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Math 231a
Homework #8

4-1. Let $\mathcal{H}$ be a vector space over $\mathbb{C}$ equipped with an inner product $\langle \cdot\,, \cdot \rangle$.

Prove: Cauchy-Schwarz

**Lemma**

The inner product is conjugate-linear in its second argument.

Proof

Assume $f, g \in \mathcal{H}$ and $\lambda \in \mathbb{C}$.

$$\langle f, \lambda g\rangle = \overline{\langle \lambda g, f\rangle} = \overline{\lambda\langle g, f\rangle} = \bar{\lambda}\,\overline{\langle g, f\rangle} = \bar{\lambda}\,\langle f, g\rangle$$

Proof

Assume $f, g \in \mathcal{H}$ and $\lambda \in \mathbb{C}$.

$$\begin{aligned}
\langle f + \lambda g, f + \lambda g\rangle &= \langle f, f\rangle + \langle f, \lambda g\rangle + \langle \lambda g, f\rangle + \langle \lambda g, \lambda g\rangle \\
&= \|f\|^2 + \bar{\lambda}\langle f, g\rangle + \lambda\langle g, f\rangle + \lambda\bar{\lambda}\|g\|^2 \\
&= \|f\|^2 + \bar{\lambda}\langle f, g\rangle + \lambda\langle g, f\rangle + |\lambda|^2\,\|g\|^2 \\
&\geq 0
\end{aligned}$$

Now, let $\lambda = -\frac{\langle f,g\rangle}{\|g\|^2} \in \mathbb{C}$:

$$\begin{aligned}
0 &\leq \|f\|^2 - \frac{\overline{\langle f, g\rangle}}{\|g\|^2}\langle f, g\rangle - \frac{\langle f, g\rangle}{\|g\|^2}\langle g, f\rangle + \frac{|\langle f, g\rangle|^2}{\|g\|^4}\|g\|^2 \\
&= \|f\|^2 - \frac{\overline{\langle f, g\rangle}}{\|g\|^2}\langle f, g\rangle - \frac{\langle f, g\rangle}{\|g\|^2}\overline{\langle f, g\rangle} + \frac{|\langle f, g\rangle|^2}{\|g\|^2} \\
&= \|f\|^2 - \frac{|\langle f, g\rangle|^2}{\|g\|^2} - \frac{|\langle f, g\rangle|^2}{\|g\|^2} + \frac{|\langle f, g\rangle|^2}{\|g\|^2} \\
&= \|f\|^2 - \frac{|\langle f, g\rangle|^2}{\|g\|^2}
\end{aligned}$$

and then:

$$\begin{aligned}
\|f\|^2 - \frac{|\langle f, g\rangle|^2}{\|g\|^2} &\geq 0 \\
\|f\|^2\|g\|^2 - |\langle f, g\rangle|^2 &\geq 0 \\
\|f\|^2\|g\|^2 &\geq |\langle f, g\rangle|^2
\end{aligned}$$

$\therefore |\langle f, g\rangle| \leq \|f\|\|g\|$

Prove: The triangle inequality

Proof

Assume $f, g \in \mathcal{H}$

$$\begin{array}{rcl} \|f+g\|^2 & = & \langle f+g, f+g\rangle \\ & = & \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle \\ & = & \|f\|^2 + \langle f,g\rangle + \langle g,f\rangle + \|g\|^2 \\ & \leq & \|f\|^2 + \|f\|\|g\| + \|g\|\|f\| + \|g\|^2 \\ & = & \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 \\ & = & (\|f\| + \|g\|)^2 \end{array}$$

$\therefore \|f+g\| \leq \|f\| + \|g\|$

4-2. Prove: $\forall f, g \in \mathcal{H}, |\langle f,g\rangle| = \|f\|\|g\|$ and $g \neq 0 \implies f = cg$ for some scalar $c$.

**Lemma**

$f \perp g \implies \forall c \in \mathbb{C}, f \perp cg$

Proof

Assume $f \perp g$
$\langle f,g\rangle = 0$
$\langle f,cg\rangle = \bar{c}\langle f,g\rangle = \bar{c}\cdot 0 = 0$
$\therefore f \perp cg$

Proof

Assume $f, g \in \mathcal{H}$ such that $g \neq 0$ and $|\langle f,g\rangle| = \|f\|\|g\|$
Let $h = f - \frac{\langle f,g\rangle}{\|g\|^2} g$

$$\begin{array}{rcl} \langle h,g\rangle & = & \langle f - \dfrac{\langle f,g\rangle}{\|g\|^2} g, g\rangle \\ & = & \langle f,g\rangle - \dfrac{\langle f,g\rangle}{\|g\|^2}\langle g,g\rangle \\ & = & \langle f,g\rangle - \dfrac{\langle f,g\rangle}{\|g\|^2}\|g\|^2 \\ & = & \langle f,g\rangle - \langle f,g\rangle \\ & = & 0 \end{array}$$

So $h \perp g$, and thus $h \perp \frac{\langle f,g\rangle}{\|g\|^2} g$

$$\begin{aligned}
f &= h + \frac{\langle f,g\rangle}{\|g\|^2} g \\
\|f\|^2 &= \|h + \frac{\langle f,g\rangle}{\|g\|^2} g\|^2 \\
&= \|h\|^2 + \|\frac{\langle f,g\rangle}{\|g\|^2} g\|^2 \\
&= \|h\|^2 + \frac{|\langle f,g\rangle|^2}{\|g\|^4}\|g\|^2 \\
&= \|h\|^2 + \frac{|\langle f,g\rangle|^2}{\|g\|^2} \\
\|f\|^2 &= \|h\|^2 + \|f\|^2 \\
\|h\|^2 &= 0
\end{aligned}$$

Thus $h = 0$.
Letting $c = \frac{\langle f,g\rangle}{\|g\|^2}$ we get the result: $0 = f - cg$
$\therefore f = cg$

8-4: Prove: $\ell^2(\mathbb{Z})$ is complete and separable.

