Interval estimation: Confidence intervals

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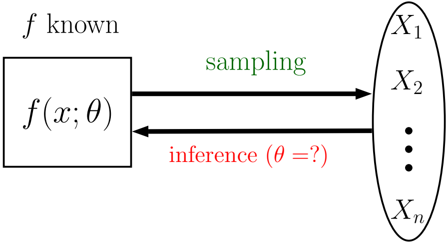
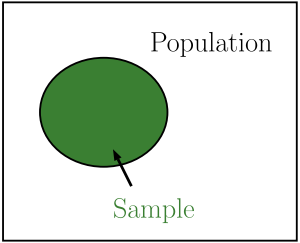
Introduction

Last time we started considering the new setting in which **we only know the distribution type**, not the values of its parameter (say θ).

The new goal is to use a random sample to infer about the population parameter. This is called **statistical inference**.



f known
$f(x;\theta)$
sampling
inference (θ =?)
X_1
X_2
X_n

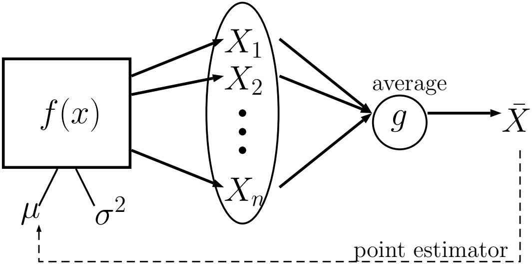


Population
Sample

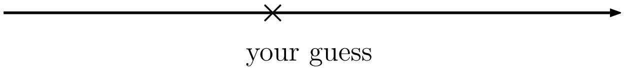
We also mentioned three kinds of inference tasks (all about an unknown population parameter θ):

- **Point estimation**: What is a single (best) guess of the value of θ?
- **Interval estimation**: In what interval does θ lie "with high probability"?
- **Hypothesis testing**: It is claimed that $\theta = \theta_0$. How do you test the hypothesis based on a random sample from the population?

Recall that mathematically, a **point estimator** $\hat{\theta}$ of θ is a (reasonable) statistic used to estimate θ.



For any specific realization of the random sample, the corresponding value of $\hat{\theta}$ is called a point estimate of θ.



Limitations with point estimation:

- Typically, the probability that a point estimator attains the true value of the parameter is zero.

  For example, for a random sample of size n from the normal population $N(\mu, \sigma^2)$, the point estimator $\bar{X}$ of the population mean μ is a continuous random variable $\bar{X} \sim N(\mu, \sigma^2/n)$. Thus, $P(\bar{X} = \mu) = 0$.

- Point estimators (even the ones that are unbiased and have least variance) provide no error information.

Question: Can we make the coverage probability much higher than 0?

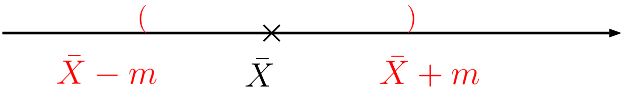
The answer is yes (by using an interval around $\bar{X}$). One extreme case is

$$P(\mu \in (-\infty, +\infty) = 1)$$

but it is useless.

A favorable solution is to find a "short" interval with "high" coverage probability:

$$P(\mu \in (\bar{X} - m, \bar{X} + m)) = 1 - \alpha \qquad \text{(for some small } \alpha\text{)}.$$



Rewrite as

$$P(\bar{X} - m < \mu < \bar{X} + m) = 1 - \alpha.$$

In the equation,

- μ: population mean (unknown parameter to be estimated)
- $\bar{X}$: sample mean (statistic)
- m: half width (fixed scalar, to be found)
- $1 - \alpha$: coverage probability (specified by user)
- $(\bar{X} - m, \bar{X} + m)$: interval estimator (random)

Task: Given α, find m.

