Polar Cauchy-Riemann Equations

Theorem

Let $f(z) = u(r, \theta) + iv(r, \theta)$ be differentiable on a domain D. The first partial derivatives $(u_r, u_\theta, v_r, v_\theta)$ exist in D and satisfy the polar form of the Cauchy-Riemann equations:

$ru_r = v_\theta$ and $u_\theta = -rv_r$

so that:

$$f'(z) = e^{-i\theta}(u_r + iv_r) = \frac{1}{r}e^{-i\theta}(v_\theta - iu_\theta)$$

Proof

Assume $z = re^{i\theta} \in D$

Let $\Delta z = (\Delta r)e^{-i\theta}$:

$$\begin{aligned}
f'(z) &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\
&= \lim_{\Delta r \to 0} \frac{[u(r + \Delta r, \theta) + iv(r + \Delta r, \theta)] - [u(r, \theta) + iv(r, \theta)]}{(\Delta r)e^{-i\theta}} \\
&= e^{i\theta} \lim_{\Delta r \to 0} \left[\frac{u(r + \Delta r, \theta) - u(r, \theta)}{\Delta r} + i\frac{v(r + \Delta r, \theta) - v(r, \theta)}{\Delta r}\right] \\
&= e^{i\theta}(u_r + iv_r)
\end{aligned}$$

Let $\Delta z = r\Delta(e^{i\theta}) = r\left(\frac{d}{d\theta}e^{i\theta}\right)\Delta\theta = ire^{i\theta}\Delta\theta$:

$$\begin{aligned}
f'(z) &= \lim_{\Delta \theta \to 0} \frac{[u(r, \theta + \Delta\theta) + iv(r, \theta + \Delta\theta)] - [u(r, \theta) + iv(r, \theta)]}{ire^{i\theta}\Delta\theta} \\
&= -i\frac{1}{r}e^{-i\theta} \lim_{\Delta \theta \to 0} \left[\frac{u(r, \theta + \Delta\theta) - u(r, \theta)}{\Delta\theta} + i\frac{v(r, \theta + \Delta\theta) - v(r, \theta)}{\Delta\theta}\right] \\
&= -i\frac{1}{r}e^{-i\theta}(u_\theta + iv_\theta) \\
&= \frac{1}{r}e^{-i\theta}(v_\theta - iu_\theta)
\end{aligned}$$

Thus, in order for the limit to exist:

$$f'(z) = e^{i\theta}(u_r + iv_r) = \frac{1}{r}e^{-i\theta}(v_\theta - iu_\theta)$$

$\therefore ru_r = v_\theta$ and $u_\theta = -rv_r$

Corollary

Let $f(z) = u(r, \theta) + iv(r, \theta)$ be differentiable on a domain D.

$$f'(z) = e^{-i\theta} f_r = -i \frac{1}{r} e^{-i\theta} f_\theta$$

Proof

The polar CR equations hold in D
$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta} f_r$
$f'(z) = \frac{1}{r} e^{-i\theta}(v_\theta - iu_\theta) = -i\frac{1}{r} e^{-i\theta}(u_\theta + iv_\theta) = -i\frac{1}{r} e^{-i\theta} f_\theta$

Theorem

Let $f(z) = u(r, \theta) + iv(r, \theta)$ be defined in some domain D with $r \neq 0$:

$u_r, u_\theta, v_r, v_\theta$ exist, are continuous, and satisfy CR in $D \implies f$ is differentiable in D

Proof

Let $x = r\cos\theta$ and $y = r\sin\theta$
$r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \frac{y}{x}$

$$r_x = \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{r}$$

$$r_y = \frac{2y}{2\sqrt{x^2 + y^2}} = \frac{y}{r}$$

$$\theta_x = \frac{1}{1 + \frac{y^2}{x^2}} \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2} = -\frac{y}{r^2}$$

$$\theta_y = \frac{1}{1 + \frac{y^2}{x^2}} \left(\frac{1}{x}\right) = \frac{1}{1 + \frac{y^2}{x^2}} \left(\frac{x}{x^2}\right) = \frac{x}{x^2 + y^2} = \frac{x}{r^2}$$

$$u_x = u_r r_x + u_\theta \theta_x = u_r \frac{x}{r} - u_\theta \frac{y}{r^2}$$

$$u_y = u_r r_y + u_\theta \theta_y = u_r \frac{y}{r} + u_\theta \frac{x}{r^2}$$

$$v_x = v_r r_x + v_\theta \theta_x = v_r \frac{x}{r} - v_\theta \frac{y}{r^2}$$

$$v_y = v_r r_y + v_\theta \theta_y = v_r \frac{y}{r} + v_\theta \frac{x}{r^2}$$

But $ru_r = v_\theta$ and $u_\theta = -rv_r$

$$u_x = \left(\frac{1}{r}v_\theta\right)\frac{x}{r} - (-rv_r)\frac{y}{r^2} = v_r\frac{y}{r} + v_\theta\frac{x}{r^2} = v_y$$

$$v_x = \left(-\frac{1}{r}u_\theta\right)\frac{x}{r} - (ru_r)\frac{y}{r^2} = -\left(u_r\frac{y}{r} + u_\theta\frac{x}{r^2}\right) = -u_y$$

u_x, u_y, v_x, v_y exist, are continuous, and the CR equation hold

$\therefore f$ is differentiable in D with $r \neq 0$