Direct Product

<u>**Definition**</u>

Let G and H be groups and define the binary operation on $G \times H$ by:

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$$

The binary algebraic structure $G \times H$ is called the *direct product* of G and H.

<u>**Theorem**</u>

Let G and H be groups. The direct product $G \times H$ is a group.

<u>Proof</u>

Assume $(g_1, h_1), (g_2, h_2), (g_3, h_3) \in G \times H$

$$\begin{aligned}
[(g_1, h_1)(g_2, h_2)](g_3, h_3) &= (g_1 g_2, h_1 h_2)(g_3 h_3) \\
&= ((g_1 g_2) g_3, (h_1 h_2) h_3) \\
&= (g_1 (g_2 g_3), h_1 (h_2 h_3)) \\
&= (g_1, h_1)(g_2 g_3, h_2 h_3) \\
&= (g_1, h_1)[(g_2, h_2)(g_3 h_3)]
\end{aligned}$$

$\therefore G \times H$ is associative under the operation.

$(e_G, e_H) \in G \times H$
Assume $(g, h) \in G \times H$
$(g, h)(e_G, e_H) = (g e_G, h e_H) = (g, h)$
$(e_G, e_H)(g, h) = (e_G g, e_H h) = (g, h)$
$\therefore G \times H$ has identity (e_G, e_H).

Assume $(g, h) \in G \times H$
$(g^{-1}, h^{-1}) \in G \times H$
$(g, h)(g^{-1}, h^{-1}) = (g g^{-1}, h h^{-1}) = (e_G, e_H)$
$(g^{-1}, h^{-1})(g, h) = (g^{-1} g, h^{-1} h) = (e_G, e_H)$
$\therefore G \times H$ is closed under inverses.

$\therefore G \times H$ is a group.

This definition can be expanded to multiple groups:

<u>**Definition**</u>

Let $G_1, \ldots G_n$ be a finite collection of groups and define the binary operation on $G_1 \times \cdots \times G_n$ as component-wise multiplication. The binary algebraic structure of $G_1 \times \cdots \times G_n$, denoted $\prod_{k=1}^{n} G_k$, is called the *direct product* of the G_k.

When the groups in question are additive/abelian, then the binary algebraic structure is called a *direct sum* and is defined by:

$$\bigoplus_{k=1}^{n} G_i = G_1 \oplus \cdots \oplus G_n$$

Theorem

Let $G_1, \ldots, G_n$ be a collection of groups. The direct product $\prod_{k=1}^{n} G_k$ is a group.

Proof

Assume $g, h, i \in \prod_{k=1}^{n} G_k$
Assume $1 \leq k \leq n$
$g_k, h_k, i_k \in G_k$
$(g_k h_k) i_k = g_k (h_k i_k)$
$\therefore \prod_{k=1}^{n} G_k$ is associative under the operation.

$e_k \in G_k$
$g_k e_k = e_k g_k = g_k$
$\therefore \prod_{k=1}^{n} G_k$ has identity $(e_1, \ldots, e_n)$.

$g_k^{-1} \in G_k$
$g_k g_k^{-1} = g_k^{-1} g_k = e_k$
$\therefore \prod_{k=1}^{n} G_k$ is closed under inverses.

$\therefore \prod_{k=1}^{n} G_k$ is a group.

When the number of groups is infinite (countable or not):

Definition

Let $\{G_i \mid i \in I\}$ be a family of groups and define:

$$\prod_{i \in I} G_i = \{g : I \to \bigcup_{i \in I} G_i \mid g(i) = g_i \in G_i\}$$

and define the binary operation on $\prod_{i \in I} G_i$ by:

$$(gh)(i) = g(i)h(i)$$

The binary algebraic structure $\prod_{i \in I} G_i$ is called the *direct product* of $\{G_i \mid i \in I\}$.

Theorem

Let $\{G_i \mid i \in I\}$ be a family of groups. The direct product $\prod_{i \in I} G_i$ is a group.

