$ and so $\vec{x} - \vec{y} \perp \vec{z}$.

$\therefore \vec{x} - \vec{y} \perp S$.

Such a $\vec{y}$ is called the *orthogonal projection* of $\vec{x}$ onto S:

$$y = \operatorname{proj}_S \vec{x}$$

Resulting in a mapping: $\operatorname{proj}_S : H \to S$.

Corollary

Let H be Hilbert space and let S be a closed subspace of H. Every $\vec{x} \in H$ can be written uniquely as $\vec{x} = \vec{y} + \vec{z}$ where $\vec{y} \in S$ and $\vec{z} \in S^\perp$.

Thus, $H = S \oplus S^\perp$.

Proof

Assume $\vec{x} \in H$.
Since S is a closed subspace, S is closed and convex.
Thus a nearest point $\vec{y} \in S$ to $\vec{x}$ exists.
But $\vec{z} = \vec{x} - \vec{y} \perp S$ and so $\vec{z} \in S^\perp$.

Therefore $\vec{x} = \vec{y} + \vec{z}$, thus proving existence.

Now, assume $\vec{x} = \vec{y} + \vec{z} = \vec{y}\,' + \vec{z}\,'$ where $\vec{y}, \vec{y}\,' \in S$ and $\vec{z}, \vec{z}\,' \in S^\perp$.
Let $\vec{w} = \vec{y} - \vec{y}\,' = \vec{z}\,' - \vec{z}$.
But $\vec{y} - \vec{y}\,' \in S$ and $\vec{z}\,' - \vec{z} \in S^\perp$.
And so $\vec{w} \in S$ and $\vec{w} \in S^\perp$.

$\therefore \vec{w} = \vec{0}$ and thus $\vec{y} = \vec{y}\,'$ and $\vec{z} = \vec{z}\,'$, thus proving uniqueness.

Theorem

Let H be a Hilbert space and let S be a closed subspaceof H:

$$S^{\perp\perp} = S$$

Proof

$\subseteq$ Assume $\vec{x} \in S^{\perp\perp}$

$\exists\, \vec{y} \in S$ and $\vec{z} \in S^\perp$ such that $\vec{x} = \vec{y} + \vec{z}$.
So $\vec{y} \in S^{\perp\perp}$.
$\vec{x} - \vec{y} = \vec{z}$.
So $\vec{z} \in S^{\perp\perp}$.
So $\vec{z} \in S^\perp$ and $\vec{z} \in S^{\perp\perp}$, and so $\vec{z} = \vec{0}$.

$\therefore \vec{x} = \vec{y} \in S$

$\therefore S^{\perp\perp} = S^{\perp}$

$\supseteq$ Assume $\vec{x} \in S$.

$\vec{x} \perp S^{\perp}$

$\therefore \vec{x} \in S^{\perp\perp}$

In general, in a Hilbert space H, $S^{\perp\perp}$ is the closure of S.