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Math 275A
Homework #14

Theorem: 8.18

Let X be a topological space. Each component of X is connected, closed, and not contained in any strictly larger connected subset of X.

Proof. Assume that U is a component of X and that $p \in U$. By definition, $U = \bigcup_{\alpha \in \lambda} U_\alpha$ where U_α is a connected subset of X containing p. Now, for each U_α, $U_\alpha \cap \{p\} \neq \emptyset$ and $\{p\}$ is trivially connected. Therefore U is connected.

Assume that V is a connected component of X such that $U \subset V$. This means that $p \in V$ and so, by definition, $U = U \cup V$. But this is ony true if $V \subset U$. Therefore $U = V$.

Now, since U is connected, $\bar{U}$ is connected. But $p \in \bar{U}$ and so, by definition, $U = U \cup \bar{U}$. But this is only true if $\bar{U} \subset U$. Therefore $U = \bar{U}$ and hence U is closed. ∎

Theorem: 8.35

A path connected topological space is connected.

Proof. Assume that X is a path connected topological space and ABC that X is disconnected. This means that there exists $A, B \subset X$ such that $A \sqcup B = X$ where A, B are open and non-empty. So assume that $x \in A$ and $y \in B$. Since X is path connected, there exists some continuous $f : [0,1] \to X$ such that $f(0) = x \in A$ and $f(1) = y \in B$. This mean that $[0,1] = f^{-1}(A) \cup f^{-1}(B)$ where neither $f^{-1}(A)$ nor $f^{-1}(B)$ are empty. Furthermore, since A and B are disjoint, $f^{-1}(A)$ and $f^{-1}(B)$ must also be disjoint, contradicting the connectedness of $[0,1]$. Therefore X is connected. ∎

Example: Exercise 8.37

The closure of the topologist's sine curve is connected but not path connected.

The topologists sine curve is given by:

$$S = \left\{ \left(x, \sin \frac{1}{x} \right) \,\middle|\, x \in (0,1) \right\}$$

and its closure is given by:

$$\bar{S} = S \cup \{(1, \sin(1)\} \cup \{(0,y) \,|\, y \in [-1,1]\}$$

Note that $\bar{S}$ was already shown to be connected.

ABC that $\bar{S}$ is path connected and assume that $p \in S$. This means that there exists a path in $\bar{S}$ such that $f(0) = p$ and $f(1) = (0,0)$. Let $f(t) = (x(t), y(t))$. Note that since f is continuous,

$x(t)$ and $y(t)$ are also continuous. Now, defined $U = \{t \in [0,1] \,|\, x(t) > 0\}$. Thus, for all $t \in U$, $f(t) \in S$ and $y(t) = \frac{1}{x(t)}$.

Next, since $U \subset [0,1]$, U is bounded and thus has a sup. So let $t_* = \sup U$. Note that t_* is the final value of t at which the path jumps to the y-axis part of $\bar{S}$ and stays there on the way to $(0,0)$. So $x(t_*) = 0$. Let $b = y(t_*)$ and let select $\epsilon > 0$ such that:

$$\epsilon < \begin{cases} 1-b, & b < 1 \\ \frac{1}{2}, & b = 1 \end{cases}$$

Now, since f is continuous, there exists $\delta > 0$ such that for all $t \in [0,1]$, if $|t - t_*| < \delta$ then $\|f(t) - f(t_*)\| < \epsilon$. Note that $[t_* - \delta, t_*]$ is connected and compact. Furthermore, f is continuous. Hence $f[t_* - \delta, t_*]$ is connected and compact, and thus must be an interval. So let $x([t_* - \delta, t_*) = [0, x_0]$ for some $x_0 \in (0,1]$. This means that for every $x \in (0, x_0]$ there exists some $t \in [t_* - \delta, t_*]$ such that $f(t) \in S$, meaning $f(t) = (x, \sin\frac{1}{x})$.

