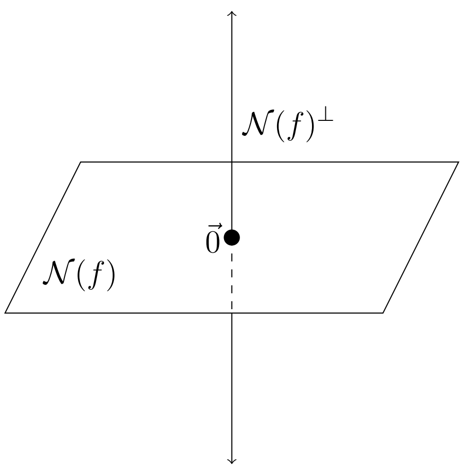
Riesz Representation Theorem

Lemma

Let H be a Hilbert space and let $f \in H'$ such that $f \not\equiv 0$:

$$\dim \mathcal{N}(f)^{\perp} = 1$$



Proof

Since $f \not\equiv 0$, $\mathcal{N}(f)$ is a proper subset of H and thus $\mathcal{N}(f)^{\perp}$ is not trivial, so fix a $\vec{x} \in \mathcal{N}(f)^{\perp}$ such that $\vec{x} \neq \vec{0}$.
Assume $\vec{y} \in \mathcal{N}(f)^{\perp}$ such that $\vec{y} \neq \vec{0}$.
Thus $f(\vec{x}), f(\vec{y}) \neq 0$; otherwise $\vec{x}, \vec{y} \in \mathcal{N}(f)$.
So $\exists\, \alpha \in \mathbb{C}$ such that $f(\vec{y}) = \alpha f(\vec{x})$.
$f(\vec{y}) - \alpha f(\vec{x}) = 0$
$f(\vec{y} - \alpha \vec{x}) = 0$
Thus $\vec{y} - \alpha \vec{x} \in \mathcal{N}(f)$.
But $\mathcal{N}(f)^{\perp}$ is subspace of H, and so $\vec{y} - \alpha \vec{x} \in \mathcal{N}(f)^{\perp}$ also.
Therefore $\vec{y} - \alpha \vec{x} = 0$ and thus $\vec{y} = \alpha \vec{x}$.

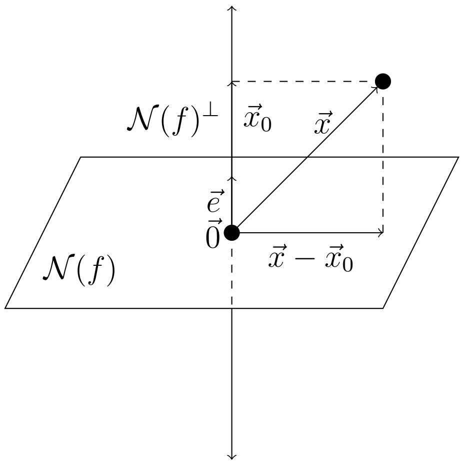
Theorem: Riesz

Let H be a Hilbert space and let $f \in H'$. There exists a unique element $\vec{y} \in H$ such that $\forall\, \vec{x} \in H$:

$$f(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$$

Moreover, $\|f\| = \|\vec{y}\|$.

Proof



Let $\vec{e} \in \mathcal{N}(f)$ such that $\|\vec{e}\| = 1$.
Assume $\vec{x} \in H$.
Let $\vec{x}_0 = \operatorname{proj}_{N(f)^\perp} \vec{x}$.
By lemma, $\dim N(f)^\perp = 1$.
And so $\vec{e}$ is an orthonormal basis for $\dim N(f)^\perp$.
Thus $\vec{x}_0 = \langle \vec{x}, \vec{e} \rangle \vec{e}$.

But $\vec{x} - \vec{x}_0 \in \mathcal{N} f$ and so $f(\vec{x} - \vec{x}_0) = 0$.
$f(\vec{x}) - f(\vec{x}_0) = 0$
$f(\vec{x}) = f(\vec{x}_0) = f(\langle \vec{x}, \vec{e} \rangle \vec{e}) = \langle \vec{x}, \vec{e} \rangle f(\vec{e}) = \left\langle \vec{x}, \overline{f(\vec{e})}\vec{e} \right\rangle$

Therefore $f(x) = \langle \vec{x}, \vec{y} \rangle$ where $\vec{y} = \overline{f(\vec{e})}\vec{e}$.

The fact that $\|f\| = \|y\|$ was previously proven.