Cavallaro, Jeffery
Math 221a
Homework #5

1.6.2

a) Prove: S_n is generated by the $(n-1)$ transpositions $(12), (13), (14) \ldots, (1n)$.

Assume $1 \leq i < j \leq n$
Assume $1 \leq m \leq n$

$$((1i)(1j)(1i))(m) = \begin{cases} j, & m = i \\ i, & m = j \\ 1, & m = 1 \\ m, & m \neq 1 \text{ and } m \neq i \text{ and } m \neq j \end{cases}$$

$$(ij)(m) = \begin{cases} j, & m = i \\ i, & m = j \\ 1, & m = 1 \\ m, & m \neq 1 \text{ and } m \neq i \text{ and } m \neq j \end{cases}$$

Thus $(1i)(1j)(1i) = (ij)$
So any transposition (ij) can be generated from the $(n-1)$ transpositions $(1x)$
Assume $\sigma \in S_n$
σ can be written as a sequence of disjoint cycles
Each cycle can be written as a sequence of transpositions (ij), $1 \leq i < j \leq n$
Therefore, σ can be generated.

b) Prove: S_n is generated by the $(n-1)$ transpositions $(12), (23), (34), \ldots, (n-1\ n)$.

Assume $1 \leq i < n$
Assume $1 \leq m \leq n$

$$((1\ i-1)(i-1\ i)(1\ i-1))(m) = \begin{cases} i, & m = 1 \\ 1, & m = i \\ i-1, & m = i-1 \\ m, & m \neq 1 \text{ and } m \neq i-1 \text{ and } m \neq i \end{cases}$$

$$(1i)(m) = \begin{cases} i, & m = 1 \\ 1, & m = i \\ i-1, & m = i-1 \\ m, & m \neq 1 \text{ and } m \neq i-1 \text{ and } m \neq i \end{cases}$$

Thus, $(1\ j-1)(j-1\ j)(1\ j-1)=(1j)$
So any transposition $(1i)$ can be generated from the (n-1) transpositions $(i-1\ i)$
Therefore, by part (a), any $\sigma \in S_n$ can be generated.

1.6.3

Let $\sigma=(i_1 i_2 \cdots i_r) \in S_n$ and $\tau \in S_n$.
Prove: $\tau\sigma\tau^{-1}$ is the r-cycle $(\tau(i_1)\tau(i_2)\cdots\tau(i_r))$

Claim: $\tau(i_j i_k)\tau^{-1}=(\tau(i_j)\tau(i_k))$ where $0 \leq j < k \leq r$

Case 1: $\tau(i_j)=i_j$ and $\tau(i_k)=i_k$

Assume $i_s \neq i_j, i_k$
$(\tau(i_j i_k)\tau^{-1})(i_s)=i_s$
$(\tau(i_j i_k)\tau^{-1})(i_j)=i_k=\tau(i_k)$
$(\tau(i_j i_k)\tau^{-1})(i_k)=i_j=\tau(i_j)$
$\therefore \tau(i_j i_k)\tau^{-1}=(\tau(i_j)\tau(i_k))$

Case 2: $\tau^{-1}(i_s)=i_j$ and $\tau(i_k)=i_k$
$i_s=\tau(i_j)$
$(\tau(i_j i_k)\tau^{-1})(i_s)=\tau(i_k)$
$(\tau(i_j i_k)\tau^{-1})(\tau(i_k))=\tau(i_j)$
$\therefore \tau(i_j i_k)\tau^{-1}=(\tau(i_j)\tau(i_k))$

Case 3: $\tau^{-1}(i_s)=i_j$ and $\tau^{-1}(i_t)=i_k$
$i_s=\tau(i_j)$
$(\tau(i_j i_k)\tau^{-1})(i_s)=\tau(i_k)$
$(\tau(i_j i_k)\tau^{-1})(\tau(i_k))=\tau(i_j)$
$i_t=\tau(i_k)$
$(\tau(i_j i_k)\tau^{-1})(i_t)=\tau(i_j)$
$(\tau(i_j i_k)\tau^{-1})(\tau(i_j))=\tau(i_k)$
$\therefore \tau(i_j i_k)\tau^{-1}=(\tau(i_j)\tau(i_k))$

