Cavallaro, Jeffery
Math 231b
Homework #2

3.8.4

(a) Let $E = \mathcal{C}^1[a,b]$, the space of all continuously differentiable complex-valued functions on $[a,b]$. For $f, g \in E$ define:

$$\langle f, g \rangle = \int_a^b f'(x)\overline{g'(x)}dx$$

Is $\langle \cdot, \cdot \rangle$ an inner product in E?

No. For a counterexample, consider $f(x) = 1$, and so $f'(x) = 0$:

$\langle f, f \rangle = \int_a^b 0 = 0$

So $\langle f, f \rangle = 0$; however, $f \not\equiv 0$.

Therefore, $\langle \cdot, \cdot \rangle$ is not an inner product in E.

(b) Let $F = \{f \in \mathcal{C}^1[a,b] \mid f(a) = 0\}$. Is $\langle \cdot, \cdot \rangle$ an inner product in F? Is F a Hilbert space?

The additional limitation excludes all non-zero constant functions. And so:

$\langle f, f \rangle = \int_a^b f'\,\overline{f'} = \int_a^b |f'|^2$

which is ≥ 0, with equality only when $f' \equiv 0$, which can only happen now when $f \equiv 0$.

$\langle f, g \rangle = \int_a^b f'\,\overline{g'} = \overline{\int_a^b \overline{f'}\,g} = \overline{\langle f, g \rangle}$

holds because f, g, f', g' are all continuous.

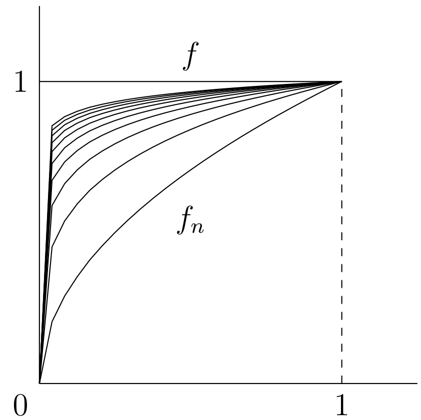
$\langle \alpha f + \beta g, h \rangle = \int_a^b (\alpha f' + \beta g')\overline{h'} = \alpha \int_a^b f'\,\overline{h'} + \beta \int_a^b g'\,\overline{h'} = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$

holds due to the linearity of the integral.

Therefore F is an inner product space.

However, F is not a Hilbert space.

As a counterexample, consider the sequence $f_n = t^{\frac{1}{2n}}$ on $[0,1]$.



$$f_n' = \frac{1}{2n} t^{\frac{1-2n}{2n}}$$

Note that $\forall\, t \in [0,1]$, $f_n(t), f_n'(t) \geq 0$.

Claim: f_n is Cauchy in the inner product induced norm:

$$\begin{aligned}
\|f_n - f_m\|^2 &= \int_0^1 \left(\frac{1}{2n} t^{\frac{1-2n}{2n}} - \frac{1}{2m} t^{\frac{1-2m}{2m}} \right)^2 \\
&= \int_0^1 \left(\frac{1}{4n^2} t^{\frac{1-2n}{n}} - \frac{1}{2nm} t^{\frac{1-2n}{2n} + \frac{1-2m}{2m}} + \frac{1}{4m^2} t^{\frac{1-2m}{m}} \right) \\
&= \int_0^1 \left(\frac{1}{4n^2} t^{\frac{1-2n}{n}} - \frac{1}{2nm} t^{\frac{m+n-4nm}{mn}} + \frac{1}{4m^2} t^{\frac{1-2m}{m}} \right) \\
&= \left[\frac{1}{4n^2} \left(\frac{n}{1-n} \right) t^{\frac{1-n}{n}} - \frac{1}{2nm} \left(\frac{mn}{m+n-3nm} \right) t^{\frac{m+n-3nm}{nm}} + \frac{1}{4m^2} \left(\frac{m}{1-m} \right) t^{\frac{1-m}{m}} \right]_0^1 \\
&= \frac{1}{4n(1-n)} - \frac{1}{2(m+n-3nm)} + \frac{1}{4m(1-m)} \\
&\to 0
\end{aligned}$$

Thus, f_n is Cauchy.

Claim: $f_n \to f$ in the inner product induced norm, where $f = 1$.

$$\begin{aligned}
\|f_n - 1\|^2 &= \int_0^1 (f_n' - 0)^2 \\
&= \int_0^1 \frac{1}{4n^2} t^{\frac{1-2n}{n}} \\
&= \left. \frac{1}{4(1-n)} t^{\frac{1-n}{n}} \right|_0^1 \\
&= \frac{1}{4(1-n)} \\
&\to 0
\end{aligned}$$

Thus, $f_n \to f$.

However, $f(0) = 1 \neq 0$, and so $f \notin F$.

Therefore, F is not complete in the inner product induced norm and hence is not a Hilbert space.

