$\mathbf{L^2}$ Norm

Theorem

$\|\cdot\|$ is a proper norm for L^2

Proof

N1: Assume $f \in L^2$

$\|f\| = 0 \iff f = 0 \;\; a.e. \qquad (property)$
Note that, once again, we are dealing with equivalence classes, where:
$f \sim g$ means $f = g \;\; a.e.$

N2: Assume $f \in L^2$ and $\alpha \in \mathbb{C}$

$$\|\alpha f\| = \left(\int |\alpha f|^2\right)^{\frac{1}{2}} = \left(|\alpha|^2 \int |f|^2\right)^{\frac{1}{2}} = |\alpha| \left(\int |f|^2\right)^{\frac{1}{2}} = |\alpha| \, \|f\|$$

N3: Assume $f, g \in L^2$

$$\begin{array}{rcl}
\|f+g\|^2 & = & \int |f+g|^2 \\
& = & \int (f+g)\overline{(f+g)} \\
& = & \int (f+g)(\bar{f}+\bar{g}) \\
& = & \int (f\bar{f} + g\bar{g} + f\bar{g} + g\bar{f}) \\
& = & \int |f|^2 + \int |g|^2 + \int f\bar{g} + \int g\bar{f}) \\
& = & \|f\|^2 + \|g\|^2 + \langle f, g\rangle + \langle g, f\rangle \\
& \leq & \|f\|^2 + \|g\|^2 + \|f\|\|g\| + \|g\|\|f\| \\
& = & \|f\|^2 + \|g\|^2 + 2\|f\|\|g\| \\
& = & (\|f\| + \|g\|)^2
\end{array}$$

$\therefore \|f+g\| \leq \|f\| + \|g\|$