Topological Spaces

Definition: Topology

Let X be a set. To say that $\mathscr{T}$ is a *topology* on X means that $\mathscr{T}$ is a collection of subsets of X such that:

1. $\emptyset \in \mathscr{T}$
2. $X \in \mathscr{T}$
3. $U, V \in \mathscr{T} \implies U \cap V \in \mathscr{T}$
4. $\{U_\alpha : \alpha \in \lambda\}$ such that $U_\alpha \in \mathscr{T} \implies \bigcup_{\alpha \in \lambda} U_\alpha \in \mathscr{T}$.

A *topological space* is a tuple $(X, \mathscr{T})$ where X is a set and $\mathscr{T}$ is a topology on X.

Definition: Open Set

Let $(X, \mathscr{T})$ be a topological space and let $U \subset X$. To say that U is an *open set* in $(X, \mathscr{T})$ means that $U \in \mathscr{T}$.

Theorem

The intersection of a finite number of open sets is open.

Proof. Let $(X, \mathscr{T})$ be a topological space and Let $\{U : 1 \leq i \leq n\}$ be a finite collection of open sets in $(X, \mathscr{T})$.

Induction on n.

Base Case: $n = 1$

$$\bigcap_{i=1}^{1} U_i = U_1 \in \mathscr{T}$$

Inductive Hypothesis: Assume $\bigcap_{i=1}^{n} U_i \in \mathscr{T}$.

Inductive Step: Consider $n + 1$.

$$\bigcap_{i=1}^{n+1} U_i = \bigcap_{i=1}^{n} U_i \cap U_{n+1}$$

But $\bigcap_{i=1}^{n} U_i \in \mathscr{T}$ (inductive assumption) and $U_{n+1} \in \mathscr{T}$ (assumption).

$\therefore \bigcap_{i=1}^{n+1} U_i \in \mathscr{T}$ (property 3).

Therefore, by the principle of induction, $\bigcap_{i=1}^{n} U_i \in \mathscr{T}$. ∎

Note that this proof does not work for an infinite number of open sets because induction is only valid for a finite number of steps.

Theorem

Let $X, \mathscr{T}$ be a topological space and let $U \subset X$:

$$U \in \mathscr{T} \iff \forall\, x \in U, \exists\, U_x \in \mathscr{T}, x \in U_x \subset U$$

Proof.

$\implies$ Assume $U \in \mathscr{T}$.

Assume $x \in U$. So $x \in U \subset U$.

$\impliedby$ Assume $\forall\, x \in U, \exists U_x \in \mathscr{T}, x \in U_x \subset U$.

Claim: $\bigcup_{x \in U} U_x = U$

First, assume that $y \in \bigcup_{x \in U} U_x$. Therefore $\exists\, x \in U$ such that $y \in U_x \subset U$.

Next, assume that $y \in U$. This means that $\exists\, U_y \in \mathscr{T}$ such that $y \in U_y \subset U$, and therefore $y \in \bigcup_{x \in U} U_x$.

Thus, U is an arbitrary union of open sets and hence is open.

$\therefore U \in \mathscr{T}$

■

Definition

Let $(X, \mathscr{T})$ be a topological space and let $x \in X$. To say that a set $U \subset X$ is a *neighborhood* of x means that $x \in U$ and $U \in \mathscr{T}$.

Corollary

Let $(X, \mathscr{T})$ be a topological space and let $U \subset X$. U is an open set iff every point in U has a neighborhood that lies within U.

Definition: Open Ball

Let $p \in \mathbb{R}^n$. The *open ball* around p of radius $\epsilon > 0$ is given by:

$$B(p, \epsilon) = \{x \in X \mid d(p, x) < \epsilon\}$$

where $d(p, x)$ is the Euclidean distance from p to x given by:

$$d(p, x) = \sqrt{\sum_{i=1}^{n} (p_i - x_i)^2}$$

Definition: Topologies

Standard: $(\mathbb{R}^n, \mathscr{T}_{\text{std}})$

$$U \in \mathscr{T} \iff \forall x \in U, \exists \epsilon_x > 0, B(x, \epsilon_x) \subset U$$