Theorem 0.1. Assume $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ where μ is unknown, but σ^2 is known. For any given $0 < \alpha < 1$, we have

$$m = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}.$$

Proof. The equation on the preceding slide is equivalent to

$$P(-m < \bar{X} - \mu < m) = 1 - \alpha, \quad \text{or} \quad P\left(-\frac{m}{\sigma/\sqrt{n}} < Z < \frac{m}{\sigma/\sqrt{n}}\right) = 1 - \alpha.$$

This implies that

$$\frac{m}{\sigma/\sqrt{n}} = z_{\alpha/2}, \qquad \text{and accordingly,} \quad m = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}.$$

Interval estimator

We have just obtained that

$$P\left(\mu \in \left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)\right) = 1 - \alpha.$$

Definition 0.1. We call the interval estimator

$$\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \equiv \bar{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

a $1-\alpha$ **random interval** for μ. The quantity $m = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ is called the **margin of error** of the point estimator $\bar{X}$.

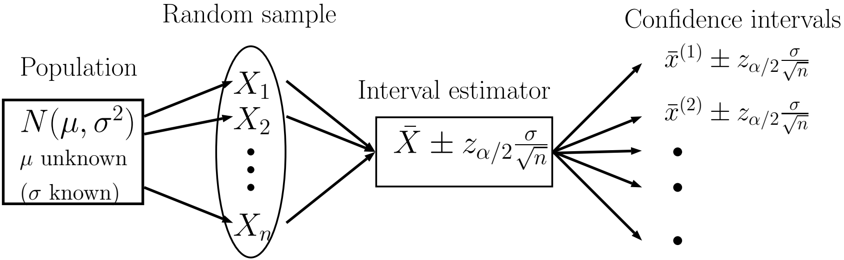
Remark. If $\alpha = 0.05$ (i.e., $1-\alpha = 0.95$), then $m = 1.96\frac{\sigma}{\sqrt{n}}$.

Confidence interval

Definition 0.2. For any specific sample $X_1 = x_1, \ldots, X_n = x_n$ (along with the observed value $\bar{x}$ of $\bar{X}$), the interval estimate

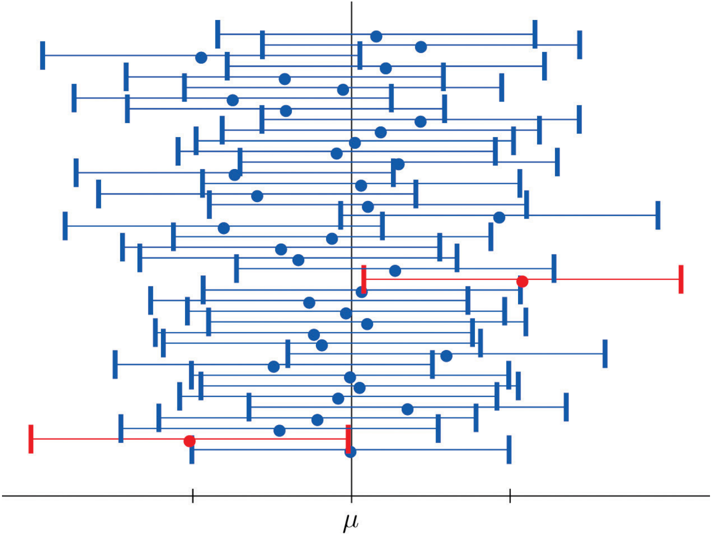
$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

is called a $1 - \alpha$ **confidence interval** for μ. In this setting, $1 - \alpha$ is called the **confidence level**.



Example 0.1. Recall the brown egg example where $n = 12, \bar{x} = 65.5$ and $\sigma = 2$, a 95% confidence interval is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 65.5 \pm 1.96 \cdot \frac{2}{\sqrt{12}} = 65.5 \pm 1.1 = (64.4, 66.6).$$


μ

Interpretations of confidence intervals

We can say that

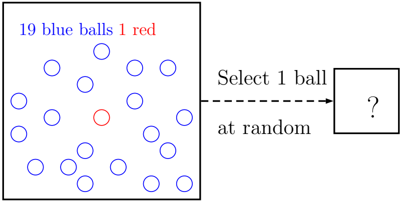
- $(64.4, 66.6)$ is a 95% confidence interval for μ, or
- We are 95% confident that the true μ is contained by this interval (i.e., between 64.4 and 66.6 grams).

