Limit Laws

All limit laws in $\mathbb{R}$ are accepted as fact.

Theorem

Let:
$f(z) = f(x+iy) = u(x,y) + iv(x,y)$
$z_0 = x_0 + iy_0$
$w_0 = u_0 + iv + 0$

$$\lim_{z \to z_0} f(z) = w_0 \iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} (x,y) = v_0$$

Proof

$|z - z_0| = |(x+iy) - (x_0 + iy_0)| = |(x - x0) + i(y - y_0)| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$

$\implies$ Assume $\lim_{z \to z_0} f(z) = w_0$

Assume $\epsilon > 0$
$\exists\, \delta > 0, 0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$

Assume $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$
Thus $0 < |z - z_0| < \delta$, and so, by assumption: $|f(z) - w_0| < \epsilon$
$|f(z) - w_0| = |(u + iv) - (u_0 + iv_0)| = |(u - u_0) + i(v - v_0)| < \epsilon$
$|u - u_0| \leq |(u - u_0) + i(v - v_0)| < \epsilon$ and
$|v - v_0| \leq |(u - u_0) + i(v - v_0)| < \epsilon$

$\therefore \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$

$\impliedby$ Assume $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$

Assume $\epsilon > 0$
$\exists\, \delta_1 > 0, 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1 \implies |u - u_0| < \frac{\epsilon}{2}$ and
$\exists\, \delta_2 > 0, 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2 \implies |v - v_0| < \frac{\epsilon}{2}$

Let $\delta = \min\{\delta_1, \delta_2\}$
Assume $0 < |z - z_0| < \delta$
Thus $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$, and so, by assumption:
$|u - u_0| < \frac{\epsilon}{2}$ and $|v - v_0| < \frac{\epsilon}{2}$

$$\begin{aligned}
|f(z) - w_0| &= |(u + iv) - (u_0 + iv_0)| \\
&= |(u - u_0) - i(v - v_0)| \\
&\leq |u - u_0| + |v - v_0| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Theorem

1). $\forall\, c \in \mathbb{C}, \lim_{z \to z_0} c = c$

2). $\lim_{z \to z_0} z = z_0$

3). $\lim_{z \to z_0} f(z) = w_0 \implies \lim_{z \to z_0} |f(z)| = |w_0|$

4). $\lim_{z \to z_0} f(z) = w_0 \implies \lim_{z \to z_0} [-f(z)] = -w_0$

5). $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} g(z) = w_1 \implies \lim_{z \to z_0} [f(z) + g(z)] = w_0 + w_1$

6). $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} g(z) = w_1 \implies \lim_{z \to z_0} [f(z) - g(z)] = w_0 - w_1$

7). $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} g(z) = w_1 \implies \lim_{z \to z_0} [f(z)g(z)] = w_0 w_1$

8). $\lim_{z \to z_0} f(z) = w_0$ and $w_0 \neq 0 \implies \lim_{z \to z_0} \frac{1}{f(z)} = \frac{1}{w_0}$

9). $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} g(z) = w_1$ and $w_1 \neq 0 \implies \lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{w_1}$

Proof

1). Assume $c \in \mathbb{C}$

Assume $\epsilon > 0$
Assume $\delta > 0$
Assume $0 < |z - z_0| < \delta$
$|c - c| = |0| = 0 < \epsilon$

2). Assume $\epsilon > 0$

Let $\delta = \epsilon$
Assume $0 < |z - z_0| < \delta$
$|z - z_0| < \delta = \epsilon$

3). Assume $\lim_{z \to z_0} f(z) = w_0$

Assume $\epsilon > 0$
$\exists\, \delta > 0, 0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$
$||f(z)| - |w_0|| = ||f(z)| - |-w_0|| \leq |f(z) + (-w_0)| = |f(z) - w_0| < \epsilon$

4). Assume $\lim_{z\to z_0} f(z) = w_0$

Assume $\epsilon > 0$
$\exists\, \delta > 0, 0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$
Assume $0 < |z - z_0| < \delta$
$|-f(z) - (-w_0)| = |w_0 - f(z)| = |f(z) - w_0| < \epsilon$

5). Assume $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} g(z) = w_1$

$$\begin{aligned}
w_0 + w_1 &= (u_0 + iv_0) + (u_1 + iv_1) \\
&= (u_0 + u_1) + i(v_0 + v_1) \\
&= \lim_{(x,y)\to(x_0,y_0)} (u_f + u_g) + i \lim_{(x,y)\to(x_0,y_0)} (v_f + v_g) \\
&= \lim_{z\to z_0} [(u_f + u_g) + i(v_f + v_g)] \\
&= \lim_{z\to z_0} [(u_f + iv_f) + (u_g + iv_g)] \\
&= \lim_{z\to z_0} [f(z) + g(z)]
\end{aligned}$$

6). Assume $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} g(z) = w_1$

$\lim_{z\to z_0} [f(z) - g(z)] = \lim_{z\to z_0} [f(z) + (-g(z))] = w_0 + (-w_1) = w_0 - w_1$

7). Assume $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} g(z) = w_1$

$$\begin{aligned}
w_0 w_1 &= (u_0 + iv_0)(u_1 + iv_1) \\
&= (u_0 u_1 - v_0 v_1) + i(u_0 v_1 + v_0 u_1) \\
&= \lim_{(x,y)\to(x_0,y_0)} (u_f u_g - v_f v_g) + i \lim_{(x,y)\to(x_0,y_0)} (u_f v_g - v_f u_g) \\
&= \lim_{z\to z_0} [(u_f u_g - v_f v_g) + i(u_f v_g - v_f u_g)] \\
&= \lim_{z\to z_0} [(u_f + iv_f)(u_g + iv_g)] \\
&= \lim_{z\to z_0} [f(z) g(z)]
\end{aligned}$$

8). Assume $\lim_{z\to z_0} f(z) = w_0$ and $w_0 \neq 0$

$$\begin{aligned}
\frac{1}{w_0} &= \frac{1}{u_0 + iv_0} \\
&= \frac{u_0}{u_0^2 + v_0^2} - \frac{v_0}{u_0^2 + v_0^2} \\
&= \lim_{(x,y)\to(x_0,y_0)} \left(\frac{u}{u^2 + v^2}\right) - i \lim_{(x,y)\to(x_0,y_0)} \left(\frac{v}{u^2 + v^2}\right) \\
&= \lim_{z\to z_0} \frac{u}{u^2 + v^2} - i \frac{v}{u^2 + v^2} \\
&= \lim_{z\to z_0} \frac{1}{f(z)}
\end{aligned}$$

9). Assume $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} g(z) = w_1$

$$\begin{aligned} \lim_{z \to z_0} \frac{f(z)}{g(z)} &= \lim_{z \to z_0} \left[f(z) \cdot \frac{1}{g(z)} \right] \\ &= w_0 \left(\frac{1}{w_1} \right) \\ &= \frac{w_0}{w_1} \end{aligned}$$

Using simple induction proofs:

Theorem

1). $\lim_{z \to z_0} f_k(z) = w_k \implies \lim_{z \to z_0} \sum_{k=1}^{n} f_k(z) = \sum_{k=1}^{n} w_k$

2). $\lim_{z \to z_0} f_k(z) = w_k \implies \lim_{z \to z_0} \prod_{k=1}^{n} f_k(z) = \prod_{k=1}^{n} w_k$

3). $\forall\, n \in \mathbb{Z}, \lim_{z \to z_0} z^n = z_0^n$

4). $P(z) = \sum_{k=0}^{n} c_k z^k \implies \lim_{z \to z_0} P(z) = \sum_{k=0}^{n} c_k z_0^k$