Reduction Formulas

Theorem

Let $n \in \mathbb{N}$:

1). $\cos^{2n+1}\theta = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \cos[(2n+1-2k)\theta]$

2). $\cos^{2n}\theta = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \cos[2(n-k)\theta]$

3). $\sin^{2n+1}\theta = \frac{(-1)^n}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} (-1)^k \sin[(2n+1-2k)\theta]$

4). $\sin^{2n}\theta = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} (-1)^k \cos[2(n-k)\theta]$

Proof

Assume $m \in \mathbb{N} - \{1\}$.

There exists $n \in \mathbb{N}$ such that:

$$m = \begin{cases} 2n+1, & m \text{ odd} \\ 2n, & m \text{ even} \end{cases}$$

$$\begin{aligned} \cos^m\theta &= \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^m \\ &= \frac{1}{2^m}\left(e^{i\theta}+e^{-i\theta}\right)^m \\ &= \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k} e^{i(m-k)\theta}e^{-ik\theta} \\ &= \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k} e^{i(m-2k)\theta} \end{aligned}$$

case 1: m odd ($m = 2n+1$)

Consider the symmetry and the possible signs of the even number of binomial coefficients:

$$\begin{array}{cccccc} \binom{2n+1}{0} & \cdots & \binom{2n+1}{n} & \binom{2n+1}{n+1} & \cdots & \binom{2n+1}{2n+1} \\ + & & + & + & & + \end{array}$$

The k coefficient is equal to the $2n+1-k$ coefficient and all signs are positive.

$$\begin{aligned} \cos^{2n+1}\theta &= \frac{1}{2^{2n+1}}\sum_{k=0}^{n}\binom{2n+1}{k}\left[e^{i(2n+1-2k)\theta} + e^{i[2n+1-2(2n+1-k)]\theta}\right] \\ &= \frac{1}{2^{2n+1}}\sum_{k=0}^{n}\binom{2n+1}{k}\left[e^{i(2n+1-2k)\theta} + e^{i(-2n-1+2k)\theta}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{2n+1}{k} \left[e^{i(2n+1-2k)\theta} + e^{-i(2n+1-2k)\theta}\right] \\
&= \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \left[\frac{e^{i(2n+1-2k)\theta} + e^{-i(2n+1-2k)\theta}}{2}\right] \\
&= \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \cos[(2n+1-2k)\theta]
\end{aligned}
$$

<u>case 2: m even ($m = 2n$)</u>

Consider the symmetry and the possible signs of the odd number of binomial coefficients:

$$
\begin{array}{ccccccc}
\binom{2n}{0} & \cdots & \binom{2n}{n-1} & \binom{2n}{n} & \binom{2n}{n+1} & \cdots & \binom{2n}{2n} \\
+ & & + & + & + & & +
\end{array}
$$

The k coefficient is equal to the $2n-k$ coefficient and all signs are positive.

$$
\begin{aligned}
\cos^{2n}\theta &= \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n}{k} \left[e^{i(2n-2k)\theta} + e^{i[2n-2(2n-k)]\theta}\right] \\
&= \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n}{k} \left[e^{i(2n-2k)\theta} + e^{i(-2n+2k)\theta}\right] \\
&= \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n}{k} \left[e^{i(2n-2k)\theta} + e^{-i(2n-2k)\theta}\right] \\
&= \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}} \sum_{k=0}^{n} \binom{2n}{k} \left[\frac{e^{i(2n-2k)\theta} + e^{-i(2n-2k)\theta}}{2}\right] \\
&= \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}} \sum_{k=0}^{n} \binom{2n}{k} \cos[2(n-k)\theta]
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
\sin^m\theta &= \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^m \\
&= \frac{1}{(2i)^m}\left(e^{i\theta} - e^{-i\theta}\right)^m \\
&= \frac{1}{(2i)^m} \sum_{k=0}^{m} \binom{m}{k} e^{i(m-k)\theta}\left(-e^{-i\theta}\right)^k \\
&= \frac{1}{(2i)^m} \sum_{k=0}^{m} \binom{m}{k} (-1)^k e^{i(m-k)\theta} e^{-ik\theta}
\end{aligned}
$$

$$= \frac{1}{(2i)^m}\sum_{k=0}^{m}\binom{m}{k}(-1)^k e^{i(m-2k)\theta}$$

<u>case 1: m odd $(m = 2n+1)$</u>

Consider the symmetry and the possible signs of the even number of binomial coefficients:

$$\begin{array}{ccccccc} \binom{2n+1}{0} & \cdots & \binom{2n+1}{n} & \binom{2n+1}{n+1} & \cdots & \binom{2n+1}{2n+1} \\ + & & + & - & & - \\ + & & - & + & & - \end{array}$$

The k coefficient is equal to the $2n+1-k$ coefficient and the signs alternate.

