First Ring Isomorphism Theorem

Theorem

Let $\phi : R \to S$ be a homomorphism of rings:

$$\ker(\phi) \trianglelefteq R$$

Proof

From group theory, we know that $\ker(\phi)$ is an additive subgroup of R
Assume $r \in R$
Assume $k \in \ker(\phi)$
$\phi(rk) = \phi(r)\phi(k) = \phi(r) \cdot 0 = 0$
$rk \in \ker(\phi)$, so $\ker(\phi)$ is a left ideal in R
$\phi(kr) = \phi(k)\phi(r) = 0 \cdot \phi(r) = 0$
$kr \in \ker(\phi)$, so $\ker(\phi)$ is a right ideal in R

Therefore, by the ideal test, $\ker(\phi) \trianglelefteq R$.

Theorem

Let R be a ring and $I \trianglelefteq R$. R/I is a ring with operations:

$$\begin{aligned} (a+I)+(b+I) &= (a+b)+I \\ (a+I)(b+I) &= ab+I \end{aligned}$$

Proof

From group theory, we already know that R/I is an additive group

Assume $a, a' \in R$ are representative from the same coset
$a - a' \in I$
Similarly, assume $b, b' \in R$ are representative from the same coset
$b - b' \in I$

$(a - a')b = ab - a'b \in I$
So $ab + I = a'b + I$
Likewise, $a'(b - b') = a'b - a'b' \in I$
So $a'b + I = a'b' + I$
Thus, by transitivity, $ab + I = a'b' + I$

Now, since all operations are on representative, we inherit all of the properties of the operations of R, including multiplicative associativity and the distributive rules

Therefore R/I is a ring.

Theorem

Let R be a ring and $I \trianglelefteq R$, and let $\phi : I \to R/I$ be the canonical homomorphism:

$$\ker(\phi) = I$$

Thus, every ideal is the kernel of some homomorphism.

Proof

Note that I is the identity for R/I

$x \in \ker(\phi) \iff \phi(x) = I \iff x + I = I \iff x \in I$

$\therefore \ker(\phi) = I$

Theorem: First (Fundamental) Ring Isomorphism Theorem

Let $\phi : R \to S$ be a homomorphism of rings:

$$R/\ker\phi \simeq \phi[R]$$

Proof

From group theory, using the canonical injection homomorphism and the first (fundamental) group isomorphism theorem, we have:

$$\begin{tikzcd}
R \arrow[r, two heads] \arrow[d, two heads, "\iota"'] & \phi[R] \\
R/\ker(\phi) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

So we already know that $\mu : R/\ker\phi \to \phi[R]$ is an isomorphism of groups where:

$$r + \ker(\phi) \mapsto \phi(r)$$

Assume $x, y \in R$

$$\begin{aligned}
\mu((x+\ker(\phi))(y+\ker(\phi))) &= \mu(xy+\ker(\phi)) \\
&= \phi(xy) \\
&= \phi(x)\phi(y) \\
&= \mu(x+\ker(\phi))\mu(y+\ker(\phi))
\end{aligned}$$

Thus μ preserves multiplication

Therefore μ is a ring isomorphism and $R/\ker\phi \simeq \phi[R]$.