Rank

Definition: Nullity and Rank

Let $A \in M_{n,m}$:

- The *null space* of A is given by:

$$\text{Null}(A) = \{\vec{x} \in \mathbb{C}^n \mid A\vec{x} = \vec{0}\}$$

- The *nullity* of A is given by:

$$\text{nullity}(A) = \dim \text{Null}(A)$$

- The *range* of A, denoted $\text{range}(A)$, is the space spanned by the columns of A.
- The *rank* of A is given by:

$$\text{rank}(A) = \dim \text{range}(A)$$

Theorem: Dimension Theorem

Let $A \in M_{m,n}$:

$$n = \text{rank}(A) + \text{nullity}(A)$$

where $\text{rank}(A)$ equals the number of pivots in the REF of A and $\text{nullity}(A)$ equals the number of free variables in the REF of A.

Note that the transformation $T : \mathbb{C}^n \to \mathbb{C}^m$ defined by $T(\vec{x}) = A\vec{x}$ is a linear transformation.

Theorem

Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be the linear transformation defined by $T(\vec{x}) = A\vec{x}$ for some $A \in M_{m,n}$:

T is injective $\iff$ $\text{Null}(A)$ is trivial

Proof

$\implies$ Assume T is injective

Assume $T(\vec{x}) = T(\vec{y})$
$T(\vec{x}) - T(\vec{y}) = \vec{0}$
$T(\vec{x} - \vec{y}) = \vec{0}$
But T is injective, so $\vec{x} = \vec{y}$
Thus, $T(\vec{0}) = \vec{0}$
Furthermore, since T is injective, no other element in the domain may map to $\vec{0}$

Therefore $\text{Null}(A)$ is trivial.

$\Longleftarrow$ Assume $\text{Null}(A)$ is trivial

Assume $T(\vec{x}) = T(\vec{y})$
$T(\vec{x}) - T(\vec{y}) = \vec{0}$
$T(\vec{x} - \vec{y}) = \vec{0}$
But the null space is trivial, and so $\vec{x} - \vec{y} = \vec{0}$
and so $\vec{x} = \vec{y}$

Therefore T is injective.

Theorem

Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be the linear transformation defined by $T(\vec{x}) = A\vec{x}$ for some $A \in M_{m,n}$:

T is injective $\iff$ T is surjective

Proof

T is injective
$\iff$ the null space is trivial
$\iff$ $\text{nullity}(A) = 0$
$\iff$ $\text{rank}(A) = n - 0 = n$
$\iff$ the column space of A spans C^n
$\iff$ T is surjective.

Lemma

Let $A \in M_{m,k}$ and $B \in M_{k,n}$:

$$\text{rank}(AB) \leq \text{rank}(A)$$

Proof

$\text{range}(AB) = \{(AB)\vec{x} | \vec{x} \in \mathbb{C}^n\} = \{A(B\vec{x}) | \vec{x} \in \mathbb{C}^n\} \subseteq \text{range}(A)$

$\therefore \text{rank}(AB) \leq \text{rank}(A)$

Theorem

Let $A \in M_{m,n}$:

$$\text{rank}(A) = \text{rank}(A^T)$$

Thus, the dimension of the column space equals the dimension of the row space.

Proof

Let $\mathrm{rank}(A) = r \leq n$
Thus, only r of the n columns of A are linearly independent
So construct $B \in M_{m,r}$ from the linearly independent columns of A
Assume $A = BX$ for some $X \in M_{r,n}$
$\mathrm{rank}(A^T) = \mathrm{rank}((BX)^T) = \mathrm{rank}(X^T B^T) \leq \mathrm{rank}(X^T)$
But $X^T \in M_{n,r}$ and so $\mathrm{rank}(X^T) \leq r$
So $\mathrm{rank}(A^T) \leq r = \mathrm{rank}(A)$
But since $(A^T)^T = A$, $\mathrm{rank}(A) \leq \mathrm{rank}(A^T)$

$\therefore \mathrm{rank}(A) = \mathrm{rank}(A^T)$