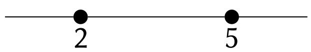
Arbitrarily Close

The difference between algebra and calculus is a concept called *arbitrarily close*. You have already seen this concept at work with things like asymptotes and the infinite repeating decimal form of some rational numbers (e.g., $0.11111\ldots = \frac{1}{9}$). To develop this concept more formally, we need to establish the concepts of *distance* and *arbitrarily small* first.

Distance

Consider two points on the number line, say 2 and 5:



To find the distance between two points, we subtract the destination from the source:

$$d(2, 5) = 5 - 2 = 3$$

How about the distance from 5 to 2?:

$$d(5, 2) = 2 - 5 = -3$$

But distance should be non-negative and should be the same, no matter which direction you go (note that this is different from *displacement*, which is signed). So how do we ignore the negative sign?

Recall from Section 0.1, Real Numbers: Order and Absolute Value, p7:

Definition: Distance

Let $a, b \in \mathbb{R}$ be two real numbers on the real number line. The *distance* between a and b, which is the same as the distance between b and a, is given by:

$$|b - a| = |a - b|$$

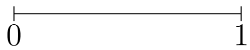
In fact, this is the only time when subtraction is commutative: when it is in an absolute value.

Arbitrarily Small

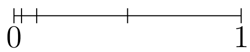
We have already dealt with the concept of ∞ as an *arbitrarily large* value. Instead, we introduce ε, the Greek letter epsilon, to refer to an *arbitrarily small* value. This means that ε is not some particular value, instead it refers to a process of getting progressively smaller but not quite 0, just like infinity refers to the process of getting progressively larger.

For example, if I ask you for a small value and you say 0.1. But I say, no smaller, so you say 0.001, but I want it smaller still so you say 0.0000001, but again, I say smaller, and so on in an infinite pattern. Note that the number of steps is arbitrarily large; however, the values of ε get arbitrarily small.

But how do I know that I can alway pick a smaller value greater than 0? Consider a subset of the number line from 0 to 1:

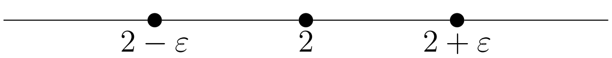


How many real values are there between 0 and 1? An infinite number! In fact, between any two real numbers there exists and infinite number of real numbers, so we can always pick one: 0.5, 0.1, 0.01, 0.000001, etc.



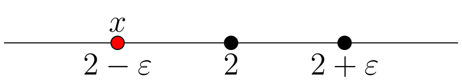
Arbitrarily Small Interval

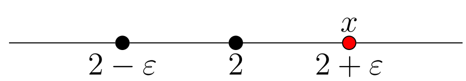
Now, suppose we have a fixed point a, say 2, and some arbitrarily small ε. We use these values to construct the following interval: $[2-\varepsilon, 2+\varepsilon]$, where each endpoint is distance ε from the base value of 2:



We now want to locate other values x in relation to our base value and our interval. There are three possibilities:

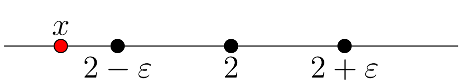
1. x *coincides* with one of the endpoints and thus is exactly ε away from the base point: $|x-a| = \varepsilon$.

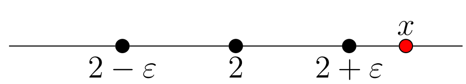
   

   This corresponds to absolute value linear equations as presented in Section 1.5.

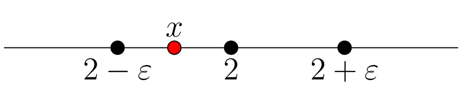
2. x is *outside* the interval and thus is greater than ε away from the base point: $|x-a| > \varepsilon$.

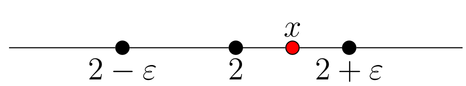
   

   This corresponds to absolute value linear inequalities as presented in Section 1.6.

3. x is *inside* the interval and thus is less than ε away from the base point: $|x-a| > \varepsilon$.

   This also corresponds to absolute value linear inequalities as presented in Section 1.6, and it is in fact the case that we are interested in for our definition of *arbitrarily close*.

Arbitrarily Close

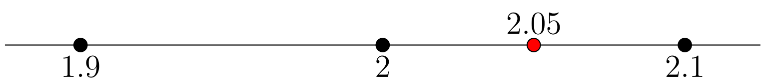
Consider the interval represented by the absolute value inequality:

$$|x - a| < \varepsilon$$

As ε gets arbitrarily small, denoted by: $\varepsilon \to 0$, the values of x that we can choose inside the interval get closer and closer to the base value a. Since $\varepsilon > 0$ we are never forced onto a, just *arbitrarily close* to it!

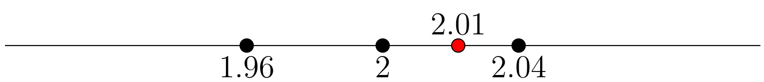
Example

1. Let our base point $a = 2$ and start with $\varepsilon = 0.1$. Pick $x = 2.05$:



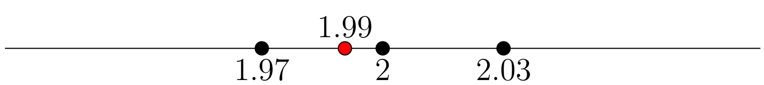
$$|2.05 - 2| = 0.05 < 0.1 = \varepsilon$$

2. Now, let $\varepsilon = 0.04$ and pick $x = 2.01$:



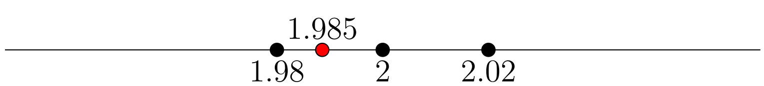
$$|2.01 - 2| = 0.01 < 0.04 = \varepsilon$$

3. Note that x need not always be on the same side. This time, let $\varepsilon = 0.03$ and pick $x = 1.99$:



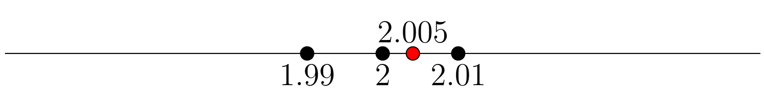
$$|1.99 - 2| = 0.01 < 0.03 = \varepsilon$$

4. Nor does each step have to get closer: it just needs to be within the new smaller interval. This time, let $e = 0.02$ and pick $x = 1.985$:



$$|1.985 - 2| = 0.015 < 0.02 = \varepsilon$$

5. But eventually, we are forced to pick smaller values. This time, let $e = 0.01$ and select $x = 2.005$:



$$|2.005 - 2| = 0.005 < 0.01 = \varepsilon$$

Thus, the x values that can be selected get *arbitrarily close* to the base value a as ε gets *arbitrarily small*. This can be stated in any of the following ways:

1. x gets arbitrarily close to a as e gets arbitrarily small
2. x *converges* to a as $e \to 0$
3. $|x - a| < \varepsilon$
4. $x \to a$ as $e \to 0$