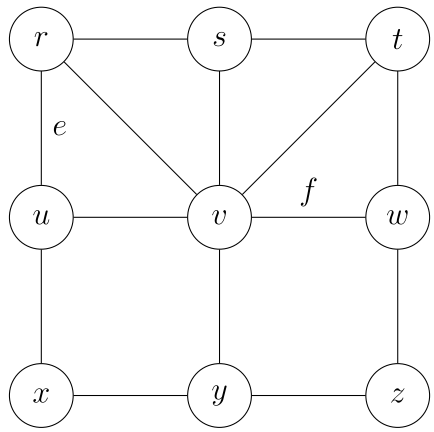
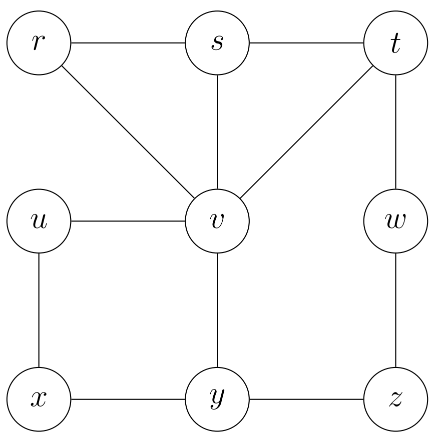
1.2: Connected Graphs

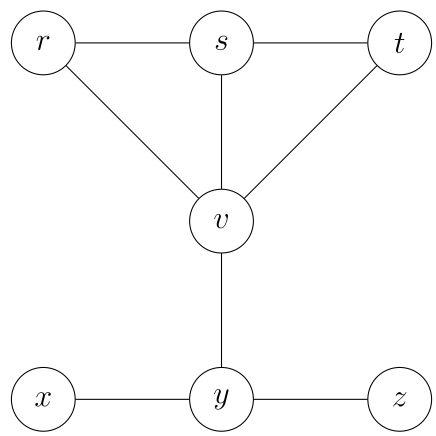
11. Let G be the graph in Figure 1.20, let $X = \{e, f\}$, where $e = ru$ and $f = vw$, and let $U = \{u, w\}$. Draw the subgraphs $G - X$ and $G - U$ of G.



G



$G - X$



$G - U$

12. For the graph G of Figure 1.20, give an example of each of the following or explain why no such example exists.

    (a) An $x - y$ walk of length 6.

    $$(x, u, r, v, w, z, y)$$

    (b) A $v - w$ trail that is not a $v - w$ path.

    $$(v, s, t, v, w)$$

    (c) An $r - z$ path of length 2.

This is not possible because $d(r, z) = 3$.

(d) An $x - z$ path of length 3.

This is not possible. If the first move, is to u, the $d(u, z) = 3$ so a 3-path including u is not possible. If the first move is to y, then only paths of length 1 and 3 are possible from y to z. Thus $x - z$ paths of length 2 and 4 are possible, but not of length 3.

(e) An $x - t$ path of length $d(x, t)$.

(x, y, v, t)

(f) A circuit of length 10.

$(r, u, x, y, z, w, v, t, s, v, r)$

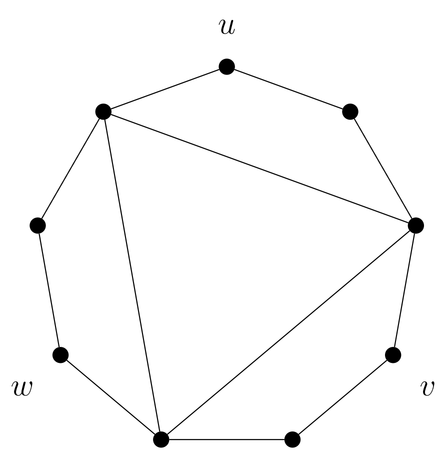
(g) A cycle of length 8.

$(r, u, x, y, z, w, t, v, r)$

(h) A geodesic whose length is $\mathrm{diam}(G)$.

