Walks

Definition: Walk

A $u - v$ *walk* W in a graph G is a finite sequence of vertices $w_i \in V(G)$ starting with $u = w_0$ and ending with $v = w_k$:

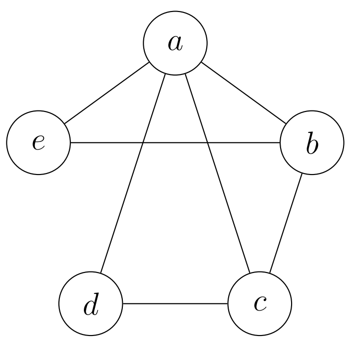
$$W = (u = w_0, w_1, \ldots, w_k = v)$$

such that $w_i w_{i+1} \in E(G)$ for $0 \leq i < k$.

To say that W is *open* means that $u \neq v$. To say that W is *closed* means that $u = v$. The *length* k of W is the number of edges traversed: $k = |W|$.

A *trivial* walk is a walk of zero length — i.e, a single vertex: $W = (u)$.

Example



$W_1 = (a, b, e, a, c)$ is open
$W_2 = (a, e, b, c, a)$ is closed

$|W_1| = |W_2| = 4$

Note that in the general case, vertices and edges are allowed to be repeated during a walk.

Definition: Special Walks

trail	An open walk with no repeating edges	(a, b, c, a, e)
path	A trail with no repeating vertices	(a, e, b, c)
circuit	A closed trail	(a, b, e, a, c, d, a)
cycle	A closed path	(a, e, b, c, a)

Notation: Concatenation

Let G be a graph and let $u, v, w \in V(G)$ such that $W_1 = (u = u_0, u_1, \ldots, u_k = v)$ is a $u - v$ walk of length k in G and $W_2 = (v = v_0, v_1, \ldots, v_\ell = w)$ is a $v - w$ walk of length ℓ in G with common endpoint v. The *concatenation* of these two walks W given by:

$$W = W_1 \cup W_2 = (u, \ldots, v, \ldots, w)$$

is a $u - w$ walk in G of length $k + \ell$.

Note that the concatenation of two paths is a walk, but not necessarily a path. In the above example, let $P_1 = (a, e, b)$ and $P_2 = (b, a, c, d)$:

$$P_1 \cup P_2 = (a, e, b, a, c, d)$$

which is not a path due to vertex a being traversed twice.

Theorem

Let G be a graph and let $u, v \in V(G)$:

$$G \text{ contains a } u - v \text{ walk of length } k \implies G \text{ contains a } u - v \text{ path of length } \ell \leq k.$$

Proof. Assume G contains a $u - v$ walk of length k.

Consider the set of all $u - v$ walks in G. Their lengths form a non-empty set of positive integers. By the well-ordering principle, there exists a $u - v$ walk P of minimal length $\ell \leq k$:

$$P = (u = w_0, \ldots, w_\ell = v)$$

Claim: P is a path.

ABC: P is not a path, and thus P has at least one repeating vertex.

Assume $w_i = w_j$ for some $0 \leq i < j \leq \ell$:

Case 1: $j = \ell$

$P' = (u = w_0, \ldots, w_i = v)$ is a $u - v$ walk in G of length $i < \ell$.

Case 2: $j < \ell$

$P' = (u = w_0, \ldots, w_i, w_{j+1}, \ldots, w_\ell = v)$ is a $u - v$ walk in G of length $\ell - (j - i) < \ell$

Both cases contradict the minimality of the length of P.

$\therefore$ P is a $u - v$ path in G of length $\ell \leq k$. ■

Definition: Connected

Let G be a graph and let $u, v \in V(G)$. To say that u and v are *connected* means that G contains a $u - v$ path.

Definition: Cycles

Let C be a cycle in a graph G:

- To say that C is a *k-cycle* means that $|C| = k$.
- To say that C is an *even* cycle means that $|C|$ is even.
- To say that C is an *odd* cycle means that $|C|$ is odd.

Note that in simple graphs, circuits and cycles must have length ≥ 3.