$\mathbf{L^2}$ Completeness

Theorem

$\forall f, g \in L^2$, let $d(f,g) = \|f - g\|$. This is a proper metric, and thus L^2 is a metric space.

Proof

M1: Assume $f, g \in L^2$

$$\|f - g\| = 0 \iff f - g = 0 \;\; a.e. \iff f = g \;\; a.e.$$

M2: Assume $f, g \in L^2$

$$\|f - g\| = \left(\int |f - g|^2\right)^{\frac{1}{2}} = \left(\int |g - f|^2\right)^{\frac{1}{2}} = \|g - f\|$$

M3: Assume $f, g, h \in L^2$

$$\|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\|$$

Theorem

L^2 is complete in its metric.

Proof

Assume (f_n) is Cauchy in L^2
Let (f_{n_k}) be the subsequence such that $\|f_{n_{k+1}} - f_{n_k}\| \leq 2^{-k}$
Let $f = f_{n_1} + \sum_{k=0}^{\infty}(f_{n_{k+1}} - f_{n_k})$
Let $g = |f_{n_1}| + \sum_{k=0}^{\infty}\left|f_{n_{k+1}} - f_{n_k}\right|$
$0 \leq |f| \leq g$

Consider the partial sum:

$$S_N(g) = |f_{n_1}| + \sum_{k=0}^{N}\left|f_{n_{k+1}} - f_{n_k}\right|$$

So, by the triangle inequality:

$$\|S_N(g)\| \leq \|f_{n_1}\| + \sum_{k=0}^{N}\|f_{n_{k+1}} - f_{n_k}\| \leq \|f_{n_1}\| + \sum_{k=0}^{N} 2^{-k} < \infty$$

As $N \to \infty$, $S_N(g) \nearrow g$, and since $S_N(g), g \geq 0$, $|S_N(g)|^2 \nearrow |g|^2$.
Thus, by the MCT, $\int |S_N(g)|^2 \nearrow \int |g|^2$
and since $\|S_N(g)\|$ converges, $\int |g|^2$ also converges
$\therefore g \in L^2$, and since $|f| \leq g$, $f \in L^2$.

So, $f < \infty \;\; a.e.$, and $S_n(f) = f_{n_{k+1}} \to f \;\; a.e.$

To show that this convergence is in L^2, consider:

$$|f_{n_k} - f|^2 \leq (|f_{n_k}| + |f|)^2 \leq (2g)^2$$

Thus, by the DCT:

$$\lim \int |f_{n_k} - f|^2 = \int \lim |f_{n_k} - f|^2 = 0$$

$\therefore \|f_{n_k} - f\| \to 0$

Finally, since L^2 is a metric space and a subsequence of every Cauchy sequence in L^2 converges in the norm, every Cauchy sequence converges in the norm.

$\therefore L^2$ is complete in its norm.