3.8.10

Show that the polarization identity:

$$\langle \vec{x}, \vec{y} \rangle = \frac{1}{4}\left[\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 + i\,\|\vec{x}+i\vec{y}\|^2 - i\,\|\vec{x}-i\vec{y}\|^2\right]$$

holds is any inner product space.

$$\begin{aligned}
\|\vec{x}+\vec{y}\|^2 &= \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle \\
&= \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle \\
&= \|\vec{x}\|^2 + \langle \vec{x}, \vec{y} \rangle + \overline{\langle \vec{x}, \vec{y} \rangle} + \|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 + 2\,\mathrm{Re}[\langle \vec{x}, \vec{y} \rangle] + \|\vec{y}\|^2
\end{aligned}$$

$$\begin{aligned}
\|\vec{x}-\vec{y}\|^2 &= \langle \vec{x}-\vec{y}, \vec{x}-\vec{y} \rangle \\
&= \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, -\vec{y} \rangle + \langle -\vec{y}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle \\
&= \|\vec{x}\|^2 - \langle \vec{x}, \vec{y} \rangle - \langle \vec{y}, \vec{x} \rangle + \|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 - \langle \vec{x}, \vec{y} \rangle - \overline{\langle \vec{x}, \vec{y} \rangle} + \|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 - (\langle \vec{x}, \vec{y} \rangle + \overline{\langle \vec{x}, \vec{y} \rangle}) + \|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 - 2\,\mathrm{Re}[\langle \vec{x}, \vec{y} \rangle] + \|\vec{y}\|^2
\end{aligned}$$

$$\begin{aligned}
\|\vec{x}+i\vec{y}\|^2 &= \langle \vec{x}+i\vec{y}, \vec{x}+i\vec{y} \rangle \\
&= \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, i\vec{y} \rangle + \langle i\vec{y}, \vec{x} \rangle + \langle i\vec{y}, i\vec{y} \rangle \\
&= \|\vec{x}\|^2 - i\,\langle \vec{x}, \vec{y} \rangle + i\,\langle \vec{y}, \vec{x} \rangle + |i|^2\,\langle \vec{y}, \vec{y} \rangle \\
&= \|\vec{x}\|^2 - i\,\langle \vec{x}, \vec{y} \rangle + i\overline{\langle \vec{x}, \vec{y} \rangle} + \|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 - i(\langle \vec{x}, \vec{y} \rangle - \overline{\langle \vec{x}, \vec{y} \rangle}) + \|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 - i2i\,\mathrm{Im}[\langle \vec{x}, \vec{y} \rangle] + \|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 + 2\,\mathrm{Im}[\langle \vec{x}, \vec{y} \rangle] + \|\vec{y}\|^2
\end{aligned}$$

$$\begin{aligned}
\|\vec{x}-i\vec{y}\|^2 &= \langle \vec{x}-i\vec{y}, \vec{x}-i\vec{y} \rangle \\
&= \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, -i\vec{y} \rangle + \langle -i\vec{y}, \vec{x} \rangle + \langle -i\vec{y}, -i\vec{y} \rangle \\
&= \|\vec{x}\|^2 + i\,\langle \vec{x}, \vec{y} \rangle - i\,\langle \vec{y}, \vec{x} \rangle + |i|^2\,\langle \vec{y}, \vec{y} \rangle \\
&= \|\vec{x}\|^2 + i\,\langle \vec{x}, \vec{y} \rangle - i\overline{\langle \vec{x}, \vec{y} \rangle} + \|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 + i(\langle \vec{x}, \vec{y} \rangle - \overline{\langle \vec{x}, \vec{y} \rangle}) + \|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 + i2i\,\mathrm{Im}[\langle \vec{x}, \vec{y} \rangle] + \|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 - 2\,\mathrm{Im}[\langle \vec{x}, \vec{y} \rangle] + \|\vec{y}\|^2
\end{aligned}$$

$\|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2 = 4\,\mathrm{Re}[\langle \vec{x}, \vec{y}\rangle]$
$\|\vec{x}+i\vec{y}\|^2 - \|\vec{x}-i\vec{y}\|^2 = 4\,\mathrm{Im}[\langle \vec{x}, \vec{y}\rangle]$

$$\begin{aligned}\frac{1}{4}\left[\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 + i(\|\vec{x}+i\vec{y}\|^2 - \|\vec{x}-i\vec{y}\|^2)\right] &= \frac{1}{4}\left(4\,\mathrm{Re}[\langle \vec{x}, \vec{y}\rangle] + 4i\,\mathrm{Im}[\langle \vec{x}, \vec{y}\rangle]\right) \\ &= \mathrm{Re}[\langle \vec{x}, \vec{y}\rangle] + i\,\mathrm{Im}[\langle \vec{x}, \vec{y}\rangle] \\ &= \langle \vec{x}, \vec{y}\rangle\end{aligned}$$

3.8.11

Show that for any $\vec{x}$ in an inner product space:

$$\|\vec{x}\| = \sup_{\|\vec{y}\|=1} |\langle \vec{x}, \vec{y}\rangle|$$

Let E be an inner product space.
Assume $\vec{x} \in E$.

By Cauchy-Schwarz:

$$\sup_{\|\vec{y}\|=1} |\langle \vec{x}, \vec{y}\rangle| \leq \sup_{\|\vec{y}\|=1} \|\vec{x}\|\,\|\vec{y}\| = \|\vec{x}\| \cdot 1 = \|\vec{x}\|$$

Also:

$$\sup_{\|\vec{y}\|=1} |\langle \vec{x}, \vec{y}\rangle| \geq \left|\left\langle \vec{x}, \frac{\vec{x}}{\|\vec{x}\|}\right\rangle\right| = \frac{1}{\|\vec{x}\|}\langle \vec{x}, \vec{x}\rangle = \frac{1}{\|\vec{x}\|}\|\vec{x}\|^2 = \|\vec{x}\|$$

So:

$\|\vec{x}\| \leq \sup_{\|\vec{y}\|=1} |\langle \vec{x}, \vec{y}\rangle| \leq \|\vec{x}\|$

$\therefore \|\vec{x}\| = \sup_{\|\vec{y}\|=1} |\langle \vec{x}, \vec{y}\rangle|$