Countable Spaces

Definition: 2^{nd} Countable

Let X be a topological space. To say that X is 2^{nd} *countable* means that X has a countable basis.

Theorem

Let X be a topological space. If X is 2^{nd} countable then X is separable.

Proof. Assume that X is 2^{nd} countable and let $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ be a countable basis for X. From each B_i, select a value x_i and construct the set $A = \{x_1, x_2, \ldots\}$. Thus $x_i \mapsto B_i$ is one-to-one and so A is countable. Now assume that $U \in \mathscr{T}$. Then there exists at least some $B_i \subset U$ and hence $U \cap A \neq \emptyset$, and so A is countable and dense in X.

Therefore X is separable. ∎

There are two ways to determine that a set is not 2^{nd} countable:

1. Show that all possible bases are uncountable.
2. Show that all countable subsets are not bases.

Example

1. Show that $\mathbb{R}_{\text{std}}$ is 2^{nd} countable (and hence separable).

   Consider the countable set $\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{Q}\}$. Since $\mathbb{Q}$ is countable, $\mathbb{Q} \times \mathbb{Q}$ is countable and hence $\mathcal{B}$ is countable. Now assume that $U \in \mathscr{T}$ and assume $x \in U$. Since x is an interior point of U, there exists some $\epsilon > 0$ such that $x \in (x - \epsilon, x + \epsilon) \subset U$. So there must exist $\delta \in \mathbb{Q}$ such that $0 < \delta < \epsilon$, and so $x \in (x - \delta, x + \delta) \subset (x - \epsilon, x + \epsilon) \subset U$. But $(x - \delta, d + \delta) \in \mathcal{B}$, and so $\mathcal{B}$ is a countable basis for $\mathbb{R}_{\text{std}}$.

   Therefore $\mathbb{R}_{\text{std}}$ is 2^{nd} countable.

2. Show that $\mathbb{R}_{LL}$ is separable but not 2^{nd} countable.

   It was already shown that $\mathbb{R}_{LL}$ is separable. So assume that $\mathcal{B}$ is a basis for $\mathbb{R}_{LL}$ and consider $U_a = [a, \infty) = \bigcup_{b>a} [a, b) \in \mathscr{T}$. Then there exists some $B_a \in \mathcal{B}$ such that $a \in B_a$. Now, assume $x, y \in \mathbb{R}$ such that $x < y$. Since $U_y \subsetneq U_x$, there exists $B_x \subset U_x$ and $B_y \subset U_y$ such that $B_x \neq B_y$. Thus, $x \mapsto B_x$ is injective and hence $\mathcal{B}$ is uncountable.

   Therefore $\mathbb{R}_{LL}$ is not 2^{nd} countable.

Theorem

Every uncountable set in a 2^{nd} countable space has a limit point.

Proof. Assume that X is a 2^{nd} countable space and assume that $A \subset X$ such that A is uncountable. Now, ABC that A has no limit points. This means that for all $a \in A$ it is the case that there exists $U \in \mathcal{U}_a$ such that $U_a \cap A = \{a\}$ and hence every $a \in A$ is an isolated point. So assume that $x, y \in A$ such that $x \neq y$. There exists $U \in \mathcal{U}_x$ and $V \in \mathcal{U}_y$ such that $U \neq V$. So for any basis $\mathcal{B}$ of X, there exists $B_x, B_y \in B$ such that $B_x \neq B_y$ and $B_x \subset U$ and $B_y \subset V$. Thus, $a \mapsto B_a$ is injective and hence $\mathcal{B}$ is uncountable, contradicting the assumption that X is 2^{nd} countable.

Therefore A contains a limit point. ■

Theorem

If X and Y are 2^{nd} countable spaces then $X \times Y$ is 2^{nd} countable.

Proof. Assume that $\mathcal{B}_X$ is a countable basis for X and B_y is a countable basis for Y.

Claim: $\mathcal{B}_X \times \mathcal{B}_Y$ is a countable basis for $X \times Y$.

$\mathcal{B}_X \times \mathcal{B}_Y$ is countable. So assume that $U \in \mathscr{T}_{X \times Y}$ and assume $(a, b) \in U$. This means that there exists $U_a \in \mathscr{T}_X$ and $V_b \in \mathscr{T}_Y$ such that $(a, b) \in U_a \times V_b \subset U$. Furthermore, there exists $B_a \in \mathcal{B}_X$ and $B_b \in \mathcal{B}_Y$ such that $(a, b) \in B_a \times B_b \subset U_a \times V_b \subset U$ and so $\mathcal{B}_X \times \mathcal{B}_Y$ is a countable basis for $X \times Y$.

Therefore $X \times Y$ is 2^{nd} countable. ■

Definition: Neighborhood Basis

Let X be a topological space and let $p \in X$. To say that a collection of sets $\{U_\alpha \in \mathcal{U}_p : \alpha \in \lambda\}$ is a *neighborhood basis* for p means that for all $U \in \mathcal{U}_p$ there exists some $U_\alpha \subset U$.

Definition: 1^{st} Countable

Let X be a topological space. To say that X is 1^{st} *countable* means that every $p \in X$ has a countable neighborhood basis.

Theorem

Let X be a topological space. If X is 2^{nd} countable then X is 1^{st} countable.

Definition: Souslin

Let X be a topological space. To say that X has the *Souslin property* means that X does not contain uncountable collection of disjoint open sets.

Theorem

$\mathbb{R}_{\text{std}}$ is Souslin.

Proof. ABC that $\mathbb{R}_{\text{std}}$ is not Souslin, meaning it does contain an uncountable collection of disjoint open sets. Let $\mathcal{U}$ be such a set. Since $\mathbb{Q}$ is countable and dense in $\mathbb{R}_{\text{std}}$, every $U \in \mathcal{U}$ contains some $r_U \in \mathbb{Q}$. So select one value from each $U \in \mathcal{U}$ to construct the set $\{r_U \in Q \mid U \in \mathcal{U}\}$. Thus $r_U \mapsto U$ is injective and hence $\mathcal{U}$ is countable, contradicting the assumption.

Therefore $\mathbb{R}_{\text{std}}$ is Souslin. ∎