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Math 275A
Homework #7

Theorem: 5.1

Let X be a topological space and let $A \subset X$. A is dense in X iff for all $U \in \mathscr{T}, U \neq \emptyset \implies U \cap A \neq \emptyset$.

Proof.

$\implies$ Assume that A is dense in X, and hence $\bar{A} = X$.

Assume that $U \in \mathscr{T}$ and assume that $U \neq \emptyset$. Since $\bar{A} = X$ it must be the case that $U \cap \bar{A} \neq \emptyset$. So assume that $x \in U \cap \bar{A}$, meaning that $x \in U$ and $x \in \bar{A}$. Therefore, since $x \in \bar{A}$, it must be the case that $U \cap A \neq \emptyset$.

$\impliedby$ Assume that $\forall U \in \mathscr{T}, U \neq \emptyset \implies U \cap A \neq \emptyset$.

Clearly, $\bar{A} \subset X$. So assume that $x \in X$. But by the assumption, $x \in \bar{A}$. Therefore $\bar{A} = X$ and hence A is dense in X.

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Example: Exercise 5.2

Show that $\mathbb{R}_{\text{std}}$ is separable. Which of the previously investigated topologies on $\mathbb{R}$ are not separable?

Consider $\mathbb{Q} \subset \mathbb{R}$ and assume that $x \in \mathbb{R} - \mathbb{Q}$. But x is the limit of some sequence in $\mathbb{Q}$ and hence $x \in \bar{\mathbb{Q}}$. This means that $\bar{\mathbb{Q}} = \mathbb{R}$ and thus $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$. Therefore $\mathbb{R}_{\text{std}}$ is separable.

For $\mathbb{R}_{LL}$, also consider $\mathbb{Q} \subset \mathbb{R}$. Assume $U \in \mathscr{T}$. This means that there exists some $a, b \in \mathbb{R}$ such that $[a, b) \subset U$. If $a \in \mathbb{Q}$ then done, so assume $a \in \mathbb{R} - \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}_{\text{std}}$, there exists $x \in \mathbb{Q}$ such that $x \in (a, b) \subset [a, b) \subset U$. Therefore $\mathbb{Q}$ is dense in $\mathbb{R}_{LL}$ as well and so $\mathbb{R}_{LL}$ is separable.

For $\mathbb{R}_{+00}$, consider $A = \{0', 0''\} \cup \mathbb{Q}^+$. Assume $U \in \mathscr{T}$. If $0' \in U$ or $0'' \in U$ then done, so assume that neither is in U. This means that there exists some $a, b \in \mathbb{R}^+$ such that $(a, b) \subset U$. Since $\mathbb{Q}$ is dense in $\mathbb{R}_{\text{std}}$, there exists $x \in \mathbb{Q}+$ such that $x \in (a, b)$. Therefore A is dense and countable in $\mathbb{R}_{+00}$ and so $\mathbb{R}_{+00}$ is separable.

Lemma

$(A \cap X) \times (B \cap Y) = (A \times B) \cap (X \times Y)$

Proof.

$$\begin{aligned}(a,b) \in (A \cap X) \times (B \cap Y) &\iff a \in A \cap X \text{ and } b \in B \cap Y \\ &\iff a \in X \text{ and } a \in X \text{ and } b \in B \text{ and } b \in Y \\ &\iff (a,b) \in A \times B \text{ and } (a,b) \in X \times Y \\ &\iff (a,b) \in (A \times B) \cap (X \times Y)\end{aligned}$$

■

Theorem: 5.5

Let X and Y be topological spaces. If X and Y are separable then $X \times Y$ is separable.

Proof. Assume that X and Y are separable. This means that there exists a countable dense $A \subset X$ and a countable dense $B \subset Y$.

Claim: $A \times B$ is countable and dense in $X \times Y$.

Since A and B are countable, $A \times B$ is countable.

Now, assume $W \in \mathscr{T}_{X \times Y}$. This means that there exists $U \in \mathscr{T}_X$ and $V \in \mathscr{T}_Y$ such that $U \times V \subset W$. But A is dense in X and so $U \cap A \neq \emptyset$. Likewise, B is dense in Y and so $V \cap B \neq \emptyset$. And so:

$$(U \cap A) \times (V \cap B) = (U \times V) \cap (A \times B) \neq \emptyset$$

Thus, $W \cap (A \times B) \neq \emptyset$ and so $A \times B$ is dense in $X \times Y$.

Therefore $A \times B$ is countable and dense in $X \times Y$ and hence $X \times Y$ is separable. ■

Theorem: 5.9

Let X be a topological space. If X is 2^{nd} countable then X is separable.

Proof. Assume that X is 2^{nd} countable and let $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ be a countable basis for X. From each B_i, select a value x_i and construct the set $A = \{x_1, x_2, \ldots\}$. Thus $x_i \mapsto B_i$ is one-to-one and so A is countable. Now assume that $U \in \mathscr{T}$. Then there exists at least some $B_i \subset U$ and hence $U \cap A \neq \emptyset$, and so A is countable and dense in X.

