Eigenvalues of Self-adjoint Operators

Theorem

Let A be a self-adjoint operator on a Hilbert space H:

> All of the eigenvalues for A are real.

Proof

Assume λ is an eigenvalue of A.
$\exists\, \vec{x} \neq \vec{0}$ such that $A\vec{x} = \lambda\vec{x}$.

$\langle A\vec{x}, \vec{x}\rangle = \langle \lambda\vec{x}, \vec{x}\rangle = \lambda \left\|\vec{x}\right\|^2$

$\langle \vec{x}, A\vec{x}\rangle = \langle \vec{x}, \lambda\vec{x}\rangle = \overline{\lambda} \left\|\vec{x}\right\|^2$

But A is self-adjoint, and so $\langle A\vec{x}, \vec{x}\rangle = \langle \vec{x}, A\vec{x}\rangle$.
$\lambda \left\|\vec{x}\right\|^2 = \overline{\lambda} \left\|\vec{x}\right\|^2$
$\lambda = \overline{\lambda}$

$\therefore \lambda \in \mathbb{R}$.

Theorem

Let A be a bounded self-adjoint operator on a Hilbert space H:

$$r(A) = \|A\|$$

Proof

It is already shown that $r(A) \leq \|A\|$, so it suffices to show that $\exists\, \lambda \in \sigma(A)$ such that $|\lambda| = \|A\|$.

It is already shown that $\|A\| = \sup\limits_{\|\vec{x}\|=1} |\langle A\vec{x}, \vec{x}\rangle|$ and $\langle A\vec{x}, \vec{x}\rangle \in \mathbb{R}$.
Thus, there exists a sequence $(\vec{x}_n)$ in H such that $\|\vec{x}_n\| = 1$ and $|\langle A\vec{x}_n, \vec{x}_n\rangle| \to \|A\|$.

Assume $\langle A\vec{x}, \vec{x}\rangle \to \lambda$ where $|\lambda| = \|A\|$.
Since $A = A^*$ and $\lambda \in \mathbb{R}$:

$$\begin{array}{rcl}
\|A\vec{x}_n - \lambda\vec{x}_n\| & = & \langle A\vec{x}_n - \lambda\vec{x}_n, A\vec{x}_n - \lambda\vec{x}_n\rangle \\
& = & \langle A\vec{x}_n, A\vec{x}_n\rangle - \langle A\vec{x}_n, \lambda\vec{x}_n\rangle - \langle \lambda\vec{x}_n, A\vec{x}_n\rangle + \langle \lambda\vec{x}_n, \lambda\vec{x}_n\rangle \\
& = & \left\|A\vec{x}_n\right\|^2 + \lambda^2 \left\|vx_n\right\|^2 - \lambda \langle A\vec{x}_n, \vec{x}_n\rangle - \lambda \langle A\vec{x}_n, \vec{x}_n\rangle \\
& = & \left\|A\vec{x}_n\right\|^2 + \lambda^2 - 2\lambda \langle A\vec{x}_n, \vec{x}_n\rangle \\
& \leq & \left\|A\right\|^2 \left\|\vec{x}_n\right\|^2 + \left\|A\right\|^2 - 2\lambda \langle A\vec{x}_n, \vec{x}_n\rangle \\
& \leq & \left\|A\right\|^2 + \left\|A\right\|^2 - 2\lambda \langle A\vec{x}_n, \vec{x}_n\rangle \\
& \leq & 2\left\|A\right\|^2 - 2\lambda \langle A\vec{x}_n, \vec{x}_n\rangle \\
& \to & 2\left\|A\right\|^2 - 2\lambda^2 \\
& = & 2\left\|A\right\|^2 - 2\left\|A\right\|^2 \\
& = & 0
\end{array}$$

Thus, $A\vec{x}_n \to \lambda\vec{x}_n$.

ABC: $\lambda \in \rho(A)$.
$\|A\| \leq |\lambda|$
$A \in \mathcal{B}(H)$ and $\|A\| \leq |\lambda| \implies A_\lambda = (A - \lambda I)^{-1} \in \mathcal{B}(H)$.

$1 = \|\vec{x}\| = \|I\vec{x}\| = \left\|(A - \lambda I)^{-1}(A - \lambda I)\vec{x}\right\| \leq \left\|(A - \lambda I)^{-1}\right\| \|(A - \lambda I)\vec{x}\| \to 0$

CONTRADICTION!

Thus, $\lambda \notin \rho(A)$ and so $\lambda \in \sigma(A)$ and $|\lambda| = \|A\|$.

$\therefore r(A) = \sup_{\|\vec{x}\|=1} \{|\lambda| \mid \lambda \in \sigma(A)\} = |\lambda| = \|A\|$.