Upper Triangular Matrices

Definition: Upper Triangular Matrix

To say that a matrix A is *upper triangular* means:

$$i > j \implies a_{ij} = 0$$

In particular, a diagonal matrix is upper triangular.

The set of all $n \times n$ upper triangular matrices is denoted by $UT(n)$.

Properties: Upper Triangular

Let $A, B \in UT(n)$:

1). $cA \in UT(n)$

2). $A + B \in UT(n)$

3). $AB \in UT(n)$

4). $\mathrm{adj}(A) \in UT(n)$

5). A invertible $\implies A^{-1} \in UT(n)$

Theorem

Let $T \in UT(n)$ such that T has distinct diagonal entries and $AT = TA$:

$$A \in UT(n)$$

Proof

Proof by induction on n.

Base case: $n = 1$

1×1 matrices are by definition UT, so nothing to prove.

Assume true for $n - 1$

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where $A_{11} \in M_{n-1}$ and let $T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$, where $T_{11} \in UT(n-1)$ and T_{11} and T_{22} have distinct diagonal entries.

$$\begin{aligned} AT &= TA \\ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} &= \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\ \begin{bmatrix} * & * \\ A_{21}T_{11} & * \end{bmatrix} &= \begin{bmatrix} * & * \\ A_{21}T_{22} & * \end{bmatrix} \\ A_{21}T_{11} &= T_{22}A_{21} \end{aligned}$$