2.1: The Degree of a Vertex

1. Give an example of the following or explain why no such example exists:

   (a) a graph of order 7 whose vertices have degrees 1, 1, 1, 2, 2, 3, 3.

   Not possible due to an odd number of odd vertices.

   (b) a graph of order 7 whose vertices have degrees 1, 2, 2, 2, 3, 3, 7.

   Not possible because $\Delta(G) \leq 7 - 1 = 6$, so a vertex with degree 7 cannot exist.

   (c) a graph of order 4 whose vertices have degrees 1, 3, 3, 3.

   Not possible because 3 of the 4 vertices are universal, and thus are all adjacent to the remaining vertex, which must also have degree $3 \neq 1$.

2. Give an example of the following or explain why no such example exists:

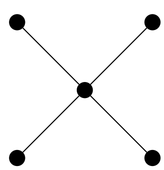
   (a) a graph that has no odd vertices.

   C_n

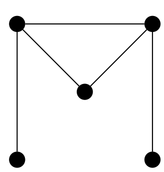
   (b) a non-complete graph, all of whose vertices have degree 3.

   $K_{3,3}$

   (c) a graph G of order 5 or more with the property that $\deg(u) \neq \deg(v)$ for every pair u, v of adjacent vertices of G.

   (d) A non-complete graph H of order 5 or more with the property that $\deg(u) \neq \deg(v)$ for every pair u, v of non-adjacent vertices in H.

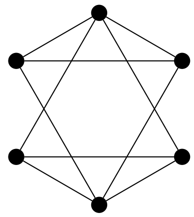
   

3. The degree of each vertex of a certain graph of order 12 and size 31 is either 4 or 6. How many vertices of degree 4 are there?

   Let $x =$ the number of vertices with degree 4:

   $$4x + 6(12 - x) = 2 \cdot 31$$
   $$4x + 72 - 6x = 62$$
   $$2x = 10$$
   $$x = 5$$

4. Give an example of a graph G of order 6 and size 10 such that $\delta(G) = 3$ and $\Delta(G) = 4$.



5. The degree of every vertex of a graph G of order 25 and size 62 is 3, 4, 5, or 6. There are two vertices of degree 4 and 11 vertices of degree 6. How many vertices of G have degree 5?

   Let $x =$ the number of vertices with degree 5:

$$
\begin{aligned}
&3(25 - 2 - x - 11) + 4 \cdot 2 + 5x + 11 \cdot 6 = 2 \cdot 62 \\
&3(12 - x) + 8 + 5x + 66 = 124 \\
&36 - 3x + 5x + 74 = 124 \\
&2x = 14 \\
&x = 7
\end{aligned}
$$

6. Prove that if a graph of order $3n$ ($n \geq 1$) has n vertices each of the degrees $n-1$,n, and $n+1$, then n is even.

   Proof.

$$
\begin{aligned}
&n(n-1) + n(n) + n(n+1) = 2m \\
&n^2 - n + n^2 + n^2 + n = 2m \\
&3n^2 = 2m
\end{aligned}
$$

   Thus, $3n^2$ must be even, and so n^2 must be even.

   $\therefore$ n must be even. ∎