Prime Factorization

In mathematics, the term "divides" when applied to integers means something very specific:

Definition: Divides

To say that one integer n *divides* another integer m, denoted $n|m$, mean that $\exists k \in \mathbb{Z}, m = kn$. In this case, n is said to be a *divisor* of m.

In arithmetic terms, this means that n divides m evenly; there is no remainder. We will study this in depth later when we encounter the so-called *division algorithm*.

This concept of division gives rise to an important subset of the integers called the prime numbers:

Definition: Prime

To say that an integer is a *prime* number means that it has only two divisors: 1 and itself. Otherwise, it is called *composite*.

The first few prime numbers are: $2, 3, 5, 7, 11, 13, 17, 19, \ldots$.

The prime numbers play a pivotal role in what is called the *Fundamental Theorem of Arithmetic*:

Theorem

Every integer can be represented uniquely as a product of powers of primes.

For example, $120 = 2^3 \cdot 3 \cdot 5$ and this representation is unique.

Prime factorization is useful when we want to add or subtract two fractions with different denominators; we need to modify one or more of the fractions so that we have a common denominator. We will then be able to use the rule:

$$\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$$

We could just multiple the denominators by each other in order to achieve such a common denominator, but that typically leads to large, error-prone numbers— especially when there are more than two fractions in our expression. Instead, we want to pick the lowest common multiple (LCM) of all the denominators.

Consider the problem: $\frac{3}{10} + \frac{7}{12} - \frac{3}{5}$

1. Perform a prime factorization of each denominator.

   $10 = 2 \cdot 5$
   $12 = 2 \cdot 6 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$
   $5 = 5$ (already prime!)

2. Take the highest power of each prime across all the factorizations.

   From 2^1, 2^2 get 2^2.

   From 3^1 we get just 3.

   From 5^1 and 5^1 we get 5.

   So the LCM of the denominators is $2^2 \cdot 3 \cdot 5 = 60$.

3. Multiply each fraction above and below the achieve the common denominator and then perform the operation.

$$
\begin{aligned}
\frac{3}{10} + \frac{7}{12} - \frac{3}{5} &= \frac{3 \cdot 6}{10 \cdot 6} + \frac{7 \cdot 5}{12 \cdot 5} - \frac{3 \cdot 12}{5 \cdot 12} \\
&= \frac{18}{60} + \frac{35}{60} - \frac{36}{60} \\
&= \frac{18 + 35 - 26}{60} \\
&= \frac{27}{60}
\end{aligned}
$$

4. Use prime factorizations on the numerator and denominator to simply the result.

   $27 = 3 \cdot 9 = 3 \cdot 3 \cdot 3 = 3^3$
   $60 = 2^2 \cdot 3 \cdot 5$.

$$
\begin{aligned}
\frac{27}{60} &= \frac{3^3}{2^2 \cdot 3 \cdot 5} \\
&= \frac{3^2}{2^2 \cdot 5} \\
&= \frac{9}{4 \cdot 5} \\
&= \frac{9}{20}
\end{aligned}
$$