Cavallaro, Jeffery
Math 221b
Homework #6

1). Show that if $f : \mathbb{C} \to \mathbb{R}$ is a ring homomorphism then f must be trivial.

This is equivalent to saying that $\ker(f) = \mathbb{C}$. Since $\ker(f)$ must be ideal in $\mathbb{C}$, and since $\mathbb{C}$ is a field, any non-zero element in $\ker(f)$ is a unit and thus would cause $\ker(f) = \mathbb{C}$. Thus, the only two choices for $\ker(f)$ are the zero ideal and $\mathbb{C}$, where the latter means that the ring homomorphism is trivial. So, we want to eliminate the zero ideal as a possibility.

So ABC that the kernel is trivial. This means that f is injective. Now, let z be a non-zero element of $\mathbb{C}$:

$$f(z) = f(1z) = f(1)f(z)$$

and since f is injective and $f(0) = 0$, neither $f(1)$ nor $f(z)$ can be 0, and so $f(1)$ is the identity in $\mathbb{R}$ and so $f(1) = 1$. But $\mathbb{C}$ and $\mathbb{R}$ are additive groups so $f(-1) = -f(1) = -1$.

Now, consider the following:

$$f(i^2) = f(i)f(i) = f(i)^2 = f(-1) = -f(1) = -1$$

Thus, there exists some $x \in \mathbb{R}$ such that $x^2 = -1$, a contradiction.

Therefore, $\ker(f) = \mathbb{C}$ and f is trivial.

2). Consider the field $K = \mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$ as an extension of $F = \mathbb{Q}$:

a). Show that $[K : F] = 6$

Consider the following field extensions:

$$\begin{array}{c} \mathbb{Q}(\sqrt[3]{2}, \sqrt{3}) \\ | \\ \mathbb{Q}(\sqrt[3]{2}) \\ | \\ \mathbb{Q} \end{array}$$

Note that all of these extensions are algebraic.

First, consider $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ and the polynomial $f(x) = x^3 - 2 \in \mathbb{Q}[x]$. By the rational root test, $f(x)$ is irreducible in $\mathbb{Q}$. Furthermore, $f(\sqrt[3]{2}) = 0$, and thus $f(x) = m_{\sqrt[3]{2},\mathbb{Q}}(x)$ and $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$:

$$\begin{tikzcd}
\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}) \arrow[d, dash] \\
\mathbb{Q}(\sqrt[3]{2}) \arrow[d, dash, "3"] \\
\mathbb{Q}
\end{tikzcd}$$

Next, consider $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3})/\mathbb{Q}(\sqrt[3]{2})$ and the polynomial $g(x) = x^2 - 3 \in \mathbb{Q}(\sqrt[3]{2})[x]$. Since $g(\sqrt{3}) = 0$, $m_{\sqrt{3},\mathbb{Q}(\sqrt[3]{2})}(x) \mid g(x)$, and so $\deg(m_{\sqrt{3},\mathbb{Q}(\sqrt[3]{2})}(x)) = 1$ or 2.

ABC: $\deg(m_{\sqrt{3},\mathbb{Q}(\sqrt[3]{2})}(x)) = 1$
This would mean that $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}) = \mathbb{Q}(\sqrt[3]{2})$ and thus $\sqrt{3} \in \mathbb{Q}(\sqrt[3]{2})$.
But that would mean that $Q \subset \mathbb{Q}(\sqrt{3}) \subset \mathbb{Q}(\sqrt[3]{2})$.
So consider $g(x) = x^2 - 3 \in \mathbb{Q}[x]$. By the rational root test, $g(x)$ is irreducible in $\mathbb{Q}$. Furthermore, $g(\sqrt{3}) = 0$, and thus $g(x) = m_{\sqrt{3},\mathbb{Q}}(x)$ and $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$. However, in order for $\mathbb{Q}(\sqrt{3}) \subset \mathbb{Q}(\sqrt[3]{2})$ it must be the case that $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] \mid [\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$, but $2 \nmid 3$ - contradiction.

