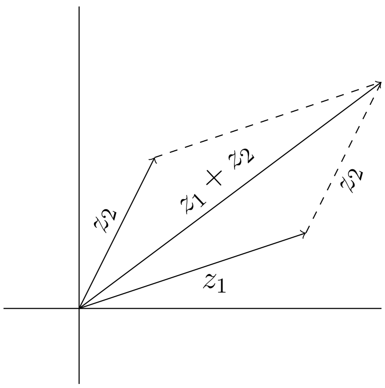
Triangle Inequality

Since complex numbers are vector-like, we can add complex numbers graphically in vector-like style:



$$|z_1 + z_1| \leq |z_1| + |z_2|$$

Note that the triangle inequality is evident from the diagram; however, it can also be proved analytically:

<u>Theorem</u>

$\forall z_1, z_2 \in \mathbb{C}$:

$$||a| - |b|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

<u>Proof</u>

Assume $z_1, z_2 \in \mathbb{C}$

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\
&= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\
&= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + \bar{z}_1 z_2 \\
&= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} \\
&= |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2) \\
&\leq |z_1|^2 + |z_2|^2 + 2\,|z_1\bar{z}_2| \\
&= |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|\bar{z}_2| \\
&= |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|z_2| \\
&= (|z_1| + |z_2|)^2 \\
|z_1 + z_2| &\leq |z_1| + |z_2|
\end{aligned}$$

$$\begin{aligned}
|z_1 - z_2|^2 &= (z_1 - z_2)\overline{(z_1 - z_2)} \\
&= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\
&= z_1\bar{z}_1 + z_2\bar{z}_2 - z_1\bar{z}_2 - \bar{z}_1 z_2 \\
&= |z_1|^2 + |z_2|^2 - (z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) \\
&= |z_1|^2 + |z_2|^2 - 2Re(z_1\bar{z}_2) \\
&\geq |z_1|^2 + |z_2|^2 - 2\,|z_1\bar{z}_2| \\
&= |z_1|^2 + |z_2|^2 - 2\,|z_1|\,|\bar{z}_2| \\
&= |z_1|^2 + |z_2|^2 - 2\,|z_1|\,|z_2| \\
&= (|z_1| - |z_2|)^2 \\
|z_1 - z_2| &\geq ||z_1| + |z_2|| \\
|z_1 - (-z_2)| &\geq ||z_1| + |-z_2|| \\
|z_1 + z_2| &\geq ||z_1| + |z_2||
\end{aligned}$$

$\therefore ||a| - |b|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

Proof (alternate)

Assume $z_1, z_2 \in \mathbb{C}$

$$\begin{aligned}
&|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |z_2| \\
&|z_1| - |z_2| \leq |z_1 + z_2| \\
&|z_2| = |z_2 + z_1 - z_1| \leq |z_2 + z_1| + |z_1| \\
&|z_2| - |z_1| = -(|z_1| - |z_2|) \leq |z_1 + z_2| \\
&\therefore ||z_1| - |z_2|| \leq |z_1 + z_2|
\end{aligned}$$

Corollary

$\forall z_1, z_2 \in \mathbb{C}$:

$$|z_1 - z_2| \leq |z_1| + |z_2|$$

Proof

Assume $z_1, z_2 \in \mathbb{C}$

$$|z_1 - z_2| = |z_1 + (-z_2)| \leq |z_1| + |-z_2| = |z_1| + |z_2|$$

Theorem

$\forall n \in \mathbb{N}$:

$$\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|$$

Proof

(by induction)

Base Case: $n = 1$

$\left|\sum_{k=1}^{1} z_k\right| = |z_1|$

$\sum_{k=1}^{1} |z_k| = |z_1|$

Assume $|\sum_{k=1}^{n} z_k| \leq \sum_{k=1}^{n} |z_k|$

Consider $\left|\sum_{k=1}^{n+1} z_k\right|$

$$\left|\sum_{k=1}^{n+1} z_k\right| = \left|\sum_{k=1}^{n} z_k + z_{n+1}\right| \leq \left|\sum_{k=1}^{n} z_k\right| + |z_{n+1}| \leq \sum_{k=1}^{n} |z_k| + |z_{n+1}| = \sum_{k=1}^{n+1} |z_k|$$

Theorem

$\forall\, z = x + iy \in \mathbb{C}$:

$$\frac{|x| + |y|}{\sqrt{2}} \leq |z| \leq |x| + |y|$$

Proof

Assume $z = x + iy \in \mathbb{C}$

$$\begin{aligned}
(|x| - |y|)^2 &= |x|^2 + |y|^2 - 2\,|x|\,|y| \\
&= 2\,|x|^2 + 2\,|y|^2 - |x|^2 - |y|^2 - 2\,|x|\,|y| \\
&= 2(|x|^2 + |y|^2) - (|x|^2 + |y|^2 + 2\,|x|\,|y|) \\
&= 2(|z|^2) - (|x| + |y|)^2 \\
&\geq 0
\end{aligned}$$

$$\begin{aligned}
(|x| + |y|)^2 &\leq 2(|z|^2) \\
|x| + |y| &\leq \sqrt{2}\,|z| \\
\frac{|x| + |y|}{\sqrt{2}} &\leq |z|
\end{aligned}$$

$$|z| = |x + iy| \leq |x| + |iy| = |x| + |i|\,|y| = |x| + (1)\,|y| = |x| + |y|$$

$\therefore \frac{|x|+|y|}{\sqrt{2}} \leq |z| \leq |x| + |y|$

Note that equality occurs in the first case when $|x| = |y|$, and in the second case when $x = 0$ or $y = 0$.