Cavallaro, Jeffery Math 221b Exam

1). Let R be a commutative ring with 1. Suppose that x is in the intersection of all maximal ideals of R. Show that x+1 is a unit in R.

Let $M = \bigcap M_i$ where M_i is maximal in R.

In particular, each M_i is proper and contains no units in R.

By assumption, $1 \in R$ and $1 \cdot 1 = 1$, so 1 is a unit in R and $\forall i, 1 \notin M_1$.

Assume $x \in M$

ABC: x + 1 is a non-unit in R

But every non-unit in R must be contained in some maximal ideal in R.

So $x + 1 \in M_i$ for some i.

But, by assumption, $x \in M_i$, and since M_i is an additive group, $-x \in M_i$.

By closure, $(-x) + (x+1) \in M_i$

$$(-x) + (x+1) = (-x+x) + 1 = 0 + 1 = 1 \in M_i$$

CONTRADICTION!

Therefore x + 1 is a unit in R.

2). Let d be a squarefree integer different from 1. Show that if $\pi \in R_d$ has norm $N(\pi) = p$ for some prime $p \in \mathbb{Z}$ then π is irreducible in R_d

Assume $\pi \in R_d$ has norm $N(\pi) = p$ for some prime $p \in \mathbb{Z}$.

Assume $\pi = ab$ for some $a, b \in R_d$.

So, by the multiplicity of the norm:

$$N(\pi) = N(ab) = N(a)N(b) = p$$

Now, by the integer criterion, since $a, b \in R_d$ it must be the case that $N(a), N(b) \in \mathbb{Z}$. But p is prime, hence the only divisors of p are p and p.

So N(a) = 1 or N(b) = 1, and thus by the unit criterion, either a or b is a unit in R_d .

Therefore, π is irreducible in R_d .

3). Explain why x^4 is irreducible over $\mathbb Q$. Find the splitting field K of x^4-2 over $\mathbb Q$ and show that $[K:\mathbb Q]=8$. Find three distinct quadratic subfields of K/Q.

1

By the rational root test, the only rational roots of x^4-2 would be from the set $\{\pm 1, \pm 2\}$; however, clearly none of these values are roots. Therefore, x^4-2 is irreducible over $\mathbb Q$.

To find the splitting field, find all of the complex roots of x^4-2 :

$$\begin{array}{rcl} x^4 - 2 & = & 0 \\ x^4 & = & 2 \\ x^4 & = & 2e^{i(2\pi n)} \\ x & = & \sqrt[4]{2}e^{i\frac{\pi}{2}n} \\ x & = & \sqrt[4]{2}, i\sqrt[4]{2}, -\sqrt[4]{2}, -i\sqrt[4]{2} \end{array}$$

Therefore, $K = \mathbb{Q}(\sqrt[4]{2}, i)$

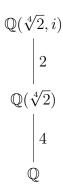
Now, consider the following extension field stack:



Since $\sqrt[4]{2}$ is a root of x^4-2 , which is irreducible in \mathbb{Q} , $m_{\sqrt[4]{2},\mathbb{Q}(\sqrt[4]{2})}(x)=x^4-2$ and thus $[\mathbb{Q}(\sqrt[4]{2}):\mathbb{Q}]=4$:



Clearly, $i \notin \mathbb{Q}(\sqrt[4]{2})$, and so $[\mathbb{Q}(\sqrt[4]{2},i):\mathbb{Q}(\sqrt[4]{2})] \neq 1$. But note that i is a root of $x^2+1\in\mathbb{Q}(\sqrt[4]{2})$, and since $\mathbb{Q}(\sqrt[4]{2},i)$ is a UFD, the only factorization of x^2+1 in $\mathbb{Q}(\sqrt[4]{2},i)$ is $(x+i)(x-i)\notin\mathbb{Q}(\sqrt[4]{2})$. Thus, x^2+1 is irreducible in $\mathbb{Q}(\sqrt[4]{2})$ and $m_{i,\mathbb{Q}(\sqrt[4]{2})}(x)=x^2+1$ and so $[\mathbb{Q}(\sqrt[4]{2},i):\mathbb{Q}(\sqrt[4]{2})]=2$:



Therefore,
$$[\mathbb{Q}(\sqrt[4]{2},i):\mathbb{Q}]=[\mathbb{Q}(\sqrt[4]{2},i):\mathbb{Q}(\sqrt[4]{2})][\mathbb{Q}(\sqrt[4]{2}):\mathbb{Q}]=2\cdot 4=8$$

Three quadratic subfields over $\ensuremath{\mathbb{Q}}$ can be constructed as follows:

| subfield | min poly |
|-------------------|-----------|
| Q(i) | $x^2 + 1$ |
| $Q(\sqrt[4]{4})$ | $x^2 - 2$ |
| $Q(i\sqrt[4]{4})$ | $x^2 + 2$ |

Note that all of the stated minimum polynomials are irreducible in $\mathbb Q$ by the rational root test.