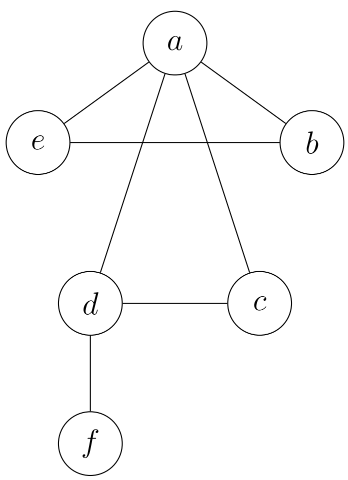
Subgraphs

Definition: Subgraph

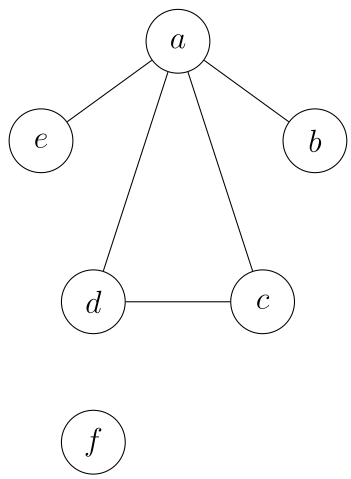
Let G and H be graphs:

- To say that H is a *subgraph* of G, denoted by $H \subseteq G$, means that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
- To say that H is a *proper* subgraph of G, denoted by $H \subset G$, means that $H \subseteq G$ but $H \neq G$. Thus, $V(H) \subset V(G)$ or $E(H) \subset E(G)$.
- To say that H is a *spanning* subgraph of G means that $V(H) = V(G)$ and $E(H) \subseteq E(G)$.

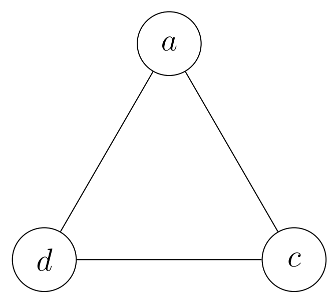
Example



G



H



F

$$F \subset H \subset G$$

H is a spanning subgraph of G, but not F because $b, e, f \in V(G)$; however, $b, e, f \notin V(F)$.

Definition: Induced

Let G be a graph and let $S \subseteq V(G), S \neq \emptyset$. The subgraph of G *induced* by S, denoted by $G[S]$, is a graph H such that $V(H) = S$ and for all $e \in E(G)$, $e \in E(H)$ iff the endpoints of e are contained in S. Such a graph H is called an induced subgraph of G:

$H \subseteq G$ and $H = G[V(H)]$

In the case of a simple graph, H is an induced subgraph of G means:

1. $V(H) = S$
2. $E(H) = E(G) \cap \mathcal{P}_2(V(H))$

In other words, $u, v \in V(H)$ and $uv \in E(G) \implies uv \in E(H)$.

In the above example, F is an induced subgraph of G; however, H is not because $b.e, d, f \in V(H)$ and $be, df \in E(G)$ but $be, df \notin E(H)$.

Definition: Edge-induced

Let G be a graph and let $X \subseteq E(G), X \neq \emptyset$. The subgraph of G *edge-induced* by X, denoted by $G[X]$, is a graph H such that:

1. $V(H) = \{v \in V(G) \,|\, \exists\, e \in X, v$ is incident to $e\}$
2. $E(H) = X$

Such a graph H is called an edge-induced subgraph of G:

$H \subseteq G$ and $H = G[E(H)]$

Note that in the above example, F is an edge-induced subgraph of G; however, H is not because $f \in V(H)$ but there is no edge in $E(H)$ that is incident to f.

Notation

$G - v$	$v \in V(G)$	The proper induced subgraph $G\,[V(G) - \{v\}]$
$G - S$	$S \subset V(G)$	The proper induced subgraph $G\,[V(G) - S]$
$G - e$	$e \in E(G)$	The proper spanning subgraph of G with edge e removed.
$G - X$	$X \subseteq E(G)$	The proper spanning subgraph of G with all edges in X removed.
$G + e$	$e \notin E(G)$	The graph with vertices $V(G)$ and edges $E(G) \cup \{e\}$, of which G is a proper spanning subgraph.