Closed Sets

Notation

Let $(X, \mathscr{T})$ be a topological space and $p \in X$:

$$\mathcal{U}_p = \{U \in \mathscr{T} \mid p \in U\}$$

Definition: Limit Point

Let $(X, \mathscr{T})$ be a topological space, $A \subset X$, and $p \in X$. To say that p is a *limit point* of A means:

$$\forall\, U \in \mathcal{U}_p, (U - \{p\}) \cap A \neq \emptyset$$

Example

Let $X = \mathbb{R}$ and $A = (1, 2)$. Verify that 0 is a limit point of A in the indiscrete and cofinite topologies but not in the standard nor discrete topologies.

Indiscrete: Since $0 \notin (1, 2)$, it follows that $(\mathbb{R} - \{0\}) \cap (1, 2) = (1, 2) \neq \emptyset$.

Therefore 0 is a limit point of $(1, 2)$.

Cofinite: Assume $U \in \mathscr{T}$.

This means that $U = \mathbb{R} - X$ where X is some finite set. But $(1, 2)$ is uncountable and so:

$$\begin{aligned}
(U - \{0\}) \cap (1, 2) &= U \cap (1, 2) \\
&= (\mathbb{R} - X) \cap (1, 2) \\
&= (\mathbb{R} \cap (1, 2)) - (X \cap (1, 2)) \\
&= (1, 2) - (X \cap (1, 2)) \\
&\neq \emptyset
\end{aligned}$$

Therefore 0 is a limit point of $(1, 2)$.

Standard: Let $\epsilon = \frac{1}{2}$.

$B(0, \frac{1}{2}) \cap (1, 2) = \emptyset$

Therefore 0 is not a limit point of $(1, 2)$.

Discrete: Consider $[0, 1] \in \mathscr{T}$.

$0 \in [0, 1]$ but $[0, 1] \cap (1, 2) = \emptyset$.

Therefore 0 is not a limit point of $(1, 2)$.

Theorem

Let $(X, \mathscr{T})$ be a topological space, $A \subset X$, and $p \in X$ but $p \notin A$. p is not a limit point of A iff there exists $U \in \mathcal{U}_p$ such that $U \cap A = \emptyset$.

Proof. If $p \notin A$ then the definition of a limit point becomes: p is a limit point of A iff for all $U \in \mathcal{U}_p, U \cap A \neq \emptyset$. Negating both sides of the equivalence yields an equivalent proposition and gives the desired result. ∎

Definition: Isolated Point

Let $(X, \mathscr{T})$ be a topological space, $A \subset X$, and $p \in X$. To say that p is an *isolated point* in A means that $p \in A$ and p is not a limit point of A.

Theorem

Let $(X, \mathscr{T})$ be a topological space, $A \subset X$, and $p \in X$. If p is an isolated point in A then there exists $U \in \mathscr{T}$ such that $U \cap A = \{p\}$.

Proof. Assume that p is an isolated point in A. This means that $p \in A$ and p is not a limit point of A. Thus, there exists $U \in \mathcal{U}_p$ such that $(U - \{p\}) \cap A = \emptyset$. But $p \in U$ and $p \in A$.

Therefore $U \cap A = \{p\}$. ∎

Example

Give examples of sets A in various topological spaces $(X, \mathscr{T})$ with:

1. A limit point of A that is an element of A.

   Let $X = \mathbb{R}$ and $A = (-1, 1)$.

   For standard, discrete, indiscrete, cofinite, and cocountable: $p = 0$.

2. A limit point of A that is not an element of A.

   Let $X = \mathbb{R}$ and $A = (-1, 1)$.

   For standard, indiscrete, cofinite, and cocountable: $p = 1$. For discrete, no such limit points can exist because if $p \notin A$ then $\{p\} \in \mathscr{T}$ and $\{p\} \cap A = \emptyset$.

3. An isolated point in A.

   Let $X = \mathbb{R}$ and $A = \mathbb{Z}$.

   For standard, discrete, indiscrete, cofinite, and cocountable: $p = 0$.

