Dominating Inequalities

Example

Is there a matrix $A \in M_4$ such that $\mathrm{Sp}(A) = \{-6, -6, -6, 12\}$ with diagonal entries $\{6, 6, 6, -12\}$?

No, because $\mathrm{tr}(A)$ must equal the sum of the diagonals and the sum on the eigenvalues, but $-6 \neq 6$.

Example

Is there a matrix $A \in M_4$ such that $\mathrm{Sp}(A) = \{2, 5, 5, 10\}$ with diagonal entries $\{3, 3, 5, 11\}$?

Here, the trace test works out because $22 = 22$; however, recall that $\lambda_1 \leq a_{ii} \leq \lambda_n$, and $11 > 10$, so no.

Theorem

Let $A \in M_n$ be Hermitian with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and diagonal entries $d_1 \leq \cdots \leq d_n$, which can be in any desired order on the diagonal. $\forall\, m$ such that $1 \leq m \leq n$:

$$\sum_{k=1}^{m} \lambda_k \leq \sum_{k=1}^{m} d_k$$

with guaranteed equality at $m = n$ by the trace test.

The $\{\lambda_k\}$ are said to be *majorized* or *dominated* by the $\{d_k\}$.

Proof

Proof by induction on n

Base Case: $A \in M_1$

$\lambda_1 = d_1$

Assume the $\{d_k\}$ dominate the $\{\lambda_k\}$ for $A \in M_{n-1}$

Consider $A \in M_n$

Use permutation matrices to sort the diagonal entries from low to high
Note that the eigenvalues do not change

Let $A = \left[\begin{array}{ccc|c} d_1 & & * & \\ & \ddots & & * \\ * & & d_{n-1} & \\ \hline & * & & d_n \end{array}\right]$

Let $B = A_{n-1} = \begin{bmatrix} d_1 & & * \\ & \ddots & \\ * & & d_{n-1} \end{bmatrix}$

Let $\mathrm{Sp}(B) = \{\mu_1, \ldots, \mu_{n-1}\}$

By the inductive assumption, the $\{d_k\}$ dominate the $\{\mu_k\}$
By the interlacing theorem: $l_k \leq \mu_k$ for $1 \leq k \leq n-1$
Thus, the $\{d_k\}$ dominate the $\{\lambda_k\}$ for $1 \leq k \leq n-1$
By the trace theorem, equality happens at $k = n$

Therefore, $\forall\, m$ such that $1 \leq m \leq n$:

$$\sum_{k=1}^{m} \lambda_k \leq \sum_{k=1}^{m} d_k$$

with guaranteed equality at $m = n$ by the trace test.