Other Operations

Definition

Subtraction in $\mathbb{C}$ is defined as follows:

$$\begin{aligned} z_1 - z_2 &= z_1 + (-z_2) \\ &= (x_1, y_1) + (-x_2, -y_2) \\ &= (x_1 - x_2, y_1 - y_2) \\ &= (x_1 - x_2) + i(y_1 - y_2) \end{aligned}$$

Theorem

$$z^{-1} = \frac{1}{z}$$

Proof

$$\begin{aligned} \frac{1}{z} &= \frac{1}{x+iy} \\ &= \left(\frac{1}{x+iy}\right)\left(\frac{x-iy}{x-iy}\right) \\ &= \frac{x-iy}{x^2+y^2} \end{aligned}$$

Theorem

$$(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$$

Proof

$(z_1 z_2)(z_1^{-1} z_2^{-1}) = (z_1 z_1^{-1})(z_2 z_2^{-1}) = 1 \cdot 1 = 1$
So, $z_1 z_2$ and $z_1^{-1} z_2^{-2}$ are multiplicative inverses.
But multiplicative inverses are unique.
$\therefore (z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$

Definition

Division in $\mathbb{C}$ is defined as follows:

$$\frac{z_1}{z_2} = z_1 z_2^{-1}$$

$$\begin{aligned} &= z_1\left(\frac{1}{z_2}\right) \\ &= (x_1, y_1)\left(\frac{x_2}{x_2^2+y_2^2}, \frac{-y_2}{x_2^2+y_2^2}\right) \\ &= \left(\frac{x_1x_2+y_1y_2}{x^2+y^2}, \frac{y_1x_2-x_1y_2}{x_2^2+y_2^2}\right) \end{aligned}$$

We can also get the same result as follows:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1+iy_1}{x_2+iy_2} \\ &= \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} \\ &= \frac{(x_1x_2+y_1y_2)+i(y_1x_2-y_2x_1)}{x_2^2+y_2^2} \\ &= \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}+i\frac{y_1x_2-y_2x_1}{x_2^2+y_2^2} \end{aligned}$$

Theorem

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) = \frac{z_1z_2}{z_3z_4}$$

Proof

$$\begin{aligned} \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) &= (z_1z_3^{-1})(z_2z_4^{-1}) \\ &= (z_1z_2)(z_3^{-1}z_4^{-1}) \\ &= (z_1z_2)(z_3z_4)^{-1} \\ &= \frac{z_1z_2}{z_3z_4} \end{aligned}$$

Example

$$\left(\frac{2i}{1+i}\right)^4 = \left[\left(\frac{2i}{1+i}\right)\left(\frac{1-i}{1-i}\right)\right]^4 = \left[\frac{2i(1-i)}{2}\right]^4 = (1+i)^4 = [(1+i)^2]^2 = (2i)^2 = -4$$

Example

$$\frac{5}{(1-i)(2-i)(3-i)} = \frac{5(1+i)(2+i)(3+i)}{2\cdot 5\cdot 10} = \frac{(1+i)(5+5i)}{20} = \frac{(1+i)^2}{4} = \frac{2i}{4} = \frac{1}{2}i$$

Theorem

$$\sqrt{z} = \left(\frac{\sqrt{x^2+y^2}+x}{2}\right)^{\frac{1}{2}} + i\left(\frac{\sqrt{x^2+y^2}-x}{2}\right)^{\frac{1}{2}}$$

Proof

$z = x + iy$
$z^{\frac{1}{2}} = (x+iy)^{\frac{1}{2}} = u + iv$
$x + iy = (u+iv)^2 = u^2 - v^2 + i2uv$
$x = u^2 - v^2$
$y = 2uv$
$x^2 = u^4 + v^4 - 2u^2v^2$
$y^2 = 4u^2v^2$
$x^2 + y^2 = u^4 + v^4 + 2u^2v^2 = (u^2+v^2)^2$
$u^2 + v^2 = \sqrt{x^2+y^2}$
$u^2 - v^2 = x$
$2u^2 = \sqrt{x^2+y^2} + x$
$u = \left(\frac{\sqrt{x^2+y^2}+x}{2}\right)^{\frac{1}{2}}$
$2v^2 = \sqrt{x^2+y^2} - x$
$v = \left(\frac{\sqrt{x^2+y^2}-x}{2}\right)^{\frac{1}{2}}$
$\therefore \sqrt{z} = \left(\frac{\sqrt{x^2+y^2}+x}{2}\right)^{\frac{1}{2}} + i\left(\frac{\sqrt{x^2+y^2}-x}{2}\right)^{\frac{1}{2}}$

The binomial theorem also holds:

Theorem

$\forall z_1, z_2 \in \mathbb{C}$ and $n \in \mathbb{N}$:

$$(z_1+z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}$$

The proof is the same as it is for $\mathbb{R}$.