Math-42 Worksheet #5

Nested Quantifiers

1. An important definition that you learn in calculus is the definition of the limit L of a function $f(x)$ at $x = a$:

   $$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, |x - a| < \delta \to |f(x) - L| < \epsilon$$

   Negate this proposition to say that L is not the limit of $f(x)$ at $x = a$.

2. There is a similar definition of continuity of a function:

   $$(f(a) \text{ exists}) \wedge (\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, |x - a| < \delta \to |f(x) - f(a)| < \epsilon)$$

   Negate this proposition to say that $f(x)$ is not continuous at $x = a$.

3. Rewrite each of the following real number axioms using quantified propositions. For example, the commutative property of addition states that two real numbers can be added in any order:

   $$\forall a, b \in \mathbb{R}, a + b = b + a$$

   (a) Commutative multiplication: two real numbers can be multiplied in any order.

   (b) Associative addition: when adding three numbers, either the first two can be added first or the last two can be added first.

   (c) Associative multiplication: when multiplying three numbers, either the first two can be multiplied first or the last two can be multiplied first.

   (d) Additive identity: there exists $0 \in \mathbb{R}$ such that for any real number a, $a + 0 = a$.

   (e) Multiplicative identity: there exists $1 \in \mathbb{R}$ such that for any real number a, $a \cdot 1 = a$.

   (f) Additive inverse: for every real number a there exists a real number $-a$ such that when added together you get the additive identity.

   (g) Multiplicative inverse: for every real number a, if $a \neq 0$ then there exists a real number a^{-1} such that when multiplied together you get the multiplicative identity.

   (h) Distributive: for all real numbers a, b, and c, $a(b + c) = ab + ac$.

4. What is the difference between these two quantified propositions?:

$$\exists\, 0 \in \mathbb{R}, \forall\, a \in \mathbb{R}, a + 0 = a$$
$$\forall\, a \in \mathbb{R}, \exists\, 0 \in \mathbb{R}, a + 0 = a$$

5. Look at your definitions for the additive and multiplicative identities. Convince yourself that these definitions say nothing about 0 and 1 being unique identities—in other words, is it possible that there exists some other $z \in \mathbb{R}$ such that $z \neq 0$ and for all $a \in \mathbb{R}, a + z = a$? Forget about what you think you know and only pay attention to what the definitions say (hint: consider the use of the existential quantifier). In fact, uniqueness is something that we must prove (and we will do so later).

6. Likewise, do your definitions for the additive and multiplicative inverses say anything about uniqueness?

7. You probably learned that there is a form of the existential quantifier that requires uniqueness: $\exists!$. We tend not to use this quantifier because it does not negate nicely.

   (a) What would be the negation of:

   $$\exists!\, a \in \mathbb{R}, a^2 = 0$$

   (b) Instead, uniqueness is expressed using a conjection (and) that indicates both *existence* and *uniqueness*. First, write the proposition that says that there exists some real number a such that $a^2 = 0$.

   (c) Now, state the uniqueness of that a by saying that for any real number b, if $b^2 = 0$ then it must be the case that a and b are the same (equal).

   (d) Put these two propositions together in a conjuction for a proposition equivalent to the one using the unique existential quantitier.

8. Let $K(x, y)$ be the predicate, "x knows y," where the domain of x and y is a particular set of people P. Rewrite each of the following statements as quantified propositions. Note that you will need to understand the previous problem on uniqueness in order to rewrite a couple of these statements:

   (a) Everybody knows Jeff.

   (b) Everybody knows somebody.

(c) There is somebody whom everybody knows.

(d) Nobody knows everybody.

(e) There is somebody whom Jeff does not know.

(f) There is somebody whom no one knows.

(g) There is exactly one person whom everybody knows.

(h) There are exactly two people whom Jeff knows.

(i) Everyone knows themselves.

(j) There is someone who knows no one besides themselves.