Rules

We now are going to develop our mathematical system for what we can do with the real numbers. We are going to establish a set of axioms for algebraic manipulation of expressions and equations. Everything you do must be traceable to these rules — don't make things up!

Expressions

Definition: Expression

An expression is a syntactic combination of constants, variables, and operators such that once values are selected for the variables, the expression can be evaluated using the rules of arithmetic to yield a single value.

An expression, no matter how complicated, is just a number!

Arithmetic Precedence Rules:

1). $()$

2). Exponentiation (R to L)

3). Multiplication (L to R)

4). Addition (L to R)

Example

(Do this example by hand, and then using a calculator)

$x + 2^{x^y} - 3(x + y) + y$

Let $x = 2$ and $y = 3$

$$\begin{aligned} 2 + 2^{2^3} - 3(2 + 3) + 3 &= 2 + 2^8 - 3(5) + 3 \\ &= 2 + 256 - 15 + 3 \\ &= 258 - 15 + 3 \\ &= 243 + 3 \\ &= 246 \end{aligned}$$

Recursive Construction:

1). Constant

2). Variable

3). (e)

4). $e_1^{e_2}$

5). $f(x)$

6). $e_1 e_2$ (products)

7). $e_1 + e_2$ (terms)

An important skill is the ability to identify the factors in an expression.

Example

$(x+1)xy^2(z-1)$ has 4 factors: $(x+1), x, y^2, (z-1)$

From now on, a statement like $a \in \mathbb{R}$ should be taken to mean that a is any expression for a real number.

What matters in the final, evaluated value (point on the number line). Thus, two expressions that result in the same value are said to be equal.

Properties: Equality

Let $a, b, c \in \mathbb{R}$:

1). Reflexive

$$a = a$$

2). Symmetric

$$a = b \implies b = a$$

3). Transitive

$$a = b \text{ and } b = c \implies a = c$$

Axiom: Substitution Principle

If two algebraic expressions are equal then one can syntactically replace the other.

This is the basis for algebraic simplification: a simpler expression can replace a more complicated expression when they are equal.

Example

When $x = 5$, $3x + 1$ is replaced with $3(5) + 1$ is replaced with $15 + 1$ is replaced with 16.

And now, the 10 rules/axioms. Any mathematic system that follows these 10 axioms is called a field. Everything that we do is mathematical systems involving the real numbers (including calculus) can be traced to one of these original axioms.

(Ref: page 12)

Note: Closure of the binary operators is assumed.

Properties: Field Axioms

1). Commutative Addition (CA)

$$\forall\, a, b \in \mathbb{R}, a + b = b + a$$

2). Associative Addition (AA)

$$\forall\, a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$$

3). Additive Identity (A0)

$$\exists\, 0 \in \mathbb{R}, \forall\, a \in \mathbb{R}, a + 0 = 0 + a = a$$

4). Additive Inverse (AI)

$$\forall\, a \in \mathbb{R}, \exists\, -a \in \mathbb{R}, a + (-a) = (-a) + a = 0$$

5). Commutative Multiplication (CM)

$$\forall\, a, b \in \mathbb{R}, ab = ba$$

6). Associative Multiplication (AM)

$$\forall\, a, b, c \in \mathbb{R}, (ab)c = a(bc)$$

7). Multiplicative Identity (M1)

$$\exists\, 1 \in \mathbb{R}, \forall\, a \in \mathbb{R}, a1 = 1a = a$$

8). Multiplicative Inverse (MI)

$$\forall\, a \in \mathbb{R} - \{0\}, \exists\, a^{-1} \in \mathbb{R}, aa^{-1} = a^{-1}a = 1$$

9). Left Distributive (LD)

$$\forall\, a, b, c \in \mathbb{R}, a(b + c) = ab + ac$$

10). Right Distributive (LD)

$$\forall\, a, b, c \in \mathbb{R}, (a+b)c = ac + bc$$

Be careful not to mix the rules of addition and multiplication, and do not make up rules of your own.