Thus $\deg(m_{\sqrt{3},\mathbb{Q}(\sqrt[3]{2})}(x)) = 2$ and so $[\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}) : \mathbb{Q}(\sqrt[3]{2})] = 2$:

$$\begin{tikzcd}
\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}) \arrow[d, dash, "2"] \\
\mathbb{Q}(\sqrt[3]{2}) \arrow[d, dash, "3"] \\
\mathbb{Q}
\end{tikzcd}$$

Therefore, $[\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}) : \mathbb{Q}(\sqrt[3]{2})][\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 2 \cdot 3 = 6$

b). Find a primitive element α for K/F.

Let $\alpha = \sqrt{3} + \sqrt[3]{2} \in K/F$.
We need to show that $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{3} + \sqrt[3]{2})$. Consider the factors:

$a_1(x) = [x - (\sqrt{3} + \sqrt[3]{2})]$
$a_2(x) = [x - (-\sqrt{3} + \sqrt[3]{2})]$
$a_3(x) = [x - (\sqrt{3} + \omega\sqrt[3]{2})]$
$a_4(x) = [x - (-\sqrt{3} + \omega\sqrt[3]{2})]$
$a_5(x) = [x - (\sqrt{3} + \omega^2\sqrt[3]{2})]$
$a_6(x) = [x - (-\sqrt{3} + \omega^2\sqrt[3]{2})]$

All of these factors need to be applied in order to obtain a polynomial with coefficients in $\mathbb{Q}$:

$$f(x) = a_1(x)a_2(x)a_3(x)a_4(x)a_5(x)a_6(x) = x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23$$

But $\mathbb{Q}(\sqrt{3}+\sqrt[3]{2})$ is a UFD, and so $f(x)$ is irreducible in $\mathbb{Q}$. Also, $f(\sqrt{3}+\sqrt[3]{2}) = 0$, and so $f(x) = m_{\sqrt{3}+\sqrt[3]{2},\mathbb{Q}}(x)$, and thus $[\mathbb{Q}(\sqrt{3}+\sqrt[3]{2}) : \mathbb{Q}] = 6$. But $\sqrt{3}+\sqrt[3]{2} \in \mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$, so $\mathbb{Q} \subset \mathbb{Q}(\sqrt{3}+\sqrt[3]{2}) \subset \mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$

Therefore $\mathbb{Q}(\sqrt{3}+\sqrt[3]{2}) = \mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$.

3). Suppose K/F is an algebraic extension of fields. Prove that if R is a ring with $F \subseteq R \subseteq K$ then R must also be a field.

It suffices to show that R^* is closed under multiplicative inverses.

Assume $\alpha \in R^*$
Since K/F is an algebraic extension and $\alpha \in K$, α is algebraic over F. Thus, there is guaranteed to be an algebraic extension $F(\alpha)/F$ such that $F \subseteq F(\alpha) \subseteq R \subseteq K$. But $F(\alpha)$ is a field and so $\alpha^{-1} \in F(\alpha) \in R$.

4). Let K/F be a field extension and supposed $\alpha \in K$ is algebraic over F. Show that if $\deg(m_{\alpha,F}(x))$ is odd then $F(\alpha^2) = F(\alpha)$.

If $\alpha \in F$ then done, so AWLOG: $\alpha \notin F$.

Assume $\deg(m_{\alpha,F}(x))$ is odd, and thus $[F(\alpha) : F]$ is odd.
ABC: $F(\alpha^2) \neq F(\alpha)$.
This means that $\alpha \notin F(\alpha^2)$, but $F(\alpha)$ is a ring so $\alpha^2 \in F(\alpha)$. So consider the following extensions:

$$\begin{tikzcd} F(\alpha) \arrow[d, dash] \\ F(\alpha^2) \arrow[d, dash] \\ F \end{tikzcd}$$

Since α is a root of $x^2 - \alpha^2$, this means that $[F(\alpha) : F(\alpha^2)] \leq 2$. But since $\alpha \notin F(\alpha^2)$ this means that $[F(\alpha) : F(\alpha^2)] \geq 2$. Thus $[F(\alpha) : F(\alpha^2)] = 2$:

$$\begin{tikzcd}[arrows=dash]
F(\alpha) \arrow[d, "2"] \\
F(\alpha^2) \arrow[d] \\
F
\end{tikzcd}$$

Thus, $[F(\alpha) : F]$ has a power of 2 in it and must be even - contradiction.

