Commute Conditions

Definition: Simultaneously Triangularizable

Let $A, B \in M_n$. To say that A and B are simultaneously triangularizable means there exists unitary matrix U such that:

$$U^*AU = T_A \text{ and } U^*BU = T_B$$

where $T_A, T_B \in UT(n)$.

Theorem

Let $A, B \in M_n$:

$AB = BA \implies A$ and B are simultaneously triangularizable.

Definition

Let $S, T \subseteq \mathbb{C}$:

$$S - T = \{s - t \mid s \in S \text{ and } t \in T\}$$

Theorem

Let $A, B \in M_n$:

$$AB = BA \implies \sigma(A - B) \subseteq \sigma(A) - \sigma(B)$$

Example

Thus, $\sigma(A - B) \nsubseteq \sigma(A) - \sigma(B) \implies AB \neq BA$

$A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\sigma(A) = \{1, 2\}$ and $\sigma(B) = 0, 1$

$\sigma(A) - \sigma(B) = \{1 - 0, 1 - 1, 2 - 0, 2 - 1\} = \{0, 1, 1, 2\} = \{0, 1, 2\}$

$A - B = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$

$p_{A-B}(t) = \det \begin{bmatrix} t - 2 & 1 \\ -2 & t \end{bmatrix} = t(t - 2) + 2 = t^2 - 2t + 2$

$\sigma(A - B) = \{1 \pm 2i\} \nsubseteq \{0, 1, 2\} = \sigma(A) - \sigma(B)$

So A and B do not commute:

$\begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -2 & 0 \end{bmatrix}$

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}3 & 1\\-2 & 0\end{bmatrix} = \begin{bmatrix}3 & 1\\0 & 0\end{bmatrix}$$

$AB \neq BA$

Proof

Assume $AB = BA$
A and B are simultaneously triangularizable
Let U be the necessary unitary matrix
$U^*(A-B)U = U^*AU - U^*BU = T_A - T_B$
So $\sigma(A-B)$ is from the diagonal entries of $T_A - T_B$, which is a subset of all possible differences, represented by $\sigma(A) - \sigma(B)$

$\therefore \sigma(A-B) \subseteq \sigma(A) - \sigma(B)$

Theorem

Let $A, B \in M_n$:

$AB = BA \implies A$ and B have a common eigenvector.

Proof

Let $\lambda \in \sigma(A)$ with eigenvector $\vec{x}$
$A\vec{x} = \lambda\vec{x}$
Consider $B\vec{x}$

Case 1: $B\vec{x} = 0$

$\vec{x} \in \mathrm{Eig}_B(0)$

Therefore $\vec{x}$ is a common eigenvector.

Case 2: $B\vec{x} \neq 0$

$A(B\vec{x}) = BA\vec{x} = B(\lambda\vec{x}) = \lambda(B\vec{x})$
So $B\vec{x} \in \mathrm{Eig}_A(\lambda)$
Thus $\{B\vec{x} \mid \vec{x} \in \mathrm{Eig}_A(\lambda)\} \subseteq \mathrm{Eig}_A(\lambda)$
Let $\{\vec{x}_1, \dots, \vec{x}_r\}$ be a basis for $\mathrm{Eig}_A(\lambda)$
Extend the set to a basis for $\mathbb{C}^n : \{\vec{x}_1, \dots, \vec{x}_r, \vec{x}_{r+1}, \dots, \vec{x}_n\}$
AWLOG that this is an orthonormal basis (otherwise use Gram-Schmidt)
Let $U = \begin{bmatrix}\vec{x}_1 & \cdots & \vec{x}_r & \vec{x}_{r+1} & \dots & \vec{x}_n\end{bmatrix}$, thus a unitary matrix

$$BU = \begin{bmatrix}B\vec{x}_1 & \cdots & B\vec{x}_r & B\vec{x}_{r+1} & \cdots & B\vec{x}_n\end{bmatrix}$$

But note that:

$$B\vec{x}_k \in \begin{cases}\mathrm{Eig}_A(\lambda) & 1 \leq k \leq r\\ \mathbb{C}^n & r+1 \leq k \leq n\end{cases}$$

and so:

$$\begin{aligned}
BU &= \begin{bmatrix}\vec{x}_1 & \cdots & \vec{x}_r & \vec{x}_{r+1} & \dots & \vec{x}_n\end{bmatrix} \begin{bmatrix} k_{1,1} & k_{1,2} & \cdots & k_{1,r} & k_{1,r+1} & \cdots & k_{1,n} \\ k_{2,1} & k_{2,2} & \cdots & k_{2,r} & k_{2,r+1} & \cdots & k_{2,n} \\ & & \vdots & & & \vdots & \\ k_{r,1} & k_{r,2} & \cdots & k_{r,r} & k_{r,r+1} & \cdots & k_{r,n} \\ 0 & 0 & \cdots & 0 & k_{r+1,r+1} & \cdots & k_{r+1,n} \\ & & \vdots & & & \vdots & \\ 0 & 0 & \cdots & 0 & k_{n,r+1} & \cdots & k_{n,n} \end{bmatrix} \\
&= U \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix}
\end{aligned}$$

where $C_{11}, C_{12} \in M_r$ and $C_{22} \in M_{n-(r+1)}$

Select an eigenvector $\vec{z}$ of C_{11} with respect to eigenvalue μ

$C_{11}\vec{z} = \mu\vec{z}$

Consider $U \begin{bmatrix} \vec{z} \\ 0 \end{bmatrix}$

$U \begin{bmatrix} \vec{z} \\ 0 \end{bmatrix}$ is a linear combination of $\{\vec{x}_1, \dots, \vec{x}_r\}$ and is thus in $\mathrm{Eig}_A(\lambda)$

But also:

$$\begin{aligned}
B\left(U \begin{bmatrix} \vec{z} \\ 0 \end{bmatrix}\right) &= U \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} \vec{z} \\ 0 \end{bmatrix} \\
&= U \begin{bmatrix} C_{11}\vec{z} \\ 0 \end{bmatrix} \\
&= U \begin{bmatrix} \mu\vec{z} \\ 0 \end{bmatrix} \\
&= \mu\left(U \begin{bmatrix} \vec{z} \\ 0 \end{bmatrix}\right)
\end{aligned}$$

Therefore $U \begin{bmatrix} \vec{z} \\ 0 \end{bmatrix} \in \mathrm{Eig}_B(\mu)$