Lab 5: The Cross Product

Before we learn about relations and then functions, we need to add one more set operation to our repertoire: the *cross product*. Before defining the cross product, we need to define what we mean by an *ordered pair*. An ordered pair is exactly what it sounds like: two numbers with a specific order. We represent ordered pairs like this:

$$(a, b)$$

The two numbers are in parenthesis and separated by a comma. In this case, a comes first, followed by b — the order is significant!

In order for two ordered pairs to be equal, both of their corresponding components must be equal:

$$(a, b) = (c, d) \iff a = c \text{ and } b = d$$

Note that (a, b) is not equal to (b, a) unless $a = b$. For example:

$$\begin{aligned} (1,3) &= (1,3) \\ (1,3) &\neq (3,1) \\ (1,3) &\neq (1,2) \\ (1,3) &\neq (2,3) \\ (1,1) &= (1,1) \end{aligned}$$

Now, given two sets A and B, we define their cross product as:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Thus, we make a set of ordered pairs consisting of all possible elements from A paired up with all possible elements of B.

When the sets are small and finite, it is easy to list out the elements of the cross product. For example, when $A = \{1, 2, 3\}$ and $B = \{10, 20\}$:

$$A \times B = \{(1, 10), (1, 20), (2, 10), (2, 20), (3, 10), (3, 20)\}$$

Note that since A has 3 elements and B has 2 elements, $A \times B$ has $3 \cdot 2 = 6$ elements.

Now you try. Let $A = \{1, 2, 3, 4\}$ and $B = \{\pi, \sqrt{2}\}$:

$$A \times B =$$

Your cross product set should have $4 \cdot 2 = 8$ elements.

When the sets involved are infinite, it is not possible to list all of the possible elements; however, we should be able to recognize what the elements look like:

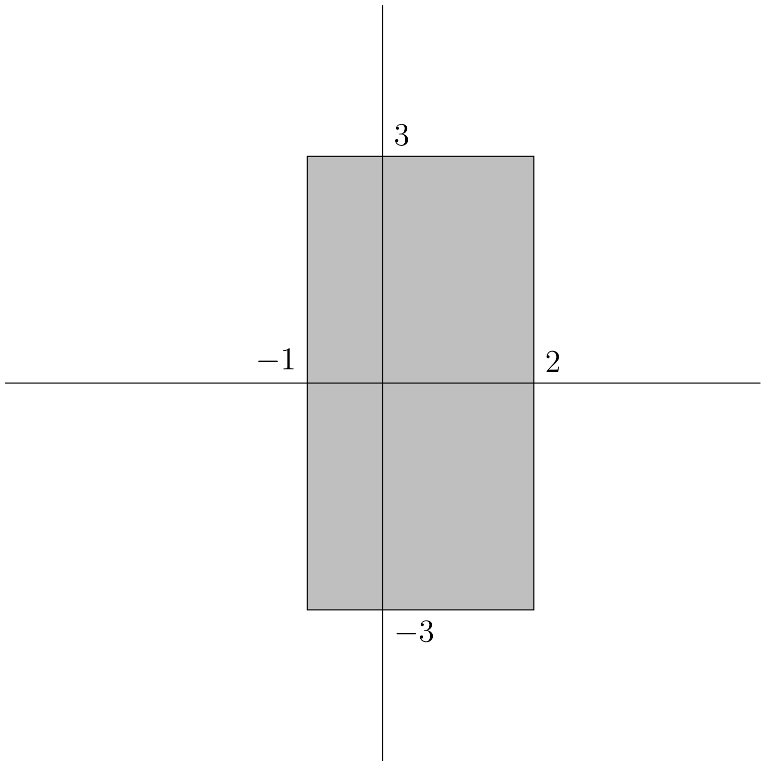
List 3 possible elements of the set $\mathbb{N} \times \mathbb{Q}$:

1).

2).

3).

And what about cross products of intervals? These will map out regions in the plane. For example, $[-1, 2] \times [-3, 3]$ would be as follows:



Here we overlay the first interval along the x-axis and the second interval along the y-axis. Note that every point within the filled rectangle defined by these intervals represents an element in the cross product of the intervals. For example: $(0, 0)$ is an element of the cross product; however $(3, 1)$ is not.

If $A = [-1, 2]$ and $B = [-3, 3]$ as above, determine whether each of the following points is or is not in $A \times B$. Be sure to use '$\in$' and '$\notin$' to indicate your choice:

1). $(1, 1) \quad A \times B$

2). $(-2, 0) \quad A \times B$

3). $(-1, -3) \quad A \times B$

4). $(3, 3) \quad A \times B$

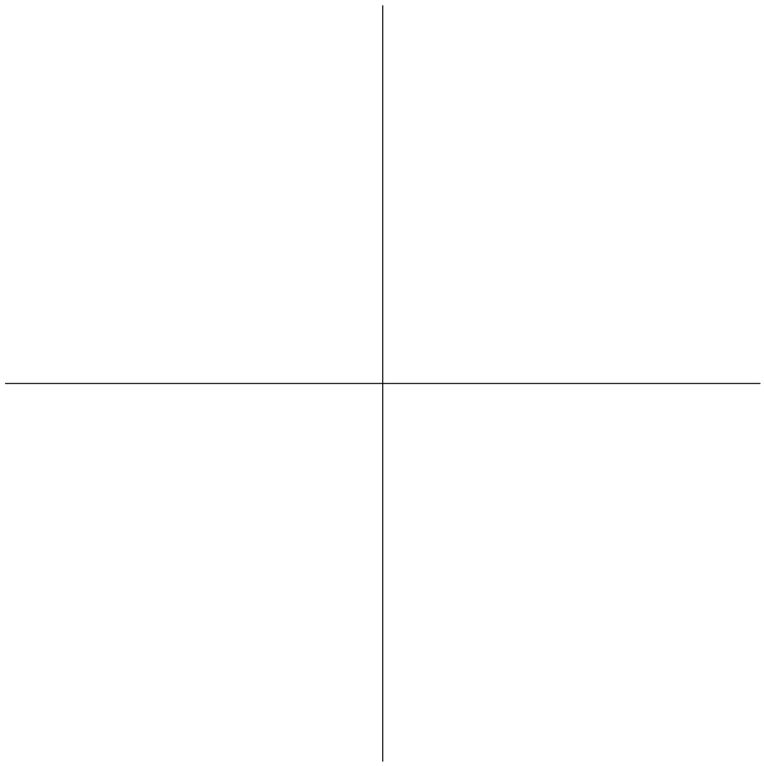
The cross product does not have to be a contiguous, continuous region. The following problem demonstrates this:

Let:

$$A = [-1, 2] \cup [3, 4]$$

$$B = [-3, 2]$$

Draw $A \times B$ below:



Indicate whether the following points are in or not in $A \times B$:

1). $(-2, 2) \quad A \times B$

2). $(0, 0) \quad A \times B$

3). $(\frac{5}{2}, 1) \quad A \times B$

4). $(3, 2) \quad A \times B$