Vector Spaces

Definition: Vector Space

Let $\mathbb{F}$ be a field (usually either $\mathbb{R}$ or $\mathbb{C}$) called *scalars* and let E be a set of objects called *vectors* that is equipped with two binary operators:

$$\begin{array}{lll} \text{Vector Addition:} & + : E \times E \to E & \text{where } (\vec{x}, \vec{y}) \mapsto \vec{x} + \vec{y} \\ \text{Scalar Multiplication:} & \cdot : \mathbb{F} \times E \to E & \text{where } (\alpha, \vec{x}) \mapsto \alpha\vec{x} \end{array}$$

To say that E is a *vector space* (over $\mathbb{F}$) means that the following axioms hold $\forall\, \vec{x}, \vec{y} \in E$ and $\forall\, \alpha, \beta \in \mathbb{F}$:

1). $(E, +)$ is an abelian group

  a). Closure

  b). Commutative

  c). Associative

  d). $\exists\, \vec{z}, \vec{x} + \vec{z} = \vec{y}$

2). Left Distributive: $\alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$

3). Right Distributive: $(\alpha + \beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$

4). Associative Multiplication: $\alpha(\beta\vec{x}) = (\alpha\beta)\vec{x}$

5). Multiplicative Identity: $1\vec{x} = \vec{x}$

Theorem: Additive Identity

Let E be a vector space over a scalar field $\mathbb{F}$:

$$\exists\, \vec{0} \in E, \forall\, \vec{x} \in E, \vec{x} + \vec{0} = \vec{x}$$

Moreover, $\vec{0}$ is unique.

This $\vec{0}$ is called the *additive identity* for E.

Proof

Assume $\vec{x} \in E$.
Since $(E, +)$ is an abelian group, $\exists\, \vec{0} \in E, \vec{x} + \vec{0} = \vec{x}$.
Assume $\vec{y} \in E$.
$\exists\, \vec{z} \in E, \vec{x} + \vec{z} = \vec{y}$
$\vec{y} + \vec{0} = (\vec{x} + \vec{z}) + \vec{0} = \vec{x} + (\vec{z} + \vec{0}) = \vec{x} + (\vec{0} + \vec{z}) = (\vec{x} + \vec{0}) + \vec{z} = \vec{x} + \vec{z} = \vec{y}$

Now, assume $\vec{0}, \vec{0}\,' \in E$ are both additive identities.
$\vec{0} + \vec{0}\,' = \vec{0}$
$\vec{0} + \vec{0}\,' = \vec{0}\,' + \vec{0} = \vec{0}\,'$
$\therefore \vec{0} = \vec{0}\,'$

Theorem

Let E be a vector space over a scalar field $\mathbb{F}$:

$$\forall \vec{x}, \vec{y} \in E, \exists!\, \vec{z} \in E, \vec{x} + \vec{z} = \vec{y}$$

(i.e., the $\vec{z}$ is unique)

Proof

Assume $\vec{x}, \vec{y} \in E$.
Since E is an abelian group, $\exists\, \vec{z} \in E, \vec{x} + \vec{z} = \vec{y}$.
Assume $\exists\, \vec{z}\,' \in E, \vec{x} + \vec{z}\,' = \vec{y}$.
$\exists\, \vec{u} \in E, \vec{z}\,' + \vec{u} = \vec{z}$
$\vec{y} = \vec{x} + \vec{z} = \vec{x} + (\vec{z}\,' + \vec{u}) = (\vec{x} + \vec{z}\,') + \vec{u} = \vec{y} + \vec{u}$
Thus $\vec{u}$ is the unique additive identity and:
$\therefore \vec{z} = \vec{z}\,' + \vec{u} = \vec{z}\,' + \vec{0} = \vec{z}\,'$

Theorem: Additive Inverses

Let E be a vector space over a scalar field $\mathbb{F}$:

$$\forall \vec{x} \in E, \exists(-\vec{x}) \in E, \vec{x} + (-\vec{x}) = \vec{z}$$

Moreover, $(-\vec{x})$ is unique.

$(-\vec{x})$ is called the *additive inverse* for $\vec{x}$.

Proof

Assume $\vec{x} \in E$.
Since E is an abelian group, $\exists\, (-\vec{x}) \in E, \vec{x} + (-\vec{x}) = \vec{0}$.
By previous theorem, $(-\vec{x})$ is unique.

Notation

$\vec{x} + (-\vec{y}) = \vec{x} - \vec{y}$

Theorem: Vector Space Properties

Let E be a vector space over a scalar field $\mathbb{F}$. $\forall\, \vec{x} \in E$ and $\forall\, \lambda \in \mathbb{F}$:

1). $\vec{0} = -\vec{0}$

2). $0\vec{x} = \vec{0}$

3). $\lambda\vec{0} = \vec{0}$

4). $0 - \vec{x} = -\vec{x}$

5). $(-1)x = -x$

6). $\lambda\vec{x} = 0 \implies \lambda = 0$ or $\vec{x} = \vec{0}$

Proof

1). $-\vec{0} = -\vec{0} + \vec{0} = \vec{0} + (-\vec{0}) = \vec{0}$

2). $0\vec{x} = (0+0)\vec{x} = 0\vec{x} + 0\vec{x}$
Thus $0\vec{x}$ is the unique additive identity, and:
$\therefore 0\vec{x} = \vec{0}$

3). $\lambda\vec{0} = \lambda(\vec{0} + \vec{0}) = \lambda\vec{0} + \lambda\vec{0}$
Thus $\lambda\vec{0}$ is the unique additive identity, and:
$\therefore \lambda\vec{0} = \vec{0}$

4). $0 - \vec{x} = 0 + (-\vec{x}) = (-\vec{x}) + \vec{0} = (-\vec{x})$

5). $\vec{x} + (-1)\vec{x} = 1\vec{x} + (-1)\vec{x} = [1 + (-1)]\vec{x} = 0\vec{x} = \vec{0}$
This $(-1)\vec{x}$ is the unique additive inverse for $\vec{x}$, and:
$\therefore (-1)\vec{x} = (-\vec{x})$

6). Assume $\lambda\vec{x} = \vec{0}$.
Trivial if $\vec{x} = 0$, so AWLOG $\vec{x} \neq 0$
ABC: $\lambda \neq 0$

$$\begin{aligned} \lambda\vec{x} &= \vec{0} \\ \frac{1}{\lambda}(\lambda\vec{x}) &= \frac{1}{\lambda} \cdot \vec{0} \\ (\frac{1}{\lambda} \cdot \lambda)\vec{x} &= \vec{0} \\ 1\vec{x} &= \vec{0} \\ \vec{x} &= \vec{0} \end{aligned}$$

CONTRADICTION!
$\therefore \lambda = 0$

Examples

1). $E = \{\vec{0}\}$ is called the trivial vector space.

2). $E = \mathbb{R}^n$ with $\mathbb{F} = \mathbb{R}$ equipped with component-wise addition and scalar multiplication.

3). $E = \mathbb{C}^n$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ equipped with component-wise addition and scalar multiplication.

4). Let E be a vector space over a field F and let X be a non-empty set. The set of functions:

$$\mathcal{F} = \{f : X \to E\}$$

equipped with the standard operations:

$$(f+g)(x) = f(x) + g(x)$$

$$(\lambda f)(x) = \lambda f(x)$$

is a vector space.