Normal Subgroups

<u>Definition</u>

Let $H \leq G$. To say that H is a *normal* subgroup of G, denoted $H \triangleleft G$, means:

$$\forall\, g \in G, h \in H, ghg^{-1} \in H$$

<u>Theorem</u>

Let $\phi : G \to G'$ be a homomorphism of groups and $K = \ker(\phi)$:

$$K \triangleleft G$$

<u>Proof</u>

Assume $g \in G$ and $k \in K$
$\phi(gkg^{-1}) = \phi(gg^{-1}) = \phi(e) = e'$
$\therefore gkg^{-1} \in K$

<u>Theorem</u>

Let $H \leq G$. TFAE:

1). $H \triangleleft G$

2). $\forall\, g \in G, gHg^{-1} = H$

3). $\forall\, g \in G, gH = Hg$

<u>Proof</u>

Assume $g \in G$

$1 \implies 2$: Assume $H \triangleleft G$

Assume $a \in gHg^{-1}$
$\exists\, h \in H, a = ghg^{-1} \in H$
$\therefore a \in H$

Assume $a \in H$
$\exists\, h \in H, h = gag^{-1}$
$a = g^{-1}hg = g^{-1}h(g^{-1})^{-1}$
$\therefore a \in gHg^{-1}$

$2 \implies 3$: Assume $gHg^{-1} = H$

Assume $g' \in gH$
$\exists\, h' \in H, g' = gh'$
$gh'g^{-1} \in H$
$\exists\, h \in H, gh'g^{-1} = h$
$gh' = hg$
$g' = hg$
$\therefore g' \in Hg$

Assume $g' \in Hg$
$\exists\, h' \in H, g' = h'g$
$g^{-1}h'g \in H$
$\exists\, h \in H, g^{-1}h'g = h$
$h'g = gh$
$g' = gh$
$\therefore g' \in gH$

$3 \implies 1$: Assume $gH = Hg$

$$\begin{aligned}&\forall\, h \in H, \exists\, h' \in H, gh = h'g\\&ghg^{-1} = h'\\&ghg^{-1} \in H\\&\therefore H \lhd G\end{aligned}$$

Corollary

Let $H \leq G$:

$$G \text{ abelian} \implies H \lhd G$$

Proof

Assume $g \in G$
Assume $h \in H$
$ghg^{-1} = gg^{-1}h = eh = h$
$ghg^{-1} \in H$
$\therefore H \lhd G$

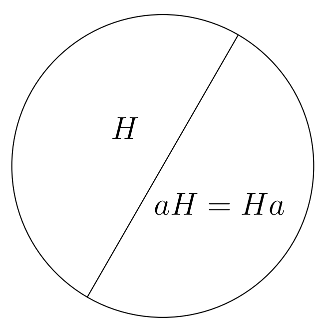
Theorem

Let $H \leq G$:

$$(G : H) = 2 \implies H \lhd G$$

Proof

Assume $(G : H) = 2$
Assume $a \in G, a \notin H$
H and aH are the two distinct left cosets
H and Ha are the two distinct right cosets
$aH = Ha$
$\therefore H \lhd G$



Example

$(S_4 : A_4) = 2$

A_4 elements are even
$(12)A_4$ elements are odd
$(12)A_4$=$A_4(12)$
$A_4 \lhd S_4$