Uncountable Sets

Theorem: Cantor Diagonalization

The set of real numbers $\mathbb{R}$ is uncountable.

Proof. ABC that $(0, 1)$ is countable. This means that there exists some bijection $f : \mathbb{N} \to (0, 1)$. Let a_{ij} be j^{th} decimal digit of the i^{th} number:

$$\begin{aligned}
f(1) &= 0.a_{11}a_{12}a_{13}a_{14}a_{15}\cdots \\
f(2) &= 0.a_{21}a_{22}a_{23}a_{24}a_{25}\cdots \\
f(3) &= 0.a_{31}a_{32}a_{33}a_{34}a_{35}\cdots \\
f(4) &= 0.a_{41}a_{42}a_{43}a_{44}a_{45}\cdots \\
f(5) &= 0.a_{51}a_{52}a_{53}a_{54}a_{55}\cdots \\
\vdots &= \vdots
\end{aligned}$$

If $f(n)$ is rational with more than one representation, for example: $0.4\bar{9} = 0.5\bar{0}$, then the repeating 0 case is selected.

Now, let $b = b_1b_2b_3b_4b_5\cdots$ where:

$$b_i = \begin{cases} 1, & a_{ii} \neq 1 \\ 2, & a_{ii} = 1 \end{cases}$$

So b never contains a 0 or 9 digit and thus the non-unique cases are avoided. This means that $b \in (0, 1)$ but $b \notin f(\mathbb{N})$, contradicting the bijectiveness of f. Thus, $(0, 1)$ is uncountable. But $(0, 1) \subset \mathbb{R}$.

Therefore $\mathbb{R}$ is uncountable. ■

Definition: Power Set

Let A be a set. The *power set* of A, denoted by 2^A, is the set of all subsets of A.

Example

Let $A = \{a, b, c\}$:

$$2^A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Theorem

Let A be a finite set:

$$|2^A| = 2^{|A|}$$

Proof. For each $B \in 2^A$, for each $a \in A$, either $a \in B$ or $a \notin B$: 2 possibilities. Therefore, since there are $|A|$ elements in A:

$$|2^A| = 2^{|A|}$$

■

<u>Theorem</u>

Let A be a set. There exists an injection from A to 2^A.

Proof. Consider $f : A \to 2^A$ defined by $f(a) = \{a\} \subset A$. This is an injection from A to 2^A. ■

<u>Theorem</u>

Let A be a set and let P be the set of all functions from A to the two-point set $\{0, 1\}$:

$$|P| = |2^A|$$

Proof. Consider the function $f : P \to 2^A$ defined by $f(p) = B$ such that:

$$p(a) = \begin{cases} 0, & a \notin B \\ 1, & a \in B \end{cases}$$

Claim: f is a bijection.

Assume $f(p_1) = f(p_2) = B$. Assume $a \in A$. If $a \notin B$ then $p_1(a) = p_2(a) = 0$. If $a \in B$ then $p_1(a) = p_2(a) = 1$. So $\forall\, a \in A, p_1(a) = p_2(a)$. Thus, by definition, $p_1 = p_2$ and therefore f is injective.

Now, assume $B \in 2^A$. Since $B \subset A$, for each $a \in A$, a is either not in B or in B. So define $p : A \to \{0, 1\}$ as above. Thus $p \in P$ and $f(p) = B$. Therefore f is surjective.

Therefore f is a bijection and thus $|P| = |2^A|$. ■

<u>Theorem</u>

Let B be the set of all bit strings of infinite length.

$$|B| = |2^{\mathbb{N}}|$$

Proof. Let P be the set of all functions from $\mathbb{N}$ to the two-point set $\{0, 1\}$. Consider the function $f : P \to B$ defined by $f(p) = b$ such that $b = b_1 b_2 b_3 \cdots$ and $p(i) = b_i$.

Claim: f is a bijection.

Assume $f(p_1) = f(p_2) = b$. Assume $i \in \mathbb{N}$. If $b_i = 0$ then $p_1(i) = p_2(i) = 0$. If $b_i = 1$ then $p_1(i) = p_2(i) = 1$. So $\forall\, i \in \mathbb{N}, p_1(i) = p_2(i)$. Thus, by definition, $p_1 = p_2$ and therefore f is injective.

Now, assume $b \in B$. For each $i \in \mathbb{N}$, b_i is either 0 or 1. So define $p : \mathbb{N} \to \{0, 1\}$ as above. Thus $p \in P$ and $f(p) = b$. Therefore f is surjective.

Thus f is a bijection and $|P| = |B|$. But, by the previous theorem, $|P| = |2^{\mathbb{N}}|$.

$\therefore |B| = |2^{\mathbb{N}}|$ ■

<u>Theorem: Cantor Power Set</u>

Let A be a set:

$$|A| \neq |2^A|$$

Proof. Let $f : A \to 2^A$ and ABC that f is bijective. For all $a \in A$ let $f(a) = B_a$. This means that either $a \notin B_a$ or $a \in B_a$. Now, construct $B \in 2^A$ as follows:

$$B = \{a \in A \,|\, a \notin f(a)\}$$

Note that if $a \notin B_a$ then $a \in B$ and if $a \in B_a$ then $a \notin B$ and so $\forall\, a \in A, B_a \neq B$. Thus, $B \in 2^A$ but $B \notin f(A)$, contradicting the bijectiveness of f.

$\therefore |A| \neq |2^A|$ ■