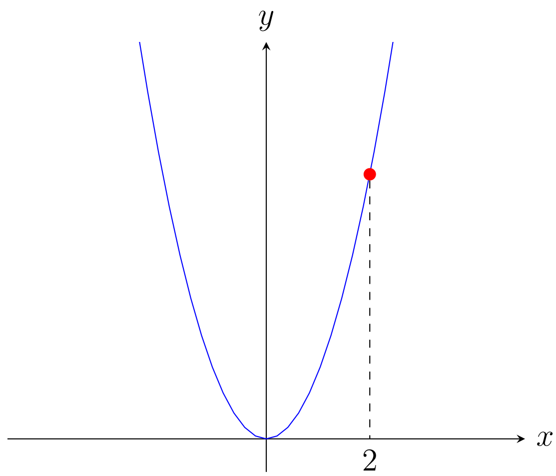
Limits

Example

Consider the standard function $f(x) = x^2$:



What happens to $f(x)$ as $x \to 2$ (but $x \neq 2$)?

x	$f(x)$
2.1	4.41
2.01	4.0401
2.001	4.004001
2.0001	4.00040001
2	???
1.9999	3.99960001
1.999	3.996001
1.99	3.9601
1.9	3.61

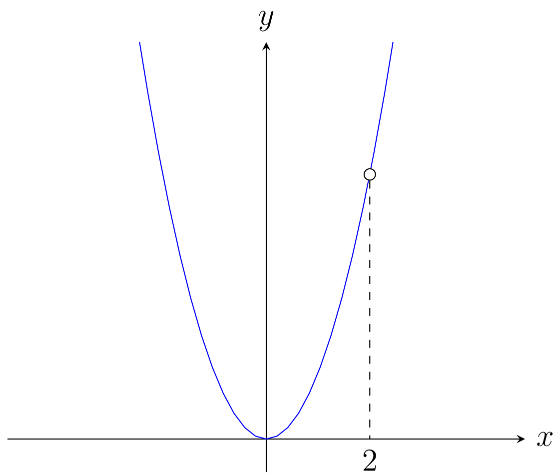
It appears that $f(x) \to 4$ as $x \to 2$ (from either direction).

In the previous example, it turns out that $f(x)$ is actually defined at $x = 2$ and furthermore, $f(2) = 4$. This special case will be used later as a formal definition of *continuity*. However, as previously stated, we don't actually care about the function value at $x = 2$. In fact, the function might not even be defined at the x value in question.

Example

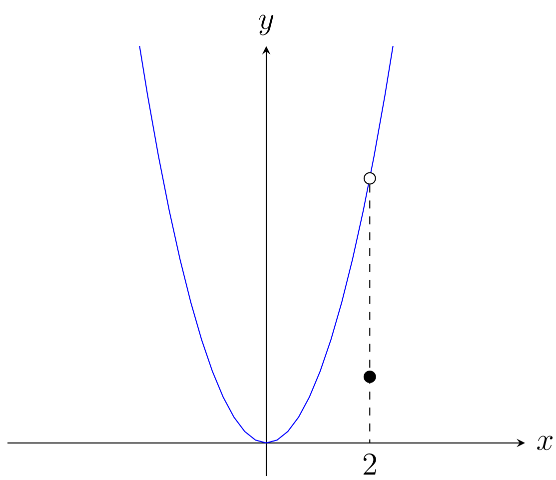
Consider the rational function:

$$f(x) = \frac{x^2(x-2)}{x-2}$$



Now, as $x \to 2$, the above table of values still applies and so it appears that $f(x) \to 4$ as $x \to 2$ (from either direction) even though $f(2)$ is not defined. To reiterate, we do not care what actually happens at $x = 2$. In fact, let's define $f(2) = 1$:

$$f(x) = \begin{cases} \frac{x^2(x-2)}{x-2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

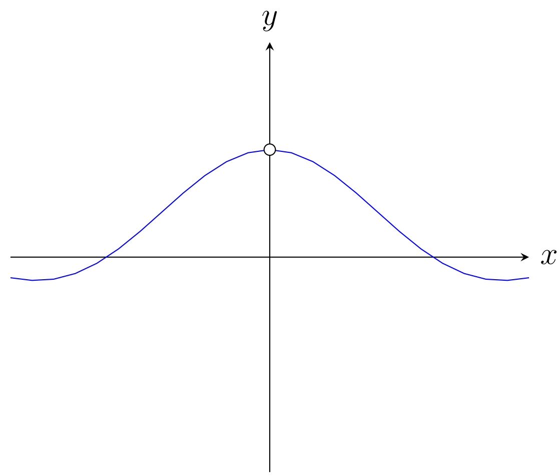


Still, $f(x) \to 4$ as $x \to 2$, regardless of the fact that $f(2) = 1$. Once again, we do not care about the function at $x = 2$; we only care what happens arbitrarily close to $x = 2$.

