Math-19 Homework #16 Solutions

Problems

1). Prove that the following is an identity:

$$\frac{1}{\csc x + \cot x} + \frac{1}{\csc x - \cot x} = 2\csc x$$

$$\begin{aligned}
\frac{1}{\csc x + \cot x} + \frac{1}{\csc x - \cot x} &= \frac{(\csc x - \cot x) + (\csc x + \cot x)}{\csc^2 x - \cot^2 x} \\
&= \frac{2\csc x}{1} \\
&= 2\csc x
\end{aligned}$$

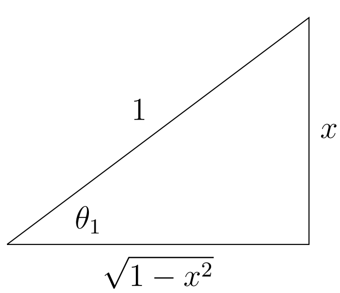
2). : Write the following as a function of x with no trig functions:

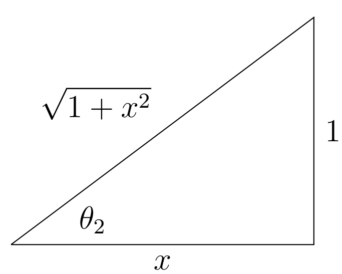
$$\sin\left(\sec^{-1}\frac{1}{\sqrt{1-x^2}} + \csc^{-1}\sqrt{1+x^2}\right)$$

Let:

$$\theta_1 = \sec^{-1}\frac{1}{\sqrt{1-x^2}}$$

$$\theta_2 = \csc^{-1}\sqrt{1+x^2}$$





$$\begin{aligned}
\sin(\theta_1 + \theta_2) &= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \\
&= \left(\frac{x}{1}\right)\left(\frac{x}{\sqrt{1+x^2}}\right) + \left(\frac{\sqrt{1-x^2}}{1}\right)\left(\frac{1}{\sqrt{1+x^2}}\right) \\
&= \frac{x^2 + \sqrt{1-x^2}}{\sqrt{1+x^2}}
\end{aligned}$$

3). Write the following as a single sine function. Note that you can use approximate value (i.e., your calculator) for the various coefficient and angle calculation. Use 4 decimal places.

$$\cos\left(x+\frac{\pi}{3}\right)+\sin\left(x-\frac{\pi}{4}\right)$$

First, use the sum and difference formulas in order to get simple sine and cosine terms:

$$\begin{aligned}
\cos\left(x+\frac{\pi}{3}\right)+\sin\left(x-\frac{\pi}{4}\right) &= \left(\cos x\cos\frac{\pi}{3}-\sin x\sin\frac{\pi}{3}\right)+\left(\sin x\cos\frac{\pi}{4}-\cos x\sin\frac{\pi}{4}\right)\\
&= \frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x+\frac{1}{\sqrt{2}}\sin x-\frac{1}{\sqrt{2}}\cos x\\
&= \left(\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{2}\right)\sin x+\left(\frac{1}{2}-\frac{1}{\sqrt{2}}\right)\cos x\\
&= -0.1589\sin x-0.2071\cos x
\end{aligned}$$

Next, calculate the new amplitude:

$A=\sqrt{(-0.1589)^2+(-0.2071)^2}=0.2610$

and then divide it out:

$A\left(\frac{-0.1589}{A}\sin x+\frac{-0.2071}{A}\cos x\right)$

We want a single sine function of the form:

$A\sin(x+\phi)=A(\sin x\cos\phi+\cos x\sin\phi)$

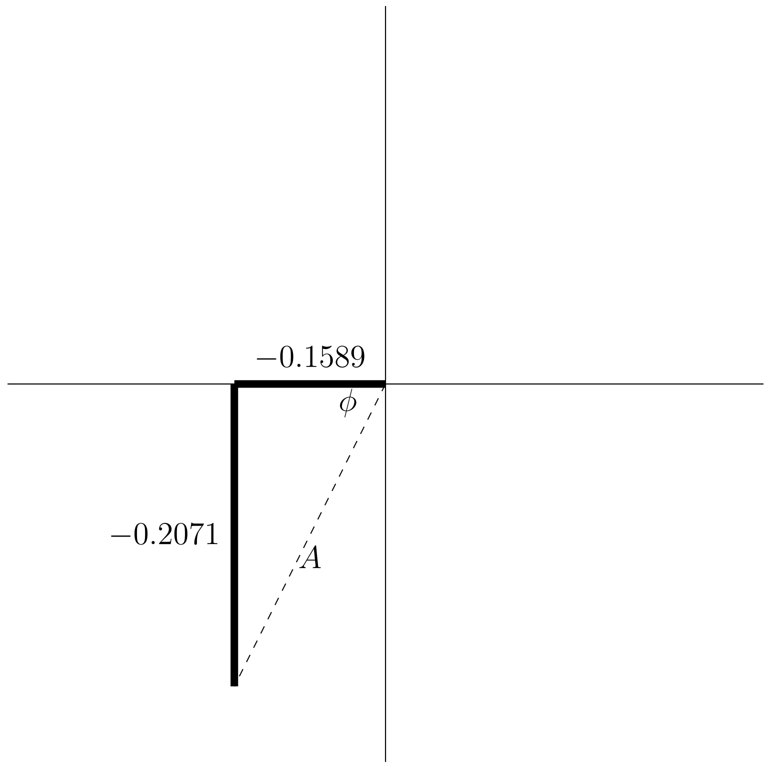
And so:

$\cos\phi=\frac{-0.1589}{A}\qquad\sin\phi=\frac{-0.2071}{A}$

And thus:

$\tan\phi=\frac{\sin\phi}{\cos\phi}=\frac{-0.2071}{-0.1589}$

Next, draw a triangle for ϕ in the proper quadrant. Note that since the horizontal ($\cos$) and vertical ($\sin$) sides are negative, ϕ will be in QIII:



Now calculate ϕ using $\tan^{-1}$ on your calculator. Be careful! Your calculator is going to give you the angle in the reduced domain, in this case QI; however, the actual angle is in QIII. So, you must add π to the value you get from your calculator to get the correct angle in QIII:

$$\phi = tan^{-1}\frac{-0.2071}{-0.1589} + \pi \approx 4.0579 \approx 232.5^\circ$$

Finally, putting it all together we get:

$$\cos\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{4}\right) = 0.2610\sin(x + 4.0579) = 0.2610\sin(x + 232.5^\circ)$$

Generally, we prefer that $\phi \in (-\pi, \pi]$, so subtract 2π:

$$\cos\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{4}\right) = 0.2610\sin(x - 2.2253) = 0.2610\sin(x - 127.5^\circ)$$

4). Find *all* possible solutions for x:

$$2\sin^2 x + (\sqrt{3} - 4)\sin x - 2\sqrt{3} = 0$$

$$(2\sin x + \sqrt{3})(\sin x - 2) = 0$$

$\sin x = 2$ has no solutions

$$\sin x = -\frac{\sqrt{3}}{2}$$

This occurs in QIII and QIV with a reference angle of $\frac{\pi}{3}$

$x = -\frac{\pi}{3} + 2\pi k$ or $x = -\frac{2\pi}{3} + 2\pi k$

5). Find *all* possible solutions for x:

$$\sin 2x + \cos x = 0$$

$\sin 2x + \cos x = 2 \sin x \cos x + \cos x = (2 \sin x + 1) \cos x = 0$

$\cos x = 0$ or $\sin x = -\frac{1}{2}$

$x = \frac{\pi}{2} + \pi k$

or

$x = -\frac{\pi}{6} + 2\pi k$ or $x = -\frac{5\pi}{6} + 2\pi k$