Discrete: $(X, 2^X)$

$$\forall U \subset X, U \in \mathscr{T}$$

Indiscrete: $(X, \{\emptyset, X\})$

$$\mathscr{T} = \{\emptyset, X\}$$

Cofinite: $(X, \mathscr{T})$

$$U \in \mathscr{T} \iff U = \emptyset \text{ or } X - U \text{ is finite}$$

Cocountable: $(X, \mathscr{T})$

$$U \in \mathscr{T} \iff U = \emptyset \text{ or } X - U \text{ is countable}$$

Example

Verify that $\mathscr{T}_{\text{std}}$ is a topology on R^n.

1. $\emptyset \in \mathscr{T}_{\text{std}}$ (vacuously).
2. Assume $x \in R^n$.

   Let $e_x = 1$. Since R^n includes everything it must be the case that $B(x, 1) \subset R^n$.

   Therefore $R^n \in \mathscr{T}_{\text{std}}$.
3. Assume $U, V \in \mathscr{T}_{\text{std}}$.

   Assume $x \in U \cap V$. This means that $x \in U$ and $x \in V$. So there exists ϵ_U, ϵ_V such that $B(x, \epsilon_U) \subset U$ and $B(x, \epsilon_V) \subset V$. Let $\epsilon = \min\{\epsilon_U, \epsilon_V\}$. Thus $B(x, \epsilon) \subset U \cap V$.

   Therefore $U \cap V \in \mathscr{T}_{\text{std}}$.
4. Assume $\{U_\alpha : \alpha \in \lambda\}$ is a family of sets such that $U_\alpha \in \mathscr{T}_{\text{std}}$.

   Let $U = \bigcup_{\alpha \in \lambda} U_\alpha$ and assume $x \in U$. This means that there exists $\alpha \in \lambda$ such that $x \in U_\alpha$. Furthermore, there exists $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset U_\alpha \subset U$.

   Therefore $U \in \mathscr{T}_{\text{std}}$.

Example

Verify that the discrete, indiscrete, cofinite, and cocountable topologies are indeed topologies for any set X.

Discrete

1. $\emptyset \subset X$ and so $\emptyset \in \mathscr{T}$.
2. $X \subset X$ and so $X \in \mathscr{T}$.
3. Assume $U, V \in \mathscr{T}$.

   $U, V \subset X$ and so $U \cap V \subset X$.

   Therefore $U \cap V \in \mathscr{T}$.
4. Assume $\{U_\alpha : \alpha \in \lambda\}$ such that $U_\alpha \in \mathscr{T}$.

   $\forall\, \alpha \in \lambda, U_\alpha \subset X$ and so $\bigcup_{\alpha\in\lambda} U_\alpha \subset X$.

   Therefore $\bigcup_{\alpha\in\lambda} U_\alpha \in \mathscr{T}$.

Indiscrete

1. By definition, $\emptyset \in \mathscr{T}$.
2. By definition, $X \in \mathscr{T}$.
3. Assume $U, V \in \mathscr{T}$.

$$\begin{aligned} \emptyset \cap \emptyset &= \emptyset \in \mathscr{T} \\ \emptyset \cap X &= \emptyset \in \mathscr{T} \\ X \cap \emptyset &= \emptyset \in \mathscr{T} \\ X \cap X &= X \in \mathscr{T} \end{aligned}$$

   Therefore $U \cap V \in \mathscr{T}$.
4. Assume $\{U_\alpha : \alpha \in \lambda\}$ such that $U_\alpha \in \mathscr{T}$.

$$\begin{aligned} \emptyset \cup \emptyset &= \emptyset \in \mathscr{T} \\ \emptyset \cup X &= X \in \mathscr{T} \\ X \cup \emptyset &= X \in \mathscr{T} \\ X \cup X &= X \in \mathscr{T} \end{aligned}$$

   Therefore $\bigcup_{\alpha\in\lambda} U_\alpha \in \mathscr{T}$.

Cofinite

1. By definition, $\emptyset \in \mathscr{T}$.
2. $X - X = \emptyset$, which is finite. Therefore $X \in \mathscr{T}$.
3. Assume $U, V \in \mathscr{T}$.

