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Math 229
Midterm Exam

It was previously proven that $\operatorname{rank} A = \operatorname{rank} A^T$. Note that this is the same as saying the dimension of the column space of A equals the dimension of the row space of A.

1). Let $A \in M_n$:

a). Prove that if A has a $k \times k$ submatrix B such that $\det(B) \neq 0$ then $\operatorname{rank}(A) \geq k$

Assume A has a $k \times k$ submatrix B such that $\det(B) \neq 0$. By the IVT, B is invertible and so its k columns are linearly independent. This means that the dimension of the column space of B is k, which is also the dimension of the row space of B. Now expand the $k \times 1$ column vectors of B into the original $n \times 1$ column vectors of A and construct a new $n \times k$ matrix C consisting of these k column vectors from A. The dimension of the row space of C is unchanged, and so the dimension of the column space of C is also unchanged, meaning that the k columns of C are still linearly independent and so $\operatorname{rank} C = k$. Finally, extend C back to the full matrix A. A has at least k linearly independent column vectors (from C).

Therefore, $\operatorname{rank}(A) \geq k$.

b). Let $n \geq 2$. Prove that:

$$\operatorname{rank}(\operatorname{adj}(A)) = \begin{cases} n, & \operatorname{rank}(A) = n \\ 1, & \operatorname{rank}(A) = n-1 \\ 0, & \operatorname{rank}(A) \leq n-2 \end{cases}$$

By previously-proved theorem:

$$A(\operatorname{adj}(A)) = (\det(A))I$$

Now, consider the three cases:

Case 1: $\operatorname{rank}(A) = n$

A has n linearly independent columns and so, by the IVT, A is invertible and:

$$\operatorname{adj}(A) = (\det(A))A^{-1}$$

But, since A is invertible, $\det(A) \neq 0$ and we have:

$$(\det(A))A^{-1})\left(\frac{1}{\det(A)}A\right) = I$$

Thus $(\det(A))A^{-1}$ is invertible, meaning that $\operatorname{adj}(A)$ is also invertible. So the n columns of $\operatorname{adj}(A)$ are linearly independent.

Therefore $\operatorname{rank}(\operatorname{adj}(A)) = n$

Case 2: $\operatorname{rank}(A) = n - 1$

The n columns of A form a linearly dependent set, and by the IVT, A is not invertible and so $\det(A) = 0$. Thus we have:

$$A(\operatorname{adj}(A)) = 0$$

But from the definition of matrix multiplication, the columns of $\operatorname{adj}(A)$ are a subset of the null space of A, which has dimension $n - (n-1) = 1$. And so the dimension of the column space of $\operatorname{adj} A$ is either 0 or 1.

Since $\operatorname{rank}(A) = n - 1$, there exists an $n - 1$ subset of the columns of A that form a linearly independent set. AWLOG that removing the j^{th} column results in such a linearly independent set. Construct a new $n \times (n-1)$ matrix B from the $n - 1$ linearly independent columns of A. Note that the dimension of the column space of B is still $n - 1$, and so the dimension of the row space of B is also $n - 1$. Thus, there exists an $n - 1$ subset of the rows of B that also forms a linearly independent set. AWLOG that removing the i^{th} row of B forms such a linearly independent set. Construct a new $(n-1) \times (n-1)$ matrix C by dropping the i^{th} row of B. Now, the $n - 1$ rows of C are linearly independent and so the dimension of the row space of C is $n - 1$, and thus the dimension of the column space of C must also be $n - 1$. This means that the $n - 1$ columns of C are linearly independent and thus $\det(C) \neq 0$. But $C = A_{ij}$, and thus A has at least one non-zero minor.

Therefore, $\operatorname{adj}(A)$ is not the zero matrix and $\operatorname{rank}(\operatorname{adj}(A)) = 1$.