Example

Consider the function:

$$f(x) = \frac{\sin x}{x}$$



As $x \to 0$:

x	$f(x)$
1	0.841471
0.1	0.998334
0.01	0.999983
0	$???$
-0.01	0.999983
-0.1	0.998334
-1	0.841471

It appears that $f(x) \to 1$ as $x \to 0$.

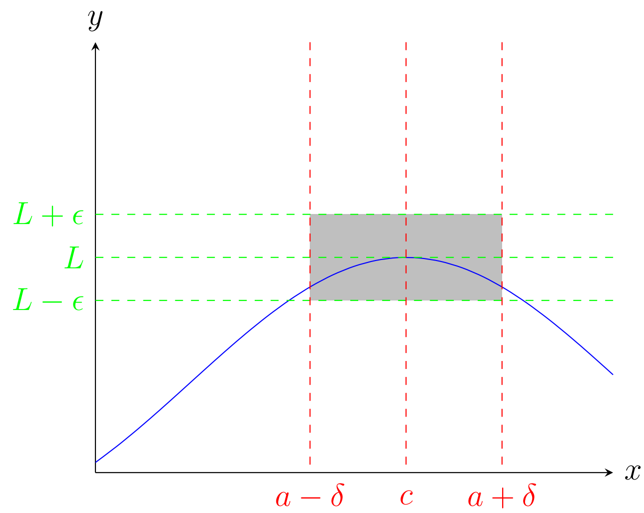
In the previous two examples, when the functions are evaluated at the point in question the result is $\frac{0}{0}$, which is one of the so-called *indeterminate forms* ($\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 1^{\infty}$). When the resulting form is indeterminate, additional effort is required to determine the actual behavior arbitrarily close to the point.

Definition: Limit of a Function at a Point

To say that L is the *limit* of a function $f(x)$ at $x = a$, denoted by $\lim_{x \to a} f(x) = L$, means that $f(x) \to L$ as $x \to a$ (but $x \neq a$):

$$\forall\, \epsilon > 0, \exists\, \delta > 0, \forall\, x \in \mathbb{R}, 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

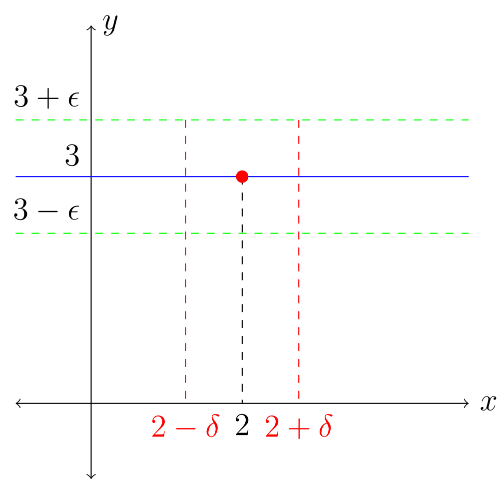
Select an $\epsilon > 0$ and then find a $\delta > 0$ such that $f(x)$ is completely contained in the bounding box $(a - \delta, a + \delta) \times (L - \epsilon, L + \epsilon)$.



It is tempting to think that $\epsilon \to 0$ forces $\delta \to 0$ and that the bounding box converges on the point a, L; however, this is not always the case.

Example

Consider the constant function $f(x) = 3$ and $a = 2$:



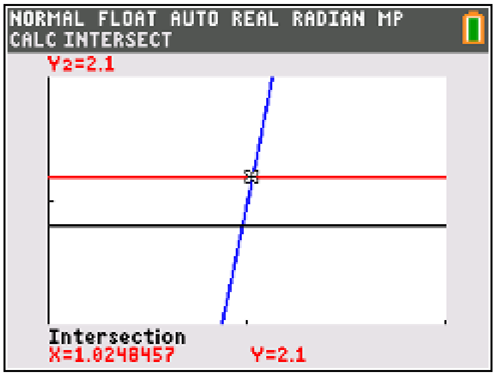
For any ϵ, any δ is sufficient.

