Positive Semidefinite Matrices

Definition: Positive Semidefinite

To say that A is *positive semidefinite* means $\forall\, \vec{x} \in \mathbb{C}^n$:

$$\vec{x}^* A \vec{x} \geq 0$$

Note that A positive semidefinite $\implies A$ Hermitian.

Also note that A positive definite $\implies A$ positive semidefinite,

Properties: Positive Semidefinite

1). $A \in M_n$ positive semidefinite $\implies \operatorname{Sp}(A) \subseteq [0, \infty)$

Assume A is positive semidefinite
Assume $\vec{x} \in \mathbb{C}^n$ such that $\vec{x} \neq \vec{0}$
$\vec{x}^* A \vec{x} \geq 0$
Let $\vec{x} \in \operatorname{Eig}_A(\lambda)$ such that $\vec{x}$ is a unit vector
$\vec{x}^* A \vec{x} = \vec{x}^* \lambda \vec{x} = \lambda \vec{x}^* \vec{x} = \lambda \geq 0$

2). $A \in M_n$ positive semidefinite $\implies a_{ii} \geq 0$

Assume A is positive semidefinite
$\vec{e}_i^* A \vec{e}_i = a_{ii} \geq 0$

3). $A \in M_n$ positive semidefinite $\implies \forall\, S \in GL(n), S^* A S$ positive semidefinite

Assume A is positive semidefinite
Assume $\vec{x} \in \mathbb{C}^n$ such that $\vec{x} \neq \vec{0}$
$\vec{x}^*(S^* A S)\vec{x} = (\vec{x}^* S^*) A (S\vec{x}) = (S\vec{x})^* A (S\vec{x}) = \vec{y}^* A \vec{y} \geq 0$

$\therefore S^* A S$ is positive semidefinite.

4). $A \in M_n$ positive semidefinite $\implies$ any principle submatrix B of A is positive semidefinite

Assume A is positive semidefinite
AWLOG: B is a leading principle submatrix, otherwise permute and note property (3)
Assume $\vec{x} \in \mathbb{C}^k$ for $1 \leq k \leq n$

$$\begin{bmatrix} \vec{x}^* & 0 \end{bmatrix} \left[\begin{array}{c|c} B & * \\ \hline * & * \end{array}\right] \begin{bmatrix} \vec{x} \\ 0 \end{bmatrix} = \vec{x}^* B \vec{x} \geq 0$$

$\therefore B$ is positive semidefinite.

Theorem

Let $A \in M_n$. A positive semidefinite $\iff A$ Hermitian and $\operatorname{Sp}(A) \subseteq [0, \infty)$

Proof

$\implies$ Assume A is positive semidefinite

A is also Hermitian
By property (1), $\forall\, \lambda \in \mathrm{Sp}(A), \lambda \geq 0$

$\impliedby$ Assume A is Hermitian and $\mathrm{Sp}(A) \subseteq [0, \infty)$

Assume $\lambda \in \mathrm{Sp}(A)$
Let $\vec{x}$ be a unit eigenvector associated with λ
$\vec{x} \neq 0$
$\vec{x}^* A\vec{x} = \vec{x}^* \lambda \vec{x} = \lambda \vec{x}^* \vec{x} = \lambda \geq 0$

$\therefore A$ is positive semidefinite.

Theorem

Let $A \in M_n$. A positive semidefinite $\implies$ A Hermitian and $\det A_k \geq 0$ for all $1 \leq k \leq n$, where A_k is the $k \times k$ leading principle submatrix of A.

Proof

Assume A is positive semidefinite
A is Hermitian
Assume $1 \leq k \leq n$
A_k is positive semidefinite
Assume $\lambda \in \sigma(A_k)$
$\lambda \geq 0$
$\det A_k = \prod_{i=1}^{k} \lambda_i(A_k) \geq 0$

Note that the converse is not true for positive semidefinite. Consider the following counterexample:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

A is Hermitian and $\det A_k = 0$; however, A is not positive semidefinite due to the negative eigenvalue.