Internal Weak Direct Product

Definition

Let $\{N_i \mid i \in I\}$ be a family of normal subgroups of a group G such that:

1). $G = \left\langle \bigcup_{i\in I} N_i \right\rangle$

2). $\forall\, k \in I, N_k \cap \bigcup_{i\neq k} N_i = \{e\}$

G is said to be the *internal weak direct product* of $\{N_i \mid i \in I\}$.

If G is additive/abelian then G is said to be the *internal direct sum* of $\{N_i \mid i \in I\}$.

Theorem

Let G be an internal weak direct product of $\{N_i \mid i \in I\}$:

$$G \simeq \prod_{i\in I}^{w} N_i$$

Proof

Assume $a \in \prod_{i\in I}^{w} N_i$
$a(i) = e$ except for a finite number of i
Let $I_a = \{i \in I \mid a(i) \neq e\}$
I_a is a finite set
$\forall\, i \in I, N_i \triangleleft G$
Assume $i, j \in I, i \neq j$
Assume $n_i \in N_i$ and $n_j \in N_j$
$n_i n_j = n_j n_i$
So, $\prod_{i\in I_a} a(i) \in G$ is well-defined

Let $\phi : \prod_{i\in I}^{w} N_i \to G$ be defined by $\phi(a) = \prod_{i\in I_a} a(i)$
Assume $a, b \in \prod_{i\in I}^{w} N_i$
$\phi(ab) = \prod_{i\in I_a\cap I_b} (ab)(i) = \prod_{i\in I_a\cap I_b} a(i)b(i) = \prod_{i\in I_a} a(i) \prod_{i\in I_b} b(i) = \phi(a)\phi(b)$
$\therefore\, \phi$ is a homomorphism.

Assume $a \in \prod_{i\in I}^{w} N_i, \phi(a) = e$
$\prod_{i\in I_a} a(i) = e$
Assume $k \in I_a$
$a(k) \prod_{i\neq k} a(i) = e$
$a(k)^{-1} = \prod_{i\neq k} a(i)$
$a(k)^{-1} \in N_k$
$\prod_{i\neq k} a(i) \in \left\langle \bigcup_{i\neq k} N_i \right\rangle$
But $N_k \cap \left\langle \bigcup_{i\neq k} N_i \right\rangle = \{e\}$
So $a(k)^{-1} = e = a(k)$ and thus the kernel of ϕ is trivial
$\therefore\, \phi$ is one-to-one.

Assume $g \in G$
Since $G = \left\langle \bigcup_{i \in I} N_i \right\rangle$ and the N_i commute,
$g = \prod_{i \in I_a} a_i$ where $a_i \in N_i$ for some finite set I_a
$\iota_i(a_i) \in \prod_{i \in I}^{w} N_i$
But $\prod_{i \in I}^{w} N_i$ is a group, so by closure: $\prod_{i \in I_a} \iota_i(a_i) \in \prod_{i \in I}^{w} N_i$
$\phi\left(\prod_{i \in I_a} \iota_i(a_i)\right) = \prod_{i \in I_a} \phi\iota_i(a_i) = \prod_{i \in I_a} a_i = g$
$\therefore \phi$ is onto.

$\therefore \phi$ is an isomorphism.

Corollary

Let G be a group with normal subgroups $N_1, N_2 \ldots, N_r$ such that:

1). $G = N_1 N_2 \cdots N_r$

2). $\forall\, 1 \leq k \leq r, N_k \cap N_1 \cdots N_{k-1} N_{k+1} \cdots N_r = \{e\}$

$G \simeq N_1 \times N_2 \times \cdots \times N_r$