Continuity

Definition: Continuity

Let X and Y be topological spaces. To say that $f : X \to Y$ is *continuous* means that for every $U \in \mathscr{T}_Y$, $f^{-1}(U) \in \mathscr{T}_X$.

Definition: Neighborhood

Let X be a topological space and $p \in X$. To say that $U_p \subset X$ is a *neighborhood* of p means that $p \in U_p$ and $U_p \in \mathscr{T}$.

Notation

Let X be a topological space and $p \in X$. $\mathcal{N}_p =$ the set of all neighborhoods of p in X.

Lemma

Let X and Y be topological spaces and let $f : X \to Y$. For all $B \subset Y$:

$$X - f^{-1}(B) = f^{-1}(Y - B)$$

Proof. Assume $A \subset Y$.

$$\begin{aligned} x \in X - f^{-1}(B) &\iff x \notin f^{-1}(B) \\ &\iff f(x) \notin B \\ &\iff f(x) \in Y - B \\ &\iff x \in f^{-1}(Y - B) \end{aligned}$$

■

Theorem

Let X and Y be topological spaces and let $f : X \to Y$. TFAE:

1. f is continuous.
2. For every closed set $K \subset Y$, $f^{-1}(K)$ is closed in X.
3. For all $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.
4. For every $x \in X$ and $V \in \mathcal{N}_{f(x)}$ there exists $U \in \mathcal{N}_x$ such that $f(U) \subset V$.

Proof.

$1 \implies 2$ Assume that f is continuous.

Assume that $K \subset Y$ is closed, and so $Y - K \in \mathscr{T}_Y$. Since f is continuous, $f^{-1}(Y - K) \in \mathscr{T}_X$. Now, applying the lemma, $f^{-1}(Y - K) = X - f^{-1}(K) \in \mathscr{T}_X$. Therefore $f^{-1}(K)$ is closed.

$2 \implies 3$ Assume that for every closed set $K \subset Y$, $f^{-1}(K)$ is closed in X.

Assume $A \subset X$. Since $\overline{f(A)}$ is closed, by the assumption, $f^{-1}(\overline{f(A)})$ is closed. Furthermore, since $f(A) \subset \overline{f(A)}$, it must be the case that $f^{-1}(f(A)) = A \subset f^{-1}(\overline{f(A)})$. But $\bar{A}$ is the smallest closed set containing A, and so $\bar{A} \subset f^{-1}(\overline{f(A)})$. Therefore $f(\bar{A}) \subset f(f^{-1}(\overline{f(A)})) \subset \overline{f(A)}$.

$3 \implies 4$ Assume that for all $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.

Assume $x \in X$ and $V \in \mathcal{N}_{f(x)}$. Note that $Y - V$ is closed. Now, let $U = f^{-1}(V)$ and so $x \in U$ and $f(U) = f(f^{-1}(V) \subset V$.

WTS: U open.

ABC that $X - U$ is not closed. This means that there exists $p \in \overline{X - U}$ but $p \notin X - U$. And so, by the assumption and the lemma:

$$f(p) \in f(\overline{X-U}) \subset \overline{f(X-U)} = \overline{f(X - f^{-1}(V))} = \overline{f(f^{-1}(Y-V))} \subset \overline{Y-V} = Y - V$$

This means that $p \in f^{-1}(Y - V) = X - f^{-1}(V) = X - U$, contradicting the assumption that $p \notin X - U$. Thus $X - U$ contains all of its limit points and is closed. Therefore U is open.

$4 \implies 1$ Assume that for every $x \in X$ and $V \in \mathcal{N}_{f(x)}$ there exists $U \in \mathcal{N}_x$ such that $f(U) \subset V$.

Assume $V \in \mathscr{T}_Y$ and assume $p \in f^{-1}(V)$. Thus $f(p) \in V \in \mathcal{N}_{f(p)}$. Now, by the assumption, there exists $U \in \mathcal{N}_p$ such that $f(U) \subset V$, and hence $f^{-1}(f(U)) = U \subset f^{-1}(V)$. This means that p is an interior point of $f^{-1}(V)$ and hence $f^{-1}(V)$ is open. Therefore f is continuous.

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Theorem

Let X and Y be topological spaces and let $y_0 \in Y$. The constant map $f : X \to Y$ defined by $f(x) = y_0$ is continuous.

Proof. Assume that $V \in \mathscr{T}_Y$. If $y_0 \in V$ then $f^{-1}(V) = X$. Otherwise, $f^{-1}(V) = \emptyset$. In either case, $f^{-1}(V) \in \mathscr{T}_X$. Therefore f is continuous. ■

Theorem

Let Y be a topological space and let X be a subspace of Y. The inclusion map $i : X \to Y$ defined by $i(x) = x$ is continuous.

Proof. Assume $V \in \mathscr{T}_Y$. Then $i^{-1}(V) = V \cap X \in \mathscr{T}_X$. Therefore i is continuous. ■

Theorem

Let X and Y be topological spaces and let $f : X \to Y$ be continuous. For all $A \subset X$, $f|_A$ is continuous.

