Converting from Decimal to Rational Form

Rational numbers can be encountered in three possible forms:

- Fractional
- Finite Decimal
- Infinitely-repeating Decimal

When we see one of the decimal forms, we should be able to convert it to the fractional form.

Fractional Form

We define the set of rational numbers as follows:

$$\mathbb{Q} = \left\{ \frac{p}{q} \middle| p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

In other words, a rational number is any real number that can be expressed as an integer divided by an integer, with the denominator not equal to 0. Note that this is not the same thing as a fraction: all rational numbers can be expressed as fractions, but not all fractions are rational numbers. For example, consider the fraction $\frac{\pi}{2}$; this is not rational because the numerator is not an integer. However, $\frac{\sqrt{9}}{2}$ is a rational number because $\sqrt{9} = 3 \in \mathbb{Z}$. In other words, evaluate complicated numerators and demoninators before determining whether or not they are integers.

Finite Decimal Form

Rational numbers can be expressed as decimal numbers where the number of digits is finite. For example, the value:

$$12.345$$

has a finite number of digits and is therefore a rational number. We convert it to fractional form by using the digits in the numerator and then dividing by the appropriate power of 10:

$$\frac{12345}{100000}$$

Of course, this fraction can be reduced to $\frac{2469}{20000}$, if desired.

Note that a rational number with a finite number of decimal digits actually contains a repeating pattern of 0's after the last digit:

$$12.3450000\ldots$$

which we represent by drawing a bar over the repeating pattern:

$$12.345\overline{0}$$

So the finite case is actually a simple form of the repeating case.

Repeating Decimal Form

Rational numbers can be expressed as decimal numbers where the fractional part contains an infinite, repeating digit sequence. For example:

$$0.\overline{3} = 0.33333\ldots$$

To convert such a number to fractional form, we procede as follows.

1). Let $x = 0.\overline{3}$.

2). Multiply x by a power of 10 so that one set of repeating digits moves to the left of the decimal point. In this case: $10x = 3.\overline{3}$.

3). Now subtract the two equations. Note that the repeating part on the right of the decimal point cancels and we have: $9x = 3$.

4). Solve for x and reduce: $x = \frac{3}{9} = \frac{1}{3}$.

When a set of non-repeating digits appears before the repeating sequence, we adjust to procedure to capture the non-repeating digits to the left of the decimal point. For example, to convert $3.298\overline{145}$:

1). Let $x = 3.298\overline{145}$

2). Multiple by a power of 10 to capture the non-repeating digits only: $1000x = 3298.\overline{145}$

3). Multiply by a power of 10 to capture both the non-repeating digits and one set of the repeating digits: $1000000x = 3298145.\overline{145}$

4). Subtract the two equations to cancel the repeating part to the right of the decimal point: $999000x = 3294847$.

5). Finally, solve and reduce (if possible): $x = \frac{3294847}{999000}$.