Cardinality

Definition: Cardinality

To say that two sets A and B have the same *cardinality*, denoted by $|A| = |B|$, means that there exists a bijection $f : A \to B$.

Notation

Let $n \in \mathbb{N}$:

$$[n] = \{1, \ldots, n\}$$

Definition: Finite

To say that a set A is *finite* means that either $A = \emptyset$ or there exists a bijection $f : A \to [n]$ for some $n \in \mathbb{N}$. For the empty set: $|\emptyset| = 0$. For a non-empty finite set: $|A| = n$. If A is not finite then it is *infinite*.

Definition: Countable

To say that a set A is *countable* means that either A is finite or $|A| = |\mathbb{N}|$. If A is not countable then it is *uncountable*.

Theorem

$|\mathbb{Z}| = |\mathbb{N}|$

Proof. Let $f : \mathbb{N} \to \mathbb{Z}$ be the bijection that enumerates the elements in $\mathbb{Z}$ as follows:

$$\begin{aligned} f(1) &= 0 \\ f(2) &= -1 \\ f(3) &= 1 \\ f(4) &= -2 \\ f(5) &= 2 \\ &\vdots \end{aligned}$$

Therefore $|\mathbb{Z}| = |\mathbb{N}|$. ∎

Theorem

$|2\mathbb{N}| = |\mathbb{N}|$

Proof. Let $f : \mathbb{N} \to 2\mathbb{N}$ be the bijection that enumerates the elements in $2\mathbb{N}$ as follows:

$$\begin{aligned}
f(1) &= 2 \\
f(2) &= 4 \\
f(3) &= 6 \\
f(4) &= 8 \\
f(5) &= 10 \\
&\vdots
\end{aligned}$$

Therefore $|2\mathbb{N}| = |\mathbb{N}|$. ∎

Theorem

Every subset of N is countable.

Proof. Assume $U \subset \mathbb{N}$. If U is finite then done, so assume that U is infinite. By the well-ordering principle, there exists some least element $u_1 \in U$ and $U = \{u_1, u_2, u_3, \ldots\}$ can be ordered such that $u_1 < u_2 < u_3 < \cdots$. So let $f : \mathbb{N} \to U$ be defined by $f(i) = u_i$. Thus, f is a bijection that enumerates the elements in U.

Therefore U is countable. ∎

Corollary

Let A and B be sets such that $A \subset B$. If B is countable then A is countable.

Proof. Assume that B is countable. If B is finite then A must be finite and thus countable, so assume that B is infinite. This means that there exists a bijection $f : B \to \mathbb{N}$, and so $f_A : A \to \mathbb{N}$ must be a bijection to a subset of N. But all subsets of N are countable.

Therefore A is countable. ∎

Corollary

Let A and B be sets such that $A \subset B$. If A is uncountable then B is uncountable.

Theorem

Every infinite set has a countably infinite subset.

Proof. Assume that X is an infinite set. If X is countable then done, so assume that X is uncountable. Select $x_1 \in X$ and let $U = \{x_1, x_2, x_3, \ldots\}$ where $x_i \in X$ and x_{i+1} is selected from $X - \left(\bigcup_{j=1}^{i} \{x_j\}\right)$. Now, let $f : \mathbb{N} \to U$ be defined by $f(i) = u_i$. Thus, f is a bijection that enumerates the elements in U.

Therefore $U \subset X$ and U is countable. ∎

Theorem

A set if finite if and only if every injection on the set is bijective.

Proof. Let X be a set. X is finite iff for every injection $f : X \to X$ it is the case that $|f(X)| = |X|$ iff every injection on X is bijective. ■

Theorem

A set is infinite if and only if there exists an injection from the set to a proper subset of itself.

Proof. X is finite if and only if every injection on X is bijective, is equivalent to: X is infinite if and only if there exists an injection on X that is not bijective, is equivalent to: X is infinite if and only if there exists an injection on X that is not surjective, is equivalent to: X is infinite if and only if there exists an injection on X to a proper subset of itself. ■

Example

If X is infinitely countable then all infinite subsets are also countable, so assume that X is uncountable. An injection to a proper subset can be constructed as follows:

1. Select an infinitely countable subset of X and call it U.
2. Construct an infinitely countable subset of U and call it S. Note that $|S| = |U|$.
3. Select a bijection $g : U \to S$.
4. Construct the set $A = S \cup (X - U)$. Note $A \subsetneq X$, since it does not contain the elements in $U - S$.
5. Define the injection $f : X \to A$ as follows:

$$f(x) = \begin{cases} g(x), & x \in U \\ x, & x \notin U \end{cases}$$

Theorem

The union of two countable sets is countable.

