LIMITS IN FINITE COMPLEMENT AND COUNTABLE COMPLEMENT TOPOLOGIES

Consider $\mathbb{R}$ with the finite complement topology and let (x_n) be a sequence of real numbers. When does (x_n) converge and to what limit(s)? Consider first the following example.

Example. Let $x_n = n$. We claim that $x_n \to x$, for every real number x. To prove this, let $x \in \mathbb{R}$ be arbitrary and let U be an arbitrary open set (in the finite complement topology) which contains x. Then $U = \mathbb{R} - F$, for some finite set F with $x \notin F$. Since F is finite, it cannot contain all terms of the sequence (x_n). Thus there exists N such that $x_n \notin F$, for all $n > N$. This implies that $x_n \in U$, for all $n > N$. Therefore, $x_n \to x$. $\square$

Let's see if we can generalize this example. Note that all we need is for (x_n) to have infinitely many different terms.

Proposition. *Let (x_n) be a sequence of real numbers such that the set $S = \{x_1, x_2, x_3, \ldots\}$ is infinite. Then for every $x \in \mathbb{R}$, $x_n \to x$.*

The proof is exactly the same as in the example.

What if $S = \{x_1, x_2, x_3, \ldots\}$ is finite? Then clearly there exists x such that $x_n = x$ for infinitely many x. It is not hard to see that in this case $x_n \to x$.

Proposition. *Let (x_n) be a sequence of real numbers such that the set $S = \{x_1, x_2, x_3, \ldots\}$ is finite. Then $x_n \to x$, where x is any number which occurs infinitely often in (x_n).*

Now consider the countable complement topology on $\mathbb{R}$. Let us us start with an example.

Example. Let $x_n = 1/n$. In the standard topology, $x_n \to 0$. However, if we let $U = \mathbb{R} - \{1/n : n \in \mathbb{N}\}$, then U is a neighborhood of 0 in the countable complement topology and does not contain any x_n's! Therefore, $x_n \not\to 0$ in this topology. Does (x_n) have any limits? Note that U is also a neighborhood of any other x, different from all x_n's, so x_n does not converge to that x as well. Similarly, for each k, $V_k = \mathbb{R} - \{1/n : n \neq k\}$ is a neighborhood of x_k which does not contain any other terms from the sequence, so $x_n \not\to 1/k$. We conclude that in the countable complement topology (x_n) has no limits. $\square$

Let's see if we can generalize this example. Let (x_n) be a sequence in $\mathbb{R}$ and assume that (x_n) is *eventually constant*. That is, assume there exist $x \in \mathbb{R}$ and N such that $x_n = x$ for all $n > N$. Than any neighborhood (in the countable complement topology) of x will contain x_n for $n > N$, i.e., $x_n \to x$. If (x_n) is not eventually constant, it will have no limits; the proof is analogous to that in the example above.

Proposition. *The only sequences in $\mathbb{R}$ which are convergent in the countable complement topology are the eventually constant sequences.*

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