4. A point not in A that is not a limit point of A.

   Let $X = \mathbb{R}$ and $A = \mathbb{N}$.

   For standard, discrete, indiscrete, cofinite, and cocountable: $p = 0$.

Notation

Let $(X, \mathscr{T})$ be a topological space and $A \subset X$:

$$A' = \{x \in X \mid x \text{ is a limit point of } A\}$$

Definition: Closure

Let $(X, \mathscr{T})$ be a topological space and let $A \subset X$. The *closure* of A in X, denoted by $\bar{A}$, is given by:

$$\bar{A} = A \cup A'$$

Definition: Closed

Let $(X, \mathscr{T})$ be a topological space and let $A \subset X$. To say that A is *closed* means that $\bar{A} = A$. Thus, A contains all of its limit points.

Example

Which sets are closed in a set X with the following topologies?:

Discrete: All $A \subset X$.

If $p \notin A$ then it cannot be a limit point for A (see above), and therefore each A contains all of its limit points. Thus every $A \subset X$ is actually clopen.

Indiscrete: Only $\emptyset$ and X.

Assume $p \in X$. Since $p \notin \emptyset$, $(X - \{p\}) \cap \emptyset = \emptyset$ and so p is not a limit point for $\emptyset$. Since X contains everything then it must contain its limit points. For any other $A \subset X$, assume $p \neq A$. Then: $(\mathbb{R} - \{p\}) \cap A = A \neq \emptyset$ and thus p is a limit point for A not in A and therefore A is not closed.

Cofinite: $\emptyset$, X, and all finite sets.

Assume $p \in X$ and $U \in \mathscr{T}$ such that $p \in U$. Since $p \notin \emptyset$, $(U - \{p\}) \cap \emptyset = \emptyset$ and so p is not a limit point for $\emptyset$. Since X contains everything then it must contain its limit points.

Now, assume A is finite and $p \notin A$. Let $U = X - A \in \mathscr{T}$. Then:

$$\begin{aligned}(U - \{p\}) \cap A &= U \cap A \\ &= (X - A) \cap A \\ &= (X \cap A) - (A \cap A) \\ &= A - A \\ &= \emptyset\end{aligned}$$

Thus p is not a limit point for A and therefore A is closed.

Finally, assume A is infinite and $p \notin A$. Assume $U \in \mathscr{T}$ and $p \in U$. But $U = X - F$ for some finite set F. Then:

$$\begin{aligned}(U - \{p\}) \cap A &= U \cap A \\ &= (X - F) \cap A \\ &= (X \cap A) - (F \cap A) \\ &= A - (F \cap A) \\ &\neq \emptyset\end{aligned}$$

Thus p is a limit point for A not in A and therefore A is not closed.

Cocountable: $\emptyset$, X, and all countable sets.

Assume $p \in X$ and $U \in \mathscr{T}$ such that $p \in U$. Since $p \notin \emptyset$, $(U - \{p\}) \cap \emptyset = \emptyset$ and so p is not a limit point for $\emptyset$. Since X contains everything then it must contain its limit points.

Now, assume A is countable and $p \notin A$. Let $U = X - A \in \mathscr{T}$. Then:

$$\begin{aligned}(U - \{p\}) \cap A &= U \cap A \\ &= (X - A) \cap A \\ &= (X \cap A) - (A \cap A) \\ &= A - A \\ &= \emptyset\end{aligned}$$

Thus p is not a limit point for A and therefore A is closed.

Finally, assume A is uncountable and $p \notin A$. Assume $U \in \mathscr{T}$ and $p \in U$. But $U = X - C$ for some countable set C. Then:

$$\begin{aligned}(U - \{p\}) \cap A &= U \cap A \\ &= (X - C) \cap A \\ &= (X \cap A) - (C \cap A) \\ &= A - (C \cap A) \\ &\neq \emptyset\end{aligned}$$

Thus p is a limit point for A not in A and therefore A is not closed.

