Alternating Groups

Definition

The subgroup of S_n consisting of the even permutations of n letters, denoted A_n, is called the *alternating group* on n letters:

$$A_n = \{\sigma \in S_n \mid \sigma \text{ is an even}\}$$

Theorem

$A_n \leq S_n$

Proof

Assume $\sigma, \tau \in A_n$

σ and τ can each be expressed as an even number of transpositions
$\sigma\tau$ can be expressed as an even number of transpositions
$\therefore A_n$ is closed under the operation.

$()$ has length 0 and so $() \in A_n$
$\therefore A_n$ has an identity.

Assume σ^{-1} can be expressed by listing the same transpositions in reverse order
So σ^{-1} has an even number of transpositions
$o^{-1} \in A_n$
$\therefore A_n$ is closed under inverses

$\therefore A_n \leq S_n$

Example

$S_2 = \{(), (12)\}$	$\lvert S_2\rvert = 2! = 2$
$A_2 = \{()\}$	$\lvert A_2\rvert = \frac{\lvert S_2\rvert}{2} = \frac{2}{2} = 1$
$S_3 = \{(), (12), (13), (23), (123), (132)\}$	$\lvert S_3\rvert = 3! = 6$
$A_3 = \{(), (123), (132)\}$	$\lvert A_3\rvert = \frac{\lvert S_3\rvert}{2} = \frac{6}{2} = 3$
$S_4 = \{(), 6 \cdot (ab), 8 \cdot (abc), 6 \cdot (abcd), 3 \cdot (ab)(cd)\}$	$\lvert S_4\rvert = 4! = 1 + 6 + 8 + 6 + 3 = 24$
$A_4 = \{(), 8 \cdot (abc), 3 \cdot (ab)(cd)\}$	$\lvert A_4\rvert = \frac{\lvert S_4\rvert}{2} = \frac{24}{2} = 12$

Theorem

$|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$

Proof

Let $B_n = \{\tau \in S_n \mid \tau \text{ is odd}\}$
$(1, 2) \in B_n$
Let $\phi : A_n \to B_n$ be defined by $\phi(\sigma) = (1, 2)\sigma$

Assume $\phi(\sigma) = \phi(\tau)$
$(1, 2)\sigma = (1, 2)\tau$
$\sigma = \tau$
$\therefore \phi$ is one-to-one.

Assume $\tau \in B_n$
Let $\sigma = (1, 2)\tau$
$\phi(\sigma) = (1, 2)(1, 2)\tau = ()\tau = \tau$
$\therefore \phi$ is one-to-one.

So ϕ is a bijection and $|A_n| = |B_n|$
But $|S_n| = |A_n| + |B_n| = |A_n| + |A_n| = 2\,|A_n|$
$\therefore |A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$

Theorem

A_n can be generated by 3-cycles.

Proof

Assume $\sigma \in A_n$
σ is composed of an even number of transpositions

Case 1: $(ab)(cd)$

$$(ab)(cd) = (acb)(acd)$$

Case 2: $(ab)(ac)$

$$(ab)(ac) = (acb)$$

Therefore, each pair of transpositions can be condensed into a single 3-cycle.