Topology

Definition: Topology

A *topology* $\mathcal{T}$ on a set X is a subset of $\mathcal{P}(X)$ such that:

1). $\emptyset, X \in \mathcal{T}$

2). $\mathcal{T}$ is closed under the operation of *arbitrary* union.

3). $\mathcal{T}$ is closed under the operation of *finite* intersection.

The ordered pair $(X, \mathcal{T})$ is called a *topological space*.

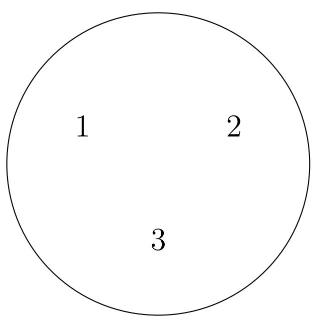
Definition: Open

Let $(X, \mathcal{T})$ be a topological space:

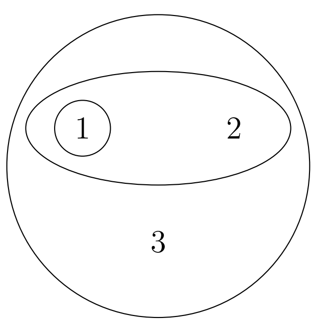
- To say that $U \subset X$ is an *open set* means $U \in \mathcal{T}$.
- To say that $U \subset X$ is a *closed set* means $X - U \in \mathcal{T}$.

Example

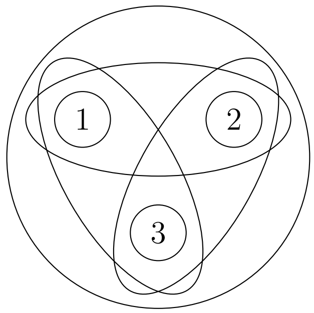
Let $X = \{1, 2, 3\}$:



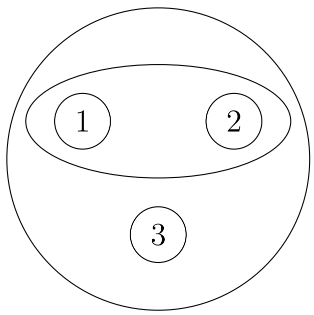
This is a topology on X.



This is a topology on X.



This is a topology on X.



This is *not* a topology on X because $\{1\}, \{3\} \in \mathcal{T}$; however $\{1\} \cup \{3\} = \{1, 3\} \notin \mathcal{T}$.

<u>Definition: Discrete</u>

Let $(x, \mathcal{T})$ be a topological space:

- $\mathcal{T} = \{\emptyset, x\}$ is called the *trivial* or *indiscrete* topology.
- $\mathcal{T} = \mathcal{P}(X)$ is called the *discrete* topology.

<u>Theorem</u>

Let X be a set. The discrete topology $\mathcal{T}$ on X is a topology.

<u>Proof</u>

1). $\emptyset, X \subset X$
$\therefore \emptyset, X \in \mathcal{P}(X) = \mathcal{T}$

2). Assume $\mathcal{A} \subset \mathcal{P}(X)$.
Let $S = \bigcup\limits_{A \in \mathcal{A}} A$.
$S \subset X$
$\therefore S \in \mathcal{P}(X) = \mathcal{T}$

3). Assume $\mathcal{A} = \{A_i \mid i \in \{1, \ldots, n\}\} \subset \mathcal{P}(X)$.
Let $T = \bigcap\limits_{i=1}^{n} A_i$.
$T \subset X$
$\therefore T \in \mathcal{P}(X) = \mathcal{T}$

$\therefore \mathcal{T}$ is a topology on X.