Sequences

Definition: Sequence

Let $(X, \mathscr{T})$ be a topological space. A *sequence* in X, denoted by $(x_i)_{i\in\mathbb{N}}$ or just (x_i), is a function $f : \mathbb{N} \to X$ where $f(i) = x_i$.

Definition: Limit

Let $(X, \mathscr{T})$ be a topological space, (x_i) a sequence in X, and $p \in X$. To say that p is the *limit* of (x_i), also stated as (x_i) *converges* to p and denote by $x_i \to p$, means:

$$\forall\, U \in \mathcal{U}_p, \exists N \in \mathbb{N}, i > N \implies x_i \in U$$

Theorem

Let $(X, \mathscr{T})$ be a topological space, $A \subset X$, and $p \in X$:

$$\{x_i \,|\, i \in \mathbb{N}\} \subset A \text{ and } x_i \to p \implies p \in \bar{A}$$

Proof. Assume that $\{x_i \,|\, i \in \mathbb{N}\} \subset A$ and $x_i \to p$. Assume that $U \in \mathcal{U}_p$. This means that there exists some $N \in \mathbb{N}$ such that for all $i > N$ it is the case that $x_i \in U$. But $x_i \in A$ also, and so $U \cap A \neq \emptyset$.

Therefore $p \in \bar{A}$. ∎

Theorem

Let $(\mathbb{R}^n, \mathscr{T})$ be the standard topology, $A \subset \mathbb{R}^n$, and $p \in X$ be a limit point of A. There exists a sequence of points in A that converge to p.

Proof. Let $U_i = B\left(p, \epsilon_i\right)$ where $\epsilon_i = \frac{1}{i}$ for $i \in \mathbb{N}$. Note that $\epsilon_i = \frac{1}{i} \to 0$ as $i \to \infty$. Also note that $U_i \cap A \neq \emptyset$ because p is a limit point of A, so select $x_i \in U_i \cap A$. Thus, all of the $x_i \in A$.

Claim: $(x_i)_{i\in\mathbb{N}}$ is a sequence in A converging to p.

Assume $U \in \mathcal{U}_p$. Then there exists some $\epsilon > 0$ such that $B(p, \epsilon) \subset U$. Since the $\epsilon_i \to 0$, there exists some $\epsilon_N < \epsilon$. Assume $i > N$. This means that $e_i < e_N < e$ and so $x_i \in U_i \subset U_N \subset U$ and therefore $x_i \in U$. ∎

Example

Find an example of a topological space and a convergent sequence in that space for which the limit of the sequence is not unique.

Consider $(\mathbb{R}, \mathscr{T})$ with the indiscrete topology and consider any random sequence of points x_i. Since $\mathbb{R}$ is the only non-empty open set, any $p \in \mathbb{R}$ is a suitable limit for (x_i).