MATH 231B, FALL 2017
HOMEWORK 5 SOLUTIONS

1. (Sec. 4.12, ex. 2) Consider the bilinear functional on H defined by

$$\phi(x, y) = \langle Ax, y \rangle.$$

The quadratic form corresponding to ϕ is

$$\Phi(x) = \phi(x, x) = \langle Ax, x \rangle.$$

Since $Ax \perp x$, for all x, it follows that $\Phi = 0$. By the Polarization Identity (Theorem 4.3.7) we have

$$4\phi(x, y) = \Phi(x + y) - \Phi(x - y) + i\Phi(x + iy) - i\Phi(x - iy),$$

so $\phi(x, y) = 0$, for all $x, y \in H$. Therefore, $Ax = 0$, for all x, which implies that $A = 0$. $\square$

2. (Sec. 4.12, ex. 3) Consider the linear operator on $\mathbb{C}^2$ given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then A is clearly bounded and $\|A\| \neq 0$. However, $A^2 = 0$, so $\left\|A^2\right\| = 0 \neq \|A\|^2$. $\square$

3. (Sec. 4.12, ex. 6) (a) Since T is required to be linear, T is forced to be defined by

$$T\left(\sum_{n=1}^{\infty} \alpha_n e_n\right) = \sum_{n=1}^{\infty} \alpha_n \lambda_n e_n.$$

Thus T is clearly unique.

Remark. For this definition to make sense, T has to commute with limits, i.e., it must be continuous thus bounded. Furthermore, for the series $\sum_{n=1}^{\infty} \alpha_n \lambda_n e_n$ to converge to an element of H, the sequence $(\alpha_n \lambda_n)$ has to be in ℓ^2, for every ℓ^2-sequence (α_n). This will be the case only if (λ_n) is bounded. So part (a) works only under the assumption that the sequence (λ_n) is bounded.

(b) Assume $|\lambda_n| \leq M$, for some $M > 0$ and all $n \geq 1$. Then (with the notation as above) by Parseval's identity:

$$\begin{aligned}
\|Tx\|^2 &= \left\|\sum_{n=1}^{\infty} \alpha_n \lambda_n e_n\right\|^2 \\
&= \sum_{n=1}^{\infty} |\alpha_n \lambda_n|^2 \\
&\leq M^2 \sum_{n=1}^{\infty} |\alpha_n|^2 \\
&= M^2 \|x\|^2,
\end{aligned}$$

so $\|T\| \leq M$.

Now assume T is bounded but (λ_n) is not. Then $|\lambda_{n_k}| \to \infty$, for some subsequence (λ_{n_k}). Thus $\|Te_{n_k}\| = \|\lambda_{n_k} e_{n_k}\| = |\lambda_{n_k}| \to \infty$, contradicting the assumption that $\|T\| < \infty$.

(c) It follows from part (b) that $\|T\| \leq L$, where $L = \sup\{|\lambda_n| : n \geq 1\}$. Let us show that $\|T\| = L$. The basic properties of supremum yield a subsequence (λ_{n_j}) such that $\left|\lambda_{n_j}\right| \to L$, as $j \to \infty$. Since

$$\left\|Te_{n_j}\right\| = \left\|\lambda_{n_j} e_{n_j}\right\| = \left|\lambda_{n_j}\right| \to L,$$

as $j \to \infty$, by the definition of the operator norm it follows that $\|T\| = L$. This completes the proof. □

4. (Sec. 4.12, ex. 8) T is given by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.$$

Since

$$A^* = \overline{A}^T = A^T = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \neq A,$$

T is clearly not self-adjoint. □