Banach-Steinhaus Theorem

Theorem: Principle of Uniform Boundedness

Let:

- X be a Banach space.
- Y be a normed space.
- $\mathcal{T} \subseteq \mathcal{B}(X, Y)$
- $\forall \vec{x} \in X, \exists\, M_x > 0, \forall T \in \mathcal{T}, \|Tx\| \leq M_x \|\vec{x}\|$

$\exists\, M > 0, \forall T \in \mathcal{T}, \|T\| \leq M$

Proof

Let $A_n = \{x \in X \mid \forall T \in \mathcal{T}, \|T\vec{x}\| \leq n\}$.

Claim: $\displaystyle\bigcup_{n=1}^{\infty} A_n = X$

($\subseteq$): Assume $\vec{x} \in \displaystyle\bigcup_{n=1}^{\infty} A_n$

Clearly, $\vec{x} \in X$.

($\supseteq$): Assume $\vec{x} \in X$

$\exists\, M_x > 0, \forall T \in \mathcal{T}, \|T\vec{x}\| \leq M_x \|\vec{x}\| \leq n$
Thus $\exists\, n \in \mathbb{N}$ such that $\vec{x} \in A_n$.

Therefore $\vec{x} \in \displaystyle\bigcup_{n=1}^{\infty} A_n$

$\therefore \displaystyle\bigcup_{n=1}^{\infty} A_n = X$

Clearly, $A_n \subset A_{n+1}$.

Let $A_{n,T} = \{\vec{x} \in X \mid \|T\vec{x}\| \leq n\}$.

Claim: $A_{n,T}$ is closed.

Assume $(\vec{x}_k)$ is a sequence in $A_{n,T}$ such that $\vec{x}_k \to \vec{x} \in X$.
But T is bounded, and thus continuous, so $\vec{x}_k \to \vec{x} \implies T\vec{x}_k \to T\vec{x}$.
$\|T\vec{x}_k\| = \|(T\vec{x}_k - T\vec{x}) + T\vec{x}\| \to \|T\vec{x}\| \leq n$

Therefore, $\vec{x} \in A_{n,T}$ and thus $A_{n,T}$ is closed.

But $A_n = \displaystyle\bigcap_{T \in \mathcal{T}} A_{n,T}$, an intersection of closed sets.

Therefore, A_n is closed.

Now, X is Banach by assumption, and thus is a Baire space.
So by the Baire Category Theorem, $\exists N \in \mathbb{N}$ such that A_N has a non-empty interior.
Thus, $\exists \vec{x}_0 \in A_N$ and $r > 0$ such that $\overline{B}(\vec{x}_0, r) \subset A_N$.
And so $\forall \vec{x} \in \overline{B}(\vec{x}_0, r), \forall T \in \mathcal{T}, \|T\vec{x}\| \leq N$.

Assume $T \in \mathcal{T}$.
Assume $\vec{x} \in X$ such that $\|\vec{x}\| \leq r$.
So $\vec{x} + \vec{x}_0 \in \overline{B}(\vec{x}_0, r)$.
$\|T\vec{x}\| = \|T(\vec{x} + \vec{x}_0 - \vec{x}_0)\| = \|T(\vec{x} + \vec{x}_0) - T(\vec{x}_0)\| \leq \|T(\vec{x} + \vec{x}_0)\| + \|T(\vec{x}_0)\| \leq 2N$

Assume $\|\vec{u}\| = 1$.
$\|T\vec{u}\| = \left\|T\left(\frac{1}{r}(r\vec{u})\right)\right\| = \frac{1}{r}\|T(r\vec{u})\| \leq \frac{2N}{r}$
Let $M = \frac{2N}{r}$.
$\therefore \|T\| = \sup\limits_{\|\vec{x}\|=1} \|T\vec{u}\| \leq M$.

Corollary

Let:

- X be a Banach space.
- Y be a normed space.
- (T_n) be a sequence in $\mathcal{B}(X, Y)$.
- $\forall \vec{x} \in X, T\vec{x} = \lim\limits_{n\to\infty} T_n\vec{x}$ exists.

T is a linear, bounded map.

Proof

Assume $\vec{x}, \vec{y} \in X$ and $\alpha, \beta \in \mathbb{F}$:

$$\begin{aligned} T(\alpha\vec{x} + \beta\vec{y}) &= \lim_{n\to\infty} T_n(\alpha\vec{x} + \beta\vec{y}) \\ &= \lim_{n\to\infty} [\alpha T_n\vec{x} + \beta T_n\vec{y}] \\ &= \alpha \lim_{n\to\infty} T_n\vec{x} + \beta \lim_{n\to\infty} T_n\vec{y} \\ &= \alpha T\vec{x} + \beta T\vec{y} \end{aligned}$$

Therefore, T is linear.

Since $(T_n\vec{x})$ converges, it is bounded.
Thus, $\forall \vec{x} \in X, \exists M_x > 0, \forall n \in \mathbb{N}, \|T_n\vec{x}\| \leq M_x \|\vec{x}\|$.
So, all the conditions for uniform boundedness are satisfied.

Thus, $\exists\, M > 0, \forall\, n \in \mathbb{N}, \|T_n\| \leq M$.
And so $\forall\, \vec{x} \in X$ such that $\|\vec{x}\| = 1$, $\|T_n\vec{x}\| \leq M$.

Therefore, $\|T\vec{x}\| < M$ and thus T is bounded.

Note that this does not imply that $T_n \to T$.