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Math 231b
Homework #5

4.12.2

If A is an operator on a complex Hilbert space H such that $A\vec{x} \perp \vec{x}$ for every $\vec{x} \in H$, show $A \equiv 0$.

By assumption: $\forall\, \vec{x} \in H, \langle A\vec{x}, \vec{x}\rangle = 0$.

Let $\varphi(\vec{x}, \vec{y}) = \langle A\vec{x}, \vec{y}\rangle$ be a bilinear functional on H with quadratic form:

$$\Phi(\vec{x}) = \varphi(\vec{x}, \vec{x}) = \langle A\vec{x}, \vec{x}\rangle = 0$$

Applying the polarization identity $\forall\, \vec{x}, \vec{y} \in H$:

$$4\varphi(\vec{x}, \vec{y}) = \Phi(\vec{x} + \vec{y}) - \Phi(\vec{x} - \vec{y}) + i\Phi(\vec{x} + i\vec{y}) - i\Phi(\vec{x} - i\vec{y})$$

But by closure: $\vec{x} + \vec{y}, \vec{x} - \vec{y}, \vec{x} + i\vec{y}, \vec{x} - i\vec{y} \in H$.
And so $\Phi(\vec{x} + \vec{y}) = \Phi(\vec{x} - \vec{y}) = \Phi(\vec{x} + i\vec{y}) = \Phi(\vec{x} - i\vec{y}) = 0$

Thus $\forall\, \vec{x}, \vec{y} \in H$ it must be the case that:

$$4\varphi(\vec{x}, \vec{y}) = 4\,\langle A\vec{x}, \vec{y}\rangle = 0$$

or $\langle A\vec{x}, \vec{y}\rangle = 0$.

But this only holds $\forall\, \vec{y} \in H$ if $A\vec{x} = 0$.
But this only holds $\forall\, \vec{x} \in H$ if $A \equiv 0$.

$\therefore A \equiv 0$.

4.12.3

Give an example of a bounded operator A such that $\left\|A^2\right\| \neq \|A\|^2$.

Let $E = \mathbb{R}^2$ and let $A(u) = A(x, y) = (y, 0)$. Note that this corresponds to the matrix:

$$[A]_e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

A is clearly bounded (triangle inequality) and linear (matrix).

$\|A\| = \sup\limits_{\|u\|=1} \|Au\| = \sup\limits_{\|u\|=1} |y| = 1$ and so $\|A\|^2 = 1$

But $A^2 = 0$ and so $\left\|A^2\right\| = 0$

$\therefore \left\|A^2\right\| \neq \|A\|^2$

Let $E = \mathbb{R}$ and let $A(x) = x + 1$.

$\|A\| = \sup_{|x|=1} |A(x)| = \sup_{|x|=1} |x+1| = 1$

$\|A\|^2 = 1^2 = 1$

4.12.6

Let $(\vec{e}_n)$ be a complete orthonormal sequence in a Hilbert space H and let (λ_n) be a sequence of scalars.

(a) Show that there exists a unique (linear) operator T on H such that $T\vec{e}_n = \lambda_n \vec{e}_n$.

Note that H is either finite dimensional or separable infinite dimensional, and so all (linear) operators on H can be represented by (infinite) matrix multiplication.

Assume $S\vec{e}_n = T\vec{e}_n = \lambda_n \vec{e}_n$.

$$S\vec{e}_n - T\vec{e}_n = (S-T)\vec{e}_n = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (s_{ij} - t_{ij}) e_{n,j} \vec{e}_i = \vec{0}$$

But $\|\vec{e}_n\| = 1$ and thus $\vec{e}_n \neq \vec{0}$.
And so $s_{ij} - t_{ij} = 0$, and thus $s_{ij} = t_{ij}$.

$\therefore S = T$.

(b) Show that T is bounded iff (λ_n) is bounded.

Since $\|\vec{e}_n\| = 1$:

$$\|T\vec{e}_n\| = \|\lambda_n \vec{e}_n\| = |\lambda_n| \|\vec{e}_n\| = |\lambda_n|$$

$\Longrightarrow$ Assume T is bounded.

$\exists\, M > 0$ such that $\|T\vec{e}_n\| = |\lambda_n| \leq M \|\vec{e}_n\| = M$

Therefore (λ_n) is bounded.

$\Longleftarrow$ Assume (λ_n) is bounded.

$\exists\, M > 0$ such that $|\lambda_n| \leq M$.
$\|T\vec{e}_n\| = |\lambda_n| \leq M = M \|\vec{e}_n\|$.

