Center

Definition

Let G be a group:

1). Let $a \in G$. G_a is the subset of G consisting of all elements in G that commute with a:

$$G_a = \{g \in G \mid ga = ag\}$$

2). Let $S \subseteq G, S \neq \emptyset$. G_S is the subset of G consisting of all elements in G that commute with all elements in S:

$$G_S = \{g \in G \mid \forall\, s \in S, gs = sg\}$$

Theorem

Let G be a group and $a \in G$:

$$G_a \leq G$$

Proof

Assume $x, y \in G_a$
$G_a \subseteq G$, so $x, y \in G$
G is a group, so $y^{-1} \in G$
Assume $g \in G$
$xgy = gxy = gyx = ygx$
$y^{-1}xgy = y^{-1}ygx$
$xy^{-1}gy = gx$
$xy^{-1}gyy^{-1} = gxy^{-1}$
$xy^{-1}g = gxy^{-1}$
$xy^{-1} \in G_a$
$\therefore$ by the subgroup test, $G_a \leq G$.

Theorem

Let G be a group and $S \subseteq G, S \neq \emptyset$:

$$G_S \leq G$$

Proof

Assume $x, y \in G_S$
$G_S \subseteq G$, so $x, y \in G$
G is a group, so $y^{-1} \in G$
Assume $s \in S$
$xsy = sxy = syx = ysx$
$y^{-1}xsy = y^{-1}ysx$

$xy^{-1}sy = sx$
$xy^{-1}syy^{-1} = sxy^{-1}$
$xy^{-1}s = sxy^{-1}$
$xy^{-1} \in G_S$
$\therefore$ by the subgroup test, $G_S \leq G$.

Definition

The *center* of a group G, denoted $Z(G)$, is the subgroup of G whose elements commute with all elements in G:

$$Z(G) = G_G$$

Theorem

Let G be a group. $Z(G)$ is abelian.

Proof

Assume $a, b \in Z(G)$
$Z(G) \subseteq G$, so $a, b \in G$
$ab = ba$
$\therefore$ $Z(G)$ is abelian.