Assume $(u_n)_{n=1}^{\infty}$ is Cauchy in $\|\cdot\|$, where $u_n = (\ldots, u_{n,-2}, u_{n,-1}, u_{n,0}, u_{n,1}, u_{n,2}, \ldots)$.

Claim 1: $\forall k \in \mathbb{Z}, (u_{n,k})$ is Cauchy in $|\cdot|$

<u>Proof</u>

ABC: $\exists k, (u_{n,k})$ is not Cauchy in $|\cdot|$
$\exists \epsilon_0 > 0, \forall N \in \mathbb{N}, \exists n, m > N, |u_{n,k} - u_{m,k}| \geq \epsilon_0$
Assume $0 < \epsilon < \epsilon_0^2$
Assume $N \in \mathbb{N}$, thus selecting an $n, m > N$.
$\|u_n - u_m\| < \epsilon$

However:

$$\begin{aligned}
\|u_n - u_m\| &= \sum_{j=-\infty}^{\infty} |u_{n,j} - u_{m,j}|^2 \\
&= |u_{n,k} - u_{m,k}|^2 + \sum_{j\neq k} |u_{n,j} - u_{m,j}|^2 \\
&\geq \epsilon_0^2 + \sum_{j\neq k} |u_{n,j} - u_{m,j}|^2
\end{aligned}$$

$$\begin{aligned} &\geq \epsilon + \sum_{j \neq k} |u_{n,j} - u_{m,j}|^2 \\ &\geq \epsilon \end{aligned}$$

Contradiction!

$\therefore \forall k \in \mathbb{Z}, (u_{n,k})$ is Cauchy in $|\cdot|$

And since $\mathbb{C}$ is complete,
$\forall k \in \mathbb{Z}, u_{n,k} \to u_k \in \mathbb{C}$ as $n \to \infty$, meaning
$u_n \to u$ where $u = (\ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots)$

Claim 2: $u \in \ell^2(\mathbb{Z})$

<u>Proof</u>

Assume $\epsilon > 0$
$\exists N, \forall k, n > N \implies |u_k - u_{n,k}| < \frac{\epsilon}{2N+1}$
Assume $n > N$

$$\begin{aligned} \sum_{k=-N}^{N} |u_k|^2 &= \sum_{k=-N}^{N} |u_k - u_{n,k} + u_{n,k}|^2 \\ &\leq \sum_{k=-N}^{N} |u_k - u_{n,k}| + \sum_{k=-N}^{N} |u_{n,k}|^2 \end{aligned}$$

But $u_n \in \ell^2(\mathbb{Z})$, so $\sum_{k=-N}^{N} |u_{n,k}|^2 \leq \sum_{k=-\infty}^{\infty} |u_{n,k}|^2 = M < \infty$. So:

$$\begin{aligned} \sum_{k=-N}^{N} |u_k|^2 &\leq \sum_{k=-N}^{N} |u_k - u_{n,k}| + M \\ &= \sum_{k=-N}^{N} \frac{\epsilon}{2N+1} + M \\ &= \sum_{k=-N}^{N} \frac{\epsilon}{2N+1} + M \\ &= (2N+1)\left(\frac{\epsilon}{2N+1}\right) + M \\ &= \epsilon + M \\ &< \infty \end{aligned}$$

Thus, letting $k \to \infty, \sum_{k=-\infty}^{\infty} |u_k|^2 < \infty$.
$\therefore u \in \ell^2(\mathbb{Z})$.

Claim 3: $u_n \to u$ in $\|\cdot\|$

<u>Proof</u>

Assume $\epsilon > 0$
$\exists N, n, m > N \implies \|u_n - u_m\| \leq \epsilon$
As $m \to \infty, u_m \to u$ and so:
$\|u_n - u\| \leq \epsilon$
$\therefore u_n \to u$ in $\|\cdot\|$

Claim 4: $\ell^2(\mathbb{Z})$ is separable.

Define $e_i \in \ell^2(\mathbb{Z})$ such that $e_{ij} = \delta_{ij}$.
Note that $e_i \in \ell^2(\mathbb{Z})$ because $\|e_i\| = \sum_{k=-\infty}^{\infty} |e_{i,k}|^2 = 1$.
Clearly, $\bigcup_i e_i$ is a countable subset of $\ell^2(\mathbb{Z})$.
Assume $u$ is a linear combination of some finite subset of $\bigcup_i e_i$.
Let $N \in \mathbb{N}$ such that $\forall\, |k| \geq N, u_k = 0$.

$$\sum_{k=-\infty}^{\infty} |u_k|^2 \leq \sum_{k=-N}^{N} |u_k|^2 < \infty$$

since it is a finite sum.
So $u \in \ell^2(\mathbb{Z})$.
Assume $\epsilon > 0$
Let $v = u + \frac{\epsilon}{2} e_N$.
Since $\ell^2(\mathbb{Z})$ is a vector space, $v \in \ell^2(\mathbb{Z})$ as well, and:

$$\|u - v\| = \|\frac{\epsilon}{2} e_N\| = \frac{\epsilon}{2}\|e_N\| = \frac{\epsilon}{2} \cdot 1 = \frac{\epsilon}{2} < \epsilon$$

$\therefore \bigcup_i e_i$ is dense in $\ell^2(\mathbb{Z})$ and thus $\ell^2(\mathbb{Z})$ is separable.