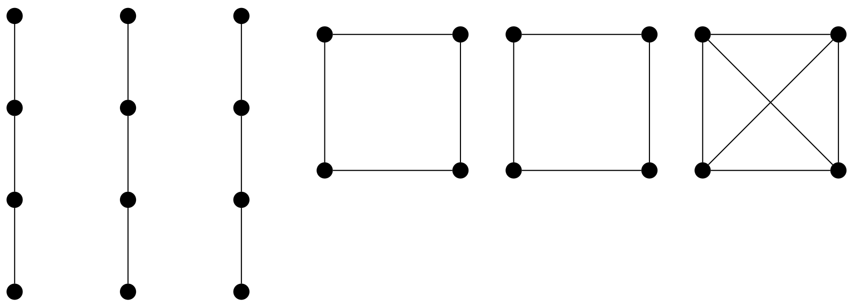
1.3: Common Classes of Graphs

21. Draw the graph $3P_4 \cup 2C_4 \cup K_4$.



22. Let G be a disconnected graph. By Theorem 1.11, $\overline{G}$ is connected. Prove that if u and v are any two vertices of $\overline{G}$, then $d_{\overline{G}}(u, v) = 1$ or $d_{\overline{G}}(u, v) = 2$. Therefore, if G is a disconnected graph, then $\operatorname{diam}(\overline{G}) \leq 2$.

Proof. Assume $u, v \in V(G)$.

Case 1: $uv \notin E(G)$

$\therefore uv \in E(\overline{G})$ and thus $d_{\overline{G}}(u, v) = 1$.

Case 2: $uv \in E(G)$

This means that u and v are in the same component in G. Furthermore, $uv \notin E(\overline{G})$. However, since G is disconnected, there exists a distinct vertex w in a different component in G, and so $uw, vw \in E(\overline{G})$. Consider the path (u, w, v). This is a $u - v$ path in $\overline{G}$ of length 2.

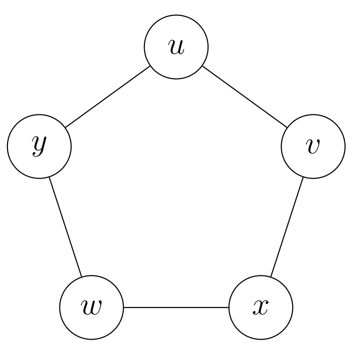
$\therefore d_{\overline{G}}(u, v) = 2$

$\therefore \operatorname{diam}(\overline{G}) \leq 2$ ■

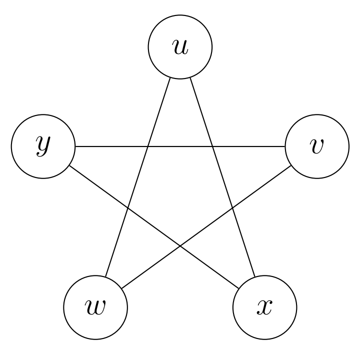
23. Consider the following question: For a given positive integer k, does there exist a connected graph G whose complement $\overline{G}$ is also connected and contains four distinct vertices u, v, x, y for which $d_G(u, v) = k = d_{\overline{G}}(x, y)$?

(a) Show that the answer to this question is yes if $k = 1$ or $k = 2$.

For $k = 1$:

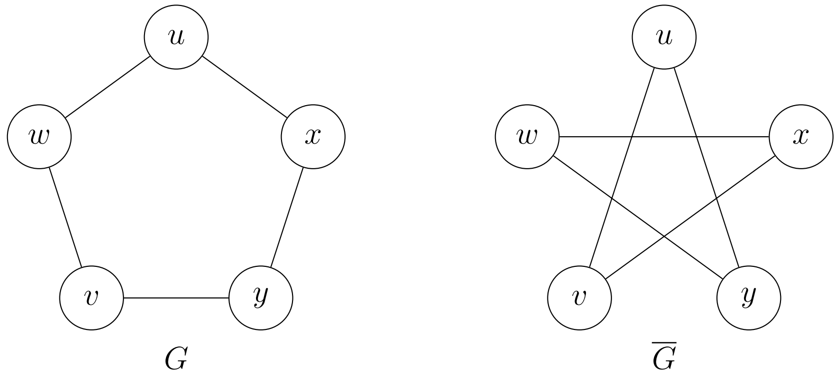


G



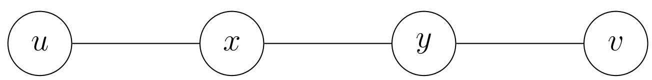
$\overline{G}$

For $k = 2$:

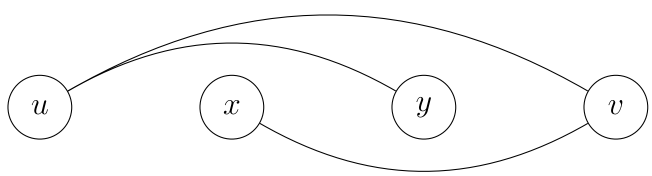


(b) Find the largest value of k for with the answer to this question is yes.

