Mathematical Logic

When we speak in mathematical terms, our use of language needs to be as precise as possible. In short, clarity is good and ambiguity is bad. We use logical constructs to describe *exactly* what we mean, so a short introduction to these constructs is in order.

Statements

The most basic logical construct is the *statement*, which is defined as follows:

Definition: Statement

A *statement* is a fact that is *unambiguously* either *true* or *false*.

Thus, $2 < 3$ is a fact that is unambiguously *true* and is therefore a statement. Likewise, $3 < 2$ is a fact that is unambiguously *false* and is therefore also a statement. But, $x < 2$, assuming that we know nothing about the value of x, could be true or false, so it is *not* a statement. However, if we claim that x has the value 1 then $x < 2$ becomes unambiguously true and is now a statement.

We tend to use capital letters, starting with P, to represent statements and the '$:=$' symbol to indicate definitions. For example:

$$P \coloneqq 2 < 3$$

would indicate that P is defined to be the true statement $2 < 3$.

Logical Operators

Single statements can be combined into *compound* statements using logical operators. There are three logical operators commonly in use:

1). NOT

2). AND

3). OR

NOT

The NOT operator flips the truth about a statement: $not\ P$ is true when P is false and false when P is true. For example, if $P \coloneqq 2 < 3$ then P is true and $not\ P$, which would be $2 \geq 3$, is false.

We can capture the effect of a logical operator using something called a *truth table*. The truth table for the NOT operator applied to a statement P would be as follows (T=true and F=false):

P	$not\ P$
T	F
F	T

Note that the left side lists all of the possible truth states of P and the right side lists the resulting truth states of $not\ P$

AND

The AND operator combines two statements P and Q, such that $P\ and\ Q$ is true only when P and Q are both true; it is false whenever P, Q, or both are false. The truth table for AND is as follows:

P	Q	$P\ and\ Q$
T	T	T
T	F	F
F	T	F
F	F	F

For example, let:

$$\begin{aligned} P &\coloneqq 2 < 3 \\ Q &\coloneqq 3 < 4 \\ R &\coloneqq 5 < 4 \end{aligned}$$

Then $P\ and\ Q$ is true (since both P and Q are true), but $P\ and\ R$ is false because even though P is true, R is false.

OR

The OR operator combines two statements P and Q, such that $P\ or\ Q$ is true whenever P, Q, or both are true. In fact, it is only false when both P and Q are false. The truth table is as follows:

P	Q	$P\ or\ Q$
T	T	T
T	F	T
F	T	T
F	F	F

For example, assuming the above definitions of P, Q, and R, $P\ or\ Q$ and $P\ or\ R$ are both true. However, $(not\ Q)\ or\ R$ is false, because both $not\ Q$ and R are false.

Note that this type of OR is often referred to as an *inclusive* OR, since the OR statement is true when both statements are true. There is another type of OR called an *exclusive* OR, which is false when both statements are true, and thus has the following truth table:

P	Q	$P\ xor\ Q$
T	T	F
T	F	T
F	T	T
F	F	F

Unless explicitly stated otherwise, we assume that all of our OR statements are inclusive,

Operator Precedence

When a compound statement contains multiple operators then we need to pay attention to operator precedence, just like we do with the arithmetic operators plus and multiply. For logical operators, the order of precedence is:

1). NOT

2). AND

3). OR

When consecutive operators are the same then NOT is evaluated from right to left, and AND and OR are evaluated from left to right. We use parentheses if we need to override this precedence, again, just like in arithmetic.

For example, the compound statement:

$$P\ and\ Q\ or\ not\ not\ P\ and\ not\ Q$$

would be evaluated as follows:

$$(P\ and\ Q)\ or\ (not\ (not\ P)\ and\ (not\ Q))$$

Assuming the definitions of P and Q above, this would be a true statement: $(P\ and\ Q)$ is true, and thus the OR statement is true. Make sure that you can see why this is true.

Implication

Mathematical systems start with a small collection of relatively simple facts and use those facts to discover new, more complicated facts. This is done by implication, thus making implication the most important logical construct in mathematics.

Implication is an if-then statement of the form:

if *(hypothesis)* then *(consequence)*

The hypothesis and consequence are statements. We make the claim that if the hypothesis is true (know facts), then the consequence must be true (new facts). Note that such implication must have a proof that supports it. Discovery of such proofs is the main task in higher mathematics.

As a simple example, consider the implication:

if $x = 1$ then $x < 2$

We claim that $x = 1$ is a true statement, and can thus conclude that $x < 2$; the implication is a true statement. A more complicated example would be:

if $x = 1$ and $y = 2$ then $x + y = 3$

Now, the hypothesis is a compound AND statement. We are claiming that the hypothesis is true, so both parts of the AND must be true, so the consequence does follow; indeed, $x+y = 1+2 = 3$ and the implication is a true statement. But consider this example:

if $x = 1$ or $y = 2$ then $x + y = 3$

Now, the hypothesis is an OR statement, so only one of its parts need be true. If they are both true then the consequence follows; however, if $x = 1$ is true but $y = 2$ is false, then the consequence does not follow. We cannot conclude that the consequence is true in all cases, so the implication is false.

Often, an arrow is used to indication implication: $P \to Q$.

Implication actually has a truth table associated with it:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Note that an implication is only false when the hypothesis is true and the consequence is false. Thus, the equivalent logic statement for implication is: $(notP) or Q$. The reason why the implication is always true when the hypothesis is false is important when dealing with quantifiers, which will be discussed later.

