Inner Product Induced Norm

Theorem

Let E be a inner product space over a field $\mathbb{F}$. E is also a normed space with norm:

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$$

Proof

Assume $\vec{x}, \vec{y} \in E$ and $\lambda \in \mathbb{F}$:

1). Positivity follows from positivity of the inner product.

2). Homogeneity

$$\|\lambda\vec{x}\| = \sqrt{\langle \lambda\vec{x}, \lambda\vec{x} \rangle} = \sqrt{|\lambda|^2 \langle \vec{x}, \vec{x} \rangle} = |\lambda| \sqrt{\langle \vec{x}, \vec{x} \rangle} = |\lambda| \|\vec{x}\|$$

3). Sub-additivity

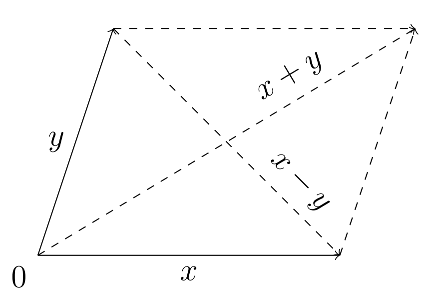
$$\begin{aligned}
\|\vec{x}+\vec{y}\|^2 &= \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle \\
&= \langle \vec{x}, \vec{x}, + \rangle \langle \vec{x}, \vec{y}, + \rangle \langle \vec{y}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle \\
&= \|\vec{x}\|^2 + \langle \vec{x}, \vec{y}, + \rangle \overline{\langle \vec{x}, \vec{y} \rangle} + \|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 + 2\,\mathrm{Re}[\langle \vec{x}, \vec{y} \rangle] + \|\vec{y}\|^2 \\
&\leq \|\vec{x}\|^2 + 2\,|\langle \vec{x}, \vec{y} \rangle| + \|\vec{y}\|^2 \\
&\leq \|\vec{x}\|^2 + 2\,\|\vec{x}\|\,\|\vec{y}\| + \|\vec{y}\|^2 \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2 \\
\therefore \|\vec{x}+\vec{y}\| &\leq \|\vec{x}\| + \|\vec{y}\|
\end{aligned}$$

Thus, every inner product space is also a normed space; however, the converse is not always true.

Theorem: Parallelogram Law

Let E be an inner product space. $\forall, \vec{x}, \vec{y} \in E$:

$$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2\left(\|\vec{x}\|^2 + \|\vec{y}\|^2\right)$$



Proof

Assume $\vec{x}, \vec{y} \in E$:

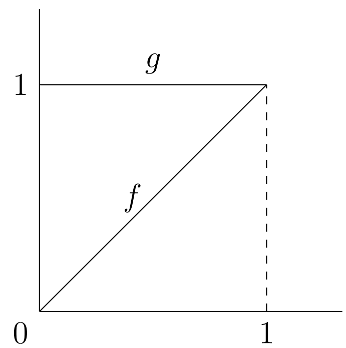
$$\begin{aligned}
\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 &= \langle \vec{x}+\vec{y}, \vec{x}+\vec{y}\rangle + \langle \vec{x}-vy, \vec{x}-\vec{y}\rangle \\
&= \langle \vec{x}, \vec{x}\rangle + \langle \vec{x}, \vec{y}\rangle + \langle \vec{y}, \vec{x}\rangle + \langle \vec{y}, \vec{y}\rangle + \langle \vec{x}, \vec{x}\rangle + \langle \vec{x}, -\vec{y}\rangle + \langle -\vec{y}, \vec{x}\rangle + \langle \vec{y}, \vec{y}\rangle \\
&= \|\vec{x}\|^2 + \langle \vec{x}, \vec{y}\rangle + \langle \vec{y}, \vec{x}\rangle + \|\vec{y}\|^2 + \|\vec{x}\|^2 - \langle \vec{x}, \vec{y}\rangle - \langle \vec{y}, \vec{x}\rangle + \|\vec{y}\|^2 \\
&= 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2 \\
&= 2\left(\|\vec{x}\|^2 + \|\vec{y}\|^2\right)
\end{aligned}$$

Every norm on an inner product space must satisfy this law.

Examples

1). $\mathcal{C}[0,1]$ and $\|f\|_\infty$

Let $f(t) = 1$ and $g(t) = t$



$\|f\| = \max\limits_{t\in[0,1]} |f| = \max\limits_{t\in[0,1]} |1| = 1$

$\|g\| = \max\limits_{t\in[0,1]} |g| = \max\limits_{t\in[0,1]} |t| = 1$

$\|f+g\| = \max\limits_{t\in[0,1]} |f+g| = \max\limits_{t\in[0,1]} |1+t| = 2$

$\|f-g\| = \max\limits_{t\in[0,1]} |f-g| = \max\limits_{t\in[0,1]} |1-t| = 1$

$\|f+g\|^2 + \|f-g\|^2 = 1^2 + 1^2 = 1 + 1 = 2$

$2(\|f\|^2 + \|g\|^2) = 2(1^2+1^2) = 2(1+1) = 4$

$2 \neq 4$, therefore not an inner product space.