Case 3: $\operatorname{rank}(A) \leq n - 2$

Since the dimension of the column space of A is $n - 2$, any $n - 1$ subset of the columns of A forms a linearly dependent set. Discard the j^{th} column of A and form a new $n \times (n-1)$ matrix B from the remaining columns of A. Since the dimension of the column space of B is still $n - 2$, the dimension of the row space must also be $n - 2$. Thus, any $n - 1$ subset of the rows of B form a linearly dependent set. Discard the i^{th} row of B and form a new $(n-1) \times (n-1)$ matrix C from the remaining rows of B. Since the dimension of the row space of C is still $n - 2$, the rank of the column space of C is also $n - 2$ and thus the $n - 1$ columns of C are linearly dependent and $\det(C) = 0$. But $C = A_{ij}$, and thus A has no non-zero minors.

Therefore, $\operatorname{rank}(\operatorname{adj}(A)) = 0$

2). Given a real n-vector $\vec{a}^T = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$ with $n \geq 2$, defined the $n \times n$ matrix:

$$M(\vec{a}) = \begin{bmatrix} a_i - a_j \end{bmatrix}$$

a). Find a necessary and sufficient condition on the vector $\vec{a}$ such that $M(\vec{a}) = 0$.

Assume $M(\vec{a}) = 0$. By selecting any row, say the i^{th} row, of $M(\vec{a})$, we get a system of n equations:

$$a_i - a_j = 0$$

for all $1 \leq j \leq n$. Thus, for all $1 \leq j \leq n$, $a_i = a_j$, and thus by transitivity, all of the components of $\vec{a}$ must be equal.

Likewise, by selecting any column, say the j^{th} column, of $M(\vec{a})$, we get a system of n equations:

$$a_i - a_j = 0$$

for all $1 \leq i \leq n$. Thus, for all $1 \leq i \leq n$, $a_i = a_j$, and thus by transitivity, all of the components of $\vec{a}$ must be equal.

Clearly, if all of the components of $\vec{a}$ equal, then the difference between any two components will always be 0.

$\therefore M(\vec{a}) = 0 \iff \vec{a} = \begin{bmatrix} a & a & \dots & a \end{bmatrix}$ where $a \in \mathbb{R}$.

b). Write $M(\vec{a}) = AB$ for some $n \times 2$ matrix A and some $2 \times n$ matrix B.

Let $A = \begin{bmatrix} \vec{a} & 1 \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ a_2 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -\vec{a}^T \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -a_1 & -a_2 & \cdots & -a_n \end{bmatrix}$

$[AB]_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} = a_i \cdot 1 + 1 \cdot (-a_j) = a_i - a_j$

c). Compute $\mathrm{Sp}(M(\vec{a}))$ in terms of $\vec{a}$.

By a previously proven theorem: $p_{AB}(t) = t^{n-2} p_{BA}(t)$, and so:

$$BA = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -a_1 & -a_2 & \cdots & -a_n \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ a_2 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{bmatrix} = \begin{bmatrix} \sum a_k & n \\ -\sum (a_k^2) & -\sum a_k \end{bmatrix}$$

The characteristic polynomial of BA is calculated as follows:

$$\begin{aligned} p_{BA}(t) &= (t - \sum a_k)(t + \sum a_k) + n \sum (a_k^2) \\ &= t^2 - (\sum a_k)^2 + n \sum (a_k^2) \\ &= t^2 - \left((\sum a_k)^2 - n \sum (a_k^2) \right) \end{aligned}$$

and therefore:

$$\begin{aligned}
\lambda_1 &= \sqrt{(\sum a_k)^2 - n \sum (a_k^2)} \\
\lambda_2 &= -\sqrt{(\sum a_k)^2 - n \sum (a_k^2)} \\
\lambda_3 - \lambda_n &= 0
\end{aligned}$$

$$\mathrm{Sp}(M(\vec{a})) = \left\{ 0^{(n-2)}, \pm\sqrt{(\sum a_k)^2 - n \sum (a_k^2)} \right\}$$

3). Let $A \in M_n$

a). Prove that AA^* and A^*A are unitary similar.

