Complex Field

Theorem

$\mathbb{C}$ is a field with:

- additive identity: $0 = (0, 0)$
- additive inverse: $-z = (-x, -y)$
- multiplicative identity: $1 = (1, 0)$
- multiplicative inverse: $z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$

Proof

1). Additive Commutivity

$$
\begin{aligned}
z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\
&= (x_1 + x_2) + i(y_1 + y_2) \\
&= (x_2 + x_1) + i(y_2 + y_1) \\
&= (x_2 + iy_2) + (x_1 + iy_1) \\
&= z_2 + z_1
\end{aligned}
$$

2). Additive Associativity

$$
\begin{aligned}
(z_1 + z_2) + z_3 &= [(x_1 + iy_1) + (x_2 + iy_2)] + (x_3 + iy_3) \\
&= [(x_1 + x_2) + i(y_1 + y_2)] + (x_3 + iy_3) \\
&= [(x_1 + x_2) + x_3] + i[(y_1 + y_2) + y_3] \\
&= [x_1 + (x_2 + x_3)] + i[y_1 + (y_2 + y_3)] \\
&= (x_1 + iy_1) + [(x_2 + x_3) + i(y_2 + y_3)] \\
&= (x_1 + iy_1) + [(x_2 + iy_2) + (x_3 + iy_3)] \\
&= z_1 + (z_2 + z_3)
\end{aligned}
$$

3). Additive Identity

$$
\begin{aligned}
z + 0 &= (x + iy) + (0 + i0) \\
&= (x + 0) + i(y + 0) \\
&= x + iy \\
&= z
\end{aligned}
$$

4). Additive Inverse

$$\begin{aligned}
z + (-z) &= 0 \\
(x + iy) + (u + iv) &= 0 + i0 \\
(x + u) + i(y + v) &= 0 + i0 \\
x + u &= 0 \\
y + v &= 0 \\
u &= -x \\
v &= -y
\end{aligned}$$

$$\therefore \forall z \in \mathbb{C}, \exists (-z) = (-x, -y), z + (-z) = 0$$

5). Multiplicative Commutivity

$$\begin{aligned}
z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\
&= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \\
&= (x_2 x_1 - y_2 y_1) + i(x_2 y_1 + x_1 y_2) \\
&= (x_2 + iy_2)(x_1 + iy_1) \\
&= z_2 z_1
\end{aligned}$$

6). Multiplicative Associativity

$$\begin{aligned}
(z_1 z_2) z_2 &= [(x_1 + iy_1)(x_2 + iy_2)](x_3 + iy_3) \\
&= [(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)](x_3 + iy_3) \\
&= [(x_1 x_2 - y_1 y_2) x_3 - (x_1 y_2 + x_2 y_1) y_3] + i[(x_1 x_2 - y_1 y_2) y_3 + (x_1 y_2 + x_2 y_1) x_3] \\
&= (x_1 x_2 x_3 - y_1 y_2 x_3 - x_1 y_2 y_3 - x_2 y_1 y_3) + i(x_1 x_2 y_3 - y_1 y_2 y_3 + x_1 y_2 x_3 + x_2 y_1 x_3) \\
&= [x_1 (x_2 x_3 - y_2 y_3) - y_1 (x_2 y_3 + x_3 y_2)] + i[x_1 (x_2 y_3 + x_3 y_2) + y_1 (x_2 x_3 - y_2 y_3)] \\
&= (x_1 + iy_1)[(x_2 x_3 - y_2 y_3) + i(x_2 y_3 + x_3 y_2)] \\
&= (x_1 + iy_1)[(x_2 + iy_2)(x_3 + iy_3)] \\
&= z_1 (z_2 z_3)
\end{aligned}$$

7). Multiplicative Identity

$$\begin{aligned}
1z &= (1 + i0)(x + iy) \\
&= (x - 0) + i(y + 0) \\
&= x + iy \\
&= z
\end{aligned}$$

8). Multiplicative Inverse

$$\begin{aligned}
zz^{-1} &= 1 \\
(x+iy)(u+iv) &= 1+i0 \\
(xu-yv)+i(xv+yu) &= 1+i0 \\
xu-yv &= 1 \\
yu+xv &= 0 \\
u &= \frac{\begin{vmatrix} 1 & -y \\ 0 & x \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = \frac{x}{x^2+y^2} \\
v &= \frac{\begin{vmatrix} x & 1 \\ y & 0 \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = \frac{-y}{x^2+y^2}
\end{aligned}$$

$$\therefore \forall z \in \mathbb{C}-\{0\}, \exists z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right), zz^{-1} = 1$$

9). Distribution

$$\begin{aligned}
z_1(z_2+z_3) &= (x_1+iy_1)[(x_2+iy_2)+(x_3+iy_3)] \\
&= (x_1+iy_1)[(x_2+x_3)+i(y_2+y_3)] \\
&= [x_1(x_2+x_3)-y_1(y_2+y_3)]+i[x_1(y_2+y_3)+y_1(x_2+x_3)] \\
&= (x_1x_2+x_1x_3-y_1y_2-y_1y_3)+i(x_1y_2+x_1y_3+y_1x_2+y_1x_3) \\
&= [(x_1x_2-y_1y_2)+i(x_1y_2+y_1x_2)]+[(x_1x_3-y_1y_2)+i(x_1y_2+y_1x_3)] \\
&= (x_1+iy_1)(x_2+iy_2)+(x_1+iy_1)(x_3+iy_3) \\
&= z_1z_2+z_1z_3
\end{aligned}$$

Since $\mathbb{C}$ is a field, all of the following properties also hold:

1). Uniqueness

a). The additive identity (0) is unique.

b). Additive inverses ($-z$) are unique.

c). The multiplicative identity (1) is unique.

d). Multiplicative inverses (z^{-1}) are unique.

2). Properties of 0

a). $0 = -0$

b). $z0 = 0$

c). $z_1 z_2 = 0 \iff z_1 = 0$ or $z_2 = 0$

3). Properties of Additive Inverses

a). $-z = (-1)z$

b). $-(-z) = z$

c). $(-z_1)z_2 = z_1(-z_2) = -(z_1 z_2)$

d). $(-z_1)(-z_2) = z_1 z_2$

e). $-(z_1 + z_2) = -z_1 + (-z_2)$