Fundamental Groups of Topological Spaces

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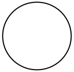
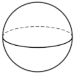
Motivation

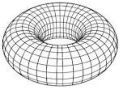
- Determining whether two topological spaces are homeomorphic is hard.
- Determining non-homeomorphism is easier: find a non-preserved topological property.
- Some spaces are problematic:



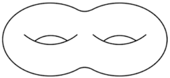
S^1



S^n for $n \geq 2$



$T = S^1 \times S^1$



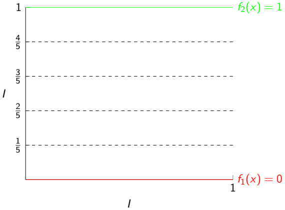
$T \# T$

- All are compact, but none are homeomorphic.

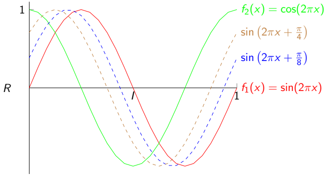
Homotopy

- Let $I = [0, 1] \subset \mathbb{R}$ imbued with the subspace topology.
- A continuous function $F : X \times I \to Y$ between continuous functions $f_1, f_2 : X \to Y$.
- $F(x, 0) = f_1(x)$ and $F(x, 1) = f_2(x)$.
- f_1 and f_2 are homotopic ($f_1 \simeq f_2$).
- If f_2 is a constant function then f_1 is called nulhomotopic.
- A continuous deformation of f_1 into f_2 via a parameterized family of continuous functions.

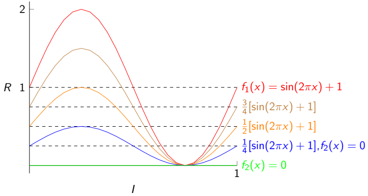
Homotopy Examples



$$F(x,t) = \pi_I(x,t) = t$$



$$F(x,t) = \sin\left(2\pi x + t\frac{\pi}{2}\right)$$

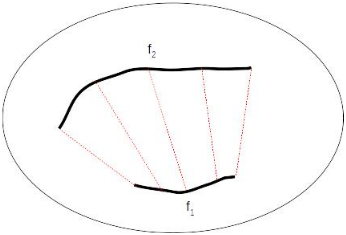


$$F(x,t) = (1-t)[\sin(2\pi x) + 1].$$

Homotopy Properties

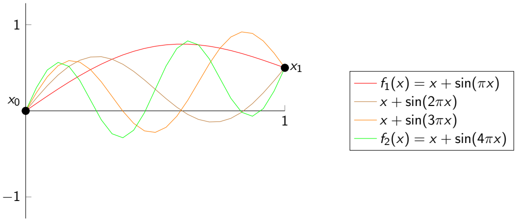
- Homotopic is an equivalence relation.
- Let $[f]$ denote the equivalence class of continuous functions that are homotopic to f.
- If $Y \subset \mathbb{R}^n$ is convex then any two continuous $f_1, f_2 : X \to Y$ are homotopic via the straight-line homotopy:

$$F(x,t) = (1-t)f_1(x) + tf_2(x)$$



Path Homotopy

- f_1 and f_2 are paths in X with the same initial (x_0) and final (x_1) points.
- $F : I \times I \to X$
- $F(x, 0) = f_1(x)$ and $F(x, 1) = f_2(x)$
- $F(0, t) = x_0$ and $F(1, t) = x_1$



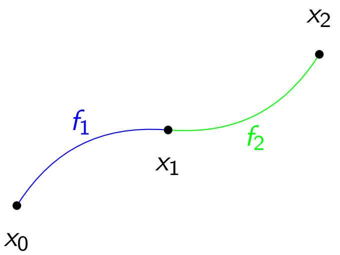
$$F(s, t) = s + \sin[(3t + 1)\pi s]$$

Product

- f_1 is a path from x_0 to x_1.
- f_2 is a path from x_1 to x_2.
- Concatenates the two paths:

$$f_1 * f_2 = \begin{cases} f_1(2t), & t \in \left[0, \frac{1}{2}\right] \\ f_2(2t-1), & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

- Continuous by the pasting lemma.



Product Groupoid

- $e_{x_0}(t) = x_0$
- $\bar{f}(t) = f(1-t)$ (reverse path)
- $*$ is a partial function on X: only works when $f_1(1) = f_2(0)$.
- Forms a groupoid.
- Associative: $([f] * [g]) * [h]$ is defined if and only if $[f] * ([g] * [h])$ is defined and if defined then they are equal.
- Identity: $[e_{x_0}] * [f] = [f]$ and $[f] * [e_{x_1}] = [f]$.
- Inverse: $[f] * [\bar{f}] = [e_{x_0}]$ and $[\bar{f}] * [f] = [e_{x_1}]$.

Fundamental Group

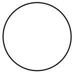
- A path that starts and ends at x_0 is called a loop based at x_0.
- For a topological space X, select some $x_0 \in X$.
- Select all loops based at x_0.
- The homotopic equivalence classes and $*$ form a group: $\pi_1(X, x_0)$ with identity $[e_{x_0}]$.
- For all $x_0, x_1 \in X$, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.
- Topologically invariant (up to isomorphism).
- Only addresses the path component containing x_0.
- If $\pi_1(X, x_0)$ is not isomorphic to $\pi_1(Y, y_0)$ then X and Y are not homeomorphic.

Simply Connected

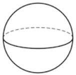
- For all $x_0 \in X$, $\pi_1(X, x_0) = \{[e_{x_0}]\}$ (trivial).
- Denoted by $\pi_1(X, x_0) = 0$.
- Any two paths with the same initial and final points are homotopic.
- Any convex subspace of $\mathbb{R}^n$ is simply connected.
- In particular, all open balls in $\mathbb{R}^n$ are simply connected: $\pi_1(B(p, r), x_0) = 0$.

Non-homeomorphic Spaces

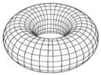
- $\pi_1(S^1, x_0) \sim \mathbb{Z}$ (times around the circle)



- $\pi_1(S^n, x_0) = 0$ for $n \geq 2$



- $\pi_1(T = S^1 \times S^1, x_0) \sim \mathbb{Z} \times \mathbb{Z}$ (abelian)



- $\pi_1(T \# T)$ is not abelian