Subfields

Definition

Let F be a field. To say that F' is a *subfield* of F, denoted $F' \leq F$, means:

1). $F' \subseteq F$

2). F' is a field under the induced operations

$F' < F$ means F' is a subfield of F but $F' \neq F$.

Theorem

Let F be a field and $F' \leq F$. $1' = 1$.

Proof

$\langle F, \cdot \rangle$ is a group, so $1'$ is unique
$\forall\, f' \in F', f'1' = 1'f' = f'$
But $f', 1' \in F$
$\langle F - \{0\}, \cdot \rangle$ is a group and thus has uniform, unique identity 1
$\therefore 1' = 1$

Theorem: Subfield Test

Let F be a field and let G be a non-empty subset of F. G is a subfield of F iff the following are true:

1). $\forall\, a, b \in G, a - b \in G$

2). $\forall\, a, b \in G^*, ab^{-1} \in G^*$

Proof

$\implies$ Assume $G \leq F$
$\langle G, + \rangle \leq \langle F, + \rangle$
$\langle G^*, \cdot \rangle \leq \langle F^*, \cdot \rangle$
Assume $a, b \in G$
$\langle G, + \rangle$ is a group, so $-b \in G$
$a - b \in G$ (closure)
$\langle G^*, \cdot \rangle$ is a group, so $b^{-1} \in G$
$ab^{-1} \in G$ (closure)
$\therefore 1$ and 2 hold.

$\impliedby$ Assume 1 and 2 hold
By the subgroup test, $\langle G, + \rangle \leq \langle F, + \rangle$
By the subgroup test, $\langle G*, \cdot \rangle \leq \langle F*, \cdot \rangle$
The distributive laws are inherited from F
$\therefore G \leq F$.

Theorem

Let F be a field and let $G = \bigcap_{i \in I} F_i$ where $F_i \leq F$:

$$G \leq F$$

Proof

$\langle G, + \rangle \leq \langle F, + \rangle$
$\langle G, \cdot \rangle \leq \langle F, \cdot \rangle$
The distributive laws are inherited from F

$\therefore G \leq F$