Continuous Random Variables

Definition: Continuous Random Variable

To say that a random variable X is *continuous* means that its range is an interval or a union of intervals.

Definition: Probability Density Function

The *Probability Density Function* (pdf) of a continuous random variable X is a function $f : \mathbb{R} \to \mathbb{R}$ such that:

1. $\forall\, x \in \mathbb{R}, f(x) \geq 0$
2. $\displaystyle\int_{-\infty}^{\infty} f(x)dx = 1$
3. $\forall\, a, b \in \mathbb{R}$ such that $a \leq b$:

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

Properties: Probability Density Function

Let X be a continuous random variable:

- $\mathrm{Range}(X) = \{x \mid f(x) > 0\}$
- $x \notin \mathrm{Range}(X) \iff f(x) = 0$
- $P(X = c) = \int_c^c f(x)dx = 0$

Since the probability of a single value is 0, the endpoints of an interval do not matter:

$$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$$

Definition: Cumulative Distribution Function

The *Cumulative Distribution Function* (cdf) of a continuous random variable X is a function $F : \mathbb{R} \to \mathbb{R}$ such that:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

The *complementary cdf*, denoted $\bar{F}(x)$, is given by:

$$\bar{F}(x) = 1 - F(x)$$

Properties: Cumulative Distribution Function

Let X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$:

- $\displaystyle\lim_{x \to -\infty} F(x) = 0$

- $\lim_{x \to \infty} F(x) = 1$
- $F(x)$ is non-decreasing and continuous on $\mathbb{R}$.
- $P(a < X) = 1 - F(a)$
- $P(a < X < b) = F(b) - F(a)$
- $F'(x) = f(x)$

Definition: Median

The *median* of a continuous random variable X with pdf $f(x)$ and cdf $F(x)$, denoted $\tilde{\mu}$, is given by:

$$\frac{1}{2} = F(\tilde{\mu}) = \int_{-\infty}^{\tilde{\mu}} f(x)dx$$

Definition: Expected Value

The *expected value (mean)* of a continuous random variable X with pdf $f(x)$, denoted μ_X or $E(X)$, is given by:

$$\mu_X = E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

Similarly, for some function $h(x)$:

$$\mu_{h(X)} = E(h(X)) = \int_{-\infty}^{\infty} h(x)f(x)dx$$

Definition: Variance

The *variance* of a continuous random variable X with pdf $f(x)$ and expected value μ, denoted σ^2 or $V(X)$, is given by:

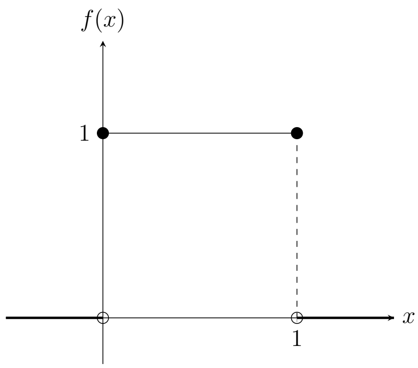
$$\sigma^2 = V(x) = E((X-\mu)^2) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = E(X^2) - \mu^2 = E(X^2) - E(X)^2$$

The *standard deviation*, denoted σ, is given by:

$$\sigma = \sqrt{\sigma^2} = \sqrt{V(X)}$$

Example

Let $f(x) = 1, 0 \leq x \leq 1$ be a pdf for a continuous random variable X:

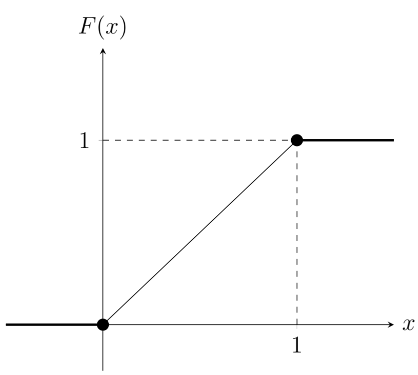


Since this pdf is simple, we can use area to calculate probabilities:

