Sets

Notation: Element

Let A be a set. The notation $a \in A$ indicates that *element* a is in set A.

Definition: Subset

Let A and B be sets. To say that A is a *subset* of B, denoted by $A \subset B$, means that:

$$a \in A \implies a \in B$$

In particular, a set is a subset of itself ($A \subset A$) and the empty set $\emptyset$ is a subset of every other set.

Definition: Equality

Let A and B be sets. To say that A is *equal* to B, denoted by $A = B$, means that:

$$a \in A \iff a \in B$$

or alternatively:

$$A \subset B \text{ and } B \subset A$$

Definition: Proper

Let A and B be sets. To say that A is a *proper* subset of B, denoted by $A \subsetneq B$, means that $A \subset B$ but $A \neq B$. Thus, $B \not\subset A$, meaning $\exists\, b \in B, b \notin A$.

Definition: Operations

Let A, B, and X be sets such that $A, B \subset X$:

Union: $A \cup B = \{x \in X \,|\, x \in A \text{ or } x \in B\}$

Intersection: $A \cap B = \{x \in X \,|\, x \in A \text{ and } x \in B\}$

Complement: $X - A = \{x \in X \,|\, x \notin A\}$

When X is understood, $X - A$ can be denoted by A^C.

Theorem: DeMorgan

Let A_1, A_2, and X be sets such that $A_1, A_2 \subset X$:

$$(A_1 \cup A_2)^C = A_1^C \cap A_2^C$$
$$(A_1 \cap A_2)^C = A_1^C \cup A_2^C$$

Proof. Assume $x \in X$:

$$\begin{aligned} x \in (A_1 \cup A_2)^C &\iff x \notin A_1 \cup A_2 \\ &\iff x \notin A_1 \text{ and } x \notin A_2 \\ &\iff x \in A_1^C \text{ and } x \in A_2^C \\ &\iff x \in A_1^C \cap A_2^C \end{aligned}$$

$\therefore (A_1 \cup A_2)^C = A_1^C \cap A_2^C$

$$\begin{aligned} x \in (A_1 \cap A_2)^C &\iff x \notin A_1 \cap A_2 \\ &\iff x \notin A_1 \text{ or } x \notin A_2 \\ &\iff x \in A_1^C \text{ or } x \in A_2^C \\ &\iff x \in A_1^C \cup A_2^C \end{aligned}$$

$\therefore (A_1 \cap A_2)^C = A_1^C \cup A_2^C$

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Notation

Let X be a set and let $\{A_\alpha : \alpha \in \lambda\}$ be a family of sets such that $A_\alpha \subset X$:

$$\bigcup_{\alpha \in \lambda} A_\alpha = \{x \in X \mid \exists\, \alpha \in \lambda, x \in A_\alpha\}$$
$$\bigcap_{\alpha \in \lambda} A_\alpha = \{x \in X \mid \forall\, \alpha \in \lambda, x \in A_\alpha\}$$

Theorem: General DeMorgan

Let X be a set and let $\{A_\alpha : \alpha \in \lambda\}$ be a family of sets such that $A_\alpha \subset X$:

$$\left(\bigcup_{\alpha \in \lambda} A_\alpha\right)^C = \bigcap_{\alpha \in \lambda} A_\alpha^C$$
$$\left(\bigcap_{\alpha \in \lambda} A_\alpha\right)^C = \bigcup_{\alpha \in \lambda} A_\alpha^C$$

Proof. Assume $x \in X$:

$$\begin{aligned} x \in \left(\bigcup_{\alpha\in\lambda} A_\alpha\right)^C &\iff x \notin \bigcup_{\alpha\in\lambda} A_\alpha \\ &\iff \forall\, \alpha \in \lambda, x \notin A_\alpha \\ &\iff \forall\, \alpha \in \lambda, x \in A_\alpha^C \\ &\iff x \in \bigcap_{\alpha\in\lambda} A_\alpha^C \end{aligned}$$

$$\therefore \left(\bigcup_{\alpha\in\lambda} A_\alpha\right)^C = \bigcap_{\alpha\in\lambda} A_\alpha^C$$

$$\begin{aligned} x \in \left(\bigcap_{\alpha\in\lambda} A_\alpha\right)^C &\iff x \notin \bigcap_{\alpha\in\lambda} A_\alpha \\ &\iff \exists\, \alpha \in \lambda, x \notin A_\alpha \\ &\iff \exists\, \alpha \in \lambda, x \in A_\alpha^C \\ &\iff x \in \bigcup_{\alpha\in\lambda} A_\alpha^C \end{aligned}$$

$$\therefore \left(\bigcap_{\alpha\in\lambda} A_\alpha\right)^C = \bigcup_{\alpha\in\lambda} A_\alpha^C$$

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