Therefore X is separable. ■

Example: Exercise 5.10

1. Show that $\mathbb{R}_{\text{std}}$ is 2^{nd} countable (and hence separable).

   Consider the countable set $\mathcal{B} = \{(a,b) \,|\, a, b \in \mathbb{Q}\}$. Since $\mathbb{Q}$ is countable, $\mathbb{Q} \times \mathbb{Q}$ is countable and hence $\mathcal{B}$ is countable. Now assume that $U \in \mathscr{T}$ and assume $x \in U$. Since x is an interior point of U, there exists some $\epsilon > 0$ such that $x \in (x - \epsilon, x + \epsilon) \subset U$. So there must exist $\delta \in \mathbb{Q}$ such that $0 < \delta < \epsilon$, and so $x \in (x-\delta, x+\delta) \subset (x-\epsilon, x+\epsilon) \subset U$. But $(x-\delta, d+\delta) \in \mathcal{B}$, and so $\mathcal{B}$ is a countable basis for $\mathbb{R}_{\text{std}}$.

Therefore $\mathbb{R}_{\text{std}}$ is 2^{nd} countable.

2. Show that $\mathbb{R}_{LL}$ is separable but not 2^{nd} countable.

   It was already shown that $\mathbb{R}_{LL}$ is separable. So assume that $\mathcal{B}$ is a basis for $\mathbb{R}_{LL}$ and consider $U_a = [a, \infty) = \bigcup_{b>a} [a, b) \in \mathscr{T}$. Then there exists some $B_a \in \mathcal{B}$ such that $a \in B_a$. Now, assume $x, y \in \mathbb{R}$ such that $x < y$. Since $U_y \subsetneq U_x$, there exists $B_x \subset U_x$ and $B_y \subset U_y$ such that $B_x \neq B_y$. Thus, $x \mapsto B_x$ is injective and hence $\mathcal{B}$ is uncountable.

   Therefore $\mathbb{R}_{LL}$ is not 2^{nd} countable.

<u>**Theorem: 5.11**</u>

Every uncountable set in a 2^{nd} countable space has a limit point.

Proof. Assume that X is a 2^{nd} countable space and assume that $A \subset X$ such that A is uncountable. Now, ABC that A has no limit points. This means that for all $a \in A$ it is the case that there exists $U \in \mathcal{U}_a$ such that $U_a \cap A = \{a\}$ and hence every $a \in A$ is an isolated point. So assume that $x, y \in A$ such that $x \neq y$. There exists $U \in \mathcal{U}_x$ and $V \in \mathcal{U}_y$ such that $U \neq V$. So for any basis $\mathcal{B}$ of X, there exists $B_x \subset U$ and $B_y \subset V$ such that $B_x \neq B_y$. Thus, $a \mapsto B_a$ is injective and hence $\mathcal{B}$ is uncountable, contradicting the assumption that X is 2^{nd} countable.

Therefore A contains a limit point. ∎

<u>**Theorem: 5.13**</u>

If X and Y are 2^{nd} countable spaces then $X \times Y$ is 2^{nd} countable.

Proof. Assume that $\mathcal{B}_X$ is a countable basis for X and B_y is a countable basis for Y.

Claim: $\mathcal{B}_X \times \mathcal{B}_Y$ is a countable basis for $X \times Y$.

$\mathcal{B}_X \times \mathcal{B}_Y$ is countable. So assume that $U \in \mathscr{T}_{X \times Y}$ and assume $(a, b) \in U$. This means that there exists $U_a \in \mathscr{T}_X$ and $V_b \in \mathscr{T}_Y$ such that $(a, b) \in U_a \times V_b \subset U$. Furthermore, there exists $B_a \in \mathcal{B}_X$ and $B_b \in \mathcal{B}_Y$ such that $(a, b) \in B_a \times B_b \subset U_a \times V_b \subset U$ and so $\mathcal{B}_X \times \mathcal{B}_Y$ is a countable basis for $X \times Y$.

Therefore $X \times Y$ is 2^{nd} countable. ∎

<u>**Theorem: 5.21**</u>

$\mathbb{R}_{\text{std}}$ is Souslin.

Proof. ABC that $\mathbb{R}_{\text{std}}$ is not Souslin, meaning it does contain an uncountable collection of disjoint open sets. Let $\mathcal{U}$ be such a set. Since $\mathbb{Q}$ is countable and dense in $\mathbb{R}_{\text{std}}$, every $U \in \mathcal{U}$ contains some $r_U \in \mathbb{Q}$. So select one value from each $U \in \mathcal{U}$ to construct the set $\{r_U \in Q \,|\, U \in \mathcal{U}\}$. Thus $r_U \mapsto U$ is injective and hence $\mathcal{U}$ is countable, contradicting the assumption.

Therefore $\mathbb{R}_{\text{std}}$ is Souslin. ■