Define a sequence x_n in [0,1] by:

$$x_n = \frac{1}{2n\pi + \frac{\pi}{2}}$$

Note that $x_n \to 0$ and:

$$\sin\frac{1}{x_n} = \sin\left(2n\pi + \frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$

But since $x_n \to 0$, there exists $N \in \mathbb{N}$ such that for all $x_n < x_0$ for all $n > N$. And so there exists $t_n \in [t_* - \delta, t_*)$ such that:

$$f(t_n) = \left(x_n, \sin\frac{1}{x_n}\right) = (x_n, 1)$$

Thus:

$$\|f(t_n) - f(t_*)\| = \|(x_n, 1) - (0, b)\| \geq 1 - b > \epsilon$$

This contradicts the continuity of f. Therefore $\bar{S}$ is not path connected.

Theorem: 8.38

Let X and Y be topological spaces. If X and Y are path connected then $X \times Y$ is path connected.

Proof. Assume that X and Y are path connected and assume that $(x_1, y_1), (x_2, y_2) \in X \times Y$. This means that there must exist a path f from x_1 to x_2 and a path g from y_1 to y_2. Now, defined $h : [0,1] \to X \times Y$ as $h(t) = (f(t), g(t))$. But $\pi_X \circ h = f$ and $\pi_Y \circ h = g$ are by definition continuous, and thus h is continuous. Furthermore, $h(0) = (f(0), g(0)) = (x_1, y_1)$ and $h(1) = (f(1), g(1)) = (x_2, y_2)$, and so h is a path between (x_1, y_1) and (x_2, y_2). Therefore $X \times Y$ is path connected. ∎

Examples: Exercise 9.1

Show that the following are all metrics on $\mathbb{R}^n$:

1. The *Euclidean metric* defined by:

$$d(x,y) = \|x-y\| = \sqrt{\sum_{k=1}^{n}(x_k - y_k)^2}$$

Positive Definition:

$$\begin{aligned}
&(x_k - y_k)^2 \geq 0 \\
&\sum (x_k - y_k)^2 \geq 0 \\
&\sqrt{\sum (x_k - y_k)^2} \geq 0 \\
&d(x,y) \geq 0
\end{aligned}$$

$$\begin{aligned}
d(x,y) = 0 &\iff \sqrt{\sum (x_k - y_k)^2} = 0 \\
&\iff \sum (x_k - y_k)^2 = 0 \\
&\iff (x_k - y_k)^2 = 0 \\
&\iff x_k - y_k = 0 \\
&\iff x_k = y_k \\
&\iff x = y
\end{aligned}$$

Symmetric:

$$d(x,y) = \sqrt{\sum (x_k - y_k)^2} = \sqrt{\sum (y_k - x_k)^2} = d(y,x)$$

Triangle Inequality:

$$\begin{aligned}
[d(x,y)]^2 &= \sum (x_k - y_k)^2 \\
&= \sum |(x_k - z_k) + (z_k - y_k)|^2 \\
&\leq \sum (|x_k - z_k| + |z_k - y_k|)^2 \\
&= \sum (|x_k - z_k|^2 + |z_k - y_k|^2 + 2|x_k - z_k||z_k - y_k|) \\
&= \sum |x_k - z_k|^2 + \sum |z_k - y_k|^2 + 2\sum |x_k - z_k||z_k - y_k|
\end{aligned}$$

Now, by the Cauchy-Schwarz inequality:

$$\begin{aligned}
\sum |x_k - z_k||z_k - y_k| &\leq \sqrt{\left(\sum (x_k - z_k)^2\right)\left(\sum (z_k - y_k)^2\right)} \\
&= \sqrt{[d(x,z)]^2 [d(z,y)]^2} \\
&= d(x,z)d(z,y)
\end{aligned}$$

and so:

$$[d(x,y)]^2 \leq [d(x,z)]^2 + [d(z,y)]^2 + 2d(x,z)d(z,y) = [d(x,z)+d(z,y)]^2$$

Therefore $d(x,y) \leq d(x,z) + d(z,y)$.

