Vector Subspaces

Definition: Vector Subspace

Let E be a vector space over a scalar field $\mathbb{F}$ and let $E' \subseteq E$. To say that E' is a *subspace* of E means that E' is also a vector space using the same scalar field and operations as E.

To say that E' is a *proper* subspace of E means that E' is a subspace of E; however, $E' \neq E$.

Theorem: Subspace Test

Let E be a vector space over a scalar field $\mathbb{F}$ and let $E' \subseteq E$:

E' is a subspace of $E \iff \forall\, \vec{x}, \vec{y} \in E'$ and $\forall\, \alpha, \beta \in \mathbb{F}$, $\alpha\vec{x} + \beta\vec{y} \in E'$.

Proof

$\Longrightarrow$ Assume E' is a subspace of E.

Assume $\vec{x}, \vec{y} \in E'$ and $\alpha, \beta \in \mathbb{F}$.
By closure, $\alpha\vec{x} \in E'$ and $\beta\vec{y} \in E'$.
$\therefore$ by closure, $\alpha\vec{x} + \beta\vec{y} \in E'$.

$\Longleftarrow$ Assume $\forall\, \vec{x}, \vec{y} \in E'$ and $\forall\, \alpha, \beta \in \mathbb{F}$, $\alpha\vec{x} + \beta\vec{y} \in E'$.

Since $E' \subseteq E$, E' inherits most of the necessary properties from E; only closure and $\exists\, \vec{z} \in E', \vec{x} + \vec{z} = \vec{y}$ need to be proven.

Assume $\vec{x}, \vec{y} \in E'$
$1\vec{x} + 1\vec{y} = \vec{x} + \vec{y} \in E'$
$\therefore$ E' is closed under vector addition.

Assume $\alpha \in \mathbb{F}$.
$\alpha\vec{x} + 0\vec{y} = \alpha\vec{x} \in E'$
$\therefore$ E' is closed under scalar multiplication.

By closure, $\exists\, \vec{z} \in E', (-1)\vec{x} + 1\vec{y} = (-\vec{x}) + \vec{y} = \vec{z}$:

$$\begin{aligned}
\vec{x} + ((-\vec{x}) + \vec{y}) &= \vec{x} + \vec{z} \\
(\vec{x} + (-\vec{x})) + \vec{y} &= \vec{x} + \vec{z} \\
\vec{0} + \vec{y} &= \vec{x} + \vec{z} \\
\vec{y} + \vec{0} &= \vec{x} + \vec{z} \\
\vec{y} &= \vec{x} + \vec{z}
\end{aligned}$$

Examples

Let Ω be an open subset of $\mathbb{R}$ or $\mathbb{C}$. The following are all subspaces of $\mathcal{F}$, the vector space of complex-valued functions on Ω:

1). $C(\Omega) =$ the space of all continuous functions on Ω.

2). $C^k(\Omega) =$ the space of all functions with continuous partial derivatives of order up to and including k on Ω.

3). $C^\infty =$ the space of all infinitely-differentiable functions on Ω.

4). $C^\omega =$ the space of all analytic functions (i.e., functions that have a power series representation) on Ω.

5). $P(\Omega) =$ the space of all polynomial functions in n variables on Ω.

Note that:

$$C^0(\Omega) \supset C^1(\Omega) \supset C^2(\Omega) \supset \ldots \supset C^\infty(\Omega) \supset C^\omega(\Omega)$$

Also note that:

$$P(\Omega) \subseteq C^\omega(\Omega)$$

Theorem

Let E_1 and E_2 be two vector spaces over a scalar field $\mathbb{F}$:

$E_1 \cap E_2$ is a subspace of both E_1 and E_2.

Proof

$E_1 \cap E_2 \subseteq E_1$ and $E_1 \cap E_2 \subseteq E_2$.

Assume $\alpha, \beta \in \mathbb{F}$.
Assume $\vec{x}, \vec{y} \in E \cap E'$.
$\vec{x}, \vec{y} \in E_1$ and $\vec{x}, \vec{y} \in E_2$.
By closure, $\alpha\vec{x} + \beta\vec{y} \in E_1$ and $\alpha\vec{x} + \beta\vec{y} \in E_2$,
$\alpha\vec{x} + \beta\vec{y} \in E_1 \cap E_2$

$\therefore$ by the subspace test, $E_1 \cap E_2$ is a subspace of both E_1 and E_2.

Theorem

Let E_1 and E_2 be subspaces of a vector space E such that $E_1 \subseteq E_2$:

E_1 is a subspace of E_2.

Proof

$E_1 = E_1 \cap E_2$
But by previous theorem, $E_1 \cap E_2$ is a subspace of E_2.

$\therefore$ E_1 is a subspace of E_2.

Example

$P(\Omega)$ and $C^\omega(\Omega)$ are subspaces of $C^\infty(\Omega)$.

$P(\Omega) \subset C^\omega(\Omega)$.

$\therefore P(\Omega)$ is a subspace of $C^\omega(\Omega)$.