Cavallaro, Jeffery
Math 275A
Homework #3

## Lemma

Let $(X, \mathscr{T})$ be a topological space, $A \subset X$, and $p \in X$:

$$p \in \bar{A} \iff \forall U \in \mathcal{U}_p, U \cap A \neq \emptyset$$

*Proof.* By definition, $p \in \bar{A}$ iff $p \in A$ or $\forall U \in \mathcal{U}_p, (U - \{p\}) \cap A \neq \emptyset$. Assume that $U \in \mathcal{U}_p$. If $p \in A$ then $p \in U \cap A \neq \emptyset$. If $p \notin A$ then $(U - \{p\}) \cap A = U \cap A$. In either case: $p \in A$ or $\forall U \in \mathcal{U}_p, (U - \{p\}) \cap A \neq \emptyset$ is logically equivalent to $\forall U \in \mathcal{U}_p, U \cap A \neq \emptyset$. ■

## Theorem: 2.16

Let $(X, \mathscr{T})$ be a topological space:

1. $\emptyset$ is closed.
2. $X$ is closed.
3. The union of finitely many closed sets is closed.
4. Let $\{A_\alpha : \alpha \in \lambda\}$ be a family of closed sets. $\bigcap_{\alpha\in\lambda} A_\alpha$ is closed.

*Proof.*

1. $X$ is open, so $X - X = \emptyset$ is closed.
2. $\emptyset$ is open, so $X - \emptyset = X$ is closed.
3. $X - \bigcup_{i=1}^n A_i = \bigcap_{i=1}^n (X - A_i)$.

   But the $X - A_i$ are open and thus $X - \bigcup_{i=1}^n A_i$ is open.

   Therefore $\bigcup_{i=1}^n A_i$ is closed.
4. $X - \bigcap_{\alpha\in\lambda} A_\alpha = \bigcup_{\alpha\in\lambda}(X - A_\alpha)$.

   But the $X - A_\alpha$ are open and thus $X - \bigcap_{\alpha\in\lambda} A_\alpha$ is open.

   Therefore $\bigcup_{\alpha\in\lambda} A_\alpha$ is closed.

■

## Notation

Let $(X, \mathscr{T})$ be a topological space and $A \subset X$:

$$\mathcal{C} = \{B \subset X \mid B \text{ is closed}\}$$
$$\mathcal{C}_A = \{B \in \mathcal{C} \mid A \subset B\}$$

**Theorem: 2.20**

Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. The closure of $A$ equals the intersection of all closed sets containing $A$:

$$\bar{A} = \bigcap \mathcal{C}_A$$

Thus, $\bar{A}$ is the smallest closed set containing $A$.

*Proof.* Since $A \subset \bar{A}$ and $\bar{A}$ is closed, $\bar{A} \in \mathcal{C}_A$ and so:

$$\bar{A} \supset \bigcap \mathcal{C}_A$$

ABC:

$$\bar{A} \supsetneq \bigcap \mathcal{C}_A$$

This means that there exists some $B' \in \mathcal{C}_A$ such that:

$$\bar{A} \supsetneq \bar{A} \cap B' \supset A$$

where $\bar{A} \cap B' \in \mathcal{C}$.

This would imply that there exists some closed set containing $A$ with less limit points of $A$ than $\bar{A}$, which contradicts the definition of $\bar{A}$.

Therefore, $\bar{A} = \bigcap \mathcal{C}_A$. ∎

**Theorem: 2.22**

Let $(X, \mathscr{T})$ be a topological space and $A, B \subset X$:

1. $A \subset B \implies \bar{A} \subset \bar{B}$
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$

*Proof.*

1. Assume $A \subset B$.

   Assume $p \in \bar{A}$. This means that:

   $$\forall U \in \mathcal{U}_p, U \cap A \neq \emptyset$$

   But $A \subset B$ and so

   $$\forall U \in \mathcal{U}_p, U \cap B \neq \emptyset$$

   meaning that $p \in \bar{B}$ as well.

   Therefore $\bar{A} \subset \bar{B}$.

2. $(\subset)$ Since $A \subset \bar{A}$ and $B \subset \bar{B}$:

$$A \cup B \subset \bar{A} \cap \bar{B}$$

But $\bar{A} \cap \bar{B}$ is closed and the smallest closed set containing $A \cup B$ is $\overline{A \cup B}$. Therefore:

$$A \cup B \subset \overline{A \cup B} \subset \bar{A} \cup \bar{B}$$

$(\supset)$ Since $A \subset A \cup B$:

$$\bar{A} \subset \overline{A \cup B}$$

and similarly:

$$\bar{B} \subset \overline{A \cup B}$$

Therefore:

$$\bar{A} \cup \bar{B} \subset \overline{A \cup B}$$

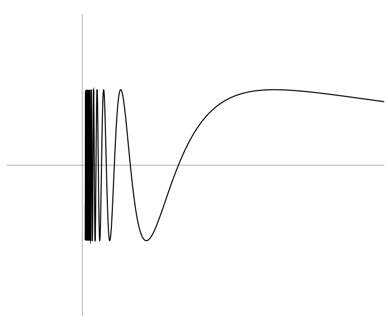
■

**Example: Exercise 2.24**

Let $(R^2, \mathscr{T})$:

