Zero Divisors

Definition

Let R be a ring and $a \in R$ such that $a \neq 0$:

- To say that a is a *left zero divisor* of R means:

$$\exists\, b \in R, b \neq 0 \text{ and } ab = 0$$

- To say that a is a *right zero divisor* of R means:

$$\exists\, b \in R, b \neq 0 \text{ and } ba = 0$$

- To say that a is a *zero divider* of R means that a is a left or a right zero divisor of R.
- To say that a is a *two-sided zero divider* of R means that a is a left and a right zero divisor of R.

Note that for a two-sided zero divisor: $ax = ya = 0$, where x need not equal y, unless R is commutative.

Example

$\mathbb{Z}_{12}$

0-divisors	**units**
$2 \cdot 6 = 0$	$1 \cdot 1 = 1$
$3 \cdot 4 = 0$	$5 \cdot 5 = 1$
$8 \cdot 3 = 0$	$7 \cdot 7 = 1$
$9 \cdot 4 = 0$	$11 \cdot 11 = 1$
$10 \cdot 6 = 0$	

Theorem

$z \in \mathbb{Z}_n$ is a zero divisor $\iff (z, n) \neq 1$

Proof

$\implies$ Assume $(z, n) = 1$
Assume $\exists\, s \in \mathbb{Z}_n, zs = 0$
$n \mid zs$
But $n \nmid z$, so $n \mid s$
Thus, $s = 0$
$\therefore z$ is not a zero divisor.

$\impliedby$ Assume $(u, n) \neq 1$
Let $(z, n) = d > 1$
$d \mid z$ and $d \mid n$
$z\frac{n}{d} = n\frac{z}{d} = 0$
But $z, \frac{n}{d} \neq 0$
$\therefore z$ is a zero divisor.

Corollary

$\forall\, a \in \mathbb{Z}_n$, exactly one of the following is true:

1). $a = 0$

2). a is a unit

3). a is a zero divisor

Corollary

p prime $\implies$ $\mathbb{Z}_p$ has no zero divisors.

Theorem

Let R be a ring. The cancellation laws hold in R iff R has no zero divisors.

Proof

$\implies$ Assume the cancellation laws hold in R

Assume $a, b \in R, ab = 0$

Case 1: $a \neq 0$
a0=0
ab=a0
b=0
$\therefore$ a is not a zero divisor.

Case 2: $b \neq 0$
0b=0
ab=0b
a=0
$\therefore$ b is not a zero divisor.

$\therefore$ R has no zero divisors.

$\impliedby$ Assume R has no zero divisors

Assume $a, b, c \in R$ such that $a \neq 0$ and $ab = ac$
$ab - ac = 0$
$a(b - c) = 0$
Since $a \neq 0$ and R has no zero divisors, $b - c = 0$
$b = c$

$\therefore$ the left cancellation law holds in R.

Assume $a, b, c \in R$ such that $a \neq 0$ and $ba = ca$
$ba - ca = 0$
$(b - c)a = 0$
Since $a \neq 0$ and R has no zero divisors, $b - c = 0$
$b = c$

$\therefore$ the right cancellation law holds in R.

$\therefore$ the cancellation laws hold in R.

Theorem

Let R be ring with no zero divisors. $\forall\, a, b \in R$, the equations $ax = b$ and $xa = b$ each have at most one solution.

Proof

Assume $a, b \in R$

Assume $ax = b$ has two solutions: x_1 and x_2
$ax_1 = ax_2$
But the left cancellation law holds in R
$\therefore x_1 = x_2$.

Assume $xa = b$ has two solutions: x_1 and x_2
$x_1 a = x_2 a$
But the right cancellation law holds in R
$\therefore x_1 = x_2$.

Note that when R is a ring with unity $1 \neq 0$ and a is a unit in R then the unique solution to $ax = b$ is given by $x = a^{-1}b$. Likewise, the unique solution to $xa = b$ is $x = ba^1$. If R is commutative then these two solutions are the same.

Notation

Let F be a field. $\forall\, a, b \in F$, since $a^{-1}b = ba^{-1}$, this element in F is denoted by $\frac{b}{a}$.