Bases

Definition: Basis

Let $(X, \mathscr{T})$ be a topological space and let $\mathcal{B} \subset \mathscr{T}$. To say that $\mathcal{B}$ is a *basis* for $\mathscr{T}$ means that:

$$\forall U \in \mathscr{T}, \exists \mathcal{B}_U \subset \mathcal{B}, U = \bigcup \mathcal{B}_U$$

The elements of $\mathcal{B}$ are called *basis elements* or *basic open sets*.

Theorem

Let $(X, \mathscr{T})$ be a topological space and let $\mathcal{B} \subset \mathscr{T}$. $\mathcal{B}$ is a basis for $\mathscr{T}$ iff:

$$\forall U \in \mathscr{T}, \forall p \in U, \exists V \in \mathcal{B}, p \in V \subset U$$

Proof.

$\Longrightarrow$ Assume that $\mathcal{B}$ is a basis for $\mathscr{T}$.

Assume $U \in \mathscr{T}$. This means that there exists $\mathcal{B}_U \subset \mathcal{B}$ such that $U = \bigcup \mathcal{B}_U$. Now, assume that $p \in U$. Thus, $p \in \bigcup \mathcal{B}_U$ and therefore there exists some $V \in \mathcal{B}_U$ such that $p \in V \subset U$.

$\Longleftarrow$ Assume $\forall U \in \mathscr{T}, \forall p \in U, \exists V \in \mathcal{B}, p \in V \subset U$

Assume that $U \in \mathscr{T}$. For each $p \in U$, choose a set $V_p \in \mathcal{B}$ such that $p \in V_p \subset U$. Thus $U = \bigcup_{p \in U} V_p$ and so every $U \in \mathscr{T}$ is generated by $\mathcal{B}$. Therefore $\mathcal{B}$ is a basis for $\mathscr{T}$.

$\blacksquare$

Example

Let $\mathscr{T}$ be the standard topology on $\mathbb{R}$. Show that the following are bases for $\mathscr{T}$:

1. $\mathcal{B}_1 = \{(a, b) \subset \mathbb{R} \mid a, b \in \mathbb{Q}\}$

   All the (a, b) are open and hence $\mathcal{B}_1 \subset \mathscr{T}$. So assume $U \in \mathscr{T}$ and assume $p \in U$. Since U is open, there exists an open ball $B(p, \epsilon) \subset U$. Since there exists an infinite number of rationals between any two reals, select two rationals $a \in (p - \epsilon, p)$ and $b \in (p, p + \epsilon)$. Thus $(a, b) \in \mathcal{B}_1$ and $p \in (a, b) \subset U$. Therefore, by the previous theorem, $\mathcal{B}_1$ is a basis for $\mathscr{T}$.

2. $\mathcal{B}_2 = \{(a, b) \cup (c, d) \subset \mathbb{R} \mid a, b, c, d \in \mathbb{R} - \mathbb{Q}$ and $a < b < c < d\}$

   All the $(a, b) \cup (c, d)$ are unions of open sets, so they are open as well and hence $\mathcal{B}_2 \subset \mathscr{T}$. So assume $U \in \mathscr{T}$ and assume $p \in U$. Since U is open, there exists an open ball $B(p, \epsilon) \in U$. Since there exists an infinite number of irrationals between any two real numbers, select four irrationals as follows:

$$
\begin{aligned}
&a \in (p-\epsilon, p) \\
&b \in (p, p+\epsilon) \\
&c \in (b, p+\epsilon) \\
&d \in (c, p+\epsilon)
\end{aligned}
$$

Thus $a < b < c < d$ and so $(a,b) \cup (c,d) \in \mathcal{B}_2$. Furthermore, $p \in (a,b) \cup (c,d) \subset U$. Therefore, by the previous theorem, $\mathcal{B}_2$ is a basis for $\mathscr{T}$.

<u>Theorem</u>

Let X be a set and let $\mathcal{B} \subset 2^X$. $\mathcal{B}$ is a basis for some topology $\mathscr{T}$ on X iff:

1. $\forall\, p \in X, \exists\, V_p \in \mathcal{B}, p \in V_p$
2. $\forall\, U, V \in \mathcal{B}, \forall\, p \in U \cap V, \exists\, W \in \mathcal{B}, p \in W \subset U \cap V$

Proof.

$\implies$ Assume $\mathcal{B}$ is a basis for some topology $\mathscr{T}$ on X.

