Euler's Formula

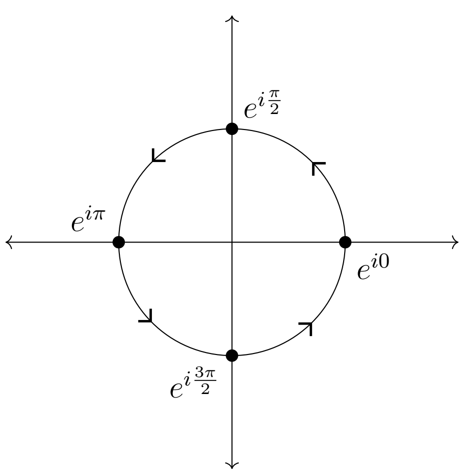
Theorem

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Proof

$$\begin{aligned}
\cos\theta &= \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} i^{2n}\frac{\theta^{2n}}{(2n)!} \\
\sin\theta &= \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} i^{2n}\frac{\theta^{2n+1}}{(2n+1)!} \\
i\sin\theta &= \sum_{n=0}^{\infty} i^{2n+1}\frac{\theta^{2n+1}}{(2n+1)!} \\
\cos\theta + i\sin\theta &= \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!} \\
&= e^{i\theta}
\end{aligned}$$

Example: Unit Circle



$$\begin{aligned}
&e^{i0} = \cos 0 + i\sin 0 = 1 + 0i = 1 \\
&e^{i\frac{\pi}{2}} = \cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2} = 0 + 1i = i \\
&e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0i = -1 \\
&e^{i\frac{3\pi}{2}} = \cos\tfrac{3\pi}{2} + i\sin\tfrac{3\pi}{2} = 0 - 1i = -i
\end{aligned}$$

Definition

The so-called *Existence of God* equation is given by:

$$e^{i\pi} + 1 = 0$$

Corollary

$$e^{-i\theta} = \cos\theta - i\sin\theta = \frac{1}{e^{i\theta}} = \overline{e^{i\theta}}$$

Proof

$$e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

$$\begin{aligned} e^{-i\theta} &= \cos\theta - i\sin\theta \\ &= \frac{(\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta)}{\cos\theta + i\sin\theta} \\ &= \frac{\cos^2\theta + \sin^2\theta}{e^{i\theta}} \\ &= \frac{1}{e^{i\theta}} \end{aligned}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta = \overline{\cos\theta + i\sin\theta} = \overline{e^{i\theta}}$$

Theorem

1). $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

2). $\frac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i(\theta_1-\theta_2)}$

3). $\left(e^{i\theta}\right)^n = e^{in\theta}, n \in \mathbb{Z}$

Proof

1).

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) \\ &= e^{i(\theta_1+\theta_2)} \end{aligned}$$

2).

$$\frac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i\theta_1}e^{-i\theta_2} = e^{i(\theta_1-\theta_2)}$$

3). Assume $n \in \mathbb{Z}$.

case 1: $n \geq 0$

Base: $n = 0$

$$\left(e^{i\theta}\right)^0 = 1$$

Assume $\left(e^{i\theta}\right)^n = e^{in\theta}$

$$\left(e^{i\theta}\right)^{n+1} = e^{i\theta}\left(e^{i\theta}\right)^n = e^{i\theta}e^{in\theta} = e^{i(n+1)\theta}$$

case 2: $n < 0$

$$\left(e^{i\theta}\right)^n = \left(e^{i\theta}\right)^{(-1)(-n)} = \left(e^{-i\theta}\right)^{-n} = e^{-i(-n)\theta} = e^{in\theta}$$

Theorem

Let $z = x + iy$:

1). $\left|e^{i\theta}\right| = 1$

2). $|e^z| = e^x$

3). $|e^{iz}| = e^{-y}$

Proof

1). $\left|e^{i\theta}\right| = |\cos\theta + i\sin\theta| = \cos^2\theta + \sin^2\theta = 1$

2). $|e^z| = |e^{x+iy}| = |e^x e^{iy}| = |e^x|\,|e^{iy}| = e^x \cdot 1 = e^x$

3). $|e^{iz}| = \left|e^{i(x+iy)}\right| = |e^{-y+ix}| = |e^{-y}e^{ix}| = |e^{-y}|\,|e^{ix}| = e^{-y} \cdot 1 = e^{-y}$

Theorem

1). $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \cosh(i\theta)$

2). $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = -i\sinh(i\theta)$

3). $\cos(i\theta) = \cosh\theta$

4). $\sin(i\theta) = i\sinh\theta$

Proof

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta \end{aligned}$$

1).

$$\begin{aligned} 2cos\theta &= e^{i\theta} + e^{-i\theta} \\ cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ cos\theta &= \cosh(i\theta) \end{aligned}$$

2).

$$\begin{aligned} 2isin\theta &= e^{i\theta} - e^{-i\theta} \\ sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \\ sin\theta &= \frac{1}{i}\sinh(i\theta) \\ sin\theta &= -i\sinh(i\theta) \end{aligned}$$

3).

$$\cos(i\theta) = \cosh(i^2\theta) = \cosh(-\theta) = \cosh\theta$$

4).

$$\sin(i\theta) = -i\sinh(i^2\theta) = -i\sinh(-\theta) = i\sinh\theta$$