(r, v, y, z)

13. (a) Give an example of a connected graph G containing three vertices $u, v,$ and w such that $d(u, v) = d(u, w) = d(v, w) = \mathrm{diam}(G) = 3$.



(b) Does the question in (a) suggest another question?

How does one draw a graph G containing r vertices such that the distance between any two of the vertices is $r = \mathrm{diam}(G)$?

Start with a (r^2)-cycle and evenly space the r vertices at every r^{th} position. Then add an inner r-cycle that includes each vertex position just before the r vertices.

14. For a graph G, a component of G has been defined as (1) a connected subgraph of G that is not a proper subgraph of any other connected subgraph of G and has been described as (2) a subgraph of G induced by the vertices in an equivalence class resulting from the equivalence relation defined in Theorem 1.7. Show that these two interpretations of components are equivalent.

Theorem

Let G be a graph and let G_i be a subgraph of G. TFAE:

(a) G_i is a component of G.

(b) G_i is induced by an equivalence class of the connectedness relation.

Proof.

$\Longrightarrow$ Assume G_i is a component of G.

So G_i is a maximal connected induced subgraph of G.

ABC: $V(G_i)$ is not an equivalence class of the connectedness relation.

Thus, $V(G_i)$ must be a proper subset of some equivalence class V_i and $G[V_i]$ is an connected induced subgraph of G such that $G_i \subset G[V_i]$, contradicting the maximality of G_i.

$\therefore G_i$ is induced by an equivalence class of the connectedness relation.

$\Longleftarrow$ Assume G_i is induced by an equivalence class of the connectedness relation.

By definition, G_i is a connected subgraph of G.

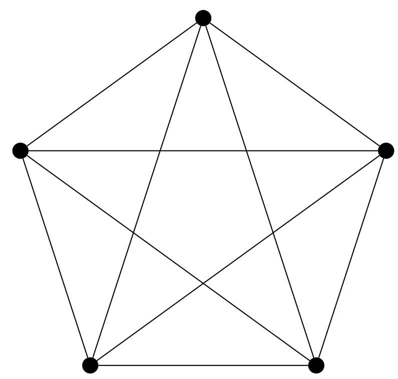
ABC: G_i is not maximal.

Thus, G_i is a proper subgraph of some connected subgraph H of G and $V(G_i) \subset V(H)$, contradicting the definition of $V(G_i)$ as an equivalence class.

$\therefore G_i$ is a component of G.

∎

15. Draw all connected graphs of order 5 in which the distance between every two distinct vertices is odd. Explain why you know that you have drawn all such graphs.



K_5

This is the only possibility. Since $n = 5$, on 1-paths and 3-paths are possible. But only 1-paths are possible, since a 3-path would contain a 2-path.

16. Let $P = (u = v_0, v_1, \ldots, v_k = v), k \geq 1$ be a $u - v$ geodesic in a connected graph G. Prove that $d(u, v_i) = i$ for each integer i with $1 \leq i \leq k$.

Theorem

Let G be a graph with $u, v \in V(G)$ and let $P = (u = v_0, v_1, \ldots, v_k = v)$ be a $u - v$ geodesic in G. For all i such that $0 \leq i \leq k$:

$$d(u, v_i) = i$$

Proof. Assume $0 \leq i \leq k$.

Since $(u = v_0, v_1, \ldots, v_i)$ is a $u - v_i$ path of length i in G, it must be that case that $d(u, v_i) \leq i$.

ABC: There exists a shorter $u - v_i$ path in G: $(u = w_0, w_1, \ldots, w_\ell = v_i)$ for $\ell < i$.

Let $W = (u = w_0, w_1, \ldots, w_\ell = v_i, \ldots v_k = v)$. W is a $u - v$ walk in G of length:

$$\ell + (k - i) = k - (i - \ell) < k$$

So there exists a $u - v$ path in G of length $< k$, contradicting the minimality of P.

$\therefore d(u, v_i) = i$ ■

17. (a) Prove that if P and Q are two longest paths in a connected graph, then P and Q have at least one vertex in common.

Theorem

Let G be a connected graph and let P and Q be two longest paths in G, both of length k:

P and Q have at least one vertex in common.

