# Matrix Nearness Problem

**Definition: Distance**

Let $|||\cdot|||$ be a matrix norm on $M_n$, $S \subseteq M_n$, and $A \notin S$. The *distance* from $A$ to $S$ is given by:

$$d(A) = \inf_{XinS}\{|||A - X|||\}$$

Questions:

1). Compute $d(A)$

2). Does $X_A \in S$ exist such that $d(A) = |||X - X_A|||$?

3). If $X_A$ exists, is it unique?

**Lemma**

Let $|||\cdot|||$ be a matrix norm on $M_n$ and $A \in M_n$:

$$|||I - A||| < 1 \implies A \text{ is invertible}$$

Proof

Assume $|||I - A||| < 1$
Let $A = I - (I - A)$

$A^{-1} = [I - (I - A)]^{-1} = \sum_{k=0}^{\infty}(I - A)^k$

Check for absolute convergence under the norm:

$\left|\left|\left|\sum_{k=0}^{\infty}(I - A)^k\right|\right|\right| \le \sum_{k=0}^{\infty} |||I - A|||^k$

which converges for $|||I - A||| < 1$.

**Theorem: Best Singular Approximation**

Let $|||\cdot|||$ be the operator norm and let $S$ be the collection of all singular matrices. Fix $A \notin S$:

$$d(A) = s_n$$

the smallest singular value.

Proof

Consider the SVD for $A$:

$$A = V \begin{bmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{bmatrix} W^*$$

$$A^{-1} = (W^*)^{-1} \begin{bmatrix} \frac{1}{s_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{s_n} \end{bmatrix} V^{-1} = W \begin{bmatrix} \frac{1}{s_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{s_n} \end{bmatrix} V^*$$

Note that this is the SVD for $A^{-1}$, so $|||A^{-1}||| = \frac{1}{s_n}$.

Assume $B \in S$
$A^{-1}B \in S$
By the CP of the above lemma:

$$\begin{aligned} \left|\left|\left|I - A^{-1}B\right|\right|\right| &\geq 1 \\ \left|\left|\left|A^{-1}(A - B\right|\right|\right| &\geq 1 \\ \left|\left|\left|A^{-1}\right|\right|\right| \, |||A - B||| &\geq 1 \\ \frac{1}{s_n} \, |||A - B||| &\geq 1 \\ |||A - B||| &\geq s_n \end{aligned}$$

Now, let $X_A = V \begin{bmatrix} s_1 & & & 0 \\ & \ddots & & \\ & & s_n & \\ 0 & & & 0 \end{bmatrix} W^* \in S$

$|||A - X_A||| = \left|\left|\left| V \begin{bmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 0 & \\ 0 & & & s_n \end{bmatrix} W^* \right|\right|\right| = s_n$

$\therefore d(A) = s_n$