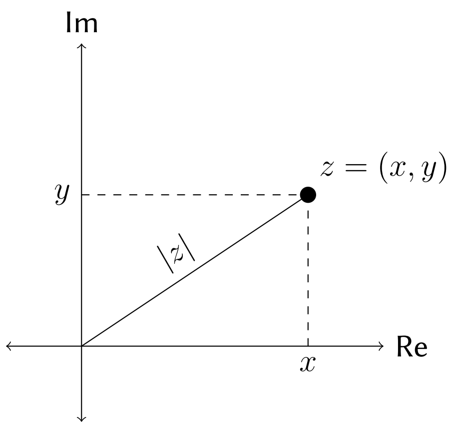
Modulus

Definition

Let $z = x + iy \in \mathbb{C}$. The *modulus* of z is given by:

$$|z| = \sqrt{x^2 + y^2}$$



$$|z| = \sqrt{x^2 + y^2} \geq 0$$

Note that $|z|$ measures the distance from z to the origin. Thus, if $|z_1| < |z_2|$ then z_1 is closer to the origin than z_2.

Theorem

$\forall z \in \mathbb{C}$:

1). $|Re(z)| \leq |z|$

2). $|Im(z)| \leq |z|$

3). $|z| \leq |Re(z)| + |Im(z)|$

Proof

Assume $z = x + iy \in \mathbb{C}$
$x = Re(z)$
$y = Im(z)$

$|z|^2 = x^2 + y^2 = |x|^2 + |y|^2 = [Re(z)]^2 + [Im(z)]^2$

$|z|^2 \leq |Re(z)|^2$
$\therefore |Re(z)| \leq |z|$

$|z|^2 \leq |Im(z)|^2$
$\therefore |Im(z)| \leq |z|$

$|z| = |x + iy| \leq |x| + |iy| = |x| + |i|\,|y| = |x| + 1 \cdot |y| = |x| + |y|$
$\therefore |z| \leq |Re(z)| + |Im(z)|$

Properties

1). $|-z| = |z|$

2). $|\bar{z}| = |z|$

3). $z\bar{z} = |z|^2$

4). $|z_1 z_2| = |z_1|\,|z_2|$

5). $\left|\frac{1}{z}\right| = \frac{1}{|z|}$

6). $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Proof

1). $|-z| = |-x - iy| = \sqrt{(-x)^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|$

2). $|\bar{z}| = |x - iy| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|$

3). $z\bar{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 - (-1)y^2 = x^2 + y^2 = |z|^2$

4). $|z_1 z_2|^2 = (z_1 z_2)\overline{(z_1 z_2)} = z_1 z_2 \bar{z}_1 \bar{z}_2 = (z_1 \bar{z}_1)(z_2 \bar{z}_2) = |z_1|^2\,|z_2|^2$
$\therefore |z_1 z_2| = |z_1|\,|z_2|$

5). $\left|\frac{1}{z}\right|^2 = \left(\frac{1}{z}\right)\overline{\left(\frac{1}{z}\right)} = \left(\frac{1}{z}\right)\left(\frac{1}{\bar{z}}\right) = \frac{1}{z\bar{z}} = \frac{1}{|z|^2}$
$\therefore \left|\frac{1}{z}\right| = \frac{1}{|z|}$

6). $\left|\frac{z_1}{z_2}\right| = \left|z_1\frac{1}{z_2}\right| = |z_1|\left|\frac{1}{z_2}\right| = |z_1|\,\frac{1}{|z_2|} = \frac{|z_1|}{|z_2|}$

Example

$$\begin{aligned}
\left|\frac{i(1-i)^3}{(\sqrt{2}+2i)^4}\right| &= \left|\frac{i(1-3i-3+i)}{4+16\sqrt{2}i-48-32\sqrt{2}i+16}\right| \\
&= \left|\frac{i(-2-2i)}{-28-16\sqrt{2}i}\right| \\
&= \left|\frac{i(1+i)}{14+8\sqrt{2}i}\right| \\
&= \left|\frac{-1+i}{14+8\sqrt{2}i}\right| \\
&= \left|\left(\frac{-1+i}{14+8\sqrt{2}i}\right)\left(\frac{14-8\sqrt{2}i}{14-8\sqrt{2}i}\right)\right| \\
&= \left|\frac{(-1+i)(14-8\sqrt{2}i}{196+128}\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left|\frac{-14+8\sqrt{2}+i(14+8\sqrt{2})}{324}\right| \\
&= \left|\frac{-7+4\sqrt{2}+i(7+4\sqrt{2})}{162}\right| \\
&= \frac{1}{162}\sqrt{(-7+4\sqrt{2})^2+(7+4\sqrt{2})^2} \\
&= \frac{1}{162}\sqrt{(49+32-56\sqrt{2})+(49+32+56\sqrt{2})} \\
&= \frac{1}{162}\sqrt{162} \\
&= \frac{9\sqrt{2}}{162} \\
&= \frac{\sqrt{2}}{18}
\end{aligned}
$$

Theorem

$\forall n \in \mathbb{N}$:

$$|z^n| = |z|^n$$

Proof

(by induction)

Base Case: $n = 1$

$$\left|z^1\right| = |z| = |z|^1$$

Assume $|z^n| = |z|^n$

Consider $|z^{n+1}|$:

$$\left|z^{n+1}\right| = |z^n z| = |z^n|\,|z| = |z|^n\,|z| = |z|^{n+1}$$

Theorem

$\forall z, a \in C$:

$$|z+a|^2 = |z|^2 + |a|^2 + 2Re(\bar{a}z)$$

Proof

$$
\begin{aligned}
|z+a|^2 &= (z+a)\overline{(z+a)} \\
&= (z+a)(\bar{z}+\bar{a}) \\
&= z\bar{z} + a\bar{a} + z\bar{a} + \bar{z}a
\end{aligned}
$$

$$
\begin{aligned}
&= |z|^2 + |a|^2 + z\bar{a} + \overline{z\bar{a}} \\
&= |z|^2 + |a|^2 + 2Re(\bar{a}z)
\end{aligned}
$$

Corollary

$\forall z \in \mathbb{C}, a \in \mathbb{R}$:

$$|z+a|^2 = |z|^2 + a^2 + 2aRe(z)$$