Trigonometric Fourier Series

Theorem

The orthonormal sequence $\varphi_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$ for $n \in \mathbb{Z}$ is complete in $L^2[-\pi.\pi]$. Thus, $\forall\, f \in L^2[-\pi.\pi]$, f can be written as:

$$f \sim \sum_{n=-\infty}^{\infty} \langle f, \varphi_n \rangle \varphi_n$$

Proving completeness is non-trivial and requires Fejér kernels.

Note that this is convergence in the L^2 sense:

$$\left\| f - \sum_{n=-N}^{N} \langle f, \varphi_n \rangle \varphi_n \right\| \to 0$$

$$\int_{-\pi}^{\pi} \left| f(t) - \sum_{n=-N}^{N} \langle f, \varphi_n \rangle e^{int} \right|^2 dt \to 0$$

Meaning $S_n \xrightarrow{L^2} f$.

However, L^2 convergence does not guarantee pointwise convergence.

Theorem: Carleson

Fourier series of L^2 periodic functions converge pointwise a.e.