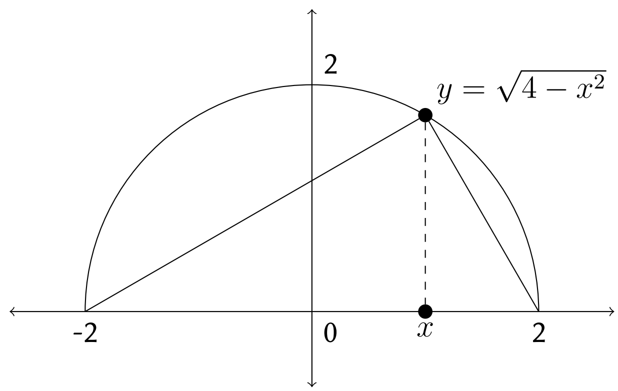
Math-71 Sections 02, 03, 60

Homework #5 Solutions

Problem

A triangle is inscribed inside a semi-circle of radius 2 as shown below:



Find the maximum possible area of the inscribed triangle. Be sure to prove why it is a maximum by either a plausible explanation or by using the first or second derivative test.

The equation for a circle with center at the origin and radius 2 is given by:

$$x^2 + y^2 = 4$$

Solving for y and taking the upper semicircle only:

$$y = \sqrt{4 - x^2}$$

Recall that the area of a triangle is given by:

$$A = \frac{1}{2}bh$$

In this case, the base b is a constant value of 4. The height h for a given x value is $y(x) = \sqrt{4 - x^2}$. Thus, the equation for the area of the triangle as a function of x is given by:

$$A(x) = \frac{1}{2}(4)\sqrt{4 - x^2} = 2(4 - x^2)^{\frac{1}{2}}$$

where the domain for x is $[-2, 2]$.

We want to maximize the value of $A(x)$ with respect to x, so we determine the derivative:

$$A'(x) = 2\left[\frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x)\right] = -\frac{2x}{\sqrt{4 - x^2}}$$

The first derivative has critical points at $x = 0$ (zero) and $x = \pm 2$ (poles).

Note that $x = \pm 2$ are both the endpoints and the zeros of the original function $A(x)$. Since $A(x) \geq 0$ these two points must be absolute minimums, so we can discard them.

Let's use test points (the second derivative is too messy) to prove that there is a relative maximum at $x = 0$. Since the denominator of $A'(x)$ is always positive, the sign is completely determined by the numerator. For a test point a little less than $x = 0$ it is the case that $A'(x) > 0$. For a test point a little greater than $x = 0$ it is the case that $A'(x) < 0$. Thus, there is a relative maximum at $x = 0$. Due to the absense of any more critical points and the fact that the endpoints are absolute minimums, we can conclude that we have an absolute maximum at $x = 0$.

Therefore, the maximum area is:

$$A(0) = 2\sqrt{4 - 0^2} = 2 \cdot 2 = 4$$