Orthonormal Basis

Definition: Orthonormal Basis

Let E be an inner product space over $\mathbb{C}$ and let $B = \{\vec{x}_n \mid n \in \mathbb{N}\}$ be an orthonormal system in E. To say that B is an *orthonormal basis* for E means $\forall\, \vec{x} \in E$, $\vec{x}$ can be written uniquely as:

$$\vec{x} = \sum_{n=1}^{\infty} \alpha_n \vec{x}_n$$

for some $\alpha_n \in \mathbb{C}$ and distinct $\vec{x}_n \in B$.

Examples

1). $H = L^2[-\pi, pi]$ with $\varphi_n(t) = \dfrac{1}{\sqrt{2\pi}}, \dfrac{1}{\sqrt{\pi}} \sin nt, \dfrac{1}{\sqrt{\pi}} \cos nt$ for all $n \in \mathbb{N}$.

2). $H = L^2[-\pi, pi]$ with $\varphi_n(t) = \dfrac{1}{\sqrt{2\pi}} e^{-nt}$ for all $n \in \mathbb{Z}$.

Theorem

Let E be an inner product space over $\mathbb{C}$. Every complete orthonormal sequence in E is an orthonormal basis for E.

Proof

Assume $(\vec{x}_n)$ is a complete orthonormal sequence in E. Assume $\vec{x} \in E$. Since $(\vec{x}_n)$ is complete:

$$\vec{x} = \sum_{n=1}^{\infty} \langle \vec{x}, \vec{x}_n \rangle \vec{x}_n$$

Thus establishing existence where $\alpha_n = \langle \vec{x}, \vec{x}_n \rangle$.

Now, assume $\vec{x} = \sum_{n=1}^{\infty} \alpha_k \vec{x}_n = \sum_{n=1}^{\infty} \beta_k \vec{x}_n$.

$$0 = \|\vec{x} - \vec{x}\|^2 = \left\| \sum_{n=1}^{\infty} \alpha_n \vec{x}_n - \sum_{n=1}^{\infty} \beta_n \vec{x}_n \right\|^2 = \left\| \sum_{n=1}^{\infty} (\alpha_n - \beta_n) \vec{x}_n \right\|^2 = \sum_{n=1}^{\infty} |\alpha_n - \beta_n|$$

But this is only true for $\alpha_n - \beta_n = 0$.

$\therefore \alpha_n = \beta_n$, thus establishing uniqueness.

Corollary

Let E be an inner product space and let $(\vec{x}_n)$ be a complete orthonormal sequence in E:

$\mathrm{Span}\{\vec{x}_1, \vec{x}_2, \ldots\}$ is dense in E.

Proof

Let $S = \operatorname{Span}\{\vec{x}_1, \vec{x}_2, \ldots\}$.
Assume $\vec{x} \in E$.
Since $(\vec{x}_n)$ is complete:

$$\vec{x} = \sum_{n=1}^{\infty} \langle \vec{x}, \vec{x}_n \rangle \vec{x}_n$$

Let $S_n = \displaystyle\sum_{k=1}^{n} \langle \vec{x}, \vec{x}_k \rangle \vec{x}_k$.
(S_n) is a sequence in S and $S_n \to \vec{x} \in E$.

Therefore S is dense in E.

Theorem

Let H be a Hilbert space and let $(\vec{x}_n)$ be an orthonormal sequence in H:

$$(\vec{x}_n) \text{ is complete} \iff ((\forall\, n \in \mathbb{N}, \vec{x} \perp \vec{x}_n) \implies \vec{x} = \vec{0})$$

Proof

$\implies$ Assume $(\vec{x}_n)$ is complete.

Assume $\forall\, n \in \mathbb{N}, \vec{x} \perp \vec{x}_n$.
$\forall\, n \in \mathbb{N}, \langle \vec{x}, \vec{x}_n \rangle = 0$
But $(\vec{x}_n)$ is complete, so $\vec{x} = \displaystyle\sum_{n=1}^{\infty} \langle \vec{x}, \vec{x}_n \rangle \vec{x}_n = \vec{0}$.

$\impliedby$ Assume $(\forall\, n \in \mathbb{N}, \vec{x} \perp \vec{x}_n) \implies \vec{x} = \vec{0}$.

Assume $\vec{x} \in H$.
Since H is Hilbert: $\displaystyle\sum_{n=1}^{\infty} \langle \vec{x}, \vec{x}_n \rangle \vec{x}_n = \vec{x}_* \in H$.
WTS: $\vec{x}_* = \vec{x}$.
Assume $n \in \mathbb{N}$:

$$\begin{aligned}
\langle \vec{x}_* - \vec{x}, \vec{x}_n \rangle &= \langle \vec{x}_*, \vec{x}_n \rangle - \langle \vec{x}, \vec{x}_n \rangle \\
&= \left\langle \sum_{k=1}^{\infty} \langle \vec{x}, \vec{x}_k \rangle \vec{x}_k, \vec{x}_n \right\rangle - \langle \vec{x}, \vec{x}_n \rangle \\
&= \langle \langle \vec{x}, \vec{x}_n \rangle \vec{x}_n, \vec{x}_n \rangle - \langle \vec{x}, \vec{x}_n \rangle \\
&= \langle \vec{x}, \vec{x}_n \rangle \langle \vec{x}_n, \vec{x}_n \rangle - \langle \vec{x}, \vec{x}_n \rangle \\
&= \langle \vec{x}, \vec{x}_n \rangle \|\vec{x}_n\|^2 - \langle \vec{x}, \vec{x}_n \rangle \\
&= \langle \vec{x}, \vec{x}_n \rangle \cdot 1 - \langle \vec{x}, \vec{x}_n \rangle \\
&= \langle \vec{x}, \vec{x}_n \rangle - \langle \vec{x}, \vec{x}_n \rangle \\
&= 0
\end{aligned}$$

And so, by assumption, $\vec{x}_* - \vec{x} = \vec{0}$.

$\therefore \vec{x}_* = \vec{x}$ and thus $(\vec{x}_n)$ is complete.

Theorem: Parseval's Formula

Let H be a Hilbert space and let $(\vec{x}_n)$ be an orthonormal sequence in H:

$$(\vec{x}_n) \text{ is complete} \iff \forall\, \vec{x} \in H, \|\vec{x}\|^2 = \sum_{n=1}^{\infty} |\langle \vec{x}, \vec{x}_n \rangle|^2$$

Proof

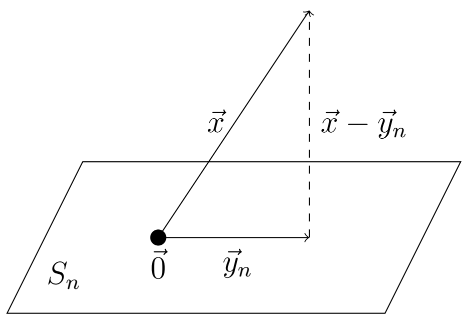
$\implies$ Assume $(\vec{x}_n)$ is complete.

Assume $\vec{x} \in H$:

$$\begin{aligned}
\vec{x} &= \sum_{n=1}^{\infty} \langle \vec{x}, \vec{x}_n \rangle\, \vec{x}_n \\
\|\vec{x}\|^2 &= \langle \vec{x}, \vec{x} \rangle \\
&= \left\langle \sum_{n=1}^{\infty} \langle \vec{x}, \vec{x}_n \rangle\, \vec{x}_n, \sum_{m=1}^{\infty} \langle \vec{x}, \vec{x}_m \rangle\, \vec{x}_m \right\rangle \\
&= \sum_{n=1}^{\infty} \langle \langle \vec{x}, \vec{x}_n \rangle\, \vec{x}_n, \langle \vec{x}, \vec{x}_n \rangle\, \vec{x}_n \rangle \\
&= \sum_{n=1}^{\infty} |\langle \vec{x}, \vec{x}_n \rangle|^2 \langle \vec{x}_n, \vec{x}_n \rangle \\
&= \sum_{n=1}^{\infty} |\langle \vec{x}, \vec{x}_n \rangle|^2 \|\vec{x}_n\|^2 \\
&= \sum_{n=1}^{\infty} |\langle \vec{x}, \vec{x}_n \rangle|^2 \cdot 1 \\
&= \sum_{n=1}^{\infty} |\langle \vec{x}, \vec{x}_n \rangle|^2
\end{aligned}$$

$\impliedby$ Assume $\forall\, \vec{x} \in H, \|\vec{x}\|^2 = \sum_{n=1}^{\infty} |\langle \vec{x}, \vec{x}_n \rangle|^2$

Assume $\vec{x} \in H$.
Let $S_n = \{\vec{x}_1, \ldots, \vec{x}_n\}$.
Let $\vec{y}_n = \mathrm{proj}_{S_n} \vec{x} = \sum_{k=1}^{n} \langle \vec{x}, \vec{x}_n \rangle\, \vec{x}_n$.
And so $\|\vec{y}_n\| = \sum_{k=1}^{n} |\langle \vec{x}, \vec{x}_n \rangle|^2$.



Note that $\vec{x}-\vec{y}_n \perp \vec{y}_n$. And so:

$$\begin{aligned}
\|\vec{x}\|^2 &= \|(\vec{x}-\vec{y}_n)+\vec{y}_n\|^2 \\
\|\vec{x}\|^2 &= \|\vec{x}-\vec{y}_n\|^2+\|\vec{y}_n\|^2 \\
\|\vec{x}\|^2-\|\vec{y}_n\|^2 &= \|\vec{x}-\vec{y}_n\|^2 \\
\|\vec{x}\|^2-\sum_{k=1}^{n}|\langle\vec{x},\vec{x}_k\rangle|^2 &= \left\|\vec{x}-\sum_{k=1}^{n}\langle\vec{x},\vec{x}_k\rangle\,\vec{x}_k\right\| \to 0
\end{aligned}$$

$\therefore \vec{x}=\sum_{n=1}^{\infty}\langle\vec{x},\vec{x}_n\rangle\,\vec{x}_n$ and thus $(\vec{x}_n)$ is complete.