Order Topology

Definition: Partial Ordering

Let X be a set and let $\leq$ be a binary operator on X. So say that $\leq$ is a *partial ordering* on X means that for all $x, y, z \in X$, $\leq$ is:

Reflexive: $x \leq x$

Antisymmetric: $x \leq y$ and $y \leq x \implies x = y$

Transitive: $x \leq y$ and $y \leq z \implies x \leq z$

A set with such a partial ordering, denoted by $(X, \leq)$, is called a *poset*.

Definition: Comparable

Let $(X, \leq)$ be a poset. To say that $x, y \in X$ are *comparable* means that $x \leq y$ or $y \leq$.

Example: Subset

Let X be a set. $\subset$ is a partial ordering on 2^X, since some elements of 2^X are not comparable.

Notation

Let $(X, \leq)$ be a poset and let $a \in X$:

$$a_< = \{x \in X \,|\, x < a\}$$
$$a_> = \{x \in X \,|\, x > a\}$$
$$(a, b) = \{x \in X \,|\, a < x < b\} = a_> \cap b_<$$

Definition: Total Ordering

Let $(X, \leq)$ be a poset. To say that $\leq$ is a *total ordering* on X means that every $x, y \in X$ is comparable.

Definition: Order Topology

Let $(X, \leq)$ be a total ordering and let:

$$B = \{a_< \,|\, a \in X\} \cup \{a_> \,|\, a \in X\} \cup \{(a, b) \,|\, a, b \in X\}$$

B is a basis for a topology $\mathscr{T}$ on X called the *order* topology.

Definition: Lexicographic Order

Let $A, \leq_A$ and $B, \leq_B$ be totally orderings. The *lexicographic* or *dictionary* ordering on $A \times B$ is given by:

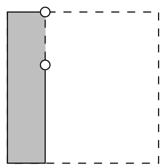
$$(a_1, b_1) < (a_2, b_2) \iff a_1 < a_2 \text{ or } a_1 = a_2 \text{ and } b_1 < b_2$$

Definition: Lexicographically Ordered Square

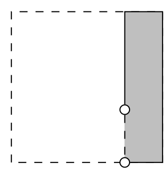
The square $[0,1] \times [0,1]$ with the lexicographic order and its associated order topology is called the *lexicographically ordered square*.

Example

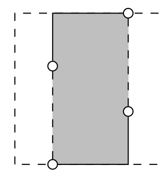
Draw pictures of various open sets in the lexicographically ordered square.



$a_<$



$a_>$



(a, b)

Example

Let $\mathcal{S}$ be the lexicographically ordered square and $A = \left\{ \left(\frac{1}{n}, 0 \right) \,\middle|\, n \in \mathbb{N} \right\} \subset \mathcal{S}$. Determine the closure of A.

$\bar{A} = A \cup \{(0, y) \,|\, y \in [0, 1]\}$