Matrix Norm

Definition

To say that $|||\cdot||| : M_n \to \mathbb{R}$ is a *matrix norm* means that it satisfies the following five properties $\forall\, A, B \in M_n$ and $\forall\, c \in \mathbb{C}^n$:

1). Nonnegativity: $|||A||| \geq 0$

2). Positivity: $|||A||| = 0 \iff A = 0$

3). Homogeneity: $|||cA||| = |c|\; |||A|||$

4). Subadditivity: $|||A + B||| \leq |||A||| + |||B|||$

5). Submultiplicativity: $|||AB||| \leq |||A|||\; |||B|||$

Some matrix norms are just extensions of the vector norms by placing the matrix columns end-to-end to form a giant vector. These norms can still use the double-bar notation. Since these norms already satisfy the first four norm properties, only submultiplicativity needs to be checked.

1). ℓ_1 Matrix Norm

$$\|A\|_1 = \sum_{i,j=1}^{n} |a_{ij}|$$

$$\begin{aligned}
\|AB\|_1 &= \sum_{i=1}^{n}\sum_{j=1}^{n} |(AB)_{ij}| \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \left|\sum_{k=1}^{n} a_{ik}b_{kj}\right| \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} |a_{ik}b_{kj}| \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\sum_{k=1}^{n} |a_{ik}|\right)\left(\sum_{k=1}^{n} |b_{kj}|\right) \\
&= \left(\sum_{i=1}^{n}\sum_{k=1}^{n} |a_{ik}|\right)\left(\sum_{j=1}^{n}\sum_{k=1}^{n} |b_{kj}|\right) \\
&= \|A\|_1 \|B\|_1
\end{aligned}$$

2). ℓ_2 (Frobenius) Matrix Norm

$$\|A\|_2 = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2} = \sqrt{\operatorname{tr}(A^*A)}$$

$$\begin{aligned}
\|AB\|_2^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} |(AB)_{ij}|^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \left|\sum_{k=1}^{n} a_{ik}b_{kj}\right|^2 \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} |a_{ik}|^2\,|b_{kj}|^2 \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\sum_{k=1}^{n} |a_{ik}|^2\right)\left(\sum_{k=1}^{n} |b_{kj}|^2\right) \\
&= \left(\sum_{i=1}^{n}\sum_{k=1}^{n} |a_{ik}|^2\right)\left(\sum_{j=1}^{n}\sum_{k=1}^{n} |a_{kj}|^2\right) \\
&= \|A\|_2\,\|B\|_2
\end{aligned}$$

3). ℓ_∞ Matrix Norm

$$\|A\|_\infty = \max_{1\leq i,j\leq n} |a_{ij}|$$

This is NOT a matrix norm. Consider the following counterexample:

$$A = B = J_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$\|A\|_\infty = \|B\|_\infty = 1$, so $\|A\|_\infty \|B\|_\infty = 1 \cdot 1 = 1$

$$AB = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

$\|AB\|_\infty = 3 \neq 1$