Graph Operations

1. Union

Definition: Union

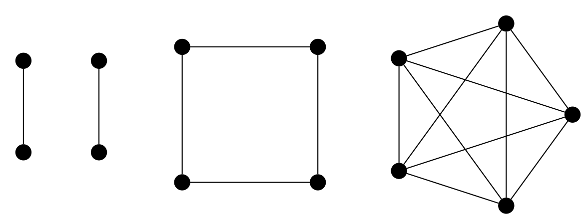
Let G and H be two disjoint graphs. The *union* of G and H, denoted by $G \cup H$, is the disconnected graph such that:

$$V(G \cup H) = V(G) \cup V(H)$$
$$E(G \cup H) = E(G) \cup E(H)$$

When $G = H$, the alternate notation $2G$ can be used.

Example

Let $G_1 = G_2 = P_2$, $G_4 = C_4$, and $G_4 = K_5$:



$$G = 2P_2 \cup C_4 \cup K_5$$

2. Join

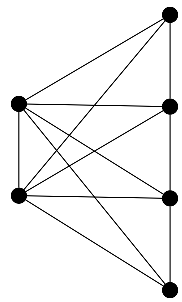
Definition: Join

Let G and H be two disjoint graphs. The *join* of G and H, denoted $G + H$, is the graph such that:

$$V(G + H) = V(G) \cup V(H)$$
$$E(G + H) = E(G) \cup E(H) \cup \{uv \,|\, u \in G \text{ and } v \in H\}$$

Example

Let $G_1 = P_2$ and $G_2 = P_4$:



$G = P_2 + P_4$

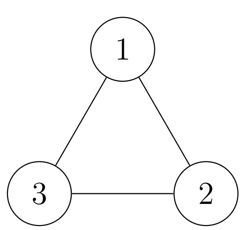
3. Product

<u>**Definition: Product**</u>

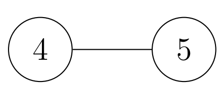
Let G and H be two disjoint graphs. The *product* of G and H, denoted $G \times H$, is the graph such that:

$$V(G \times H) = V(G) \times V(H)$$
$$E(G \times H) = \{\{(u,v),(x,y)\} \mid u = x \text{ and } vy \in E(H) \text{ or } v = y \text{ and } ux \in E(G)\}$$

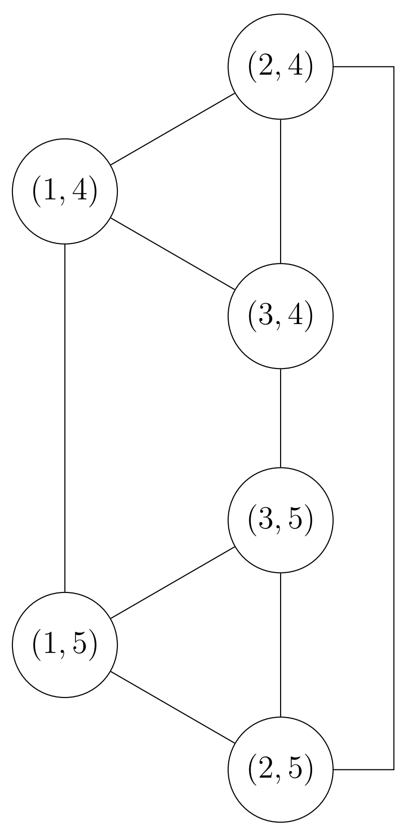
<u>**Example**</u>



C_3



P_2



$G = C_3 \times P_2$

4. Complement

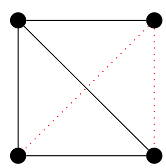
Definition: Complement

Let G be a graph. The *complement* of G, denoted by $\overline{G}$, is the graph:

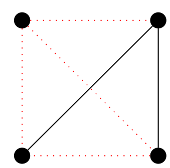
$$V(\overline{G}) = V(G)$$
$$E(\overline{G}) = \mathcal{P}_2\left(V(G)\right) - E(G)$$

In other words, $\forall\, u, v \in V(\overline{G}), uv \in E(\overline{G}) \iff uv \notin (G)$.

Example



G



$\overline{G}$

Theorem

Let G be a graph:

G is disconnected $\implies$ $\overline{G}$ is connected and $\operatorname{diam}(\overline{G}) \leq 2$.

Proof. Assume G is disconnected.

Thus, G contains two or more components. Now assume $u, v \in V(G)$ and so $u, v \in V(\overline{G})$.

Case 1: $uv \notin E(G)$

$\therefore uv \in E(\overline{G})$ and $d_{\overline{G}}(u, v) = 1$.

Case 2: $uv \in E(G)$

This means that u and v are in the same component in G. Furthermore, $uv \notin E(\overline{G})$. However, since G is disconnected, there exists a distinct vertex w in a different component in G, and so $uw, vw \in E(\overline{G})$. Consider the path (u, w, v). This is a $u - v$ path in $\overline{G}$ of length 2.

$\therefore u$ and v are connected and $d_{\overline{G}}(u, v) = 2$.

$\therefore u$ and v are connected in $\overline{G}$ and $\operatorname{diam}(\overline{G}) \leq 2$. ∎