Proof

Assume $g, h \in \prod_{i \in I} G_i$
$(gh)(i) = g(i)h(i) = g_i h_i$
But $g_i, h_i \in G_i$
So $(gh)(i) = g_i h_i \in G_i$
so $gh \in \prod_{i \in I} G_i$
$\therefore \prod_{i \in I} G_i$ is closed under the operation.

Assume $g, h, k \in \prod_{i \in I} G_i$
Assume $i \in I$
$[(gh)k](i) = [(gh)(i)]k(i) = [g(i)h(i)]k(i) = g(i)[h(i)k(i)] = g(i)[(hk)(i)] = [g(hk)](i)$
$\therefore \prod_{i \in I} G_i$ is associative under the operation.

$e_i \in G_i$
$\exists\, e \in \prod_{i \in I} G_i, e(i) = e_i$
$(ge)(i) = g(i)e(i) = g_i e_i = g_i = g(i)$
$(ee)(i) = e(i)g(i) = e_i g_i = g_i = g(i)$
$\therefore \prod_{i \in I} G_i$ has identity e.

$g(i) = g_i \in G_i$
$g_i^{-1} \in G_i$
$\exists\, g^{-1} \in \prod_{i \in I} G_i, g^{-1}(i) = g_i^{-1}$
$(gg^{-1})(i) = g(i)g^{-1}(i) = g_i g_i^{-1} = e_i = e(i)$
$(g^{-1}g)(i) = g^{-1}(i)g(i) = g_i^{-1} g_i = e_i = e(i)$
$\therefore \prod_{i \in I} G_i$ is closed under inverses.

$\therefore \prod_{i \in I} G_i$ is a group.

Definition

Let $\prod_{i \in I} G_i$ be a direct product of groups and $\forall\, k \in I$ define the map $\pi_k : \prod_{i \in I} G_i \to G_k$ by:

$$\pi_k(g) = g(k)$$

The π_k are called the *canonical projections* of the direct product.

Theorem

Let $\prod_{i \in I} G_i$ be a direct product of groups and π_k be the canonical projections for the direct product. π_k is an onto homomorphism.

Proof

Assume $g, h \in \prod_{i \in I} G_i$
Assume $k \in I$
$\pi_k(gh) = (gh)(k) = g(k)h(k) = \pi_k(g)\pi_k(h)$
$\therefore \pi_k$ is a homomorphism.

Assume $g_k \in G_k$
$\exists\, g \in \prod_{i\in I} G_i, g(k) = g_k$
$\pi_k(g) = g(k) = g_k$
$\therefore \pi_k$ is a onto.

Theorem

Let $\{\phi_i : G_i \to H_i \mid i \in I\}$ be a family of a group homomorphisms and let $\phi = \prod_{i\in I} \phi_i$ be the map $\phi : \prod_{i\in I} G_i \to \prod_{i\in I} H_i$ defined by $\phi(g) = \prod_{i\in I} \phi_i(g(i))$.

1). ϕ is a homomorphism of groups

2). $\ker(\phi) = \prod_{i\in I} \ker(\phi_i)$

3). $\phi\left[\prod_{i\in I} G_i\right] = \prod_{i\in I} \phi_i[G_i]$

4). ϕ is one-to-one iff all of the ϕ_i are one-to-one

5). ϕ is onto iff all of the ϕ_i are onto

Proof

1). Assume $a, b \in \prod_{i\in I} G_i$

$$\begin{aligned} \phi(ab) &= \prod_{i\in I} \phi_i((ab)(i)) \\ &= \prod_{i\in I} \phi_i(a(i)b(i)) \\ &= \prod_{i\in I} \phi_i(a(i))\phi_i(b(i)) \\ &= \prod_{i\in I} \phi_i(a(i)) \prod_{i\in I} \phi_i(b(i)) \\ &= \phi(a)\phi(b) \end{aligned}$$

$\therefore \phi$ is a homomorphism.