Because of commutativity and associativity, factors in products and products in terms can be evaluated in any order — but don't mix them!

Example

$1 + 2(3)(5) + (-2) + 3$

Properties: Zero

(Ref: page 14)

1). The additive identity is unique

2). $-0 = 0$

3). $\forall\, a \in \mathbb{R}, a0 = 0a = 0$

4). $\forall\, a, b \in \mathbb{R}, ab = 0 \implies a = 0$ or $b = 0$

Properties: Negatives

(Ref: page 13)

1). Additive inverses are unique

2). $-a = (-1)a$

3). $-(-a) = a$

4). $-(ab) = (-a)b = a(-b)$

5). $(-a)(-b) = ab$

6). $-(a+b) = (-a) + (-b)$

Don't assume that $-a$ is a negative number when you don't know exactly what a is. For example, a could be -2, in which case $-a = -(-2) = 2$, which is positive.

Definition: Subtraction

Subtraction is a syntactic convenience given by:

$$a - b = a + (-b)$$

Warnings about subtraction:

1). The negative sign applies to the syntactic element immediately following it:

$$1 - x + 2 = 1 + (-x) + 2$$

$$1 - (x + 2) = 1 - x - 2$$

2). It does not follow the rules:

$$2 - 1 \neq 1 - 2$$

$$(2 - 3) - 4 \neq 2 - (3 - 4)$$

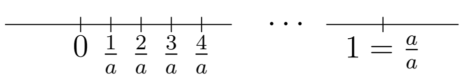
3). Be very careful with mixing subtraction and substitution:

Let $y = x - 5$

$4 - y = 4 - (x - 5) = 4 - x + 5 = 4 + (-x) + 5 = 9 - x$

Definition: Inverse

$\forall\, a \in \mathbb{R} - \{0\}, a^{-1} = \frac{1}{a}$



$a\left(\frac{1}{a}\right) = 1$

Don't assume that a is simple; it may be a fraction as well:

$$(a^{-1})^{-1} = \frac{1}{\frac{1}{a}} = a$$

Definition: Division

Division is a syntactic convenience given by:

$$\frac{a}{b} = a\left(\frac{1}{b}\right)$$

Example

$\frac{a+b}{c} = \frac{1}{c}(a + b) = \left(\frac{1}{c}\right) a + \left(\frac{1}{c}\right) b = \frac{a}{c} + \frac{b}{c}$

$\frac{a+b}{a} = \frac{1}{a}(a + b) = \left(\frac{1}{a}\right) a + \left(\frac{1}{a}\right) b = \frac{a}{a} + \frac{b}{a} = 1 + \frac{b}{a}$

Example

How many factors are there in:

$$\frac{(x+1)y^2}{z(w-1)^3}$$

There are 4: $x-1, y^2, \frac{1}{z}, \frac{1}{(w-1)^3}$

$$\frac{xy^2}{zw^3} = (x+1)(y^2)\left(\frac{1}{z}\right)\left[\frac{1}{(w-1)^3}\right]$$

Properties: Fractions

(Ref: page 14)

$\forall\, a, b, c, d \in \mathbb{R}$ such that $b, d \neq 0$:

1). $a \neq 0 \implies \frac{0}{a} = 0$

2). $\frac{a}{0}$ is undefined, or ∞, or indeterminate if $a = 0$

3). $\frac{a}{b} = \frac{c}{d} \iff ad = bc$

4). $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$

5). $\frac{-a}{-b} = \frac{a}{b}$

6). $c \neq 0 \implies \frac{a}{b} = \frac{ac}{bc}$

7). $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$

8). $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{db}$

9). $\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$

10). $c \neq 0 \implies \left(\frac{a}{b}\right) \div \left(\frac{c}{d}\right) = \frac{\frac{a}{b}}{\frac{c}{d}} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc}$