Schur Triangularization

Theorem

Let $A \in M_n$. There exists a unitary matrix U such that A is unitary similar with a $T \in UT(n)$ such that the diagonal entries of T are the eigenvalues of A is a prescribed order.

Proof

By induction on n:

Base case: $n = 1$

$$\begin{bmatrix}1\end{bmatrix} \begin{bmatrix}a\end{bmatrix} \begin{bmatrix}1\end{bmatrix} = \begin{bmatrix}a\end{bmatrix} \in UT(1)$$

Assume true for $n - 1$

Consider $A \in M_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$

Each λ_k is associated with an unit eigenvector $\vec{x}_k$
Extend the $\vec{x}_k$ to an orthonormal basis for $\mathbb{C}^n$: $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$
Let $U_1 = \begin{bmatrix}\vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n\end{bmatrix}$
Thus, U_1 is unitary and:

$$\begin{aligned}
AU_1 &= \begin{bmatrix}A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n\end{bmatrix} \\
&= \begin{bmatrix}\lambda_1\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n\end{bmatrix} \\
&= \begin{bmatrix}\vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n\end{bmatrix} \begin{bmatrix}\lambda_1 & * \\ 0 & A_2\end{bmatrix} \\
&= U_1 \begin{bmatrix}\lambda_1 & * \\ 0 & A_2\end{bmatrix} \\
U_1^* A U_1 &= \begin{bmatrix}\lambda_1 & * \\ 0 & A_2\end{bmatrix}
\end{aligned}$$

Where $A_2 \in M_{n-1}$
So by the inductive assumption, there exists unitary matrix U_2 such that:

$$U_2^* A_2 U_2 = \begin{bmatrix}\lambda_2 & & * \\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$$

Let $U = U_1 \begin{bmatrix}1 & 0 \\ 0 & U_2\end{bmatrix}$

U is also unitary and we have:

$$
\begin{aligned}
U^*AU &= \left(U_1 \begin{bmatrix} 1 & 0 \\ 0 & U_2 \end{bmatrix}\right)^* A \left(U_1 \begin{bmatrix} 1 & 0 \\ 0 & U_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 1 & 0 \\ 0 & U_2^* \end{bmatrix} (U_1^* A U_1) \begin{bmatrix} 1 & 0 \\ 0 & U_2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & U_2^* \end{bmatrix} \begin{bmatrix} \lambda_1 & * \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U_2 \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & * \\ 0 & U_2^* A_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U_2 \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & * \\ 0 & U_2^* A U_2 \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}
\end{aligned}
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