Separable Hilbert Spaces

Definition: Separable

Let H be a Hilbert space. To say that H is *separable* means that H contains a complete orthonormal sequence.

Examples

1). ℓ^2 is separable with (e_n)

2). $L^2[-\pi,\pi]$ is separable with $\left(\dfrac{1}{\sqrt{2\pi}}, \dfrac{1}{\sqrt{\pi}}\sin(nt), \dfrac{1}{\sqrt{\pi}}\cos(nt)\right)_{n\in\mathbb{N}}$

3). Let:

$$H = \left\{ f : \mathbb{R} \to \mathbb{C} \,\middle|\, f(x) \neq 0 \text{ for only countably many } x \in \mathbb{R} \text{ and } \sum_{x \mid f(x)\neq 0} |f(x)|^2 < \infty \right\}$$

Let $\langle f, g\rangle = \displaystyle\sum_{\vec{x} \mid f(x)g(x)\neq 0} f(x)\overline{g(x)}$.

Claim: H is complete.

Assume (f_n) is a Cauchy sequence in H.
Let S_n be the support of f_n.
Since S_n is countable, let $S_n = \{t_1, t_2, t_3, \ldots\}$.
Let (x_n) be a sequence in $\mathbb{C}$ where $x_{n,k} = f(t_k)$.

$$\|f_n\|_H = \sum_{x \mid f(x)\neq 0} |f(x)|^2 = \sum_{k=1}^{\infty} |f(t_k)|^2 = \sum_{k=1}^{\infty} |x_{n,k}|^2 = \|x_n\|_{\ell^2} < \infty$$

And so (x_n) is a sequence in ℓ^2.

Furthermore, $\|x_n - x_m\|_{\ell^2} = \|f_n - f_m\|_H$, and so (x_n) is Cauchy in ℓ^2.
But ℓ^2 is complete, so $x_n \to x \in \ell^2$.

Now, let $S = \displaystyle\bigcup_{n=1}^{\infty} S_n$.
Note that S is countable, since all the S_n are countable.
Define $f : \mathbb{R} \to \mathbb{C}$ as $f(t_k) = x_k, k \in \mathbb{N}$ and $f(t) = 0, t \notin S$.
Thus, $f \in H$ and $\|x - x_n\| = \|f - f_n\| \to 0$.

Therefore H is complete.

Claim: H is not separable.

Assume (f_n) be an orthonormal sequence in H.
Construct $f \in H$ such that $f \not\equiv 0$ and $\forall\, n \in \mathbb{N}, \langle f, f_n\rangle = 0$.
Let $S_n = \{x \in \mathbb{R} \mid f_n(x) \neq 0\}$.

Let $S = \bigcup_{n \in N} S_n = \{x \in \mathbb{R} \mid \exists\, n \in \mathbb{N}, f_n(x) \neq 0\}$.
So $\mathbb{R} \setminus S = \{x \in \mathbb{R} \mid \forall\, n \in \mathbb{N}, f_n(x) = 0\}$.
Assume $x_0 \in \mathbb{R} \setminus S$.
Let $f(x) = \begin{cases} 1, & x = x_0 \\ 0, & x \neq x_0 \end{cases}$.
$\langle f, f_n \rangle = f(x_0) f_n(x_0) = 0$.
But $f(x) \not\equiv 0$.

Therefore (f_n) is not complete.

Theorem

Let H be a finite dimensional inner product (hence Hilbert) space.

H is separable.

Proof

Assume $\dim H = N$.
Assume $S = \{\vec{x}_1, \ldots, \vec{x}_N\}$ is an orthonormal basis for H.
Thus, $(\vec{x}_n)$ is an orthonormal sequence in H.
Assume $x \in H$.
$\exists\, \alpha_n \in \mathbb{F}$ such that $x = \sum_{n=1}^{N} \alpha_n \vec{x}_n$

$$\langle \vec{x}, \vec{x}_n \rangle = \left\langle \sum_{k=1}^{N} \alpha_k \vec{x}_k, \vec{x}_n \right\rangle = \langle \alpha_n \vec{x}_n, \vec{x}_n \rangle = \alpha_n.$$

$$\sum_{n=1}^{N} \langle x, \vec{x}_n \rangle \vec{x}_n = \sum_{n=1}^{N} \alpha_n \vec{x}_n = \vec{x}$$

Therefore $(\vec{x}_n)$ is complete and thus H is separable.