For $k \geq 3$, it must be the case that $uv \notin E(G)$. Furthermore, it must be the case that $xy \in E(G)$; otherwise, $xy \in E(\overline{G})$ and $d_{\overline{G}}(x,y) = 1$. If G contains any vertex w such that $wx, wy \notin E(G)$ then there is always a (x, w, y) path of length 2 in $\overline{G}$. Thus u and v must be adjacent to x and y, but neither can be adjacent to both, which would mean that $d_G(u,v) = 2$. So G must contain a 4-path as follows:



In $\overline{G}$, this path would become:

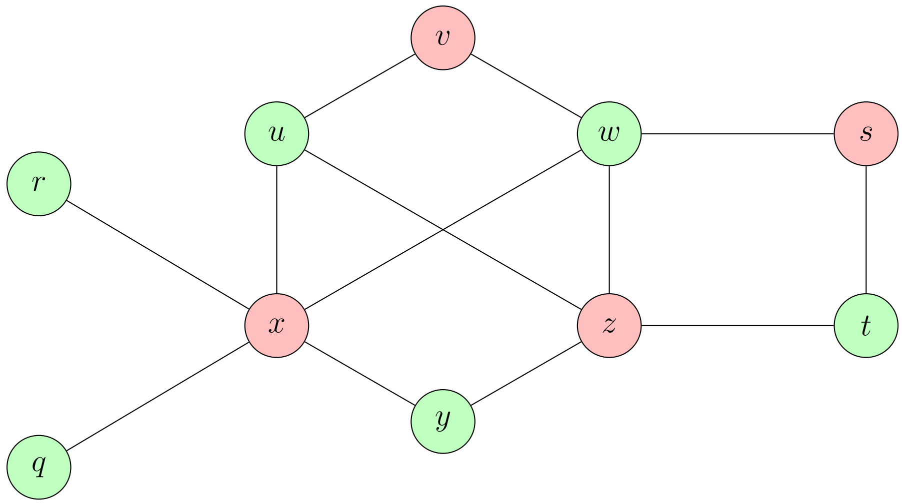


Thus, $d_{\overline{G}}(x,y) \leq 3$.

Therefore, the maximum such $k = 3$.

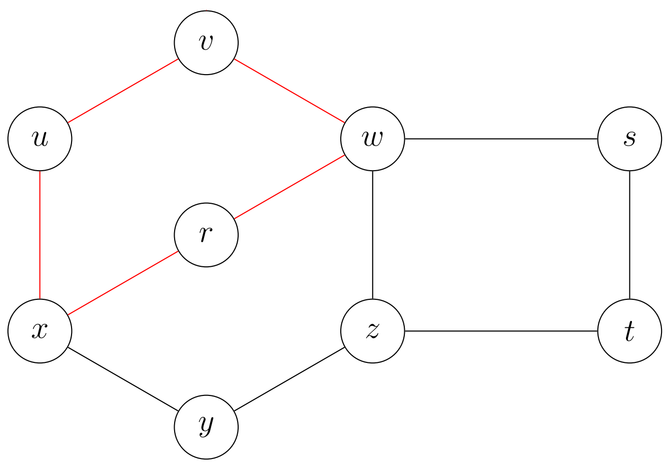
24. Determine whether the graphs G_1 and G_2 of Figure 1.34 are bipartite. If a graph is bipartite, then redraw it indicating the partite sets; if not, then give an explanation as to why the graph is not bipartite.

G_1 is bipartite:



G_1

G_2 is not bipartite since it contains an odd cycle: (u, v, w, r, x, u).



G_1

25. Let G be a graph of order 5 or more. Prove that at most one of G and $\overline{G}$ is bipartite.

    Proof. AWLOG: G is bipartite.

    Since $n = 5$, there must be at least 3 vertices in at least one of the partite sets. Since these vertices are not adjacent in G, they will all be adjacent in $\overline{G}$, thus forming a 3-cycle.

