Functions

Definition

Let $S \subseteq \mathbb{C}$. A *function* f is a rule that assigns to each $z \in S$ a value $w \in \mathbb{C}$, denoted:

$$w = f(z)$$

where w is called the value of the function f at z.

The set S is called the *domain* of f. When no domain is explicitly stated then the domain is assumed to be as large as possible.

$w = f(x+iy) = u(x,y) + iv(i,y)$
$u(x,y) = Re(w)$
$v(x,y) = Im(w)$

$w = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$
$u(r,\theta) = Re(w)$
$v(r,\theta) = Im(w)$

Example

$f(z) = z^2$

Domain: $\mathbb{C}$

$$\begin{aligned} f(x+iy) &= (x+iy)^2 \\ &= x^2 - y^2 + i2xy \\ u &= x^2 - y^2 \\ v &= 2xy \end{aligned}$$

$$\begin{aligned} f(re^{i\theta}) &= (re^{i\theta})^2 \\ &= r^2 e^{i2\theta} \\ &= r^2 \cos 2\theta + ir^2 \sin 2\theta \\ u &= r^2 \cos 2\theta \\ v &= r^2 \sin 2\theta \end{aligned}$$

When $Im(w) = 0$ then f is called a real-valued function of a complex variable.

Example

$f(x) = |z| = \sqrt{x^2 + y^2}$

$u = \sqrt{x^2 + y^2}$
$v = 0$

Example

Let $Z = f(z) = \frac{z-1}{z+1}$

Find: $\operatorname{Arg} Z$

$$\begin{aligned}
Re(Z) &= \frac{Z + \overline{Z}}{2} \\
&= \frac{1}{2}\left[\frac{z-1}{z+1} + \overline{\left(\frac{z-1}{z+1}\right)}\right] \\
&= \frac{1}{2}\left[\frac{z-1}{z+1} + \frac{\overline{z}-1}{\overline{z}+1}\right] \\
&= \frac{1}{2}\left[\frac{(z-1)(\overline{z}+1) + (z+1)(\overline{z}-1)}{(z+1)(\overline{z}+1)}\right] \\
&= \frac{1}{2}\left[\frac{z\overline{z} + z - \overline{z} - 1 + z\overline{z} - z + \overline{z} - 1}{z\overline{z} + z + \overline{z} + 1}\right] \\
&= \frac{1}{2}\left[\frac{2\left|z\right|^2 - 2}{\left|z\right|^2 + z + \overline{z} + 1}\right] \\
&= \frac{\left|z\right|^2 - 1}{\left|z\right|^2 + (z + \overline{z}) + 1} \\
&= \frac{\left|z\right|^2 - 1}{\left|z\right|^2 + 2Re(z) + 1} \\
&= \frac{x^2 + y^2 - 1}{x^2 + y^2 + 2x + 1}
\end{aligned}$$

$$\begin{aligned}
Im(Z) &= \frac{Z - \overline{Z}}{2i} \\
&= \frac{1}{2i}\left[\frac{z-1}{z+1} - \overline{\left(\frac{z-1}{z+1}\right)}\right] \\
&= \frac{1}{2i}\left[\frac{z-1}{z+1} - \frac{\overline{z}-1}{\overline{z}+1}\right] \\
&= \frac{1}{2i}\left[\frac{(z-1)(\overline{z}+1) - (z+1)(\overline{z}-1)}{(z+1)(\overline{z}+1)}\right] \\
&= \frac{1}{2i}\left[\frac{z\overline{z} + z - \overline{z} - 1 - z\overline{z} + z - \overline{z} + 1}{z\overline{z} + z + \overline{z} + 1}\right] \\
&= \frac{1}{2i}\left[\frac{2z - 2\overline{z}}{\left|z\right|^2 + z + \overline{z} + 1}\right]
\end{aligned}$$

$$
\begin{aligned}
&= 2\left[\frac{\frac{z-\overline{z}}{2i}}{|z|^2+(z+\overline{z})+1}\right] \\
&= \frac{2Im(z)}{|z|^2+2Re(z)+1} \\
&= \frac{2y}{x^2+y^2+2x+1}
\end{aligned}
$$

$$
Z = \frac{x^2+y^2-1}{x^2+y^2+2x+1} + i\frac{2y}{x^2+y^2+2x+1}
$$

$$
\operatorname{Arg} Z = \tan^{-1}\frac{Im(Z)}{Re(Z)} = \tan^{-1}\frac{2y}{x^2+y^2-1}
$$