Closure

Definition: Closure

Let E be a normed space and $S \subseteq E$. The *closure* of S, denoted $\operatorname{cl}(S)$ or $\overline{S}$, is the intersection of all closed subsets $V \subseteq E$ that contain S:

$$\operatorname{cl}(S) = \bigcap \{V \subseteq E \mid V \text{ is closed and } S \subseteq V\}$$

Properties

Let E be a normed space and $S \subseteq E$:

1). $cl(S)$ is closed.

2). $cl(S)$ is the smallest closed subset of E containing S.

Proof

1). By definition, $\operatorname{cl}(S)$ is an intersection of closed sets.

Therefore, $\operatorname{cl}(S)$ is closed.

2). Assume $S \subseteq V \subseteq E$ such that V is closed.

By definition, $\operatorname{cl}(S) \subseteq \operatorname{cl}(S) \cap V$.

$\therefore \operatorname{cl}(S) \subseteq V$.

Theorem

Let E be a normed space and $S \subseteq E$. The closure of S is the set of all limit points of S in E.

Proof

Let $\overline(S)$ be the closure of S and S' be the set of all limit points of S.
WTS: $\overline(S) = S'$

($\subset$) ABC: $\overline{S} \not\subseteq S'$.

There exists elements in $\overline{S}$ that are not limit points of S.
Let T be the set of all such non-S limit points in $\overline{S}$.
Let $V = \overline{S} \setminus T$.
V is still closed, but $S \subseteq V \subset \overline{S}$.
CONTRADICTION! (of the minimality of $\overline{S}$)

$\therefore \overline{S} \subseteq S'$

($\supset$) Assume $\vec{x} \in S'$

But $S \subseteq \overline{S}$ and $\overline{S}$ is closed.
So $\vec{x} \in \overline{S}$.

$\therefore S' \subseteq \overline{S}$.

Examples

1). $\mathrm{cl}(\mathbb{Q}) = \mathbb{R}$

2). $\mathrm{cl}(\mathcal{P}[a,b]) = \mathcal{C}[a,b]$

Definition: Dense

Let E be a normed space and $S \subseteq E$. To say that S is *dense* in E means:

$$\mathrm{cl}(S) = E$$

Theorem

Let E be a normed space and $S \subseteq E$. TFAE:

1). S is dense in E.

2). $\forall \vec{x} \in E, \exists (\vec{x}_n)$ in S such that $\vec{x}_n \to \vec{x}$.

3). Every non-empty, open subset of E contains an element of S.

In summary, every element in E is arbitrarity close to some element in S.

Proof

$(1 \iff 2)$

$$\begin{aligned} S \text{ is dense in } E &\iff \text{Every element in } E \text{ is a limit point of } S \\ &\iff \forall \vec{x} \in E, \exists (\vec{x}_n) \text{ in } S \text{ such that } \vec{x}_n \to \vec{x} \end{aligned}$$

$(2 \implies 3)$ Assume $\forall \vec{x} \in E, \exists (\vec{x}_n)$ in S such that $\vec{x}_n \to \vec{x}$.

Assume U is a non-empty, open subset of E.
Assume $\vec{x} \in U$.
Since U is open, $\exists \epsilon > 0, B(\vec{x}, \epsilon) \subseteq U$.
By assumption: $\exists (\vec{x}_n)$ in S such that $\vec{x}_n \to \vec{x}$.
So $\exists N > 0$ sufficiently large such that $\|\vec{x}_N - \vec{x}\| < \epsilon$.
And so $\vec{x}_N \in B(\vec{x}, \epsilon)$.

$\therefore \vec{x}_N \in U$.

$(3 \implies 2)$ Assume that every non-empty, open subset of E contains an element of S.

Assume $\vec{x} \in E$.
Construct a sequence $(\vec{x}_n)$ in S such that $\vec{x}_n \in B(\vec{x}, \dfrac{1}{n})$.
Assume $\epsilon > 0$.
Let $N = \dfrac{1}{\epsilon}$, and so $\dfrac{1}{N} = \epsilon$.
Assume $n > N$.
$\|\vec{x}_n - \vec{x}\| < \dfrac{1}{N} = \epsilon$.

$\therefore \vec{x}_n \to \vec{x}$.