# Limits

## Definition

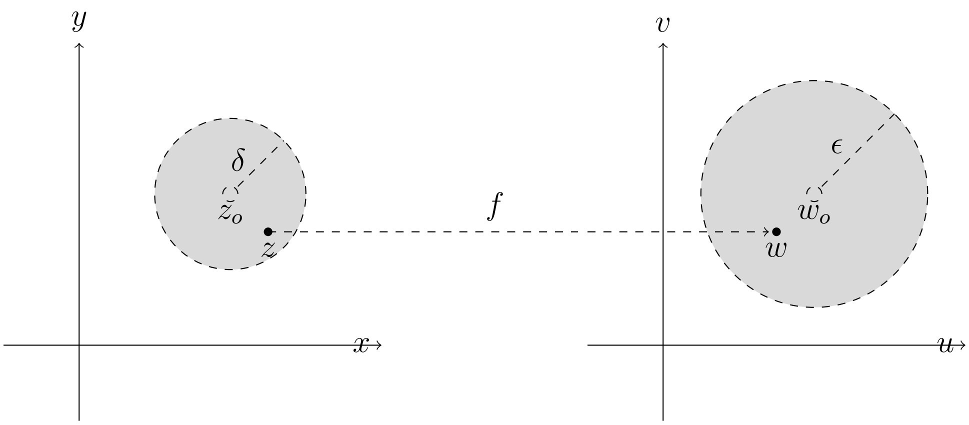
To say that:

$$\lim_{z \to z_0} f(z) = L$$

means:

$$\forall\, \epsilon > 0, \exists\, \delta > 0, 0 < |z - z_0| < \delta \implies |f(z) - L| < \epsilon$$

For all $z$ in an arbitrarily small deleted neighborhood for $z_0$, there exists an image in the arbitrarily small neighborhood for $w_0$.




## Example

Prove:

$$\lim_{z \to i} \frac{iz}{3} = -\frac{1}{3}$$

Assume $\epsilon > 0$
Let $\delta = 3\epsilon > 0$
Assume $0 < |z - i| < \delta$

$$\begin{aligned} \left|\frac{iz}{3} + \frac{1}{3}\right| &= \left|\frac{i}{3}(z - i)\right| \\ &= \left|\frac{1}{3}\right| |z - i| \\ &< \frac{\delta}{3} \\ &= \epsilon \end{aligned}$$

$$\begin{aligned} \left|\frac{iz}{3} + \frac{1}{3}\right| &= \left|\frac{i}{3}(z - i)\right| \\ &= \left|\frac{1}{3}\right| |z - i| \\ &< \frac{\delta}{3} \\ \epsilon &= \frac{\delta}{3} \\ \delta &= 3\epsilon \end{aligned}$$

**Example**

Prove:

$$\lim_{z \to z_0} z^2 = z_0^2$$

Assume $\epsilon > 0$
Let $\delta = \sqrt{\epsilon + |z_0|^2} - |z_0| > 0$
Assume $0 < |z - z_0| < \delta$

$$\begin{aligned}
\left|z^2 - z_0^2\right| &= |(z - z_0)(z + z_0)| \\
&= |z - z_0|\,|z + z_0| \\
&= |z - z_0|\,|z - z_0 + 2z_0| \\
&< \delta\,|\delta + 2z_o| \\
&= \delta^2 + 2z_0\delta \\
&= (\delta^2 + 2z_0\delta + |z_0|^2) - |z_0|^2 \\
&= (\delta + |z_0|)^2 - |z_0|^2 \\
&= \left[\left(\sqrt{\epsilon + |z_0|^2} - |z_0|\right) + |z_0|\right]^2 - |z_0|^2 \\
&= \left(\sqrt{\epsilon + |z_0|^2}\right)^2 - |z_0|^2 \\
&= \epsilon + |z_0|^2 - |z_0|^2 \\
&= \epsilon
\end{aligned}$$

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&< \delta\,|\delta + 2z_o| \\
&= \delta^2 + 2z_0\delta \\
&= (\delta^2 + 2z_0\delta + |z_0|^2) - |z_0|^2 \\
&= (\delta + |z_0|)^2 - |z_0|^2 \\
\epsilon &= (\delta + |z_0|)^2 - |z_0|^2 \\
(\delta + |z_0|)^2 &= \epsilon + |z_0|^2 \\
\delta + |z_0| &= \sqrt{\epsilon + |z_0|^2} \\
\delta &= \sqrt{\epsilon + |z_0|^2} - |z_0|
\end{aligned}$$

