Math-42 Worksheet #12

Sequences and Summations

1. Determine the first 5 terms (starting from a_0) of each of the following sequences:
   (a) $a_n = n$
   (b) $a_n = \frac{1}{n}$
   (c) $a_n = 2n - 3$
   (d) $a_n = (-1)^n \frac{1}{2^n}$
   (e) $a_n = (n+1)^2 - n^2$

2. Determine a closed form for each of the following sequences:
   (a) $1, 4, 7, 10, 13, \ldots$
   (b) $-2, 3, 8, 13, 18, \ldots$
   (c) $1, 4, 9, 16, 25, \ldots$
   (d) $1, 8, 27, 81, 243, \ldots$
   (e) $2, 6, 18, 54, 162, \ldots$

3. Determine the first 5 terms of each of the following sequences:
   (a) $a_n = 2a_{n-1} + 1, a_0 = 3$
   (b) $a_n = a_{n-1}^2 - 4, a_0 = 0$
   (c) $a_n = 2a_{n-1} + 3a_{n-2}, a_0 = 1, a_1 = -1$
   (d) $a_n = \frac{a_{n-2}}{a_{n-1}}, a_0 = 1, a_1 = 2$

4. Calculate the following sums:
   (a) $\sum_{k=3}^{7} k$

(b) $\sum_{k=-1}^{4} k^2$

(c) $\sum_{k=1}^{10}(-1)^n$

(d) $\sum_{k=0}^{5}(-1)^n\frac{1}{2^n}$

(e) $\sum_{k=0}^{10} 3(2^k)$

(f) $\sum_{k=10}^{20}(-2)^k$

(g) $\sum_{k=1}^{n}(3k^2+5k-2)$

(h) $\sum_{k=1}^{\infty}\frac{2}{3^k}$

(i) $\sum_{k=0}^{\infty}\frac{2}{(-3)^k}$

(j) $\sum_{k=3}^{\infty}\left(-\frac{1}{2}\right)^k$

5. A *telescoping* sum is a special type of sum where most of the terms cancel each other out. For example, consider the sum:

   $$\sum_{k=1}^{100}[(k+1)^2-k^2]$$

   Note that the terms of this sum are:

   $$(2^2-2^1)+(2^3-2^2)+(2^4-2^3)+\cdots+(2^{100}-2^{99})+(2^{101}-2^{100})$$

   Due to cancellation, the only terms left are:

   $$2^{101}-2^1$$

   The general form of a telescoping sum is:

   $$\sum_{k=m}^{n}(a_{k+1}-a_k)=a_{n+1}-a_m$$

Calculate each of the following telescoping sums:

(a) $\displaystyle\sum_{k=1}^{10}\left(\frac{1}{k+1}-\frac{1}{k}\right)$

(b) $\displaystyle\sum_{k=5}^{49}\left(\frac{1}{k}-\frac{1}{k-1}\right)$

(c) $\displaystyle\sum_{k=1}^{100}[(k+1)^2-k^2]$

(d) $\displaystyle\sum_{k=1}^{100}[k^3-(k-1)^3]$

6. An important identity is Gauss's Formula:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Prove this formula starting with sum:

$$\sum_{k=1}^{n}[(k+1)^2-k^2]$$

7. Prove the following:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Which telescoping sum do you start with this time?