Statistics and their distributions

– Math 161a, Spring 2019, San Jose State University

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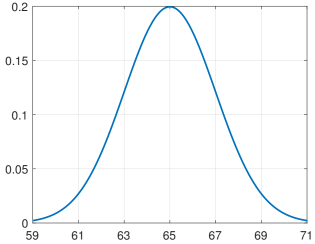
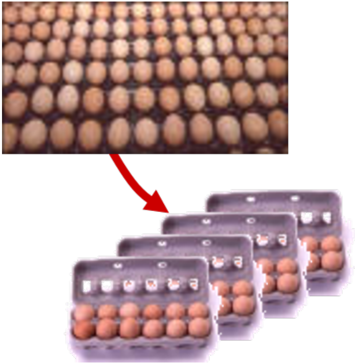
Outline

Introduction

Suppose that the weights (in grams) of all brown eggs produced at a local farm have a normal distribution: $X \sim N(65, 2^2)$.



Those eggs are divided into cartons of 12 each, to be sold on the market.



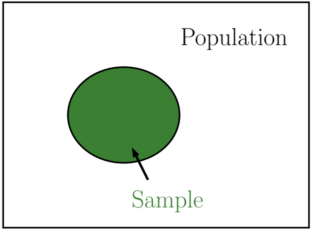
You can randomly select a carton and measure the weights of all the eggs in it. Let $\bar{X}$ be their **average weight**.

$\bar{X}$ clearly may vary from carton to carton, thus it is a (continuous) random variable.

Question: What is the distribution of $\bar{X}$?

The above problem is about the **sampling distribution of a statistic**.

- **Population**: all brown eggs produced at the local farm
- **Sample**: a carton of 12 eggs
- **Statistic**: average weight of eggs in the carton



To study the distribution of $\bar{X}$, we denote individual weights of the selected eggs as $X_1, \ldots, X_{12}$. We can then obtain a formula for $\bar{X}$:

$$\bar{X} = \frac{X_1 + \cdots + X_{12}}{12}.$$

What we know about $X_1, \ldots, X_{12}$:

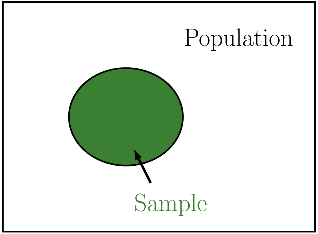
They are independent random variables, and all have the identical distribution

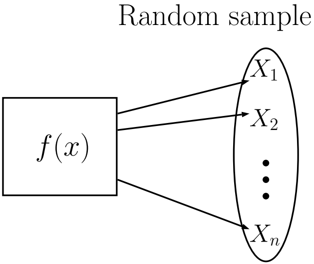
$$X_1, \ldots, X_{12} \stackrel{\text{iid}}{\sim} N(65, 2^2)$$

Such *identically and independently distributed (iid)* random variables are called a **random sample** (of size 12) from the normal distribution $N(65, 2^2)$.

Remark. There are two kinds of samples that are associated to each other:

- physical sample (carton of 12 eggs) from a physical population (all brown eggs produced at the farm)
- (hypothetical) random sample from a distribution





Random sample

Definition 0.1. More generally, a collection of n random variables $X_1, \ldots, X_n$ is called a random sample if

(1) They are identically distributed according to pmf/pdf $f(x)$, and

(2) They are independent.

In short, we write $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} f(x)$.

Remark. Random samples are very common, e.g., heights/salaries of randomly selected individuals from a large population, final exam scores of this class (regarded as a sample of all the 161A students), etc.

Sometimes, there may be no physical sample associated to the random sample.

Example 0.1. Suppose you toss a coin (with probability of heads p) independently for n times, and let $X_1, \ldots, X_n$ denote the numerical outcomes of single trials: 1 (heads) or 0 (tails). This constitutes a random sample from the Bernoulli(p) distribution because

$$X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} \text{Bernoulli}(p).$$

Example 0.2. Let $X_1, \ldots, X_n$ represent n repeated measurements of an object's length/weight. They can be thought of as a random sample from a normal distribution

$$X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$$

where

- μ: true length/weight (if the measurement process is unbiased)
- σ^2: variance of the measurement error.