Let $A = UDV$ be the SVD for A, where U, V are unitary and D is diagonal containing the singular values of A.
$A^* = (UDV)^* = V^*D^*U^*$
$AA^* = (UDV)(V^*D^*U^*) = UDD^*U^*$
But D and D^* are both diagonal, so $DD^* = D^*D$ and:
$AA^* = UD^*DU^*$
Similarly, $A^*A = (V^*D^*U^*)(UDV) = V^*D^*DV$ and so $D^*D = VA^*AV^*$
Now, plugging the second equation into the first:
$AA^* = U(VA^*AV^*)U^* = (UV)A^*A(V^*U^*) = (UV)A^*A(UV)^*$
But the product of unitary matrices is also unitary, so UV is unitary

Therefore AA^* is unitary similar to A^*A.

b). Use (a) to prove that $\operatorname{rank}(AA^* - \lambda I) = \operatorname{rank}(A^*A - \lambda I)$ for any λ.

It was previously proven that the ranks of similar matrices are equal, so it suffices to show that $AA^* - \lambda I$ and $A^*A - \lambda I$ are similar.

Since AA^* and A^*A are unitary similar, there exists unitary U such that $AA^* = UA^*AU^*$, and so:

$$AA^* - \lambda I = UA^*AU^* - \lambda I = UA^*AU^* - \lambda UU^* = U(A^*A - \lambda I)U^*$$

and thus $AA^* - \lambda I$ is unitary similar to $A^*A - \lambda I$
But unitary similar matrices are similar.

$\therefore \operatorname{rank}(AA^* - \lambda I) = \operatorname{rank}(A^*A - \lambda I)$

4). Let $A, B \in M_3$

a). List all possible Jordan matrices (up to permutation) of A if $\sigma(A) = \{\lambda, \mu\}$.

$$J_{21} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \qquad J_{22} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{bmatrix} \qquad J_{23} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \qquad J_{24} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

b). List all possible Jordan matrices (up to permutation) of A if $\sigma(A) = \{\lambda\}$.

$$J_{11} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \qquad J_{12} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \qquad J_{13} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

c). Let $A, B \in M_3$ such that $g_A(\lambda) = g_B(\lambda)$ for all $\lambda \in \sigma(A) = \sigma(B)$. Prove: $A \sim B$.

There are three cases that need to be investigated:

i. $\sigma(A) = \{\lambda\}$ and thus $a_A(\lambda) = 3 \quad (J_{11}, J_{12}, J_{13})$

ii. $\sigma(A) = \{\lambda, \mu\}$ with $a_A(\lambda) = 1$ and $a_A(\mu) = 2 \quad (J_{21}, J_{22})$

iii. $\sigma(A) = \{\lambda, \mu\}$ with $a_A(\lambda) = 2$ and $a_A(\mu) = 1 \quad (J_{23}, J_{24})$

The goal is to show for each case that $J_A = J_B$, from which we can thus conclude that $A \sim B$.

Start with case 1. Since $g_A(\lambda) = \dim \mathrm{Null}(A - \lambda I)$:

$$\begin{aligned} r_0 &= \mathrm{rank}(A - \lambda I)^0 = \mathrm{rank}(I_3) = 3 \\ r_1 &= \mathrm{rank}(A - \lambda I) = 3 - g_A(\lambda) \\ r_2 &= \mathrm{rank}(A - \lambda I)^2 = ? \\ r_3 &= n - a_A(\lambda) = 3 - 3 = 0 \\ r_4 &= n - a_A(\lambda) = 3 - 3 = 0 \end{aligned}$$

Since $J_A \sim A$, we can compute the cases for r_2 from the possible Jordan forms:

$$(J_{11} - \lambda I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(J_{12} - \lambda I)^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(J_{13} - \lambda I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And so:

$$\begin{aligned} r_0 &= 3 \\ r_1 &= 3 - g_A(\lambda) \\ r_2 &= 0 \text{ or } 1 \\ r_3 &= 0 \\ r_4 &= 0 \end{aligned}$$

Assume $g_A(\lambda) = 1$, and so $r_1 = 2$:

$$b_1 = r_0 - 2r_1 + r_2 = 3 - 2(2) + r_2 = -1 + r_2$$

This forces $r_2 = 1$ and so:

$$\begin{aligned} b_1 &= 0 \\ b_2 &= r_1 - 2r_2 + r_3 = 2 - 2(1) + 0 = 0 \\ b_3 &= r_2 - 2r_3 + r_4 = 1 - 2(0) + 0 = 1 \end{aligned}$$

