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Math 275A
Homework #10

Lemma

Let X and Y be topological spaces and let $f : X \to Y$. For all $B \subset Y$:

$$X - f^{-1}(B) = f^{-1}(Y - B)$$

Proof. Assume $A \subset Y$.

$$\begin{aligned} x \in X - f^{-1}(B) &\iff x \notin f^{-1}(B) \\ &\iff f(x) \notin B \\ &\iff f(x) \in Y - B \\ &\iff x \in f^{-1}(Y - B) \end{aligned}$$

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Theorem: 7.1

Let X and Y be topological spaces and let $f : X \to Y$. TFAE:

1. f is continuous.
2. For every closed set $K \subset Y$, $f^{-1}(K)$ is closed in X.
3. For all $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.
4. For every $x \in X$ and $V \in \mathcal{N}_{f(x)}$ there exists $U \in \mathcal{N}_x$ such that $f(U) \subset V$.

Proof.

$1 \implies 2$ Assume that f is continuous.

Assume that $K \subset Y$ is closed, and so $Y - K \in \mathscr{T}_Y$. Since f is continuous, $f^{-1}(Y - K) \in \mathscr{T}_X$. Now, applying the lemma, $f^{-1}(Y - K) = X - f^{-1}(K) \in \mathscr{T}_X$. Therefore $f^{-1}(K)$ is closed.

$2 \implies 3$ Assume that for every closed set $K \subset Y$, $f^{-1}(K)$ is closed in X.

Assume $A \subset X$. Since $\overline{f(A)}$ is closed, by the assumption, $f^{-1}(\overline{f(A)})$ is closed. Furthermore, since $f(A) \subset \overline{f(A)}$, it must be the case that $f^{-1}(f(A)) = A \subset f^{-1}(\overline{f(A)})$. But $\bar{A}$ is the smallest closed set containing A, and so $\bar{A} \subset f^{-1}(\overline{f(A)})$. Therefore $f(\bar{A}) \subset f(f^{-1}(\overline{f(A)})) \subset \overline{f(A)}$.

$3 \implies 4$ Assume that for all $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.

Assume $x \in X$ and $V \in \mathcal{N}_{f(x)}$. Note that $Y - V$ is closed. Now, let $U = f^{-1}(V)$ and so $x \in U$ and $f(U) = f(f^{-1}(V) \subset V$.

WTS: U open.

ABC that $X - U$ is not closed. This means that there exists $p \in \overline{X - U}$ but $p \notin X - U$. And so, by the assumption and the lemma:

$$f(p) \in f(\overline{X-U}) \subset \overline{f(X-U)} = \overline{f(X - f^{-1}(V))} = \overline{f(f^{-1}(Y-V))} \subset \overline{Y-V} = Y - V$$

This means that $p \in f^{-1}(Y - V) = X - f^{-1}(V) = X - U$, contradicting the assumption that $p \notin X - U$. Thus $X - U$ contains all of its limit points and is closed. Therefore U is open.

$4 \implies 1$ Assume that for every $x \in X$ and $V \in \mathcal{N}_{f(x)}$ there exists $U \in \mathcal{N}_x$ such that $f(U) \subset V$.

Assume $V \in \mathscr{T}_Y$ and assume $p \in f^{-1}(V)$. Thus $f(p) \in V \in \mathcal{N}_{f(p)}$. Now, by the assumption, there exists $U \in \mathcal{N}_p$ such that $f(U) \subset V$, and hence $f^{-1}(f(U)) = U \subset f^{-1}(V)$. This means that p is an interior point of $f^{-1}(V)$ and hence $f^{-1}(V)$ is open. Therefore f is continuous.

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Theorem: 7.2

Let X and Y be topological spaces and let $y_0 \in Y$. The constant map $f : X \to Y$ defined by $f(x) = y_0$ is continuous.

Proof. Assume that $V \in \mathscr{T}_Y$. If $y_0 \in V$ then $f^{-1}(V) = X$. Otherwise, $f^{-1}(V) = \emptyset$. In either case, $f^{-1}(V) \in \mathscr{T}_X$. Therefore f is continuous. ■

Theorem: 7.7

Let X and Y be topological spaces such that $D \subset X$ is dense and Y is Hausdorff. Let $f : X \to Y$ and $g : X \to Y$ be continuous such that $\forall\, d \in D, f(d) = g(d)$. Then $\forall\, x \in X, f(x) = g(x)$.

Proof. ABC that there exists $x \in X$ such that $f(x) \neq g(x)$. Now, since Y is Hausdorff, there exists $U \in \mathcal{N}_{f(x)}$ and $V \in \mathcal{N}_{g(x)}$ such that $U \cap V = \emptyset$. Furthermore, since f and g are continuous, $f^{-1}(U) \in \mathcal{N}_x$ and $g^{-1}(V) \in \mathcal{N}_x$. Since $x \in f^{-1}(U)$ and $x \in g^{-1}(V)$, this means that $f^{-1}(U) \cap g^{-1}(V) \neq \emptyset$, and so, since D is dense in X, there must exists $d \in D$ such that $d \in f^{-1}(U) \cap g^{-1}(V)$. But this means that $f(d) \in U \cap V$, contradicting the assumption that U and V are disjoint. Therefore $\forall\, x \in X, f(x) = g(x)$. ■