Ring Isomorphisms

Definition

Let R and R' be rings. To say that $\phi : R \to R'$ is a *ring isomorphism* means:

1). ϕ is a bijection

2). ϕ is a ring homomorphism

If so, then R and R' are said to be isomorphic, denoted $R \simeq R'$.

Lemma

Let $\phi : R \to R'$ be an isomorphism of rings. $\phi^{-1} : R' \to R$ exists and is also an isomorphism of rings.

Proof

ϕ is bijective, so ϕ^{-1} exists and is bijective

Assume $x, y \in R'$
$\exists\, a \in R, \phi(a) = x$
$\exists\, b \in R, \phi(b) = y$

$\phi^{-1}(x) = a$
$\phi^{-1}(y) = b$

$\phi^{-1}(x + y) = \phi^{-1}(\phi(a) + \phi(b)) = \phi^{-1}(\phi(a + b)) = (\phi^{-1}\phi)(a + b) = a + b = \phi^{-1}(x) + \phi^{-1}(y)$

$\phi^{-1}(xy) = \phi^{-1}(\phi(a)\phi(b)) = \phi^{-1}(\phi(ab)) = (\phi^{-1}\phi)(ab) = ab = \phi^{-1}(x)\phi^{-1}(y)$

Lemma

Let $\phi : R \to R'$ and $\mu : R' \to R''$ be isomorphisms of rings. $\mu\phi : R \to R''$ is also an isomorphism of rings.

Proof

Assume $(\mu\phi)(a) = (\mu\phi)(b)$
$\mu(\phi(a)) = \mu(\phi(b))$
μ is a bijection, so μ^{-1} exists
$\phi(a) = \phi(b)$
ϕ is a bijection, so ϕ^{-1} exists
$a = b$
$\therefore\ \mu\phi$ is one-to-one.

Assume $z \in R''$
$\exists\, y \in R', \mu(y) = z$
$\exists\, x \in R, \phi(x) = y$
$(\mu\phi)(x) = \mu(\phi(x)) = \mu(y) = z$

$\therefore \mu\phi$ is onto, and thus a bijection.

Assume $a, b \in R$

$(\mu\phi)(a+b) = \mu(\phi(a+b)) = \mu(\phi(a)+\phi(b)) = \mu(\phi(a)) + \mu(\phi(b)) = (\mu\phi)(a) + (\mu\phi)(b)$

$(\mu\phi)(ab) = \mu(\phi(ab)) = \mu(\phi(a)\phi(b)) = \mu(\phi(a))\mu(\phi(b)) = (\mu\phi)(a)(\mu\phi)(b)$

$\therefore \mu\phi$ is a homomorphism, and thus an isomorphism.

Theorem

Isomorphism is an equivalence relation.

Proof

Reflexive: Assume R is a ring

Let $\phi = \iota_R$
ι_R is a ring isomorphism on R
$\therefore R \simeq R$

Symmetric: Assume $R \simeq R'$

There exists ring isomorphism $\phi : R \to R'$
So there exists ring isomorphism $\phi^{-1} : R' \to R$ (lemma)
$\therefore R' \simeq R$

Transitive: Assume $R \simeq R'$ and $R' \simeq R''$

There exists ring isomorphism $\phi : R \to R'$
There exists ring isomorphism $\mu : R' \to R''$
So there exists ring isomorphism $\mu\phi : R \to R''$ (lemma)
$\therefore R \simeq R''$