2). Let $e \in \prod_{i\in I} H_i, \forall\, i \in I, e(i) = e_H$
e is the identity element for $\prod_{i\in I} H_i$

$$\begin{aligned} a \in \ker(\phi) &\iff \phi(a) = e \\ &\iff \forall\, i \in I, \phi(a(i)) = e(i) = e_H \\ &\iff \forall\, i \in I, a(i) \in \ker(\phi_i) \\ &\iff a \in \prod_{i\in I} \ker(\phi_i) \end{aligned}$$

3).

$$
\begin{aligned}
b \in \phi\left[\prod_{i\in I} G_i\right] &\iff \exists\, a \in \prod_{i\in I} G_i, \phi(a) = b \\
&\iff \forall\, i \in I, \phi_i(a(i)) = b(i) \\
&\iff \forall\, i \in I, b(i) \in \phi_i[G_i] \\
&\iff b \in \prod_{i\in I} \phi_i[G_i]
\end{aligned}
$$

4).

$$
\begin{aligned}
\phi \text{ is one-to-one} &\iff \phi(a) = \phi(b) \implies a = b \\
&\iff \forall\, i \in I, \phi_i(a(i)) = \phi_i(b(i)) \implies a(i) = b(i) \\
&\iff \forall\, i \in I, \phi_i \text{ is one-to-one}
\end{aligned}
$$

5).

$$
\begin{aligned}
\phi \text{ is onto} &\iff b \in \prod_{i\in I} H_i \implies \exists\, a \in \prod_{i\in I} G_i, \phi(a) = b \\
&\iff \forall\, i \in I, b(i) \in H_i \implies \exists\, a(i) \in G_i, \phi(a(i)) = b(i) \\
&\iff \forall\, i \in I, \phi_i \text{ is onto}
\end{aligned}
$$

<u>Theorem</u>

Let $\{G_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ be families of groups such that $\forall\, i \in I, N_i \triangleleft G_i$:

$$\prod_{i\in I} N_i \triangleleft \prod_{i\in I} G_i$$

<u>Proof</u>

Assume $n \in \prod_{i\in I} N_i$
Assume $g \in \prod_{i\in I} G_i$
Assume $i \in I$
$n(i) \in N_i$
$g(i) \in G_i$
$N_i \triangleleft G_i$
$g(i)(n(i))g^{-1}(i) \in N_i$
$gng^{-1} \in \prod_{i\in I} N_i$
$\therefore \prod_{i\in I} N_i \triangleleft \prod_{i\in I} G_i$

Theorem

Let $\{G_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ be families of groups such that $\forall\, i \in I, N_i \triangleleft G_i$:

$$\prod_{i \in I} G_i / \prod_{i \in I} N_i \simeq \prod_{x \in I} G_i / N_i$$

Proof

Assume $i \in I$
Let $\pi_i : G_i \to G_i/N_i$ be the (onto) canonical homomorphism
$\ker(\pi_i) = N_i$
Let $\pi : \prod_{i \in I} G_i \to \prod_{i \in I} G_i/N_i$ be defined by $\pi(g) = \prod_{i \in I} \pi_i(g(i)$

By the previous theorem:

1). π is a (onto) homomorphism

2). $\ker(\pi) = \prod_{i \in I} \ker(\pi_i) = \prod_{i \in I} N_i$

Thus, we have the following setup per the FIT:

$$\begin{tikzcd}
\prod_{i \in I} G_i \arrow[r, "\pi"] \arrow[d] & \prod_{i \in I} G_i/N_i \\
\prod_{i \in I} G_i / \prod_{i \in I} N_i \arrow[ur, "\phi"'] &
\end{tikzcd}$$

Thus, there exists an isomorphism $\phi : \prod_{i \in I} G_i / \prod_{i \in I} N_i \to \prod_{i \in I} G_i/N_i$

$\therefore \prod_{i \in I} G_i / \prod_{i \in I} N_i \simeq \prod_{i \in I} G_i/N_i$.