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Math 221b
Exam

1). Let R be a commutative ring with 1. Suppose that x is in the intersection of all maximal ideals of R. Show that $x + 1$ is a unit in R.

Let $M = \bigcap M_i$ where M_i is maximal in R.
In particular, each M_i is proper and contains no units in R.
By assumption, $1 \in R$ and $1 \cdot 1 = 1$, so 1 is a unit in R and $\forall\, i, 1 \notin M_1$.
Assume $x \in M$
ABC: $x + 1$ is a non-unit in R
But every non-unit in R must be contained in some maximal ideal in R.
So $x + 1 \in M_i$ for some i.
But, by assumption, $x \in M_i$, and since M_i is an additive group, $-x \in M_i$.
By closure, $(-x) + (x + 1) \in M_i$
$(-x) + (x + 1) = (-x + x) + 1 = 0 + 1 = 1 \in M_i$
CONTRADICTION!

Therefore $x + 1$ is a unit in R.

2). Let d be a squarefree integer different from 1. Show that if $\pi \in R_d$ has norm $N(\pi) = p$ for some prime $p \in \mathbb{Z}$ then π is irreducible in R_d

Assume $\pi \in R_d$ has norm $N(\pi) = p$ for some prime $p \in \mathbb{Z}$.
Assume $\pi = ab$ for some $a, b \in R_d$.
So, by the multiplicity of the norm:

$$N(\pi) = N(ab) = N(a)N(b) = p$$

Now, by the integer criterion, since $a, b \in R_d$ it must be the case that $N(a), N(b) \in \mathbb{Z}$.
But p is prime, hence the only divisors of p are p and 1.
So $N(a) = 1$ or $N(b) = 1$, and thus by the unit criterion, either a or b is a unit in R_d.

Therefore, π is irreducible in R_d.

3). Explain why x^4 is irreducible over $\mathbb{Q}$. Find the splitting field K of $x^4 - 2$ over $\mathbb{Q}$ and show that $[K : \mathbb{Q}] = 8$. Find three distinct quadratic subfields of K/Q.

By the rational root test, the only rational roots of $x^4 - 2$ would be from the set $\{\pm 1, \pm 2\}$; however, clearly none of these values are roots. Therefore, $x^4 - 2$ is irreducible over $\mathbb{Q}$.

To find the splitting field, find all of the complex roots of $x^4 - 2$:

$$\begin{aligned} x^4 - 2 &= 0 \\ x^4 &= 2 \\ x^4 &= 2e^{i(2\pi n)} \\ x &= \sqrt[4]{2}e^{i\frac{\pi}{2}n} \\ x &= \sqrt[4]{2}, i\sqrt[4]{2}, -\sqrt[4]{2}, -i\sqrt[4]{2} \end{aligned}$$

Therefore, $K = \mathbb{Q}(\sqrt[4]{2}, i)$

Now, consider the following extension field stack:

$$\begin{tikzcd} \mathbb{Q}(\sqrt[4]{2}, i) \arrow[d, dash] \\ \mathbb{Q}(\sqrt[4]{2}) \arrow[d, dash] \\ \mathbb{Q} \end{tikzcd}$$

Since $\sqrt[4]{2}$ is a root of $x^4 - 2$, which is irreducible in $\mathbb{Q}$, $m_{\sqrt[4]{2}, \mathbb{Q}(\sqrt[4]{2})}(x) = x^4 - 2$ and thus $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 4$:

$$\begin{tikzcd} \mathbb{Q}(\sqrt[4]{2}, i) \arrow[d, dash] \\ \mathbb{Q}(\sqrt[4]{2}) \arrow[d, dash, "4"] \\ \mathbb{Q} \end{tikzcd}$$

Clearly, $i \notin \mathbb{Q}(\sqrt[4]{2})$, and so $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}(\sqrt[4]{2})] \neq 1$. But note that i is a root of $x^2 + 1 \in \mathbb{Q}(\sqrt[4]{2})$, and since $\mathbb{Q}(\sqrt[4]{2}, i)$ is a UFD, the only factorization of $x^2 + 1$ in $\mathbb{Q}(\sqrt[4]{2}, i)$ is $(x + i)(x - i) \notin \mathbb{Q}(\sqrt[4]{2})$. Thus, $x^2 + 1$ is irreducible in $\mathbb{Q}(\sqrt[4]{2})$ and $m_{i, \mathbb{Q}(\sqrt[4]{2})}(x) = x^2 + 1$ and so $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}(\sqrt[4]{2})] = 2$:

$$\begin{tikzcd}
\mathbb{Q}(\sqrt[4]{2}, i) \arrow[d, dash, "2"] \\
\mathbb{Q}(\sqrt[4]{2}) \arrow[d, dash, "4"] \\
\mathbb{Q}
\end{tikzcd}$$

Therefore, $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}(\sqrt[4]{2})][\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 2 \cdot 4 = 8$

Three quadratic subfields over $\mathbb{Q}$ can be constructed as follows:

subfield	min poly
$Q(i)$	$x^2 + 1$
$Q(\sqrt[4]{4})$	$x^2 - 2$
$Q(i\sqrt[4]{4})$	$x^2 + 2$

Note that all of the stated minimum polynomials are irreducible in $\mathbb{Q}$ by the rational root test.