Therefore, $F(\alpha^2) = F(\alpha)$.

5). Find the splitting field $K \subseteq \mathbb{C}$ for $x^3 - 2$ over $\mathbb{Q}$ and determine all subfields of K.

First, find all roots of $x^3 - 2$ in $\mathbb{C}$:

$$x^3 = 2 = 2e^{i2\pi n}$$

$$x = \sqrt[3]{2}e^{i\frac{2\pi}{3}n}$$

$$x = \sqrt[3]{2}, \sqrt[3]{2}\omega, \sqrt[3]{2}\omega^2$$

And therefore $x^3 - 2$ splits in $Q(\sqrt[3]{2}, \omega)$.

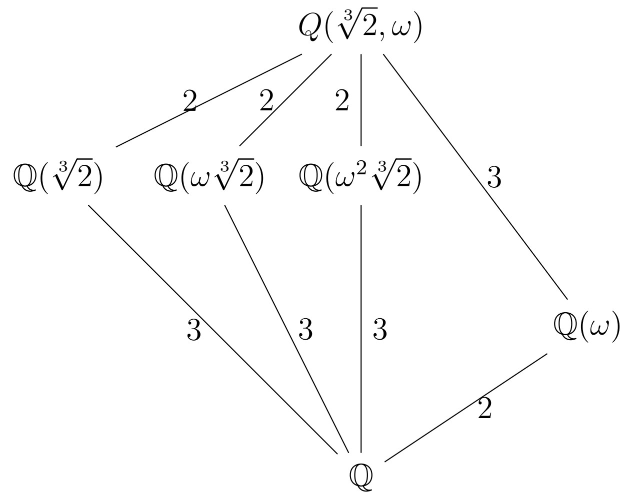
We have already showed that $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$, so consider the field extensions:

$$\begin{tikzcd}[arrows=dash]
Q(\sqrt[3]{2}, \omega) \arrow[d] \\
\mathbb{Q}(\sqrt[3]{2}) \arrow[d, "3"] \\
\mathbb{Q}
\end{tikzcd}$$

Since $x^3 - 2$ has 3 roots in $Q(\sqrt[3]{2}, \omega)$ we know that $[Q(\sqrt[3]{2}, \omega) : \mathbb{Q}] \leq 3! = 6$. But clearly $\omega \notin \mathbb{Q}(\sqrt[3]{2})$ and so $\mathbb{Q}(\sqrt[3]{2})$ is properly contained in $Q(\sqrt[3]{2}, \omega)$. Since $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ must divide $[Q(\sqrt[3]{2}, \omega) : \mathbb{Q}]$, which can only be 4, 5, or 6, this forces $[Q(\sqrt[3]{2}, \omega) : \mathbb{Q}] = 6$.

The subfields are derived from the roots. Guard against duplicates:

$\sqrt[3]{2}, \sqrt[3]{4}$
ω, ω^2
$\omega\sqrt[3]{2}, \omega^2\sqrt[3]{4}$
$\omega^2\sqrt[3]{2}, \omega\sqrt[3]{4}$


$Q(\sqrt[3]{2}, \omega)$
$\mathbb{Q}(\sqrt[3]{2})$
$\mathbb{Q}(\omega\sqrt[3]{2})$
$\mathbb{Q}(\omega^2\sqrt[3]{2})$
$\mathbb{Q}(\omega)$
$\mathbb{Q}$
2
2
2
3
3
3
3
2