$$\begin{aligned}
\sin^{2n+1}\theta &= \frac{1}{(2i)^{2n+1}}\sum_{k=0}^{n}\binom{2n+1}{k}\left[(-1)^k e^{i(2n+1-2k)\theta} + (-1)^{k+1}e^{i[2n+1-2(2n+1-k)]\theta}\right] \\
&= \frac{1}{(2i)^{2n+1}}\sum_{k=0}^{n}\binom{2n+1}{k}(-1)^k\left[e^{i(2n+1-2k)\theta} - e^{i(-2n-1+2k)\theta}\right] \\
&= \frac{1}{(2i)^{2n+1}}\sum_{k=0}^{n}\binom{2n+1}{k}(-1)^k\left[e^{i(2n+1-2k)\theta} - e^{-i(2n+1-2k)\theta}\right] \\
&= \frac{1}{(2i)^{2n}}\sum_{k=0}^{n}\binom{2n+1}{k}(-1)^k\left[\frac{e^{i(2n+1-2k)\theta} - e^{-i(2n+1-2k)\theta}}{2i}\right] \\
&= \frac{1}{2^{2n}}\frac{1}{i^{2n}}\sum_{k=0}^{n}\binom{2n+1}{k}(-1)^k\sin[(2n+1-2k)\theta] \\
&= \frac{(-1)^n}{2^{2n}}\sum_{k=0}^{n}\binom{2n+1}{k}(-1)^k\sin[(2n+1-2k)\theta]
\end{aligned}$$

<u>case 2: m even $(m = 2n)$</u>

Consider the symmetry and the possible signs of the odd number of binomial coefficients:

$$\begin{array}{ccccccc} \binom{2n}{0} & \cdots & \binom{2n}{n-1} & \binom{2n}{n} & \binom{2n}{n+1} & \cdots & \binom{2n}{2n} \\ + & & + & - & + & & + \\ + & & - & + & - & & + \end{array}$$

The k coefficient is equal to the $2n-k$ coefficient and all signs match, but alternate.

$$\begin{aligned}
\sin^{2n}\theta &= \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{(2i)^{2n}}\sum_{k=0}^{n}\binom{2n}{k}(-1)^k\left[e^{i(2n-2k)\theta} + e^{i[2n-2(2n-k)]\theta}\right] \\
&= \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{(2i)^{2n}}\sum_{k=0}^{n}\binom{2n}{k}(-1)^k\left[e^{i(2n-2k)\theta} + e^{i(-2n+2k)\theta}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{(2i)^{2n}}\sum_{k=0}^{n}\binom{2n}{k}(-1)^k\left[e^{i(2n-2k)\theta} + e^{-i(2n-2k)\theta}\right] \\
&= \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\left(\frac{1}{i^{2n}}\right)\sum_{k=0}^{n}\binom{2n}{k}(-1)^k\left[\frac{e^{i(2n-2k)\theta} + e^{-i(2n-2k)\theta}}{2}\right] \\
&= \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n}\binom{2n}{k}(-1)^k\cos[2(n-k)\theta]
\end{aligned}
$$

Example

$$
\begin{aligned}
\cos^2\theta &= \frac{1}{4}\binom{2}{1} + \frac{1}{2}\sum_{k=0}^{0}\binom{2}{k}\cos(2-2k)\theta \\
&= \frac{1}{4}(2) + \frac{1}{2}\binom{2}{0}\cos 2\theta \\
&= \frac{1}{2} + \frac{1}{2}\cos 2\theta \\
&= \frac{1+\cos 2\theta}{2}
\end{aligned}
$$

$$
\begin{aligned}
\sin^2\theta &= \frac{1}{4}\binom{2}{1} - \frac{1}{2}\sum_{k=0}^{0}\binom{2}{k}\cos(2-2k)\theta \\
&= \frac{1}{4}(2) - \frac{1}{2}\binom{2}{0}\cos 2\theta \\
&= \frac{1}{2} - \frac{1}{2}\cos 2\theta \\
&= \frac{1-\cos 2\theta}{2}
\end{aligned}
$$

$$
\begin{aligned}
\cos^3\theta &= \frac{1}{4}\sum_{k=0}^{1}\binom{3}{k}\cos[(3-2k)\theta] \\
4\cos^3\theta &= \binom{3}{0}\cos 3\theta + \binom{3}{1}\cos\theta \\
4\cos^3\theta &= \cos 3\theta + 3\cos\theta \\
\cos 3\theta &= 4\cos^3\theta - 3\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
\sin^3\theta &= -\frac{1}{4}\sum_{k=0}^{1}\binom{3}{k}(-1)^k \sin[(3-2k)\theta] \\
-4\sin^3\theta &= \binom{3}{0}\sin 3\theta - \binom{3}{1}\sin\theta \\
-4\sin^3\theta &= \sin 3\theta - 3\sin\theta \\
\sin 3\theta &= 3\sin\theta - 4\sin^3\theta
\end{aligned}
$$