

Introduction

There are some problems that algebra alone cannot solve.

Definition: Rate-of-change Problem

A *rate-of-change* problem seeks to determine how much one quantity changes with respect to a change in another quantity.

Examples: Rate-of-change Problems with respect to Time

- The velocity (speed) of a moving object: change in position (miles per hour, feet per second).
- The rate at which a product is produced during a chemical reaction (grams per second).
- The rate of radioactive decay (grams per year).
- The rate of population growth (members per year).
- The rate of change in the price of a stock during a particular trading day (dollars per hour).

Examples: Rate-of-change Problems with respect to Other Quantities

- The change in gravitational force applied to the earth with respect to changing distance from the sun (newtons per kilometer).
- The change in magnetic force applied to an iron nail with respect to changing distance from a magnet (newtons per centimeter).
- Elasticity of demand: the change in the quantity of a commodity sold with respect to a change in price (units per dollar).

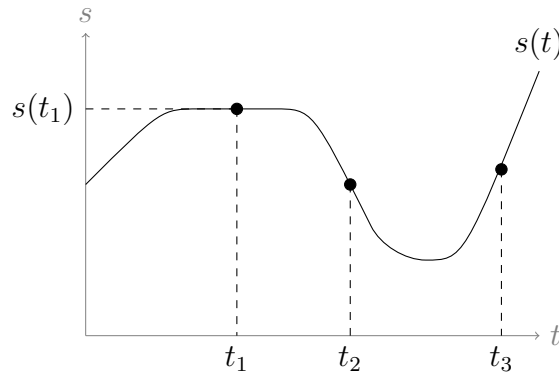
Why do we care?

- Was a car exceeding the speed limit when it passed a checkpoint?
- Is a rocket moving fast enough to escape the earth's gravity ($V_e \approx 11 \text{ km/s}$)?
- Which chemical reaction produces product the fastest?
- Is a radioactive substance safe for humans?
- Has a population's reproduction rate fallen beneath the replacement level?
- Will an increase in the price of a product produce more revenue?

Algebraically, these situations are modeled by a function $y = f(x)$. The quantity measured is represented by the dependent variable y and the quantity causing the change is represented by the independent variable x . Note that the variable names can change to represent the actual quantities in the problem. Furthermore, the function name usually matches the independent variable name.

Example

The position s of a moving body with respect to time t is modeled by $s = s(t)$:



Determining the body's position s at some time t is easy: $s(t)$. However, how the position is changing at a particular time t is a completely different question:

time	position
t_1	constant
t_2	decreasing
t_3	increasing

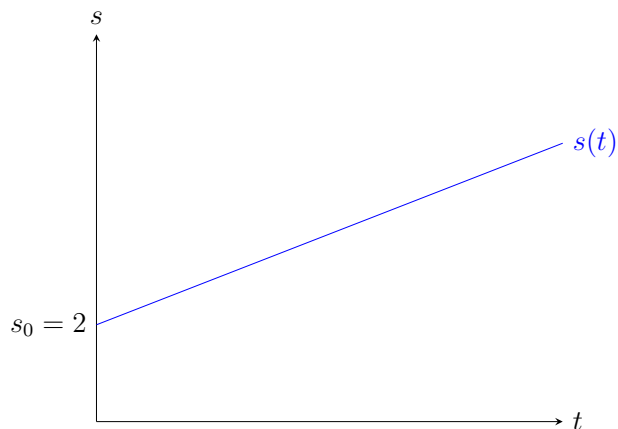
The goal of rate-of-change problems is to quantify the magnitude of such changes.

This is easy when the model is linear: $y = f(x)$ is a line.

Example

Consider a body moving in a straight line at constant velocity 0.75 ft/sec with initial position $s(0) = 2$ ft. The equation of motion is given by:

$$s(t) = \frac{3}{4}t + 2$$



The equation of motion is linear and the velocity is the slope of the line.

From a unit analysis standpoint:

$$\left(\frac{\text{ft}}{\text{sec}}\right) \text{sec} + \text{ft} = \text{ft}$$

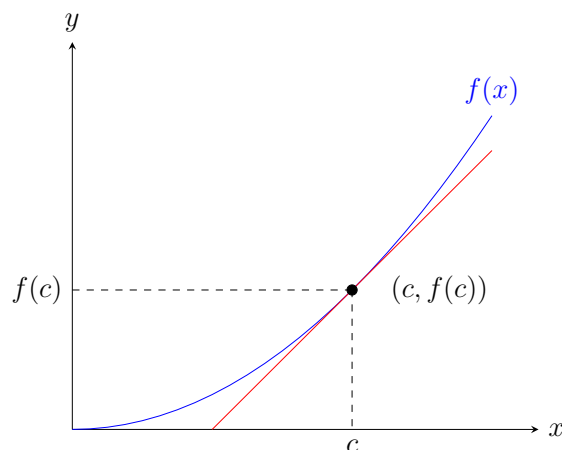
Thus, the change in position with respect to a change in time at any time t is simply the slope of the line. In fact, this is exactly how the slope of a line is calculated:

$$v = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1}$$

But what happens when the function is not linear?

Definition: Rate-of-change of a Function at a Point

Let $f(x)$ be a function. The *rate-of-change* of the function at a point $(c, f(c))$ is the slope of the tangent line to the function at that point.



We use this definition because it works. In the previous example, the tangent line at $(t_1, f(t_1))$ appears to be horizontal (slope=0), indicating that the function is constant. At $(t_2, f(t_2))$ the tangent has a negative slope, indicating that the function is decreasing. At $(t_3, f(t_3))$ the tangent has a positive slope, indicating that the function is increasing. Furthermore, the steeper the function, the steeper the tangent line.

Unfortunately, there is no way using just algebra to determine the slope of a tangent line to a general curve at a point.