Nilpotence

Definition

Let R be a ring and $a \in R$. To say that a is *nilpotent* in R means: $\exists\, n \in \mathbb{Z}^+, a^n = 0$.

Theorem

Let R be a commutative ring and $N = \{a \in R \mid a \text{ is nilpotent in } R\}$.
N is closed under addition.

Proof

Assume $a, b \in N$
$\exists\, n \in \mathbb{Z}^+, a^n = 0$
$\exists\, m \in \mathbb{Z}^+, b^m = 0$
$(a+b)^{n+m} = \sum_{k=0}^{n+m} \binom{n+m}{k} \cdot a^{n+m-k} b^k$
For $0 \leq k \leq m, a^{n+m-k} = a^n a^r = 0a^r = 0$
For $m \leq k \leq n+m, b^k = b^m b^s = 0b^s = 0$
$(a+b)^{n+m} = 0$
$a + b \in N$

$\therefore N$ is closed under addition.

Theorem

Let $\phi : R \to R'$ be a homomorphism of rings:

a nilpotent in $R \implies \phi(a)$ nilpotent in R'

Proof

Assume a is nilpotent in R
$\exists\, n \in \mathbb{Z}^+, a^n = 0$
$\phi(a^n) = \phi(0) = 0'$
$\phi(a^n) = \phi(a)^n$
$\phi(a)^n = 0'$

$\therefore \phi(a)$ is nilpotent in R'.

Theorem

Let R be a ring:

R has no non-zero nilpotent elements $\iff (x^2 = 0 \iff x = 0)$.

Proof

$\Longrightarrow$ Assume R has no non-zero nilpotent elements

$\Longrightarrow$ Assume $x \neq 0$

$x^2 \neq 0$, otherwise x would be nilpotent (contradiction)

$\Longleftarrow$ Assume $x = 0$

$x^2 = 0^2 = (0)(0) = 0$

$\therefore x^2 = 0 \iff x = 0$

$\Longleftarrow$ Assume $x^2 = 0 \iff x = 0$

ABC: $x \neq 0$ is nilpotent in R
Let $n \in \mathbb{Z}^+$ be the smallest n such that $x^n = 0$

Case 1: n even

$\left(x^{\frac{n}{2}}\right)^2 = 0$
But by minimality of n, $x^{\frac{n}{2}} \neq 0$
Contradiction!

Case 2: n odd

Case A: $n = 1$

$x = 0$
Contradiction!

Case B: $n > 1$

$\left(x^{\frac{n+1}{2}}\right)^2 = 0$
But by minimality of n, $x^{\frac{n+1}{2}} \neq 0$
Contradiction!

So x is not nilpotent in R
$\therefore$ R contains no non-zero nilpotent elements.