Proof. Let A and B be two countable sets. Since it is possible that $A \cap B \neq \emptyset$ define new sets as follows:

$$A' = A$$
$$B' = B - A$$

Note that $A \cup B = A' \cup B'$ and $A' \cap B' = \emptyset$. If A' or B' is finite then the finite set(s) can be enumerated first, followed by any countably infinite set, so assume that neither A' nor B' are finite. Let $A' = \{a_1, a_2, \ldots\}$ and $B' = \{b_1, b_2, \ldots\}$ and define $f : \mathbb{N} \to A' \cup B'$ as follows:

$$\begin{aligned} f(1) &= a_1 \\ f(2) &= b_1 \\ f(3) &= a_2 \\ f(4) &= b_2 \\ &\vdots \end{aligned}$$

Thus, f is a bijection that enumerates the elements of $A' \cup B'$ and hence $A' \cup B'$ is countable.

$\therefore A \cup B$ is countable. ■

Lemma

Let $\{U_i : i \in N\}$ be a countably infinite number of countably infinite sets such that the U_i are pairwise disjoint. Then:

$$U = \bigcup_{i \in \mathbb{N}} U_i$$

is countable.

Proof. Let $U_i = \{u_{ij} : j \in \mathbb{N}\}$ and arrange the U_i as the rows of a matrix. Note that the u_{ij} are distinct and in one-to-one correspondence with the elements of U.

Now, enumerate the u_{ij} along the diagonals as follows:

$$\begin{array}{cccccc} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & \cdots \\ u_{21} & u_{22} & u_{23} & u_{24} & u_{25} & \cdots \\ u_{31} & u_{32} & u_{33} & u_{34} & u_{35} & \cdots \\ u_{41} & u_{42} & u_{43} & u_{44} & u_{45} & \cdots \\ u_{51} & u_{52} & u_{53} & u_{54} & u_{55} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

This is a one-to-one correspondence between the u_{ij} and $\mathbb{N}$ and hence the u_{ij} are countable.

Therefore, U is countable. ■

Theorem

The union of countably many countable sets is countable.

Proof. Let $A = \bigcup_{i \in I} A_i$ be a union of countably many countable sets. In order to remove duplicates from the A_i (elements in the intersections of two or more A_i), let:

$$A'_1 = A_1$$
$$A'_i = A_i - \bigcup_{j=1}^{i-1} A_j$$

Note that the A'_i are pairwise disjoint and $A = \bigcup_{i \in I} A'_i$

Now arrange the A'_i as the rows of a matrix B. This means that the b_{ij} are distinct and in one-to-one correspondence with the elements of A. Let U be a matrix consisting of a countably infinite number of rows and columns as described in the preceding lemma. There is an injection between the rows in B and the rows in U. Furthermore, there is an injection between the columns of each row B_i and its corresponding row U_i. Thus, there is a one-to-one correspondence between the elements of B, and hence the elements of A, and the elements of some subset C of the elements of U. But the subset of a countable set is countable and so C is countable.

Therefore A is countable. ■

<u>Theorem</u>

The set $\mathbb{Q}$ is countable.

Proof. Let $\{Q_i : i \in \mathbb{N}\}$ be a family of sets where $Q_i = \left\{ \frac{p}{i} \,\middle|\, p \in \mathbb{Z} \right\}$. Note that:

$$\mathbb{Q} = \bigcup_{i \in \mathbb{N}} Q_i$$

But $\{Q_i : i \in \mathbb{N}\}$ is a countable number of countable sets, and hence is countable.

Therefore, Q is countable. ■

<u>Theorem</u>

The set of all finite subsets of a countable set is countable.

Proof. Assume that A is a countable set. Let A_i be the set of all finite subsets of A such that $|A_i| = i \in \mathbb{N}$ and let $\{A_i : i \in \mathbb{N}\}$ be the family of all such sets. Note that $B = \bigcup_{i \in \mathbb{N}} A_i$ is a union of a countable number of finite (countable) sets.

Therefore, B is countable. ■