$\mathbf{L^2}$ Inner Product

Definition

To say that $\langle \cdot\,, \cdot \rangle$ is an inner product on a vector space V with field F means that $\langle \cdot\,, \cdot \rangle$ is:

I1: Linear in the first argument

$\forall u, v, w \in V$ and $\forall\, \alpha \in F$:

$$\begin{array}{rcl} \langle u+v, w\rangle & = & \langle u, w\rangle + \langle v, w\rangle \\ \langle \alpha u, v\rangle & = & \alpha\langle u, v\rangle \end{array}$$

I2: Conjugate-symmetric (hermitian)

$\forall u, v \in V,\ \overline{\langle u, v\rangle} = \langle v, u\rangle$

I3: Positive-definite

$\forall v \in V,\ \langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0 \iff v = 0$

Theorem

$\langle \cdot\,, \cdot \rangle$ is a proper inner product for L^2

Proof

I1: Assume $f, g, h \in L^2$ and $\alpha \in \mathbb{C}$

$\langle f+g, h\rangle = \int (f+g)\bar{h} = \int f\bar{h} + \int g\bar{h} = \langle f, h\rangle + \langle g, h\rangle$

$\langle \alpha f, g\rangle = \int (\alpha f)\bar{g} = \alpha \int f\bar{g} = \alpha\langle f, g\rangle$

I2: Assume $f, g \in L^2$

$\overline{\langle f, g\rangle} = \overline{\int f\bar{g}} = \int \bar{f} g = \langle g, f\rangle$

I3: Assume $f \in L^2$

$\langle f, f\rangle = \int f\bar{f} = \int |f|^2 = \|f\|^2 \geq 0$

$\langle f, f\rangle = 0 \iff \|f\|^2 = 0 \iff f = 0\ \ a.e.$