Proof. Assume $A \subset X$ and assume V is open in Y. Since f is continuous, $f^{-1}(V)$ is open in X. Furthermore, by definition of the subspace topology, $f|_A^{-1}(B) = f^{-1}(B) \cap A$ is open in A. Therefore $f|_A$ is continuous. ∎

Definition: Continuous

Let X and Y be topological spaces and $f : X \to Y$. To say that f is *continuous* at a point $x \in X$ means that for all $V \in \mathcal{N}_{f(x)}$ there exists $U \in \mathcal{N}_x$ such that $f(U) \subset V$. Thus, to say that f is continuous means that it is continuous at each $x \in X$.

Theorem

A function $f : \mathbb{R}_{\text{std}} \to \mathbb{R}_{\text{std}}$ is continuous iff for every $x \in \mathbb{R}$ and $\epsilon > 0$ there exists $\delta > 0$ such that for every $y \in \mathbb{R}$:

$$d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon$$

Proof.

$\implies$ Assume that f is continuous.

Assume $x \in \mathbb{R}$ and $\epsilon > 0$. Let $V = B(f(x), \epsilon) \in \mathcal{N}_{f(x)}$. Since f is continuous, there exists $U \in \mathcal{N}_x$ such that $f(U) \subset V$. But, since U is open, there exists $\delta > 0$ such that $B(x, \delta) \subset U$. Now, assume $y \in \mathbb{R}$ such that $d(x, y) < \delta$. This means $y \in B(x, \delta) \subset U \subset f^{-1}(V)$. Therefore $f(y) \in V$ and thus $d(f(x), f(y)) < \epsilon$.

$\impliedby$ Assume for every $x \in \mathbb{R}$ and $\epsilon > 0$ there exists $\delta > 0$ such that for every $y \in \mathbb{R}$:

$$d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon$$

Assume $x \in \mathbb{R}$ and $V \in \mathcal{N}_{f(x)}$. Since $f(x)$ is an interior point of V, there exists $\epsilon > 0$ such that $B(f(x), \epsilon) \subset V$. But by the assumption, this means that there exists $\delta > 0$ such that $U = B(x, \delta) \subset f^{-1}(V)$. Therefore $f(U) \subset V$ and thus f is continuous.

∎

Lemma

Let X be a 1^{st} countable topological space, $A \subset X$, and $p \in A$. There exists a sequence (a_n) in A such that $a_n \to p$ iff $p \in \bar{A}$.

Proof.

$\Longrightarrow$ Assume that there exists a sequence (a_n) in A such that $a_n \to p$.

Assume that $U \in \mathcal{N}_p$. This means that there exists some $N \in \mathbb{N}$ such that for all $n > N$, $a_n \in U$. But $a_n \in A$ also, so $U \cap A \neq \emptyset$. Therefore $p \in \bar{A}$.

$\Longleftarrow$ Assume that $p \in \bar{A}$.

This means that for all $U \in \mathcal{N}_p$ it must be the case that $U \cap A \neq \emptyset$. Now, since X is 1^{st} countable, assume that $\{B_k : k \in N\}$ is a countable neighborhood basis for p. Define the collection $\{U_n : n \in \mathbb{N}\}$ such that:

$$U_n = \bigcap_{k=1}^{n} B_k$$

Note that each U_n is a finite intersection of open sets and so $U_n \in \mathcal{N}_p$. Furthermore, since $p \in U_n$ and $p \in \bar{A}$, it must be the case that $U_n \cap A \neq \emptyset$. So select $a_n \in U_n \cap A$. Therefore (a_n) is in sequence in A and $a_n \to p$.

■

Theorem

Let X and Y be topological spaces such that X is 1^{st} countable. $f : X \to Y$ is continuous iff for every convergent sequence $x_n \to x$ in X, $f(x_n) \to f(x)$ in Y.

Proof.

$\Longrightarrow$ Assume that f is continuous.

Assume that $f(x_n) \not\to f(x)$. This means that there exists a $V \in \mathcal{N}_{f(x)}$ such that for all $N \in \mathbb{N}$ there exists an $n > N$ such that $f(x_n) \notin V$. So $f(x_n) \in Y - V$ and hence $x_n \in f^{-1}(Y - V) = X - f^{-1}(V)$. Thus $x_n \notin f^{-1}(V)$. But f is continuous, so $f^{-1}(V) \in \mathcal{N}_x$. Let $U = f^{-1}(V)$. Therefore, there exists $U \in \mathcal{N}_x$ such that for all $N \in \mathbb{N}$ there exists $n > N$ such that $x_n \notin U$, and thus $x_n \not\to x$.

$\Longleftarrow$ Assume for all sequences (x_n) in A, $x_n \to x$ implies $f(x_n) \to f(x)$.

