Components

Definition: Component

Let G be a graph and let $\mathscr{G}$ be the set of all connected subgraphs of G. To say that a graph $H \in \mathscr{G}$ is a *component* of a G means that H is not a subgraph of any other connected subgraph of G:

$$\forall\, F \in \mathscr{G} - \{H\}, H \not\subset F$$

The number of distinct components in G is denoted by:

$$k = k(G)$$

For a connected graph: $k(G) = 1$.

Notation: Union

Let G be a graph and let $\mathscr{G} = \{G_i : 1 \leq i \leq k\}$ for some $k \in \mathbb{N}$ be a set of subgraphs of G such that each vertex and each edge in G is present in exactly one G_i:

$$G = \bigcup_{1 \leq i \leq k} G_i$$

Note that the components of a graph are suitable choices for the G_i.

Lemma

Let G be a graph and let G_i be a component of G:

> G_i is an induced subgraph of G.

Proof. By definition, G_i is a maximal connected subgraph of G.

ABC: G_i is not an induced subgraph of G.

Thus, G_i is missing some edges that when added would result in a connected induced subgraph H of G. But then $G_i \subset H$, contradicting the maximality of G_i.

$\therefore$ G_i is an induced subgraph of G. ∎

Theorem

Let G be a graph and define a relation $\sim$ on $V(G)$ as follows:

$$\forall\, u, v \in V(G), u \sim v \iff u \text{ and } v \text{ are connected}$$

$\sim$ is an equivalence relation.

Proof. Assume $u, v, w \in V(G)$:

R: u is connected to u by the trivial path (u).

$\therefore u \sim u$

S: Assume $u \sim v$:

There exists a $u - v$ path in G.
And so, going in reverse order, there exists a $v - u$ path in G.

$\therefore v \sim u$

T: Assume $u \sim v$ and $v \sim w$:

There exists a $u - v$ path and a $v - w$ path in G.
So there exists a $u - w$ walk in G.
And thus there must exist a $u - w$ path in G.

$\therefore u \sim w$

■

Theorem

Let G be a graph and let G_i be a subgraph of G. TFAE:

1. G_i is a component of G.
2. G_i is induced by an equivalence class of the connectedness relation.

Proof.

$\Longrightarrow$ Assume G_i is a component of G.

So G_i is a maximal connected induced subgraph of G.

ABC: $V(G_i)$ is not an equivalence class of the connectedness relation.

Thus, $V(G_i)$ must be a proper subset of some equivalence class V_i and $G[V_i]$ is an connected induced subgraph of G such that $G_i \subset G[V_i]$, contradicting the maximality of G_i.

$\therefore G_i$ is induced by an equivalence class of the connectedness relation.

$\Longleftarrow$ Assume G_i is induced by an equivalence class of the connectedness relation.

By definition, G_i is a connected subgraph of G.

ABC: G_i is not maximal.

Thus, G_i is a proper subgraph of some connected subgraph H of G and $V(G_i) \subset V(H)$, contradicting the definition of $V(G_i)$ as an equivalence class.

$\therefore G_i$ is a component of G.

■

Corollary

Let G be a graph with k components. Each vertex and each edge in G belong to exactly one component of G.

Proof. Each $v \in V(G)$ is in exactly one equivalence class and hence in exactly one component. Furthermore, the endpoints of each edge are also in the same equivalence class and thus must exist in the same component, forcing the edge into that component as well. ■

Corollary

Let G be a graph with k components and let $u, v \in V(G)$ such that $u \in G_i$ and $v \in G_j$ for $1 \leq i, j \leq k$:

$$i \neq j \implies uv \notin E(G)$$

Proof. Assume $uv \in E(G)$.

This means that u and v are connected and thus must be in the same component.

$\therefore i = j$ ■