We <u>cannot</u> say that

- The probability that μ is contained by this interval is 0.95,

as both μ and this interval are fixed and there is only one outcome: "contain" or "not contain". We just do not know which one is true (when μ is unknown).

Confidence is not probability!



- Probability describes the chance of selecting a blue ball before you actually do it (or if you do it many times)
- Confidence is, after you selected one ball, how certain you believe the ball you got is blue (without looking at it).

Relationship between m and n, α

(m: margin of error, n: sample size, $1-\alpha$: confidence level)

$$m = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

- The larger the sample size n, the smaller the margin of error m (the shorter the confidence interval);
- The larger the confidence level $1-\alpha$, the bigger the margin of error m (the wider the confidence interval).

Example 0.2 (Continuation of the brown egg example)**.** For another sample from the same normal distribution but with a larger size, say $n = 48$, a 95% confidence interval is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 65.5 \pm 1.96 \cdot \frac{2}{\sqrt{48}} = 65.5 \pm 0.55.$$

How large the sample size must be in order for the margin of error to be 0.2 (at level 95%)?

$$n = \left(z_{\alpha/2}\frac{\sigma}{m}\right)^2 = \left(1.96 \cdot \frac{2}{0.2}\right)^2 = 384.2.$$

The smallest sample size thus is 385.

Example 0.3 (Continuation of the brown egg example)**.** Using the same sample, a 99% confidence interval is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 65.5 \pm 2.576 \cdot \frac{2}{\sqrt{12}} = 65.5 \pm 1.5 = (64.0, 67.0),$$

and a 90% confidence interval is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 65.5 \pm 1.645 \cdot \frac{2}{\sqrt{12}} = 65.5 \pm 0.95$$

Remark. 99% CI $>$ (longer than) 95% CI $>$ 90% CI

i-Clicker Quiz 9 (extra credit)

Which of the following interpretations of a confidence interval $(64.0, 67.0)$ constructed at level 99% for the population mean μ is INCORRECT?

A. We don't know whether this interval contains the true value of μ.

B. The probability that μ is contained by this interval is 0.99.

C. We are 99% confident that μ is contained by this interval.

D. If we generate many more such intervals, roughly 99% of them will capture the true value of μ.

E. None of the above

What if we do not know σ?

Assuming a normal population $N(\mu, \sigma^2)$, with both μ, σ^2 unknown, we can still construct a $1-\alpha$ confidence intervals for

(1) μ

(2) σ^2

We present the details next.

Confidence interval for μ (when σ is unknown)

Recall when σ is needed for deriving a $1-\alpha$ confidence interval for μ:

We started with

$$P(\bar{X} - m < \mu < \bar{X} + m) = 1 - \alpha$$

and got (after rearranging terms)

$$P(-m < \bar{X} - \mu < m) = 1 - \alpha.$$

In order to solve for m, we then standardized $\bar{X} \sim N(\mu, \sigma^2/n)$ by **using the known** σ:

$$P\left(-\frac{m}{\sigma/\sqrt{n}} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{m}{\sigma/\sqrt{n}}\right) = 1 - \alpha.$$

When σ is unknown, we can use its estimator S in place of σ: Dividing all sides of the inequalities in the equation

$$P(-m < \bar{X} - \mu < m) = 1 - \alpha.$$

by $S/\sqrt{n}$ gives that

$$P\left(-\frac{m}{S/\sqrt{n}} < \frac{\bar{X} - \mu}{S/\sqrt{n}} < \frac{m}{S/\sqrt{n}}\right) = 1 - \alpha$$

To determine m, we need to know the distribution of the middle quantity, which has a t **distribution with** $n - 1$ **degrees of freedom**:

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t(n-1).$$

Student's t distributions

Definition 0.3. The t distribution with ν degrees of freedom is a continuous distribution whose pdf has the following form

$$f(x) = C\left(1 + x^2/\nu\right)^{-\frac{\nu+1}{2}}, \quad -\infty < x < \infty.$$

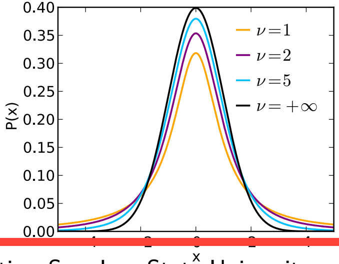
(1) The graph is also symmetric, unimodal and bell-shaped.

