Free Abelian Groups

Definition

Let G be an abelian group and $X \subseteq G$. To say that X is a basis for G means:

- 1). $G = \langle X \rangle$
- 2). $\forall x_k \in X \text{ distinct and } n_k \in \mathbb{N}$:

$$\sum_{k=1}^{n} n_k x_k = 0 \iff \forall n_k = 0$$

Example

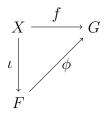
$$G = \bigoplus_{i=1}^{n} Z$$

Let
$$x_k = \begin{cases} 1, i = k \\ 0, i \neq k \end{cases}$$

Theorem

Let F be an abelian group. TFAE:

- 1). F has a non-empty basis.
- 2). F is the internal direct sum of a family of infinite cyclic groups.
- 3). F is isomorphic to a direct sum of copies of \mathbb{Z} .
- 4). There exists a non-empty set X and function $\iota:X\to F$ such that given an abelian group G and a function $f:X\to G$, there exists a unique homomorphism $\phi:F\to G$ such that $f=\phi\iota$.



A group ${\cal F}$ that satisfies these properties is called a $\it free$ abelian group.

Proof

 $1 \rightarrow 2$: Assume F has a non-empty basis X

Assume $x \in X$ $nx = 0 \implies n = 0$ Thus, $\langle x \rangle$ is infinite cyclic Since F is abelian, $\langle x \rangle \triangleleft F$ Since X is a basis, there exists a set $\{x_k\} \subseteq X$ such that $F = \langle \bigcup_{k=1}^n \langle x_k \rangle \rangle$

Assume
$$z \in X$$

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ABC: $\langle z \rangle \cap \left\langle \bigcup_{x_k \neq z} \langle x_k \rangle \right\rangle \neq \{0\}$
 $\exists n \in \mathbb{N}, nz = \sum_{k=1}^{n} n_k x_k$
 $nz + \sum_{k=1}^{n} n_k x_k = 0$; however, $n \neq 0$

$$\exists n \in \mathbb{N}, nz = \sum_{k=1}^{n} n_k x_k$$

$$nz + \sum_{k=1}^{n} n_k \overline{x_k} = 0$$
; however, $n \neq 0$

Contradiction (of basis)!

So
$$\langle z \rangle \cap \left\langle \bigcup_{x_k \neq z} \left\langle x_k \right\rangle \right\rangle = \{0\}$$

 \therefore F is an internal direct sum of the $\langle x_k \rangle$.

 $2 \rightarrow 3$: Assume F is the internal direct sum of a family of infinite cyclic groups

Let
$$F = \sum F_k$$
 be such a sum

$$F \simeq \bigoplus \overline{F_k}$$

$$F_k \simeq Z$$

There exists an isomorphism $\phi_k: F_k \to Z$

Therefore there exists an isomorphism $\phi: F \to \bigoplus \mathbb{Z}$.

 $3 \to 1$ Assume F is isomorphic to a direct sum of copies of \mathbb{Z}

Let $\{u_k\}$ be the standard basis for $\bigoplus \mathbb{Z}$

There exists isomorphism $\phi: \bigoplus \mathbb{Z} \to F$

Let
$$x_k = \phi_k(u_k)$$

Let
$$X = \{x_k\}$$

Let
$$f_i = \phi(z_i)$$

$$f_i=\phi(\sum n_k u_k)=\sum \phi_k(n_k u_k)=\sum n_k \phi_k(u_k)=\sum n_k x_x$$
 Thus, $F=\langle X\rangle$

Now, assume $\sum n_k x_k = 0$

$$\phi^{-1}(\sum n_k x_k) = \sum \phi_k^{-1}(n_k x_k) = \sum n_k \phi^{-1}(x_k) = \sum n_k u_k = 0$$
 This, $\forall n_k = 0$

Therefore, X is a basis for F.

 $1 \rightarrow 4$: Assume F has a non-empty basis X

Let $\iota: X \to F$ be the canonical injection homomorphism

Assume G is an abelian group and $f: F \to G$

Assume $u \in F$

$$u = \sum_{k=1}^{n} n_k x_k$$

Assume
$$u = \sum_{k=1}^{n} m_k x_k$$

Assume
$$u=\sum_{k=1}^n n_k x_k$$

Assume $u=\sum_{k=1}^n m_k x_k$

$$\sum_{k=1}^n n_k x_k - \sum_{k=1}^n m_k x_k = \sum_{k=1}^n (n_k - m_k) x_k = 0$$
But $n_k - m_k = 0$ since X is a basis for F

But
$$n_k - m_k = 0$$
 since X is a basis for F

So the representation for each $u \in F$ is unique wrt basis X

Let $\phi: F \to G$ be defined by $\phi(u) = \phi(\sum_{k=1}^n n_k x_k) = \sum_{k=1}^n n_k f(x_k)$

Since the representation for each $u \in F$ is unique, ϕ is well-defined

Assume $u,v\in F$

$$\phi(u+v) = \phi(\sum_{k=1}^{n} n_k x_k + \sum_{j=1}^{n} m_j x_j)$$

$$= \phi(\sum_{k=1}^{n} (n_k + m_k) x_k)$$

$$= \sum_{k=1}^{n} (n_k + m_k) f(x_k)$$

$$= \sum_{k=1}^{n} n_k f(x_k) + \sum_{k=1}^{n} m_k f(x_k)$$

$$= \phi(u) + \phi(v)$$

Therefore ϕ is a homomorphism

Assume that there is another such homomorphism ψ Since X generates F So the action of ψ on F is completely determined by ψ on X Assume $x\in X$

$$\psi(x) = \psi(\iota(x)) = (\psi\iota)(x) = f(x) = (\phi\iota)(x) = \phi(\iota(x)) = \phi(x)$$