Sufficient Conditions for Differentiability

The Cauchy-Riemann equations holding is a necessary condition for differentiability; however, it is not sufficient.

Example

Let:

$$f(z) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} + i\frac{x^3+y^3}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Is it differentiable at $z = 0$?

$$u_x(0,0) = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x - 0} = \lim_{x\to 0} \frac{x - 0}{x} = \lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$$

$$v_y(0,0) = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{0 - y} = \lim_{y\to 0} \frac{y - 0}{y} = \lim_{y\to 0} \frac{y}{y} = \lim_{y\to 0} 1 = 1$$

$$v_x(0,0) = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x - 0} = \lim_{x\to 0} \frac{x - 0}{x} = \lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$$

$$u_y(0,0) = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{0 - y} = \lim_{y\to 0} \frac{-y - 0}{y} = \lim_{y\to 0} \frac{-y}{y} = \lim_{y\to 0} -1 = -1$$

$u_x = v_y$ and $v_x = -u_y$

$$f'(z) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{f(z) - 0}{z} = \lim_{z\to 0} \frac{f(z)}{z}$$

$(x, 0) \to (0, 0)$:

$$f'(z) = \lim_{x\to 0} \frac{x + ix}{x - 0} = \lim_{x\to 0} \frac{x + ix}{x} = \lim_{x\to 0} 1 + i = 1 + i$$

$(0, y) \to (0, 0)$:

$$f'(z) = \lim_{x\to 0} \frac{-y + iy}{y - 0} = \lim_{x\to 0} \frac{-y + iy}{y} = \lim_{x\to 0} -1 + i = -1 + i$$

Thus CR holds; however $f'(z)$ DNE.

Theorem

Let $f(z) = u(x, y) + iv(x, y)$ be defined in some domain D:

u_x, u_y, v_x, v_y exist, are continuous, and satisfy CR in $D \implies f$ is differentiable in D.

Proof

Assume u_x, u_y, v_x, v_y exist, are continuous, and satisfy CR in D
Assume $z_0 = x_0 + iy_0 \in D$
There exists some $N_\epsilon(z_0) \in D$ where the assumptions hold
Let $\Delta z = \Delta x + i\Delta y$ where $0 < |\Delta z| < \epsilon$
SInce the first order partials exist:
$du = u_x dx + u_y dy$ and $dv = v_x dx + v_y dy$
Since the first order partials are continuous:
$\Delta u = u_x \Delta x + u_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ and
$\Delta v = v_x \Delta x + v_y \Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y$
where the epsilons go to 0 as $(\Delta x, \Delta y) \to (0, 0)$

$$\begin{aligned}
\Delta w &= \Delta u + i\Delta v \\
&= u_x \Delta x + u_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y + i(v_x \Delta x + v_y \Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y)
\end{aligned}$$

Because CR holds, let $u_y = -v_x$ and $v_y = u_x$:

$$\begin{aligned}
\Delta w &= u_x \Delta x - v_x \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y + i(v_x \Delta x + u_x \Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y) \\
&= u_x(\Delta x + i\Delta y) + v_x(i\Delta x - \Delta y) + (\epsilon_1 + \epsilon_3)\Delta x + (\epsilon_2 + \epsilon_4)\Delta y \\
&= u_x(\Delta x + i\Delta y) + iv_x(\Delta x + i\Delta y) + (\epsilon_1 + \epsilon_3)\Delta x + (\epsilon_2 + \epsilon_4)\Delta y \\
&= u_x(\Delta z) + iv_x(\Delta z) + (\epsilon_1 + \epsilon_3)\Delta x + (\epsilon_2 + \epsilon_4)\Delta y \\
\frac{\Delta w}{\Delta z} &= u_x + iv_x + (\epsilon_1 + \epsilon_3)\frac{\Delta x}{\Delta z} + (\epsilon_2 + \epsilon_4)\frac{\Delta y}{\Delta z}
\end{aligned}$$

But $|\Delta x| \leq |\Delta z|$ and $|\Delta y| \leq |\Delta z|$ and so:

$$\left|(\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z}\right| \leq |\epsilon_1 + i\epsilon_3| \leq |\epsilon_1| + |\epsilon_3|$$

$$\left|(\epsilon_2 + i\epsilon_4)\frac{\Delta x}{\Delta z}\right| \leq |\epsilon_2 + i\epsilon_4| \leq |\epsilon_2| + |\epsilon_4|$$

and therefore:

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = u_x + iv_x + 0 + 0 = u_x + iv_x = f_x$$

and:

$$f'(z) = v_y - iu_y = -i(u_y + iv_y) = -if_y$$

Theorem

$f'(z)$ exists $\implies f_{\bar{z}} = 0$

Proof

Assume $f'(z)$ exists

$$\begin{aligned}
f_{\bar{z}} &= f_x x_{\bar{z}} + f_y y_{\bar{z}} \\
f_x &= u_x + iv_x \\
f_y &= u_y + iv_y \\
x_{\bar{z}} &= \frac{\partial}{\partial \bar{z}}\left(\frac{z+\bar{z}}{2}\right) = \frac{1}{2} \\
y_{\bar{z}} &= \frac{\partial}{\partial \bar{z}}\left(\frac{z-\bar{z}}{2i}\right) = -\frac{1}{2i} = \frac{1}{2}i \\
f_{\bar{z}} &= \frac{1}{2}(u_x + iv_x) + \frac{1}{2}i(u_y + iv_y) \\
&= \frac{1}{2}(u_x + iv_x) + \frac{1}{2}(-v_y + iu_y) \\
&= \frac{1}{2}(u_x - v_y) + i\frac{1}{2}(v_x + u_y) \\
&= \frac{1}{2}(0) + i\frac{1}{2}(0) \\
&= 0 + i0 \\
&= 0
\end{aligned}$$

Example

$f(z) = z + \bar{z} = 2x$

We already know that this is only differentiable at $z = 0$. Otherwise:

$$f_{\bar{z}} = 1 \neq 0$$

$u = 2x$ and $v = 0$, so the the first-order partials exist and are continuous; however, CR only holds at $z = 0$.