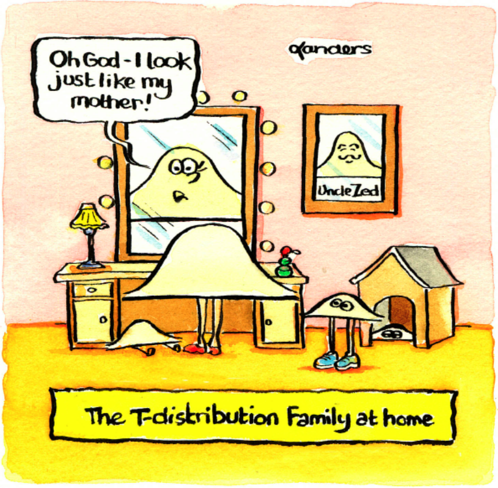
(2) $\mathrm{E}(X) = 0$

(3) $\mathrm{Var}(X) = \frac{\nu}{\nu-2}$ (when $\nu > 2$).

(4) $t(\nu) \to N(0,1)$ as $\nu \to +\infty$.

(5) t has thicker tails than N(0,1).




Oh God - I look just like my mother!
UncleZed
The T-distribution Family at home

Confidence interval for μ (when σ unknown)

Theorem 0.2. A $1-\alpha$ confidence interval for μ in the case of a normal population

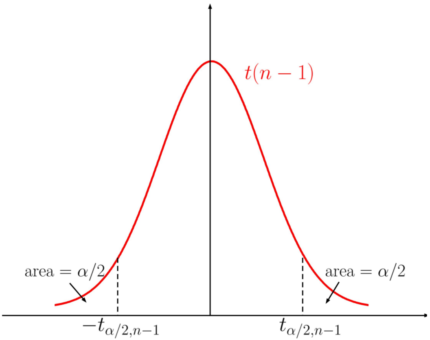
$$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2),$$

where σ is unknown, is

$$\bar{x} \pm t_{\alpha/2,n-1} \frac{s}{\sqrt{n}}.$$

Remark. Compare with:

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \text{ (when } \sigma \text{ known).}$$



Example 0.4. In the brown egg example, we selected a sample of 12 eggs (in a carton) and obtained that $\bar{x} = 65.5$ and $s^2 = 4.69$. Assuming normal population (with unknown variance), we obtain a 95% confidence interval

$$\bar{x} \pm t_{\alpha/2,n-1} \frac{s}{\sqrt{n}} = 65.5 \pm t_{0.025,11} \frac{\sqrt{4.69}}{\sqrt{12}} = 65.5 \pm 2.201 \sqrt{\frac{4.69}{12}} = 65.5 \pm 1.4.$$

Remark. Previously, when $\sigma = 2$ was used, we obtained the following 95% confidence interval

$$\bar{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = 65.5 \pm 1.96 \cdot \frac{2}{\sqrt{12}} = 65.5 \pm 1.1,$$

which is shorter. Why?

Confidence interval for σ^2

Assume the same setting of a random sample from a normal population:

$$X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2),$$

where neither μ nor σ^2 is known.

We already know that

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

is an (unbiased) estimator for σ^2.

We can further use S^2 to construct a $1 - \alpha$ confidence interval for σ^2.

The chi-square distribution

Definition 0.4. The chi-square distribution with k degrees of freedom, denoted $\chi^2(k)$, is a continuous distribution with pdf of the form

$$f(x) = C\left(\frac{x}{2}\right)^{k/2-1} e^{-\frac{x}{2}}, \quad x > 0.$$

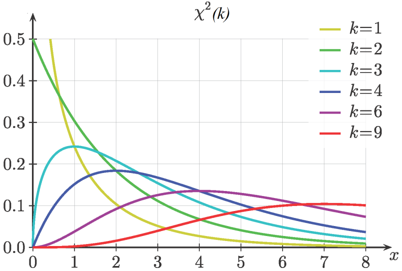
Properties:

(1) If $Z_1, \ldots, Z_k \overset{\text{iid}}{\sim} N(0,1)$, then

$$X = Z_1^2 + \cdots + Z_k^2 \sim \chi^2(k).$$

(2) $\mathrm{E}(X) = k$.