1. Topologist's Sine Curve

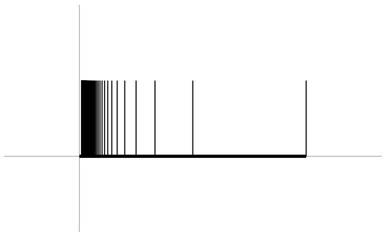
$$S = \left\{ \left( x, \sin\left(\frac{1}{x}\right) \right) \middle| x \in (0,1) \right\}$$



$$\bar{S} = S \cup \{(1, \sin(1))\} \cup \{(0, y) \mid y \in [-1, 1]\}$$

2. Topologists Comb

$$C = \{(x, 0) \mid x \in [0,1]\} \cap \bigcup_{n=1}^{\infty} \left\{ \left( \frac{1}{n}, 1) \right) \middle| y \in [0,1] \right\}$$



$\bar{C} = C \cup \{(0, y) \,|\, y \in [0, 1]\}$

**Theorem: 2.26**

Let $(X, \mathscr{T})$ be a topological space, $A \subset X$, and $p \in X$. $p$ is an interior point of $A$ iff there exists $U \in \mathscr{T}$ such that $p \in U \subset A$.

*Proof.* $p \in \mathrm{Int}(A) \iff p \in \bigcup \mathcal{U}_A \iff \exists U \in \mathcal{U}_A, p \in U \subset A$ ∎

**Theorem: 2.28**

Let $(X, \mathscr{T})$ be a topological space and let $A \subset X$. $\mathrm{Int}(A)$, $\mathrm{Bd}(A)$, and $\mathrm{Int}(X - A)$ are disjoint sets whose union is $X$.

*Proof.* Assume that $p \in \mathrm{Int}(A)$. This means that there exists $U \in \mathcal{U}_A$ such that $p \in U \subset A$. Now ABC that $p \in \mathrm{Bd}(A)$. This means that $p \in \overline{X - A}$ and so for all $U \in \mathcal{U}_p, U \cap (X - A) \neq \emptyset$. This contradicts the fact that there exists a $U \in \mathcal{U}_p$ that is a subset of $A$.

Therefore $\mathrm{Int}(A) \cap \mathrm{Bd}(A) = \emptyset$.

Similarly, assume that $p \in \mathrm{Int}(X - A)$. This means that there exists $U \in \mathcal{U}_{X-A}$ such that $p \in U \subset (X - A)$. Now ABC that $p \in \mathrm{Bd}(A)$. This means that $p \in \bar{A}$ and so for all $U \in \mathcal{U}_p, U \cap A \neq \emptyset$. This contradicts the fact that there exists a $U \in \mathcal{U}_p$ that is a subset of $X - A$.

Therefore $\mathrm{Int}(X - A) \cap \mathrm{Bd}(A) = \emptyset$.

Finally, note that for all $U \in \mathcal{U}_p$, $U$ cannot be a subset of both $A$ and $X - A$.

Therefore $\mathrm{Int}(A) \cap \mathrm{Int}(X - A) = \emptyset$.

Clearly, $\mathrm{Int}(A) \cup \mathrm{Int}(X - A) \cup \mathrm{Bd}(A) \subset X$. Assume that $p \in X$. If $p \in \mathrm{Int}(A)$ or $p \in \mathrm{Int}(X - A)$ then done, so assume that $X$ is in neither. This means that for all $U \in \mathcal{U}_p$, $U \cap A \neq \emptyset$ and $U \cap (X - A) \neq \emptyset$, and thus $p \in \bar{A}$ and $p \in \overline{X - A}$.

Therefore, $p \in \mathrm{Bd}(A)$. ∎

**Theorem: 2.30**

Let $(X, \mathscr{T})$ be a topological space, $A \subset X$, and $p \in X$:

$$\{x_i \,|\, i \in \mathbb{N}\} \subset A \text{ and } x_i \to p \implies p \in \bar{A}$$

*Proof.* Assume that $\{x_i \,|\, i \in \mathbb{N}\} \subset A$ and $x_i \to p$. Assume that $U \in \mathcal{U}_p$. This means that there exists some $N \in \mathbb{N}$ such that for all $i > N$ it is the case that $x_i \in U$. But $x_i \in A$ also, and so $U \cap A \neq \emptyset$.

Therefore $p \in \bar{A}$. ∎

**Theorem: 2.31**

Let $(\mathbb{R}^n, \mathscr{T})$ be the standard topology, $A \subset \mathbb{R}^n$, and $p \in X$ be a limit point of $A$. There exists a sequence of points in $A$ that converge to $p$.

*Proof.* Select $x_1 \in A$ such that $x_1 \neq p$. Note that $x_1 \in B(p, 2|x_1 - p|)$. Now select $x_{i+1} \in B(p, |x_i - p|) \cap A$, which cannot be empty since $p$ is a limit point of $A$. These $x_i$ fulfill the requirements for a sequence $(x_i)$ in $A$ that converges to $p$. ∎

**Example: Exercise 2.32**

Find an example of a topological space and a convergent sequence in that space for which the limit of the sequence is not unique.

Consider $(\mathbb{R}, \mathscr{T})$ with the indiscrete topology and consider any random sequence of points $x_i$. Since $\mathbb{R}$ is the only non-empty open set, any $p \in \mathbb{R}$ is a suitable limit for $(x_i)$.