Natural Numbers

Definition

The set of *Natural Numbers*, denoted $\mathbb{N}$, are the whole numbers starting from 1 and continuing to infinity:

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

Notation

$[n] = \{1, 2, 3, \ldots, n\}$

Peano Axioms

N1: $1 \in \mathbb{N}$.

N2: $n \in \mathbb{N} \implies (n+1) \in \mathbb{N}$.

N3: 1 is not the successor of any $n \in \mathbb{N}$.

N4: $n, m \in \mathbb{N}$ have the same successor $\implies n = m$.

N5: $S \subseteq \mathbb{N}$ and $1 \in S$ and $(n \in S \implies (n+1) \in S) \implies S = \mathbb{N}$.

Note that N5 is the basis for mathematical induction.

Closure Property

$\forall n, m \in \mathbb{N}$:

1). $n + m \in \mathbb{N}$

2). $nm \in \mathbb{N}$

Theorem

$\sum_{k=1}^{n}(2k-1) = n^2$

Proof

By induction:

Base: $n = 1$

$$\sum_{k=1}^{n}(2k-1) = 2(1) - 1 = 2 - 1 = 1$$

$$1^2 = 1$$

Inductive Assumption

Assume $\sum_{k=1}^{n}(2k-1) = n^2$.

$$\begin{aligned}
\sum_{k=1}^{n+1}(2k-1) &= 2(n+1) - 1 + \sum_{k=1}^{n}(2k-1) \\
&= 2n + 2 - 1 + n^2 \\
&= n^2 + 2n + 1 \\
&= (n+1)^2
\end{aligned}$$