$\therefore \tau(i_j i_k)\tau^{-1}=(\tau(i_j)\tau(i_k))$

Now:

$$\begin{aligned}
\tau\sigma\tau^{-1} &= \tau(i_1 i_2 \cdots i_r)\tau^{-1} \\
&= \tau(i_1 i_2)(i_2 i_3)\cdots(i_{r-1} i_r)\tau^{-1} \\
&= \tau(i_1 i_2)\tau^{-1}\tau(i_2 i_3)\tau^{-1}\tau\cdots\tau^{-1}\tau(i_{r-1} i_r)\tau^{-1} \\
&= (\tau(i_1)\tau(i_2))(\tau(i_2)\tau(i_3))\cdots(\tau(i_{r-1})\tau(i_r)) \\
&= (\tau(i_1)\tau(i_2)\cdots\tau(i_r))
\end{aligned}$$

1.6.3

a) Prove: $S_n = \langle (12), (12\cdots n) \rangle$

Using problem (3):

$$\tau(12)\tau^{-1} = (\tau(1), \tau(2)) = (23)$$
$$\tau(23)\tau^{-1} = (\tau(2), \tau(3)) = (34)$$
$$\vdots$$
$$\tau(i-1\ i)\tau^{-1} = (\tau(i-1), \tau(i)) = (i\ i+1)$$
$$\vdots$$
$$\tau(n-2\ n-1)\tau^{-1} = (\tau(n-2), \tau(n-1)) = (n-1\ n)$$

Thus, all possible transpositions of the form $(i-1\ i)$ can be generated, and therefore, by problem (2b), all of S_n can be generated.

b) Prove: $S_n = \langle (12), (23\cdots n) \rangle$

Using problem (3):

$$\tau(12)\tau^{-1} = (\tau(1), \tau(2)) = (13)$$
$$\tau(13)\tau^{-1} = (\tau(1), \tau(3)) = (14)$$
$$\vdots$$
$$\tau(1\ i)\tau^{-1} = (\tau(1), \tau(i)) = (1\ i+1)$$
$$\vdots$$
$$\tau(1\ n-1)\tau^{-1} = (\tau(1), \tau(n-1)) = (1\ n)$$

Thus, all possible transpositions of the form $(1\ \ i)$ can be generated, and therefore, by problem (2a), all of S_n can be generated.

1.8.1

Prove that the following are not direct products of their proper subgroups.

In order for a group to be a direct product of its proper subgroups, the following must hold:

1). The group must be generated by a collection of its proper, normal, almost disjoint subgroups.

2). The group must be isomorphic to the external cross product of those subgroups.

a) S_3

Although S_3 is generated by its 3-element subgroup $\{(), (123), (132)\}$ and any of its 2-element subgroups, e.g., $\{(), (13)\}$, and although those subgroups are almost disjoint, Only the 3-element subgroup is normal; none of the 2-element subgroups are normal.

Moreover, since the 2 and 3 element subgroups are necessarily abelian, their external direct product must also be abelian; however, S_3 is not abelian and thus not isomorphic to the external direct product.

Therefore S_3 is not a direct product of any of its proper subgroups.

b) $\mathbb{Z}_{p^n}$

By Lagrange, all of the proper subgroups of $\mathbb{Z}_{p^n}$ must be of order p^k where $1 \leq k < n$. Since $\mathbb{Z}_{p^n}$ is abelian, all of its subgroups must also be abelian, so all of the proper subgroups are normal. However, we know that these subgroups form a chain:

$$G_p \leq G_{p^2} \leq \cdots \leq G_{p^{n-1}}$$

Thus, these subgroups are not almost disjoint.

Moreover, all of the elements in all of these subgroups are multiples of p, in other words $\equiv 0 \pmod p$. Any other elements are relatively prime with p^n and thus are generators of $\mathbb{Z}_{p^n}$ and are not members of any proper subgroup. Thus, elements congruent to 1 thru $p-1 \pmod p$ can never be generated by the proper subgroups.

Therefore $\mathbb{Z}_{p^n}$ is not the direct product of any of its proper subgroups.

c) $\mathbb{Z}$

All of the proper subgroups of $\mathbb{Z}$ are the $n\mathbb{Z}$ groups. Since $\mathbb{Z}$ is abelian, all of its subgroups must also be abelian, so all of the proper subgroups are normal.

Assume $h\mathbb{Z}$ and $k\mathbb{Z}$ are two such subgroups, $h \neq k$. Then:

$$hk \in k\mathbb{Z}$$

$$kh \in h\mathbb{Z}$$

But since the subgroups are normal:

$$hk = kh$$

Thus, $hk \in h\mathbb{Z} \cup k\mathbb{Z}$, so the two subgroups are not almost disjoint.

Therefore $\mathbb{Z}$ is not the direct product of any of its proper subgroups.

