Isometric Operators

Definition: Isometric

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. To say that T is an *isometric* operator means $\forall\, \vec{x} \in H$:

$$\|T\vec{x}\| = \|\vec{x}\|$$

An isometric operator preserves the norm.

Theorem

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$:

$$T \text{ is isometric } \iff T^*T = I$$

Proof

Assume $\vec{x} \in H$.

$\|T\vec{x}\| = \|\vec{x}\|$

$\implies$ Assume T is isometric.

$$\begin{aligned} \|T\vec{x}\|^2 &= \|\vec{x}\|^2 \\ \langle T\vec{x}, T\vec{x}\rangle &= \langle \vec{x}, \vec{x}\rangle \\ \langle T^*T\vec{x}, \vec{x}\rangle &= \langle \vec{x}, \vec{x}\rangle \end{aligned}$$

$$\therefore T^*T = I$$

$\impliedby$ Assume $T^*T = I$.

$$\begin{aligned} \|T\vec{x}\|^2 &= \langle T\vec{x}, T\vec{x}\rangle \\ &= \langle T^*T\vec{x}, \vec{x}\rangle \\ &= \langle I\vec{x}, \vec{x}\rangle \\ &= \langle \vec{x}, \vec{x}\rangle \\ &= \|\vec{x}\|^2 \\ \|T\vec{x}\| &= \|\vec{x}\| \end{aligned}$$

Therefore T is isometric.

Theorem

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be isometric. $\forall \vec{x}, \vec{y} \in H$:

$$\langle T\vec{x}, T\vec{y}\rangle = \langle \vec{x}, \vec{y}\rangle$$

In other words, an isometric operator preserves the inner product.

Proof

Assume $\vec{x}, \vec{x} \in H$.

$\langle T\vec{x}, T\vec{y} \rangle = \langle T^*T\vec{x}, \vec{y} \rangle = \langle I\vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$

Corollary

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be isometric. $\forall \vec{x}, \vec{y} \in H$:

$$\vec{x} \perp \vec{y} \iff T\vec{x} \perp T\vec{y}$$

Proof

Assume $\vec{x}, \vec{y} \in H$.

$\vec{x} \perp \vec{y} \iff \langle \vec{x}, \vec{y} \rangle = 0 \iff \langle T\vec{x}, T\vec{y} \rangle = 0 \iff T\vec{x} \perp T\vec{y}$

Theorem

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be isometric:

T is a Hilbert space isomorphism between H and $\mathcal{R}(T)$.

Proof

$T\vec{x} = 0 \iff \|T\vec{x}\| = 0 \iff \|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$

Thus, the kernel is trivial and T is one-to-one.

By definition, T is onto $\mathcal{R}(T)$

Therefore, T is bijective.

Also by definition: $\langle T\vec{x}, T\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$.

Therefore T is a Hilbert space isomorphism between H and $\mathcal{R}(T)$.

Note that although isometric T is necessarily one-to-one, it need not be onto.

Example

Let $H = \ell^2$ and $T(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, z_3, \ldots)$.

T is isometric; however, it is not onto.