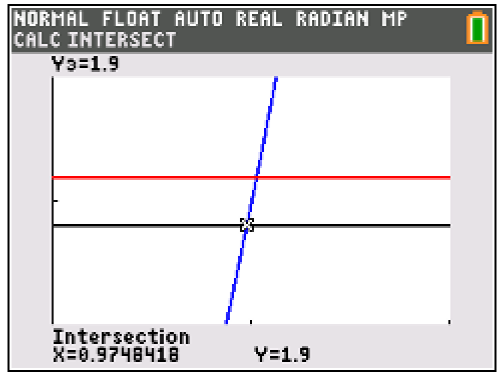
Example

Consider the function $f(x) = x^2 + 2x - 1$ and note that $\lim_{x \to 1} f(x) = 2$. Find a suitable δ to two decimal places for $\epsilon = 0.1$.

Although this can be done analytically, the algebra tends to get messy. A convenient shortcut is to use a graphing calculator. The general procedure is as follows:

1. Graph the function and mark the ϵ-neighborhood around the limit by graphing the constant functions $y = 2 + 0.1 = 2.1$ and $y = 2 - 0.1 = 1.9$. Adjust the Window so that there is sufficient separation to see all three graphs.





2. Use the *intersection* function to determine the minimum and maximum x values around $x = 1$ such that the graph of the function is completely within the marked ϵ-neighborhood.

$$x_1 = 0.9748418$$
$$x_2 = 1.0248457$$

$$0.9748418 < x < 1.0248457$$

3. Calculate the distance of each endpoint from $x = 1$:

$$\delta_1 = 1.024845 - 1 = 0.0248457$$
$$\delta_2 = 1 - 0.9748418 = 0.0251582$$

4. Select the smaller of the two distances for δ:

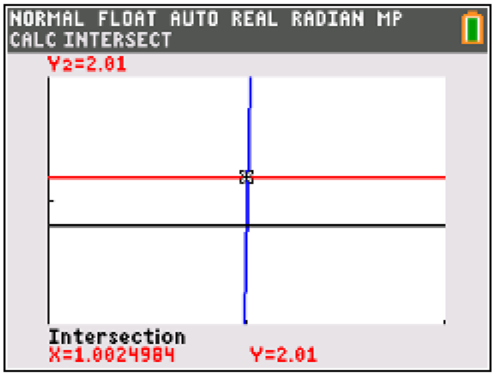
$$\delta = \min\{\delta_1, \delta_2\} = 0.0248457$$

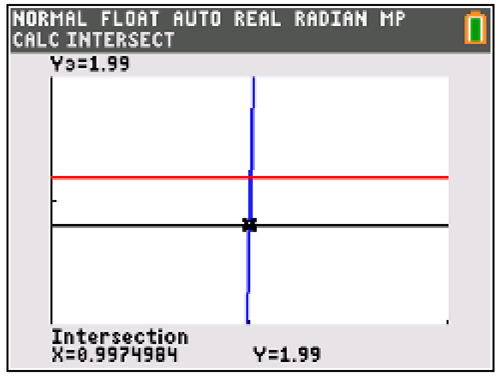
5. Be sure to round down to stay within the selected interval.

$$\delta = 0.024$$

Therefore, if $|x - 1| < 0.024$ then $|f(x) - 2| < 0.1$.

Find a suitable δ to four decimal places for $\epsilon = 0.01$.





$$\delta_1 = 1.0024984 - 1 = 0.0024984$$
$$\delta_2 = 1 - 0.9974984 = 0.0025016$$

$$\delta = \min\{\delta_1, \delta_2\} = 0.0024984$$

$\delta = 0.0024.$

Therefore, if $|x - 1| < 0.0024$ then $|f(x) - 2| < 0.01$.

<u>**Example**</u>

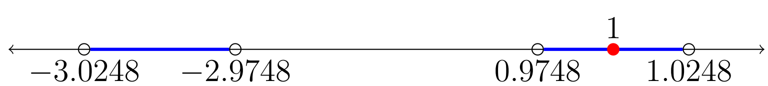
Solve the previous problem for $\epsilon = 0.1$ analytically.