Specific realizations of a random sample

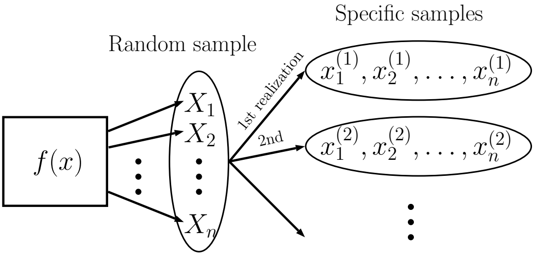
Example 0.3. Suppose you *actually* bought a carton of $n = 12$ eggs from the farm and measure their weights individually. Then you may obtain a data set like the following (called **specific sample**):

$$x_1 = 65.4,\ x_2 = 65.0,\ x_3 = 64.8,\ x_4 = 65.1,\ x_5 = 64.8,\ x_6 = 64.4,$$
$$x_7 = 65.0,\ x_8 = 65.1,\ x_9 = 65.5,\ x_{10} = 64.8,\ x_{11} = 64.8,\ x_{12} = 65.2$$

Notation. We use lowercase letters to represent specific values of the random variables in a random sample.

Remark. If we realize the sampling process again, then we may obtain a different set of weights. For example,

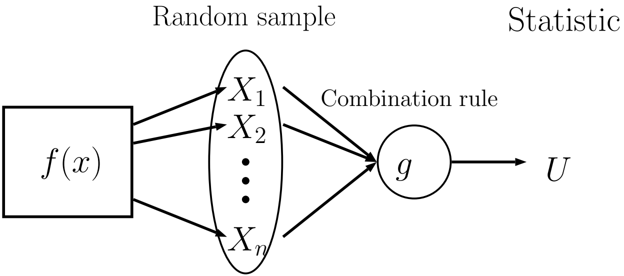
$$x_1 = 65.6,\ x_2 = 64.3,\ x_3 = 64.2,\ x_4 = 65.4,\ x_5 = 64.9,\ x_6 = 64.4,$$
$$x_7 = 65.2,\ x_8 = 65.2,\ x_9 = 65.0,\ x_{10} = 64.7,\ x_{11} = 64.5,\ x_{12} = 65.1$$



Statistic

Definition 0.2. Mathematically, a statistic is just a function of a random sample:

$$U = g(X_1, X_2, \ldots, X_n)$$



Remark. Depending on purpose, different statistics may be defined on a random sample. Two common ones are

- Sample mean

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- Sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

Other examples of statistics include sampling minimum, median, and maximum.

Statistics are random variables

Clearly, for different realizations of the sampling process, the values of the statistic may vary. For the eggs example (and the statistic $\bar{X}$),

(1) One realization ($\bar{x} = 64.992$):

$$x_1 = 65.4,\ x_2 = 65.0,\ x_3 = 64.8,\ x_4 = 65.1,\ x_5 = 64.8,\ x_6 = 64.4,$$
$$x_7 = 65.0,\ x_8 = 65.1,\ x_9 = 65.5,\ x_{10} = 64.8,\ x_{11} = 64.8,\ x_{12} = 65.2$$

(2) Second realization ($\bar{x} = 64.875$) :

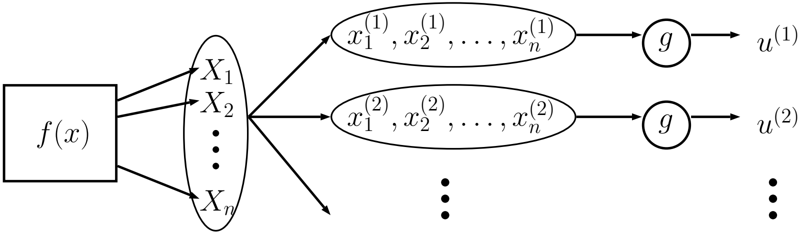
$$x_1 = 65.6,\ x_2 = 64.3,\ x_3 = 64.2,\ x_4 = 65.4,\ x_5 = 64.9,\ x_6 = 64.4,$$
$$x_7 = 65.2,\ x_8 = 65.2,\ x_9 = 65.0,\ x_{10} = 64.7,\ x_{11} = 64.5,\ x_{12} = 65.1$$

Sampling distribution of a statistic

Definition 0.3. The probabilistic distribution of a statistic (as a random variable)

$$U = g(X_1, X_2, \ldots, X_n)$$

is called the *sampling distribution* of the statistic.



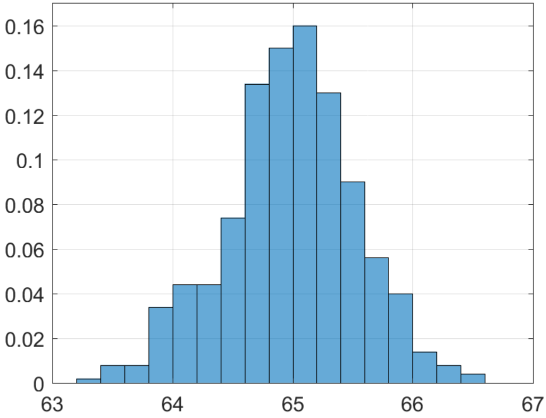
Simulation

We selected 500 cartons of eggs randomly from the farm (through computer simulation) and computed their average weights. Below show the first 50 specific values of $\bar{X}$:

