Expected value and variance

– Math 161a, Spring 2019, San Jose State University

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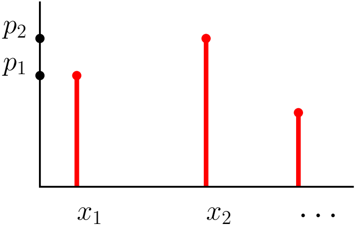
Section 3.3 expected value and variance

Introduction

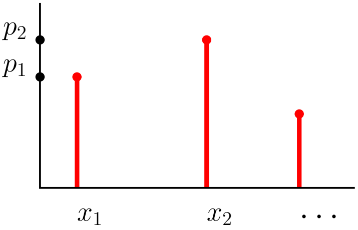
We have introduced the **pmf** of discrete random variable X which describes the probability distribution of X in terms of

- **Range**: the set of all possible values that X may take, and
- **Frequency**: individual probability $P(X = x_i)$ for each x_i in range.



We next present two ways of summarizing the distribution of X:

- **Expectation**: center of distribution (also mean value of X over many trials)
- **Variance**: spread of distribution



Definition of expected value

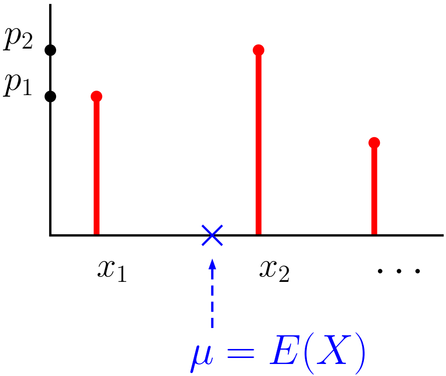
Def 0.1. Let X be a discrete random variable with pmf

x	x_1	x_2	$\cdots$
$P(X = x)$	p_1	p_2	$\cdots$

The expected value of X is defined as

$$\mu = \mathrm{E}(X) = \sum_i \underbrace{x_i}_{\text{value}} \cdot \underbrace{p_i}_{\text{prob}}$$

(If the sum is not finite, then we say the expected value does not exist).



Expected value and variance

Interpretation: $\mathrm{E}(X)$ represents the mean value of X over a large number (say N) of repetitions of the experiment:

$$\begin{array}{cccccccc} & x_1 & & x_1 & & & & x_1 \\ \underline{\hspace{3em}} & \underline{\hspace{3em}} & \underline{\hspace{3em}} & \underline{\hspace{3em}} & \underline{\hspace{3em}} & \underline{\hspace{3em}} & \underline{\hspace{3em}} & \underline{\hspace{3em}} \\ 1 & 2 & & & & & & N \end{array}$$

- Each value x_i in the range of X should occur about Np_i times
- The sum of all the x_i is about $x_i \cdot Np_i$
- The overall sum of all the N values of X is about $\sum_i x_i N p_i$

The mean value of X is thus (about) $\longleftarrow$ the larger N, the closer

$$\frac{1}{N} \sum_i x_i N p_i = \sum_i x_i p_i.$$

Ex 0.1 (Flip a coin with probability of getting heads equal to p)**.** Let $X = 1$ (heads) or 0 (tails). Find $\mathrm{E}(X)$.

Ex 0.2 (Toss a fair die)**.** Let X denote the number. Find $\mathrm{E}(X)$.

Ex 0.3. Let X be a random variable with pmf

$$f(x) = \frac{1}{x(1+x)}, \quad x = 1, 2, \ldots$$

Show that the expectation does not exist.

Remark.

- Expectation is only a summary of the distribution (in terms of its center)
- Expected value is not necessarily achievable by the random variable
- Expectation may be infinite (we say that it does not exist)

Some “on average” jokes

A statistician confidently tried to cross a river that was 1 meter deep on average.

He drowned.

A mathematician, a physicist and a statistician went hunting for deer.

When they chanced upon one buck lounging about, the mathematician fired first, missing the buck's nose by a few inches.

The physicist then tried his hand, and missed the tail by a wee bit.

The statistician started jumping up and down saying "We got him! We got him!"

"Every American should have above average income, and my Administration is going to see they get it." (Bill Clinton on campaign trail)

With one foot in a bucket of ice water, and one foot in a bucket of boiling water, you are, on the average, comfortable.

The great majority of people have more than the average number of legs. Amongst the 57 million people in Britain there are probably 5,000 people who have only one leg. Therefore the average number of legs is

$$(5,000 \times 1 + 56,995,000 \times 2)/57,000,000 = 1.9999123$$

Expected value of a function of X

Ex 0.4 (Toss a fair die). Let X denote the number. What is $\mathrm{E}(X^2)$? $\mathrm{E}(e^X)$?

