Degree

Definition: Neighbor

Let G be a graph and let $u, v \in V(G)$. To say that u is a *neighbor* of v (and vice-versa) means that $uv \in E(G)$.

Thus, neighbor vertices are adjacent.

Definition: Neighborhood

Let G be a graph and let $u \in V(G)$. The *neighborhood* of u, denoted by $N(u)$, is the set of all neighbors of u in G:

$$N(u) = \{v \in V(G) \,|\, uv \in E(G)\}$$

Note that for simple graphs, a vertex is never a neighbor of itself.

Definition: Degree

Let G be a graph and let $u \in G$. The *degree* of u, denoted by $\deg_G(u)$ or $\deg(u)$, is the cardinality of the neighborhood of u:

$$\deg(u) = |N(u)|$$

The degree of a vertex can be viewed as the number of neighbor vertices or the number of incident edges.

Notation

Let G be a graph of order n and let $u \in V(G)$:

$$\delta(G) = \min_{v \in V(G)} \deg(v)$$
$$\Delta(G) = \max_{v \in V(G)} \deg(v)$$

and so:

$$0 \le \delta(G) \le \deg(u) \le \Delta(G) \le n - 1$$

Definition: Vertex Types

Let G be a graph of order n and let $u \in V(G)$:

$\deg(u)$	TYPE
0	isolated
1	pendant, end, leaf
$n-1$	universal
even	even
odd	odd

Theorem: First Theorem of Graph Theory

Let G be a graph of size m:

$$\sum_{v \in V(G)} \deg(v) = 2m$$

Proof. When summing all the degrees, each edge is counted twice: once for each endpoint. ■

Corollary

Let $G = B(U, W)$ be a bipartite graph of order m:

$$\sum_{u \in U} \deg(u) = \sum_{w \in W} \deg(w) = m$$

Proof. Each edge joins a vertex in U with a vertex in W, and so:

$$\sum_{u \in U} \deg(u) = \sum_{w \in W} \deg(w)$$

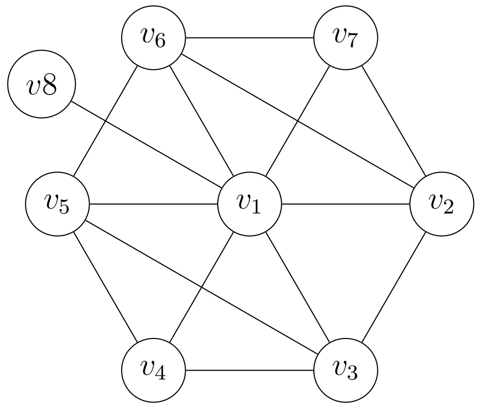
This means that:

$$\sum_{u \in U} \deg(u) + \sum_{w \in W} \deg(w) = 2\sum_{u \in U} \deg(u) = 2m$$

$$\therefore \sum_{u \in U} \deg(u) = \sum_{w \in W} \deg(w) = m$$

■

Example



vertex	degree	type
v_1	7	universal,odd
v_2	4	even
v_3	4	even
v_4	3	odd
v_5	4	even
v_6	4	even
v_7	3	odd
v_8	1	pendant,odd
total	30	

$n = 8$ $\quad m = 15 = \frac{30}{2}$
$\delta(G) = 1$ $\quad \Delta(G) = 7$
$\mathrm{diam}(G) = 2$

Theorem

Let G be a graph. G has an even number of odd vertices.

Proof. Partition $V(G)$ into two sets:

$$V_1 = \{v \in V(G) \mid \deg(v) \text{ is odd}\}$$
$$V_2 = \{v \in V(G) \mid \deg(v) \text{ is even}\}$$

and let:

$$n_o = \sum_{v \in V_1} \deg(v)$$
$$n_e = \sum_{v \in V_2} \deg(v)$$

By the FTGT:

$$n_o + n_e = 2m$$

which is even. But n_e is even and so n_o must also be even.

$\therefore |V_1|$ is even. ∎

Theorem

Let G be a graph of order n such that $\Delta(G) = n - 1$. The following are all true:

1. $n > 1 \implies \delta(G) > 0$
2. G is connected

3. $\operatorname{diam}(G) \leq 2$

Proof. Assume $u \in G$ such that $\deg(u) = n - 1$.

First, assume $n = 1$. This means that $G = E_1$, which is connected by definition with $\operatorname{diam}(G) = 0 \leq 2$.

Now, assume that $n > 1$.

This means that u is adjacent to all of the other vertices in G. and so there are no isolated vertices.

$\therefore \delta(G) \geq 1 > 0$

Next assume $n > 1$ and assume $v, w \in V(G)$ such that $v \neq w$.

Case 1: $u \in \{v, w\}$

AWLOG: $u = v$.

But $uw \in E(G)$ and so u and w are adjacent and thus connected with $d(u, w) = 1$.

Case 2: $u \notin \{v, w\}$

Case a: $vw \in E(G)$

So v is adjacent, and thus connected, to w with $d(v, w) = 1$.

Case b: $vw \notin E(G)$

Consider the path (v, u, w). This is a $v - w$ path in G of length 2.

$\therefore G$ is connected and $\operatorname{diam}(G) \leq 2$. ■

<u>Theorem</u>

Let G be a graph:

$\exists\, u, v \in V(G), \deg(u) = \deg(v)$

Proof.

Case 1: $\delta(G) = 0$

Thus, there is at least one isolated vertex and so $\Delta(G) \leq n - 2$. So for all $v \in V(G)$:

$$0 \leq \deg(v) \leq n - 2$$

Case 2: $\delta(G) \neq 0$

Thus, there are no isolated vertices and so $\Delta(G) \leq n - 1$. So for all $v \in V(G)$:

$$1 \leq \deg(v) \leq n - 1$$

In either case, there are n vertices and $n - 1$ possible degree values.

Therefore, by the PHP, at least two vertices must have the same degree. ■

Theorem

Let G be graph of order n such that $\forall\, u, v \in V(G), \deg(u) + \deg(v) \geq n - 1$:

G is connected.

Proof. Assume $u, v \in V(G)$.

Case 1: Assume $uv \in E(G)$.

So u and v are adjacent, and thus connected, with $d(u, v) = 1$.

Case 2: Assume $uv \notin E(G)$.

By PIE:

$$|N(u) \cup N(v)| = |N(u)| + |N(v)| - |N(u) \cap N(v)|$$

Now, since $uv \notin E(G)$, it must be the case that $N(u) \cup N(v) \subseteq V(G) - \{u, v\}$ and so:

$$|N(u) \cup N(v)| \leq n - 2$$

Furthermore, by assumption, $|N(u)| + |N(v)| \geq n - 1$. And so:

$$|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \geq (n - 1) - (n - 2) = 1$$

Thus, u and v are adjacent to at least one common vertex $w \in V(G)$. This means that there exists a (u, w, v) path in G of length 2.

$\therefore G$ is connected and $\operatorname{diam}(G) \leq 2$. ∎

Corollary

Let G be a graph of order n such that $\delta(G) \geq \frac{n-1}{2}$:

G is connected.

Proof. Assume $u, v \in V(G)$:

$$\deg(u) + \deg(v) \geq \frac{n-1}{2} + \frac{n-1}{2} = n - 1$$

$\therefore G$ is connected. ∎