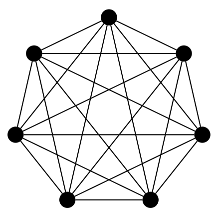
    $\therefore \overline{G}$ is not bipartite. ■

26. Suppose that the vertex set of a graph G is a (finite) set of integers. Two vertices x and y are adjacent if $x + y$ is odd. To which well-known class of graphs is G a member?

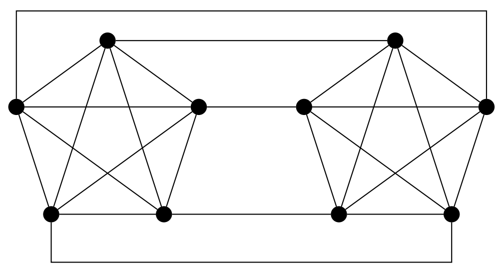
Such a graph would be a bipartite graph, partitioning the vertices into even and odd partite sets. This is because: E+E=E (not adjacent), O+O=E (not adjacent), and O+E=E+O=O (adjacent).

27. For the following pairs G, H of graphs, draw $G + H$ and $G \times H$.

(a) $G = K_5$ and $H = K_2$

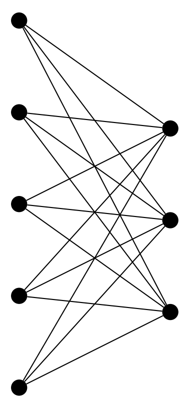


$K_5 + K_2 = K_7$

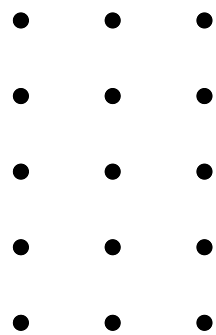


$K_5 \times K_2$

(b) $G = \overline{K_5}$ and $H = \overline{K_3}$

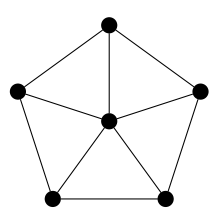


$\overline{K_5} + \overline{K_3} = K_{5,3}$

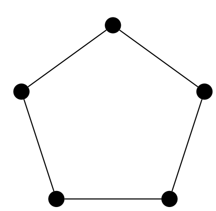


$\overline{K_5} \times \overline{K_3} = E_{15}$

(c) $G = C_5$ and $H = K_1$



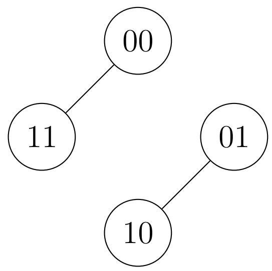
$C_5 + K_1$



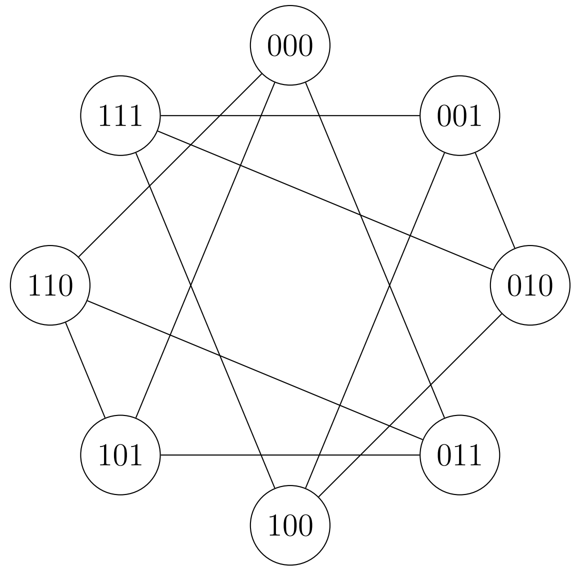
$C_5 \times K_1 = C_5$

28. We have seen that for $n \geq 1$, the n-cube Q_n is that graph whose vertex set is the set of n-bit strings, where two vertices of Q_n are adjacent if they differ in exactly one coordinate.

(a) For $n \geq 2$, define the graph R_n to be that graph whose vertex set is the set of n-bit strings, where two vertices of R_n are adjacent if they differ in exactly two coordinates. Draw R_2 and R_3.