1. $P(X < -1) = 0$
2. $P(X = 0.2) = 0$
3. $P(X < 0.2) = 0.2 \cdot 1 = 0.2$
4. $P(0.2 < X < 0.5) = (0.5 - 0.2) \cdot 1 = 0.3$
5. $P(X > 0.6) = (1 - 0.6) \cdot 1 = 0.5$

To find the cdf:

$$F(x) = \int_0^x dt = t|_0^x = x$$



To find the median:

$$\frac{1}{2} = F(\tilde{\mu}) = \tilde{\mu}$$

To find the expected value:

$$E(X) = \int_0^1 x \cdot 1dx = \int_0^1 xdx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$$

To find the variance and standard deviation:

$$E(X^2) = \int_0^1 x^2 \cdot 1 dx = \int_0^1 x^2 dx = \frac{1}{3}x^2\Big|_0^1 = \frac{1}{3}$$

$$\sigma^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

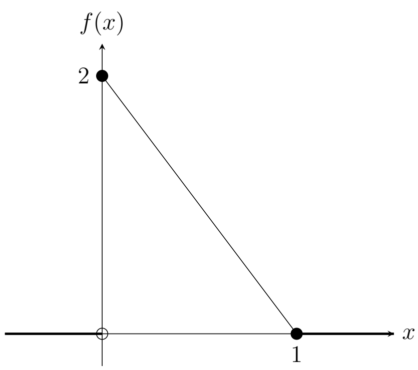
$$\sigma = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}} = \frac{1}{6}\sqrt{3}$$

Example

Let $f(x) = c(1-x), 0 < x < 1$ be the pdf for a continuous random variable X. We first need to find an appropriate value for c:

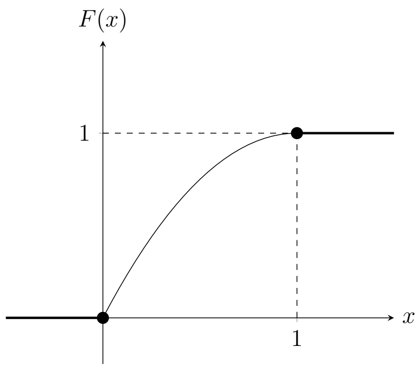
$$\int_0^1 c(1-x)dx = c\left(x - \frac{1}{2}x^2\right)\Big|_0^1 = \frac{1}{2}c = 1$$
$$c = 2$$

And so $f(x) = 2(1-x)$:



To find the cdf:

$$F(x) = \int_0^x 2(1-t)dt = \int_0^x (2-2t)dt = (2t - t^2)\big|_0^x = 2x - x^2 = x(2-x)$$



To find the median:

$$\frac{1}{2} = F(\tilde{\mu}) = 2\tilde{\mu} - \tilde{\mu}^2$$
$$\tilde{\mu}^2 - 2\tilde{\mu} + \frac{1}{2} = 0$$
$$2\tilde{\mu}^2 - 4\tilde{\mu} + 1 = 0$$
$$\tilde{\mu} = \frac{4 \pm \sqrt{16-8}}{4} = 1 \pm \frac{1}{2}\sqrt{2} = 1 \pm \frac{1}{\sqrt{2}}$$

Honoring the domain:

$$\tilde{\mu} = 1 - \frac{1}{\sqrt{2}} = 0.2929$$

To find the expected value:

$$E(X) = \int_0^1 x \cdot 2(1-x)dx = \int_0^1 (2x - 2x^2)dx = \left(x^2 - \frac{2}{3}x^2\right)\Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$

To find the expected value:

$$E(X) = \int_0^1 x \cdot 2(1-x)dx = \int_0^1 (2x - 2x^2)dx = \left(x^2 - \frac{2}{3}x^3\right)\Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$

To find the variance and standard deviation:

$$E(X^2) = \int_0^1 x^2 \cdot 2(1-x)dx = \int_0^1 (2x^2 - 2x^3)dx = \left(\frac{2}{3}x^3 - \frac{1}{2}x^4\right)\Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$\sigma^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

$$\sigma = \sqrt{\frac{1}{18}} = \frac{1}{3\sqrt{2}} = \frac{1}{6}\sqrt{2}$$