Math-42 Worksheet #17

Strong Induction and Well-ordering

1. Assume that you have an unlimited number of 2-cent and 3-cent stamps and the package that you want to mail requires a postage of n cents, where $n \geq 2$. Prove that this is always possible using:
   (a) Simple induction. (Hint: proof by cases may be useful here.)
   (b) Strong induction.

2. Now, assume that you have only 3-cent and 8-cent stamps.
   (a) For what n can you make all postages for greater than or equal to n.
   (b) Prove this using strong induction.

3. Use strong induction to prove that every positive integer is a product of one or more primes.

4. Prove using the well-ordering principle that there are no integers between 0 and 1. (Hint: contradiction).

5. Show that the well-ordering principle does not hold for the set of integers $\mathbb{Z}$.

6. The well-ordering principle is crucial to proving the *existence* part of the division algorithm:

   For every $n, d \in \mathbb{Z}$ where $d > 0$ there exists *unique* $q, r \in \mathbb{Z}$ such that $n = dq + r$ and $0 \leq r < d$.

   (a) Start by definition the set $S = \{n - ds \mid s \in \mathbb{Z}\}$ and then define $T = \{t \in S \mid t \geq 0\}$. Show that $T \neq \emptyset$.
   (b) Apply the well-ordering principle to T. Call the minimum value r.
   (c) Prove by contradiction that $0 \leq r < d$. Since a choice of r results in a corresponding choice for q, this proves existence. See if you can prove uniqueness.