Idempotence

Definition

Let R be a ring and $a \in R$. To say that a is *idempotent* in R means: $a^2 = a$.

Theorem

Let R be a commutative ring and $I = \{a \in R \mid a \text{ is idempotent in } R\}$.
I is closed under multiplication.

Proof

Assume $a, b \in I$
$a^2 = a$
$b^2 = b$
$(ab)^2 = a^2b^2 = ab$
$ab \in I$

$\therefore I$ is closed under multiplication.

Theorem

Let $\phi : R \to R'$ be a homomorphism of rings:

a idempotent in $R \implies \phi(a)$ idempotent in R'

Proof

Assume a is idempotent in R
$a^2 = a$
$\phi(a^2) = \phi(a)$
$\phi(a^2) = \phi(a)^2$
$\phi(a)^2 = \phi(a)$

$\therefore \phi(a)$ is idempotent in R'.