Cycles

Definition

To say that $\sigma \in S_n$ is a *cycle* means that it has at most one orbit that contains more than one element.

The *length* of a cycle is the number of elements in the cycle.

To say that two cycles are *disjoint* means that they contain no common elements.

Theorem

Every permutation σ of a finite set is a product of disjoint cycles.

Proof

Let $B_1, B_2, \ldots, B_r$ be the orbits of σ
The B_k are equivalence classes and are thus disjoint
Let $\mu_k(x) = \begin{cases} \sigma(x), & x \in B_k \\ x, & x \notin B_k \end{cases}$
$\sigma = \prod_{k=1}^{r} \mu_k$
But the μ_k are disjoint cycles
$\therefore \sigma$ is a product of disjoint cycles.

Corollary

Composition of disjoint cycles is commutative.

Theorem

The order of a cycle of length n is n.

Corollary

The order of a permutation is the least common multiple of the orders of its disjoint cycles.

Theorem

Let σ be a cycle represented by $(i_1 i_2 \ldots i_{n-1} i_n)$. The inverse of σ is given by:

$$\sigma^{-1} = (i_n i_{n-1} \ldots i_2 i_1)$$