# Vector Norm

**Definition: Vector Norm**

Let $V$ be a vector space over a field $F$. To say that a function $\| \cdot \| : V \to \mathbb{R}$ is a *vector norm* means that it satifies the following four properties $\forall\, \vec{x}, \vec{y} \in V$ and $\forall\, c \in F$:

1). Nonnegativity: $\|\vec{x}\| \geq 0$

2). Positivity: $\|\vec{x}\| = 0 \iff \vec{x} = 0$

3). Homogeneity: $\|c\vec{x}\| = |c|\, \|\vec{x}\|$

4). Subadditivity: $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ (triangle inequality)

**Theorem**

Let $V$ be a vector space and let $\vec{x} \in V$:

$$\| - \vec{x}\| = \|\vec{x}\|$$

Proof

$\| - \vec{x}\| = \|(-1)\vec{x}\| = |-1|\, \|\vec{x}\| = 1 \cdot \|\vec{x}\| = \|\vec{x}\|$

**Theorem**

The nonnegativity property of the norm can be derived from the other three properties.

Proof

Assume $V$ be a vector space over a field $F$
Assume $\vec{x} \in V$

$$\begin{aligned} \|\vec{x} - \vec{x}\| &\leq \|\vec{x}\| + \| - \vec{x}\| \\ \|\vec{0}\| &\leq \|\vec{x}\| + \|\vec{x}\| \\ 0 &\leq 2\|\vec{x}\| \\ \therefore \|\vec{x}\| &\geq 0 \end{aligned}$$

**Theorem**

Let $V$ be a vector space over a field $F$. $\forall\, \vec{x}, \vec{y} \in V$:

$$|\|x\| - \|y\|| \leq \|\vec{x} - \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

Proof

$\|\vec{y}\| = \|\vec{x} + (\vec{y} - \vec{x})\| \leq \|\vec{x}\| + \|\vec{y} - \vec{x}\| = \|\vec{x}\| + \|\vec{x} - \vec{y}\|$
$\|\vec{y}\| - \|\vec{x}\| \leq \|\vec{x} - \vec{y}\|$
$-(\|\vec{x}\| - \|\vec{y}\|) \leq \|\vec{x} - \vec{y}\|$

$\|\vec{x}\| = \|\vec{y} + (\vec{x} - \vec{y})\| \leq \|\vec{y}\| + \|\vec{x} - \vec{y}\|$
$\|\vec{x}\| - \|\vec{y}\| \leq \|\vec{x} - \vec{y}\|$

$\pm(\|\vec{x}\| - \|\vec{y}\|) \leq \|\vec{x} - \vec{y}\|$
$|\|\vec{x}\| - \|\vec{y}\|| \leq \|\vec{x} - \vec{y}\|$

$\|\vec{x} - \vec{y}\| \leq \|\vec{x}\| + \| - \vec{y}\| = \|\vec{x}\| + \|\vec{y}\|$

$\therefore |\|x\| - \|y\|| \leq \|\vec{x} - \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

**Example**

Let $V = \mathbb{C}^n$:

1). $\ell_2$ (Euclidean) Norm

$\|\vec{x}\|_2 = \left(\sum_{k=1}^n |x_k|^2\right)^{\frac{1}{2}} = \vec{x}^*\vec{x}$ (standard norm)

2). $\ell_1$ Norm

$\|\vec{x}\|_1 = \sum_{k=1}^n |x_k|$

3). $\ell_\infty$ Norm

$\|\vec{x}\|_1 = \max\{|x_k| \mid 1 \leq k \leq n\}$

4). $\ell_p$ Norm ($1 \leq p \leq \infty$)

$\|\vec{x}\|_p = (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}}$

5). $k$-norm ($k \in \mathbb{Z}^+$

$\|\vec{x}\|_{[k]} = \sum_{i=1}^k |x_{j_i}|$ where the components have been permuted such that $|x_{j_i}| \geq \left|x_{j_{i+1}}\right|$