2. The *box metric* defined by:

$$d(x,y) = \max_{1\leq k\leq n}\{|x_k - y_k|\}$$

Positive Definition:

$$\begin{aligned}&|x_k - y_k| \geq 0\\&\max\{x_k - y_k\} \geq 0\\&d(x,y) \geq 0\end{aligned}$$

$$\begin{aligned}d(x,y) = 0 &\iff \max\{|x_k - y_k|\} = 0\\&\iff |x_k - y_k| = 0\\&\iff x_k - y_k = 0\\&\iff x_k = y_k\\&\iff x = y\end{aligned}$$

Symmetric:

$$d(x,y) = \max\{|x_k - y_k|\} = \max\{|y_k - x_k|\} = d(y,x)$$

Triangle Inequality:

$$\begin{aligned}d(x,y) &= \max\{|x_k - y_k|\}\\&= \max\{|(x_k - z_k) + (z_k - y_k)|\}\\&\leq \max\{|x_k - z_k| + |z_k - y_k|\}\\&\leq \max\{|x_k - z_k|\} + \max\{|z_k - y_k|\}\\&= d(x,z) + d(z,y)\end{aligned}$$

3. The *taxi-cab metric* defined by:

$$d(x,y) = \sum_{k=1}^{n}|x_k - y_k|$$

Positive Definition:

$$\begin{aligned}&|x_k - y_k| \geq 0\\&\sum|x_k - y_k| \geq 0\\&d(x,y) \geq 0\end{aligned}$$

$$
\begin{aligned}
d(x,y)=0 &\iff \sum|x_k-y_k|=0\\
&\iff |x_k-y_y|=0\\
&\iff x_k-y_k=0\\
&\iff x_k=y_k\\
&\iff x=y
\end{aligned}
$$

Symmetric:

$$d(x,y)=\sum|x_k-y_k|=\sum|y_k-x_k|=d(y,x)$$

Triangle Inequality:

$$
\begin{aligned}
d(x,y)&=\sum|x_k-y_k|\\
&=\sum|(x_k-z_k)+(z_k-y_k)|\\
&\leq\sum(|x_k-z_k|+|z_k-y_k|)\\
&=\sum|x_k-z_k|+\sum|z_k-y_k|\\
&=d(x,z)+d(z,y)
\end{aligned}
$$

Show that when $n\geq 2$, these metrics are different.

Consider $(0,0),(3,4)\in\mathbb{R}^2$:

$$
\begin{aligned}
&d_E=\sqrt{(3-0)^2+(4-0)^2}=5\\
&d_B=\max\{(3-0),(4-0)\}=4\\
&d_T=(3-0)+(4-0)=7
\end{aligned}
$$

Example: Exercise 9.2

Let X be a compact topological space and let $\mathcal{C}(X)$ denote the set of continuous functions $f:X\to\mathbb{R}$. We can endow $\mathcal{C}(X)$ with a metric:

$$d(f,g)=\sup_{x\in X}\{|f(x)-g(x)|\}$$

This distance is sometimes denoted $\|f-g\|$. Check that d is a well-defined metric on $\mathcal{C}(X)$.

Note that for any $f\in\mathcal{C}(X)$, since X is compact and $f:X\to f(X)$ is surjective, $f(X)$ is compact and thus bounded. Therefore, all sups are finite.

