Compact Sets

Definition: Cover

Let X be a topological space and $A \subset X$, and let $\mathcal{U} = \{U_\alpha : \alpha \in \lambda\}$ be a collection of subsets of X. To say that $\mathcal{U}$ is a *cover* of A means that $A \subset \bigcup \mathcal{U}$. To say that $\mathcal{U}$ is an *open* cover of A means that $\mathcal{U}$ is a cover composed of open sets. A subset of $\mathcal{U}$ that still covers A is called a *subcover* of A. A subcover of open sets is called an *open subcover*.

Definition: Compact

Let X be a topological space. To say that X is *compact* means that every open cover of X has a finite subcover.

Theorem

$\mathbb{R}_{\text{std}}$ is not compact.

Proof. Let $\mathbb{C} = \{(-a, a) : a \in \mathbb{R}\}$. This is an open cover of $\mathbb{R}$. So ABC that $\mathbb{C}$ contains a finite subcover. But this would mean that there is some maximum a such that only $(-a, a) \subsetneq \mathbb{R}$ is covered, violating the assumption that there exists a finite subcover.

Therefore $\mathbb{R}_{\text{std}}$ is not compact. ∎

Theorem

Let $A \subset \mathbb{R}_{\text{std}}$. If A is compact then A has a maximum point.

Proof. If A is finite then trivial, so assume that A is infinite. ABC that A has no maximum point. This means that for all $a \in A$ there exists $b_a \in A$ such that $b_a > a$. So let $\{(-\infty, b_a) : a \in A\}$ be an open cover for A. Since A is compact, there exists a finite subcover $U = \{(-\infty, b_{a_k}) : 1 \leq k \leq n\}$. Let $c = \max\{b_{a_k}\}$, and so $\bigcup U = (-\infty, c)$. Thus $c \in A$ but $c \notin U$, contradicting the assumption that U is a finite subcover.

Therefore A has a maximum point. ∎

Theorem

If X is a compact space then every infinite subset of X has a limit point.

Proof. Assume that X is a compact set and assume that $A \subset X$ is infinite. Now, ABC that A has no limit points, and so all $a \in A$ are isolated points. So let $\mathcal{U} = \{U_a : a \in A\}$ be an open cover of A such that the $U_a \cap A = \{a\}$. Thus the U_a are disjoint and so $a \mapsto U_a$ is bijective. Hence $\mathcal{U}$ is an infinite cover and no finite subcover is possible, violating the compactness of A.

Therefore A has a limit point. ∎

Definition: Finite Intersection Property

To say that a collection of sets has the *infinite intersection property* means that every finite subcollection has a non-empty intersection.

Theorem

Let X be a topological space. X is compact iff every collection of closed subsets of X with the finite intersection property has a non-empty intersection.

Proof.

$\implies$ Assume that X is compact.

Assume that $\mathcal{A} = \{A_\alpha : \alpha \in \lambda\}$ is a collection of closed subsets of X with the finite intersection property. Now, ABC that $\bigcap_{\alpha\in\lambda} A_\alpha = \emptyset$. But since the A_α are closed, the A_α^C are open and $\bigcup_{\alpha\in\lambda} A_\alpha^C = X$ is an open cover for X. Furthermore, since X is compact, there exists a finite subcover $A_{\alpha_1}^C \cup \cdots \cup A_{a_n}^C = X$. Thus, $A_{\alpha_1} \cap \cdots \cap A_{a_n} = \emptyset$ is a finite subcollection of $\mathcal{A}$ with empty intersection, contradicting the finite intersection property of $\mathcal{A}$.

Therefore, every collection of closed subsets of X with the finite intersection property has a non-empty intersection.

$\impliedby$ Assume that every collection of closed subsets of X with the finite intersection property has a non-empty intersection.

Assume that $\mathcal{U} = \{U_\alpha : \alpha \in \lambda\}$ is an open cover of X and ABC that $\mathcal{U}$ contains no finite subcover. This means that for all finite subcollections $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\} \subset \mathcal{U}$ there exists $x \in X$ such that $x \notin U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ and hence $x \in U_{\alpha_1}^C \cap \cdots \cap U_{\alpha_n}^C$ and so $U_{\alpha_1}^C \cap \cdots \cap U_{\alpha_n}^C \neq \emptyset$. This shows that $\{U_\alpha^C : \alpha \in \lambda\}$ is a collection of closed sets with the finite intersection property, and so by assumption, $\bigcap_{\alpha\in\lambda} U_\alpha^C \neq \emptyset$. But this means that $\bigcup_{\alpha\in\lambda} U_\alpha \neq X$, contradicting the assumption that $\mathcal{U}$ is a cover for X, and so $\mathcal{U}$ must contain a finite subcover.

Therefore X is compact.

$\blacksquare$

Theorem

Let X be a topological space. X is compact iff for all $U \in \mathscr{T}$ and all collections of closed sets $\mathcal{K} = \{K_\alpha : \alpha \in \lambda\}$ such that $\bigcap \mathcal{K} \subset U$, there exists a finite subcollection of $\mathcal{K}$ whose intersection $K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \subset U$.

Proof.

$\implies$ Assume that X is compact.

Assume that $U \in \mathscr{T}$ and $\mathcal{K} = \{K_\alpha : \alpha \in \lambda\}$ is a collection of closed sets such that $\bigcap_{\alpha\in\lambda} K_\alpha \subset U$. Let $U_\alpha = K_a^C \in \mathscr{T}$. This means that $\bigcup_{\alpha\in\lambda} U_\alpha \supset U^C$ and so $\mathcal{U} = \{U\} \cup \{U_\alpha : \alpha \in \lambda\}$ is an open cover for X, which must contain a finite subcover. Now, note that $\bigcap_{\alpha\in\lambda} K_\alpha \subset U$ but $\bigcap_{\alpha\in\lambda} K_\alpha \not\subset \bigcup_{\alpha\in\lambda} U_\alpha$, so any finite subcover must contain U and some finite subcollection of the U_α. So assume that $U \cup U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} = X$ is such a finite subcover. Therefore $U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \supset U^C$ and hence $K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \subset U$.

