Introduction

There are some problems that algebra alone cannot solve:

1. The slope of a tangent line to a curve
2. The area under a curve
3. Infinite sequences and series

Slope of a Tangent Line to a Curve

Definition: Rate-of-change Problem

A *rate-of-change* problem seeks to determine how much one quantity changes with respect to a change in another quantity.

Examples: Rate-of-change Problems with respect to Time

- The velocity (speed) of a moving object (miles per hour, feet per second).
- The rate at which a product is produced during a chemical reaction (grams per second).
- The rate of radioactive decay (grams per year).
- The rate of population growth (members per year).
- The rate of change in the price of a stock during a particular trading day (dollars per hour).

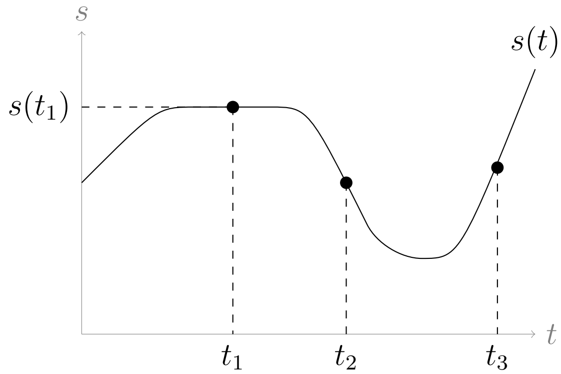
Examples: Rate-of-change Problems with respect to Other Quantities

- The change in gravitational force applied to the earth with respect to changing distance from the sun (newtons per kilometer).
- The change in magnetic force applied to an iron nail with respect to changing distance from a magnet (newtons per centimeter).
- Elasticity of demand: the change in the quantity of a commodity sold with respect to a change in price (units per dollar).

Algebraically, these situations are modeled by a function $y = f(x)$. The quantity measured is represented by the dependent variable y and the quantity causing the change is represented by the independent variable x. Note that the variable names can change to represent the actual quantities in the problem. Furthermore, the function name usually matches the independent variable name.

Example

The position s of a moving body with respect to time t is modeled by $s = s(t)$:



Determining the body's position s at some time t is easy: $s(t_1)$. However, how the position is changing at a particular time t is a completely different question:

time	position
t_1	constant
t_2	decreasing
t_3	increasing

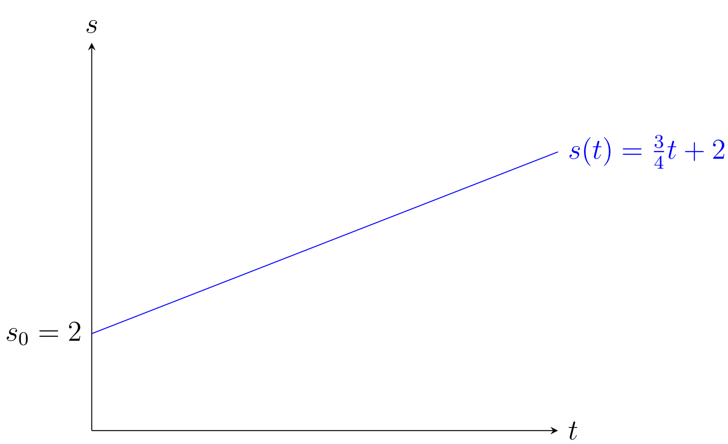
The goal of rate-of-change problems is to quantify the magnitude of such changes.

This is easy when the model is linear: $y = f(x)$ is a line.

Example

Consider a body moving in a straight line at constant velocity 0.75 ft/sec with initial position $s(0) = 2$ ft. The equation of motion is given by:

$$s(t) = \frac{3}{4}t + 2$$



The equation of motion is linear and the velocity is the slope of the line.

From a unit analysis standpoint:

$$\left(\frac{\mathsf{ft}}{\mathsf{sec}}\right)\mathsf{sec} + \mathsf{ft} = \mathsf{ft}$$

Thus, the change in position with respect to a change in time at any time t is simply the slope of the line. In fact, this is exactly how the slope of a line is calculated:

$$v = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1}$$

But what happens when the function is not linear?

Definition: Rate-of-change of a Function at a Point

Let $f(x)$ be a function. The *rate-of-change* of the function at a point $(c, f(c))$ is the slope of the tangent line to the function at that point.

We use this definition because it works. In the above example, the tangent line at $(t_1, f(t_1))$ appears to be horizontal (slope=0), indicating that the function is constant. At $(t_2, f(t_2))$ the tangent has a negative slope, indicating that the function is decreasing. At $(t_3, f(t_3))$ the tangent has a positive slope, indicating that the function is increasing. Furthermore, the steeper the function, the steeper the tangent line.

Unfortunately, there is no way using just algebra to determine the slope of a tangent line to a general curve at a point.

Why do we care?

1. Was a car exceeding the speed limit when it passed a checkpoint?
2. Which chemical reaction produces product the fastest?
3. Is a radioactive substance safe for humans?
4. Has a population's reproduction rate fallen beneath the replacement level?
5. Will an increase in the price of a product produce more revenue?

Area Under a Curve

Definition: Summation Problem

A *summation problem* seeks to determine the total amount of a quantity based on its rate of change over a particular range of values (region) of the other quantity.

For summation problems, a rate-of-change function $f(x)$ and a region of the independent variable x are given.