Positive Definition:

$$
\begin{aligned}
&|f(x)-g(x)|\geq 0\\
&\sup\{|f(x)-g(x)|\}\geq 0\\
&d(f,g)\geq 0
\end{aligned}
$$

$$\begin{aligned} d(f,g)=0 &\iff \sup\{|f(x)-g(x)|\}=0 \\ &\iff |f(x)-g(x)|=0 \\ &\iff f(x)-g(x)=0 \\ &\iff f(x)=g(x) \\ &\iff f=g \end{aligned}$$

Symmetric:

$$d(f,g)=\sup\{|f(x)-g(x)|\}=\sup\{|g(x)-f(x)|\}=f(g,f)$$

Triangle Inequality:

$$\begin{aligned} d(f,g) &= \sup\{|f(x)-g(x)|\} \\ &= \sup\{|(f(x)-h(x))+(h(x)-g(x))|\} \\ &\leq \sup\{|f(x)-h(x)|+|h(x)-g(x)|\} \\ &\leq \sup\{|f(x)-h(x)|\}+\sup\{|h(x)-g(x)|\} \\ &= d(f,h)+d(h,g) \end{aligned}$$

Lemma

Let X be a metric space with metrics d_1 and d_2. If there exists $\alpha, \beta > 0$ such that for all $x, y \in X$:

$$\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_1(x,y)$$

then d_1 and d_2 generate the same topology.

Proof. Let B_1 denote a ball using d_1 and let B_2 denote a ball using d_2. Assume that $x \in X$ and $\epsilon > 0$.

First, assume that $y \in B_2(x, \epsilon)$. This means that $d_2(x,y) < \epsilon$, and so $d_1(x,y) < \frac{\epsilon}{\alpha}$. Hence $y \in B_1\left(x, \frac{\epsilon}{\alpha}\right)$, and so $B_2(x,\epsilon) \subset B_1\left(x, \frac{\epsilon}{\alpha}\right)$.

Next, assume that $y \in B_1(x,\epsilon)$. This means that $d_1(x,y) < \epsilon$, and so $d_2(x,y) < \beta\epsilon$. Hence $y \in B_2(x, \beta\epsilon)$, and so $B_1(x,\epsilon) \subset B_2(x,\beta\epsilon)$.

Now, assume that $B_1 \in \mathscr{T}_1$. For every $x \in B_1$ there exists $B_{2_x} \in \mathscr{T}_2$ such that $B_{2_x} \subset B_1$. Thus, $\mathscr{T}_2$ generates $\mathscr{T}_1$. Likewise, assume that $B_{2_x} \in \mathscr{T}_2$. For every $x \in B_2$ there exists $B_{1_x} \in \mathscr{T}_1$ such that $B_{1_x} \in \mathscr{T}_1$. Thus $\mathscr{T}_1$ generates $\mathscr{T}_2$.

Therefore $\mathscr{T}_1 = \mathscr{T}_2$. ∎

Example: Exercise 9.4

Show that the Euclidean metric, box metric, and taxicab metric generate the same topology as the product topology on n copies of $\mathbb{R}$.

$$\begin{aligned}
d_E(x,y) &= \sqrt{\sum (x_k - y_k)^2} \\
&\leq \sqrt{\sum \max\{(x_k - y_k)^2\}} \\
&= \sqrt{n \cdot \max\{(x_k - y_k)^2\}} \\
&= \sqrt{n} \max\{|x_k - y_k|\} \\
&= \sqrt{n} \cdot d_B(x,y)
\end{aligned}$$

Also:

$$\begin{aligned}
d_E(x,y) &= \sqrt{\sum (x_k - y_k)^2} \\
&\geq \sqrt{\max\{(x_k - y_k)^2\}} \\
&= \max\{\sqrt{(x_k - y_k)^2}\} \\
&= \max\{|x_k - y_k|\} \\
&= d_B(x,y)
\end{aligned}$$

So $d_B(x,y) \leq d_E(x,y) \leq \sqrt{n} d_B(x,y)$ and thus $\mathscr{T}_B = \mathscr{T}_E$.

