Fundamental Groups of Topological Spaces

Every topological space has a fundamental group. Homeomorphic spaces have isomorphic fundamental groups. This allows for an extra technique for determining that two spaces are not homeomorphic when all of the other topological properties seem to be preserved by continuous functions. For example:

1. $\mathbb{R}^2$ and $\mathbb{R}^3$
2. S^2, T, and $T\#T$

Definition: Group

A *group* $(G, *)$ is a set G with binary operator $*$ such that the following axioms are satisfied:

1. Associative: $(a * b) * c = a * (b * c)$
2. Identity: $a * e = e * a$
3. Inverse: $a * a' = a' * a = e$

Definition: Abelian Group

An *abelian* group $(G, *)$ is a group whose binary operator is commutative:

$$a * b = b * a$$

Definition: Homomorphic

To say that two groups $(G, *)$ and $(G', *')$ are *homomorphic* means that there exists a function $\phi : G \to G'$ such that:

$$\phi(x * y) = \phi(x) *' \phi(y)$$

Such a ϕ is called a *homomorphism*.

Properties: Homomorphism

Let $f : G \to G'$ be a homomorphism:

1. The *kernel* of f is the preimage of the identity of G': $f^{-1}(e')$, and it is a subgroup of G.
2. The image of f: $f(G)$, is a subgroup of G'.
3. If f is injective (kernel=$\{e\}$) then it is called a *monomorphism*.
4. If f is surjective then it is called an *epimorphism*.
5. If f is bijective then it is called an *isomorphism*.

Definition: Isomorphism

To say that two groups $(G, *)$ and $(G', *')$ are *isomorphic* means that there exists an isomorphism $\phi : G \to G'$.

Definition: Coset

Let H be a subgroup of a group G. The *left cosets* of H in G are defined as:

$$xH = \{xh \,|\, h \in H\}$$

for all $x \in X$. The left cosets of H form a partition of G. Similarly, the *right cosets* Hx also form a partition of G.

Definition: Normal

To say that a subgroup H of a group G is *normal* in G means for all $x \in G$ and all $h \in H$, $xhx^{-1} \in H$. In this case, for all $x \in G$, $xH = Hx$.

Definition: Quotient

Let H be a normal subgroup of a group G and define the binary operator:

$$(xH)(yH) = (xy)H$$

The partition of G by H with this operation forms a group denoted by G/H and called the *quotient group* of G by H.

Properties: Quotient

Let G/H be the quotient group of G by H:

1. The function $f : G \to G/H$ defined by $f(x) = xH$ is an epimorphism with kernel H.

Notation

$I = [0, 1] \subset \mathbb{R}$ imbued with the subspace topology.

Definition: Homotopy

Let X and Y be topological spaces and let $f_1, f_2 : X \to Y$ be continuous. To say that f_1 and f_2 are *homotopic*, denoted by $f_1 \simeq f_2$, means that there exists a continuous function $F : X \times I \to Y$ such that $F(x, 0) = f_1(x)$ and $F(x, 1) = f_2(x)$. Such a function F is called a *homotopy* between f_1 and f_2.

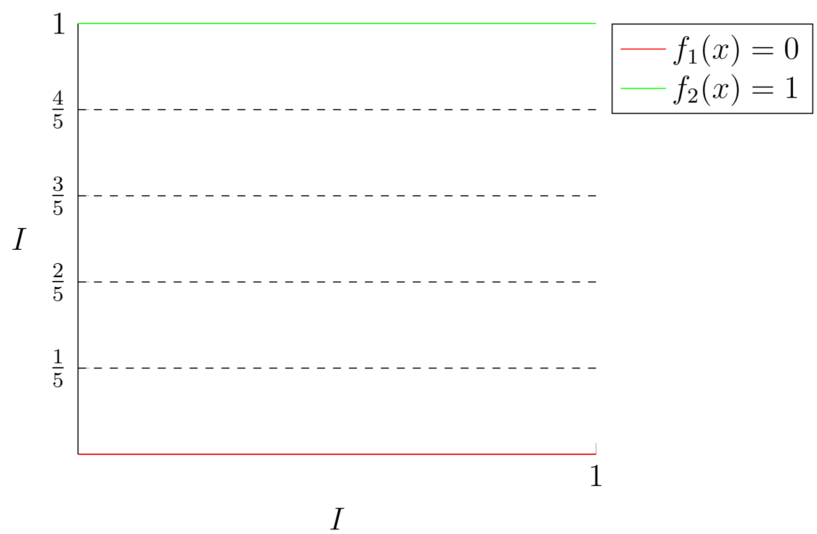
Definition: Nulhomotopic

Let X and Y be topological spaces and let $f_1 : X \to Y$ be continuous. To say that f_1 is *nulhomotopic* means that there exists a constant function $f_2 : X \to Y$ such that $f_1 \simeq f_2$.

A homotopy can be viewed as a continuous deformation of f_1 into f_2 via a parameterized family of continuous functions.

