Sequences

Note that R^n and C^n can be viewed as function spaces where:

$$R^n = \{f : \{1, \ldots, n\} \to \mathbb{R}\}$$

$$C^n = \{f : \{1, \ldots, n\} \to \mathbb{C}\}$$

As $n \to \infty$ we get the vector space of all real/complex sequences with component-wise vector addition and scalar multiplication:

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots)$$

$$\lambda(x_1, x_2, \ldots) = (\lambda x_1, \lambda x_2, \ldots)$$

Notation

(x_n) is the sequence whose n^{th} term is x_n.

$\{x_n \mid n \in \mathbb{N}\}$ is the set of all elements in the sequence (x_n).

Let $\mathcal{C}$ be the set of all complex sequences in $\mathbb{C}$. The following are subspaces of $\mathcal{C}$:

- Bounded sequences in $\mathcal{C}$.
- Converging sequences in $\mathcal{C}$.
- Sequences in $\mathcal{C}$ whose partial sums corresponding series converge.

Definition: ℓ^p

Let $p \in \mathbb{N}$:

$$\ell^p = \left\{ (z_n)_{n \in \mathbb{N}} \middle| \sum_{n=1}^{\infty} |z_n|^p < \infty \right\}$$

Definition: Convex Function

Let $f(x)$ be a real function defined on an open interval I. To say that f is *convex* (or *concave up*) on I means $\forall\, s, t \in I$ and $\forall\, \lambda \in \mathbb{R}$:

$$f\left((1-\lambda)s + \lambda t\right) \leq (1-\lambda) f(s) + \lambda t$$

Theorem: Young's Inequality

Let $a, b, p, q \in \mathbb{R}$ such that $a, b > 0$, $1 \leq p, q < \infty$, and $\dfrac{1}{p} + \dfrac{1}{q} = 1$:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof

Let $a = e^{\frac{s}{p}}$ and $b = e^{\frac{t}{q}}$ for some $s, t \in \mathbb{R}$.

$$ab = e^{\frac{s}{p}} e^{\frac{t}{q}} = e^{\frac{s}{p}+\frac{t}{q}}$$

But $f(x) = e^x$ is convex (concave up) everywhere and so:

$$ab \leq \frac{1}{p}e^s + \frac{1}{q}e^t = \frac{a^p}{p} + \frac{b^q}{q}$$

Theorem: Hölder's Inequality

Let $p, q \in \mathbb{R}$ such that $1 \leq p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and let $(x_n), (y_n) \in \ell^p$:

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |y_n|^q\right)^{\frac{1}{q}}$$

Proof

Let $A = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$ and $B = \left(\sum_{n=1}^{\infty} |y_n|^q\right)^{\frac{1}{q}}$.
Since $(x_n), (y_n) \in \ell^p, 0 \leq A, B < \infty$.
If either sequence is the zero sequence then trivial, so assume both are non-zero.
Let $a = \dfrac{|x_n|}{A}$ and $b = \dfrac{|y_n|}{B}$.
Since $A, B > 0$, and hence $a, b > 0$, so applying Young:

$$ab = \frac{|x_n y_n|}{AB} \leq \frac{1}{p}\left(\frac{|x_n|}{A}\right)^p + \frac{1}{q}\left(\frac{|y_n|}{B}\right)^q$$

Summing both sides:

$$\frac{1}{AB}\sum_{n=1}^{\infty} |x_n y_n| \leq \frac{1}{pA^p}\sum_{n=1}^{\infty} |x_n|^p + \frac{1}{qB^q}\sum_{n=1}^{\infty} |y_n|^q = \frac{1}{pA^p}A^p + \frac{1}{qB^q}B^q = \frac{1}{p} + \frac{1}{q} = 1$$

Therefore:

$$\sum_{n=1}^{\infty} |x_n y_n| \leq AB = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |y_n|^q\right)^{\frac{1}{q}}$$

Theorem: Minkowski's Inequality

Let $p \in \mathbb{R}$ such that $1 \leq p < \infty$ and let $(x_n), (y_n) \in \ell^p$:

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}$$

Proof

For $p = 1$, Minkowski reduces to the triangle inequality (trivial), so AWLOG $p > 1$.

$$\begin{aligned}\sum_{n=1}^{\infty} |x_n + y_n|^p &= \sum_{n=1}^{\infty} |x_n + y_n| \, |x_n + y_n|^{p-1} \\ &\leq \sum_{n=1}^{\infty} (|x_n| + |y_n|) \, |x_n + y_n|^{p-1} \\ &= \sum_{n=1}^{\infty} |x_n| \, |x_n + y_n|^{p-1} + \sum_{n=1}^{\infty} |y_n| \, |x_n + y_n|^{p-1}\end{aligned}$$

Since $p > 1$, $\exists \, q > 1$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, so applying Hölder:

$$\sum_{n=1}^{\infty} |x_n + y_n|^p \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{(p-1)q}\right)^{\frac{1}{q}}$$

But $(p-1)q = 1$, and so:

$$\sum_{n=1}^{\infty} |x_n + y_n|^p \leq \left(\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}\right) \left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{q}}$$

Finally, dividing both sides by $\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{q}}$ and noting that $1 - \dfrac{1}{q} = \dfrac{1}{p}$:

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}$$

Theorem

Let $\mathcal{C}$ be the vector space consisting of all complex sequences.

ℓ^p is a subspace of $\mathcal{C}$.

Proof

Clearly, $\ell^p \subset \mathcal{C}$.

Assume $(x_n) \in \ell^p$ and $\lambda \in \mathbb{C}$.

$$\left(\sum_{n=1}^{\infty} |\lambda x_n|^p\right)^{\frac{1}{p}} = |\lambda|^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} < \infty$$

$(\lambda x_n) \in \ell^p$

$\therefore \ell^p$ is closed under scalar multiplication.

Assume $(y_n) \in \ell^p$.
By Minkowski:

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}} < \infty$$

$(x_n + y_n) \in \ell^p$

$\therefore \ell^p$ is closed under vector addition.

$\therefore \ell^p$ is a subspace of $\mathcal{C}$.