Lemma

Let $(X, \mathscr{T})$ be a topological space, $A \subset X$, and $p \in X$:

$$p \in \bar{A} \iff \forall U \in \mathcal{U}_p, U \cap A \neq \emptyset$$

Proof. By definition, $p \in \bar{A}$ iff $p \in A$ or $\forall U \in \mathcal{U}_p, (U - \{p\}) \cap A \neq \emptyset$. Assume that $U \in \mathcal{U}_p$. If $p \in A$ then $p \in U \cap A \neq \emptyset$. If $p \notin A$ then $(U - \{p\}) \cap A = U \cap A$. In either case: $p \in A$ or $\forall U \in \mathcal{U}_p, (U - \{p\}) \cap A \neq \emptyset$ is logically equivalent to $\forall U \in \mathcal{U}_p, U \cap A \neq \emptyset$. ∎

Theorem

Let $(X, \mathscr{T})$ be a topological space. For all $A \subset X$:

$$\bar{\bar{A}} = \bar{A}$$

Proof. $\bar{\bar{\emptyset}} = \bar{\emptyset} = \emptyset$ is vacuously true, so assume $A \neq \emptyset$.

$\bar{A} \subset \bar{\bar{A}}$ by definition, so assume $p \in \bar{\bar{A}}$. This means that for all $U \in \mathcal{U}_p, U \cap \bar{A} \neq \emptyset$. So assume that $U \in \mathcal{U}_p$ and $x \in U \cap \bar{A}$, meaning $x \in U$ and $x \in \bar{A}$. But this is only true if $U \cap A \neq \emptyset$ and so $p \in \bar{A}$.

Therefore $\bar{\bar{A}} = \bar{A}$. ∎

Theorem

Let $(X, \mathscr{T})$ be a topological space. For all $A \subset X$, A is closed iff $X - A$ is open.

Proof. X is closed iff $X - X = \emptyset$ is open is true, so assume that $A \neq X$.

$\Longrightarrow$ Assume A is closed.

Assume $p \in X - A$. Since $p \notin A$, p is not a limit point of A. Thus, there exists a neighborhood U_p of p such $U_p \cap A = \emptyset$. But this means that $U_p \subset X - A$.

Therefore $X - A$ is open.

$\Longleftarrow$ Assume $X - A$ is open.

Assume $p \in X - A$. So there exists a neighborhood U_p of p such that $U_p \subset X - A$. But this means that $U_p \cap A = \emptyset$ and hence p is not a limit point of A. Thus A contains all of its limit points.

Therefore A is closed.

∎

Theorem

Let $(X, \mathscr{T})$ be a topological space, $U \subset X$ open, and $A \subset X$ closed. $U - A$ is open and $A - U$ is closed.

Proof.

1. $U - A = U \cap (X - A)$. But U and $X - A$ are both open.

   Therefore $U - A$ is open.

2. $X - (A - U) = X - (A \cap (X - U)) = (X - A) \cap (X - (X - U)) = (X - A) \cap U$. But $X - A$ and U are both open and so $X - (A - U)$ is open.

   Therefore $A - U$ is closed.

■

Theorem

Let $(X, \mathscr{T})$ be a topological space:

1. $\emptyset$ is closed.
2. X is closed.
3. The union of finitely many closed sets is closed.
4. Let $\{A_\alpha : \alpha \in \lambda\}$ be a family of closed sets. $\bigcap_{\alpha\in\lambda} A_\alpha$ is closed.

Proof.

1. X is open, so $X - X = \emptyset$ is closed.
2. $\emptyset$ is open, so $X - \emptyset = X$ is closed.
3. $X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n}(X - A_i)$.

   But the $X - A_i$ are open and thus $X - \bigcup_{i=1}^{n} A_i$ is open.

   Therefore $\bigcup_{i=1}^{n} A_i$ is closed.
4. $X - \bigcap_{\alpha\in\lambda} A_\alpha = \bigcup_{\alpha\in\lambda}(X - A_\alpha)$.

   But the $X - A_\alpha$ are open and thus $X - \bigcap_{\alpha\in\lambda} A_\alpha$ is open.

   Therefore $\bigcup_{\alpha\in\lambda} A_\alpha$ is closed.

