Polynomials

For matrices, any field will work (e.g., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, any finite field); however, the default scalar field for this class will be $\mathbb{C}$.

Give $z = a + ib \in \mathbb{C}$, there are two important operations:

1). Conjugation: $\overline{z} = a - ib$

2). Modulus: $|z| = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}}$

An alternate representation using Euler's formula:

$$z = |z|\, e^{i\theta} = |z|\,(\cos\theta + i\sin\theta)$$

Definition

A *polynomial* in a complex variable z is of the form:

$$p(z) = \sum_{k=0}^{n} a_k z^k, \qquad a_k \in \mathbb{C}, a_n \neq 0$$

The *degree* of $p(z) = n$.

To say that $p(z)$ is *monic* means $a_n = 1$.

Definition

To say that z_0 is a *zero* of the polynomial $p(z)$ means:

$$p(z_0) = 0$$

Theorem: Fundamental Theorem of Algebra

Every non-constant polynomial has at least one zero.

Corollary

A monic polynomial of degree $n \geq 1$ has exactly n zeros, including multiplicity:

$$p(z) = \sum_{k=0}^{n} a_k z^k = \prod_{k=1}^{n}(z - \lambda_k)$$

Theorem

Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree $n \geq 1$:

$$a_{n-k} = (-1)^k \sum_{\mathcal{P}_k[n]} \prod_{i=1}^{k} \lambda_i$$

where $\mathcal{P}_k[n]$ are all k subsets of $\{1, 2, \ldots, n\}$.

In particular:

$$\begin{aligned} a_{n-1} &= -\sum_{k=1}^{n} \lambda_k \\ a_0 &= (-1)^n \prod_{k=1}^{n} \lambda_k \end{aligned}$$

Corollary

Let $p(z)$ be a polynomial with real coefficients:

$$p(z_0) = 0 \implies p(\overline{z_0}) = 0$$

In other words, all complex zeros must occur in conjugate pairs.

Proof

Assume $p(z_0) = 0$
$\overline{p(z_0)} = \overline{0} = 0$

$\therefore p(\overline{z_0}) = 0$