Example: Vertical Shift

Let $X = Y = I$ and let $f_1(x) = 0$ and $f_2(x) = 1$. Consider $F : I \times I \to I$ defined by $F(x, k) = \pi_I(x, k) = k$. Note that the projection map is known to be continuous:



Lemma

Let X and Y be topological spaces and let $f : X \to Y$ be continuous. For all $A \subset X$, $f|_A$ is continuous.

Proof. Assume $A \subset X$ and assume V is open in Y. Since f is continuous, $f^{-1}(V)$ is open in X. Furthermore, by definition of the subspace topology, $f|_A^{-1}(V) = f^{-1}(V) \cap A$ is open in A. Therefore $f|_A$ is continuous. ∎

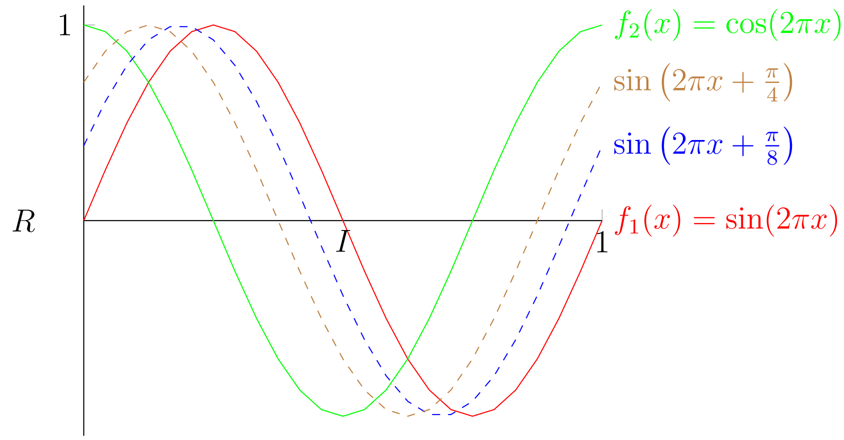
Corollary

Let X and Y be topological spaces, let $f_1, f_2 : X \to Y$ be continuous, and let $F : X \times I \to Y$ be a homotopy between f_1 and f_2. For all $k \in I$, define $f_k(x) = F(x, k) = F|_{X \times \{k\}}$. All of the f_k are continuous.

Proof. Since F is a homotopy, it is continuous. Therefore $F|_{X \times \{k\}} = F(x, k) = f_k(x)$ is continuous. ∎

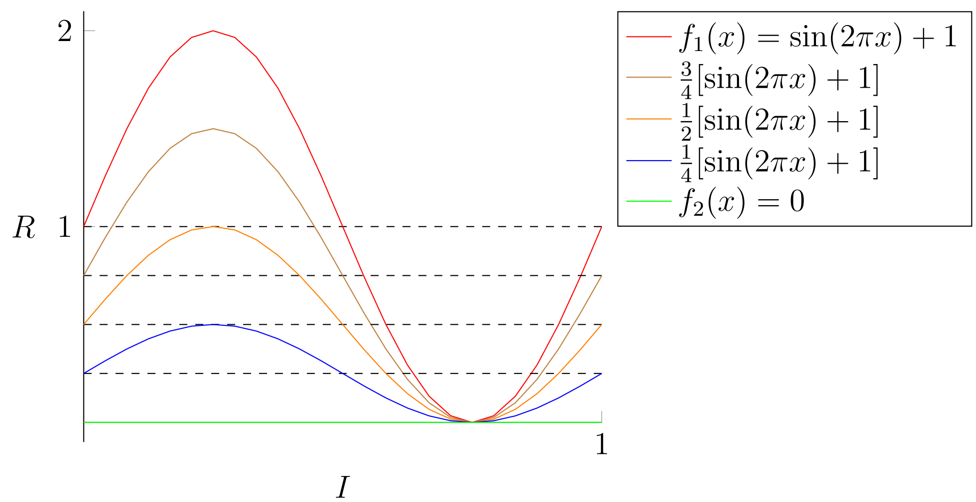
Example: Horizontal Shift

Let $X = I$ and and $Y = \mathbb{R}$, and let $f_1(x) = \sin(2\pi x)$ and $f_2(x) = \cos(2\pi x)$. Consider $F : I \times I \to \mathbb{R}$ defined by $F(x,k) = \sin\left(2\pi x + k\frac{\pi}{2}\right)$.



Example: Scaling

Let $X = I$ and and $Y = \mathbb{R}$, and let $f_1(x) = \sin(2\pi x) + 1$ and $f_2(x) = 0$. Consider $F : I \times I \to \mathbb{R}$ defined by $F(x,k) = (1-k)[\sin(2\pi x) + 1]$.



Based on the past examples it is tempting to make the theorem and equivalence. But, a function of a product space is not necessarily continuous if it is continuous in each variable separately.

Example

Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by:

$$F(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Note that F is continuous in both x and y separately, but is discontinuous at $(0,0)$ along the line $y = x$:

$$F(x,x) = \begin{cases} \frac{1}{2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Lemma

Let X and Y be topological spaces and let $h : X \times Y \to X \times Y$ be defined by $h(x,y) = (f(x), g(y))$ where $f : X \to X$ and $g : Y \to Y$ are continuous. Then h is continuous.

