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Math 221b
Homework #3

1). Let R be a ring and let I and J be ideals in R:

a). Prove: $I \cap J$ and $I + J$ are both ideals in R.

From group theory, we already know that $I \cap J$ is an additive subgroup of R. Furthermore, since R is an additive abelian group, $I \cap J$ is an additive abelian subgroup of R.

Assume $a \in I \cap J$
$a \in I$ and $a \in J$
Assume $b \in R$
But I is an ideal in R, so $ab, ba \in I$
Similarly, $ab, ba \in J$
So $ab, ba \in I \cap J$

Therefore, by the ideal test, $I \cap J$ is an ideal in R

From group theory, we know that $I + J = I \vee J$ (join) when either subgroup is normal in R. But since R is abelian, all subgroups are normal. Therefore $I + J$ is an additive subgroup of R. Furthermore, since R is an additive abelian group, $I + J$ is an additive abelian subgroup of R.

Now, assume $a \in I + J$
By definition, there exists $i \in I$ and $j \in J$ such that $a = i + j$
Assume $b \in R$
$ab = (i + j)b = ib + jb$
But I is an ideal, so $ib \in I$
Similarly, $jb \in J$
Thus, $ab = ib + jb \in I + J$

$ba = b(i + j) = bi + bj$
But I is an ideal, so $bi \in I$
Similarly, $bj \in J$
Thus $ba = bi + bj \in I + J$

Therefore, by the ideal test, $I + J$ is an ideal in R.

b). Prove that there is an isomorphism of rings:

$$I/(I \cap J) \simeq (I + J)/J$$

From part (a) we know that $I + J$ is a ring
Since $0 \in I$ we have $J \subseteq I + J$
J is a ring and is thus an additive abelian subgroup of $I + J$

Assume $a \in J$
Assume $b \in I + J$

There exists $i \in I$ and $j \in J$ such that $b = i + j$
$ab = a(i + j) = ai + aj$
But J is an ideal in R, so $ai, aj \in J$
So by closure, $ab = ai + aj \in J$

$ba = (i + j)a = ia + ja$
But J is an ideal in R, so $ia, ja \in J$
So by closure, $ba = ia + ja \in J$

Thus, by the ideal test, J is an ideal in $I + J$, and therefore $(I + J)/J$ is a factor ring.

Now, consider $\phi : I \to (I + J)/J$ defined by $\phi(i) = i + J$.

Assume $i, i' \in I$
$\phi(i + i') = (i + i') + J = (i + J) + (i' + J) = \phi(i) + \phi(i')$
$\phi(ii') = (ii') + J = (i + J)(i' + J) = \phi(i)\phi(i')$

Therefore ϕ is a ring homomorphism.

Now, assume $a \in (I + J)/J$
There exists $b \in (I + J)$ such that $a = b + J$
But, there exists $i \in I$ and $j \in J$ such that $b = i + j$
So, $a = (i + j) + J$
Now, since J is the additive identity for $(I + J)/J$:
$\phi(i) = i + J = (i + J) + J$
And since $j \in J$:
$\phi(i) = (i + J) + (j + J) = (i + j) + J$

Therefore, ϕ is surjective.

Now, consider $i \in I$ such that $\phi(i) = i + J = J$
This means that $i \in J$ as well, so $\ker(\phi) = I \cap J$

Therefore, by the first fundamental theorem:

$$I/(I \cap J) \simeq (I + J)/J$$

2). Let R be a commutative ring with $1 \neq 0$ and suppose S is a multiplicatively-closed subset of $R \setminus \{0\}$ containing no zero divisors. Define $\sim$ on $R \times S$ by $(a, b) \sim (c, d) \iff ad = bc$.

a). Prove: $\sim$ is an equivalence relation.

R: Assume $(a, b) \in R \times S$

$ab = ab$ and so, by definition, $(a, b) \sim (a, b)$

Therefore $\sim$ is reflexive.

S: Assume $(a, b) \sim (c, d)$

$ad = bc$
But R is commutative, so $da = cb$

Furthermore, equality is symmetric, so $cb = da$
Thus, by definition, $(c, d) \sim (a, b)$

Therefore $\sim$ is symmetric.

