Harmonic Functions

Definition

To say that a real-valued function $h(x, y)$ is *harmonic* in a domain D means that the first and second partial derivatives exist and are continuous in D, and:

$$\nabla^2 h = h_{xx} + h_{yy} = 0$$

This is known as the *Laplace's equation*.

Theorem

Let D be a domain:

$f(z) = u + iv$ analytic in $D \implies u$ and v harmonic in D

Proof

$u_x = v_y$ and $v_x = -u_y$

$u_{xx} = v_{yx}$ and $v_{xy} = -u_{yy}$
But since the partials are continuous, $v_{xy} = v_{yx}$
$u_{xx} = -u_{yy}$
$\therefore u_{xx} + u_{yy} = 0$

$u_{xy} = v_{yy}$ and $v_{xx} = -u_{yx}$
But since the partials are continuous, $u_{xy} = u_{yx}$
$v_{yy} = -v_{xx}$
$\therefore v_{xx} + v_{yy} = 0$

Example

$f(z) = z^2 = (x^2 - y^2) + i2xy$

$f(z)$ is entire

$u_x = 2x$ and $u_{xx} = 2$
$u_y = -2y$ and $u_{yy} = -2$
$u_{xx} + u_{yy} = 2 - 2 = 0$
$\therefore u$ is harmonic

$v_x = 2y$ and $v_{xx} = 0$
$v_y = 2x$ and $y_{yy} = 0$
$v_{xx} + v_{yy} = 0 + 0 = 0$
$\therefore v$ is harmonic

Note that the converse is *not* true!

Example

$f(z) = x + i(x^2 - y^2)$

$u_x = 1$ and $u_{xx} = 0$
$u_y = 0$ and $u_{yy} = 0$
$u_{xx} + u_{yy} = 0 + 0 = 0$
$\therefore u$ is harmonic

$v_x = 2x$ and $v_{xx} = 2$
$v_y = -2y$ and $v_{yy} = -2$
$v_{xx} + v_{yy} = 2 - 2 = 0$
$\therefore v$ is harmonic

$$\begin{aligned} f(z) &= \frac{z+\overline{z}}{2} + i\left[\left(\frac{z+\overline{z}}{2}\right)^2 - \left(\frac{z-\overline{z}}{2i}\right)^2\right] \\ &= \frac{z+\overline{z}}{2} + i\left[\frac{(z^2+2z\overline{z}+\overline{z}^2)+(z^2-2z\overline{z}+\overline{z}^2)}{4}\right] \\ &= \frac{z+\overline{z}}{2} + i\left(\frac{2z^2+2\overline{z}^2}{4}\right) \\ &= \frac{1}{2}(z+\overline{z}+iz^2+i\overline{z}^2) \end{aligned}$$

$$\frac{df}{d\overline{z}} = \frac{1}{2}(1+i2\overline{z}) = \frac{1}{2} + i\overline{z} \neq 0$$

$\therefore f(z)$ is analytic nowhere

So, u and v harmonic is necessary but not sufficient.

Definition

To say that v is a *harmonic conjugate* of u on a domain D means that u and v are harmonic and CR holds.

Theorem

Let D be a domain:

$f(z) = u + iv$ analytic in $D \iff v$ is a harmonic conjugate of u in D

Proof

$\Longrightarrow$ Assume $f(z) = u + iv$ analytic in D

u and v are harmonic
CR holds
$\therefore v$ is a harmonic conjugate of u

$\Longleftarrow$ Assume v is a harmonic conjugate of u

u and v are harmonic
The partials of u and v exist and are continuous
CR holds
$\therefore f$ is analytic