Similarly:

$$\begin{aligned}
d_T(x,y) &= \sum |x_k - y_k| \\
&\leq \sum \max\{x_k - y_k\} \\
&= n \cdot \max\{x_k - y_k\} \\
&= n \cdot d_B(x,y)
\end{aligned}$$

Also:

$$\begin{aligned}
d_T(x,y) &= \sum |x_k - y_k| \\
&\geq \max\{x_k - y_k\} \\
&= d_B(x,y)
\end{aligned}$$

So $\frac{1}{n} d_T(x,y) \leq d_B(x,y) \leq d_T(x,y)$ and thus $\mathscr{T}_T = \mathscr{T}_B$.

Therefore $\mathscr{T}_E = \mathscr{T}_B = \mathscr{T}_T$.

Now, consider a basis element $U = \prod_{k=1}^n U_k \in \mathbb{R}^n$ and assume that $p \in U$. Then there exists some $\epsilon > 0$ such that $p \in \prod_{k=1}^n (p - \epsilon, p + \epsilon)$. But $B(p, \epsilon) \subset \prod_{k=1}^n (p - \epsilon, p + \epsilon)$ and so $\mathscr{T}_E$ generates $\mathscr{T}_{\mathbb{R}^n}$. Similarly, consider a basis element $B(p,r) \in \mathbb{R}^n$ and assume that $a \in B(p,r)$. Then there exists some $\epsilon > 0$ such that $B(a,\epsilon) \in B(p,r)$. But $\prod_{k=1}^n \left(a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2}\right) \subset B(a,\epsilon)$ and so $\mathscr{T}_{\mathbb{R}^n}$ generates T_E.

Therefore $\mathscr{T}_E = \mathscr{T}_B = \mathscr{T}_T = \mathscr{T}_{\mathbb{R}^n}$.

Lemma

Let (X, d) be a metric space and let $p \in X$ and $A \subset X$ such that $p \notin A$ and A is closed:

$$\operatorname{dist}(p, A) = \inf \{d(a, p) \,|\, a \in A\} > 0$$

Proof. Since A is closed and $p \notin A$, p is not a limit point of A. Thus, there exists $\epsilon > 0$ such that $B(p, \epsilon) \cap A = \emptyset$ and so for all $a \in A$ the distance from p to a is at least ϵ.

Therefore, $\inf \{d(a, p) \,|\, a \in A\} > \epsilon > 0$. ∎

Theorem: 9.8

A metric space is Hausdorff, regular, and normal.

Proof. Let (X, d) be a metric space and let $p \in X$ and $A \subset X$ such that $p \notin A$ and A is closed. Then there exists some $\epsilon > 0$ such that for all $a \in A$, $d(p, a) > \epsilon$. Let $\delta = \frac{\epsilon}{3}$ and consider $U = B(p, \delta)$ and open set V generated by $\{B(a, \delta_a) \,|\, a \in A, \delta_a < \delta\}$. Thus, for every point $x \in U$ and $y \in V$, $d(x, y) \geq \delta$ and so $U \cap V = \emptyset$.

Therefore (X, d) is regular, and hence also Hausdorff.

Now, assume that $A, B \subset (X, d)$ such that A and B are closed and $A \cap B = \emptyset$. Then for every $a \in A$ there exists $B(a, \epsilon_a)$ such that $B(a, \epsilon_a) \cap B = \emptyset$. Likewise, for every $b \in B$ there exists $B(b, \epsilon_b)$ such that $B(b, \epsilon_b) \cap A = \emptyset$. So let $\delta_a = \frac{\epsilon_a}{3}$ and let $\delta_b = \frac{\epsilon_b}{3}$ and consider the families of open sets $U_a = B(a, \delta_a)$ and $V_b = B(b, \delta_b)$. Let:

$$U = \bigcup_{a \in A} U_a \supset A$$

$$V = \bigcup_{b \in B} V_b \supset B$$

Now, assume that $a \in A$ and $b \in B$:

$$d(a, b) \geq \max\{\epsilon_a, \epsilon_b\} > \max\{\delta_a, \delta_b\}$$

Thus $U_a \cap V_b = \emptyset$ and hence $U \cap V = \emptyset$.

Therefore (X, d) is normal. ∎