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Math 221b
Homework #5

1). Show that every non-zero prime ideal in a PID R is maximal.

Assume R is a PID and $a \in R$ such that $a \neq 0$ and (a) is prime.
Since (a) is prime, by definition it must be a proper ideal.
ABC: (a) is not maximal.
Since a PID is an integral domain, there exists $b \in R$ such that (a) is a proper subset of (b) and (b) is maximal in R.
To contain is to divide, so $b \mid a$.
So $\exists\, c \in R$ such that $a = cb \in (a)$.
But (a) is prime, so $b \in (a)$ or $c \in (a)$.

Case 1: $b \in (a)$

Let $b = as$ for some $s \in R$
Assume $d \in (b)$
Let $d = br$ for some $r \in R$.
Note that by closure, $sr \in R$, and so:
$d = (as)r = a(sr) \in (a)$
Thus $(b) \subseteq (a)$
Contradiction.

Case 2: $c \in (a)$

Then $\exists\, d \in R$ such that $c = ad$.
So $a = (ad)b = a(db)$ and so $db = 1$.
Thus b is a unit and $(b) = R$.
Contradiction.

Therefore (a) is maximal.

A shorter proof might be:

Assume R is a PID and $a \in R$ such that $a \neq 0$ and (a) is prime.
Since (a) is prime, by definition it must be a proper ideal.
Since R is a PID, prime and irreducible are the same thing.
So a is prime and irreducible and thus has no non-trivial factorization.
To divide is to contain, so since a has no non-trivial divisors there is no containing proper ideal other than R.
Therefore (a) is maximal.

2). Let R be a ring and $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ be a chain of ideals in R. Prove:

$$I = \bigcup_{k=1}^{\infty} I_k \trianglelefteq R$$

Clearly, I is a non-empty subset of R

Assume $a, b \in I$
$\exists\, i, j \in \mathbb{Z}^+$ such that $a \in I_i$ and $b \in I_j$
AWLOG: $I_i \subseteq I_j$
So also $a \in I_j$
I_j is a group so $(-b) \in I_j$ and by closure $a - b \in I_j$
But $I_j \subseteq I$, so $a - b \in I$

Therefore, by the subgroup test, I is a subgroup of R. Furthermore, since R is an additive abelian group, so is I.

Since $a \in I_i$ and $I_i \subseteq I$ we have $a \in I$ as well
Assume $r \in R$
I_i is an ideal, and so $ar \in I_i$ and thus $ar \in I$
Likewise, $ra \in I$

Therefore, $I \trianglelefteq R$.

3). Show that an integral domain R is Noetherian iff every ideal is finitely-generated.

$\Longrightarrow$ Assume R is Noetherian:

Assume $I \trianglelefteq R$.
Assume $a_1 \in I$.
If $(a_1) = I$ then I is finitely-generated, so done.
Otherwise, choose $a_2 \in I \setminus (a_1)$.
If $(a_1, a_2) = I$ then I is finitely-generated, so done.
Continue in this fashion as long as the generated ideal does not equal I, which creates the chain:

$$(a_1) \subset (a_1, a_2) \subset (a_1, a_2, a_3) \subset \cdots$$

But R is Noetherian, so there exists $k \in \mathbb{Z}^+$ such that the chain stabilizes after k steps. At that point, $(a_1, \ldots, a_k) = I$, otherwise, another step could be performed.

Therefore, I is finitely-generated with k generators.

$\Longleftarrow$ Assume every ideal in R is finitely-generated:

Assume $\mathcal{C}$ is an ascending chain of ideals in R:

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

By the result of problem (2):

$$I = \bigcup_{k=1}^{\infty} I_k \trianglelefteq R$$

Furthermore, by assumption, I is finitely-generated, so let $I = (a_1, a_2, \ldots, a_n)$.
For each a_i in the generating set, pick a ideal in the chain where it occurs and identify that ideal by I_{k_i}.
Let $k = \max\{k_i\}$
So by the I_k ideal, all generators have been included and the chain must thus stabilize.

Therefore R is Noetherian.

4). Show that $\mathbb{Z}[\omega] = R_{-3}$ (the ring of Eisenstein integers) is a Euclidean domain with Euclidean function:

$$N(a + b\omega) = a^2 - ab + b^2$$

Let $a, b \in \mathbb{Z}[\omega]$. By the division algorithm and working in $\mathbb{Q}[\omega]$ (the field of fractions), we have:

$$\frac{a}{b} = q + \frac{r}{b}$$

$$\frac{a}{b} - q = \frac{r}{b}$$

We want q to be close to $\frac{a}{b}$ such that:

$$N(\frac{a}{b} - q) = N(\frac{r}{b}) < 1$$

so that we get the desired condition for $N(r) < N(b)$. So, let $\frac{a}{b} = n_1 + n_2\omega$ and $q = q_1 + q_2\omega$ and try the condition:

$$|n_1 - q_1| \leq \frac{1}{2} \text{ and } |n_2 - q_2| \leq \frac{1}{2}$$

Now, calculate the resulting norm:

$$\begin{aligned}
N(\frac{a}{b} - q) &= N((n_1 + n_2\omega) - (q_1 + q_2\omega)) \\
&= N((n_1 - q_1) + (n_2 - q_2)\omega) \\
&= (n_1 - q_1)^2 + (n_2 - q_2)^2 - (n_1 - q_1)(n_2 - q_2) \\
&\leq \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) \\
&= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \\
&= \frac{3}{4} \\
&< 1
\end{aligned}$$

Thus resulting in the desired condition.

Therefore, $\mathbb{Z}[\omega]$ is a Euclidean domain under the norm function.

5). Let R be a Euclidean domain with a nice Euclidean function $d : R^* \to \mathbb{N}_0$. Show that if $a \mid b$ and $d(a) = d(b)$ for some $a, b \in R$ then a and b are associates.

Since a divides b, let $\frac{a}{b} = q \in R$. Now, since d is multiplicative:

$$d\left(\frac{a}{b}\right) = \frac{d(a)}{d(b)} = 1$$

So $d(q) = 1$ and thus q is a unit in R. So:

$$a = qb \text{ and } b = q^{-1}a$$

Therefore a and b are associates.