Since $X \in \mathscr{T}$, (1) must hold by the previous theorem. So assume $U, V \in \mathcal{B}$. If $U \cap V = \emptyset$ then (2) is vacuously true, so assume that $U \cap V \neq \emptyset$ and assume $p \in U \cap V$. Since $U, V \in \mathcal{B} \subset \mathscr{T}$, both U and V are open and hence $U \cap V$ is open. So there exists some $U_p \in \mathscr{T}$ such that $p \in U_p \subset U \cap V$. But U_p is generated by $\mathcal{B}$, and so there must exist some $W \in \mathcal{B}$ such that $p \in W \subset U \cap V$. Therefore (2) holds as well.

$\impliedby$ Assume that properties (1) and (2) hold.

WTS: $\mathscr{T} = \left\{ U \subset 2^X \,\middle|\, \exists\, \mathcal{B}_U \subset \mathcal{B}, U = \bigcup \mathcal{B}_U \right\}$ is a topology on X.

First, consider $\emptyset$. Since $\emptyset \subset \mathcal{B}$ and $\emptyset$ is the union of no sets, $\emptyset \in \mathscr{T}$.

Next, consider X. By (1), X is generated by $\mathcal{B}$ and hence $X \in \mathscr{T}$.

Next, assume $U, V \in \mathscr{T}$ and let U be generated by $\mathcal{B}_U \subset \mathcal{B}$ and let V be generated by $\mathcal{B}_V \subset \mathcal{B}$, specifically:

$$
\begin{aligned}
U &= \bigcup \mathcal{B}_U = \bigcup_{\alpha \in A} B_\alpha \\
V &= \bigcup \mathcal{B}_V = \bigcup_{\lambda \in \Lambda} B_\lambda
\end{aligned}
$$

Then:

$$
U \cap V = \bigcup \mathcal{B}_U \cap \bigcup \mathcal{B}_V = \bigcup_{\alpha \in A} B_\alpha \cap \bigcup_{\lambda \in \Lambda} B_\lambda = \bigcup_{\alpha \in A, \lambda \in \Lambda} (B_\alpha \cap B_\lambda)
$$

But by (2), each of the $B_\alpha \cap B_\lambda$ is generated by some subset of $\mathcal{B}$, and hence $U \cap V$ is generated by some subset of $\mathcal{B}$. Therefore $U \cap V \in \mathscr{T}$.

Finally, assume that $\{U_\alpha : \alpha \in A\} \subset \mathscr{T}$ and let $U = \bigcup_{\alpha \in A} U_\alpha$. But each U_α is generated by some subset of $\mathcal{B}$ and hence U is generated by some subset of $\mathcal{B}$. Therefore $U \in \mathscr{T}$.

Therefore $\mathscr{T}$ is a topology on X.

∎

Theorem: Lower Limit Topology

Let $\mathcal{B} = \{[a,b) \mid a, b \in \mathbb{R}\}$. $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$ called the *lower limit* topology and denoted by $\mathbb{R}_{LL}$. It is also known as the *Sorgenfrey line*, denoted by $\mathbb{R}^1_{bad}$.

Proof. Assume that $p \in \mathbb{R}$. There exists $\epsilon > 0$ such that $B(p, \epsilon) \subset \mathbb{R}$. Let $a = p - \epsilon$ and $b = p + \epsilon$. Thus $p \in [a, b) \in \mathcal{B}$.

Now, assume that $U, V \in \mathcal{B}$. Let $U = [a, b)$ and $V = [c, d)$:

Case 1: $b \leq c$ or $a \geq d$.

The $U \cap V = \emptyset$ and property (2) holds vacuously.

Case 2: $U \subset V$ or $V \subset U$.

AWLOG that $U \subset V$. Then $U \cap V = U$ and so property (2) holds trivially.

Case 3: Otherwise.

AWLOG that $a < c$. Then $U \cap V = [c, b) \in \mathcal{B}$ and so property (2) holds.

Therefore $\mathbb{R}_{LL}$ is a topology on $\mathbb{R}$. ∎

Theorem

On $\mathbb{R}$, $\mathscr{T}_{std} \subseteq \mathscr{T}_{LL}$.

Proof. Consider the basis $B = \{(a,b) \subset \mathbb{R} \mid a, b \in \mathbb{R}\}$ for T_{std} and assume $(a, b) \in \mathcal{B}$. Then:

$$(a, b) = \bigcup_{x \in (a,b)} [x, b)$$

Thus everything in T_{std} is generated by B, which is generated by $\mathscr{T}_{LL}$. Therefore, $\mathscr{T}_{std} \subset \mathscr{T}_{LL}$. However, $[a, b)$ is not open in T_{std}. Therefore $\mathscr{T}_{std} \subsetneq \mathscr{T}_{LL}$. ∎

Definition: Finer

Let X be a set and let $\mathscr{T}$ and $\mathscr{T}'$ be two topologies on X. To say that $\mathscr{T}'$ is *finer* than $\mathscr{T}$ means that $\mathscr{T} \subset \mathscr{T}'$. Also, T is *coarser*. Furthermore, if $\mathscr{T} \neq \mathscr{T}'$ then the terms *strictly* finer or coarser are used.

