Exponential Expressions

<u>Definition</u>

An exponential expression is an expression of the form a^b, where a, the *base* and b, the exponent, are real expressions.

How an exponential expression is interpreted depends on what type of real number b is.

$b \in \mathbb{N}$

$$a^1 = a$$
$$a^2 = aa$$
$$a^3 = aaa$$
$$a^n = aa \cdots a \text{ } (n \text{ times})$$

$b \in \mathbb{Z}$

$b = 0$

By definition:

$$a \neq 0 \implies a^0 = 1$$

$b < 0$

$$a^{-1} = \frac{1}{a}$$
$$a^{-2} = \frac{1}{a} \cdot \frac{1}{a}$$
$$a^{-3} = \frac{1}{a} \cdot \frac{1}{a} \cdot \frac{1}{a}$$
$$a^{-n} = \frac{1}{a} \cdot \frac{1}{a} \cdots \frac{1}{a} \text{ } (n \text{ times})$$

$b \in \mathbb{Q}$

Definition

Let $n \in \mathbb{N}, n > 1$ and $a \in \mathbb{R}$:

$\sqrt[n]{a}$ is the number that when raised to the n^{th} power equals a

$$b = \sqrt[n]{a} \iff a^n = b$$

When $n = 2$ the n is omitted.

The root sign is called a *radical* and the expression inside the root is called the *radicand*.

Example

$\sqrt{1} = 1$ because $1^2 = 1$
$\sqrt{4} = 2$ because $2^2 = 4$
$\sqrt{9} = 3$ because $3^2 = 9$
$\sqrt{16} = 4$ because $4^2 = 16$

$\sqrt[3]{1} = 1$ because $1^3 = 1$
$\sqrt[3]{8} = 2$ because $2^3 = 8$
$\sqrt[3]{27} = 3$ because $3^3 = 27$
$\sqrt[3]{64} = 4$ because $4^3 = 64$

When n is odd, there aren't any limitations because you can always take the odd root of any real number:

$\sqrt[3]{-1} = -1$ because $(-1)^3 = 1$
$\sqrt[3]{-8} = -2$ because $(-2)^3 = 8$
$\sqrt[3]{-27} = -3$ because $(-3)^3 = 27$
$\sqrt[3]{-64} = -4$ because $(-4)^3 = 64$

Note the placement of the parentheses: $(-a)^n \neq -a^n$ because the exponent has higher precedence that the minus sign:

$(-2)^2 = 4$ but $-2^2 = -4$

When n is even, the radicand must be ≥ 0. For example, you cannot take the square root of a negative number because a number squared is always nonnegative:

$$\sqrt{-1} \notin \mathbb{R}$$

because $x^2 = -1$ has no solutions in $\mathbb{R}$.

When n is even, $\sqrt[n]{a}$ refers to the *principle* root, which is the nonnegative root. $-\sqrt[n]{a}$ refers to the negative root.

Example

$$\sqrt{4} = 2$$

$$-\sqrt{4} = -2$$

Definition

Let $p, q \in \mathbb{Z}$ such that $q > 1$ and let $a \in \mathbb{R}$:

$$\sqrt[q]{a} = a^{\frac{1}{q}}$$

$$\sqrt[q]{a^p} = a^{\frac{p}{q}}$$

Example

$$\sqrt{100} = 100^{\frac{1}{2}} = 10$$
$$\sqrt[3]{100} = \sqrt[3]{10^2} = 10^{\frac{2}{3}}$$
$$\sqrt[3]{\frac{1}{100}} = \sqrt[3]{\frac{1}{10} \cdot \frac{1}{10}} = \sqrt[3]{10^{-2}} = 10^{-\frac{2}{3}}$$

$b \in \mathbb{R}$

What in the heck does something like 2^π mean?

Using a calculator:

$$2^\pi = 8.82498$$

Build a table of approximations using rational exponents that get arbitrarity close to the exact value of 2^π:

π	2^π
3	8.00000
3.1	8.57419
3.14	8.81524
3.141	8.82135
3.1415	8.82441
3.14159	8.82496

Properties

Properties

$\forall\, a, b, c \in \mathbb{R}$ (not 0 when appropriate):

1). $a \neq 0 \implies a^0 = 1$

2). $a^{-1} = \frac{1}{a}$

3). $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$

4). $a^{-b} = (a^{-1})^b = \left(\frac{1}{a}\right)^b = \frac{1}{a^b}$

5). $a^{-b} = \left(a^b\right)^{-1} = \frac{1}{a^b}$

6). $a^b a^c = a^{b+c}$

7). $\frac{a^b}{a^c} = a^b \cdot \frac{1}{a^c} = a^b a^{-c} = a^{b-c}$

8). $(ab)^c = a^c b^c$

9). $\left(\frac{a}{b}\right)^c = \left(a \cdot \frac{1}{b}\right)^c = a^c \left(\frac{1}{b}\right)^c = a^c \left(\frac{1}{b^c}\right) = \frac{a^c}{b^c}$

10). $\left(\frac{a}{b}\right)^{-c} = \left[\left(\frac{a}{b}\right)^{-1}\right]^c = \left(\frac{b}{a}\right)^c = \frac{b^c}{a^c}$

11). $\left(a^b\right)^c = a^{bc}$

We need to be careful when rational exponents cancel:

Case 1: $(a^{\frac{1}{n}})^n$, n odd

No problems, sinced we can always take an odd root of any real number:

$$(a^{\frac{1}{n}})^n = a$$

Case 2: $(a^{\frac{1}{n}})^n$, n even

We can assume that $a \geq 0$, since we cannot take an even root of a negative number:

$$(a^{\frac{1}{n}})^n = a$$

Case 3: $(a^n)^{\frac{1}{n}}$, n odd

No problems with odd roots:

$$(a^n)^{\frac{1}{n}} = a$$

Case 4: $(a^n)^{\frac{1}{n}}$, n even

Always take the principle root:

$$(a^n)^{\frac{1}{n}} = |a|$$