$\Longleftarrow$ Assume that for all $U \in \mathscr{T}$ and all collections of closed sets $\mathcal{K} = \{K_\alpha : \alpha \in \lambda\}$ such that $\bigcap \mathcal{K} \subset U$, there exists a finite subcollection of $\mathcal{K}$ whose intersection $K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \subset U$.

Assume that $\mathcal{U} = \{U_\alpha : \alpha \in \lambda\}$ is an open cover for X. Now, assume $U_{\alpha_0} \in \mathcal{U}$. This means that $U_{\alpha_0} \cup \bigcup_{\alpha\neq\alpha_0} U_\alpha = X$. Let $K_\alpha = U_\alpha^C$, and so the K_α are closed. Then $K_{\alpha_0} \cap \bigcap_{\alpha\neq\alpha_0} K_\alpha = \emptyset$ and hence $\bigcap_{\alpha\neq\alpha_0} K_\alpha \subset U_{\alpha_0}$. Furthermore, by the assumption, there exists a finite subcollection $\{K_{\alpha_1}, \ldots, K_{\alpha_n}\}$ such that $K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \subset U_{\alpha_0}$ and so $U_{\alpha_1} \cup \ldots \cup U_{\alpha_n} \supset K_{\alpha_0}$. Therefore $U_{\alpha_0} \cup U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} = X$ is a finite subcover, hence X is compact.

■

Theorem

Every closed subspace of a compact space is compact.

Proof. Assume that X is a compact topological space and A is a closed subspace of X. Now, assume that $\mathcal{U}$ is an open cover of X and $\mathcal{U}_A = \{U_\alpha : \alpha \in \lambda\} \subset \mathcal{U}$ is an open cover of A. Since A is closed, let $U = A^C \in \mathscr{T}$. Thus, $U \cup \bigcup_{\alpha\in\lambda} U_\alpha = X$ is also an open cover of X. But X is compact and so this open cover contains a finite subcover. Since any such finite subcover can always include U and still be finite, let $U \cup U_{\alpha_1} \cup \ldots U_{\alpha_n} = X$ be such a finite subcover. This requires that $U_{\alpha_1} \cup \ldots U_{\alpha_n} \supset A$ be a finite subcover for A. Therefore, $(U_{\alpha_1} \cup \ldots U_{\alpha_n}) \cap A = (U_{\alpha_1} \cap A) \cup \cdots \cup (U_{\alpha_n} \cap A) = A$ is a finite open cover of the subspace A and hence A is compact. ■

Theorem

Every compact subspace of a Hausdorff space is closed.

Proof. Assume that X is Hausdorff and A is a compact subspace of X. Assume that $b \in A^C$. Since X is Hausdorff, for every $a \in A$ there exists $U_a, V_a \in \mathscr{T}_X$ such that $a \in U_a$, $b \in V_a$, and $U_a \cap V_a = \emptyset$. So let the $\{U_a : a \in A\}$ be an open cover of A in X. Thus $\{U_a \cap A : a \in A\}$ for $U_a \cap A \in \mathscr{T}_Y$ is an open cover of A in A. Now, since A is a compact subspace of X, there exists a finite subcover $(U_{a_1} \cap A) \cup \cdots \cup (U_{a_n} \cap A)$ of A in A, and hence a finite subcover $U_{a_1} \cup \cdots \cup U_{a_n}$ of A in X. Let $V = V_{a_1} \cap \cdots \cap V_{a_n}$. Note that $b \in V$ and $V \in \mathscr{T}_X$. Furthermore, since all the $U_a \cap V_a = \emptyset$, it must be the case that $V \cap (U_{a_1} \cup \cdots \cup U_{a_n}) = \emptyset$. But since $U_{a_1} \cup \cdots \cup U_{a_n} \supset A$ it must be the case that $V \subset A^C$. So b is an interior point in A^C, meaning that all the points in A^C are interior, and so $A^C \in \mathscr{T}_X$. Therefore A is closed in X. ■

Lemma

Every compact, Hausdorff space is regular.

Proof. Assume that X is compact and Hausdorff. Assume that $A \subset X$ is closed. Thus, by previous theorem, A is also compact. So assume $p \in A^C$. This means that $p \notin A$ and so, by the previous proof, there exists $U, V \in \mathscr{T}$ such that $A \subset U$ and $p \in V$ and $U \cap V = \emptyset$.

Therefore X is regular. ∎

Theorem

Every compact, Hausdorff space is normal.

Proof. Assume $A, B \subset X$ are closed. Since X is regular (by the previous lemma), for all $b \in B$ there exists $U_b, V_b \in \mathscr{T}$ such that $A \subset U_b$ and $b \in V_b$ and $U_b \cap V_b = \emptyset$. So let $V = \{V_b : b \in B\}$ be an open cover for B. But, by previous theorem, B is also compact, and so there exists a finite subcover $V_{b_1} \cup \cdots \cup V_{b_n} \supset B$. So let $U = U_{b_1} \cap \cdots \cap U_{b_n} \in \mathscr{T}$. Note that $A \subset U$ and, since all the $U_b \cap V_b = \emptyset$, $U \cap V = \emptyset$. Therefore, X is normal. ∎