Adjoint Operator

Let H be a Hilbert space and let $A \in \mathcal{B}(H)$. Fix some $\vec{y} \in H$ and define:

$$f_{\vec{y}}(\vec{x}) = \langle A\vec{x}, \vec{y} \rangle$$

f is clearly linear, and it is bounded:

$$|f_{\vec{y}}(\vec{x})| = |\langle A\vec{x}, \vec{y} \rangle| \leq \|A\vec{x}\| \, \|\vec{y}\| \leq \|A\| \, \|\vec{x}\| \, \|\vec{y}\| = (\|A\| \, \|\vec{y}\|) \, \|\vec{x}\|$$

Furthermore, by the Riesz Representation Theorem, $\exists, \vec{z} \in H$ such that:

$$f_{\vec{y}}(\vec{x}) = \langle \vec{x}, \vec{z} \rangle \text{ where } \|f_y\| = \|\vec{z}\|$$

Let $\vec{z} = A^*\vec{y}$.

Definition

Let H be a Hilbert space and let $A \in \mathcal{B}(H)$. The *adjoint* of A, denoted A^* is the uniquely-defined operator that makes the following statement true $\forall\, \vec{x}, \vec{y} \in H$:

$$\langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A^*\vec{y} \rangle$$

Example

Let $H = \mathbb{C}^N$ and let $A \in \mathcal{B}(H)$.

A can be represented by matrix multiplication.
Assume the standard basis:

$$a^*_{ij} = \langle A^* e_j, e_i \rangle = \langle e_j, Ae_i \rangle = \overline{\langle Ae_i, e_j \rangle} = \overline{a_{ji}}$$

Therefore, A^* corresponds to the conjugate transpose of the matrix corresponding to A.

Note that this also holds for H infinite dimensional and separable.

Lemma

Let H Hilbert space and let $S, T \in \mathcal{B}(H)$:

1). $(\forall\, \vec{x}, \vec{y} \in H, \langle S\vec{x}, \vec{y} \rangle = \langle T\vec{x}, \vec{y} \rangle) \iff S = T$

2). $(\forall\, \vec{x}, \vec{y} \in H, \langle \vec{x}, S\vec{y} \rangle = \langle \vec{x}, T\vec{y} \rangle) \iff S = T$

Proof

Assume $\vec{x}, \vec{y} \in H$.

$$\begin{aligned}
\langle S\vec{x}, \vec{y} \rangle = \langle T\vec{x}, \vec{y} \rangle \quad &\iff \quad \langle S\vec{x}, \vec{y} \rangle - \langle T\vec{x}, \vec{y} \rangle = 0 \\
&\iff \quad \langle S\vec{x} - T\vec{x}, \vec{y} \rangle = 0 \\
&\iff \quad \langle (S - T)\vec{x}, \vec{y} \rangle = 0 \\
&\iff \quad S - T \equiv 0 \\
&\iff \quad S = T
\end{aligned}$$

$$
\begin{aligned}
\langle \vec{x}, S\vec{y} \rangle = \langle \vec{x}, T\vec{y} \rangle \quad &\iff \quad \overline{\langle S\vec{y}, \vec{x} \rangle} = \overline{\langle T\vec{y}, \vec{x} \rangle} \\
&\iff \quad \langle S\vec{y}, \vec{x} \rangle = \langle T\vec{y}, \vec{x} \rangle \\
&\iff \quad S = T
\end{aligned}
$$

Theorem

Let H be a Hilbert space and let $A \in \mathcal{B}(H)$:

$$A^* \in \mathcal{B}(H)$$

Proof

Assume $\vec{x}, \vec{y}, \vec{z} \in H$ and $\alpha, \beta \in \mathbb{C}$:

$$
\begin{aligned}
\langle \vec{z}, A^*(\alpha\vec{x} + \beta\vec{y}) \rangle &= \langle A\vec{z}, \alpha\vec{x} + \beta\vec{y} \rangle \\
&= \overline{\alpha} \langle A\vec{z}, \vec{x} \rangle + \overline{\beta} \langle A\vec{z}, \vec{y} \rangle \\
&= \overline{\alpha} \langle \vec{z}, A^*\vec{x} \rangle + \overline{\beta} \langle \vec{z}, A^*\vec{y} \rangle \\
&= \langle z, \alpha A^*\vec{x} + \beta A^*\vec{y} \rangle
\end{aligned}
$$

$\therefore A^*(\alpha\vec{x} + \beta\vec{y}) = \alpha A^*\vec{x} + \beta A^*\vec{y}$ and thus A^* is linear.

$$
\begin{aligned}
\|A^*\vec{x}\|^2 &= \langle A^*\vec{x}, A^*\vec{x} \rangle \\
&= \langle A(A^*\vec{x}), \vec{x} \rangle \\
&\leq \|A(A^*\vec{x})\| \, \|\vec{x}\| \\
&\leq \|A\| \, \|A^*\vec{x}\| \, \|\vec{x}\|
\end{aligned}
$$

Thus $\|A^*\vec{x}\| \leq \|A\| \, \|\vec{x}\|$ with equality at $\vec{x} = 0$.

