EXAM 2

Math 161a: Appl. Prob. & Stats.
Instructor: Guangliang Chen
San Jose State University
Spring 2018

You have 75 minutes.

No books, but you are allowed to use a flash-card (provided by the instructor) as cheat sheet.

Please write legibly (unrecognizable work will receive zero credit).

You must show all necessary steps to receive full credit.

Good luck!

Name: ______________________________

1. __________

2. __________

3. __________

4. __________

5. __________

6. __________

"I have adhered to the SJSU Academic Integrity Policy in completing this exam."

Signature: ______________________

Date: ______________________

Total score: ________________ (/50 points)

List of distributions covered in class

- Bernoulli ($X \sim \text{Bernoulli}(p)$): $f_X(x) = p^x(1-p)^{1-x}$ for $x = 0, 1$
  - $\text{E}(X) = p$
  - $\text{Var}(X) = p(1-p)$
- Binomial ($X \sim \text{B}(n,p)$): $f_X(x) = \binom{n}{x}p^x(1-p)^{n-x}$ for $x = 0, 1, \ldots, n$
  - $\text{E}(X) = np$
  - $\text{Var}(X) = np(1-p)$
- HyperGeometric ($X \sim \text{HyperGeom}(N, r, n)$): $f_X(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$ for $x = 0, 1, \ldots, n$
  - $\text{E}(X) = \frac{nr}{N} = np$ (where $p = \frac{r}{N}$)
  - $\text{Var}(X) = np(1-p)\left(\frac{N-n}{N-1}\right)$
- Geometric ($X \sim \text{Geom}(p)$): $p(x) = p(1-p)^{x-1}$ for $x = 1, 2, \ldots$
  - $\text{E}(X) = \frac{1}{p}$
  - $\text{Var}(X) = \frac{1-p}{p^2}$
- Negative Binomial ($X \sim \text{NB}(p, r)$): $p(x) = \binom{x-1}{r-1}p^r(1-p)^{x-r}$ for $x = r, r+1, \ldots$
  - $\text{E}(X) = \frac{r}{p}$
  - $\text{Var}(X) = \frac{r(1-p)}{p^2}$
- Poisson ($X \sim \text{Pois}(\lambda)$): $p(x) = \frac{\lambda^x}{x!}e^{-\lambda}$ for $x = 0, 1, 2, \ldots$
  - $\text{E}(X) = \lambda$
  - $\text{Var}(X) = \lambda$
- Uniform ($X \sim \text{Unif}(a, b)$): $f(x) = \frac{1}{b-a}$ for $a < x < b$
  - cdf: $F(x) = \frac{x-a}{b-a}$ for $a < x < b$.
  - $\text{E}(X) = \frac{a+b}{2}$
  - $\text{Var}(X) = \frac{(b-a)^2}{12}$
- Normal ($X \sim \text{N}(\mu, \sigma)$): $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for $-\infty < x < \infty$.
  - $\text{E}(X) = \mu$
  - $\text{Var}(X) = \sigma^2$
- Exponential ($X \sim \text{Exp}(\lambda)$): $f(x) = \lambda e^{-\lambda x}$ for $x > 0$.
  - cdf: $F(x) = 1 - e^{-\lambda x}$ for $x > 0$.
  - $\text{E}(X) = \frac{1}{\lambda}$,
  - $\text{Var}(X) = \frac{1}{\lambda^2}$

1. *(10 pts)*. What distribution does the random variable X in each of the following questions have? Write down both the distribution name and parameter value(s) directly.

(a) A couple decides to have four kids in total. Suppose the probability of having a boy is $\frac{1}{2}$. Let $X=$ # boys the couple will have.

Answer: $X \sim B(n = 4, p = \frac{1}{2})$

(b) Another couple wants to have two daughters (so they will stop giving birth as soon as they have got two daughters). Assume the same probability of having boys $\frac{1}{2}$. Let $X=$ the total number of kids this couple will end up with.

Answer: $X \sim NB(p = \frac{1}{2}, r = 2)$

(c) Let X = the number of diamonds in a poker hand that is dealt from a well-shaffled ordinary deck of 52 cards.

Answer: $X \sim \text{HyperGeom}(N = 52, r = 13, n = 5)$

(d) Suppose that you just bought a new computer of certain brand and know that the average number of repairs that is needed for the brand over one year is 0.6. Let X = the total number of repairs that will need to be done for your computer in the coming year.

Answer: $X \sim \text{Pois}(\lambda = 0.6)$

(e) Assume the same setting as in (d), but define instead X = amount of time between the purchase of the product and the first repair.

