THE CLOSURE OF THE TOPOLOGIST'S SINE CURVE IS NOT PATH CONNECTED

Recall that the topologist's sine curve (at least one version of it) is defined by

$$S = \left\{ \left(x, \sin \frac{1}{x} \right) : 0 < x < 1 \right\}.$$

We will show that its closure, $\overline{S}$, is not path connected.

Assume that $\overline{S}$ is path connected. Let $p \in S$ be arbitrary. Then there exists a continuous path

$$f : [0, 1] \to \overline{S}$$

such that $f(0) = p$ and $f(1) = (0, 0)$. Set $f(t) = (x(t), y(t))$. Since f is continuous, so are x and y. Define

$$U = \{t \in [0, 1] : x(t) > 0\}.$$

Observe that if $t \in U$, then $f(t) \in S$, so

$$y(t) = \sin \frac{1}{x(t)}.$$

Let $t_* = \sup U$. Since $U \subset [0, 1]$, t_* is well-defined. Observe that $x(t_*) = 0$ and let $b = y(t_*)$. Without loss of generality, we can assume $b < 1$. If $b = 1$, the rest of the proof is analogous.

Let ε be any positive number such that

$$\varepsilon < 1 - b.$$

Since f is continuous, there exists $\delta > 0$ such that for all $t \in [0, 1]$, if $|t - t_*| \leq \delta$, then

$$\|f(t) - f(t_*)\| < \varepsilon. \tag{1}$$

In other words, for t sufficiently close to t_*, $f(t)$ has to be ε-close to $f(t_*) = (0, b)$. On the other hand, S and therefore $f(t)$ oscillates wildly and leaves the ε-neighborhood of $(0, b)$ infinitely many times. For instance, for an infinite sequence $t_n \to t_*-$, we have $f(t_n) = (x_n, 1)$ which is at a distance $> \varepsilon$ from $(0, b)$. I suggest you draw your own picture.

This can be made more precise in the following way.

Since $x : [0, 1] \to \mathbb{R}$ (the first component of f) is continuous and the interval $[t_* - \delta, t_*]$ is compact and connected, $x([t_* - \delta, t_*])$ is compact and connected, hence a compact interval. Recall that $x(t_*) = 0$ and $x(t) > 0$ for $t < t_*$, so

$$x([t_* - \delta, t_*]) = [0, x_0],$$

for some $x_0 \in (0, 1]$. Observe that for each $x \in (0, x_0]$ there exists $t \in [t_* - \delta, t_*)$ (depending on x) such that

$$f(t) = \left(x, \sin \frac{1}{x} \right).$$

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Next, define

$$x_n = \frac{1}{2n\pi + \frac{\pi}{2}}.$$

Observe that

$$\sin\frac{1}{x_n} = \sin(2n\pi + \frac{\pi}{2}) = \sin\frac{\pi}{2} = 1.$$

Since $x_n \to 0$, there exists N such that $x_n < x_0$, for all $n \geq N$. Thus there exists $t_n \in [t_* - \delta, t_*)$ such that

$$f(t_n) = \left(x_n, \sin\frac{1}{x_n}\right) = (x_n, 1).$$

It follows that

$$\|f(t_n) - f(t_*)\| = \|(x_n, 1) - (0, b)\| \geq 1 - b > \varepsilon,$$

which contradicts (1).

Therefore, $\overline{S}$ is not path connected.