Zeros of Analytic Functions

Theorem

Let $f(z)$ be analytic in a domain D:

$$\exists\, a \in D, \forall\, n \in \mathbb{Z} \cup \{0\}, f^{(n)}(a) = 0 \implies \forall\, z \in D, f(z) = 0$$

Theorem

Let $f(z)$ be analytic such that $|f(z)| \leq M$ in and on a circle $\overline{C}$ with center a and radius r:

$$\forall\, n \in \mathbb{Z} \cup \{0\}, \left|f^{(n)}(a)\right| \leq \frac{Mn!}{r^n}$$

Proof

By the CITFD:

$$\begin{aligned}
\left|f^{(n)}(a)\right| &= \left|\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz\right| \\
&= \left|\frac{n!}{2\pi i}\right|\left|\int_C \frac{f(z)}{(z-a)^{n+1}}dz\right| \\
&\leq \frac{n!}{2\pi}\int_c \frac{|f(z)|}{|z-a|^{n+1}}\,|dz| \\
&\leq \frac{n!}{2\pi}\int_c \frac{M}{r^{n+1}}\,|dz| \\
&\leq \frac{Mn!}{2\pi r^{n+1}}\int_c |dz| \\
&= \frac{Mn!}{2\pi r^{n+1}}(2\pi r) \\
&= \frac{Mn!}{r^n}
\end{aligned}$$

Theorem: Liouville

$f(z)$ entire and bounded $\implies$ $f(z)$ constant.

Proof

Assume $f(z) \leq M$
By the previous theorem with $n = 1$ and a in some circle C with radius r:

$$|f'(z)| \leq \frac{M}{r}$$

As $r \to \infty$, $|f'(z)| \to 0$
Thus, $\forall\, z \in \mathbb{C}, f'(z) = 0$

$\therefore f(z)$ is constant.

Definition

To say that a point x is an *accumulation point* of a set X means that $\forall \epsilon > 0, N_\epsilon(x)$ contains infinitely many points in X.

Theorem: Uniqueness

Let $f(z)$ and $g(z)$ be analytic in a domain D. If there exists a set E in which $f(z) = g(z)$ and which contains an accumulation point $z_0 \in D$ for D then $f(z) = g(z)$ in D.

Proof

Assume that such a set E exists
There exists a sequence $\{z_n\} \subset E$ such that $\lim z_n = z_0$
So $f(z_n) = g(z_n)$ in E
Let $F(z) = (f - g)(z)$, which is also analytic in D
$F(z_n) = (f - g)(z_n) = 0$ in E, and in particular:
$F(z_0) = 0$ in both E and D

Rewrite $F(z)$ as a Taylor series about z_0:

$$F(z) = \sum_{n=0}^{\infty} \frac{F^{(k)}(z_0)}{k!}(z - z_0)^k = F(z_0) + \sum_{n=1}^{\infty} \frac{F^{(k)}(z_0)}{k!}(z - z_0)^k$$

We already know that $F(z_0) = 0$, so:

$$F(z) = \sum_{n=1}^{\infty} \frac{F^{(k)}(z_0)}{k!}(z - z_0)^k$$

Let $z = z_n$:

$$F(z_n) = \sum_{n=1}^{\infty} \frac{F^{(k)}(z_0)}{k!}(z_n - z_0)^k = 0$$

$$F'(z_0)(z_n - z_0) + \sum_{n=2}^{\infty} \frac{F^{(k)}(z_0)}{k!}(z_n - z_0)^k = 0$$

But $(z_n - z_0) \neq 0$, so:

$$F'(z_0) + \sum_{n=2}^{\infty} \frac{F^{(k)}(z_0)}{k!}(z_n - z_0)^{k-1} = 0$$

As $n \to \infty$, $z_n \to z_0$, so:

$$F'(z_0) = 0$$

By repeating the process, we find that $F^{(n)}(z_0) = 0$
So, by previous theorem, $F(z) = 0$ in D

$\therefore f(z) = g(z)$ in D.

Corollary

If D is compact then $f(z)$ has a finite number of zeros in D.

Note that zeros of an analytic function must be isolated; otherwise, the function is the zero function over the domain.