Maximal Ideals

Definition

Let R be a ring and $I \trianglelefteq R$. To say that I is *maximal* means:

1). I is proper

2). $\forall\, J \trianglelefteq R, I \not\subset J$.

Example

1). Let F be a field. Only the zero ideal is maximal.

2). For Z, all ideals are of the form $n\mathbb{Z}$. Since $n\mathbb{Z} \subseteq m\mathbb{Z} \iff m \mid n$, it follows that $p\mathbb{Z}$, p prime are the maximal ideals in $\mathbb{Z}$.

3). Let p be prime. $F_p = \mathbb{Z}/p\mathbb{Z}$. Since $F_p^\times = \{a + p\mathbb{Z} \mid (a,p) = 1\}$, $F_p^\times = F_p^*$ and thus F_p is a field. Therefore, the only maximal ideal is $\{p\mathbb{Z}\}$ (the zero ideal).

Theorem

Let R be a ring with $1 \neq 0$. Every proper ideal I in R is contained in a maximal ideal in R.

Proof

Assume I is a proper ideal in R
Let $\mathcal{L}$ be the set of all proper ideals of R such that $I \subseteq J$, partially ordered by inclusion
$I \in \mathcal{L}$ so $\mathcal{L} \neq \emptyset$
Note that none of the J contain a unit (since proper)
Assume $\mathcal{C}$ is a chain in $\mathcal{L}$
Let $L = \bigcup\{J : J \in \mathcal{C}\}$
$L \trianglelefteq R$
Plus, L does not contain a unit, so L is proper
Thus, $L \in \mathcal{L}$ and L is an upper bound for $\mathcal{C}$

Therefore, by Zorn's Lemma, there exists a maximal ideal $M \in \mathcal{L}$

Theorem

Let R be a commutative ring with $1 \neq 0$ and $I \trianglelefteq R$. TFAE:

1). I is maximal in R

2). R/I is simple

3). R/I is a field

<u>Proof</u>

I is maximal in R
$\iff \forall\, J \trianglelefteq R, I \not\subset J$
$\iff R/I$ has no proper ideals
$\iff R/I$ is simple
$\iff R/I$ is a field.