Abstract Algebra

Abstract algebra is concerned with the structure imposed upon sets by one or more binary operators, and which such sets have the same structure.

Definition

Let $c \in \mathbb{R}, c > 0$:

$$\mathbb{R}_c = [0, c) = \{x \in \mathbb{R} \mid 0 \leq x < c\}$$

Definition

Let $a, b \in \mathbb{R}_c$. Addition of a and b modulo c, denoted $a +_c b$, is given by:

$$a +_c b = \begin{cases} a + b, & a + b \in \mathbb{R}_c \\ a + b - c, & a + b \notin \mathbb{R}_c \end{cases}$$

Example

$\mathbb{R}_1 = [0, 1)$:

$$0.5 +_1 0.25 = 0.75$$

$$0.5 +_1 0.5 = 1 - 1 = 0$$

$$0.5 +_1 0.75 = 1.25 - 1 = 0.25$$

The angular arithmetic in polar and complex exponential forms usually takes place in $\mathbb{R}_{2\pi}$, although other intervals such as $(-\pi, \pi]$ are sometimes used.

Let U be the locus of points on the unit circle:

$$U = \{z \in \mathbb{C} \mid |z| = 1$$

U has the following properties:

1). U is *closed* under multiplication:

$$\forall\, z_1, z_2 \in U, z_1 z_2 \in U$$

Let $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$:

$$z_1 z_2 = e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 +_{2\pi} \theta_2)} \in U$$

2). There exists a unique element $e^{i0} = 1 \in U$ such that

$$\forall\, z \in U, 1z = z1 = z$$

Such an element is called the identity element.

3). There exists a bijection $\phi : U \to \mathbb{R}_{2\pi}$ defined by:

$$\phi(z) = \phi(e^{i\theta}) = \theta \in \mathbb{R}_{2\pi}$$

4). The bijection ϕ is also a *homomorphism*:

$$\phi(z_1 z_2) = \phi\left(e^{i(\theta_1 +_{2\pi} \theta_2)}\right) = \theta_1 +_{2\pi} \theta_2 = \phi(z_1) +_{2\pi} \phi(z_2)$$

5). The equation $z \cdot z \cdot z \cdot z = 1$ in U has four solutions: $1, -1, i, -i$. Thus, the equation: $x +_{2\pi} x +_{2\pi} x +_{2\pi} x = 0$ in $\mathbb{R}_{2\pi}$ has four corresponding solutions: $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

A bijection that is also a homomorphism is called an *isomorphism*. Isomorphisms indicate that two sets have the same structure, although the names of the elements may be different.

Definition

$$\mathbb{Z}_n = \{m \in \mathbb{Z} \mid 0 \leq m < n\} = \{0, 1, 2, \ldots, n-1\}$$

Definition

The n^{th} roots of unity, denoted U_n, are given by:

$$U_n = \{z \in \mathbb{C} \mid z^n = 1\}$$

The k^{th} root, denoted ζ^k, is given by:

$$\zeta^k = e^{i\frac{2\pi k}{n}}$$

Note that $U_n \subset U$, where the members of U_n start at $(1, 0)$ and are equally spaced by $\frac{2\pi}{n}$. This results in a total of $n-1$ unique roots, corresponding to $0 \leq k < n$.

Similarly, U_n has the following properties:

1). U_n is closed under multiplication:

$$\forall\, \zeta^h, \zeta^k \in U_n, \zeta^h \zeta^k \in U_n$$

Let $\zeta^h = e^{i\frac{2\pi h}{n}}$ and $\zeta^k = e^{i\frac{2\pi k}{n}}$:

$$\zeta^h \zeta^k = e^{i\frac{2\pi h}{n}} e^{i\frac{2\pi k}{n}} = e^{i\left[\frac{2\pi(h +_n k)}{n}\right]} \in U_n$$

2). There exists a unique element $\zeta^0 = e^{i0} = 1 \in U_n$ such that

$$\forall\, \zeta^k \in U_n, \zeta^0 \zeta^k = \zeta^k \zeta^0 = \zeta^k$$

Such an element is called the identity element.

3). There exists a bijection $\phi : U_n \to \mathbb{Z}_n$ defined by:

$$\phi(\zeta^k) = \phi(e^{i\frac{2\pi k}{n}}) = k$$

4). The bijection ϕ is also a *homomorphism*:

$$\phi(\zeta^h \zeta^k) = \phi\left(e^{i\left[\frac{2\pi(h +_n k)}{n}\right]}\right) = h +_n k = \phi(h) +_n \phi(k)$$

Example

Find all solutions to the equation:

$$x +_8 x +_8 x +_8 x +_8 x +_8 x +_8 x +_8 x = 0$$

Since $\mathbb{Z}_8$ is isomorphic to U_8, consider the equation:

$$z \cdot z \cdot z \cdot z \cdot z \cdot z \cdot z \cdot z = z^8 = 1$$

The solutions to this equation are the 8^{th} roots of unity, which correspond to the eight solutions: $0, 1, 2, 3, 4, 5, 6, 7$ in $\mathbb{Z}_8$.