Joins

Definition

Let G be a group and $H, K \leq G$:

$$HK = \{hk \mid h \in H \text{ and } k \in K\}$$

The *join* of H and K, denoted $H \vee K$, is the smallest subgroup of G containing HK.

Theorem

Let G be an abelian group:

$$H, K \leq G \implies HK = H \vee K$$

Proof

Assume $H, K \leq G$
Assume $a, b \in HK$
$\exists\, h_1 \in H$ and $k_1 \in K, a = h_1 k_1$
$\exists\, h_2 \in H$ and $k_2 \in K, b = h_2 k_2$
H and K are groups, so $h_2^{-1} \in H$ and $k_2^{-1} \in K$
$H, K \subseteq G$, so $h_2, k_2 \in G$, and by closure, $b = h_2 k_2 \in G$
G is a group, so $b^{-1} = (h_2 k_2)^{-1} \in G$
$ab^{-1} = (h_1 k_1)(h_2 k_2)^{-1} = (h_1 k_1)(k_2^{-1} h_2^{-1}) = (h_1 h_2^{-1})(k_1 k_2^{-1})$
But, by closure, $h_1 h_2^{-1} \in H$ and $k_1 k_2^{-1} \in K$
$ab^{-1} \in HK$
So, by the subgroup test, $HK \leq G$
HK is the smallest subgroup containing HK
$\therefore HK = H \vee K$

Theorem

Let G be a group and $H, K \leq G$:

$$H, K \subseteq HK$$

Proof

Assume $h \in H$	Assume $k \in K$
$e \in K$	$e \in H$
$he = h \in HK$	$ek = k \in HK$
$\therefore H \subseteq HK$	$\therefore K \subseteq HK$

Corollary

Let G be a group and $H, K \leq G$:

$$H, K \leq H \vee K$$

Proof

$H, K \subseteq HK$
$HK \subseteq H \vee K$
$H, K \subseteq H \vee K$
But H and K are also groups
$\therefore H, K \leq H \vee K$

Theorem

Let G be a group and $H, K \leq G$:

$H \vee K$ is the smallest subgroup of G containing H and K.

Proof

Assume $S \leq G$ such that $H, K \leq S$
Assume $hk \in H \vee K$
$h \in H$ and $k \in K$
$h \in S$ and $k \in S$
So, by closure, $hk \in S$
$H \vee K \subseteq S$ and $H \vee K$ is a group
$\therefore H \vee K$ is the smallest subgroup of G containing H and K.