**Example**

Prove:

$$\lim_{z \to z_0} Re(z) = Re(z_0)$$

Assume $\epsilon > 0$
Let $\delta = \epsilon$
Assume $0 < |z - z_0| < \delta$

$$\begin{aligned}
|Re(z) - Re(z_0)| &= |Re(z - z_0)| \\
&\leq |z - z_0| \\
&< \delta \\
&= \epsilon
\end{aligned}$$

$$\begin{aligned}
|Re(z) - Re(z_0)| &= |Re(z - z_0)| \\
&\leq |z - z_0| \\
&< \delta \\
\epsilon &= \delta
\end{aligned}$$

## Example

Let $z = x + iy$. Prove:

$$\lim_{z \to 1-i} [x + i(2x + y)] = 1 + i$$

$x + i(2x + y) = (x + iy) + i2x = z + i2Re(z)$

WTS: $\lim_{z \to z_0} [z + i2Re(z)] = z_0 + i2Re(z_0)$

$$\begin{aligned}
|[z + i2Re(z)] - [z_0 + i2Re(z_0)]| &= |(z - z_0) + 2i[Re(z) - Re(z_0)]| \\
&= |(z - z_0) + 2i[Re(z - z_0)]| \\
&\leq |(z - z_0)| + |2iRe(z - z_0)| \\
&\leq |z - z_0| + 2\,|z - z_0| \\
&= 3\,|z - z_0| \\
&= 3\delta \\
\epsilon &= 3\delta \\
\delta &= \frac{\epsilon}{3}
\end{aligned}$$

Assume $\epsilon > 0$
Let $\delta = \frac{\epsilon}{3}$
Assume $0 < |z - z_0| < \delta$

$$\begin{aligned}
|[z + 2iRe(z)] - [z_0 + 2iRe(z_0)]| &= |(z - z_0) + 2i[Re(z) - Re(z_0)]| \\
&= |(z - z_0) + 2iRe(z - z_0)| \\
&\leq |(z - z_0)| + |2iRe(z - z_0)| \\
&\leq |z - z_0| + 2\,|z - z_0| \\
&= 3\,|z - z_0| \\
&< 3\delta \\
&= 3\left(\frac{\epsilon}{3}\right) \\
&= \epsilon
\end{aligned}$$

$$\begin{aligned}
\lim_{z \to 1-i} [x + i(2x + y)] = 1 + i &= \lim_{z \to 1-i} [z + i2Re(z)] \\
&= (1 - i) + i2Re(1 - i) \\
&= 1 - i + 2i \\
&= 1 + i
\end{aligned}$$

**Theorem**

$\lim_{z\to z_0} f(z)$ exists $\implies$ the limit is unique.

Proof

Assume $\lim_{z\to z_0} f(z)$ exists
Assume $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} f(z) = w_1$
Assume $\epsilon > 0$
$\exists\, \delta_0 > 0, 0 < |z - z_0| < \delta_0 \implies |f(z) - w_0| < \frac{\epsilon}{2}$
$\exists\, \delta_1 > 0, 0 < |z - z_0| < \delta_1 \implies |f(z) - w_1| < \frac{\epsilon}{2}$
Let $\delta = \min\{\delta_0, \delta_1\}$
Assume $0 < |z - z_0| < \delta$

$$\begin{array}{rcl} |w_0 - w_1| & = & |w_0 - f(z) + f(z) - w_1| \\ & le & |w_0 - f(z)| + |f(z) - w_1| \\ & = & |f(z) - w_0| + |f(z) - w_1| \\ & < & \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} \\ & = & \epsilon \end{array}$$

$w_0 - w_1 = 0$
$\therefore w_0 = w_1$

**Corollary**

If a limit is not unique then it does not exist.

**Example**

Prove that the following limit does not exist:

$$\lim_{z\to 0} \frac{\bar{z}}{z}$$

Path along postive real axis:

$z = x$
$\bar{z} = x$
$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{z\to 0} \frac{x}{x} = 1$

Path along postive imaginary axis:

$z = iy$
$\bar{z} = -iy$
$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{z\to 0} \frac{-iy}{iy} = -1$

The limits differ based on path
$\therefore$ the limit DNE.