Even/Odd Integers

Definition

To say that $n \in \mathbb{Z}$ is *even* means that it can be expressed as:

$$n = 2k, k \in \mathbb{Z}$$

To say that $n \in \mathbb{Z}$ is *odd* means that it can be expressed as:

$$n = 2k + 1, k \in \mathbb{Z}$$

Theorem

An odd integer n can be expressed as:

$$n = 2k - 1, k \in \mathbb{Z}$$

Proof

Assume n is odd
$\exists\, h \in \mathbb{Z}, n = 2h + 1$
Let $h = k - 1 \in \mathbb{Z}$
$n = 2h + 1 = 2(k - 1) + 1 = 2k - 1$

Theorem

An integer is either even or odd.

Proof

Assume $n \in \mathbb{Z}$

Case 1: n is positive

Proof by strong induction on n

Base: n=1,2

$1 = 2 \cdot 0 + 1$
$2 = 2 \cdot 1$
$\therefore$ 1 is odd and 2 is even.

Assume all integers from 1 to n are either even or odd.

Consider $n + 1$

Case 1: n is even

$\exists\, k \in \mathbb{Z}, n = 2k$
$n + 1 = 2k + 1$
$\therefore$ $n + 1$ is odd.

Case 2: n is odd

$\exists\, k \in \mathbb{Z}, n = 2k - 1$
$n + 1 = 2k - 1 + 1 = 2k$
$\therefore n + 1$ is even.

Case 2: $n = 0$

$0 = 2 \cdot 0$
$\therefore 0$ is even.

Case 3: n is negative

Let $m = -n$
m is positive

Case 1: m is even

$\exists\, k \in \mathbb{Z}, m = 2k$
$n = -m = -(2k) = 2(-k)$
But $(-k) \in \mathbb{Z}$
$\therefore n$ is even.

Case 2: m is odd

$\exists\, k \in \mathbb{Z}, m = 2k - 1$
$n = -m = -(2k - 1) = 2(-k) + 1$
But $(-k) \in \mathbb{Z}$
$\therefore n$ is odd.

Theorem

$\forall\, n \in \mathbb{Z}, n$ is even $\iff$ n^2 is even

Proof

$\implies$ Assume n is even

$\exists\, k \in \mathbb{Z}, n = 2k$
$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$
But, by closure, $2k^2 \in \mathbb{Z}$
$\therefore n^2$ is even.

$\impliedby$ Assume n is odd

$\exists\, k \in \mathbb{Z}, n = 2k + 1$
$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$
But, by closure, $2k^2 + 2k \in \mathbb{Z}$
$\therefore n^2$ is odd.