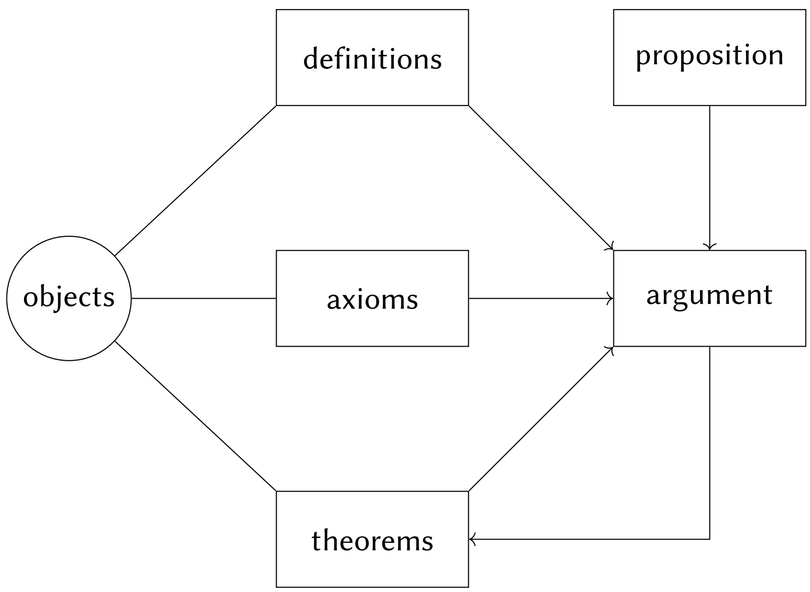
Mathematical Systems

Question: Does mathematics represent some absolute truth about the universe?

It seems to. We use mathematics to solve problems in science, technology, business, medicine, and practically every area of life. But to answer this question properly we need to define what is meant by a *mathematical system*.

Definition: Mathematical System

A *mathematical system* is composed of a set of objects, a set of *definitions* that describe the nature of the objects, a set of inherent operations (*axioms*) that can be performed on the objects, and new true propositions about the objects that are proved with logical *arguments* using the definitions, axioms, and previously proved propositions (*theorems*).



So a mathematical system expresses relative truth based on the selected definitions and axioms. Thus, we attempt to construct the definitions and axioms to reflect perceived reality.

Example: The Real Number System

The real number system is constructed as follows:

1. Specify the objects (real numbers) by definitions for natural numbers, zero, integers, rational numbers, and irrational numbers.
2. Add a definition for the notion of equality.
3. Add definitions for the binary operations of addition and multiplication.
4. Add axioms for precedence rules (multiplication before addition).

5. Add the so-called *field* axioms:

   For all real numbers a, b, and c:

   Additive Commutativity: $a + b = b + a$

   Multiplicative Commutativity: $ab = ba$

   Additive Associativity: $(a + b) + c = a + (b + c)$

   Multiplicative Associativity: $(ab)c = a(bc)$

   Additive Identity: There exists a real number 0 such that for every real number a:

   $$0 + a = a$$

   Multiplicative Identity: There exists a real number 1 such that for every real number a:

   $$1a = a$$

   Additive Inverse: For every real number a there exists a real number $-a$ such that

   $$a + (-a) = 0$$

   Multiplicative Inverse: For every non-zero real number a there exists a real number $\frac{1}{a}$ such that:

   $$a\left(\frac{1}{a}\right) = 1$$

   Distributivity: $a(b + c) = ab + ac$

6. Extend the system with theorems. For example, consider the proposition: $(b+c)a = ba+ca$:

$(b + c)a = a(b + c)$	Multiplicative Commutativity
$(b + c)a = ab + ac$	Distributivity
$(b + c)a = ba + ca$	Multiplicative Commutativity

Arguments are made using $Boolean$ logic, which is itself a mathematical system. An argument starts with the statement of a *theorem* and is accepted as either true or false based on a *proof*.