Orthogonal

Definition: Orthogonal

Let E be an inner product space and let $\vec{x}, \vec{y} \in E$. To say that $\vec{x}$ is *orthogonal* to $\vec{y}$, denoted $\vec{x} \perp \vec{y}$, means:

$$\langle \vec{x}, \vec{y} \rangle = 0$$

Properties

Let E be an inner product space. $\forall\, \vec{x}, \vec{y} \in E$:

1). $\vec{x} \perp \vec{0}$

2). $\vec{x} \perp \vec{y} \iff \vec{y} \perp \vec{x}$

Proof

Assume $\vec{x}, \vec{y} \in E$:

1). $\left\langle \vec{x}, \vec{0} \right\rangle = 0$

2). Assume $\vec{x} \perp \vec{y}$

$$\langle \vec{x}, \vec{y} \rangle = 0$$
$$\langle \vec{y}, \vec{x} \rangle = \overline{\langle \vec{x}, \vec{y} \rangle} = \overline{0} = 0$$

$$\therefore \vec{y} \perp \vec{x}$$

Assume $\vec{y} \perp \vec{x}$

$$\langle \vec{y}, \vec{x} \rangle = 0$$
$$\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle} = \overline{0} = 0$$

$$\therefore \vec{x} \perp \vec{y}$$

Theorem: Pythagorean

Let E be an inner product space and $\vec{x}, \vec{y} \in E$:

$$\vec{x} \perp \vec{y} \implies \|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$$

Proof

Assume $\vec{x} \perp \vec{y}$.

Also $\vec{y} \perp \vec{x}$ and so $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle = 0$.

$\|\vec{x} + \vec{y}\|^2 = \langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle = \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle = \|\vec{x}\|^2 + 0 + 0 + \|\vec{y}\|^2$

$\therefore \|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$

Definition: Mutually Orthogonal

Let E be an inner product space and let $S = \{\vec{x}_1, \ldots, \vec{x}_n\}$ be a non-empty subset of E. To say that S is a *mutually orthogonal* set means:

$$\forall\, i \neq j, \vec{x}_i \perp \vec{x}_j$$

Lemma

Let E be an inner product space over a field $\mathbb{F}$ and let $S = \{\vec{x}_1, \ldots, \vec{x}_n\}$ be a mutually orthogonal subset of E. $\forall\, 1 \leq r \leq n$:

$$S' = \{\vec{x}, \vec{x}_{r+1}, \ldots, \vec{x}_n\}$$

where $\vec{x} \in \mathrm{Span}\{\vec{x}_1, \ldots, \vec{x}_r\}$ is also a mutually orthogonal set.

Proof

Assume $1 \leq r \leq n$.
Assume $\vec{x} \in \mathrm{Span}\{\vec{x}_1, \ldots, \vec{x}_r\}$.
$\exists\, \lambda_k \in \mathbb{F}$ such that $\vec{x} = \displaystyle\sum_{k=1}^{r} \lambda_k \vec{x}_k$.

Assume $r + 1 \leq j \leq n$.

$$\langle \vec{x}, \vec{x}_j \rangle = \left\langle \sum_{k=1}^{r} \lambda_k \vec{x}_k, \vec{x}_j \right\rangle = \lambda_k \sum_{k=1}^{r} \langle \vec{x}_k, \vec{x}_j \rangle = \lambda_k \sum_{k=1}^{r} 0 = 0$$

Thus, $\vec{x}$ is orthogonal to all the $\vec{x}_j \mid r + 1 \leq j \leq n$.
Furthermore, all the $\vec{x}_j$ are orthogonal.

Therefore S' is a mutually orthogonal set.

Corollary

Let E be an inner product space and let $\{\vec{x}_1, \ldots, \vec{x}_n\}$ be a non-empty, mutually orthogonal subset of E:

$$\left\| \sum_{k=1}^{n} \vec{x}_k \right\|^2 = \sum_{k=1}^{n} \|\vec{x}_k\|^2$$

Proof

By induction on n:

Base case: $n = 1$

Trivial.

Assume for mutually orthogonal set $\{\vec{x}_1, \ldots, \vec{x}_n\} \subset E$:

$$\left\| \sum_{k=1}^{n} \vec{x}_k \right\|^2 = \sum_{k=1}^{n} \|\vec{x}_k\|^2$$

Consider the mutually orthogonal set $\{\vec{x}_1, \ldots, \vec{x}_n, \vec{x}_{n+1}\} \subset E$:

$$\left\|\sum_{k=1}^{n+1} \vec{x}_k\right\|^2 = \left\|\sum_{k=1}^{n} \vec{x}_k + \vec{x}_{n+1}\right\|^2 = \left\|\sum_{k=1}^{n} \vec{x}_k\right\|^2 + \|\vec{x}_{n+1}\|^2 = \sum_{k=1}^{n} \|\vec{x}_k\|^2 + \|\vec{x}_{n+1}\|^2 = \sum_{k=1}^{n+1} \|\vec{x}_k\|^2$$

Definition: Euclidean

A *Euclidean* space is a finite-dimensional real inner product space. In other words, $\langle \cdot, \cdot \rangle \in \mathbb{R}$.

Applying the Cauchy-Schwarz inequality to a Euclidean space:

$$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \, \|\vec{y}\|$$

For $\vec{x}, \vec{y} \neq \vec{0}$:

$$\frac{|\langle \vec{x}, \vec{y} \rangle|}{\|\vec{x}\| \, \|\vec{y}\|} \leq 1$$

$$-1 \leq \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\| \, \|\vec{y}\|} \leq 1$$

So let $\cos\theta = \dfrac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\| \, \|\vec{y}\|}$:

$$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \, \|\vec{y}\| \cos\theta$$