Gram-Schmidt

Theorem: Gram-Schmidt

Let E be an inner product space over a field $\mathbb{F}$ and let $\{\vec{x}_1, \ldots \vec{x}_n\}$ be an independent set in E:

$$\vec{y}_1 = \vec{x}_1$$

$$\vec{y}_n = \vec{x}_n - \sum_{k=1}^{n-1} \frac{\langle \vec{x}_n, \vec{y}_k \rangle}{\|\vec{y}_k\|} \vec{y}_k$$

form an orthogonal system in E.

Furthermore:

$$\mathrm{Span}\{\vec{x}_1, \ldots, \vec{x}_n\} = \mathrm{Span}\{\vec{y}_1, \ldots, \vec{y}_n\}$$

Thus, an equivalent orthonormal system can be generated by:

$$\left\{ \frac{\vec{y}_k}{\|\vec{y}_k\|^2} \middle| 1 \leq k \leq n \right\}$$

Proof

By induction on n.

Base case: $n = 2$

$$\begin{aligned} \langle \vec{y}_2, \vec{y}_1 \rangle &= \left\langle \vec{x}_2 - \frac{\langle \vec{x}_2, \vec{y}_1 \rangle}{\|\vec{y}_1\|^2} \vec{y}_1, \vec{y}_1 \right\rangle \\ &= \left\langle \vec{x}_2 - \frac{\langle \vec{x}_2, \vec{x}_1 \rangle}{\|\vec{x}_1\|^2} \vec{x}_1, \vec{x}_1 \right\rangle \\ &= \langle \vec{x}_2, \vec{x}_1 \rangle - \frac{\langle \vec{x}_2, \vec{x}_1 \rangle}{\|\vec{x}_1\|^2} \langle \vec{x}_1, \vec{x}_1 \rangle \\ &= \langle \vec{x}_2, \vec{x}_1 \rangle - \frac{\langle \vec{x}_2, \vec{x}_1 \rangle}{\|\vec{x}_1\|^2} \|\vec{x}_1\|^2 \\ &= \langle \vec{x}_2, \vec{x}_1 \rangle - \langle \vec{x}_2, \vec{x}_1 \rangle \\ &= 0 \end{aligned}$$

$$\therefore \vec{y}_2 \perp \vec{y}_1$$

Assume $\{\vec{y}_1, \ldots, \vec{y}_n$ is an orthogonal set.

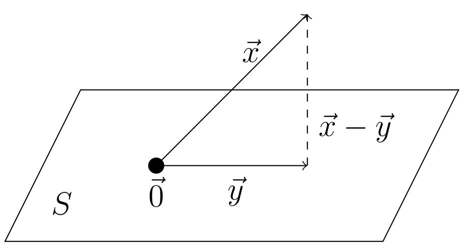
Assume $1 \leq m < n$.

$$\begin{aligned}
\langle \vec{y}_n, \vec{y}_m \rangle &= \left\langle \vec{x}_n - \sum_{k=1}^{n-1} \frac{\langle \vec{x}_n, \vec{y}_k \rangle}{\|\vec{y}_k\|^2} \vec{y}_k, \vec{y}_m \right\rangle \\
&= \langle \vec{x}_n, \vec{y}_m \rangle - \left\langle \sum_{k=1}^{n-1} \frac{\langle \vec{x}_n, \vec{y}_k \rangle}{\|\vec{y}_k\|^2} \vec{y}_k, \vec{y}_m \right\rangle \\
&= \langle \vec{x}_n, \vec{y}_m \rangle - \frac{\langle \vec{x}_n, \vec{y}_m \rangle}{\|\vec{y}_m\|^2} \langle \vec{y}_m, \vec{y}_m \rangle \\
&= \langle \vec{x}_n, \vec{y}_m \rangle - \frac{\langle \vec{x}_n, \vec{y}_m \rangle}{\|\vec{y}_m\|^2} \|\vec{y}_m\|^2 \\
&= \langle \vec{x}_n, \vec{y}_m \rangle - \langle \vec{x}_n, \vec{y}_m \rangle \\
&= 0
\end{aligned}$$

$\therefore \vec{y}_n \perp \vec{y}_m$ and thus $\{\vec{y}_1, \ldots, \vec{y}_n\}$ is an orthogonal system in E.

Why does this work?

Let E be an inner product space and let S be a finite dimensional subspace of E. Assume $\vec{x} \in E$.



$$d(\vec{x}, S) = \inf_{\vec{y} \in S} \|\vec{x} - \vec{y}\|$$

Definition

Let E be a finite dimensional inner product space where $\dim E = n$ and let S be a subspace of E with orthonormal basis $\{\vec{b}_1, \ldots, \vec{b}_r\}$ where $r \leq n$. $\forall\, \vec{x} \in E$, the *projection* of $\vec{x}$ on S, denoted $\operatorname{proj}_S \vec{x}$, is given by:

$$\operatorname{proj}_S \vec{x} = \sum_{k=1}^{r} \left\langle \vec{x}, \vec{b}_k \right\rangle \vec{b}_k$$

Lemma

Let E be a finite dimensional inner product space over a field $\mathbb{F}$ and let S be a subspace of E where $\dim S = n$. Let $\vec{x} \in E$ and $\vec{y}_0 = \operatorname{proj}_S \vec{x}$:

$$\vec{x} - \vec{y}_0 \perp S$$

Proof

Assume $\{\vec{b}_1, \ldots, \vec{b}_n\}$ is an orthonormal basis for S.
Assume $\vec{y} \in S$.

