Continuity

Definition: Continuous

Let $T : E_1 \to E_2$ be a mapping of normed spaces. To say that T is continuous at $\vec{x}$ means $\forall\,(\vec{x}_n)$ in E_1:

$$\vec{x}_n \to \vec{x} \implies T(\vec{x}_n) \to T(\vec{x})$$

To say that T is continuous means T is continuous at $\forall\,\vec{x} \in E_1$.

Theorem

Let E be a normed space. $\|\cdot\| : E \to \mathbb{R}$ is continuous.

Proof

Assume $\vec{x} \in E$.
Assume $(\vec{x}_n)$ is a sequence in E such that $\vec{x}_n \to \vec{x}$.

$$|\|\vec{x}_n\| - \|\vec{x}\|| \leq \|\vec{x}_n - \vec{x}\| \to 0$$

Therefore $\|\cdot\|$ is continuous.

Theorem

Let $L : E_1 \to E_2$ be a mapping of normed spaces and let $U \subseteq E_2$. TFAE:

1). L is continuous.

2). U open $\implies L^{-1}[U]$ open.

3). U closed $\implies L^{-1}[U]$ closed.

Proof

$1 \to 2$: Assume L is continuous.

Assume U is open.
Assume $\vec{x} \in L^{-1}[U]$.
Since U is open, $\exists\,\epsilon > 0, B(L\vec{x}, \epsilon) \subset U$.
But since L is continuous, $\exists\,\delta > 0$ such that $\forall\,\vec{x}_n \in E$:

$$\|\vec{x}_n - \vec{x}\| < \delta \implies \|L\vec{x}_n - L\vec{x}\| < \epsilon$$

But $L\vec{x}_n \in U$, so $\vec{x}_n \in L^{-1}[U]$.
Therefore $B(\vec{x}, \delta) \in L^{-1}[U]$ and thus $L^{-1}[U]$ is open.

$2 \to 1$: Assume U open $\implies L^{-1}[U]$ open.

Assume $\vec{x} \in E_1$.
And so $L\vec{x} \in E_2$.

Assume $\epsilon > 0$.
Consider the open ball $U = B(L\vec{x}, \epsilon)$.
By assumption, $L^{-1}[U]$ is also open.
But $\vec{x} \in L^{-1}[U]$, so $\exists\, \delta > 0$ such that $B(\vec{x}, \delta) \subset L^{-1}[U]$.
Assume $\vec{x}_n \in E_1$ such that $\|\vec{x}_n - \vec{x}\| < \delta$.
This means that $\vec{x}_n \in B(\vec{x}, \delta)$, and hence in $L^{-1}[U]$.
Thus, $L\vec{x}_n \in U$ and $\|L\vec{x}_n - L\vec{x}\| < e$.

Therefore, L is continuous.

$2 \implies 3$: Assume U open $\implies L^{-1}[U]$ open.

Assume U is closed.
Thus $E_2 \setminus U$ is open.
Note that $L^{-1}[U] \cap L^{-1}[E_2 \setminus U] = \emptyset$, otherwise L is not well-defined.
By assumption, $L^{-1}[E_2 \setminus U] = E_1 \setminus L^{-1}[U]$ is open.

Therefore $L^{-1}[U]$ is closed.

$3 \implies 2$: Assume U closed $\implies L^{-1}[U]$ closed.

Assume U is open.
Thus $E_2 \setminus U$ is closed.
Note that $L^{-1}[U] \cap L^{-1}[E_2 \setminus U] = \emptyset$, otherwise L is not well-defined.
By assumption, $L^{-1}[E_2 \setminus U] = E_1 \setminus L^{-1}[U]$ is closed.

Therefore $L^{-1}[U]$ is open.

Theorem

Let $L : E_1 \to E_2$ be a linear map of normed spaces:

L is continuous (everywhere in E_1) iff it is continuous at some $\vec{x}_0 \in E_1$

Proof

Assume L is continuous everywhere in E_1.

Therefore it must be continuous at some $\vec{x}_0 \in E_1$.

Assume L is continuous at some $\vec{x}_0 \in E_1$.

Assume $\vec{x} \in E_1$.
Assume $(\vec{x}_n)$ is a sequence in E_1 such that $\vec{x}_n \to \vec{x}$.
Thus, the sequence $(\vec{x}_n - \vec{x} + \vec{x}_o)$ converges to $\vec{x}_0$.
$\|L\vec{x}_n - L\vec{x}\| = \|L\vec{x}_n - L\vec{x} + L\vec{x}_0 - L\vec{x}_0\| = \|L(\vec{x}_n - \vec{x} + \vec{x}_0) - L\vec{x}_0\| \to 0$

Therefore, L is continuous at $\vec{x}$, and thus is continuous everywhere in E_1.

Theorem

Let $L : E_1 \to E_2$ be a linear map of normed spaces.

L is continuous iff L is bounded.

Proof

$\Longrightarrow$ Assume L is continuous.

ABC: L is not bounded.
$\forall\, M > 0, \exists\, \vec{x} \in E_1, \|L\vec{x}\| > M\,\|\vec{x}\|$
Let $M = n$.
So $\|L\vec{x}\| > n\,\|\vec{x}\|$ for some $\vec{x} \in E_1$.
Thus, $\left\|L\dfrac{\vec{x}}{n\,\|\vec{x}\|}\right\| > 1$.
Let $y_n = \dfrac{\vec{x}}{n\,\|\vec{x}\|}$.
$\|y_n\| = \left\|\dfrac{\vec{x}}{n\,\|\vec{x}\|}\right\| = \dfrac{1}{n}\left\|\dfrac{\vec{x}}{\|\vec{x}\|}\right\| = \dfrac{1}{n}\cdot 1 = \dfrac{1}{n} \to 0.$
But $\|L\vec{y}_n\| > 1 \neq 0$.
Therefore L is not continuous.
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Therefore, L is bounded.

$\Longleftarrow$ Assume L is bounded.

Assume $(\vec{x}_n)$ is a sequence in E_1 such that $\vec{x}_n \to \vec{0}$.
$\left\|L\vec{x}_n - L\vec{0}\right\| = \left\|L\vec{x}_n - \vec{0}\right\| = \|L\vec{x}_n\| \leq \|L\|\,\|\vec{x}_n\| \to 0$

Therefore L is continuous at $\vec{0} \in E_1$ and thus is continuous everywhere in E_1.