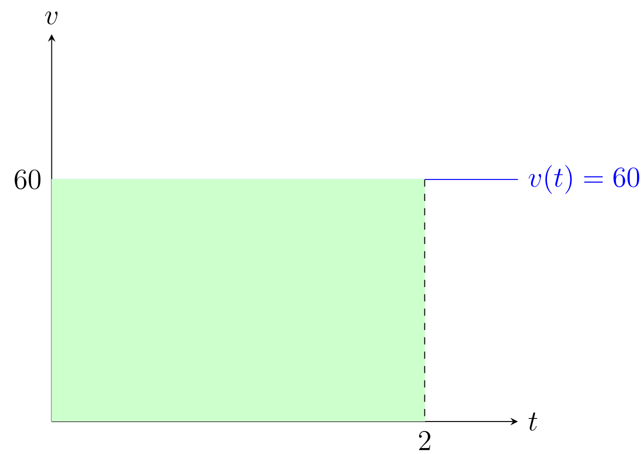
Example

A car is traveling at a constant speed of 60 mph. How far does the car travel in 2 hours?

This is simply a distance equal rate times time problem:

$$d = (60\,\mathrm{mi/hr})(2\,\mathrm{hr}) = 120\,\mathrm{mi}$$

Interpreted geometrically:

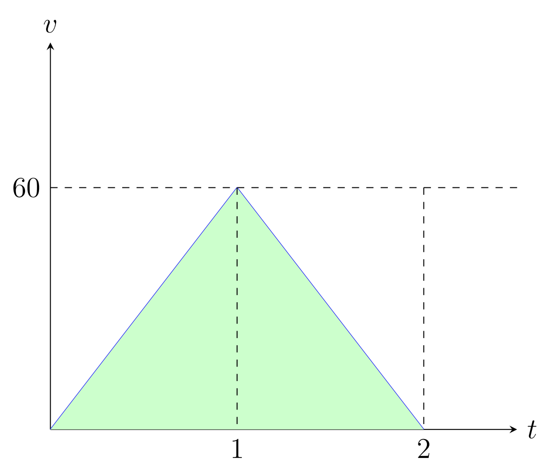


Note that the area under the rate-of-change curve in the selected region is calculated.

But what if the rate-of-change function is not constant? The total is still the area under the curve.

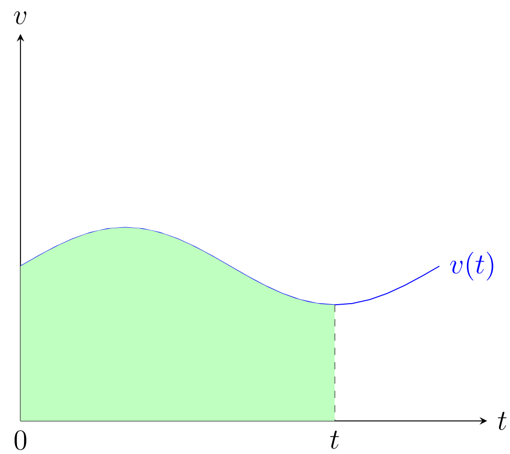
Example

A car is traveling with a speed that increases linearly from 0 mph to 60 mph after one hour, and then decreases linearly from 60 mph to 0 mph over the next hour. How far does the car travel?



$$d = \frac{1}{2}(60\,\mathsf{mi/hr})(2\,\mathsf{hr}) = 60\,\mathsf{mi}$$

But what happens if there is no convenient geometric formula for the area under the curve?



Unfortunately, there is no way using just algebra to determine the area under a general curve.

Why do we care?

1. How far does a car travel that varies its speed along the way?
2. How much product is produced during a chemical reaction that has a particular reaction rate?
3. What is the projected population given a particular growth rate?
4. Which requires less work to lift a box: pushing it up a ramp or lifting it with a pulley.

Infinite Sequences and Series

A sequence is an ordered collection of real numbers. Examples are:

1. Finite:

   $$1, 2, 3, 4, 5$$

2. Infinite with elements going to infinity:

   $$1, 2, 3, 4, 5, \ldots$$

3. Infinite with elements jumping around:

   $$1, -1, 1, -1, 1, -1, \ldots$$

4. Infinite with elements approaching a particular finite value:

   $$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$$

5. Decimal estimates of rational numbers:

$$0.1, 0.11, 0.111, 0.1111, 0.11111, \ldots$$

6. Decimal estimates of irrational numbers:

$$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots$$

The behavior of infinite sequences cannot be described using algebra alone.

A series is the sum of the terms of a sequence. Examples are:

1. Finite:

$$1 + 2 + 3 + 4 + 5 = 15$$

2. Infinite with elements going to infinity:

$$1 + 2 + 3 + 4 + 5 + \cdots \to \infty$$

3. Infinite with elements jumping around:

$$1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots \stackrel{?}{=} 0$$

4. Infinite with elements approaching a particular finite value that *diverge*:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}, \cdots \to \infty$$

5. Infinite with elements approaching a particular finite value that *converge*:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2$$

The behavior of infinite series cannot be described using algebra alone.

The behavior of infinite sequences and series are important for:

1. Determining series equivalents for some functions:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

2. Problems involving mechanical vibrations and signal processing: infinite series of harmonics:

$$f(t) = a_1 \sin \omega t + a_2 \sin 2\omega t + a_3 \sin 3\omega t + a_4 \sin 4\omega t + \cdots$$

3. Providing a rigorous theoretical basis for calculus in a real analysis class.