The direction of the implication is important. For example, consider the statement:

If $x = 2$ then $x^2 = 4$

This is certainly a true statement; however, the other direction:

If $x^2 = 4$ then $x = 2$

is not true because x could actually be equal to -2.

Equivalence

There are some implications that are true in both directions. For example, by slightly modifying the previous example:

If $x = 2$ or $x = -2$ then $x^2 = 4$

This statement is now true in both directions.

When $P \to Q$ and $Q \to P$ are both true then we say that P and Q are *equivalent*. This means that either both P and Q must be true or that both must be false. As a short cut, we represent equivalences using *if and only if* statements, usually abbreviated P iff Q or also $P \leftrightarrow Q$.

The corresponding truth table is as follows:

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

A variation on equivalence is to join a list of statements together and say that *the following are equivalent* — they are all true together or all false together. This is sometimes abbreviated as $TFAE$. For example, let n be an integer. The following are equivalent:

1). $n = 5$

2). $4 < n < 6$

3). $n - 3 = 2$

4). $n + 1 = 6$

Notice that if any one of these statements is true then they must *all* be true. If any one of them is false then they must *all* be false. In otherwords, each implies the others.

Quantifiers

Although the statement $1 < 2$ may be true, it is sort of obvious and not very interesting. Instead, what we might want to do is compare all the values in some, possibly infinite, collection of values to the number 2 — something like $x < 2$ for all considered values of x. We do this using parameterized statements and quantifiers.

Parameterized Statements

We determined that something like $x < 2$ is not a statement until we have a definite value of x. We call something like this a *parameterized* statement, where x is the parameter. When using letters to represent statements, we add the parameter as follows:

$$P(x) \coloneqq x < 2$$

Now, we need a way to provide various values of x. We will see how to identify a collection of possible x when we look at *sets*. Until then, we will assume that we have some collection of x values in which we are interested. We then apply those x values to our parameterized statement using a *quantifier*.

Universal Quantifier

We use the universal quantifier to test all of the values in our collection. The universal quantifier uses the words "for all", so we would say something like:

For all $x, x < 2$

Using mathematical syntax, we would write: $\forall x, P(x)$. The upside-down 'A' stands for "for all". This is a statement that is true if $P(x)$ is true for all of our x values. It is false if there is at least one x value for which P(x) is false. So, you can think of the universal quantifier as a shortcut for a long compound AND statement:

$$\forall x, P(x) \coloneqq P(x_1)\ and\ P(x_2)\ and\ P(x_3)\ and\ \ldots$$

and indeed we must use a universal quantifier when there are a large or even infinite number of possible x values. If we can find at least one x value such that $P(x)$ is false then the whole statement is false. We call such a failing x value a *counterexample*.

For example, let our collection of x values be the following collection of numbers: 2,4,6,8, and 11. Define $P(x)$ to be the parameterized statement: $x < 12$. We can see that $\forall x, P(x)$ is a true statement because all of the candidate values for x are indeed less than 12. But if we let $Q(x) \coloneqq x$ is an even number, then $\forall x, Q(x)$ is a false statement because the value 11 is not even and is thus a counterexample.

We mentioned before that an if-then statement is always true when the hypothesis is false. We can see why this is convenient when implication is used with the universal quantifier. Continuing with the previous example, define $R(x)$ to be the implication: if $x < 10$ then x is an even number. Now consider $\forall x, R(x)$. This time, when we get to the value 11, the hypothesis fails and the implication is true, thus preventing it from being a counterexample. In other words, the implication acts as a sort of short-circuit that filters out those values of x that do not concern us.

Existential Quantifier

We use the existential quantifier to see if there is at least one value in our collection that makes our parameterized statement true. The existential quantifier uses the words "there exists", so we would say something line:

> There exists x such that $x < 2$.

Using mathematical syntax, we would write: $\exists x, P(x)$. The backwards 'E' stands for "there exists". This is a statement that is true if $P(x)$ is true for at least one of our x values — there can be more, but there has to be at least one. It is false if $P(x)$ is false for all of our x values. So, you can think of the existential quantifier as a shortcut for a long compound OR statement:

$$\exists x, P(x) \coloneqq P(x_1) \textit{ or } P(x_2) \textit{ or } P(x_3) \textit{ or } \ldots$$

For example, let our collection of x values be the numbers 3, 4, and 5 and define $P(x) := x < 4$. We can see that $\exists x, P(x)$ is a true statement since there is at least one x value (3) that is less than 4. If we define $Q(x) := x < 5$ then $\exists x, Q(x)$ is also true since 3 and 4 are less than 5 — we only need one! But if we define $R(x) := x < 3$ then $\exists x, R(x)$ is a false statement because all of the candidate x values are greater than or equal to 3.

There is a similar quantifier denoted $\exists!$ that stands for "there exists one and only one" that we use when we are interested in the existence of a unique value. For example, let our collection of x values be the numbers 3, 4, and 5 and $P(x) := x \leq 4$. We can see that $\exists! x, P(x)$ is a false statement because both 3 and 4 are less than or equal to 4.

Nested Quantifiers

Quantifiers can be nested, mixed, and matched. For example, the statement:

> For all x there exists a y such that $x + y = 0$

would be written as:

$$\forall x, \exists y, x + y = 0$$

If we let our candidate x values be the numbers 1, 2, and 3 and the candidate y values be -1, -2, and -3, then the above nested quantifier statement is true: for each candidate value of x, there is a y value such that $x + y = 0$. If we were to add the value 4 to our possible values of x then the nested quantifier statement is false, because for the candidate x value of 4, none of the possible y values satisfies $4 + y = 0$.