Arbitrarily Close

- Everything that we can do in algebra is ultimately based on three things:
  1. The substitution principle.
  2. The closed and well-defined nature of addition and multiplication.
  3. The nine real number (field) axioms.
- But there are some problems that algebra cannot solve:
  1. The slope of a tangent line to a non-linear curve.
  2. The area under a non-linear curve.
- A new concept is needed to solve problems that algebra alone cannot solve: arbitrarily close.

$$\begin{array}{c}\text{CALCULUS}\\ \uparrow \\ \text{LIMITS} \\ \uparrow \\ \text{ARBITRARILY CLOSE}\end{array}$$

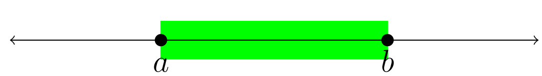
> Q: What is meant by saying that one thing is *close* to another?
>
> A: The *distance* between them is *small*.

But this is a subjective statement. In math, we want objective facts.

<u>Definition: Distance</u>

Let $a, b \in \mathbb{R}$. The *distance* from a to b is given by:

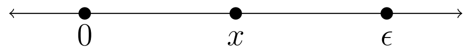
$$d(a, b) = |b - a|$$



<u>Properties: Distance</u>

1. $d(a, b) = |b - a| = |a - b| = d(b, a)$
2. $d(a, 0) = |a - 0| = |a|$

Let $\epsilon > 0$. By the density of $\mathbb{R}$, there always exists some x such that $0 < x < \epsilon$.



Example

Consider the following game:

1. Select some $\epsilon > 0$.
2. Select some $x \in (0, \epsilon)$.
3. Let $\epsilon = x$.
4. Go to step 2.

$$1$$

$$\tfrac{1}{2}$$

$$\tfrac{1}{4}$$

$$\tfrac{1}{8}$$

$$0.1$$

$$0.0001$$

$$0.00005$$

$$0.00000000001$$

$$\vdots$$

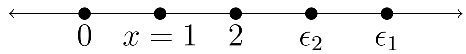
Definition: Arbitrarily Small

To say that a value x is *arbitrarily small* means that for every $\epsilon > 0, 0 < x < \epsilon$.

- This does not imply that x is assigned a particular value nor does it say that $x = 0$.
- It is indicative of an infinite argument: no matter which ϵ is selected, x is smaller (less) than ϵ (but not 0).
- It is tempting to think of ϵ constantly decreasing, pushing x continually closer to 0; however, this is imprecise: there are ways to continually decrease ϵ such that x gets no closer to 0.

Example

Consider $\epsilon > 2$ and $x = 1$. Now, for each step, move ϵ to the halfway point between the previous ϵ and 2. Although ϵ is continually getting smaller, x is not forced to move.



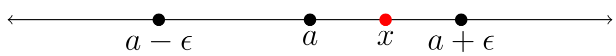
The problem is that values of $\epsilon \leq 2$ are not considered.

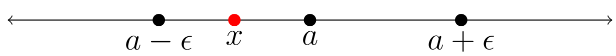
Definition: Arbitrarily Close

To say that a value $x \in \mathbb{R}$ is *arbitrarily close* to another value $a \in \mathbb{R}$, denoted by $x \to a$, means that the distance between x and a becomes arbitrarily small (but not 0):

$$\forall \epsilon > 0, 0 < |x - a| < \epsilon$$

This means that for every $\epsilon > 0$, $a - \epsilon < x < a + \epsilon$:





Definition: Neighborhood

Let $x, \epsilon \in \mathbb{R}$ such that $\epsilon > 0$. The open interval $(x - \epsilon, x + \epsilon)$ is called an ϵ-*neighbohood* of x.

Notation: One-sided

Note that $x \to a$ implies that x can approach a from either direction (from the left or from the right). When we are only interested in one direction:

$x \to a^+$	x approaches from the right of a
$x \to a^-$	x approaches from the left of a

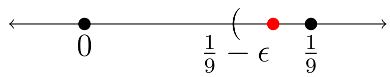
If the direction is understood then the direction indicator can be omitted. For example, if $x \in \mathbb{R}^+$ then $x \to 0^+$ can be written as just $x \to 0$, which can be used to represent the fact that x gets arbitrarily small.

Example

Recall that one of the ways of representing a rational number is a terminating infinite repeating sequence of decimal digits. For example:

$$\frac{1}{9} = 0.11111\ldots = 0.\overline{1}$$

It is easy to mark $\frac{1}{9}$ on the number line. But how does $0.\overline{1}$ correspond to this point? As each repeated digit is added, the value $0.\overline{1}$ gets *arbitrarily close* to $\frac{1}{9}$. For every $\epsilon > 0$, enough digits can be added so that the result is eventually within ϵ of $\frac{1}{9}$.



How many digits are required for $\epsilon = 0.001$?

$$0.\bar{9} - 0.9 = 0.0\bar{9} > 0.001$$

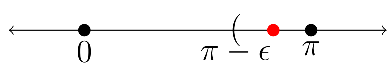
$$0.\bar{9} - 0.99 = 0.00\bar{9} > 0.001$$

$$0.\bar{9} - 0.999 = 0.000\bar{9} = 0.001$$

$$0.\bar{9} - 0.9999 = 0.0000\bar{9} = 0.0001 < 0.001$$

Example

This works for irrational numbers as well, which are represented by terminating infinite sequences of non-repeating digits. Consider $\pi = 3.1415926\ldots$. For every $\epsilon > 0$, enough digits can be added so that the result is eventually within ϵ or π.



How many digits are required for $\epsilon = 0.001$?

$$3.1415926\ldots - 3 = 0.1415926\ldots > 0.001$$

$$3.1415926\ldots - 3.1 = 0.0415926\ldots > 0.001$$

$$3.1415926\ldots - 3.14 = 0.0015926\ldots > 0.001$$

$$3.1415926\ldots - 3.141 = 0.0005926\ldots < 0.001$$

Example

Consider the real numbers $\frac{1}{7}$, π, and ϵ. How many digits in the decimal forms are required such that each value is within 0.005 and then 0.000001 of its corresponding exact value?

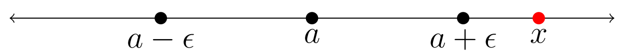
$$\frac{1}{7} = 0.14285714\ldots$$

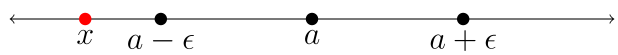
$$\pi = 3.14159265\ldots$$

$$e = 2.71828182\ldots$$

ϵ	$\frac{1}{7}$	π	e
0.0005	0.1428	3.1415	2.718
0.000001	0.142857	3.141592	2.718281

Also important is the negation: To say that $x \not\to a$ means that there exists an $\epsilon > 0$ such that $|x - a| \geq \epsilon$.





Thus, there is always some finite gap between x and a.

Example

Why isn't $24.57\bar{9}$ arbitrarily close to 24.6?

Since $24.57\bar{9} \leq 24.58$:

$$24.6 - 24.57\bar{9} \geq 24.6 - 24.58 = 0.02$$

So there exists $\epsilon = 0.02$ such that $24.6 - 24.57\bar{9} \geq \epsilon$.

Theorem

If $x \to a$ then $x = a$.

Proof. Assume that $x \neq a$. Then there exist some $\epsilon > 0$ such that $|x - a| \geq \epsilon$. Therefore $x \not\to a$. ∎

Note that the converse is not true because if $x = a$ then $|x - a| = 0$, which violates the definition of arbitrarily close.