Math-19 Section 1

Homework #5 Solutions

Problems

Consider the rational function:

$$y = \frac{2x^3 - 3x^2 - 3x + 2}{2x^3 + x^2 - 2x - 1}$$

1. Completely factor the numerator and the denominator and rewrite the function in simplified, factored form.

   Starting with the numerator, first apply the rational roots theorem:

   $$a_0 = 2 : \pm 1, \pm 2$$
   $$a_n = 2 : \pm 1, \pm 2$$
   $$\frac{a_0}{a_n} = \pm 1, \pm\frac{1}{2}, \pm 2$$

   And now the remainder theorem in order to find our first root:

   $$2(1)^3 - 3(1)^2 - 3(1) + 2 = 2 - 3 - 3 + 2 \neq 0$$
   $$2(-1)^3 - 3(-1)^2 - 3(-1) + 2 = -2 - 3 + 3 + 2 = 0$$

   And so $x + 1$ is a factor. So do the long division:

   $$\begin{array}{r} 2x^2 - 5x + 2 \\ x+1 \overline{\big) \; 2x^3 - 3x^2 - 3x + 2} \\ \underline{-2x^3 - 2x^2 \qquad\qquad} \\ -5x^2 - 3x \quad \\ \underline{5x^2 + 5x \quad} \\ 2x + 2 \\ \underline{-2x - 2} \\ 0 \end{array}$$

   And so:

   $$2x^3 - 3x^2 - 3x + 2 = (x+1)(2x^2 - 5x + 2) = (x+1)(2x-1)(x-2)$$

   Repeating the process for the denominator:

   $$a_0 = 1 : \pm 1$$
   $$a_n = 2 : \pm 1, \pm 2$$
   $$\frac{a_0}{a_n} = \pm 1, \pm\frac{1}{2}$$

   $$2(1)^3 + (1)^2 - 2(1) - 1 = 2 + 1 - 2 - 1 = 0$$

$$
\begin{array}{r|l}
 & \quad 2x^2 + 3x + 1 \\
\hline
x - 1 & 2x^3 + x^2 - 2x - 1 \\
 & -2x^3 + 2x^2 \\
\hline
 & \quad 3x^2 - 2x \\
 & \quad -3x^2 + 3x \\
\hline
 & \quad\quad x - 1 \\
 & \quad\quad -x + 1 \\
\hline
 & \quad\quad\quad 0
\end{array}
$$

$$2x^3 + x^2 - 2x - 1 = (x-1)(2x^2 + 3x + 1) = (x-1)(2x+1)(x+1)$$

And so the factored rational function is:

$$\frac{\cancel{(x+1)}(2x-1)(x-2)}{(x-1)(2x+1)\cancel{(x+1)}} = \frac{(x-2)(2x-1)}{(x-1)(2x+1)}$$

Noting that: $x \neq -1$

2. Do the long division and rewrite the function in quotient/remainder form.

$$
\begin{array}{r|l}
 & \quad\quad\quad\quad 1 \\
\hline
2x^3 + x^2 - 2x - 1 & 2x^3 - 3x^2 - 3x + 2 \\
 & -2x^3 - x^2 + 2x + 1 \\
\hline
 & \quad -4x^2 - x + 3
\end{array}
$$

Factoring out a (-1) from the remainder:

$$1 - \frac{4x^2 + x - 3}{2x^3 + x^2 - 2x - 1} = 1 - \frac{(4x-3)\cancel{(x+1)}}{(x-1)(2x+1)\cancel{(x+1)}} = 1 - \frac{4x-3}{(x-1)(2x+1)}$$

3. Where are the zeros?

$$x = \frac{1}{2}, 2$$

4. Where are the poles?

$$x = -\frac{1}{2}, 1$$

5. Where are the holes?

$$x = -1$$

6. Where are the horizontal asymptotes?

$$y = 1$$

7. Where are the vertical asymptotes?

   $$x = -\frac{1}{2}, 1$$

8. Where is the y-intercept?

   $$(0, -2)$$

9. What is the end-behavior as $x \to \infty$? (be sure to specify above or below)

   The last zero/pole is at $x = 2$, so use $x = 3$ as a test point and plug it in to the remainder portion in part (2). The result is $1 - c$ where $c > 0$ and so $f(x) \to 1^-$ (from below).

10. What is the end-behavior as $x \to -\infty$? (be sure to specify above or below)

    The first zero/pole is at $x = -\frac{1}{2}$, so use $x = -1$ as a test point and plug it in to the remainder portion in part (2). The result is $1 - c$ where $c < 0$ and so $f(x) \to 1^+$ (from above).

11. Sketch the graph. All intercepts, asymptotes, and holes must be clearly marked.