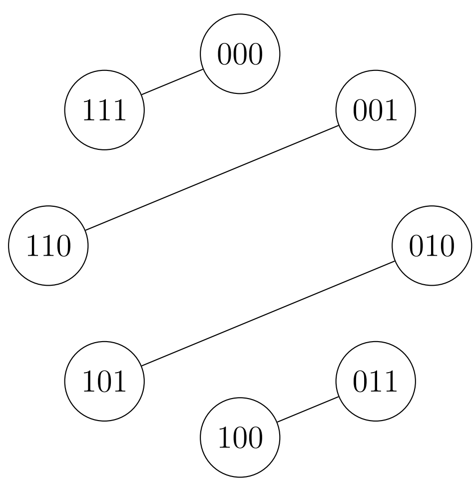


R_2

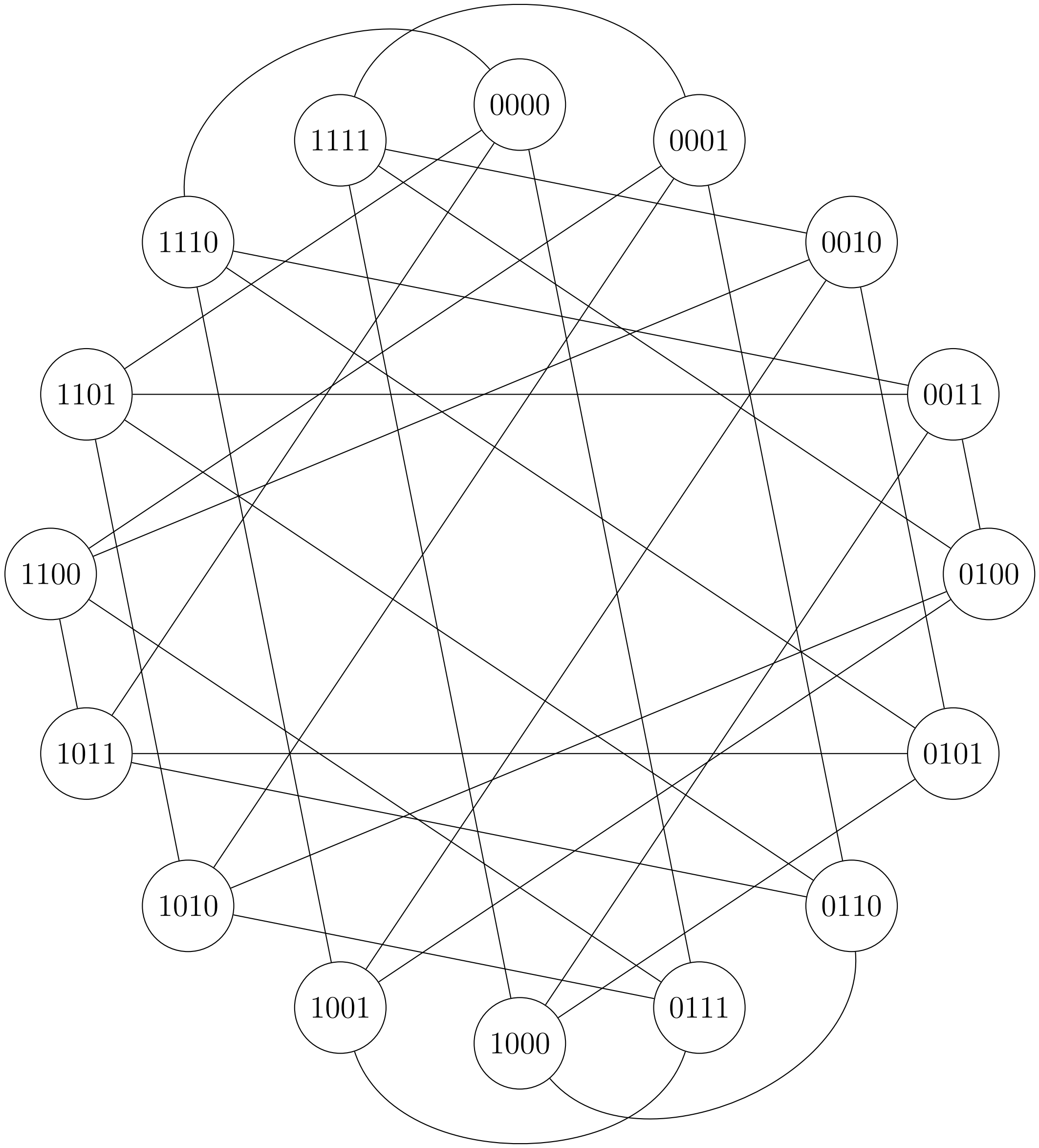


R_3

(b) For $n \geq 3$, define the graph S_n to be that graph whose vertex set is the set of n-bit strings, where two vertices of S_n are adjacent if they differ in exactly three coordinates. Draw S_3 and S_4.



S_3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S_4