Harmonic Polar Form

Theorem

Let $u(x, y)$ be harmonic:

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$$

Proof

$$\begin{array}{lll} x = r\cos\theta & x_r = \cos\theta & x_\theta = -r\sin\theta \\ y = r\sin\theta & y_r = \sin\theta & y_\theta = r\cos\theta \end{array}$$

$$u_r = u_x x_r + u_y y_r = u_x \cos\theta + u_y \sin\theta$$

$$u_\theta = u_x x_\theta + u_y y_\theta = -u_x r \sin\theta + u_y r \cos\theta$$

$$u_{rr} = (u_{xx} x_r + u_{xy} y_r)\cos\theta + (u_{yx} x_r + u_{yy} y_r)\sin\theta$$

$$= (u_{xx}\cos\theta + u_{yx}\sin\theta)\cos\theta + (u_{yx}\cos\theta + u_{yy}\sin\theta)\sin\theta$$

$$= u_{xx}\cos^2\theta + u_{yx}\sin\theta\cos\theta + u_{yx}\sin\theta\cos\theta + u_{yy}\sin^2\theta$$

$$= u_{xx}\cos^2\theta + 2u_{xy}\sin\theta\cos\theta + u_{yy}\sin^2\theta$$

$$u_{\theta\theta} = -(u_{xx} x_\theta + u_{xy} y_\theta) r\sin\theta - u_x r\cos\theta + (u_{yx} x_\theta + u_{yy} y_\theta) r\cos\theta - u_y r\sin\theta$$

$$= u_{xx} r^2 \sin^2\theta - u_{xy} r^2 \sin\theta\cos\theta - u_x r\cos\theta - u_{yx} r^2 \sin\theta\cos\theta + u_{yy} r^2\cos^2\theta - u_y r\sin\theta$$

$$= u_{xx} r^2 \sin^2\theta - 2u_{xy} r^2 \sin\theta\cos\theta + u_{yy} r^2 \cos^2\theta - r(u_x\cos\theta + u_y\sin\theta)$$

$$= u_{xx} r^2 \sin^2\theta - 2u_{xy} r^2 \sin\theta\cos\theta + u_{yy} r^2 \cos^2\theta - r u_r$$

$$\frac{1}{r^2} u_{\theta\theta} = u_{xx}\sin^2\theta - 2u_{xy}\sin\theta\cos\theta + u_{yy}\cos^2\theta - \frac{1}{r} u_r$$

$$u_{rr} + \frac{1}{r^2} u_{\theta\theta} = u_{xx} + u_{yy} - \frac{1}{r} u_r$$

$$u_{rr} + \frac{1}{r^2} u_{\theta\theta} = 0 - \frac{1}{r} u_r$$

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$$