Stable

<u>**Definition: Stable**</u>

Let $F \subseteq L \subseteq K$ be an inclusion of fields. To say that L is *stable* with respect to K/F means:

$$\forall \varphi \in \mathrm{Aut}(K/F), \varphi[L] \subseteq L$$

<u>**Theorem**</u>

Let $F \subseteq L \subseteq K$ be an inclusion of fields such that L is stable with respect to K/F:

$$\forall \varphi \in \mathrm{Aut}(K/F), \varphi[L] = L$$

Thus $\varphi|_L \in \mathrm{Aut}(L/F)$.

<u>Proof</u>

Assume $\varphi \in G$
Since L is stable, $\varphi[L] \subseteq L$
But φ is bijective, so $\varphi|_L$ is surjective

$\therefore \varphi[L] = L$

<u>**Theorem**</u>

Let $F \subseteq L \subseteq K$ be an inclusion of fields such that K/F is Galois and L is stable with respect to K/F:

L/F is Galois

<u>**Theorem**</u>

Assume $\alpha \in L \setminus F$
$\exists \varphi \in \mathrm{Aut}(K/F), \varphi(\alpha) \neq \alpha$
But L is stable, so $\varphi|_L \in \mathrm{Aut}(L/F)$ and $\varphi|_L (\alpha) \neq \alpha$

Therefore L/F is Galois.

<u>**Theorem**</u>

Let $F \subseteq L \subseteq K$ be an inclusion of fields:

1). L stable $\implies G(L) \trianglelefteq G$

2). $H \trianglelefteq G \implies F(H)$ stable.

<u>Proof</u>

1). Assume L is stable

Assume $\varphi \in G(L)$
Assume $\alpha \in L$
$\varphi(\alpha) = \alpha$

Assume $\psi \in G$
Since ψ is bijective and L is stable, $\exists\, \beta \in L, \psi(b) = \alpha$ and $\psi^{-1}(\alpha) = \beta$
Also, since $\beta \in L$, $\varphi(\beta) = \beta$

$(\psi\varphi\psi^{-1})(\alpha) = \psi(\varphi(\psi^{-1}(\alpha))) = \psi(\varphi(\beta)) = \psi(\beta) = \alpha$
Thus, $\psi\varphi\psi^{-1} \in G(L)$

$\therefore G(L) \trianglelefteq G$.

2). Assume $H \trianglelefteq G$

Assume $\varphi \in H$
Assume $\psi \in G$
Since $H \trianglelefteq G$, $\psi^{-1}\varphi\psi \in H$

Assume $\alpha \in F(H)$
$(\psi^{-1}\varphi\psi)(\alpha) = \alpha$
$(\varphi\psi)(\alpha) = \psi(\alpha)$
$\varphi(\psi(\alpha)) = \psi(\alpha)$
Thus, $\psi(\alpha) \in F(H)$

Therefore, $F(H)$ is stable.