Spectral Radius

Definition: Spectral Radius

Let $A \in M_n$. The *spectral radius* of A, denoted $\rho(A)$, is given by:

$$\rho(A) = \max_{\lambda \in \sigma(A)} \{|\lambda|\}$$

Lemma

Let $A \in M_n$ and σ_1 be the largest singular value for A:

$$\sigma_1 = \sqrt{\rho(A^*A)}$$

Proof

Let the SVD for A be as follows:

$$A = U \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} V^*$$

for unitary matrices U and V and $\sigma_k \in \mathbb{R}$ such that $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$.

$$A^*A = V \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} U^*U \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} V^* = V \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} V^*$$

Thus, the σ_k^2 are the eigenvalues for A^*A and $\rho(A^*A) = \sigma_1^2$

$\therefore \sigma_1 = \sqrt{\rho(A^*A)}$

Lemma

Let $A \in M_n$ and σ_1 be the largest singular value for A. $\forall\, \vec{x} \in \mathbb{C}^n$:

$$\|A\vec{x}\|_2 \leq \sigma_1 \|\vec{x}\|_2$$

Proof

From the previous proof:

$$A^*A = V \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} V^*$$

Assume $\vec{x} \in \mathbb{C}^n$:

$$\vec{x}^*A^*A\vec{x} = \vec{x}^*V \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} V^*\vec{x}$$

Let $\vec{y} = V^*\vec{x}$:

$$\|A\vec{x}\|_2^2 = \vec{y}^* \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} \vec{y}$$

By isometry: $\|\vec{y}\|_2 = \|V^*\vec{x}\|_2 = \|\vec{x}\|_2$, and so:

$$\begin{aligned} \|A\vec{x}\|_2^2 &= \begin{bmatrix} \bar{y}_1 & \bar{y}_2 & \ldots & \bar{y}_n \end{bmatrix} \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= \sum_{k=1}^{n} \bar{y}_k \sigma_k^2 y_k \\ &= \sum_{k=1}^{n} \sigma_k^2 |y_k|^2 \\ &\leq \sum_{k=1}^{n} \sigma_1^2 |y_k|^2 \\ &= \sigma_1^2 \sum_{k=1}^{n} |y_k|^2 \\ &= \sigma_1^2 \|\vec{y}\|_2^2 \\ &= \sigma_1^2 \|\vec{x}\|_2^2 \\ \therefore \|A\vec{x}\|_2 &\leq \sigma_1 \|\vec{x}\|_2 \end{aligned}$$

Lemma

Let $A \in M_n$ and σ_1 be the largest singular value for A. $\exists\, \vec{x} \in \mathbb{C}^n$ such that:

$$\|A\vec{x}\|_2 = \sigma_1 \|\vec{x}\|_2$$

Proof

Let the SVD for A be as follows:

$$A = U \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} V^*$$

for unitary matrices U and V and $\sigma_k \in \mathbb{R}$ such that $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$.

$$AV = U \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$$

Compare the first columns:

$$A\vec{v}_1 = \sigma_1 \vec{u}_1$$

Since $\vec{u}_1$ and $\vec{v}_1$ are a unit vectors:

$$\|A\vec{v}_1\|_2 = \|\sigma_1 \vec{u}_1\|_2 = \sigma_1 \|\vec{u}_1\|_2 = \sigma_1 \cdot 1 = \sigma_1 \|\vec{v}_1\|_2$$

Let $\vec{x} = \vec{v}_1$. Therefore, $\exists\, \vec{x} \in \mathbb{C}^n$ such that:

$$\|A\vec{x}\|_2 = \sigma_1 \|\vec{x}\|_2$$

Theorem

Let $A \in M_n$ and σ_1 be the largest singular value for A:

$$|||A|||_2 = \sqrt{\rho(A^*A)} = \sigma_1$$

Proof

By definition:

$$|||A|||_2 = \max_{\|\vec{x}\|_2 = 1} \{\|A\vec{x}\|_2\}$$

By the above lemma: $\forall\, \vec{x} \in \mathbb{C}^n$:

$$\|A\vec{x}\|_2 \leq \sigma_1 \|\vec{x}\|_2$$

And by the subsequent lemma, there exists a $\vec{x} \in \mathbb{C}^n$ such that $\|\vec{x}\|_2 = 1$ and:

$$\|A\vec{x}\|_2 = \sigma_1 \|\vec{x}\|_2 = \sigma_1 \cdot 1 = \sigma_1$$

Therefore:

$$|||A|||_2 = \max_{\|\vec{x}\|_2 = 1} \|A\vec{x}\|_2 = \sigma_1$$