Functions

Definition: Function

A *function* (*map*) from a set X to a set Y, denoted by $f : X \to Y$, is a rule that assigns to each $x \in X$ a corresponding $f(x) \in Y$. X is called the *domain* of f and Y is called the *codomain* of f.

Definition: Image

Let $f : X \to Y$ be a function and let $A \subset X$. The *image* of A under f is given by:

$$f(A) = \{f(a) \in Y \mid a \in A\}$$

Definition: Preimage

Let $f : X \to Y$ be a function and let $B \subset Y$. The *preimage* of B under f is given by:

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

When $B = \{y\}$ (a single point) then the alternate notation $f^{-1}(y)$ is often used.

Theorem

Let $f : X \to Y$ be a function and let $A, B \subset Y$:

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$
$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

Proof.

$$\begin{aligned}
x \in f^{-1}(A \cup B) &\iff f(x) \in A \cup B \\
&\iff f(x) \in A \text{ or } f(x) \in B \\
&\iff x \in f^{-1}(A) \text{ or } x \in f^{-1}(B) \\
&\iff x \in f^{-1}(A) \cup f^{-1}(B)
\end{aligned}$$

$$\therefore f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

$$\begin{aligned}
x \in f^{-1}(A \cap B) &\iff f(x) \in A \cap B \\
&\iff f(x) \in A \text{ and } f(x) \in B \\
&\iff x \in f^{-1}(A) \text{ and } x \in f^{-1}(B) \\
&\iff x \in f^{-1}(A) \cap f^{-1}(B)
\end{aligned}$$

$$\therefore f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

∎

Definition: Injection

Let $f : X \to Y$ be a function. To say that f is an *injection* (*one-to-one*) means that:

$$\forall\, a, b \in X, f(a) = f(b) \implies a = b$$

Definition: Surjection

Let $f : X \to Y$ be a function. To say that f is a *surjection* (*onto*) means that:

$$\forall\, b \in Y, \exists\, a \in X, f(a) = b$$

Definition: Bijection

Let $f : X \to Y$ be a function. To say that f is a *bijection* (*one-to-one correspondence*) means that f is both an injection and a surjection.

Theorem

Let $f : X \to Y$ be a function and let $y \in Y$. If f is injective then $f^{-1}(y)$ contains at most one point.

Proof. Assume $a, b \in f^{-1}(y)$. By definition: $f(a) = f(b) = y$. But f is injective and therefore $a = b$. ∎

Theorem

Let $f : X \to Y$ be a function and let $y \in Y$. If f is surjective then $f^{-1}(y)$ contains at least one point.

Proof. Since f is surjective, for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Therefore, by definition, $x \in f^{-1}(y)$. ∎