Cardinality

Definition

The *cardinality* of a set A, denoted $|A|$, is the number of elements in A.

This is fairly intuitive for finite sets, but what about infinite sets like $\mathbb{Z}$ and $\mathbb{R}$?

Definition

To say that two sets A and B are *equivalent*, denoted $A \approx B$, means there exists a bijection $\phi : A \to B$. When two sets are equivalent they are said to have the same cardinality, denoted $|A| = |B|$.

Notation

$$[0] = \emptyset$$

$$[n] = \{1, 2, 3, ..., n\}$$

Definition

To say that a set A is finite means $\exists\, n \in \mathbb{Z}^+, A \approx [n]$. Otherwise, A is said to be infinite.

To say that A is countable means A is finite or $A \approx Z^+$. Otherwise, A is said to be uncountable.

Note that by definition, $\mathbb{Z}^+$ is countably infinite. We denote the cardinality of $\mathbb{Z}^+$ by:

$$\left|\mathbb{Z}^+\right| = \aleph_0$$

Theorem

$\mathbb{Z}$ is countable, and in fact:

$$|\mathbb{Z}| = \aleph_0$$

Proof

Consider the following bijection mapping $\mathbb{Z}$ to $\mathbb{Z}^+$:

$\mathbb{Z}$	0	1	-1	2	-2	3	-3	…
$\mathbb{Z}^+$	1	2	3	4	5	6	7	…

Thus, $\mathbb{Z} \approx \mathbb{Z}^+$ and is therefore countable. Furtherfore:

$$|\mathbb{Z}| = \left|\mathbb{Z}^+\right| = \aleph_0$$

So, even though $\mathbb{Z}^+ \subset \mathbb{Z}$, they have the same cardinality. In fact:

Theorem

Every subset of a countable set is countable. Furthermore, if the set and the subset are infinite then they have the same cardinality.

Theorem

$\mathbb{Q}$ is countable, and in fact:

$$|\mathbb{Q}| = \aleph_0$$

Theorem

Arrange and traverse the elements of $\mathbb{Q}$ as follows:

$$\begin{array}{cccccccccc}
0 & & 1 & \rightarrow & -1 & & 2 & \rightarrow & -2 & \cdots \\
\downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & \\
\frac{1}{2} & \rightarrow & -\frac{1}{2} & & \frac{3}{2} & & -\frac{3}{2} & & \frac{5}{2} & \cdots \\
 & & & & \downarrow & & \uparrow & & \downarrow & \\
\frac{1}{3} & \leftarrow & -\frac{1}{3} & \leftarrow & \frac{2}{3} & & -\frac{2}{3} & & \frac{4}{3} & \cdots \\
\downarrow & & & & & & \uparrow & & \downarrow & \\
\frac{1}{4} & \rightarrow & -\frac{1}{4} & \rightarrow & \frac{3}{4} & \rightarrow & -\frac{3}{4} & & \frac{5}{4} & \cdots \\
 & & & & & & & & \downarrow & \\
\frac{1}{5} & \leftarrow & -\frac{1}{5} & \leftarrow & \frac{2}{5} & \leftarrow & -\frac{2}{5} & \leftarrow & \frac{3}{5} & \cdots \\
\downarrow & & & & & & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

Note that every rational number is counted once; none are missed.
Therefore, $\mathbb{Q} \approx \mathbb{Z}^+$ and $|\mathbb{Q}| = |\mathbb{Z}^+| = \aleph_0$.

Theorem: Cantor Diagonalization

$[0, 1]$ is uncountable, and thus $\mathbb{R}$ is uncountable.

Proof

ABC that $\mathbb{R}$ is countable.
Consider $[0, 1] \subset \mathbb{R}$; it must also be countable.
Let the following represent an exhaustive list of the elements in $[0, 1]$:

$$0.\underline{a_{1,1}}a_{1,2}a_{1,3}a_{1,4}a_{1,5}\ldots$$

$$0.a_{2,1}\underline{a_{2,2}}a_{2,3}a_{2,4}a_{2,5}\ldots$$

$$0.a_{3,1}a_{3,2}\underline{a_{3,3}}a_{3,4}a_{3,5}\ldots$$

$$0.a_{4,1}a_{4,2}a_{4,3}\underline{a_{4,4}}a_{4,5}\ldots$$

$$0.a_{5,1}a_{5,2}a_{5,3}a_{5,4}\underline{a_{5,5}}\ldots$$

$$\vdots$$

Now, consider a real number $x \in [0, 1]$ whose n^{th} digit differs from the n^{th} digit in the n^{th} number in the list. Such a number is not in the original list, and thus the list can never be exhaustive. Thus, $[0, 1]$ is not countable, and therefore neither is $\mathbb{R}$.

The uncountable cardinality of $\mathbb{R}$ is simply denoted by $|\mathbb{R}|$, and the cardinality of $\mathbb{R}^n$ is denoted by $|\mathbb{R}|^n$, where:

$$\aleph_0 < |\mathbb{R}| < |\mathbb{R}|^2 < \dots$$