Eigenvalues of Unitary Operators

Theorem

Let E be a normed space and let A be a unitary operator on E:

$$\lambda \text{ is an eigenvalue of } A \implies |\lambda| = 1$$

Proof

Assume λ is an eigenvalue of A. Assume $\vec{x}$ is an eigenvector of λ, and thus $\vec{x} \neq \vec{0}$. A unitary $\implies$ A preserves the norm: $\|A\vec{x}\| = \|\vec{x}\|$.

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ \|A\vec{x}\| &= \|\lambda\vec{x}\| \\ \|\vec{x}\| &= |\lambda|\,\|\vec{x}\| \\ \therefore |\lambda| &= 1 \end{aligned}$$

Lemma

Let E be a normed space and let A be a unitary operator on E. For all eigenvalues λ, μ of A:

$$\lambda \neq \mu \implies \lambda\overline{\mu} \neq 1$$

Proof

Assume λ and μ are eigenvalues of A such that $\lambda \neq \mu$.
Since A is unitary, $|\lambda| = |\mu| = 1$.
So $\exists\, \alpha, \beta \in \mathbb{R}$ such that $\lambda = e^{i\alpha}$ and $\mu = e^{i\beta}$.
ABC: $\lambda\overline{\mu} = 1$
$\frac{\lambda}{\mu} = 1$
$\lambda = \mu$
CONTRADICTION!

$\therefore \lambda\overline{\mu} \neq 1$

Theorem

Let H be a Hilbert space and let A be a unitary or self-adjoint operator on H. For all eigenvalues λ, μ of A:

$$\lambda \neq \mu \implies E_\lambda \perp E_\mu$$

Proof

Assume $\vec{x} \in E_\lambda$ and $\vec{y} \in E_\mu$.
Thus $\vec{x}, \vec{y} \neq 0$.

Case 1: $A = A^*$

$\lambda \langle \vec{x}, \vec{y} \rangle = \langle \lambda\vec{x}, \vec{y} \rangle = \langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A\vec{y} \rangle = \langle \vec{x}, \mu\vec{y} \rangle = \overline{\mu} \langle \vec{x}, \vec{y} \rangle$

$(\lambda - \mu) \langle \vec{x}, \vec{y} \rangle = 0$

But $\lambda \neq \mu$ and so $\langle \vec{x}, \vec{y} \rangle = 0$ and thus $\vec{x} \perp \vec{y}$.

$\therefore E_\lambda \perp E_\mu$

Case 2: $AA^* = A^*A = I$

$\lambda\overline{\mu} \langle \vec{x}, \vec{y} \rangle = \langle \lambda\vec{x}, \mu\vec{y} \rangle = \langle A\vec{x}, A\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$

$(\lambda\overline{\mu} - 1) \langle \vec{x}, \vec{y} \rangle = 0$

But $\lambda\overline{\mu} \neq 1$ and so $\langle \vec{x}, \vec{y} \rangle = 0$ and thus $\vec{x} \perp \vec{y}$.

$\therefore E_\lambda \perp E_\mu$