Normal Operators

Definition: Normal

Let E be a vector space and let T be a bounded operator on E. To say that T is a *normal* operator means:

$$T^*T = TT^*$$

In other words, T and its adjoint commute.

Note that if T is self-adjoint then it is clearly normal; however, the converse is not necessarily true.

Example

Let $T\vec{x} = i\vec{x}$.
$T = iI$
$T^* = (iI)^* = -iI^* = -iI$

$\therefore T \neq T^*$

$TT^* = (iI)(-iI) = (-iI)(iI) = T^*T$

Therefore T and T^* commute.

Theorem

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$:

$$T \text{ is normal } \iff \forall\, \vec{x} \in H, \|T\vec{x}\| = \|T^*\vec{x}\|$$

Proof

Assume $\vec{x} \in H$.

$\implies$ Assume T is normal.

$$\|T\vec{x}\|^2 = \langle T\vec{x}, T\vec{x}\rangle = T^*T\vec{x}, \vec{x} = TT^*\vec{x}, \vec{x} = T^*\vec{x}, T^*\vec{x} = \|T^*\vec{x}\|^2$$

$$\therefore \|T\vec{x}\| = \|T^*\vec{x}\|$$

$\impliedby$ Assume $\|T\vec{x}\| = \|T^*\vec{x}\|$.

$$\langle T\vec{x}, T\vec{x}\rangle = \langle T^*\vec{x}, T^*\vec{x}\rangle$$

$$\langle T^*T\vec{x}, \vec{x}\rangle = \langle TT^*\vec{x}, \vec{x}\rangle$$

Therefore $T^*T = TT^*$ and thus T is normal.

Theorem

Let H be a Hilbert space and let $T, A, B \in \mathcal{B}(H)$ such that $T = A + iB$:

T is normal $\iff AB = BA$.

Proof

By previous theorem, A and B are unique and self-adjoint.

$T^* = (A + iB)^* = A^* + (iB)^* = A^* - iB^* = A - iB$

$TT^* = (A + iB)(A - iB) = A^2 - iAB + iBA = B^2 = A^2 + B^2 + i(BA - AB)$

$T^*T = (A - iB)(A + iB) = A^2 - iBA + iAB = B^2 = A^2 + B^2 - i(BA - AB)$

$$\begin{aligned} T \text{ is normal} &\iff TT^* = T^*T \\ &\iff A^2 + B^2 + i(BA - AB) = A^2 + B^2 - i(BA - AB) \\ &\iff 2i(BA - AB) = 0 \\ &\iff BA - BA = 0 \\ &\iff AB = BA \end{aligned}$$

Theorem

Let H be a Hilbert space and $T \in \mathcal{B}(H)$:

T is normal $\implies \|T^n\| = \|T\|^n$

Proof

It is already known that $\|T^n\| \leq \|T\|^n$.

For $\|T^n\| \geq \|T\|^n$, proof by induction on n:

Base case: $n = 1$: Trivial.

Assume $\|T^n\| \leq \|T\|^n$

Consider $n + 1$.

Assume $\vec{x} \in H$ such that $\|\vec{x}\| = 1$.

First, note:

$$\left\|T^2\vec{x}\right\|^2 = \left\langle T^2\vec{x}, T^2\vec{x}\right\rangle = \langle T^*T\vec{x}, T^*T\vec{x}\rangle = \|T^*T\vec{x}\|^4$$

But, since $\|T^*T\| = sup_{\|\vec{x}\|=1} |\langle T^*T\vec{x}, \vec{x}\rangle|$:

$$\left\|T^2\vec{x}\right\| = \|T^*T\vec{x}\|^2 \geq \langle T^*T\vec{x}, \vec{x}\rangle = \langle T\vec{x}, T\vec{x}\rangle = \|T\vec{x}\|^2$$

Now:

$$
\begin{aligned}
\left\|T^{n+1}\vec{x}\right\| &= \|T^n T\vec{x}\| \\
&= \|T\vec{x}\| \left\| \frac{1}{\|T\vec{x}\|} T^n T\vec{x} \right\| \\
&= \|T\vec{x}\| \left\| T^n \frac{T\vec{x}}{\|T\vec{x}\|} \right\| \\
&\geq \|T\vec{x}\| \left\| T \frac{T\vec{x}}{\|T\vec{x}\|} \right\|^n \\
&= \|T\vec{x}\|^{1-n} \left\|T^2\vec{x}\right\|^n \\
&\geq \|T\vec{x}\|^{1-n} \|T\vec{x}\|^{2n} \\
&= \|T\vec{x}\|^{n+1}
\end{aligned}
$$