Span

Definition: Linear Combination

Let V be a vector space over a field $\mathbb{F}$ and let S be a non-empty subset of V. To say that $\vec{v} \in V$ is a *linear combination* of the vectors in S means $\exists \{\vec{s_1}, \dots, \vec{s_n}\} \subseteq S$ and $\exists c_1, \dots, c_n \in \mathbb{F}$ such that:

$$\vec{v} = \sum_{k=1}^{n} c_k \vec{s_k}$$

Note that linear combinations are finite sums.

When all $c_k = 0$ then the linear combination is called *trivial*.

Definition: Span

Let V be a vector space over a field $\mathbb{F}$ and let $S \subseteq V$. The *span* of S, denoted $\operatorname{span}(S)$, is the set of all possible linear combinations of S.

By definition, $\operatorname{span}(\emptyset) = \{\vec{0}\}$.

Theorem

Let V be a vector space over a field $\mathbb{F}$ and let $S \subseteq V$:

$\operatorname{span}(S)$ is a subspace of V.

Proof

If $S = \emptyset$ then by definition $\operatorname{span}(S)$ is the trivial subspace. So AWLOG $S \neq \emptyset$.

Assume $\vec{u} \in \operatorname{span}(S)$
$\exists \{\vec{s_1}, \dots, \vec{s_n}\} \subseteq S$ and $\exists c_1, \dots, c_n \in \mathbb{F}$ such that $\vec{u} = \sum_{k=1}^{n} c_k \vec{s_k}$
But $\{\vec{s_1}, \dots, \vec{s_n}\} \in V$ as well, so by closure, $\vec{u} \in V$

$\therefore \operatorname{span}(S) \subseteq V$

Now assume $\vec{v} \in \operatorname{span}(S)$
$\exists \{\vec{t_1}, \dots, \vec{t_m}\} \subseteq S$ and $\exists d_1 \dots, d_m \in \mathbb{F}$ such that $\vec{v} = \sum_{k=1}^{m} d_k \vec{t_k}$
$\vec{u} + \vec{v} = \sum_{k=1}^{n} c_k \vec{s_k} + \sum_{k=1}^{m} d_k \vec{t_k}$
After combining coefficients of common vectors, $\vec{u} + \vec{v}$ is a linear combination of S, and thus $\vec{u} + \vec{v} \in \operatorname{span}(S)$

Therefore $\operatorname{span}(S)$ is closed under vector addition.

Assume $a \in \mathbb{F}$
$a\vec{u} = a \sum_{k=1}^{n} c_k \vec{s_k} = \sum_{k=1}^{n} (ac_k) \vec{s_k}$
So $a\vec{u}$ is also a linear combination of S, and thus $a\vec{u} \in \operatorname{span}(S)$

Therefore $\operatorname{span}(S)$ is closed under scalar multiplication.

Therefore $\operatorname{span}(S)$ is a subspace of V.

Theorem

Let V be a vector space over a field $\mathbb{F}$ and let $S \subseteq V$:

$\mathrm{span}(S)$ is the smallest subspace of V containing S.

Proof

Assume W is a subspace of V such that $S \subseteq W$
Assume $\vec{v} \in \mathrm{span}(S)$
$\exists \{\vec{s_1}, \ldots, \vec{s_n}\} \subseteq S$ and $\exists c_1, \ldots, c_n \in \mathbb{F}$ such that $\vec{v} = \sum_{k=1}^{n} c_k \vec{s_k}$
But $\{\vec{s_1}, \ldots, \vec{s_n}\} \subseteq W$ as well, so by closure, $\vec{v} \in W$

Therefore $\mathrm{span}(S) \subseteq W$

Corollary

Let V be a vector space over a field $\mathbb{F}$ and let $S \subseteq V$:

$\mathrm{span}(S)$ is the intersection of all subspaces of V that contain S.

Proof

Let W be the intersection of all subspaces of V containing S
W is a subspace of V containing S
But $\mathrm{span}(S)$ is the smallest such subspace, so $\mathrm{span}(S) \subseteq W$
But by construction $W \subseteq \mathrm{span}(S)$

$\therefore \mathrm{span}(S) = W$

Theorem

Let V be a vector space over a field $\mathbb{F}$ and let U, W be subspaces of V:

$$U + W = \mathrm{span}(U \cup W)$$

Proof

$\Longrightarrow$ Assume $\vec{v} \in U + W$

There exists $\vec{u} \in U$ and $\vec{w} \in W$ such that $\vec{v} = \vec{u} + \vec{w}$
$\vec{u} \in U \cup W$ and $\vec{w} \in U \cup W$
Thus $\vec{v}$ is a linear combination of vectors in U and W

$\therefore \vec{v} \in \mathrm{span}(U \cup W)$

$\Longleftarrow$ Assume $\vec{v} \in \mathrm{span}(U \cup W)$

$\vec{v} = \sum_{k=1}^{m} c_k \vec{u_k} + \sum_{k=1}^{n} d_k \vec{w_k}$ for some $u_k \in U$, $w_k \in W$, and $c_k, d_k \in \mathbb{F}$
But by closure, $\sum_{k=1}^{m} c_k \vec{u_k} \in U$ and $\sum_{k=1}^{n} d_k \vec{w_k} \in W$

$\therefore \vec{v} \in U + W$