Inner Product

Definition: Inner Product

Let $\vec{x}, \vec{y} \in \mathbb{C}^n$. The *inner product* of $\vec{x}$ and $\vec{y}$, denoted $\langle \vec{x}, \vec{y} \rangle$, is given by:

$$\langle \vec{x}, \vec{y} \rangle = \vec{y}^* \vec{x} = \sum_{k=1}^{n} \overline{y_k} x_k$$

Theorem

Let $\vec{x}, \vec{y}, \vec{z}, \vec{u} \in \mathbb{C}^n$ and $c \in \mathbb{C}$:

1). $\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle}$

2). $\langle c\vec{x}, \vec{y} \rangle = c \langle \vec{x}, \vec{y} \rangle$

3). $\langle \vec{x}, c\vec{y} \rangle = \overline{c} \langle \vec{x}, \vec{y} \rangle$

4). $\langle \vec{x} + \vec{y}, \vec{z} + \vec{u} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{x}, \vec{u} \rangle + \langle \vec{y}, \vec{z} \rangle + \langle \vec{y}, \vec{u} \rangle$

Proof

1).

$$\langle \vec{x}, \vec{y} \rangle = \vec{y}^* \vec{x} = (\vec{x}^* \vec{y})^* = \overline{\vec{x}^* \vec{y}} = \overline{\langle \vec{y}, \vec{x} \rangle}$$

2).

$$\langle c\vec{x}, \vec{y} \rangle = \vec{y}^* (c\vec{x}) = c(\vec{y}^* \vec{x}) = c \langle \vec{x}, \vec{y} \rangle$$

3).

$$\langle \vec{x}, c\vec{y} \rangle = (c\vec{y})^* \vec{x} = \overline{c}(\vec{y}^* \vec{x}) = \overline{c} \langle \vec{x}, \vec{y} \rangle$$

4).

$$\begin{aligned} \langle \vec{x} + \vec{y}, \vec{z} + \vec{u} \rangle &= (\vec{z} + \vec{u})^* (\vec{x} + \vec{y}) \\ &= (\vec{z}^* + \vec{u}^*)(\vec{x} + \vec{y}) \\ &= \vec{z}^* \vec{x} + \vec{u}^* \vec{x} + \vec{z}^* \vec{y} + \vec{u}^* \vec{y} \\ &= \langle \vec{x}, \vec{z} \rangle + \langle \vec{x}, \vec{u} \rangle + \langle \vec{y}, \vec{z} \rangle + \langle \vec{y}, \vec{u} \rangle \end{aligned}$$

Norm

Definition: Norm

Let $\vec{x} \in \mathbb{C}^n$. The *norm* of $\vec{x}$, denoted $\|\vec{x}\|$, is given by:

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{\sum_{k=1}^{n} \overline{x_k} x_k} = \sqrt{\sum_{k=1}^{n} |x_k|^2}$$

Definition: Orthogonal

Let $\vec{x}, \vec{y} \in \mathbb{C}^n$. To say that $\vec{x}$ and $\vec{y}$ are *orthogonal* means:

$$\langle \vec{x}, \vec{y} \rangle = 0$$

Theorem: Pythagorean Theorem

Let $\vec{x}, \vec{y} \in \mathbb{C}^n$:

$\vec{x}, \vec{y}$ orthogonal $\implies \|\vec{x} + \vec{y}\| = \|\vec{x}\| + \|\vec{y}\|$

Proof

Assume $\vec{x}, \vec{y}$ orthogonal

$$\begin{aligned} \|\vec{x} + \vec{y}\| &= \langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle \\ &= \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle \\ &= \|\vec{x}\| + 0 + 0 + \|\vec{y}\| \\ &= \|\vec{x}\| + \|\vec{y}\| \end{aligned}$$

Note that the converse is only true when $\langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle = 0$, or when $\langle \vec{x}, \vec{y} \rangle = -\overline{\langle \vec{x}, \vec{y} \rangle}$, but this can only occur if the real part is zero. Thus, the converse only holds when $\langle \vec{x}, \vec{y} \rangle$ is imaginary.

Theorem

Let $\vec{x}$ and $\vec{y}$ be non-zero vectors in $\mathbb{C}^n$:

$\vec{x}$ and $\vec{y}$ are orthogonal $\implies$ $\vec{x}$ and $\vec{y}$ are linearly independent.

Proof

Assume $\vec{x}$ and $\vec{y}$ are orthogonal
$\langle \vec{x}, \vec{y} \rangle = 0$
ABC: $\vec{x}$ and $\vec{y}$ are linearly dependent
There exists non-zero $c \in \mathbb{C}$ such that $\vec{x} = c\vec{y}$
$\langle \vec{x}, \vec{y} \rangle = \langle c\vec{y}, \vec{y} \rangle = c\|\vec{y}\|^2 \neq 0$
CONTRADICTION!

Therefore, $\vec{x}$ and $\vec{y}$ must be linearly independent.

Inequalities

Theorem: Cauchy-Schwarz

Let $\vec{x}, \vec{y} \in \mathbb{C}^n$:

$$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \|\vec{y}\|$$

Proof

Note that when $\vec{x}$ and $\vec{y}$ are dependent (including one or both zero) then equality holds, so AWLOG: $\vec{x}$ and $\vec{y}$ are independent (and thus non-zero).

Let $\vec{z} = \vec{x} - \frac{\langle \vec{x}, \vec{y} \rangle}{\langle \vec{y}, \vec{y} \rangle} \vec{y}$
$\langle \vec{y}, \vec{z} \rangle = \langle \vec{z}, \vec{y} \rangle = 0$
$\vec{x} = \vec{z} + \frac{\langle \vec{x}, \vec{y} \rangle}{\langle \vec{y}, \vec{y} \rangle} \vec{y}$
$\|\vec{x}\|^2 = \|\vec{z}\|^2 + \|\frac{\langle \vec{x}, \vec{y} \rangle}{\langle \vec{y}, \vec{y} \rangle} \vec{y}\|^2 = \|\vec{z}\|^2 + \frac{|\langle \vec{x}, \vec{y} \rangle|^2}{\|\vec{y}\|^2}$
$\|\vec{x}\|^2 \geq \frac{|\langle \vec{x}, \vec{y} \rangle|^2}{\|\vec{y}\|^2}$

$\therefore |\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \|\vec{y}\|$

Theorem: Triangle Inequality

Let $\vec{x}, \vec{y} \in \mathbb{C}^n$:

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

Proof

Note that when $\vec{x}$ and $\vec{y}$ are dependent (including one or both zero) then equality holds.

$$\begin{array}{rcl}
\|\vec{x} + \vec{y}\|^2 & = & \langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle \\
& = & \|\vec{x}\|^2 + \|\vec{y}\|^2 + \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle \\
& \leq & \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\,|\langle \vec{x}, \vec{y} \rangle| \\
& \leq & \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\|\vec{x}\| \|\vec{y}\| \\
& = & (\|\vec{x}\| + \|\vec{y}\|)^2 \\
\therefore \|\vec{x} + \vec{y}\| & \leq & \|\vec{x}\| + \|\vec{y}\|
\end{array}$$