Therefore A^* is bounded by $\|A\|$.

Properties

Let H be a Hilbert space and let $A, B \in \mathcal{B}(H)$ and $\alpha \in \mathbb{C}$:

1). $(A + B)^* = A^* + B^*$

2). $(\alpha A)^* = \overline{\alpha} A^*$

3). $(A^*)^* = A$

4). $I^* = I$

5). $(AB)^* = B^* A^*$

Proof

Assume $\vec{x}, \vec{y} \in H$:

1).

$$
\begin{aligned}
\langle (A+B)^*\vec{x}, \vec{y}\rangle &= \langle \vec{x}, (A+B)\vec{y}\rangle \\
&= \langle \vec{x}, A\vec{y} + B\vec{y}\rangle \\
&= \langle \vec{x}, A\vec{y}\rangle + \langle \vec{x}, B\vec{y}\rangle \\
&= \langle A^*\vec{x}, \vec{y}\rangle + \langle B^*\vec{x}, \vec{y}\rangle \\
&= \langle A^*\vec{x} + B^*\vec{x}, \vec{y}\rangle \\
&= \langle (A^* + B^*)\vec{x}, \vec{y}\rangle
\end{aligned}
$$

$\therefore (A+B)^* = A^* + B^*$

2).

$$\langle (\alpha A)^*\vec{x}, \vec{y}\rangle = \langle \vec{x}, \alpha A\vec{y}\rangle = \overline{\alpha}\,\langle \vec{x}, A\vec{y}\rangle = \overline{\alpha}\,\langle A^*\vec{x}, \vec{y}\rangle = \langle \overline{\alpha}A^*\vec{x}, \vec{y}\rangle$$

$\therefore (\alpha A)^* = \overline{\alpha}A^*$

3).

$$\langle (A^*)^*\vec{x}, \vec{y}\rangle = \langle \vec{x}, A^*\vec{y}\rangle = \langle A\vec{x}, \vec{y}\rangle$$

$\therefore (A^*)^* = A$

4).

$$\langle I^*\vec{x}, \vec{y}\rangle = \langle \vec{x}, I\vec{y}\rangle = \langle \vec{x}, \vec{y}\rangle = \langle I\vec{x}, \vec{y}\rangle$$

$\therefore I^* = I$

5).

$$\langle (AB)^*\vec{x}, \vec{y}\rangle = \langle \vec{x}, (AB)\vec{y}\rangle = \langle \vec{x}, A(B\vec{y})\rangle = \langle A^*\vec{x}, B\vec{y}\rangle = \langle B^*(A^*\vec{x}), \vec{y}\rangle = \langle (B^*A^*)\vec{x}, \vec{y}\rangle$$

$\therefore (AB)^* = B^*A^*$

Theorem

Let H be a Hilbert space and $A \in \mathcal{B}(H)$:

1). $\|A^*\| = \|A\|$

2). $\|A^*A\| = \|A\|^2$

Proof

Assume $\vec{x} \in H$:

1). From above: $\|A^*\| \leq \|A\|$.
Also: $\|A\| = \|(A^*)^*\| \leq \|A^*\|$.

$\therefore \|A^*\| = \|A\|$

2). $\|A^*A\| \leq \|A^*\| \, \|A\| = \|A\| \, \|A\| = \|A\|^2$ Also:

$$
\begin{aligned}
\|A\|^2 &= \left[\sup_{\|\vec{x}\|=1} \|A\vec{x}\|\right]^2 \\
&= \sup_{\|\vec{x}\|=1} \|A\vec{x}\|^2 \\
&= \sup_{\|\vec{x}\|=1} \langle A\vec{x}, A\vec{x}\rangle \\
&= \sup_{\|\vec{x}\|=1} \langle A^*(A\vec{x}), \vec{x}\rangle \\
&= \sup_{\|\vec{x}\|=1} \langle (A^*A)\vec{x}, \vec{x}\rangle \\
&\leq \sup_{\|\vec{x}\|=1} \|(A^*A)\vec{x}\| \, \|\vec{x}\| \\
&\leq \sup_{\|\vec{x}\|=1} \|A^*A\| \, \|\vec{x}\| \, \|\vec{x}\| \\
&= \sup_{\|\vec{x}\|=1} \|A^*A\| \, \|\vec{x}\|^2 \\
&\leq \|A^*A\|
\end{aligned}
$$

$\therefore \|A^*A\| = \|A\|^2$