■

Example

Give an example to show that the union of infinitely many closed sets in a topological space may be a set that is not closed.

Consider the standard topology on R and the family of closed sets: $\{[-a, a] : a \in [0, 1)\}$. The union of these sets is $(-1, 1)$, which is open.

Example

Give examples of topological spaces and sets in them that are:

1. closed but not open.
2. open but not closed.
3. both open and closed.
4. neither open nor closed.

Let $X = \mathbb{R}$.

1. closed but not open.

   Standard: $[0, 1]$.

   Discrete: None

   Indiscrete: None

   Cofinite: $\{1, 2, 3\}$

   Cocountable: $\mathbb{Q}$

2. open but not closed.

   Standard: $(0, 1)$.

   Discrete: None

   Indiscrete: None

   Cofinite: $\mathbb{R} - \{1, 2, 3\}$

   Cocountable: $\mathbb{R} - \mathbb{Q}$

3. both open and closed.

   For all topologies, both $\emptyset$ and $\mathbb{R}$.

   Discrete: $(0, 1)$

4. neither open nor closed.

   Standard: $(0, 1]$

   Discrete: None

   Indiscrete: $(0, 1)$

   Cofinite: $(0, 1)$

   Cocountable: $(0, 1)$

Example

State whether each of the following sets are open, closed, both, or neither.

1. In $\mathbb{Z}$ with the cofinite topology:

   (a) $\{0, 1, 2\}$ (closed)

   (b) $\{n \in \mathbb{Z} \mid n$ is a prime number$\}$ (neither)

   (c) $\{n \in \mathbb{Z} \mid |n| \geq 10\}$ (open)

2. In $\mathbb{R}$ with the standard topology:

   (a) $(0, 1)$ (open)
   (b) $(0, 1]$ (neither)
   (c) $[0, 1])$ (closed)
   (d) $0, 1$ (closed)
   (e) $\left\{\frac{1}{n} \,\middle|\, n \in N\right\}$ (neither)

3. In $\mathbb{R}^2$ with the standard topology:
   (a) $\{(x, y) \,|\, x^2 + y^2 = 1\}$ (closed)
   (b) $\{(x, y) \,|\, x^2 + y^2 > 1\}$ (open)
   (c) $\{(x, y) \,|\, x^2 + y^2 \geq 1\}$ (closed)

Notation

Let $(X, \mathscr{T})$ be a topological space and $A \subset X$:

$$\mathcal{C} = \{B \subset X \,|\, B \text{ is closed}\}$$
$$\mathcal{C}_A = \{B \in \mathcal{C} \,|\, A \subset B\}$$

Theorem

Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. The closure of A equals the intersection of all closed sets containing A:

$$\bar{A} = \bigcap \mathcal{C}_A$$

Thus, $\bar{A}$ is the smallest closed set containing A.

Proof. Since $A \subset \bar{A}$ and $\bar{A}$ is closed, $\bar{A} \in \mathcal{C}_A$ and so:

$$\bar{A} \supset \bigcap \mathcal{C}_A$$

ABC:

$$\bar{A} \supsetneq \bigcap \mathcal{C}_A$$

This means that there exists some $B' \in \mathcal{C}_A$ such that:

$$\bar{A} \supsetneq \bar{A} \cap B' \supset A$$

where $\bar{A} \cap B' \in \mathcal{C}$.

This would imply that there exists some closed set containing A with less limit points of A than $\bar{A}$, which contradicts the definition of $\bar{A}$.

Therefore, $\bar{A} = \bigcap \mathcal{C}_A$. ∎

Example

Let $X = \mathbb{R}$ and let:

$$A = \{0\}$$
$$B = (0, 1)$$
$$C = [0, 1]$$
$$D = \{1, 2, 3\}$$

topology	$\bar{A}$	$\bar{B}$	$\bar{C}$	$\bar{D}$	$\bar{\mathbb{Z}}$	$\overline{\mathbb{R} - \mathbb{Q}}$
discrete	A	B	C	D	$\mathbb{Z}$	$\mathbb{R} - \mathbb{Q}$
indiscrete	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
cofinite	A	$\mathbb{R}$	$\mathbb{R}$	D	$\mathbb{R}$	$\mathbb{R}$
standard	A	C	C	D	$\mathbb{Z}$	$\mathbb{R}$

Theorem

Let $(X, \mathscr{T})$ be a topological space and $A, B \subset X$:

1. $A \subset B \implies \bar{A} \subset \bar{B}$
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$

Proof.