Proof. ABC: P and Q have no vertices in common.

Let $P = (u_0, u_1, \ldots, u_k)$ and $Q = (v_0, v_1, \ldots, v_k)$. Since G is connected, every u_i in P is connected to every v_j in Q. Let $R = (u_i = w_1, w_2, \ldots, w_\ell = v_j)$ be the shortest such path and AWLOG that $i \geq j$. Note that no other vertices in P or Q can exist in R, otherwise the minimality of $|R|$ is contradicted. Now, consider the path $S = (u_0, \ldots, u_i, \ldots v_j, \ldots v_k)$:

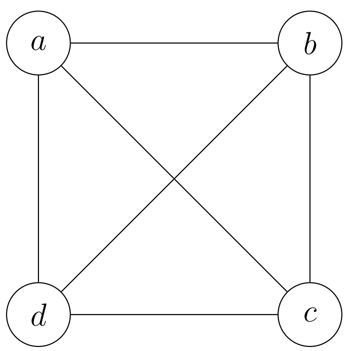
$$\begin{aligned} |S| &= i + \ell + (k - j) \\ &= k + \ell + (i - j) \\ &> k \end{aligned}$$

since $\ell > 0$ and $i - j \geq 0$, thus contradicting the maximality of $|P|$ and $|Q|$.

$\therefore$, P and Q share at least one vertex in common. ∎

(b) Prove or disprove: Let G be a connected graph of diameter k. If P and Q are two geodesics of length k in G, then P and Q have at least one vertex in common.

FALSE. Consider the following counterexample: $G = K_4$:



$\text{diam}(G) = 1$

$P_1 = (a, b)$ is geodesic and $|P_1| = 1$
$P_2 = (c, d)$ is geodesic and $|P_2| = 1$

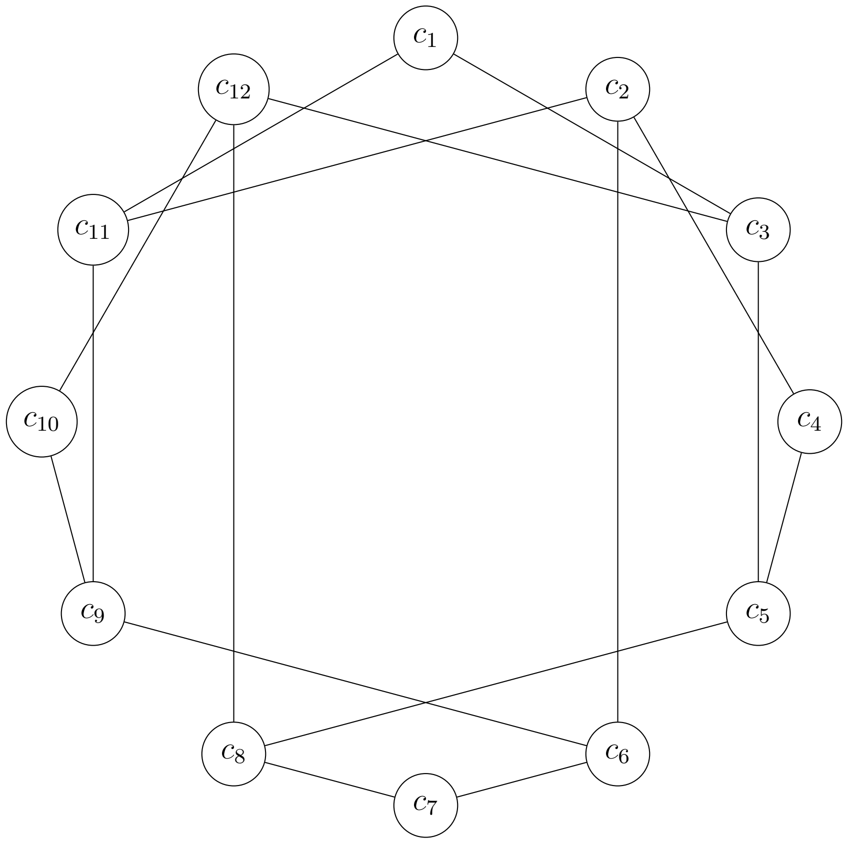
But P_1 and P_2 have no vertices in common.

