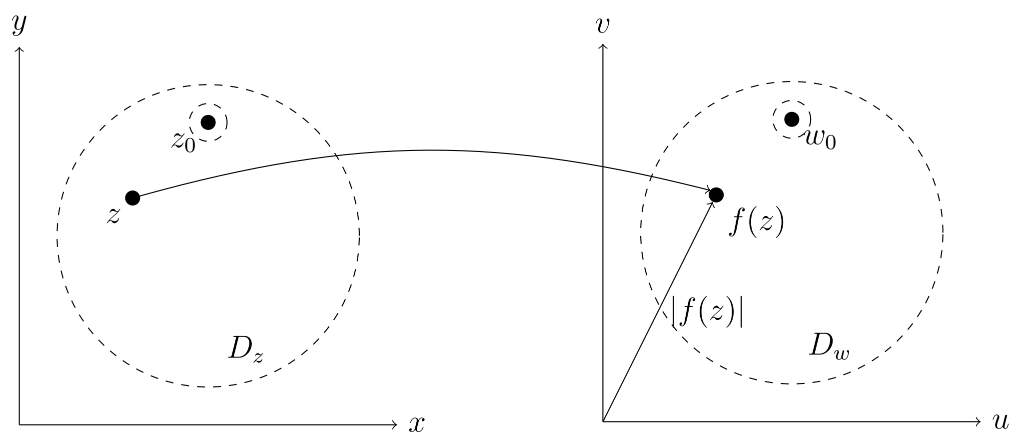
Maximum Principle

Theorem: Open Mapping

A non-constant analytic function maps an open set onto an open set and a domain onto a domain (connectedness is preserved).

Theorem: Maximum Principle

Let $f(z)$ be a non-constant analytic function in a domain D.
$|f(z)|$ has no maximum in D.



Proof

ABC: $|f(z)|$ obtains its maximum value at $z_0 \in D_z$
$\forall\, z \in D_z, |f(z)| \leq |f(z_0)|$
Let $w_0 = f(z_0)$
Since $z_0 \in D_z$ and D_z is open, there exists $N_\epsilon(z_0) \in D_z$
By the open mapping theorem, $N_\epsilon(z_0)$ is mapped to some $N_{\epsilon'}(w_0) \in D_w$
$\exists\, z_1 \in N_\epsilon(z_0)$ such that $f(z_1) = w_1 \in N_{\epsilon'}(w_0) \in D_w$
But $|w_1| > |w_0|$
CONTRADICTION!
Therefore, $f(z)$ has no maximum in D.

Corollary

Let $f(z)$ be a non-constant analytic function in region $\overline{D}$ with boundary C.
$|f(z)|$ has a maximum at some $z \in C$.

Proof

$\overline{D}$ is compact and thus $|f(z)|$ has a maximum in $\overline{D}$
But that maximum cannot occur in D
Therefore the maximum must occur on the boundary at some $z \in C$.

Note that the above theorems can also be stated in terms of a minimum as long as $\forall z \in D, f(z) \neq 0$.