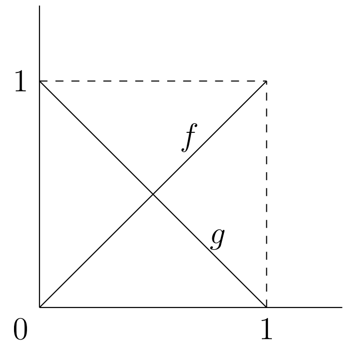
2). $L^p[a,b]$ for $1 \leq p < \infty$ and $\|f\| = \left(\int_a^b |f|\right)^{\frac{1}{p}}$

Any $f \in L^p[a,b]$ can be transformed to $[0,1]$ via:

$$g(t) = f(a + t(b-a))$$

So AWLOG: $L^p[0,1]$.

Let $f(t) = t$ and $g(t) = 1 - t$:



$$\|f\| = \left(\int_0^1 |f|^p\right)^{\frac{1}{p}} = \left(\int_0^1 t^p\right)^{\frac{1}{p}} = \left(\frac{t^{p+1}}{p+1}\bigg|_0^1\right)^{\frac{1}{p}} = \frac{1}{(p+1)^{\frac{1}{p}}}$$

$$\|g\| = \left(\int_0^1 |g|\right)^{\frac{1}{p}} = \left(\int_0^1 (1-t)^p\right)^{\frac{1}{p}} = \left(\int_1^0 (-t)^p\right)^{\frac{1}{p}} = \left(\int_0^1 t^p\right)^{\frac{1}{p}} = \frac{1}{(p+1)^{\frac{1}{p}}}$$

$$\|f+g\| = \left(\int_0^1 |f+g|^p\right)^{\frac{1}{p}} = \left(\int_0^1 1\right)^{\frac{1}{p}} = \left(t|_0^1\right)^{\frac{1}{p}} = 1$$

$$\|f-g\| = \left(\int_0^1 |f-g|^p\right)^{\frac{1}{p}} = \left(\int_0^1 |2t-1|^p\right)^{\frac{1}{p}} = \left(\int_0^{\frac{1}{2}} (1-2t)^p + \int_{\frac{1}{2}}^1 (2t-1)^p\right)^{\frac{1}{p}}$$

$$= \left(-\frac{(1-2t)^{p+1}}{2(p+1)}\bigg|_0^{\frac{1}{2}} + \frac{(2t-1)^{p+1}}{2(p+1)}\bigg|_{\frac{1}{2}}^1\right)^{\frac{1}{p}} = \frac{1}{2(p+1)} + \frac{1}{2(p+1)} = \frac{1}{(p+1)}$$

$$\|f+g\|^2 - \|f-g\|^2 = 1^2 + \left(\frac{1}{(p+1)^{\frac{1}{p}}}\right)^2 = 1 + \frac{1}{(p+1)^{\frac{2}{p}}}$$

$$2(\|f\|^2 + \|g\|^2) = 2\left[\left(\frac{1}{(p+1)^{\frac{1}{p}}}\right)^2 + \left(\frac{1}{(p+1)^{\frac{1}{p}}}\right)^2\right] = \frac{4}{(p+1)^{\frac{2}{p}}}$$

$$1 + \frac{1}{(p+1)^{\frac{2}{p}}} = \frac{4}{(p+1)^{\frac{2}{p}}}$$

$$(p+1)^{\frac{2}{p}} + 1 = 4$$

$$(p+1)^{\frac{2}{p}} = 3$$

This only has a solution at $p = 2$, and so only L^2 is an inner product space.

3). $L^\infty[a,b]$ and $\|f\|_\infty$

Once again, AWLOG: $L^\infty[0,1]$.

$$\|f\| = \max_{t\in[0,1]} |f| = \max_{t\in[0,1]} |t| = 1$$

$$\|g\| = \max_{t\in[0,1]} |g| = \max_{t\in[0,1]} |1-t| = 1$$

$$\|f+g\| = \max_{t\in[0,1]} |f+g| = \max_{t\in[0,1]} |t+(1-t)| = \max_{t\in[0,1]} 1 = 1$$

$$\|f-g\| = \max_{t\in[0,1]} |f-g| = \max_{t\in[0,1]} |t-(1-t)| = \max_{t\in[0,1]} 2t-1 = 1$$

$$\|f+g\|^2 + \|f-g\|^2 = 1^2 + 1^2 = 1+1 = 2$$

$$2(\|f\|^2 + \|g\|^2) = 2(1^2+1^2) = 2(1+1) = 4$$

$2 \neq 4$, therefore L^∞ is not an inner product space.