Theorem

Let $u(z, \overline{z})$ be a real-valued function on a domain D:

$$\nabla^2 u = 4u_{z\overline{z}}$$

Proof

Let: $\begin{array}{lll} z = x + iy & z_x = 1 & z_y = i \\ \overline{z} = x - iy & \overline{z}_x = 1 & \overline{z}_y = -i \end{array}$

$$u_x = u_z z_x + u_{\overline{z}} \overline{z}_x = u_z + u_{\overline{z}}$$

$$\begin{aligned} u_{xx} &= u_{zz} z_x + u_{z\overline{z}} \overline{z}_x + u_{\overline{z}z} z_x u_{\overline{z}\overline{z}} + \overline{z}_x \\ &= u_{zz} + u_{z\overline{z}} + u_{\overline{z}z} + u_{\overline{z}\overline{z}} \\ &= u_{zz} + u_{z\overline{z}} + u_{z\overline{z}} + u_{\overline{z}\overline{z}} \\ &= u_{zz} + 2u_{z\overline{z}} + u_{\overline{z}\overline{z}} \end{aligned}$$

$$u_y = u_z z_y + u_{\overline{z}} \overline{z}_y = iu_z - iu_{\overline{z}} = i(u_z - u_{\overline{z}})$$

$$\begin{aligned} u_{yy} &= i(u_{zz} z_y + u_{z\overline{z}} \overline{z}_y - u_{\overline{z}z} z_y - u_{\overline{z}\overline{z}} \overline{z}_y) \\ &= i(iu_{zz} - iu_{z\overline{z}} - iu_{\overline{z}z} + iu_{\overline{z}\overline{z}}) \\ &= i(iu_{zz} - iu_{z\overline{z}} - iu_{z\overline{z}} + iu_{\overline{z}\overline{z}}) \\ &= i(iu_{zz} - 2iu_{z\overline{z}} + iu_{\overline{z}\overline{z}}) \\ &= -u_{zz} + 2u_{z\overline{z}} - u_{\overline{z}\overline{z}} \end{aligned}$$

$$\nabla^z u = u_{xx} + u_{yy} = (u_{zz} + 2u_{z\overline{z}} + u_{\overline{z}\overline{z}}) - (-u_{zz} + 2u_{z\overline{z}} - u_{\overline{z}\overline{z}}) = 4u_{z\overline{z}}$$

Corollary

Let $u(z, \overline{z})$ be a real-valued,analytic function on a domain D:

$$u_{z\overline{z}} = 0$$

Proof

$u(z, \overline{z})$ is harmonic
$\nabla^2 u = 4u_{z\overline{z}} = 0$
$\therefore u_{z\overline{z}} = 0$

Note that this is consistent with the fact that for f analytic, $f_{\overline{z}} = 0$.

Theorem

Let $\phi(x, y)$ be harmonic in a domain D_z and let $w = f(z)$ be analytic in D_z such that $f'(z) \neq 0$. ϕ is harmonic in D_w.

Proof

Let $w = u + iv$. In order for ϕ to be harmonic in D_w:

$$\phi_{uu} + \phi_{vv} = 4\phi_{w\overline{w}} = 0$$

So, WTS $\phi_{w\overline{w}} = 0$

$\phi_z = \phi_w w_z + \phi_{\overline{w}} \overline{w}_z$
But in order for ϕ to be differentiable on D_w, $\phi_{\overline{w}} = 0$, so:
$\phi_z = \phi_w w_z$

$\phi_{z\overline{z}} = (\phi_{ww} w_{\overline{z}} + \phi_{w\overline{w}} \overline{w}_{\overline{z}}) w_z + \phi_w w_{z\overline{z}}$
But for f to be differentiable in D_z, $w_{\overline{z}} = 0$ and for w_z to be differentiable in D_z, $w_{z\overline{z}} = 0$, so:
$\phi_{z\overline{z}} = \phi_{w\overline{w}} \overline{w}_{\overline{z}} w_z = \phi_{w\overline{w}} \overline{w_z} w_z = \phi_{w\overline{w}} |w_z|^2 = \phi_{w\overline{w}} |f'(z)|^2$

But for ϕ harmonic on D_z, $\phi_{z\overline{z}} = 0$, so
$\phi_{w\overline{w}} |f'(z)|^2 = 0$
But $f'(z) \neq 0$ by assumption,
$\therefore \phi_{w\overline{w}} = 0$