T: Assume $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$

$ad = bc$ and $cf = de$
R is a ring, so using ring properties:
$adcf = bcde$
$adcf - bcde = 0$
$(af)(dc) - (be)(cd) = 0$
$(af)(cd) - (be)(cd) = 0$
$(af - be)(cd) = 0$
But since R has no zero-divisors: $af - be = 0$ or $cd = 0$

Case 1: $cd = 0$

By construction, $d \neq 0$, and so $c = 0$
$ad = b0 = 0$, and thus $a = 0$
Similarly, $0f = de = 0$, and thus $e = 0$
So $af = 0f = 0$ and $be = b0 = 0$
$af = be$
Thus, by definition, $(a, b) \sim (e, f)$

Case 2: $af - be = 0$

$af = be$
Thus, by definition, $(a, b) = (e, f)$

Therefore, $\sim$ is transitive.

Therefore, $\sim$ is an equivalence relation.

b). Let R_S denote the set of equivalence classes $\frac{a}{b}$ of (a, b). Prove that addition in R_S:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

is well-defined. (Given: multiplication is well-defined)

Assume (a, b) and $(c, d) \in R_S$
Assume $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$
By definition: $ab' = ba'$ and $cd' = dc'$

Consider $(1, 1) \in R_S$:

$$(a, b)(1, 1) = (a1, b1) = (a, b)$$

$$(1, 1)(a, b) = (1a, 1b) = (a, b)$$

Thus $(1, 1)$ is a multiplicative identity for R_S

By construction, $b, d \in S$ are non-zero and S has no zero divisors, so $bd \neq 0$
Thus, $\frac{bd}{bd} \in R_S$
Furthermore: $(bd)1 = 1(bd)$, so $(bd, bd) \sim (1, 1)$
Similarly: $(b'd', b'd') \sim (1, 1)$

Adding the two alternate representatives we get:

$$\frac{a'}{b'} + \frac{c'}{d'} = \frac{a'd' + b'c'}{b'd'}$$

Since multiplication is assumed to be well-defined:

$$\frac{a'}{b'} + \frac{c'}{d'} = \frac{1}{1} \cdot \frac{a'd' + b'c'}{b'd'} = \frac{bd}{bd} \cdot \frac{a'd' + b'c'}{b'd'} = \frac{(bd)(a'd' + b'c')}{(bd)(b'd')}$$

But R is a commutative ring, so using ring properties:

$$\begin{aligned}
\frac{a'}{b'} + \frac{c'}{d'} &= \frac{bda'd' + bdb'c'}{bdb'd'} \\
&= \frac{(ba')(dd') + (dc')(bb')}{(b'd')(bd)} \\
&= \frac{(ab')(dd') + (cd')(bb')}{(b'd')(bd)} \\
&= \frac{(b'd')(ad) + (b'd')(bc)}{(b'd')(bd)} \\
&= \frac{(b'd')(ad + bc)}{(b'd')(bd)} \\
&= \frac{b'd'}{b'd'} \cdot \frac{ad + bc}{db} \\
&= \frac{1}{1} \cdot \frac{ad + bc}{db} \\
&= \frac{ad + bc}{db}
\end{aligned}$$

Therefore, addition in R_S is well-defined.

c). Prove that there are exactly two prime ideals in $\mathbb{Z}_{(p)}$: one corresponding to the zero ideal and one corresponding to the prime p.

We know that the ideals of $\mathbb{Z}_{(p)}$ are of the form $p\mathbb{Z}_{(p)}$ and that they form a chain:

$$\{0\} \subset \ldots \subset p^3\mathbb{Z}_{(p)} \subset p^2\mathbb{Z}_{(p)} \subset p\mathbb{Z}_{(p)} \subset \mathbb{Z}_{(p)}$$

Assume $x, y \in \mathbb{Z}_{(p)}$

Since $\mathbb{Z}_{(p)}$ is an integral domain and thus has no zero divisors:

$$xy = 0 \implies x = 0 \text{ or } y = 0$$

Thus, the zero ideal is prime.

Now, assume $k \in \mathbb{Z}^+$

Case 1: $k > 1$:

$p^{k-1} \in p^{k-1}\mathbb{Z}_{(p)}$
$p \in p\mathbb{Z}_{(p)}$
$p^{k-1}p = p^k \in p^k\mathbb{Z}_{(p)}$
But $p^{k-1}, p \notin p^k$

Therefore $p^k\mathbb{Z}_{(p)}$ is not prime.

Case 2: $k = 1$

Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$
Since p divides neither b nor d, $xy \in p\mathbb{Z}_{(p)}$ means that one of x, y must be in $p\mathbb{Z}_{(p)}$ and the other must be in $\mathbb{Z}_{(p)}$. Otherwise, xy would fall in one of the other ideals.

Therefore, $p\mathbb{Z}_{(p)}$ is prime.