Derivative Laws

Lemma

$f(z)$ continuous at $z_0 \implies \lim_{\Delta z \to 0} f(z_0 + \Delta z) = f(z_0)$

Proof

Assume $f(z)$ is continuous at z_0:

$$\lim_{\Delta z \to 0} f(z_0 + \Delta z) = \lim_{z \to z_0} f(z_0 + (z - z_0)) = \lim_{z \to z_0} f(z) = f(z_0)$$

Theorem

Assume $f(z)$ and $g(z)$ be differentiable (and thus continuous):

1). $\frac{\mathrm{d}}{\mathrm{dz}}[c] = 0$

2). $\frac{\mathrm{d}}{\mathrm{dz}}[z] = 1$

3). $\frac{\mathrm{d}}{\mathrm{dz}}[f(z) + g(z)] = f'(z) + g'(z)$

4). $\frac{\mathrm{d}}{\mathrm{dz}}[f(z)g(z)] = f'(z)g(z) + f(z)f'(z)$

5). $g(z) \neq 0 \implies \frac{\mathrm{d}}{\mathrm{dz}}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$

6). $\frac{\mathrm{d}}{\mathrm{dz}}[cf(z)] = cf'(z)$

7). $\forall\, n \in \mathbb{Z}, (n > 0$ or $z \neq 0) \implies \frac{\mathrm{d}}{\mathrm{dz}}[z^n] = nz^{n-1}$

8). $\frac{\mathrm{d}}{\mathrm{dz}}[g(f(z))] = g'[f(z)]f'(z)$

Proof

1).

$$\lim_{\Delta z \to 0} \frac{c - c}{\Delta z} = \lim_{\Delta z \to 0} \frac{0}{\Delta z} = \lim_{\Delta z \to 0} 0 = 0$$

2).

$$\lim_{\Delta z \to 0} \frac{(z + \Delta z) - z}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = \lim_{\Delta z \to 0} 1 = 1$$

3).

$$\lim_{\Delta z \to 0} \frac{[f(z + \Delta z) + g(z + \Delta z)] - [f(z) + g(z)]}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{[f(z + \Delta z) - f(z)] + [g(z + \Delta z) - g(z)]}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z) - f(z)}{\Delta z} + \frac{g(z+\Delta z) - g(z)}{\Delta z} \right]$$
$$= f'(z) + g'(z)$$

4).

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z)g(z+\Delta z) - f(z)g(z)}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{f(z+\Delta z)g(z+\Delta z) - f(z)g(z+\Delta z) + f(z)g(z+\Delta z) - f(z)g(z)}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{[f(z+\Delta z) - f(z)]g(z+\Delta z) + f(z)[g(z+\Delta z) - g(z)]}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z) - f(z)}{\Delta z} g(z+\Delta z) + f(z) \frac{g(z+\Delta z) - g(z)}{\Delta z} \right]$$
$$= f'(z)g(z) + f(z)g'(z)$$

5).

$$\lim_{\Delta z \to 0} \frac{\frac{f(z+\Delta z)}{g(z+\Delta z)} - \frac{f(z)}{g(z)}}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{f(z+\Delta z)g(z) - f(z)g(z+\Delta z)}{g(z+\Delta z)g(z)\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{f(z+\Delta z)g(z) - f(z)g(z) + f(z)g(z) - f(z)g(z+\Delta z)}{g(z+\Delta z)g(z)\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{[f(z+\Delta z) - f(z)]g(z) - f(z)[g(z+\Delta z) - g(z)]}{g(z+\Delta z)g(z)\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{\frac{f(z+\Delta z) - f(z)}{\Delta z} g(z) - f(z) \frac{g(z+\Delta z) - g(z)}{\Delta z}}{g(z+\Delta z)g(z)}$$
$$= \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$$

6).

$$\frac{\mathrm{d}}{\mathrm{dz}} [cf(z)] = \frac{\mathrm{d}}{\mathrm{dz}} [c] \, f(z) + c \frac{\mathrm{d}}{\mathrm{dz}} [f(z)] = 0 + cf'(z) = cf'(z)$$

