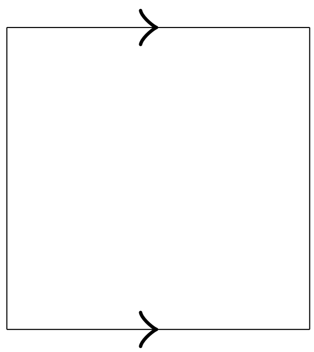
Quotient Maps

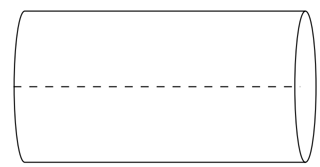
New spaces can be created from existing spaces by identifying (gluing) points in new spaces via equivalence relations.

Example: Cylinder

Identifying two edges of a square to make a cylinder:



$X = [0,1] \times [0,1]$



C

The gluing function $g : X \to C$ defined by:

$$g(x, y) = (x, \sin(2\pi y), \cos(2\pi y))$$

A set is U is open in C iff $g^{-1}(U)$ is open in X. Thus, gluing functions are automatically continuous.

However, it is easier to identify points using equivalence relations that partition the identified points.

$$C = \{\{(x, y)\} \mid x \in [0,1], y \in (0,1)\} \cup \{\{(x,0),(x,1)\} \mid x \in [0,1]\}$$

Definition: Identification Space

Let X be a topological space and let $\sim$ be an equivalence relation on X. The function $f : X \to X/\sim$ defined by $f(X) = X/\sim$ is called an *identification map*. The resulting space X^* of equivalence classes is called an *identification space*. A set U is open in X^* iff $f^{-1}(U)$ is open in X. Hence, f is continuous by definition.

Example: Möbius Band

Construct a Möbius band explicitly as an identification space of $X = [0,8] \times [0,1]$.

$$X^* = \{\{(x, y)\} \mid x \in (0,8), y \in [0,1]\} \cup \{(0,y),(8,1-y) \mid y \in [0,1]\}$$

Example: Torus

Construct a torus explicitly as:

1. An identification space of a cylinder C.

$$C = \{(R\sin\theta, R\cos\theta, \ell) \mid \theta \in [0, 2\pi), \ell \in [0, L]\}$$

$$\begin{aligned} C^* = &\{\{(R\sin\theta, R\cos\theta, \ell)\} \mid \theta \in [0, 2\pi), \ell \in (0, L)\} \cup \\ &\{\{(R\sin\theta, R\cos\theta, 0), (R\sin\theta, R\cos\theta, L)\} \mid \theta \in [0, 2\pi)\} \end{aligned}$$

2. An identification space of $X = [0, 1] \times [0, 1]$.

$$\begin{aligned} X^* = &\{\{(x, y)\} \mid x \in (0, 1), y \in (0, 1)\} \cup \\ &\{\{(x, 0), (x, 1)\} \mid x \in (0, 1)\} \cup \\ &\{\{(0, y), (1, y)\} \mid y \in [0, 1]\} \end{aligned}$$

3. An identification space of $\mathbb{R}^2$.

$$(x, y) \sim (u, v) \iff x - u \in \mathbb{Z} \text{ and } y - v \in \mathbb{Z}$$

Definition: Quotient Topology

Let X be a topological space and Y be a set, and let $f : X \to Y$ be surjective. The *quotient topology* on Y with respect to f is the collection of all $U \subset Y$ such that $f^{-1}(U) \in \mathscr{T}_X$. Thus, f is continuous by definition.

Theorem

The quotient topology actually defines a topology.

Proof. Assume X is a topological space, Y is a set, and $f : X \to Y$ is surjective.

1. $f^{-1}(\emptyset) = \emptyset \in \mathscr{T}_X$. Therefore $\emptyset \in \mathscr{T}_Y$.
2. $f^{-1}(Y) = X \in \mathscr{T}_X$. Therefore $Y \in \mathscr{T}_Y$.
3. Assume that $U, V \in \mathscr{T}_Y$. This means that $f^{-1}(U), f^{-1}(V) \in \mathscr{T}_X$ and so:

   $$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) \in \mathscr{T}_X$$

   Therefore $U \cap V \in \mathscr{T}_Y$.
4. Assume that $\{U_\alpha : \alpha \in \lambda\} \subset \mathscr{T}_Y$. This means that for all $a \in \lambda$, $f^{-1}(U_\alpha) \in \mathscr{T}_X$ and so:

   $$\bigcup_{\alpha \in \lambda} f^{-1}(U_\alpha) = f^{-1}(\bigcup_{\alpha \in \lambda} U_\alpha) \in \mathscr{T}_X$$

   Therefore $\bigcup_{\alpha \in \lambda} U_\alpha \in \mathscr{T}_Y$.

Therefore, the quotient topology on Y defines a topology. ∎

Theorem

Let X be a topological space and Y be a set, and let $f : X \to Y$ be surjective. The quotient topology on Y is the finest topology that makes f continuous.

Proof. ABC there exists some topology $\mathscr{T}$ on T that is finer than T_Y. Thus, there exists $U \in \mathscr{T}$ but $U \notin \mathscr{T}_Y$. This would mean that $f^{-1}(U)$ is not open in X, contradicting the continuity of f.

Therefore $\mathscr{T} = \mathscr{T}_Y$. ■

Theorem

Let X and Y be topological spaces and let $f : X \to Y$ be a continuous, surjective, open map. f is a quotient map.

Proof. Let $\mathscr{T}_Y^f = \{U \subset Y \mid f^{-1}(U) \in \mathscr{T}_X\}$. Since $\mathscr{T}_Y^f$ is the finest topology that makes f continuous, it must be the case that $\mathscr{T}_Y \subset \mathscr{T}_Y^f$.

WTS: $\mathscr{T}_Y^f \subset \mathscr{T}_Y$.

Assume $U \in \mathscr{T}_Y^f$. Then, by definition, $f^{-1}(U) \in \mathscr{T}_X$. But f is open and surjective, so:

$$f(f^{-1}(U)) = U \in \mathscr{T}_Y$$

Therefore $\mathscr{T}_Y^f \subset \mathscr{T}_Y$ and hence $T_Y^f = T_Y$. ■

Theorem

Let X, Y, and Z be topological spaces and let $f : X \to Y$ be a quotient map. The map $g : Y \to Z$ is continuous iff $g \circ f$ is continuous.

Proof.

$\implies$ Assume $g : Y \to Z$ is continuous.

But the composition of continuous functions is continuous.

Therefore $g \circ f$ is continuous.

$\impliedby$ Assume $g \circ f$ is continuous.

Assume $W \in \mathscr{T}_Z$, and thus $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \in \mathscr{T}_X$. But f is a quotient map, and so by definition, $g^{-1}(W) \in \mathscr{T}_Y$.

Therefore g is continuous.

■

Example

Quotient maps do not preserve Hausdorff. As a counterexample, consider $X = \mathbb{R}^{+} \times \{0, 1\}$ and the equivalence relationship $(x, 0) \sim (x, 1)$. This yields R_{+00}, which is not Hausdorff.