$$|f(x) - 2| < 0.1$$
$$|(x^2 + 2x - 1) - 2| < 0.1$$
$$|x^2 + 2x - 3| < 0.1$$
$$-0.1 < x^2 + 2x - 3 < 0.1$$

$$x^2 + 2x - 3 > -0.1$$
$$x^2 + 2x - 2.9 > 0$$
$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2.9)}}{2(1)} = -1 \pm \sqrt{3.9}$$
$$x = -2.9748, 0.9748$$
$$0^2 + 2(0) - 2.9 = -2.9 < 0$$
$$x \in (-\infty, -2.9748) \cup (0.9748, \infty)$$

$$x^2 + 2x - 3 < 0.1$$
$$x^2 + 2x - 3.1 < 0$$
$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-3.1)}}{2(1)} = -1 \pm \sqrt{4.1}$$
$$x = -3.0248, 1.0248$$
$$0^2 + 2(0) - 3.1 = -3.1 < 0$$
$$x \in (-3.0248, 1.0248)$$

$$x \in ((-\infty, -2.9748) \cup (0.9748, \infty)) \cap (-3.0248, 1.0248)$$



$$0.9748 < x < 1.0248$$
$$\delta_1 = 1 - 0.9748 = 0.0252$$
$$\delta_2 = 1.0248 - 1 = 0.0248$$

$$\delta = \min\{\delta_1, \delta_2\} = 0.0248$$

$$\delta = 0.248$$

However, proving that $\lim_{x \to c} f(x) = L$ cannot be done by example — the result must hold for all $\epsilon > 0$.

Strategy:

1. Assume that $\epsilon > 0$.
2. Rewrite $f(x) - L < \epsilon$ as $g(x-c) < \epsilon$ for $0 < |x-c| < \delta$.
3. Consider $g(\delta) = \epsilon$.
4. Solve for $\delta(\epsilon)$.
5. Show that the selected δ works.

Helpful tools:

1. $x = (x-c) + c$
2. Triangle inequality: $|a+b| < |a| + |b|$

Template:

0. Determine a suitable $\delta(\epsilon)$ on the side.
1. Assume that $\epsilon > 0$.
2. Let $\delta = \delta(\epsilon)$ previously found.
3. Show that if $0 < |x-c| < \delta$ then $f(x) - L < e$.

Example

Prove: $\lim_{x\to 1}(2x+5) = 7$

$$|(2x+5) - 7| = |2x-2| = 2|x-1| < \epsilon$$
$$2\delta = \epsilon$$
$$\delta = \frac{\epsilon}{2}$$

Assume that $\epsilon > 0$.
Let $\delta = \frac{\epsilon}{2}$.
Assume that $0 < |x-1| < \delta$.
$|f(x) - L| = |(2x+5) - 7| = |2x-2| = 2|x-2| < 2\delta = 2\frac{\epsilon}{2} = \epsilon$

Example

Prove: $\lim_{x \to 1} (x^2 + 2x - 1) = 2$

$$\begin{aligned}
|(x^2 + 2x - 1) - 2| &= |x^2 + 2x - 3| \\
&= |(x - 1)(x + 3)| \\
&= |x - 1||x + 3| \\
&= |x - 1||(x - 1) + 4| \\
&\leq |x - 1|(|x - 1| + 4) \\
&= |x - 1|^2 + 4|x - 1| \\
&< \epsilon
\end{aligned}$$

$$\delta^2 + 4\delta = \epsilon$$

$$\delta^2 + 4\delta - \epsilon = 0$$

$$\delta = \frac{-4 \pm \sqrt{4^2 - 4(1)(-\epsilon)}}{2(1)} = -2 \pm \sqrt{4 + \epsilon}$$

$$\delta = \sqrt{4 + \epsilon} - 2$$

Assume $\epsilon > 0$.
Let $\delta = \sqrt{4 + \epsilon} - 2$.
Assume that $0 < |x - 1| > \delta$

$$\begin{aligned}
|f(x) - L| &= |(x^2 + 2x - 1) - 2| \\
&= |x^2 + 2x - 3| \\
&= |(x - 1)(x + 3)| \\
&= |x - 1||x + 3| \\
&= |x - 1||(x - 1) + 4| \\
&\leq |x - 1|(|x - 1| + 4) \\
&< \delta(\delta + 4) \\
&= \delta^2 + 4\delta \\
&= (\sqrt{4 + \epsilon} - 2)^2 + 4(\sqrt{4 + \epsilon} - 2) \\
&= (4 + \epsilon) - 4\sqrt{4 + \epsilon} + 4 + 4\sqrt{4 + \epsilon} - 8 \\
&= \epsilon
\end{aligned}$$

Example

Prove: $\lim_{x \to e} \ln(x) = 1$