Theorem

Let H be a Hilbert space:

H is separable $\iff$ H contains a countable dense subset.

Proof

$\implies$ Assume H is separable.

Assume $(\vec{x}_n)$ is a complete sequence in H.
Assume $\vec{x} \in H$.

$$\vec{x} = \sum_{n=1}^{\infty} \langle \vec{x}, \vec{x}_n \rangle \vec{x}_n = \lim_{N \to \infty} \sum_{n=1}^{N} \langle \vec{x}, \vec{x}_n \rangle \vec{x}_n$$

Let $\langle \vec{x}, \vec{x}_n \rangle = \alpha_n + i\beta_n$ where $\alpha_n, \beta_n \in \mathbb{Q}$.

Let $S_n = \{(\alpha_n + i\beta_n)\vec{x}_n \mid \alpha_n, \beta_n \in \mathbb{Q}\}$.
Clearly, S_n is a countable set.
Furthermore, $\left\|\sum_{n=1}^{N} \langle \vec{x}, \vec{x}_n \rangle \vec{x}_n - \vec{x}\right\| = \left\|\sum_{n=1}^{N} (\alpha_n + i\beta_n)\vec{x}_n - \vec{x}\right\| \to 0.$
Therefore S_n is dense and countable in H.

$\Longleftarrow$ Assume $S \subset H$ is a countable dense subset.

Assume $S = \{\vec{x}_n \mid n \in \mathbb{N}\}$.
Discard any $\vec{0}$ and linearly dependent elements, resulting in a linearly independent set $Y = \{\vec{y}_n \mid n \in \mathbb{N}\}$.
Note that $\mathrm{Span}(S) = \mathrm{Span}(Y)$.
Furthermore, $\overline{\mathrm{Span}(Y)} = H$ since S is dense in H.
Now apply Gram-Schmidt to Y to produce an orthonormal set $X = \{\vec{x}_n \mid n \in \mathbb{N}\}$.
So $(\vec{x}_n)$ is an orthonormal sequence in H.
Assume $\vec{x} = \sum_{n=1}^{\infty} \alpha_n \vec{x}_n \in H$.
Assume $\forall\, n \in N, \vec{x} \perp \vec{x}_n$.
$\langle \vec{x}, \vec{x}_n \rangle = \left\langle \sum_{k=1}^{\infty} \alpha_k \vec{x}_k, \vec{x}_n \right\rangle = \alpha_n = 0$
Thus, $\vec{x} = 0$ and so $(\vec{x}_n)$ is complete.

Therefore H is separable.

Theorem

Let H be a separable Hilbert space and let $S = \{\vec{x}_n \mid n \in \mathbb{N}\} \subset H$ be a mutually orthogonal set.

S is countable.

Proof

AWLOG: S is an orthonormal set (otherwise normalize the vectors in S).
Assume $\vec{x}$ and $\vec{y}$ are two distinct elements of S.
$\|\vec{x} - \vec{y}\|^2 = \langle \vec{x} - \vec{y}, \vec{x} - \vec{y} \rangle = \|\vec{x}\|^2 - \langle \vec{x}, \vec{y} \rangle - \langle \vec{y}, \vec{x} \rangle + \|\vec{y}\|^2 = 1 - 0 - 0 + 1 = 2$
So any two elements in S are separated by a distance of $\sqrt{2}$.
Now, consider the set of disjoint open balls:

$$\left\{ B\left(\vec{x}, \frac{\sqrt{2}}{2}\right) \middle| \vec{x} \in S \right\}$$

But H is separable, so it contains a countable dense subset Z.
Thus, there is at least one element of $\vec{z}_x \in Z$ in each $B\left(\vec{x}, \frac{\sqrt{2}}{2}\right)$.
Thus, $\varphi : S \to Z$ defined in this way is a one-to-one mapping.
But $\{\vec{z}_x \mid \vec{x} \in S\}$ is a countable set, and therefore S is a countable set.