Simple Graphs

Definition: Simple Graph

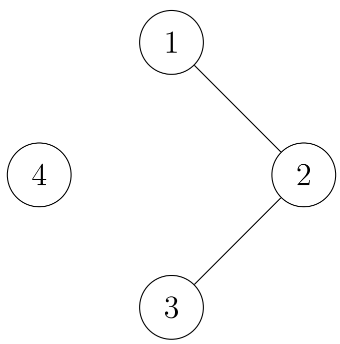
A *simple* graph $G = (V, E, \ldots)$ is a graph with a non-empty and finite set of vertices $V(G)$ and a possibly empty and finite set of edges $E(G)$ such that each edge is represented by a two-element subset of $V(G)$:

$$E(G) \subseteq \mathcal{P}_2(V(G))$$

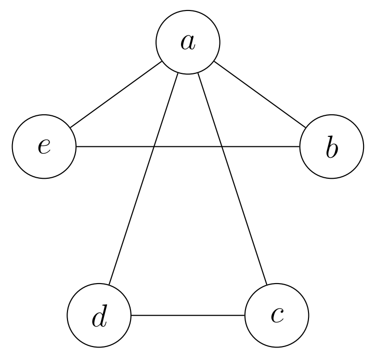
In particular, a simple graph is never the null graph, has no loops, and has no multiple edges.

Examples

$V = V(G) = \{1, 2, 3, 4\}$
$E = E(G) = \big\{\{1, 2\}, \{2, 3\}\big\}$



$V = \{a, b, c, d, e\}$
$E = \big\{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, e\}, \{c, d\}\big\}$



Notation

The edge $\{u, v\}$ is usually represented by just uv.

Definition: Isolated Vertex

Let G be a simple graph and let $u \in V(G)$. To say that v is an *isolated* vertex means that it is not an endpoint for any edge in $E(G)$:

$$\forall\, e \in E(G), v \notin e$$

In the above example, vertex 4 is an isolated vertex.

Definition: Adjacent Vertices

Let G be a simple graph and let $u, v \in V(G)$. To say that u and v are *adjacent* vertices (*neighbors*) means that they are the endpoints of some edge $e \in E(G)$:

$$\exists\, e \in E(G), e = uv$$

The edge e is said to *join* u and v. Furthermore, the edge e is said to be *incident* to u and v.

Definition: Adjacent Edges

Let G be a simple graph and let $e, f \in E(G)$. To say that e and f are *adjacent* edges means that they share an endpoint:

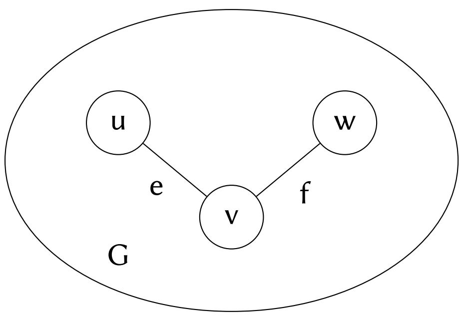
$$\exists\, v \in V(G), e \cap f = \{v\}$$

or

$$|e \cap f| = 1$$

Example

Let G be a simple graph; $u, v, w \in V(G)$; and $e, f \in E(G)$ such that $e = uv$ and $f = vw$:



- u and v are adjacent vertices (neighbors).
- u and v are joined by e.
- u and e are incident.
- e and f are adjacent edges.

Definition: Equality

To say that two simple graphs G and H are *equal*, denoted by $G = H$, means that $V(G) = V(H)$ and $E(G) = E(H)$.

Theorem

Let G be a simple graph of order n and size m:

$$m \leq \frac{n(n-1)}{2}$$

Proof. Since the graph is simple, each pair of distinct vertices has at most one edge joining them, and so the maximum number of possible edges is $\binom{n}{2}$. Hence:

$$m \leq \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

∎