$\exists\, \lambda_k \in \mathbb{F}$ such that $\vec{y} = \displaystyle\sum_{k=1}^{n} \lambda_k \vec{b}_k$

$$\begin{aligned}
\langle \vec{x} - \vec{y}_0, \vec{y} \rangle &= \langle \vec{x} - \operatorname{proj}_S \vec{x}, \vec{y} \rangle \\
&= \left\langle \vec{x} - \sum_{k=1}^{n} \left\langle \vec{x}, \vec{b}_k \right\rangle \vec{b}_k, \sum_{j=1}^{n} \lambda_j \vec{b}_j \right\rangle \\
&= \left\langle \vec{x}, \sum_{k=1}^{n} \lambda_k \vec{b}_k \right\rangle - \left\langle \sum_{k=1}^{n} \left\langle \vec{x}, \vec{b}_k \right\rangle \vec{b}_k, \sum_{j=1}^{n} \lambda_j \vec{b}_j \right\rangle \\
&= \left\langle \vec{x}, \sum_{k=1}^{n} \lambda_k \vec{b}_k \right\rangle - \sum_{k=1}^{n} \left\langle \left\langle \vec{x}, \vec{b}_k \right\rangle \vec{b}_k, \lambda_k \vec{b}_k \right\rangle \\
&= \left\langle \vec{x}, \sum_{k=1}^{n} \lambda_k \vec{b}_k \right\rangle - \sum_{k=1}^{n} \left\langle \vec{x}, \vec{b}_k \right\rangle \overline{\lambda_k} \left\langle \vec{b}_k, \vec{b}_k \right\rangle \\
&= \left\langle \vec{x}, \sum_{k=1}^{n} \lambda_k \vec{b}_k \right\rangle - \sum_{k=1}^{n} \left\langle \vec{x}, \vec{b}_k \right\rangle \overline{\lambda_k} \\
&= \left\langle \vec{x}, \sum_{k=1}^{n} \lambda_k \vec{b}_k \right\rangle - \sum_{k=1}^{n} \left\langle \vec{x}, \lambda_k \vec{b}_k \right\rangle \\
&= \left\langle \vec{x}, \sum_{k=1}^{n} \lambda_k \vec{b}_k \right\rangle - \left\langle \vec{x}, \sum_{k=1}^{n} \lambda_k \vec{b}_k \right\rangle \\
&= 0
\end{aligned}$$

$\therefore \vec{x} - \vec{y}_0 \perp S$

Theorem

Let E be a finite dimensional inner product space, let S be a subspace of E, and let $\vec{x} \in E$ and $\vec{y}_0 = \operatorname{proj}_S \vec{x}$:

$$d(\vec{x}, S) = \|\vec{x} - \vec{y}_0\|$$

Proof

Note that $\vec{y}_0 \in S$.
Assume $\vec{y} \in S$.
$\|\vec{x} - \vec{y}\|^2 = \|(\vec{x} - \vec{y}_0) + (\vec{y}_0 - \vec{y})\|^2 = \|\vec{x} - \vec{y}_0\|^2 + \|\vec{y}_0 - \vec{y}\|^2 \geq \|\vec{x} - \vec{y}_0\|^2$
So, $\forall\, \vec{y} \in S, \|\vec{x} - \vec{y}_0\| \leq \|\vec{x} - \vec{y}\|$.

$\therefore d(\vec{x}, S) = \vec{x} - \vec{y}_0$

Corollary

Let E be a finite dimensional inner product space, let S be a subspace of E, and let $\vec{x} \in E$ and $\vec{y}_0 \in S$ such that $d(\vec{x}, S) = \|\vec{x} - \vec{y}_0\|$:

$\vec{y}_0$ is unique.

Proof

Assume $\exists \vec{y}_0{}' \in S$ such that $d(\vec{x}, S) = \|\vec{x} - \vec{y}_0{}'\|$.
$|\|\vec{x} - \vec{y}_0\| - \|\vec{x} - \vec{y}_0{}'\|| \leq \|(\vec{x} - \vec{y}_0) - (\vec{x} - \vec{y}_0{}')\| = \|\vec{y}_0{}' - \vec{y}_0'\| = 0$

Therefore $\vec{y}_0{}' - \vec{y}_0' = \vec{0}$ and so $\vec{y}_0 = \vec{y}_0'$.

Thus, we can rewrite Gram-Schmidt as:

$$\vec{y}_1 = \vec{x}_1$$

$$\vec{y}_n = \vec{x}_n - \sum_{k=1}^{n-1} \operatorname{proj}_{\vec{y}_k} \vec{x}_k = \vec{x}_n - \operatorname{proj}_{\mathrm{Span}\{\vec{y}_1, \ldots, \vec{y}_{n-1}\}} \vec{x}$$