Orbits

Definition

Let σ be a permutation on a set A and let $a \in A$. The *orbit* of a under σ is given by:

$$\mathcal{O}_{a,\sigma} = \{\sigma^n(a) \mid n \in \mathbb{Z}\}$$

Theorem

Let σ be a permutation on a set A and define the relation $a \sim b$ iff $b \in \mathcal{O}_{a,\sigma}$:

$\sim$ is an equivalence relation

Proof

R: Assume $a \in A$
$\sigma^0(a) = a$
$\therefore a \sim a$

S: Assume $a \sim b$
$\exists\, n \in \mathbb{Z}, \sigma^n(a) = b$
$\sigma^{-n}(b) = a$
$\therefore b \sim a$

T: Assume $a \sim b$ and $b \sim c$
$\exists\, n, m \in \mathbb{Z}, \sigma^n(a) = b$ and $\sigma^m(b) = c$
$\sigma^m(\sigma^n(a)) = c$
$\sigma^{n+m}(a) = c$
$\therefore a \sim c$

$\therefore$ is an equivalence relation.

Thus, the orbits are the equivalence classes of the above equivalence relation.

Corollary

Let σ be a permutation on a set A and $a, b \in A$:

$$(\exists\, c \in A, c \in \mathcal{O}_{a,\sigma} \text{ and } c \in \mathcal{O}_{b,\sigma}) \implies \mathcal{O}_{a,\sigma} = \mathcal{O}_{b,\sigma}$$

Proof

Assume $\exists\, c \in A, c \in \mathcal{O}_{a,\sigma}$ and $c \in \mathcal{O}_{b,\sigma}$
$a \sim c$ and $b \sim c$
$c \sim b$
$a \sim b$
$\therefore \mathcal{O}_{a,\sigma} = \mathcal{O}_{b,\sigma}$