   $X - (U \cap V) = (X - U) \cup (X - V)$. But $X - U$ and $X - V$ are both finite and so their union is finite.

   Therefore $U \cap V \in \mathscr{T}$.

4. Assume $\{U_\alpha : \alpha \in \lambda\}$ such that $U_\alpha \in \mathscr{T}$.

   $X - \bigcup_{\alpha\in\lambda} U_\alpha = \bigcap_{\alpha\in\lambda}(X - U_\alpha)$. But for all $\alpha \in \lambda$ it is the case that $X - U_\alpha$ is finite and so their intersection is finite.

   Therefore $\bigcup_{\alpha\in\lambda} U_\alpha \in \mathscr{T}$.

Cocountable

1. By definition, $\emptyset \in \mathscr{T}$.
2. $X - X = \emptyset$, which is finite and thus countable. Therefore $X \in \mathscr{T}$.
3. Assume $U, V \in \mathscr{T}$.

   $X - (U \cap V) = (X - U) \cup (X - V)$. But $X - U$ and $X - V$ are both countable and so their union is countable.

   Therefore $U \cap V \in \mathscr{T}$.
4. Assume $\{U_\alpha : \alpha \in \lambda\}$ such that $U_\alpha \in \mathscr{T}$.

   $X - \bigcup_{\alpha\in\lambda} U_\alpha = \bigcap_{\alpha\in\lambda}(X - U_\alpha)$. But for all $\alpha \in \lambda$ it is the case that $X - U_\alpha$ is countable. Now, for some $\alpha \in \lambda$:

   $$\bigcap_{\alpha\in\lambda}(X - U_\alpha) \subset (X - U_\alpha)$$

   However, the subset of a countable set is countable.

   Therefore $\bigcup_{\alpha\in\lambda} U_\alpha \in \mathscr{T}$.

Example

Describe some of the open sets you get if $\mathbb{R}$ is endowed with the standard, discrete, indiscrete, cofinite, and cocountable topologies. Specifically, identify sets that demonstrate the differences among these topologies, that is, find sets that are open in some topologies but not in others. For each of the topologies, determine if the interval $(0, 1)$ is an open set in that topology.

Of course, $\emptyset$ and $\mathbb{R}$ are in $\mathscr{T}$ for all topologies.

Standard: All open intervals, but not closed intervals, are in $\mathscr{T}$. Therefore $(0, 1) \in \mathscr{T}$.

Discrete: All open and closed intervals are in $\mathscr{T}$. Therefore $(0, 1) \in \mathscr{T}$ and $[0, 1] \in \mathscr{T}$

Indiscrete: Nothing other than $\emptyset$ and $\mathbb{R}$. Therefore $(0, 1) \notin \mathscr{T}$.

Cofinite All open sets are of the form $\mathbb{R} - X$ where X is finite. For example: $\mathbb{R} - \{1, 2, 3\}$. Thus, the open sets are uncountable. Such sets are also open sets in all the other topologies sans indiscrete. Therefore $(0, 1) \notin \mathscr{T}$.

Cocountable All open sets are of the form $\mathbb{R} - X$ where X is countable. Thus, all open sets must include all but a countable number of irrational numbers and are thus uncountable. For example: $\mathbb{R} - \mathbb{Q}$ or $(\mathbb{R} - \mathbb{Q}) \cup \{\sqrt{2}, \sqrt{3}\}$ or $(\mathbb{R} - \mathbb{Q}) \cup \{\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots\}$. Such sets are also open in the standard and discrete topologies. Since finite sets are countable, it is the case that $\mathscr{T}_{cof} \subset \mathscr{T}_{coc}$. Therefore $(0,1) \notin \mathscr{T}$.

Example

Give an example of a topological space and a collection of open sets in that topological space that show that the infinite intersection of open sets need not be open.

Consider $(\mathbb{R}, \mathscr{T}_{\text{std}})$ and the open sets $\left\{(-\frac{1}{n}, \frac{1}{n}) \,\middle|\, n \in \mathbb{N}\right\}$. Their infinite intersection is $\{0\}$, which is not open.