7). Assume $n \in \mathbb{Z}$

Assume $n > 0$ or $z \neq 0$

Case 1: $n > 0$

Proof by induction on n:

Base: $z = 1$

$\frac{\mathrm{d}}{\mathrm{dz}}\left[z^1\right] = \frac{\mathrm{d}}{\mathrm{dz}}\left[z\right] = 1$
$1z^{1-1} = z^0 = 1$

Assume $\frac{\mathrm{d}}{\mathrm{dz}}\left[z^n\right] = nz^{n-1}$

Consider $\frac{\mathrm{d}}{\mathrm{dz}}\left[z^{n+1}\right]$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dz}}\left[z^{n+1}\right] &= \frac{\mathrm{d}}{\mathrm{dz}}\left[z^n z\right] \\
&= \frac{\mathrm{d}}{\mathrm{dz}}\left[z^n\right] z + z^n \frac{\mathrm{d}}{\mathrm{dz}}\left[z\right] \\
&= nz^{n-1}z + z^n(1) \\
&= nz^n + z^n \\
&= (n+1)z^n \\
&= (n+1)z^{(n+1)-1}
\end{aligned}$$

Case 2: $n = 0$

$\frac{\mathrm{d}}{\mathrm{dz}}\left[z^0\right] = \frac{\mathrm{d}}{\mathrm{dz}}\left[1\right] = 0$
Since $n \not> 0$, $z \neq 0$
$0z^{0-1} = 0z^{-1} = \frac{0}{z} = 0$

Case 3: $n < 0$

Let $m = -n > 0$
$\frac{\mathrm{d}}{\mathrm{dz}}\left[z^n\right] = \frac{\mathrm{d}}{\mathrm{dz}}\left[z^{-m}\right] = \frac{\mathrm{d}}{\mathrm{dz}}\left[\frac{1}{z^m}\right] = \frac{z^m(0)-(1)mz^{m-1}}{z^{2m}} = -mz^{-m-1} = nz^{n-1}$

8). Assume $f(z)$ is differentiable at z_0

Let $w_0 = f(z_0)$
Assume $g(z)$ is differentiable at w_0
Define the following function at w_0 and in some neighborhood $|w - w_0| < \epsilon$:

$$\phi(w) = \begin{cases} 0, & w = w_0 \\ \frac{g(w)-g(w_0)}{w-w_0} - g'(w_0), & |w - w_0| < \epsilon \end{cases}$$

$\lim_{w \to w_0} \phi(w) = 0 = \phi(w_0)$
Thus, ϕ is continuous at w_0
$g(w) - g(w_0) = [g'(w_o) + \phi(w)](w - w_0)$
f is continuous at z_0
$\exists\, \delta > 0, 0 < |z - z_0| < \delta \implies |w - w_0| < \epsilon$

Assume $0 < |z - z_0| < \delta$

$$g[f(z)] - g[f(z_0)] = [g'(f(z_0)) + \phi(f(z))](f(z) - f(z_0))$$
$$\frac{g[f(z)] - g[f(z_0)]}{z - z_0} = [g'(f(z_0)) + \phi(f(z))]\frac{f(z) - f(z_0)}{z - z_0}$$

Since ϕ is continuous at w_0 and f is continuous at z_0, $\phi[f(z)]$ is continuous at z_0 and:

$$\lim_{z \to z_0} \phi[f(z)] = \phi[f(z_0)] = \phi(w_0) = 0$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dz}}\left[g(f(z))\right] &= \lim_{z \to z_0} \frac{g[f(z)] - g[f(z_0)]}{z - z_0} \\
&= \lim_{z \to z_0} [g'(f(z_0)) + \phi(f(z))]\frac{f(z) - f(z_0)}{z - z_0} \\
&= [g'(f(z_0)) + 0]f'(z_0) \\
&= g'[f(z_0)]f'(z_0)
\end{aligned}$$