Orthogonality

Definition

Let $\mathcal{H}$ be a Hilbert space and let $f, g \in \mathcal{H}$. To say that f is orthogonal (perpendicular) to g, denoted $f \perp g$, means that $\langle f, g \rangle = 0$.

Lemma

$f \perp g \iff g \perp f$

Proof

$f \perp g \iff \langle f, g \rangle = 0 \iff \overline{\langle g, f \rangle} = 0 \iff \langle g, f \rangle = 0 \iff g \perp f$

Theorem

$f \perp g \implies \|f + g\|^2 = \|f\|^2 + \|g\|^2$

Proof

Assume $f \perp g$
$\langle f, g \rangle = \langle g, f \rangle = 0$

$$\begin{aligned}
\|f + g\|^2 &= \langle f + g, f + g \rangle \\
&= \langle f, f \rangle + \langle f, g \rangle + \langle g, f \rangle + \langle g, g \rangle \\
&= \|f\|^2 + 0 + 0 + \|g\|^2 \\
&= \|f\|^2 + \|g\|^2
\end{aligned}$$

Definition

Let $\mathcal{H}$ be a Hilbert space. To say that $\{e_1, e_2, \ldots\} \subset \mathcal{H}$ is orthonormal means:

$$\langle e_j, e_k \rangle = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

In other words, every element has a norm of 1 and is orthogonal to every other element.

Theorem

Let $\mathcal{H}$ be a Hilbert space, $\{e_k\}_{k=1}^{\infty} \subset \mathcal{H}$ be orthonormal, and $f = \sum a_k e_k \in \mathcal{H}$ be a finite sum.

$$\|f\|^2 = \sum |a_k|^2$$

Proof

$\|f\|^2 = \langle f, f \rangle = \langle \sum a_j e_j, \sum a_k e_k \rangle$

Each term in the linear expansion is one of the following two forms:

1). $j = k$

$$\langle a_k e_k, a_k e_k \rangle = |a_k|^2 \langle e_k, e_k \rangle = |a_k|^2 \cdot 1 = |a_k|^2$$

2). $j \neq k$

$$\langle a_j e_j, a_k e_k \rangle = a_j \overline{a_k} \langle e_j, e_k \rangle = a_j \overline{a_k} \cdot 0 = 0$$

$\therefore \|f\|^2 = \sum |a_k|^2$