Hilbert Spaces

Definition

To say that a set $\mathcal{H}$ is a *Hilbert space* means that it satisfies the following properties:

H1: $\mathcal{H}$ is a vector space over $\mathbb{C}$ (or $\mathbb{R}$).

H2: $\mathcal{H}$ is equipped with an inner product $\langle \cdot , \cdot \rangle$ that also defines a norm $\|f\| = \langle f, f\rangle^{\frac{1}{2}}$.

H3: $\mathcal{H}$ is complete in its metric $d(f, g) = \|f - g\|$.

H4: $\mathcal{H}$ is separable.

0.1 Examples:

1). $L^2(E)$

$$L^2(E) = \left\{ f \text{ supported on } E \text{ and } \int_E |f|^2 < \infty \right\}$$

$$\langle f, g \rangle = \int_E f\bar{g}$$

$$\|f\| = \left(\int_E |f|^2 \right)^{\frac{1}{2}}$$

2). $\mathbb{R}^N$ or $\mathbb{C}^N$

$$\mathbb{R}^N = \{(a_1, a_2, \ldots, a_N) \mid a_k \in \mathbb{R}\}$$

$$\mathbb{C}^N = \{(a_1, a_2, \ldots, a_N) \mid a_k \in \mathbb{C}\}$$

$$\langle a, b \rangle = \sum_{k=1}^{N} a_k \overline{b_k}$$

$$\|a\| = \left(\sum_{k=1}^{N} |a_k|^2 \right)^{\frac{1}{2}}$$

3). $\ell^2(\mathbb{Z})$

$$\ell^2(\mathbb{Z}) = \left\{ (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots) \mid a_k \in \mathbb{C} \text{ and } \sum_{k=-\infty}^{\infty} |a_k|^2 < \infty \right\}$$

$$\langle a, b \rangle = \sum_{k=-\infty}^{\infty} a_k \overline{b_k}$$

$$\|a\| = \left(\sum_{k=-\infty}^{\infty} |a_k|^2 \right)^{\frac{1}{2}}$$

4). $\ell^2(\mathbb{N})$

$$\ell^2(\mathbb{N}) = \left\{ (a_1, a_2, \dots) \mid a_k \in \mathbb{C} \text{ and } \sum_{k=1}^{\infty} |a_k|^2 < \infty \right\}$$

$$\langle a, b \rangle = \sum_{k=1}^{\infty} a_k \overline{b_k}$$

$$\|a\| = \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{\frac{1}{2}}$$

Remark: All Hilbert spaces are isomorphic to $\ell^2(\mathbb{Z})$.