And thus $J_A = J_B = J_{13}$

Assume $g_A(\lambda) = 2$, and so $r_1 = 1$:

$$b_2 = r_1 - 2r_2 + r_3 = 1 - 2r_2 + 0 = 1 - 2r_2$$

This forces $r_2 = 0$ and so:

$$\begin{aligned} b_1 &= r_0 - 2r_1 + r_2 = 3 - 2(1) + 0 = 1 \\ b_2 &= 1 \\ b_3 &= r_2 - 2r_3 + r_4 = 0 - 2(0) + 0 = 0 \end{aligned}$$

And thus $J_A = J_B = J_{12}$

Assume $g_A(\lambda) = 3$, and so $r_1 = 0$:

$$b_1 = r_0 - 2r_1 + r_2 = 3 - 2(0) + r_2 = 3 + r_2$$

This forces $r_2 = 0$ and so:

$$\begin{aligned} b_1 &= 3 \\ b_2 &= r_1 - 2r_2 + r_3 = 0 - 2(0) + 0 = 0 \\ b_3 &= r_2 - 2r_3 + r_4 = 0 - 2(0) + 0 = 0 \end{aligned}$$

And thus $J_A = J_B = J_{11}$

So in all of these cases, $J_A = J_B$ and therefore $A \sim B$.

Next, consider case 2. Since $a_A(\lambda) = 1$, the only possible indices for λ are:

$$\begin{aligned} b_1 &= 1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

So we only need to check μ.

$$\begin{aligned} r_0 &= \operatorname{rank}(A-\mu I)^0 = \operatorname{rank}(I_3) = 3 \\ r_1 &= 3 - g_A(\mu) \\ r_2 &= ? \\ r_3 &= n - a_A(\mu) = 3-2 = 1 \\ r_4 &= n - a_A(\mu) = 3-2 = 1 \end{aligned}$$

Once again, we need to calculate r_2 for the possible cases:

$$(J_{21}-\mu I)^2 = \begin{bmatrix} \lambda-\mu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} (\lambda-\mu)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(J_{22}-\mu I)^2 = \begin{bmatrix} \lambda-\mu & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} (\lambda-\mu)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And so in both cases $r_2 = 1$. Also note that since $a_A(\mu) = 2$ and $g_A(\mu) \le a_A(\mu)$, this forces $g_A(\mu) = 0$ or 1. We now have:

$$\begin{aligned} r_0 &= 3 \\ r_1 &= 3 - g_A(\mu) \\ r_2 &= 1 \\ r_3 &= 1 \\ r_4 &= 1 \end{aligned}$$

Assume $g_A(\mu) = 1$ and so $r_1 = 2$:

$$\begin{aligned} b_1 &= r_0 - 2r_1 + r_2 = 3 - 2(2) + 1 = 0 \\ b_2 &= r_1 - 2r_2 + r_3 = 2 - 2(1) + 1 = 1 \\ b_3 &= r_2 - 2r_3 + r_4 = 1 - 2(1) + 1 = 0 \end{aligned}$$

And thus $J_A = J_B = J_{22}$

Assume $g_A(\mu) = 2$ and so $r_1 = 1$:

$$\begin{aligned} b_1 &= r_0 - 2r_1 + r_2 = 3 - 2(1) + 1 = 2 \\ b_2 &= r_1 - 2r_2 + r_3 = 1 - 2(1) + 1 = 0 \\ b_3 &= r_2 - 2r_3 + r_4 = 1 - 2(1) + 1 = 0 \end{aligned}$$

And thus $J_A = J_B = J_{21}$

So in all of these cases, $J_A = J_B$ and therefore $A \sim B$.

Next consider case 3. By symmetry, this is the same as case 2, except $J_A = J_B = J_{23}$ or J_{24}

In summary, each choice of $g_A(\lambda)$ forces a particular Jordan matrix, and since $g_A(\lambda) = g_B(\lambda)$, the resulting Jordan matrices will be the same.

Therefore $A \sim B$.