Linear Maps

Notation

Let E_1 and E_2 be vector spaces and let L be a mapping from E_1 to E_2. Also, let $A \subseteq E_1$ and $B \subseteq E_2$:

- $L\vec{x} = L(\vec{x})$
- If $\vec{y} = L\vec{x}$ then $\vec{y}$ is called the *image* of $\vec{x}$ and $\vec{x}$ is called the *pre-image* of $\vec{y}$.
- $L[A] = \{L\vec{x} \mid \vec{x} \in A\} \subseteq E_2$ is called the *image* of A.
- $L^{-1}[B] = \{\vec{x} \in E_1 \mid L\vec{x} \in B\} \subseteq E_1$ is called the *pre-image* of B.
- $\mathcal{D}(L) \subseteq E_1$ is the domain of L.
- $\mathcal{R}(L) = L[\mathcal{D}(L)] \subseteq E_2$ is called the range of L.
- $\mathcal{N}(L) = \{\vec{x} \in \mathcal{D}(L) \mid L\vec{x} = \vec{0}\} \subseteq \mathcal{D}(L)$ is called the *null space (kernel)* of L.
- $\mathcal{G}(L) = \{(\vec{x}, L\vec{x}) \mid \vec{x} \in \mathcal{D}(L)\} \subseteq E_1 \times E_2$ is called the *graph* of L.

Definition: Linear

Let $L : E_1 \to E_2$ be a mapping of vector spaces over a field $\mathbb{F}$. To say that L is *linear* means $\forall\, \vec{x}, \vec{y} \in E_1$ and $\forall\, \alpha, \beta \in \mathbb{F}$:

$$L(\alpha\vec{x} + \beta\vec{y}) = \alpha L\vec{x} + \beta L\vec{y}$$

Theorem

Let $L : E_1 \to E_2$ be a linear mapping of vector spaces over a field $\mathbb{F}$ and let $\mathcal{D}(L) \subseteq E_1$:

1). $\mathcal{D}(L)$ is a subspace of E_1.

2). $\mathcal{N}(L)$ is a subspace of $\mathcal{D}(L)$.

3). $\mathcal{R}(L)$ is a subspace of E_2.

4). $\mathcal{G}(L)$ is a subspace of $E_1 \times E_2$ using component-wise operations.

Proof

1). Assume $S = \{\vec{x}_1, \dots, \vec{x}_n\} \subseteq \mathcal{D}(L)$.

Thus, by linearity, $\forall\, \alpha_k \in \mathbb{F}$:

$$\sum_{k=1}^{n} \alpha_k \vec{x}_k \in \mathcal{D}(L)$$

Therefore, $\mathcal{D}(L)$ is a subspace of E_1.

2). Assume $\vec{x}, \vec{y} \in \mathcal{N}(L)$ and $\alpha, \beta \in \mathbb{F}$.

$L(\alpha\vec{x} + \beta\vec{y}) = \alpha L\vec{x} + \beta L\vec{y} = \alpha\vec{0} + \beta\vec{0} = \vec{0} + \vec{0} = \vec{0}$
So $\alpha\vec{x} + \beta\vec{y} \in \mathcal{N}(L)$.

Therefore, $\mathcal{N}(L)$ is a subspace of $\mathcal{D}(L)$.

3). Assume $\vec{v}, \vec{w} \in \mathcal{R}(L)$ and $\alpha, \beta \in \mathbb{F}$:

$\exists\, \vec{x}, \vec{y} \in \mathcal{D}(L)$ such that $\vec{v} = L\vec{x}$ and $\vec{w} = L\vec{y}$.
$\alpha\vec{v} + \beta\vec{w} = \alpha L\vec{x} + \beta L\vec{y} = L(\alpha\vec{x} + \beta\vec{y})$
But, by closure, $\alpha\vec{x} + \beta\vec{y} \in \mathcal{D}(L)$.
And so $L(\alpha\vec{x} + \beta\vec{y}) \in \mathcal{R}(L)$.

Therefore $\mathcal{R}(L)$ is a subspace of E_2.

4). Assume $(\vec{x}, L\vec{x}), (\vec{y}, L\vec{y}) \in \mathcal{G}(L)$ and $\alpha, \beta \in \mathbb{F}$.

$\alpha(\vec{x}, L\vec{x}) + \beta(\vec{y}, L\vec{y}) = (\alpha\vec{x} + \beta\vec{y}, \alpha L\vec{x} + \beta L\vec{y}) = (\alpha\vec{x} + \beta\vec{y}, L(\alpha\vec{x} + \beta\vec{y}))$
But, by closure, $\alpha\vec{x} + \beta\vec{y} \in \mathcal{D}(L)$.
And so, $(\alpha\vec{x} + \beta\vec{y}, L(\alpha\vec{x} + \beta\vec{y})) \in \mathcal{G}(L)$.

Therefore $\mathcal{G}(L)$ is a subspace of $E_1 \times E_2$.

Theorem

Let $L : E_1 \to E_2$ be a linear map of vector spaces:

$L\vec{0} = \vec{0}$

Proof

$L\vec{0} = L(\vec{0} + \vec{0}) = L(\vec{0}) + L(\vec{0})$

$\therefore L\vec{0} = \vec{0}$