Courant-Fischer Minimax

<u>Theorem: Courant-Fischer</u>

Let $A \in M_n$ be Hermitian with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and orthonormal eigenvectors $\{\vec{u}_1, \ldots, \vec{u}_n\}$ and let $S = \operatorname{span}\{\vec{u}_{i_1}, \ldots, \vec{u}_{i_m}\}$ where $i_1 \leq \cdots \leq i_m$:

$$\lambda_k = \min_{\dim(S)=k} \left(\max_{\vec{x} \in S - \{\vec{0}\}} \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} \right)$$

Recall that if S_k are subspaces in $\mathbb{C}^n$ then:

1). $\dim(S_1 \cap S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 + S_2)$

2). $\dim(S_1 \cap S_2) \geq \dim(S_1) + \dim(S_2) - n$

3). $\dim(S_1 \cap S_2 \cap S_3) \geq \dim(S_1) + \dim(S_2) + \dim(S_3) - 2n$

<u>Proof</u>

Assume $\dim(S) = k$
Let $T = \operatorname{span}\{\vec{u}_k, \ldots . \vec{u}_n\}$
$\dim(S \cap T) \geq \dim(S) + \dim(T) - n = k + (n - k + 1) - n = 1$
Thus, $\exists\, \vec{x} \in S \cap T$ such that $\vec{x} \neq \vec{0}$
Since these $\vec{x} \in T$, by the key lemma, $\forall\, \vec{x} \in S \cap T$:

$$\lambda_k \leq \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}}$$

But since these $\vec{x} \in S$ also, $\forall \vec{x} \in S$:

$$\lambda_k \leq \max_{\vec{x} \in S - \{\vec{0}\}} \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}}$$

Now, let $S_0 = \operatorname{span}\{\vec{u}_1, \ldots, \vec{u}_k\}$
$\dim S_0 = k$
By the key lemma and its corollaries, at $\vec{x} = \vec{u}_k$:

$$\lambda_k = \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}}$$

So there exists a subspace S of dimension k at which the minimum is achieved, and therefore:

$$\lambda_k = \min_{\dim(S)=k} \left(\max_{\vec{x} \in S - \{\vec{0}\}} \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} \right)$$