Answer: $X \sim \text{Exp}(\lambda = 0.6)$

2. *(10 pts)* Suppose that X is a random variable whose pdf is given by

$$f(x) = C(4-2x), \quad 0 < x < 2.$$

(a) What is the value of C?

Answer: From

$$1 = \int_0^2 C(4-2x)\mathrm{d}x = C(4x - x^2)]_0^2 = C(4-0) = 4C$$

we obtain that $C = \frac{1}{4}$.

(b) Find $P(X > 1)$

Answer:

$$P(X > 1) = \int_1^2 \frac{1}{4}(4-2x)\mathrm{d}x = \frac{1}{4}(4x - x^2)]_1^2 = \frac{1}{4}(4-3) = \frac{1}{4}$$

(c) Find the critical value $z_{.01}$.

Answer: $z_{.01} = 2.33$ by using the standard normal table.

(d) What is the expected value of X?

Answer:

$$E(X) = \int_0^2 x \cdot \frac{1}{4}(4-2x)\mathrm{d}x = \frac{1}{4}\int_0^2 4x - 2x^2\mathrm{d}x = \frac{1}{4}\left(2x^2 - \frac{2}{3}x^3\right)]_0^2 = \frac{1}{4}\left(\frac{8}{3} - 0\right) = \frac{2}{3}.$$

3. *(10 pts)* Suppose that the total number of miles that a certain brand of auto can be driven before it would need to be junked is an exponential r.v. with an average life mileage of 250,000 miles. Smith has a used car that has been driven only 50,000 miles.

(a) If you purchase the car, what is the probability that you would get at least 200,000 more miles out of it?

Answer. Let X be the total mileage of the car in the end. Then X has an exponential distribution with parameter $\lambda = \frac{1}{250{,}000}$. Using the memoryless property, the probability that the car can be driven for at least 200,000 more miles is

$$P(X > 200,000+50,000 \mid X > 50,000) = P(X > 200,000) = \int_{200{,}000}^{\infty} \frac{1}{250,000} e^{-\frac{x}{250{,}000}} \, \mathrm{d}x = e^{-\frac{4}{5}}.$$

Note. You could select the unit as thousand of miles to simplify the numbers.

(b) Repeat under the assumption that the life-time mileage of the car is not exponentially distributed, but rather is uniformly distributed over (0, 300,000).

Answer. Under the new assumption of a uniform distribution, the pdf of X is

$$f(x) = \frac{1}{300,000}, \quad 0 < x < 300,000.$$

The probability in question is

$$\begin{aligned} P(X > 200,000 + 50,000 \mid X > 50,000) &= \frac{P(X > 250,000, X > 50,000)}{P(X > 50,000)} \\ &= \frac{P(X > 250,000)}{P(X > 50,000)} \\ &= \frac{\frac{50{,}000}{300{,}000}}{\frac{250{,}000}{300{,}000}} = \frac{1}{5}. \end{aligned}$$

4. *(10 pts)* Use the normal approximation (with continuity correction) to find the probability of getting 520 heads or more in 1000 tosses of a fair coin.

Answer. Let X be the number of heads that can be obtained from tossing a fair coin 1000 times. Then $X \sim B(n = 1000, p = \frac{1}{2})$ and it approximately has a normal distribution

$$X \sim N(\mu = np = 500, \sigma^2 = np(1-p) = 250).$$

Thus,

$$\begin{aligned}
P(X \geq 520) &= P(X > 519.5) && \text{(continuity correction)} \\
&= P\left(\frac{X - np}{\sqrt{np(1-p)}} > \frac{519.5 - 500}{\sqrt{250}}\right) \\
&\approx P(Z > 1.23) \\
&= 1 - 0.8907 = 0.1093
\end{aligned}$$

5. *(10 pts)* Let X, Y be two discrete random variables that have the following joint pmf

$y \backslash x$	0	1
-1	0.1	0.1
0	0.1	0.3
1	0.3	0.1

(a) Determine the following probabilities:

$$P(X = 0, Y = 0.1) = \mathbf{0}$$
$$P(X \leq 0, Y \leq 0) = \mathbf{0.1{+}0.1{=}0.2}$$

(b) Find the marginal distributions of X and Y.

x	0	1
p	0.5	0.5

y	p
-1	0.2
0	0.4
1	0.4

(c) What is the conditional distribution of Y given $X = 1$?

y	p
-1	0.2
0	0.6
1	0.2

(d) Are X, Y independent? State your reason clearly.

Answer. No, because $f(y \mid x = 1) \neq f_Y(y)$. Another way to see that they are not independent is to note that $f(x, y) \neq f_X(x)f_Y(y)$ for some pairs like $x = y = 0$:

$$f(0, 0) = 0.1 \neq 0.5 \cdot 0.4 = f_X(0)f_Y(0).$$