Implication

A mathematical system is extended by collecting a set of facts called *preconditions* and showing that if the preconditions hold (are true) then some other fact must also be true. This is called *implication*. This is expressed using an if-then construct.

Examples

If $x = 2$ then $x^2 = 4$.

If x is an even number then x^2 is an even number.

Definition: Implication

Let p and q be propositions. The *conditional* statement: "if p then q," also called *implication* and denoted by $p \to q$, is the proposition that is false when p is true and q is false and true otherwise. p is called the *hypothesis* or *antecedent* and q is called the *conclusion* or *consequence*.

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

Note that $p \to q = \neg p \vee q = \bar{p} + q$.

Implications can be stated in multiple ways:

- If p then q.
- p implies q.
- p is sufficient for q.
- p only if q.
- q is necessary for p.
- q when p.
- q unless $\neg p$.

Example

If $x = 2$ then $x^2 = 4$.

If $x^2 = 4$ then we cannot conclude that $x = 2$ because $x = \pm 2$.

If the hypothesis is false, then the truth value of the consequence doesn't matter.

Example

If you are in this class then you are under 7 feet tall.

If the hypothesis is false, meaning a person is not in this class, then that person may or may not be 7 feet tall.

Note that conditionals in logic are different from if statements in computer languages, which act as guards to blocks of statements.

Definition: Implication Forms

Let p and q be propositions:

- Implication: $p \to q$
- Inverse: $\neg p \to \neg q$
- Converse: $q \to p$
- Contrapositive $\neg q \to \neg p$

From the truth tables: the implication is equal to the contrapositive and the inverse is equal to the converse.

p	q	$\neg p \to \neg q$
F	F	T
F	T	F
T	F	T
T	T	T

p	q	$q \to p$
F	F	T
F	T	F
T	F	T
T	T	T

p	q	$\neg q \to \neg p$
F	F	T
F	T	T
T	F	F
T	T	T

Conditionals are unidirectional. A bidirectional implication is an equivalence.

Definition: Equivalence

Let p and q be propositions. The *biconditional* statement: "p if and only if q," also called *equivalence* and denoted by $p \leftrightarrow q$ or p iff q, is the proposition that is true when p and q have the same truth value and false otherwise.

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Note that $p \leftrightarrow q = (p \to q) \wedge (q \to p) = \neg(p \oplus q)$.

The complete preference table is as follows:

$\neg$
$\wedge$
$\vee$
$\rightarrow$
$\leftrightarrow, \oplus$