Assume $A \subset X$ and $x \in \bar{A}$, and hence $f(x) \in f(\bar{A})$. By the lemma, there exists a sequence (x_n) in A such that $x_n \to x$. Furthermore, by the assumption, $f(x_n) \to f(x)$. But $f(x_n) \in f(A)$ and so $f(x) \in \overline{f(A)}$. Therefore $f(\bar{A}) \subset \overline{f(A)}$ and thus f is continuous.

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Theorem

Let X and Y be topological spaces such that $D \subset X$ is dense and Y is Hausdorff. Let $f : X \to Y$ and $g : X \to Y$ be continuous such that $\forall\, d \in D, f(d) = g(d)$. Then $\forall\, x \in X, f(x) = g(x)$.

Proof. ABC that there exists $x \in X$ such that $f(x) \neq g(x)$. Now, since Y is Hausdorff, there exists $U \in \mathcal{N}_{f(x)}$ and $V \in \mathcal{N}_{g(x)}$ such that $U \cap V = \emptyset$. Furthermore, since f and g are continuous,

$f^{-1}(U) \in \mathcal{N}_x$ and $g^{-1}(V) \in \mathcal{N}_x$. Since $x \in f^{-1}(U)$ and $x \in g^{-1}(V)$, this means that $f^{-1}(U) \cap g^{-1}(V) \neq \emptyset$, and so, since D is dense in X, there must exists $d \in D$ such that $d \in f^{-1}(U) \cap g^{-1}(V)$. But this means that $f(d) \in U \cap V$, contradicting the assumption that U and V are disjoint. Therefore $\forall\, x \in X, f(x) = g(x)$. ∎

Lemma

Let X, Y, Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous then for all $W \subset Z$:

$$(g \circ f)^{-1}(W) = (f^{-1} \circ g^{-1})(W)$$

Proof. Assume $W \subset Z$.

$$\begin{aligned} x \in (g \circ f)^{-1}(W) &\iff (g \circ f)(x) \in W \\ &\iff g(f(x)) \in W \\ &\iff f(x) \in g^{-1}(W) \\ &\iff x \in f^{-1}(g^{-1}(W)) \\ &\iff x \in (f^{-1} \circ g^{-1})(W) \end{aligned}$$

∎

Theorem

Let X, Y, Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous then their composition $g \circ f : X \to Z$ is continuous.

Proof. Assume that f and g are continuous and $W \in \mathscr{T}_Z$. Since g is continuous, $g^{-1}(W) \in \mathscr{T}_Y$. And, since f is continuous, $f^{-1}(g^{-1}(W)) = (f^{-1} \circ g^{-1})(W) = (g \circ f)^{-1}(W) \in \mathscr{T}_X$. Therefore $g \circ f$ is continuous. ∎

Theorem: Pasting Lemma

Let X and Y be a topological spaces such that $A \cup B = X$ for A, B closed in X and $f, g : A \to Y$ continuous functions that agree on $A \cup B$. The function $h : A \cup B \to Y$ defined by $h = f$ on A and $h = g$ on B is continuous.

Proof. Assume $K \subset Y$ is closed in Y. Since f and g are continuous, $f^{-1}(K)$ and $g^{-1}(K)$ are closed in X. Now, since f and g agree on $A \cup B$:

$$h^{-1}(K) = (h^{-1}(K) \cap A) \cup (h^{-1}(K) \cap B) = f^{-1}(K) \cap g^{-1}(K)$$

which is closed in X. Therefore h is continuous. ∎

Theorem

Let X and Y be topological spaces. If X is compact and $f : X \to Y$ is continuous and surjective then Y is compact.

Proof. Assume that X is compact and $f : X \to Y$ is continuous and surjective. Assume that $\{V_\alpha : \alpha \in \lambda\}$ is an open cover for Y. Since f is continuous, each $f^{-1}(V_\alpha) \in \mathscr{T}_X$. Furthermore, since f is surjective, $f^{-1}(\bigcup_{\alpha\in\lambda} V_\alpha) = \bigcup_{\alpha\in\lambda} f^{-1}(V)$ is an open cover of X. But X is compact, so there exists a finite subcover $\{f^{-1}(V_1), \ldots, f^{-1}(V_n)\}$ of X. And since f is surjective $\{V_1, \ldots, V_n\}$ is a finite subcover for Y. Therefore Y is compact. ∎

Theorem

Let X and Y be topological spaces. If D is dense in X and $f : X \to Y$ is continuous and surjective then $f(D)$ dense in Y.

Proof. Assume that D is dense in X and $f : X \to Y$ is continuous and surjective. Assume that $V \in \mathscr{T}_Y$ and $V \neq \emptyset$. Since f is continuous, $f^{-1}(V) \in \mathscr{T}_Y$. Furthermore, since f is surjective, $f^{-1}(V) \neq \emptyset$, and since D is dense in X, $f^{-1}(V) \cap D \neq \emptyset$. Therefore $f(U) \cap f(D) \neq \emptyset$ and thus $f(D)$ is dense in Y. ∎