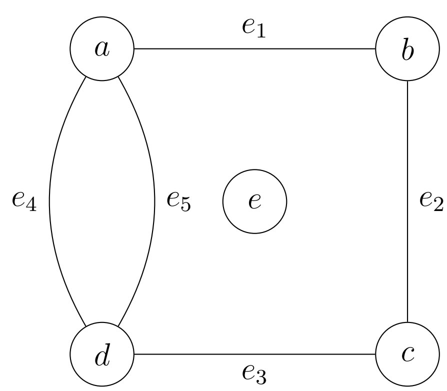
Multigraphs

Definition: Multigraph

A *multigraph* $M = (V, E, \mathscr{E}, \ldots)$ is a graph with a non-empty and finite set of vertices $V(M)$, a possibly empty and finite set of edges $E(M)$, and a function $\mathscr{E}$ that associates each edge with a two-element subset of $V(M)$:

$$\mathscr{E} : E(M) \to \mathcal{P}_2(V(M))$$

Example



$$V = V(M) = \{a, b, c, d, e\}$$
$$E = E(M) = \{e_1, e_2, e_3, e_4, e_5\}$$

$$e_1 \mapsto \{a, b\}$$
$$e_2 \mapsto \{b, c\}$$
$$e_3 \mapsto \{c, d\}$$
$$e_4 \mapsto \{a, d\}$$
$$e_5 \mapsto \{a, d\}$$

Definition: Isolated Vertex

Let M be a multigraph and let $u \in V(M)$. To say that u is an *isolated* vertex means that it is not and endpoint for any edge in $E(M)$:

$$\forall\, e \in E(M), u \notin \mathscr{E}(e)$$

In the above example, e is an isolated vertex.

Definition: Parallel Edges

Let M be a multigraph and let $e, f \in E(M)$. To say that e and f are *parallel* edges means that $\mathscr{E}$ associates e and f with the same two endpoints:

$$\mathscr{E}(e) = \mathscr{E}(f)$$

In the above example, e_4 and e_5 are parallel edges.

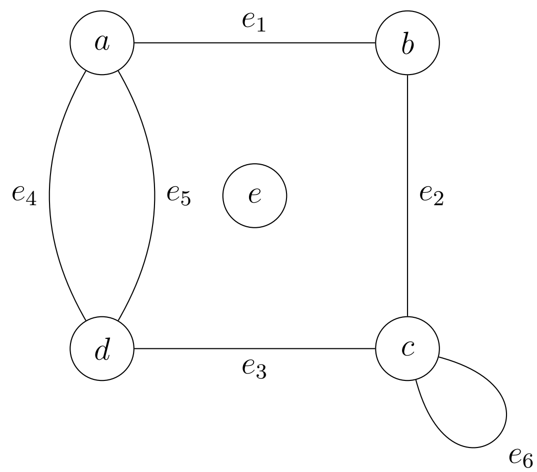
Definition: Pseudograph

A *pseudograph* P is a multigraph such that each edge is associated with either a one-element or a two-element subset of $V(P)$:

$$\mathscr{E} : E(P) \to \mathcal{P}_1(V(P)) \cup \mathcal{P}_2(V(P))$$

The single endpoint case is referred to as a *loop* edge.

Example



$$V = V(M) = \{a, b, c, d, e\}$$
$$E = E(M) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$e_1 \mapsto \{a, b\}$$
$$e_2 \mapsto \{b, c\}$$
$$e_3 \mapsto \{c, d\}$$
$$e_4 \mapsto \{a, d\}$$
$$e_5 \mapsto \{a, d\}$$
$$e_6 \mapsto \{c\}$$

In the above example, e_6 is a loop edge.

Definition: Adjacent Vertices

Let M be a multigraph or pseudograph and let $u, v \in V(M)$. To say that u and v are *adjacent* vertices (*neighbors*) means that they are the endpoints of some edge $e \in E(M)$:

$$\exists\, e \in E(M), \mathscr{E}(e) = \{u, v\}$$

The edge e is said to *join* u and v. Furthermore, the edge e is said to be *incident* to u and v.

Note that in the pseudograph case, a vertex $v \in V(M)$ is adjacent to itself when:

$$\exists\, e \in E(M), \mathscr{E}(e) = \{v, v\} = \{v\}$$

Definition: Adjacent Edges

Let M be a multigraph or pseudograph and let $e, f \in E(M)$. To say that e and f are *adjacent* edges means that they share an endpoint:

$$\exists\, v \in V(M), \mathscr{E}(e) \cap \mathscr{E}(f) = \{v\}$$

or

$$|\mathscr{E}(e) \cap \mathscr{E}(f)| = 1$$