Example

Give an example of two topologies on R such that neither is finer than the other, that is, the two topologies are not comparable.

Consider the standard and cocountable topologies on R. $(0, 1)$ is open in the standard topology; however, $\mathbb{R} - (0, 1)$ is uncountable and hence $(0, 1)$ is not in the cocountable topology. Likewise, $\mathbb{R} - \mathbb{Q}$ is open in the cocountable topology (since $\mathbb{Q}$ is countable); however, since there are an infinite number of rationals between any two irrationals, it is impossible to draw an open ball around any irrational that is a subset of the irrationals and so $\mathbb{R} - \mathbb{Q}$ is not in the standard topology. Therefore the two topologies are not comparable.

Theorem: Double-headed Snake

Let $X = \mathbb{R}_{+00} = \mathbb{R}^+ \cup 0', 0''$ where $0', 0'' \in \mathbb{R} - \mathbb{R}^+$ and $0' \neq 0''$, and let:

$$\mathcal{B} = \left\{(a, b), \{0'\} \cup (0, c), \{0''\} \cup (0, d) \,\middle|\, a, b, c, d \in \mathbb{R}^+\right\}$$

$\mathcal{B}$ forms a basis for a topology $\mathscr{T}$ on X called the *double-headed snake* topology.

Proof. Assume that $x \in \mathbb{R}_{+00}$:

Case 1: $x = 0'$

$$x \in \{0'\} \cup (0, c) \in \mathcal{B}$$

Case 2: $x = 0''$

$$x \in \{0''\} \cup (0, d) \in \mathcal{B}$$

Case 3: $x \in \mathbb{R}^+$

$$x \in \left(\frac{x}{2}, x + 1\right) \in \mathcal{B}$$

Therefore property (1) holds.

Next, assume that $U, V \in \mathcal{B}$. If $U \cap V = \emptyset$ or $U \subset V$ or $V \subset U$ then done, so assume otherwise:

Case 1: $U = (a, b)$ and $V = (c, d)$

AWLOG that $a \leq c$. Then $U \cap V = (c, d) \in \mathcal{B}$.

Case 2: $U = (a, b)$ and $V = \{0'\} \cup (0, c)$ or $V = \{0''\} \cup (0, c)$

$U \cap V = (a, \min\{b, c\}) \in \mathcal{B}$

Case 3; $U = \{0'\} \cup (0, c)$ or $V = \{0''\} \cup (0, d)$

$U \cap V = (\min\{c, d\}, \max\{c, d\}) \in \mathcal{B}$

Therefore property (2) holds.

Therefore the double-headed snake is a topology on $\mathbb{R}_{00+}$. ∎

Theorem

Let $\mathscr{T}$ be the double-headed snake topology on $\mathbb{R}_{00+}$:

1. Every point in $\mathbb{R}_{00+}$ is a closed set.
2. It is impossible for find disjoint open sets U and V such that $0' \in U$ and $0'' \in V$.

Proof.

1. Assume $x \in \mathbb{R}_{00+}$ and consider $\{x\}$. Let $A = \mathbb{R}_{00+} - \{x\}$:

   Case 1: $x = 0'$

   $$A = (\{0''\} \cup (0,2)) \cup \bigcup_{1<b\in\mathbb{R}^+}(1,b) \in \mathscr{T}$$

   Case 2: $x = 0''$

   $$A = (\{0'\} \cup (0,2)) \cup \bigcup_{1<b\in\mathbb{R}^+}(1,b) \in \mathscr{T}$$

   Case 3: $x \in \mathbb{R}^+$

   $$A = (\{0'\} \cup (0,x)) \cup (\{0''\} \cup (0,x)) \cup \bigcup_{x<b\in\mathbb{R}^+}(x,b) \in \mathscr{T}$$

   Therefore A is open and thus $\{x\}$ is closed.

2. WTS: $\forall\, U, V \in \mathscr{T}, (0' \in U \text{ and } 0'' \in V \implies U \cap V \neq \emptyset)$

   Assume $U, V \in \mathscr{T}$ and assume $0' \in U$ and $0'' \in V$. This means that $\{0'\} \cup (0,c) \subset U$ and $\{0''\} \cup (0,d) \subset V$ for some $c, d \in \mathbb{R}+$. Since U and V are generated by these and possibly other basis elements, it must be the case that:

   $$(\{0'\} \cup (0,c)) \cap (\{0''\} \cup (0,d)) = (0, \min\{c,d\}) \subset U \cap V$$

   But $(0, \min\{c,d\}) \neq \emptyset$.

   Therefore $U \cap V \neq \emptyset$.

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