1. Assume $A \subset B$.

   Assume $p \in \bar{A}$. This means that:

   $$\forall U \in \mathcal{U}_p, U \cap A \neq \emptyset$$

   But $A \subset B$ and so

   $$\forall U \in \mathcal{U}_p, U \cap B \neq \emptyset$$

   meaning that $p \in \bar{B}$ as well.

   Therefore $\bar{A} \subset \bar{B}$.

2. $(\subset)$ Since $A \subset \bar{A}$ and $B \subset \bar{B}$:

   $$A \cup B \subset \bar{A} \cap \bar{B}$$

   But $\bar{A} \cap \bar{B}$ is closed and the smallest closed set containing $A \cup B$ is $\overline{A \cup B}$. Therefore:

   $$A \cup B \subset \overline{A \cup B} \subset \bar{A} \cup \bar{B}$$

($\supset$) Since $A \subset A \cup B$:

$$\bar{A} \subset \overline{A \cup B}$$

and similarly:

$$\bar{B} \subset \overline{A \cup B}$$

Therefore:

$$\bar{A} \cup \bar{B} \subset \overline{A \cup B}$$

■

Example

Let $(X, \mathscr{T})$ be a topological space and $\{A_\alpha : \alpha \in \lambda\}$ be a family of subsets of X. It is not necessarily the case that:

$$\overline{\bigcup_{\alpha \in \lambda} A_\alpha} = \bigcup_{\alpha \in \lambda} \overline{A_\alpha}$$

Consider the counterexample where $(\mathbb{R}, \mathscr{T}_{\text{std}})$ and $A = \{[-\alpha, \alpha] \,|\, \alpha \in (0, 1)\}$:

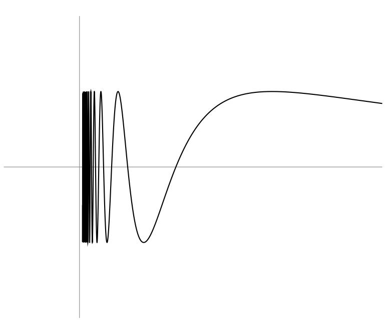
$$\overline{\bigcup_{\alpha \in \lambda} A_\alpha} = [-1, 1] \neq (-1, 1) = \bigcup_{\alpha \in \lambda} \overline{A_\alpha}$$

Example

Let $(R^2, \mathscr{T})$:

1. Topologist's Sine Curve

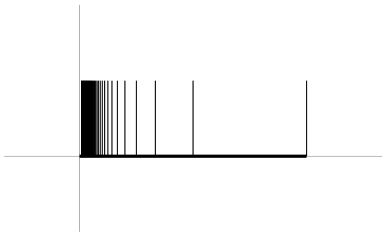
$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \middle| \, x \in (0, 1) \right\}$$



$$\bar{S} = S \cup \{(1, \sin(1))\} \cup \{(0, y) \,|\, y \in [-1, 1]\}$$

2. Topologists Comb

$$C = \{(x,0) \mid x \in [0,1]\} \cap \bigcup_{n=1}^{\infty} \left\{ \left(\frac{1}{n}, 1) \right) \middle| y \in [0,1] \right\}$$



$$\bar{C} = C \cup \{(0,y) \mid y \in [0,1]\}$$

Example

In $(\mathbb{R}, \mathscr{T}_{\text{std}})$, the Cantor set $\mathcal{C}$ is a non-empty subset of $[0,1]$ such that:

1. $\mathcal{C}$ is closed.
2. $\mathcal{C}$ contains no non-empty open intervals.
3. $\mathcal{C}$ contains no isolated points.