Theorem 0.1. *Let X be a discrete random variable and Y another random variable that depends on X (i.e., $Y = g(X)$ for some function g). Then*

$$E(Y) = \sum_i \underbrace{g(x_i)}_{value} \cdot \underbrace{p_i}_{prob.}$$

$Y = g(X)$	$g(x_1)$	$g(x_2)$	$\cdots$
X	x_1	x_2	$\cdots$
$P(X = x)$	p_1	p_2	$\cdots$

Properties of expectation

Theorem 0.2. $E(\cdot)$ *is a linear operator, that is*

- *For any* $a, b \in \mathbb{R}$, *and a random variable* X,

$$E(a \cdot X + b) = a \cdot E(X) + b.$$

- *For any two random variables* X, Y,

$$E(X + Y) = E(X) + E(Y).$$

Ex 0.5. Find the mean of X which denotes the sum of two tosses of a fair die.

Ex 0.6 (Toss a coin repeatedly for n times)**.** Let X be the total count of heads (which occur with fixed probability p). Find $\mathrm{E}(X)$.

i-Clicker activity 4 (extra credit)

Which of the following statements about the expectation of a random variable X is wrong?

A. It characterizes the center of the distribution of X

B. It represents the mean value of X over many repetitions of the experiment

C. It may not be achievable by X

D. It may not exist

E. None of the above

Definition of variance and standard deviation

Def 0.2. The variance of a discrete X which has expected value $\mu = \mathrm{E}(X)$ is defined as

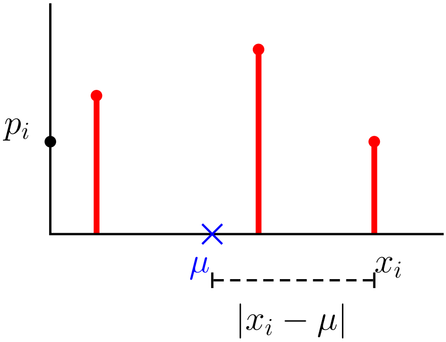
$$\sigma^2 = \mathrm{Var}(X) \stackrel{\text{def}}{=} \mathrm{E}[(X-\mu)^2] = \sum_i \underbrace{(x_i-\mu)^2}_{\text{squared deviation}} \cdot \underbrace{p_i}_{\text{prob}}$$

The square root of the variance is called the standard deviation of X:

$$\sigma = \mathrm{Std}(X) \stackrel{\text{def}}{=} \sqrt{\mathrm{Var}(X)}.$$

variance = ave. squared deviation

$$Var(X) = \Sigma(x_i-\mu)^2 p_i$$



A joke on variance, standard deviation, etc.

One day the variance and the standard deviation were engaged in a heated argument over which was the better measure of variability.

The standard deviation shouted at the variance, "**You are useless because you don't even relate to the original scale**."

The variance glared back and yelled, "**Oh yeah! You are totally worthless because you are far too radical**."

Just then the mean deviation stepped between the two and pushed them both back. In a proud voice the mean deviation proclaimed, **"You are both wrong! I am ABSOLUTELY the best measure of variability since both of you would be worth ZERO if you didn't square your deviations!!!"**

$$\mathrm{E}(|X-\mu|) = \sum_i |x_i - \mu| p_i$$

Theorem 0.3. *For any random variable X with $\mu = E(X)$,*

$$Var(X) = E(X^2) - \mu^2.$$

Proof. We prove this result in class. □

Remark. This result indicates that $\mathrm{Var}(X)$ can be calculated in three steps:

(1) $\mathrm{E}(X) = \sum x_i \cdot p_i$

(2) $\mathrm{E}(X^2) = \sum x_i^2 \cdot p_i$

(3) $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$

Ex 0.7 (Toss a coin which gives heads with fixed probability p)**.** Let X denote the numerical outcome: 1 (heads) or 0 (tails). Find $\mathrm{Var}(X)$.

Ex 0.8 (Toss 1 fair die)**.** Let X denote the result. Find $\mathrm{Var}(X)$.

Properties of variance

Theorem 0.4. *For any real numbers* a, b*, and random variable* X*,*

$$Var(aX + b) = a^2\, Var(X),$$

For independent random variables X, Y*,*

$$Var(X + Y) \overset{indep.}{=} Var(X) + Var(Y).$$

***Remark**. The second equation does not hold true for two dependent random variables, e.g.,* $Y = -X$*:*

$$Var(X+Y) = 0, \quad but\ \ Var(X)+Var(Y) = Var(X)+Var(-X) = 2\,Var(X) > 0.$$

Ex 0.9. Find the variance of X which denotes the sum of two independent tosses of a fair die.

Ex 0.10 (Toss a coin repeatedly and independently for n times)**.** Let X be the total count of heads (which occur with fixed probability p). Find $\mathrm{Var}(X)$.