18. A graph G of order 12 has vertex set $V(G) = \{c_1, c_2, \ldots, c_{12}\}$ for the twelve configurations in Figure 1.4. A "move" on this checkerboard corresponds to moving a single coin to an unoccupied square, where

(1) the gold coin can only be moved horizontally or diagonally.

(2) the silver coin can only be moved vertically or diagonally.

Two vertices c_i and c_j $(i \neq j)$ are adjacent if it is possible to move c_i to c_j by a single move.



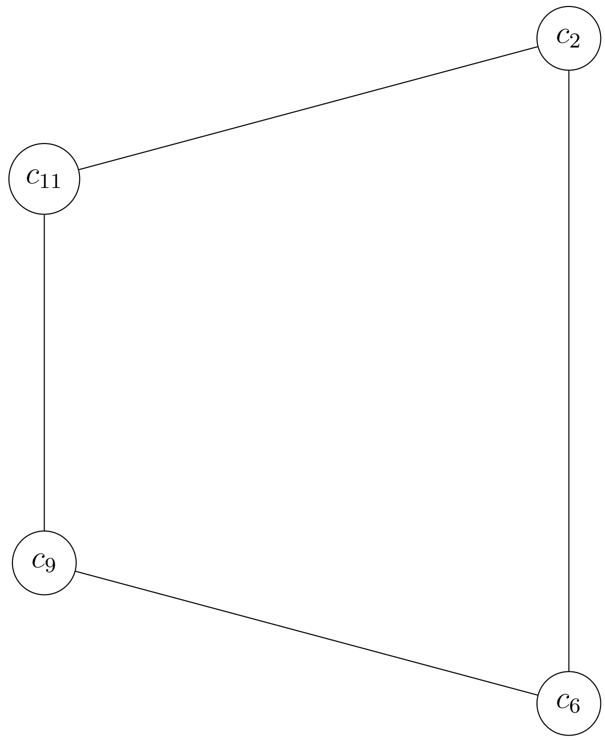
(a) What vertices are adjacent to c_1 in G?

$\{c_3, c_{11}\}$

(b) What vertices are adjacent to c_2 in G?

$\{c_4, c_6, c_{11}\}$

(c) Draw the subgraph of G induced by $\{c_2, c_6, c_9, c_{11}\}$.

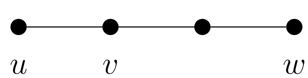


(d) Give an example of a $c_1 - c_7$ path in G.

$(c_1, c_3, c_5, c_8, c_7)$

19. Theorem 1.10 states that a graph G of order 3 of more is connected if and only if G contains two distinct vertices u and v such that $G - u$ and $G - v$ are connected. Based on this, one might suspect that the following statement is true. *Every connected graph G of order 4 or more contains three distinct vertices u, v, and w such that $G - u$, $G - v$, and $G - w$ are connected.* Is it?

    No. Consider $G = P_4$:

    Note that $G - u$ and $G - w$ are connected; however, $G - v$ is not.

20. (a) Let u and v be distinct vertices in a connected graph G. There may be several connected subgraphs of G containing u and v. What is the minimum size of a connected subgraph of G containing u and v? Explain your answer.

    The minimum subgraph is a $u - v$ geodesic. This path contains the minimum number of edges to ensure that u and v (and all intervening nodes) are connected. Thus, the minimum size is $d(u, v)$.

(b) Does the question in (a) suggest another question to you?

Let G be a connected graph and let $S \subseteq V(G)$. What is the minimum size of a connected subgraph of G containing all of the vertices in S?

This can be obtained via construction. If $|S| = 1$ then done. If $|S| = 2$ then apply part (a). If $|S| > 2$, start with two vertices as above, and then for each additional vertex, add the minimum number of edges from $E(G)$ to connect it to the existing graph. The result will be a minimum spanning tree of S using edges from $E(G)$ with size $\geq |S| - 1$.