Quadratic Number Fields

Definition: Squarefree

To say that and integer $d \neq 1$ is *squarefree* means no prime p exists such that $p^2 \mid d$.

Definition: Quadratic Number Field

Let $d \neq 1$ be a squarefree integer. The *quadratic number field* associated to d is given by:

$$\mathbb{Q}(\sqrt{d}) = \{r + s\sqrt{d} \mid r, s \in \mathbb{Q}\}$$

Note that since d is squarefree it is irrational.

When $d > 0$ then $\mathbb{Q}(\sqrt{d})$ is said to be real.

When $d < 0$ then $\mathbb{Q}(\sqrt{d})$ is said to be imaginary.

Note that $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}[\sqrt{d}]$.

Theorem

$\mathbb{Q}(\sqrt{d})$ is a field.

Proof

Assume $\alpha, \beta \in \mathbb{Q}(\sqrt{d})$
Let $\alpha = r + s\sqrt{d}$ and $\beta = u + v\sqrt{d}$ for $r, s, u, v \in \mathbb{Q}$

$\alpha + \beta = (r + s\sqrt{d}) + (u + v\sqrt{d}) = (r + u) + (s + v)\sqrt{d} \in \mathbb{Q}(\sqrt{d})$

$\alpha\beta = (r + s\sqrt{d})(u + v\sqrt{d}) = (ru + svd) + (rv + su)\sqrt{d} \in \mathbb{Q}(\sqrt{d})$

So, $\mathbb{Q}(\sqrt{d})$ is closed under the operations and is thus a subring of $\mathbb{C}$, and thus an integral domain.

Now, assume $\alpha \neq 0$
$\frac{1}{\alpha} = \frac{1}{r+s\sqrt{d}} = \frac{r-s\sqrt{d}}{r^2-ds^2} = \frac{r}{r^2-ds^2} - \frac{s}{r^2-ds^2}\sqrt{d} \in \mathbb{Q}(\sqrt{d})$

So, $\mathbb{Q}(\sqrt{d})$ is closed under inverses

Therefore $\mathbb{Q}(\sqrt{d})$ is a field.