Isomorphisms

Definition: Isomorphism

Let H_1 and H_2 be inner product (or Hilbert) spaces. To say that H_1 is *isomorphic* to H_2 means there exists a mapping $T : H_1 \to H_2$, called an inner product (or Hilbert) space isomorphism, such that:

- T is a bijection.
- $\forall\, \vec{x}, \vec{y} \in H_1, \langle T\vec{x}, T\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$.

Theorem

Every finite dimensional inner product (and hence Hilbert) space H is isomorphic to $\mathbb{C}^N$.

Proof

Assume $\dim H = N$.
Assume $\{\vec{x}_1, \dots, \vec{x}_N\}$ is an orthonormal basis for H.
Let $T : H \to \mathbb{C}^N$ be defined by:

$$T\vec{x} = T\left(\sum_{k=1}^{N} \alpha_k \vec{x}_k\right) = \sum_{k=1}^{N} \alpha_k e_k = y$$

Clearly T is bijective.

Assume $\vec{x}, \vec{y} \in H$.

$\exists\, \alpha, \beta \in \mathbb{C}$ such that $\vec{x} = \displaystyle\sum_{k=1}^{N} \alpha_k \vec{x}_k$ and $\vec{y} = \displaystyle\sum_{k=1}^{N} \beta_k \vec{x}_k$.

$\langle T\vec{x}, T\vec{y} \rangle = \left\langle \displaystyle\sum_{k=1}^{N} \alpha_k e_k, \sum_{j=1}^{N} \beta_j e_j \right\rangle = \displaystyle\sum_{k=1}^{N} \alpha_k \overline{\beta_k}$

Similarly: $\langle \vec{x}, \vec{y} \rangle = \left\langle \displaystyle\sum_{k=1}^{N} \alpha_k \vec{x}_k, \sum_{j=1}^{N} \beta_j \vec{x}_j \right\rangle = \displaystyle\sum_{k=1}^{N} \alpha_k \overline{\beta_k}$

$\therefore \langle T\vec{x}, T\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$

$\therefore T$ is an isomorphism and thus $H \sim \mathbb{C}^N$.

Theorem

Every infinite dimensional separable Hilbert space is isomorphic to ℓ^2.

Proof

Since H is separable, H contains a complete orthonormal sequence $(\vec{x}_n)$.

Assume $\vec{x} = \displaystyle\sum_{n=1}^{\infty} \langle \vec{x}, \vec{x}_n \rangle \vec{x}_n \in H$.

Define $T : H \to \ell^2$ by $T\vec{x} = (\langle \vec{x}, \vec{x}_n \rangle)$, which converges (Bessel).

T is linear due to the linearity of the inner product.

Assume $T\vec{x} = 0$.
So $\forall\, n \in \mathbb{N}$, $\langle \vec{x}, \vec{x}_n \rangle = 0$.
Thus $\vec{x} = \vec{0}$, and so the kernel of linear $T = \{\vec{0}\}$.

Therefore T is injective.

Assume $(\alpha_n) \in \ell^2$.
Let $\vec{x} = \displaystyle\sum_{n=1}^{\infty} \alpha_n \vec{x}_n$, which converges since $(\alpha_n) \in \ell^2$.
But $(\vec{x}_n)$ is complete, so $\alpha_n = \langle \vec{x}, \vec{x}_n \rangle$.
And so $T\vec{x} = (\alpha_n)$.

Therefore T is surjective.

Therefore T is a bijection.

Finally, assume $\vec{x}, \vec{y} \in H$:

$$\begin{array}{rcl}
\langle T\vec{x}, T\vec{y} \rangle & = & \langle (\langle \vec{x}, \vec{x}_n \rangle), (\langle \vec{y}, \vec{x}_n \rangle) \rangle \\
 & = & \displaystyle\sum_{n=1}^{\infty} \langle \vec{x}, \vec{x}_n \rangle \, \overline{\langle \vec{y}, \vec{x}_n \rangle} \\
 & = & \left\langle \displaystyle\sum_{n=1}^{\infty} \langle \vec{x}, \vec{x}_n \rangle \, \vec{x}_n, \sum_{m=1}^{\infty} \langle \vec{y}, \vec{x}_m \rangle \, \vec{x}_m \right\rangle \\
 & = & \langle \vec{x}, \vec{y} \rangle
\end{array}$$

Therefore T is an isomorphism and thus $H \sim \ell^2$.