Cauchy-Schwarz Inequality

Theorem: Cauchy-Schwarz

Let E be a normed space over a field $\mathbb{F}$ with an inner product induced norm. $\forall\, \vec{x}, \vec{y} \in E$:

$$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \, \|\vec{y}\|$$

with equality iff $\exists \alpha \in \mathbb{F}$ such that $\vec{x} = \alpha \vec{y}$, i.e., $\vec{x}$ and $\vec{y}$ are dependent.

Proof

If $\vec{y} = \vec{0}$ then trivial, so AWLOG: $\vec{y} \neq \vec{0}$.

Assume $\lambda \in \mathbb{F}$:

$$\begin{aligned}
0 &\leq \|\vec{x} + \lambda \vec{y}\|^2 \\
&= \langle \vec{x} + \lambda \vec{y}, \vec{x} + \lambda \vec{y} \rangle \\
&= \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, \lambda \vec{y} \rangle + \langle \lambda \vec{y}, \vec{x} \rangle + \langle \lambda \vec{y}, \lambda \vec{y} \rangle \\
&= \|\vec{x}\|^2 + \langle \vec{x}, \lambda \vec{y} \rangle + \overline{\langle \vec{x}, \lambda \vec{y} \rangle} + \|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 + 2\,\mathrm{Re}(\langle \vec{x}, \lambda \vec{y} \rangle) + |\lambda|^2 \|\vec{y}\|^2
\end{aligned}$$

Now, let $\lambda = -\dfrac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2}$

$$\begin{aligned}
0 &\leq \|\vec{x}\|^2 + 2\,\mathrm{Re}\left(\left\langle \vec{x}, -\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2} \vec{y} \right\rangle\right) + \left| -\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2} \right|^2 \|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 + 2\,\mathrm{Re}\left(-\frac{\overline{\langle \vec{x}, \vec{y} \rangle}}{\|\vec{y}\|^2} \langle \vec{x}, \vec{y} \rangle \right) + \frac{|\langle \vec{x}, \vec{y} \rangle|}{\|\vec{y}\|^4} \|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 + 2\,\mathrm{Re}\left(-\frac{|\langle \vec{x}, \vec{y} \rangle|^2}{\|\vec{y}\|^2} \right) + \frac{|\langle \vec{x}, \vec{y} \rangle|}{\|\vec{y}\|^2} \\
&= \|\vec{x}\|^2 - 2\frac{|\langle \vec{x}, \vec{y} \rangle|^2}{\|\vec{y}\|^2} + \frac{|\langle \vec{x}, \vec{y} \rangle|}{\|\vec{y}\|^2} \\
&= \|\vec{x}\|^2 - \frac{|\langle \vec{x}, \vec{y} \rangle|^2}{\|\vec{y}\|^2}
\end{aligned}$$

But, by assumption, $\vec{y} \neq \vec{0}$ and so $\|\vec{y}\| \neq 0$:

$$\begin{aligned}
\|\vec{x}\|^2 \|\vec{y}\|^2 - |\langle \vec{x}, \vec{y} \rangle|^2 &\geq 0 \\
\|\vec{x}\|^2 \|\vec{y}\|^2 &\geq |\langle \vec{x}, \vec{y} \rangle|^2 \\
\|\vec{x}\| \, \|\vec{y}\| &\geq |\langle \vec{x}, \vec{y} \rangle|
\end{aligned}$$

$\therefore\ |\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \, \|\vec{y}\|$

Equality $\iff \|\vec{x} + \lambda \vec{y}\|^2 = 0 \iff \vec{x} + \lambda \vec{y} = 0 \iff \vec{x} = -\lambda \vec{y} = \alpha \vec{y}$