Math-13 Sections 01 and 02

Homework #11 Solutions

Consider the function:

$$f(x) = x^3 - 4x^2 - 4x + 16$$

1. Using the rational roots theorem, completely factor $f(x)$.

$$
\begin{aligned}
p &= \pm 1, \pm 2, \pm 4, \pm 8, \pm 16 \\
q &= \pm 1
\end{aligned}
$$

$$c = \pm 1, \pm 2, \pm 4, \pm 8, \pm 16$$

$$
\begin{aligned}
&f(1) = 1^3 - 4(1)^2 - 4(1) + 16 = 1 - 4 - 4 + 16 \neq 0 \qquad \times \\
&f(-1) = (-1)^3 - 4(-1)^2 - 4(-1) + 16 = -1 - 4 + 4 + 16 \neq 0 \qquad \times \\
&f(2) = 2^3 - 4(2)^2 - 4(2) + 16 = 8 - 16 - 8 + 16 = 0 \qquad \checkmark
\end{aligned}
$$

$$
\begin{array}{rr}
 & x^2 - 2x \quad - 8 \\ \cline{2-2}
x - 2 \big) & x^3 - 4x^2 - 4x + 16 \\
 & - x^3 + 2x^2 \qquad\qquad \\ \cline{2-2}
 & - 2x^2 - 4x \qquad \\
 & 2x^2 - 4x \qquad \\ \cline{2-2}
 & - 8x + 16 \\
 & 8x - 16 \\ \cline{2-2}
 & 0
\end{array}
$$

$$f(x) = (x-2)(x^2 - 2x - 8) = (x-2)(x+2)(x-4)$$

2. What are the critical points of $f(x)$?

   The critical numbers are $x = \pm 2, 4$, so the critical points are $(\pm 2, 0)$ and $(4, 0)$.

3. What is the y-intercept of $f(x)$?

$$f(0) = 0^3 - 4(0)^2 - 4(0) = 16 = 0 + 0 + 0 + 16 = 16$$

   Therefore, the y-intercept is $(0, 16)$.

4. What are the critical points of $f'(x)$?

$$f'(x) = 3x^2 - 8x - 4$$

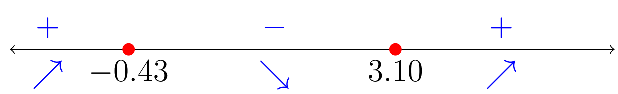
This does not factor nicely, so:

$$\begin{aligned} x &= \frac{8 \pm \sqrt{(-8)^2 - 4(3)(-4)}}{2(3)} \\ &= \frac{8 \pm \sqrt{64+48}}{6} \\ &= \frac{8 \pm \sqrt{112}}{6} \\ &= \frac{8 \pm 4\sqrt{7}}{6} \\ &= \frac{4 \pm 2\sqrt{7}}{3} \\ &\approx -0.43, 3.10 \end{aligned}$$

$$\begin{aligned} f\left(\frac{4-2\sqrt{7}}{3}\right) &\approx 16.90 \\ f\left(\frac{4+2\sqrt{7}}{3}\right) &\approx -5.05 \end{aligned}$$

Therefore, the first derivative critical points are $(-0.43, 16.90)$ and $(3.10, -5.05)$.

5. Using the first derivative test, determine the relative extrema of $f(x)$.



Using test points:

$$\begin{aligned} &f'(-1) > 0 \\ &f'(0) < 0 \\ &f'(4) > 0 \end{aligned}$$

Therefore $(-0.43, 16.90)$ is a relative maximum and $(3.10, -5.05)$ is a relative minimum.

6. Using the second derivative test, verify the relative extrema of $f(x)$.

$$f''(x) = 6x - 8$$

So $f''(-0.43) < 0$, so the function is concave down, verifying that $(-0.43, 16.90)$ is a relative maximum. Likewise, $f''(3.10) > 0$, so the function is concave up, verifying that $(3.10, -5.05)$ is a relative minimum.

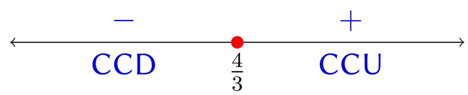
7. What are the critical points of $f''(x)$?

$$x = \frac{8}{6} = \frac{4}{3} \approx 1.33$$

$$f\left(\frac{4}{3}\right) \approx 5.93$$

Therefore, the second derivative has one critical point at $(1.33, 5.93)$.

8. Using the second derivative, prove that the critical point of $f''(x)$ is a point of inflection.



Using test points:

$$f''(0) < 0$$
$$f''(2) > 0$$

Therefore, the function is concave down to the left of the critical point and concave up to the right of the critical point. The change in concavity proves that the critical point is a point of inflection.

9. What is the end behavior of $f(x)$?

Using the leading term test, the end behavior is like x^3:

As $x \to -\infty$, $f(x) \to -\infty$.

As $x \to +\infty$, $f(x) \to +\infty$.

10. Sketch $f(x)$. For full credit, all intercepts, extrema, and points of inflection must be labeled with their coordinate values.