Closure

Definition: Closure

Let $F \subseteq L \subseteq K$ be an inclusion of fields and let $H \leq G(L)$:

- The *closure* of L is $F(G(L))$.
- The *closure* of H is $G(F(H))$.

To say that L and G are called *closed* means:

- $F(G(L)) = L$
- $G(F(H)) = H$

Thus, K/F is closed iff K/F is Galois.

Theorem

Let $F \subseteq E \subseteq L \subseteq K$ be an inclusion of fields and subgroups $\{\mathrm{id}\} = I \leq J \leq H \leq G = \mathrm{Aut}(K/F)$:

1). $G(L) \leq G(E)$

2). $F(H) \subseteq F(J)$

3). $H \leq G(F(H))$

4). $L \subseteq F(G(L))$

5). $G(L)$ is closed

6). $F(H)$ is closed

Proof

1). Assume $\varphi \in G(L)$

Since it has already been proven that $G(L), G(E) \leq G$, it suffices to show inclusion:

$\forall\, \alpha \in L, \varphi(\alpha) = \alpha$
Since $E \subseteq L, \forall\, \alpha \in E, \varphi(\alpha) = \alpha$
$\varphi \in G(E)$

$\therefore G(L) \leq G(E)$

2). Assume $\alpha \in F(H)$

$\forall\, \varphi \in H, \varphi(\alpha) = \alpha$
Since $J \subseteq H, \forall\, \varphi \in J, \varphi(\alpha) = \alpha$
So $\alpha \in F(J)$

$\therefore F(H) \subseteq F(J)$

3). Assume $\varphi \in H$

Since it has already been proven that $H, G(F(H)) \leq G$, it suffices to show inclusion.

By definition, φ fixes everything in $F(H)$
So, by definition, $\varphi \in G(F(H))$.

$\therefore H \leq G(F(H))$

4). Assume $\alpha \in L$

By definition, α is fixed by everything in $G(L)$
So, by definition, $\alpha \in F(G(L))$.

$\therefore L \subseteq F(G(L))$

5). It has already been proven that $G(L) \subseteq G(F(G(L)))$

Assume $\varphi \in G(F(G(L)))$
φ fixes everything in $F(G(L))$
$L \subseteq F(G(L))$
So φ fixes everything in L
Thus, $\varphi \in G(L)$

$\therefore G(L) = G(F(G(L)))$ and so $G(L)$ is closed.

6). It has already been proven that $F(H) \subseteq F(G(F(H)))$

Assume $\alpha \in F(G(F(H)))$
α is fixed by everything in $G(F(H))$
$H \subseteq G(F(H))$
So α is fixed by everything in H
Thus, $\alpha \in F(H)$

$\therefore F(H) = F(G(F(H)))$.