Counting

– Math 161a, Spring 2019, San José State University

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January 31, 2019

Outline

Section 2.3 Counting

Introduction

Counting is a very important task in the study of probability, as it is often needed to count the objects of a sample space, or those in a subset (i.e. event).

For example, in the setting of a finite sample space with equally likely outcomes, the formula for computing the probability of any event $E \subset S$ involves two counting questions:

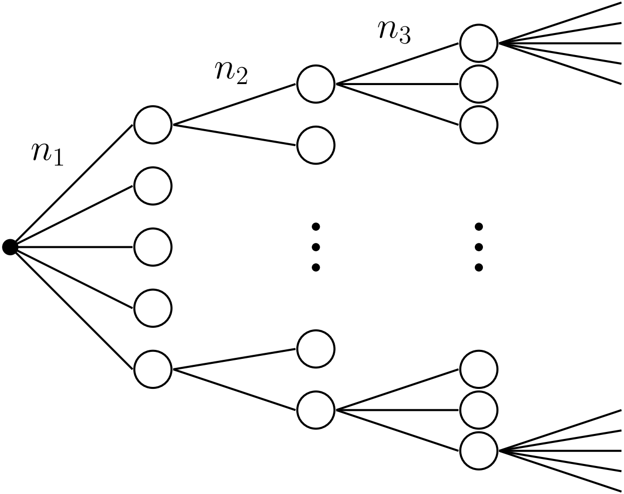
$$P(A) = \frac{|E|}{|S|}.$$

Fundamental Counting Principle

Theorem 0.1. *Suppose an experiment can be performed in a sequence of k steps, such that*

- *the first step can be done in n_1 ways, and*
- *for each result of step 1, step 2 can always be done in n_2 ways, and*
- *step 3 can always be done in n_3 ways for each combination of results of steps 1 and 2, so on and so forth.*

Then the entire experiment has a total of $n_1 n_2 \cdots n_k$ possible outcomes.


n_1
n_2
n_3

Example 0.1. A local restaurant provides 5 kinds of bread, 4 kinds of cheese, 4 kinds of meats, and 6 kinds of sauces. In how many ways can you order a sandwich?

Example 0.2. How many different CA driver licenses are there (1 capital letter followed by 7 numbers)? How many driver license numbers have all repeated digit? All distinct digits?

Example 0.3. How many ordered lists of size 3 can be made from a set $S = \{a, b, c, d\}$

(a) with repetition allowed, or

(b) with repetition not allowed?

Permutation

Briefly, permutations are <u>ordered lists</u> of all <u>distinct</u> objects, e.g.,

$$\{0, 1, 2, \ldots, 9\} \quad \longrightarrow \quad 5810,\ 1058,\ 0439,\ 7192,\ 3028,\ 1634, \ldots$$

Definition 0.1. A permutation of size r chosen from a set of n objects is an ordered list of r objects from the set (with repetition not allowed).

$$\underset{\text{position 1}}{\underline{\qquad}} \quad \underset{\text{position 2}}{\underline{\qquad}} \quad \underline{\qquad} \quad \underline{\qquad} \quad \underset{\text{position } r}{\underline{\qquad}}$$

Example 0.4. List all permutations of size $r = 3$ chosen from the set $S = \{a, b, c, d\}$. How many are there? What if $r = 4$?

Theorem 0.2. *The number of permutations of size r that can be formed from a total of n objects is*

$$P(n,r) = \underbrace{n(n-1)\cdots(n-r+1)}_{r\ integers} = \frac{n!}{(n-r)!}.$$

In particular,

$$P(n,n) = n! \quad (\#\ full\ permutations\ of\ size\ n)$$

Example 0.5. In how many different ways can 5 people be arranged in a row? Along a circle?

Example 0.6. How many 3-digit numbers are divisible by 5?

Example 0.7 (Birthday problem)**.** Find the probability p that no two people in a class of 35 have a common birthday (i.e., all have different birthdays). Assume that people's birthdays are equally likely to occur among the 365 days of the year and ignore leap years. (Answer: $.1856$.)

Combinations

Briefly, combinations are unordered collections of distinct objects, e.g.,

$$\{0, 1, 2, \ldots, 9\} \quad \longrightarrow \quad \{0, 1, 5, 8\},\ \{0, 3, 4, 9\},\ \{1, 2, 7, 9\}, \ldots$$

Definition 0.2. A combination of size r chosen from a set of n objects is an unordered selection of r objects from the set (with repetition not allowed).

Example 0.8. List all combinations of size 3 chosen from the set $S = \{a, b, c, d\}$.

Theorem 0.3. *The number of combinations of size r that can be formed from a total of n objects is*

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)! \cdot r!}.$$

Remark. To compute combinations by hand, use the following (equivalent) formula (and make cancellation as much as possible):

$$\binom{n}{r} = \frac{n \cdot (n-1) \cdots (n-r+1)}{1 \cdot 2 \cdots r}.$$

Example 0.9. Consider the problem of choosing 4 members from a group of 10 to work on a special project.

(a) Suppose two people A and B really like each other, so they must be simultaneously chosen or skipped. How many distinct four-person teams can be chosen?

(b) Suppose two people A and B really hate each other, so they cannot be both selected for the project. How many distinct four-person teams can be chosen?

Example 0.10. An urn has 5 red balls and 7 blue balls. Suppose you randomly select 5 balls from the urn. What is the probability that your hand has exactly 3 red balls?

A ordinary deck of 52 cards is divided into 4 suits (heart, diamond, spade and club) and 13 ranks (2, 3, . . . , 10, J, Q, K, A)

Example 0.11. Suppose your randomly draw 5 cards from a deck of 52. What is the probability that you have a

(a) four of a kind (4 cards of the same rank, and one side card)

(b) flush (5 cards of the same suit)