Convergence in a Vector Norm

Definition: Convergence

Let $\|\cdot\|$ be a vector norm on $\mathbb{C}^n$. To say that a sequence of vectors $\{\vec{x}_k\}$ in $\mathbb{C}^n$ *converges* with respect to the norm means $\exists\, \vec{x}_0 \in \mathbb{C}^n$ such that:

$$\lim_{k\to\infty} \|\vec{x}_k - \vec{x}_0\| = 0$$

Definition: Cauchy

Let $\|\cdot\|$ be a vector norm on $\mathbb{C}^n$. To say that a sequence of vectors $\{\vec{x}_k\}$ in $\mathbb{C}^n$ is *Cauchy* with respect to the norm means:

$$\forall\, \epsilon > 0, \exists\, N_\epsilon > 0, \forall\, i, j > N_\epsilon, \|\vec{x}_i - \vec{x}_j\| < \epsilon$$

Definition: Complete

Let $\|\cdot\|$ be a vector norm on $\mathbb{C}^n$. To say that $\mathbb{C}^n$ is *complete* with respect to the norm means that every Cauchy sequence in $\mathbb{C}^n$ converges to to some $\vec{x}_0 \in \mathbb{C}^n$.

Theorem

$\mathbb{C}^n$ is complete with respect to ℓ_∞

Proof

Assume $\{\vec{x}_k\}$ in $\mathbb{C}^n$ is Cauchy with respect to ℓ_∞
Let $\vec{x}_k(i)$ refer to the i^{th} component of $\vec{x}_k$
Assume $1 \leq i \leq n$
Assume $\epsilon_1 > 0$
$\exists\, N_{\epsilon_1} > 0, \forall\, j, k > N_{\epsilon_1}, \|\vec{x}_j - \vec{x}_k\| < \epsilon_1$
Assume $j, k > N_{\epsilon_1}$

$$|\vec{x}_j(i) - \vec{x}_k(i)| \leq \max_{1\leq i\leq n} |\vec{x}_j(i) - \vec{x}_k(i)| = \|\vec{x}_j - \vec{x}_k\|_\infty < \epsilon_1$$

So $\{\vec{x}_k(i)\}$ is Cauchy in $\mathbb{C}$
But $\mathbb{C}$ is complete, so $\{\vec{x}_k(i)\} \to \vec{x}_0(i)$ as $k \to \infty$
Assume $\epsilon > 0$
$\exists\, N_\epsilon(i) > 0, \forall\, k > N_\epsilon(i), |\vec{x}_k(i) - \vec{x}_0(i)| < \epsilon$
Let:

$$N_\epsilon = \max_{1\leq i\leq n} N_\epsilon(i)$$

Assume $k > N_\epsilon$

$$\|\vec{x}_k - \vec{x}_0\| = \max_{1\leq i\leq n} |\vec{x}_k(i) - \vec{x}_0(i)| < \epsilon$$

Therefore:

$$\lim_{k\to\infty} \|\vec{x}_k - \vec{x}_0\| = 0$$

and thus $\{\vec{x}_k\}$ converges with respect to the norm to some $\vec{x}_0 \in \mathbb{C}^n$.

Theorem

Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two norms on $\mathbb{C}^n$. There exists $c_m, c_M \in \mathbb{R}$ such that $\forall\, \vec{x} \in \mathbb{C}^n$:

$$c_m \|\vec{x}\|_\alpha \leq \|\vec{x}\|_\beta \leq c_M \|\vec{x}\|_\alpha$$

Proof

Consider $S = \{\vec{x} \in \mathbb{C}^n \mid \|\vec{x}\|_2 = 1\}$
S is compact
Let $h(\vec{x}) = \frac{\|\vec{x}\|_\beta}{\|\vec{x}\|_\alpha}$
$h(\vec{x})$ is continuous
$h[S]$ is compact in $\mathbb{R}$
Let $h[S] = [c_m, c_M]$
Assume $\vec{x} \in S$
$c_m \leq \frac{\|\vec{x}\|_\beta}{\|\vec{x}\|_\alpha} \leq c_M$
$c_m \|\vec{x}\|_\alpha \leq \|\vec{x}\|_\beta \leq c_M \|\vec{x}\|_\alpha$
Now, assume $\vec{x} \in \mathbb{C}^n$
$\frac{\vec{x}}{\|\vec{x}\|_2} \in S$
$c_m \left\|\frac{\vec{x}}{\|\vec{x}\|_2}\right\|_\alpha \leq \left\|\frac{\vec{x}}{\|\vec{x}\|_2}\right\|_\beta \leq c_M \left\|\frac{\vec{x}}{\|\vec{x}\|_2}\right\|_\alpha$

$\therefore c_m \|\vec{x}\|_\alpha \leq \|\vec{x}\|_\beta \leq c_M \|\vec{x}\|_\alpha$

Example

$c_m \|\vec{x}\|_2 \leq \|\vec{x}\|_1 \leq c_M \|\vec{x}\|_2$

Clearly, $c_m = 1$

$\sum_{k=1}^n |x_k| \leq c_M (\sum_{k=1}^n |x_k|^2)^{\frac{1}{2}}$

But by C-S:

$\sum_{k=1}^n 1 \cdot |x_k| \leq (\sum_{k=1}^n 1^2)^{\frac{1}{2}} (\sum_{k=1}^n |x_k|^2)^{\frac{1}{2}}$

So $c_M = (\sum_{k=1}^n 1^2)^{\frac{1}{2}} = \sqrt{n}$

Theorem

Let $\|\cdot\|$ be a vector norm on $\mathbb{C}^n$ and let $\{\vec{x}_k\}$ be a sequence in $\mathbb{C}^n$:

The sequence converges iff the sequence is Cauchy.

Proof

The statement holds for ℓ_∞
Assume $\vec{x} \in \mathbb{C}^n$
There exists some c_M such that $0 \leq \|\vec{x}\| \leq c_M \|\vec{x}\|_\infty$
So, by the squeeze theorem, the statement also holds for $\|\cdot\|$.

Consequences:

1). $\{\vec{x}_k\}$ Cauchy wrt $\|\cdot\|_\alpha \implies \{\vec{x}_k\}$ Cauchy wrt $\|\cdot\|_\beta$.

2). $\{\vec{x}_k\}$ converges wrt $\|\cdot\|_\alpha \implies \{\vec{x}_k\}$ converges wrt $\|\cdot\|_\beta$.

3). All $\|\cdot\|$ on $\mathbb{C}^n$ are complete.