MATH 231B, FALL 2017
HOMEWORK 7 SOLUTIONS

1. (Sec. 4.12, ex. 48) ($\Rightarrow$) Let $P : H \to F$ be compact. Assume F is infinite-dimensional. Since F is closed, it is complete, hence a Hilbert space. Since H is separable, so is F. Therefore, F admits a (complete) orthonormal sequence (f_n). As such, f_n converges weakly to zero. Hence by compactness of P, $Pf_n \to 0$, as $n \to \infty$, strongly. But $Pf_n = f_n$, since $f_n \in F$. It follows that $f_n \to 0$, which is impossible, since $\|f_m - f_n\|^2 = 2$, for all m, n. This shows that F cannot be infinite-dimensional.

   ($\Leftarrow$) Now assume that $\dim F < \infty$. Then $P : H \to F$ is finite-rank, hence compact. $\square$

2. (Sec. 4.12, ex. 49) Define $T_n : \ell^2 \to \ell^2$ by

$$T_n x = \left(\frac{x_1}{2}, \cdots, \frac{x_n}{2^n}, 0, 0, \cdots \right),$$

   where $x = (x_n)$. Since $\dim \mathscr{R}(T_n) = n$, T_n is a finite-rank operator, for all n. We will show that $T_n \to T$, which will prove that T is compact (see Corollary 4.8.13). Indeed, for any $x = (x_n) \in \ell^2$, we have:

$$\begin{aligned} \|Tx - T_n x\|^2 &= \left\| \left(0, \cdots, 0, \frac{x_{n+1}}{2^{n+1}}, \frac{x_{n+2}}{2^{n+2}}, \cdots \right) \right\|^2 \\ &= \sum_{k=n+1}^{\infty} \frac{|x_k|^2}{4^k} \\ &\le \frac{1}{4^{n+1}} \sum_{k=n+1}^{\infty} |x_k|^2 \\ &\le \frac{1}{4^{n+1}} \|x\|^2, \end{aligned}$$

   so $\|T - T_n\| \le 1/2^{n+1} \to 0$, as $n \to \infty$, as desired. $\square$

3. (Sec. 4.12, ex. 50) ($\Rightarrow$) Assume T is self-adjoint and compact. By the spectral theorem we have

$$T = \sum_{n=1}^{\infty} \lambda_n P_n,$$

   where λ_n are real and P_n are projections to finite-dimensional subspaces. Let T_n be the n^{th} partial sum of the above series. Then T_n is finite-rank and $T_n \to T$, as desired.

   ($\Leftarrow$) If T is the limit of finite-rank operators, then T is compact, by Corollary 4.8.13. $\square$

4. (Sec. 4.12, ex. 51) Let T be a compact operator, $\lambda \neq 0$ an eigenvalue of T, and E_λ the eigenspace corresponding to λ. Assume E_λ is infinite-dimensional. Since E_λ is closed (being the kernel of the continuous operator $T - \lambda I$) and separable (since the ambient Hilbert space is separable), E_λ admits a (complete) orthonormal sequence (e_n), clearly consisting of eigenvectors of T. Being an orthonormal sequence, e_n converges weakly to zero. Since T is compact, $Te_n \to 0$ (strongly).

But $Te_n = \lambda e_n$. Since $\lambda \neq 0$, $\lambda e_n \to 0$ implies $e_n \to 0$, which is impossible, since $\|e_m - e_n\|^2 = 2$, for all m, n. Thus E_λ is finite-dimensional, as claimed. $\square$