# Unity

__Definition__

To say that $R$ is a *ring with unity* means:

1). $R$ is a ring

2). $R$ has a multiplicative identity element, called *unity*, and denoted by $1$.

Note that since $\langle R, \cdot \rangle$ is a binary algebraic structure, $1 \in R$ is unique.

__Theorem__

Let $R$ be a ring with unity. $0 = 1$ iff $R$ is the trivial ring.

Proof

$\implies$ Assume $0 = 1$

Assume $a \in R$
$a0 = 0$
$a1 = a0 = a$
$\therefore a = 0$

$\impliedby$ Assume $R$ is the trivial ring

$00 = 0$
So, $0$ is unity for $R$
$R$ is a ring with unity,
But $|R| = 1$
$\therefore 0 = 1$

__Theorem__

Let $r, s \in \mathbb{N}$ such that $(r, s) = 1$:

$$Z_{rs} \simeq \mathbb{Z}_r \times \mathbb{Z}_s$$

Proof

$\mathbb{Z}_r$ and $\mathbb{Z}_s$ are cyclic with generator $1$
$\mathbb{Z}_{rs}$ is cyclic with generator $(1, 1)$
Let $\phi : \mathbb{Z}_{rs} \to \mathbb{Z}_r \times \mathbb{Z}_s$ be defined by $\phi(n) = n \cdot (1, 1)$

Assume $\phi(n) = \phi(m)$
$n \cdot (1, 1) = m \cdot (1, 1)$
But additon in $Z_{rs}$ is well-defined
$n = m$
$\therefore \phi$ is one-to-one.

Assume $x \in \mathbb{Z}_{rs}$
$\exists n \in \mathbb{N}, n \cdot (1,1) = x$
$\phi(n) = n \cdot (1,1) = x$
$\therefore \phi$ is onto, and thus a bijection.

$\phi(n+m) = (n+m) \cdot (1,1) = n \cdot (1,1) + m \cdot (1,1) = \phi(n) + \phi(m)$

$\phi(nm) = (nm) \cdot (1,1) = [n \cdot (1,1)][m \cdot (1,1)] = \phi(n)\phi(m)$

$\therefore \phi$ is a ring homomorphism, and thus a ring isomorphism.