Therefore T is bounded.

(c) For a bounded sequence (λ_n), find the norm of T.

Since (λ_n) is bounded, $|\lambda_n|$ has a supremum.
Let $\lambda = \sup |\lambda_n|$

Claim: $\|T\| = \lambda$

Since $T \in \mathcal{B}(H)$:

$\|T\vec{e}_n\| \leq \|T\| \, \|\vec{e}_n\| = \|T\| \cdot 1 = \|T\|$
$\|T\| \geq \|T\vec{e}_n\| = \|\lambda_n \vec{e}_n\| = |\lambda_n| \, \|\vec{e}_n\| = |\lambda_n| \cdot 1 = |\lambda_n|$

$\therefore \|T\| \geq \lambda$

Furthermore:

$$\begin{aligned}
\|T\| &= \sup_{\|\vec{x}\|=1} \|T\vec{x}\| \\
&= \sup_{\|\vec{x}\|=1} \left\| T \sum_{k=1}^{\infty} x_k \vec{e}_k \right\| \\
&= \sup_{\|\vec{x}\|=1} \left\| \sum_{k=1}^{\infty} x_k T \vec{e}_k \right\| \\
&= \sup_{\|\vec{x}\|=1} \left\| \sum_{k=1}^{\infty} x_k \lambda_k \vec{e}_k \right\| \\
&\leq \sup_{\|\vec{x}\|=1} \left\| \sum_{k=1}^{\infty} |\lambda_k x_k| \, \vec{e}_k \right\| \\
&= \sup_{\|\vec{x}\|=1} \left\| \sum_{k=1}^{\infty} |\lambda_k| \, |x_k| \, \vec{e}_k \right\| \\
&\leq \sup_{\|\vec{x}\|=1} \left\| \sum_{k=1}^{\infty} \lambda \, |x_k| \, \vec{e}_k \right\| \\
&= \lambda \sup_{\|\vec{x}\|=1} \left\| \sum_{k=1}^{\infty} |x_k| \, \vec{e}_k \right\|
\end{aligned}$$

But note that:

$$\begin{aligned}
\left\| \sum_{k=1}^{\infty} x_k \vec{e}_k \right\|^2 &= \left\langle \sum_{k=1}^{\infty} x_k \vec{e}_k, \sum_{k=1}^{\infty} x_k \vec{e}_k \right\rangle \\
&= \sum_{k=1}^{\infty} |x_k|^2 \\
&= \left\langle \sum_{k=1}^{\infty} |x_k| \, \vec{e}_k, \sum_{k=1}^{\infty} |x_k| \, \vec{e}_k \right\rangle \\
&= \left\| \sum_{k=1}^{\infty} |x_k| \, \vec{e}_k \right\|^2
\end{aligned}$$

So taking the absolute value of the components does not change the norm.

Hence:

$$\|T\| \leq \lambda \sup_{\|\vec{x}\|=1} \left\| \sum_{k=1}^{\infty} |x_k| \, \vec{e}_k \right\| = \lambda \sup_{\|\vec{x}\|=1} \|\vec{x}\| = \lambda \cdot 1 = \lambda$$

$\therefore \|T\| = \lambda$

4.12.8

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (x + 3y, 2x + y)$. Show that $T^* \neq T$.

From matrix theory, we know that:

$$[T]_e = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

And since T^* is just the conjugate transpose, and in this case, just the transpose of T:

$$[T^*]_e = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

And thus $T \neq T^*$.

Using the definition of the transpose, we have:

$$\langle Tu, v \rangle = \langle u, T^* v \rangle$$

So let $u = (x_1, y_1)$ and $v = (x_2, y_2)$:

$$\begin{aligned} \langle Tu, v \rangle &= \langle (x_1 + 3y_1, 2x_1 + y_1), (x_2, y_2) \rangle \\ &= x_2(x_1 + 3y_1) + y_2(2x_1 + y_1) \\ &= x_1 x_2 + 3y_1 x_2 + 2x_1 y_2 + y_1 y_2 \\ &= x_1(x_2 + 2y_2) + y_1(3x_2 + y_2) \\ &= \langle (x_1, y_1), (x_2 + 2y_2, 3x_2 + y_2) \rangle \\ &= \langle u, T^* v \rangle \end{aligned}$$

And so $T^*(x, y) = (x + 2y, 3x + y)$ (as expected) and therefore $T \neq T^*$.