Polynomials

Definition

A *polynomial* in a complex variable z of degree $n \in \mathbb{N} \bigcup \{0\}$ is given by:

$$p(z) = \sum_{k=0}^{n} a_k z^k, a_k \in \mathbb{C}, a_n \neq 0$$

The special case $p(z) = 0$ is called the *zero polynomial*.

Note that for $|z| < R$, $p(z)$ is bounded.

Example

Let $p(z) = 3 + z + z^2$ and $|z| < 2$.

$$\begin{aligned} |p(z)| &= \left|3 + z + z^2\right| \\ &\leq |3| + |z| + \left|z^2\right| \\ &= 3 + |z| + |z|^2 \\ &< 3 + 2 + 2^2 \\ &= 9 \end{aligned}$$

Theorem

Let $p(z)$ be a complex polynomial. There exists $R > 0$ such that $\forall\, |z| < R$:

$$|p(z)| > \frac{|a_n|\, R^n}{2}$$

and thus:

$$\left|\frac{1}{p(z)}\right| < \frac{2}{|a_n|\, R^n}$$

Thus, $\frac{1}{p(z)}$ is bounded for all z outside the circle $|z| = R$ for R sufficiently big.

Proof

Let $w = \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}$.
$|w| = \left|\sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}\right| \leq \sum_{k=0}^{n-1} \left|\frac{a_k}{z^{n-k}}\right| = \sum_{k=0}^{n-1} \frac{|a_k|}{\left|z^{n-k}\right|} = \sum_{k=0}^{n-1} \frac{|a_k|}{|z|^{n-k}}$
Let $|a_m| = max\{|a_k| \mid 0 \leq k \leq n\}$
$\exists c \in \mathbb{R}, |a_n| = c\,|a_m|$
Let $R = max\left\{1, \frac{2n}{c}\right\}$

Assume $|z| > R \geq 1$

$$\begin{aligned}
|w| &\leq \sum_{k=0}^{n-1} \frac{|a_k|}{|z|^{n-k}} \\
&\leq \sum_{k=0}^{n-1} \frac{|a_m|}{|z|} \\
&\leq \sum_{k=0}^{n-1} \frac{|a_n|}{c\,|z|} \\
&< \sum_{k=0}^{n-1} \frac{|a_n|}{c\frac{2n}{c}} \\
&< \sum_{k=0}^{n-1} \frac{|a_n|}{2n} \\
&= n\frac{|a_n|}{2n} \\
&= \frac{|a_n|}{2}
\end{aligned}$$

$|a_n + w| \geq ||a_n| - |w|| = \left||a_n| - \frac{|a_n|}{2}\right| = \left|\frac{|a_n|}{2}\right| = \frac{|a_n|}{2}$
$p(z) = a_n z^n + w z^n = (a_n + w)z^n$
$|p(z)| = |(a_n + w)z^n| = |a_n + w|\,|z^n| \geq \frac{|a_n|}{2}\,|z|^n > \frac{|a_n|R^n}{2}$
$\therefore \left|\frac{1}{p(z)}\right| < \frac{2}{|a_n|R^n}$