Maximum Column Sum

Definition: Maximum Column Sum

Let $A \in M_n$, the *maximum column sum* of A is given by:

$$M = \max_{1 \leq j \leq n} \left\{ \sum_{i=1}^{n} |a_{ij}| \right\}$$

Lemma

Let $A \in M_n$ and M be the maximum column sum for A. $\forall \, \vec{x} \in \mathbb{C}^n$:

$$\|A\vec{x}\|_1 \leq M \, \|\vec{x}\|_1$$

Proof

Assume $\vec{x} \in \mathbb{C}^n$:

$$\begin{aligned}
\|A\vec{x}\|_1 &= \left\| \begin{bmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \sum_{j=1}^{n} a_{2j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj}x_j \end{bmatrix} \right\|_1 \\
&= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij}x_j \right| \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}x_j| \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} |a_{ij}| \right) |x_j| \\
&\leq \sum_{j=1}^{n} M \, |x_j| \\
&= M \sum_{j=1}^{n} |x_j| \\
&= M \, \|\vec{x}\|_1 \\
\therefore \|A\vec{x}\|_1 &\leq M \, \|\vec{x}\|_1
\end{aligned}$$

Lemma

Let $A \in M_n$ and M be the maximum column sum for A. $\exists \, \vec{x} \in \mathbb{C}^n$:

$$\|A\vec{x}\|_1 = M \, \|\vec{x}\|_1$$

Proof

Let $1 \leq j \leq n$ such that:

$$\sum_{i=1}^{n} |a_{ij}| = M$$

Consider $\vec{e}_j$:

$$\|A\vec{e}_j\|_1 = \sum_{i=1}^{n} |a_{ij}| = M = M \cdot 1 = M \|\vec{e}_k\|$$

Let $\vec{x} = \vec{e}_j$. Therefore, $\exists \vec{x} \in \mathbb{C}^n$ such that:

$$\|A\vec{x}\|_1 = M \|\vec{x}\|_1$$

Theorem

Let $A \in M_n$ and M be the maximum column sum for A:

$$|||A|||_1 = M$$

Proof

By definition:

$$|||A|||_1 = \max_{\|\vec{x}\|_1 = 1} \{\|A\vec{x}\|_1\}$$

By the above lemma: $\forall \, \vec{x} \in \mathbb{C}^n$:

$$\|A\vec{x}\|_1 \leq M \|\vec{x}\|_1$$

And by the subsequent lemma, there exists a $\vec{x} \in \mathbb{C}^n$ such that $\|\vec{x}\|_1 = 1$ and:

$$\|A\vec{x}\|_1 = M \|\vec{x}\|_1 = M \cdot 1 = M$$

Therefore:

$$|||A|||_1 = \max_{\|\vec{x}\|_1 = 1} \|A\vec{x}\|_1 = M$$