65.0506	64.7592	65.0571	64.9674	65.4973	64.7503	65.0393	64.6714
65.3764	65.2525	65.2012	64.4910	65.6002	65.1868	65.0916	63.8280
65.2636	64.9638	65.2998	65.5587	63.9801	65.3903	64.9052	65.7352
64.6329	64.5109	65.7044	64.3291	65.1044	64.8036	66.0407	65.3560
65.3534	65.4668	64.7394	65.1690	64.5668	64.8478	64.0334	65.7562
64.8553	64.9939	65.6044	64.5237	64.2092	64.5860	65.2096	65.5114

64.6195 65.0312 ......

We can display all 500 mean values through a histogram



i-Clicker Quiz 8 (extra credit)

Which of the following statements is wrong about a statistic?

- It is a random variable
- It has certain distribution
- It is a numerical summary of a random sample.
- It is also a reduction of the random sample
- It must have a smaller variance than the population

The sample mean

We focus on the sample mean statistic

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

where

$$X_1, \ldots, X_n \overset{\text{iid}}{\sim} f(x)$$

and

$$\mathrm{E}(X_i) = \mu, \quad \mathrm{Var}(X_i) = \sigma^2, \text{ for all } i.$$

We present three different results for the statistic $\bar{X}$:

1. **Expectation and variance of** $\bar{X}$ (for any distribution $f(x)$)
2. **Exact distribution of** $\bar{X}$ when $f(x)$ is a normal distribution
3. **Approximate distribution of** $\bar{X}$ for nonnomral distributions in the setting of a large sample

General distributions: Expectation and variance of $\bar{X}$

Theorem 0.1. Suppose $X_1, \ldots, X_n \overset{\text{iid}}{\sim} f(x)$, with population mean $\mathrm{E}(X_i) = \mu$ and variance $\mathrm{Var}(X_i) = \sigma^2$. The mean and variance of $\bar{X}$ are

$$\mathrm{E}(\bar{X}) = \mu, \quad \mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}, \quad \mathrm{Std}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

Remark. This result does NOT concern the specific distribution of $\bar{X}$!

Proof. By linearity and independence,

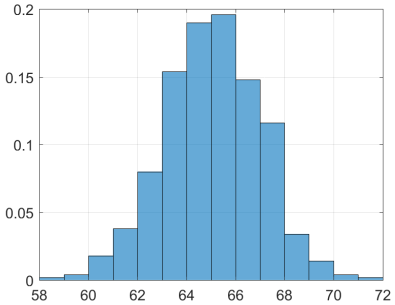
$$\mathrm{E}(\bar{X}) = \frac{1}{n}\left(\mathrm{E}(X_1) + \cdots + \mathrm{E}(X_n)\right) = \frac{1}{n}(\mu + \cdots + \mu) = \mu$$

$$\mathrm{Var}(\bar{X}) = \frac{1}{n^2}(\mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)) = \frac{1}{n^2}(\sigma^2 + \cdots + \sigma^2) = \frac{\sigma^2}{n}.$$

Remark. The theorem indicates that

- expectation of $\bar{X}$ is μ (population mean), and
- variance of $\bar{X}$ is only $1/n$ of the population variance (for single X_i)

Example 0.4. Weights of 500 single eggs (left) and average weights of 500 cartons (right), all selected at random.



Normal populations: Exact distribution of $\bar{X}$

Assume a random sample

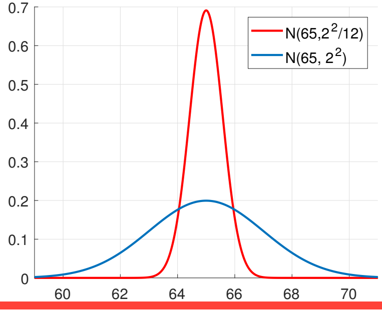
$$X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2).$$

Theorem 0.2. We have

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}).$$

This also implies that

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1).$$



Example 0.5. In the egg example, suppose the population distribution is $N(65, 2^2)$. For a random sample of size 12, what is the probability that the sample mean $\bar{X}$ is within 65 ± 1? What about an individual egg? (Answers: $.9167, .3829$)

Example 0.6. In the library elevator of a large university, there is a sign indicating a 16-person limit as well as a weight limit of 2500 lbs. When the elevator is full, we can think of the 16 people in the elevator as a random sample of people on campus. Suppose that the weight of students, faculty, and staff is normally distributed with a mean weight of 150 lbs and a standard deviation of 27 lbs. What is the probability that the total weight of a random sample of 16 people in the elevator will exceed the weight limit? (*Answer*: $.1762$)

Nonnormal populations: Approximate distribution of $\bar{X}$

Assume a random sample

$$X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} f(x) \quad \longleftarrow \text{any distribution}$$

and that the population has finite mean μ and variance σ^2.

