Category Theory

Definition: Category

A *category* C is a mathematical structure consisting of two *classes*:

1). $\operatorname{obj}(C) =$ class of *objects* of C

2). $\operatorname{mor}(C) =$ class of *morphisms* of C

Objects are typically denoted by $A, B, C, \ldots$

Morphisms are disjoint sets, one per each pair of objects, denoted $\operatorname{mor}(A, B)$ for objects $A, B \in \operatorname{obj}(C)$. An element $f \in \operatorname{mor}(A, B)$, denoted $f : A \to B$ is called a morphism from A to B and is used to define structure between the objects.

Definition: Composition

Let C be a category and $A, B, C \in \operatorname{obj}(C)$. There exists a function:

$$\operatorname{mor}(B, C) \times \operatorname{mor}(A, B) \to \operatorname{mor}(A, C)$$

defined by $(g, f) \mapsto g \circ f$, where $g \circ f$ is called the *composite* of f and g and is subject to the following two axioms:

1). *Associativity*: $\forall\, f : A \to B, g : B \to C, h : C \to D$:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

2). *identity*: $\forall\, B \in \operatorname{obj}(C), \exists\, \iota_B : B \to B$ such that:

- $\forall\, f : A \to B, \iota_B \circ f = f$
- $\forall\, g : B \to C, g \circ \iota_B = g$

Note that $\operatorname{mor}(A, A) \neq \emptyset$ because at least $i_A \in \operatorname{mor}(A, A)$.

Definition: Equivalence

Let C be a category, $A, B \in \operatorname{obj}(C)$, and $f : A \to B$. To say that f is an *equivalence* means $\exists\, g : B \to A$ such that:

- $g \circ f = \iota_A$
- $f \circ g = \iota_B$

In this case, A and B are said to be *equivalent*.

Example

Let $\mathcal{G} =$ category of groups, where $A = \operatorname{obj}(\mathcal{G})$ is a group, and $\operatorname{mor}(A, B)$ is the (possibly empty) set of homomorphisms from A to B.

Definition: Product

Let C be a category and $\mathcal{S} = \{A_i \mid i \in I\}$ be a family of objects in $\mathrm{obj}(C)$. A *product* of $\mathcal{S}$, denoted $\prod_{i \in I} A_i$, is an object $P \in \mathrm{obj}(C)$ together with a family of morphisms $\{\pi_i : P \to A_i \mid i \in I\}$ such that $\forall\, B \in \mathrm{obj}(C)$ and family of morphisms $\{\phi_i : B \to A_i \mid i \in I\}$ there exists a unique morphism $\phi : B \to P$ such that $\forall\, i \in I, \pi_i \circ \phi = \phi_i$.

$$
\begin{tikzcd}
 & A_i & \\
P \arrow[ur, "\pi_i"] \arrow[dr, "\pi_j"'] & & B \arrow[ul, "\phi_i"'] \arrow[ll, "\phi"'] \arrow[dl, "\phi_j"] \\
 & A_j &
\end{tikzcd}
$$

For the category of groups, define $\pi_i : G \to G_i$ by projection.

Theorem

Let $\{A_i \mid i \in I\}$ be a family of objects in a category C. If $\{A_i \mid i \in I\}$ has a product then that product is unique (up to equivalence).

Proof

Assume $\{A_i \mid i \in I\}$ has a product
Let $(P, \{\pi_i\})$ and $(Q, \{\phi_i\})$ be two such products
$\exists\, \phi : Q \to P, \pi_i\phi = \phi_i$
$\exists\, \pi : P \to Q, \phi_i\pi = \pi_i$
$\phi_i(\pi\phi) = \phi$, so $\pi\phi = \iota_Q$
$\pi_i(\phi\pi) = \pi_i$, so $\phi\pi = \iota_P$
$\therefore$ P and Q are equivalent.

Definition: Coproduct

Let C be a category and $\mathcal{S} = \{A_i \mid i \in I\}$ be a family of objects in $\mathrm{obj}(C)$. A *coproduct* of $\mathcal{S}$ is an object $P \in \mathrm{obj}(C)$ together with a family of morphisms $\{\pi_i : P \to A_i \mid i \in I\}$ such that $\forall\, B \in \mathrm{obj}(C)$ and family of morphisms $\{\phi_i : B \to A_i \mid i \in I\}$ there exists a unique morphism $\pi : P \to B$ such that $\forall\, i \in I, \phi_i \circ \pi = \pi_i$.

In other words, a cofactor is a factor in the opposite direction.