1.8.2

Given an example of groups H_i and K_i such that $H_1 \times K_1 \simeq H_2 \times K_2$, and no H_i is isomorphic to any K_i.

Consider:

$$\begin{aligned}
H_1 &= \mathbb{Z}_3 \\
H_2 &= \mathbb{Z}_4 \\
K_1 &= \mathbb{Z}_{20} \\
H_2 &= \mathbb{Z}_{15}
\end{aligned}$$

Note that $(3, 20) = 1$ and $(4, 15) = 1$, so:

$$H_1 \times K_1 = \mathbb{Z}_3 \times \mathbb{Z}_{20} \simeq \mathbb{Z}_{60}$$

$$H_2 \times K_2 = \mathbb{Z}_4 \times \mathbb{Z}_{15} \simeq \mathbb{Z}_{60}$$

Therefore, $H_1 \times K_1 \simeq H_2 \times K_2$; however, by cardinality, no H_i is isomorphic to any K_i.

1.8.3

Let G be additive abelian with subgroups H and K.
Prove: $G \simeq H \oplus K$ iff there exists homomorphisms $H \underset{\iota_1}{\overset{\pi_1}{\leftrightarrows}} G \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} K$ such that:

$$\begin{aligned}
\pi_1 i_1 &= 1_H \\
\pi_2 i_2 &= 1_K \\
\pi_1 i_2 &= 0 \\
\pi_2 i_1 &= 0
\end{aligned}$$

and:

$$\forall\, x \in G, (i_1\pi_1)(x) + (i_2\pi_2)(x) = x$$

$\implies$ Assume $G \simeq H \oplus K$

Since the direct product holds as defined in the category of groups, we have the commutative diagram for the product:

$$\begin{tikzcd}
G \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & H \\
K & H \times K \arrow[l, "\phi_2"] \arrow[u, "\phi_1"'] \arrow[ul, "\phi"]
\end{tikzcd}$$

where the homomorphisms are the canonical projections. Likewise, we have the commutative diagram for the co-product:

$$
\begin{tikzcd}
G \arrow[dr, "\psi"] & H \arrow[l, "\iota_1"'] \arrow[d, "\psi_1"] \\
K \arrow[u, "\iota_2"] \arrow[r, "\psi_2"'] & H \times K
\end{tikzcd}
$$

where the homomorphisms are the canonical injections.

Thus, the homomorphisms $H \overset{\pi_1}{\underset{\iota_1}{\leftrightarrows}} G \overset{\pi_2}{\underset{\iota_2}{\rightleftarrows}} K$ exist.

Assume $h \in H$:

$(\pi_1\iota_1)(h) = \pi_1(\iota_1(h)) = \pi_1(h+0) = h$
$\therefore \pi_1\iota_1 = 1_H$

$(\pi_2\iota_1)(h) = \pi_2(\iota_1(h)) = \pi_2(h+0) = 0$
$\therefore \pi_2\iota_1 = 0$

Assume $k \in K$:

$(\pi_2\iota_2)(k) = \pi_2(\iota_2(k)) = \pi_2(0+k) = k$
$\therefore \pi_2\iota_2 = 1_K$

$(\pi_1\iota_2)(k) = \pi_1(\iota_2(k)) = \pi_1(0+k) = 0 \therefore \pi_1\iota_2 = 0$

Assume $x \in G$
$x = h + k$ for some $h \in H$ and $k \in K$

$$
\begin{aligned}
(i_1\pi_1)(x) + (i_2\pi_2)(x) &= i_1(\pi_1(x)) + i_2(\pi_2(x)) \\
&= i_1(h) + i_2(k) \\
&= (h+0) + (0+k) \\
&= h + k \\
&= x
\end{aligned}
$$

$\Longleftarrow$ Assume all of that other stuff

Since $H, K \leq G$ and G is abelian, H and K are abelian
So, $H \triangleleft G$ and $K \triangleleft G$
Thus $H + K \leq G$

Assume $x \in G$
$x = (i_1\pi_1)(x) + (i_2\pi_2)(x) = i_1(\pi_1(x)) + i_2(\pi_2(x))$

Let $\pi_1(x) = h \in H$ and $\pi_2(x) = k \in K$
$x = \iota_1(h) + i_2(k) = (h + 0) + (k + 0) = h + k \in H + K$

Therefore, $H + K = G$

But, since $\pi_1 \iota_2 = 0$ and $\pi_2 \iota_1 =$, H and K have no projection into each other and thus $H \cap K = \{e\}$.

Therefore, the requirements for corollary 8.7 are met and $G \simeq H \oplus K$.