(3) $\mathrm{Var}(X) = 2k$.



Another way of defining the t distribution

We have defined Student's t distribution (with ν degrees of freedom) by specifying directly its pdf:

$$f(x) = C\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad -\infty < x < \infty.$$

Another way to define the t distribution is to use standard normal and chi-square: Let $Z \sim N(0,1)$ and $X \sim \chi^2(\nu)$ be independent random variables, then

$$\frac{Z}{\sqrt{X/\nu}} \sim t(\nu).$$

Confidence interval for σ^2

Theorem 0.3. Let

$$X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2),$$

Then

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1).$$

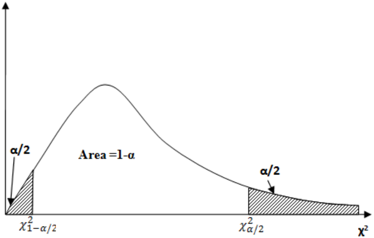
Remark. This result will be used for constructing a $1-\alpha$ confidence interval for σ^2.

Selecting constants a, b (choices not unique!) such that

$$P\left(a < \frac{(n-1)S^2}{\sigma^2} < b\right) = 1 - \alpha.$$

For example,

$$a = \chi^2_{1-\alpha/2}(n-1), \quad b = \chi^2_{\alpha/2}(n-1)$$



We then solve for σ^2:

$$\frac{(n-1)S^2}{b} < \sigma^2 < \frac{(n-1)S^2}{a}$$

i.e.,

$$\frac{(n-1)S^2}{\chi^2_{\alpha/2}(n-1)} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}(n-1)}.$$

We have obtained the following result.

Theorem 0.4. A $1-\alpha$ confidence interval for σ^2 in the case of a normal population $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ is

$$\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2}(n-1)}, \ \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}(n-1)} \right)$$

In the brown egg example, suppose we did not know the true value of σ^2. Let us find a 95% confidence interval for σ^2 based on the specific sample we have been using: $n = 12, s^2 = 4.69$.

We need to find the two χ^2 critical values:

- $\chi^2_{\alpha/2}(n-1) = \chi^2_{.025}(11) = 21.92$ (from Chi-square table)
- $\chi^2_{1-\alpha/2}(n-1) = \chi^2_{.975}(11) = 3.82$ (using software).

Therefore, a 95% confidence interval for σ^2 is

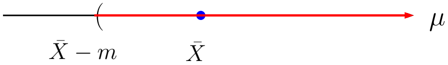
$$\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2}(n-1)}, \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}(n-1)}\right) = \left(\frac{11 \cdot 4.69}{21.92}, \frac{11 \cdot 4.69}{3.82}\right) = (2.35, 13.51).$$

One-sided confidence intervals

Sometimes there is a need for only one-sided confidence intervals:

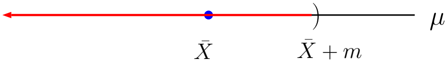
- Lower confidence bound

$$1 - \alpha = P(\mu > \bar{X} - m) = P(\mu \in (\bar{X} - m, +\infty))$$



- Upper confidence bound

$$1 - \alpha = P(\mu < \bar{X} + m) = P(\mu \in (-\infty, \bar{X} + m))$$



Assuming a random sample $X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$ with unknown μ but known σ^2. Then

- A $1-\alpha$ upper confidence bound for μ is

$$\mu < \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}}$$

- A $1-\alpha$ lower confidence bound for μ is

$$\mu > \bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}}$$

Remark. When σ is unknown, one can use the t distribution instead: in the above formulas, just change σ to s and z_α to $t_{\alpha,n-1}$.

In the brown egg example, a 95% upper confidence bound for μ is

$$\mu < \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}} = 65.5 + z_{.05}\frac{2}{\sqrt{12}} = 65.5 + 1.645\frac{2}{\sqrt{12}} = 66.45.$$

Similarly, a 95% lower confidence bound for μ is

$$\mu > \bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}} = 64.55.$$