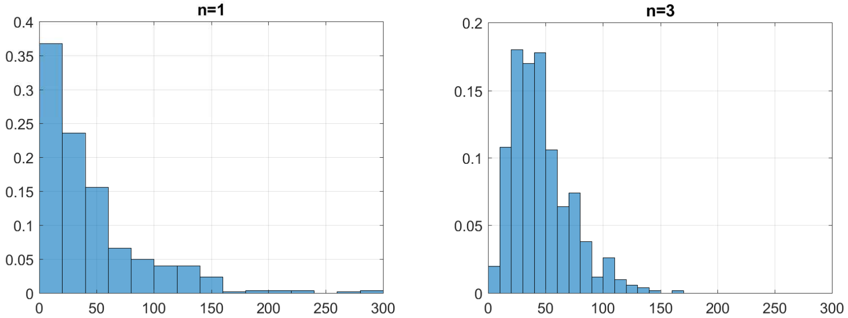
Theorem 0.3. If n **is large** (30 or greater), then

$$\bar{X} \stackrel{\text{approx.}}{\sim} N(\mu, \frac{\sigma^2}{n}), \quad \text{and} \quad \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \stackrel{\text{approx.}}{\sim} N(0, 1).$$

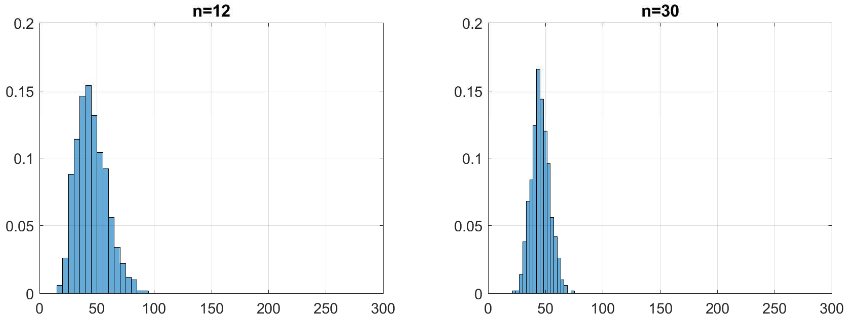
Remark. This is called the Central Limit Theorem (CLT), one of the most important results in probability and statistics.

Example 0.7. Suppose salaries of all SJSU employees follow an exponential distribution with average salary = 45K (which means that $\lambda = \frac{1}{45}$). We draw a random sample of size n from the population, and compute the sample mean $\bar{X}$.

We display the histograms of the simulated values of $\bar{X}$ through 500 repetitions for each of $n = 1, 3, 12, 30$.


n=1
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
50
100
150
200
250
300
n=3
0.2
0.15
0.1
0.05
0
0
50
100
150
200
250
300

Statistics and their distributions



Example 0.8 (Employee salary distribution, cont'd). Suppose we draw a random sample of size 30 from the population, and let $\bar{X}$ be the sample mean. Find $P(\bar{X} > 55)$.

Answer: 0.1118 (CLT), 0.1157 (exact)

The normal approximation to Binomial is a direct consequence of the CLT.

Corollary 0.4. Let $X \sim B(n, p)$. If n is large (i.e., $np, n(1-p) \geq 10$), then

$$\frac{X - np}{\sqrt{np(1-p)}} \overset{\text{approx.}}{\sim} N(0, 1)$$

Proof. Consider the experiment of tossing a coin independently for a total of n times, and denote the results by $X_1, \ldots, X_n$. Then

$$X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Bernoulli}(p), \quad \text{and} \quad X = \sum_{i=1}^{n} X_i \sim B(n, p).$$

According to the CLT,

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - p}{\sqrt{p(1-p)}/\sqrt{n}} = \frac{X - np}{\sqrt{np(1-p)}} \overset{\text{approx.}}{\sim} N(0, 1).$$

A “large-sample” joke

One day there was a fire in a wastebasket in the Dean’s office and in rushed a physicist, a chemist, and a statistician.

The physicist immediately starts to work on how much energy would have to be removed from the fire to stop the combustion. The chemist works on which reagent would have to be added to the fire to prevent oxidation.

While they are doing this, the statistician is setting fires to all the other wastebaskets in the office.

“What are you doing?” they demanded. “Well to solve the problem, obviously you need a large sample size” the statistician replies.