Proof. Assume that $W \in \mathscr{T}_{X\times Y}$. This means that $W = \bigcup_{\alpha\in\lambda}(U_\alpha \times V_\alpha)$ where the $U_\alpha \in \mathscr{T}_X$ and the $V_\alpha \in \mathscr{T}_Y$. Now:

$$\begin{aligned} h^{-1}(W) &= h^{-1}\left[\bigcup_{\alpha\in\lambda}(U_\alpha \times V_\alpha)\right] \\ &= \bigcup_{\alpha\in\lambda} h^{-1}(U_\alpha \times V_\alpha) \\ &= \bigcup_{\alpha\in\lambda} [f^{-1}(U_\alpha) \times g^{-1}(V_\alpha)] \\ &\in \mathscr{T}_{X\times Y} \end{aligned}$$

Therefore h is continuous. ∎

Corollary

Let X, Y, and $\mathbb{Z}$ be topological spaces and let $F : X \times Y \to Z$, $f : X \to X$, and $g : Y \to Y$ be continuous. The function $G : X \times Y \to Z$ defined by $G(x,y) = F(f(x), g(y))$ is continuous.

Proof. Let $h : X \times Y \to X \times Y$ be defined by $h(x,y) = (f(x), g(y))$. This means that h is continuous. Therefore $G = F \circ h$ is continuous. ∎

Corollary

Let X and Y be topological spaces and let $f_1, f_2 : X \to Y$ be homotopic. If $g : Y \to Z$ is continuous then $g \circ f_1$ is homotopic to $g \circ f_2$ in Z. In particular, if F is a homotopy between f_1 and f_2 then $g \circ F$ is a homotopy between $g \circ f_1$ and $g \circ f_2$.

Proof. Assume that F is a homotopy between f_1 and f_2 in Y. This means that $F : X \times I \to Y$ is continuous and $F(x,0) = f_1(x)$ and $F(x,1) = f_2(x)$. So, since F and g are continuous, $g \circ F$ is continuous in Z. Furthermore, $(g \circ F)(x,0) = g(F(x,0)) = g(f_1(x)) = (g \circ f_1)(x)$ and $(g \circ F)(x,1) = g(F(x,1)) = g(f_2(x)) = (g \circ f_2)(x)$. Therefore $g \circ F$ is a homotopy between $g \circ f_1$ and $g \circ f_2$. ∎

Theorem

Homotopic is an equivalence relation.

Proof. Let X and Y be topological spaces and let $f_1, f_2, f_3 : X \to Y$ be continuous:

Reflexive: Consider the function $F : X \times I \to Y$ defined by $F(x, k) = f_1(x)$.

Assume that $V \in \mathscr{T}_Y$. Since f is continuous, $f_1^{-1}(V) \in \mathscr{T}_X$. Furthermore, $F^{-1}(V) = f_1^{-1}(V) \times I \in \mathscr{T}_{X\times I}$. Thus F is continuous, $F(x, 0) = f_1(x)$, and $F(x, 1) = f_1$, and hence F is a homotopy between f_1 and f_1. Therefore $f_1 \simeq f_1$.

Symmetric: Assume that $f_1 \simeq f_2$.

This means that there exists a continuous function $F : X \times I \to Y$ between f_1 and f_2 such that $F(x, 0) = f_1(x)$ and $F(x, 1) = f_2(x)$. Let $G : X \times I \to Y$ be defined by $G(x, k) = F(x, 1 - k)$. Thus, G is continuous. Furthermore, $G(x, 0) = F(x, 1) = f_2(x)$ and $G(x, 1) = F(x, 0) = f_1(x)$. Therefore G is a homotopy between f_2 and f_1 and thus $f_2 \simeq f_1$.

Transitive: Assume that $f_1 \simeq f_2$ and $f_2 \simeq f_3$.

Assume that F is a homotopy between f_1 and f_2 and G is a homotopy between f_2 and f_3 and defined the function $G : X \times I \to Y$ as:

$$H(x, k) = \begin{cases} F(x, 2k), & k \in \left[0, \frac{1}{2}\right] \\ G(x, 2k - 1), & k \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Note that F and G agree for $k = \frac{1}{2}$, so by the pasting lemma, H is continuous. Furthermore, $H(x, 0) = F(x, 0) = f_1(x)$ and $H(x, 1) = G(x, 1) = f_3(x)$. Therefore H is a homotopy between f_1 and f_3 and thus $f_1 \simeq f_3$.

■

Notation

Let X and Y be a topological spaces and let $f : X \to Y$ be continuous. The equivalence class containing all continuous functions that are homotopic to f is denoted by $[f]$:

$$[f] = \{g : X \to Y \mid g \text{ is continuous and } g \simeq f\}$$

In particular, any two continuous functions $f, g : X \to \mathbb{R}^2$ are homotopic via the so-called *straight-line* homotopy:

$$F(x, k) = (1 - k)f(x) + kg(x)$$

It is called this because each $f(x)$ moves to its corresponding $g(x)$ along a straight line.

This can be generalized.

Definition: Convex

Let A be a subspace of R^n. To say that A is convex means that for any two points $a, b \in A$ and the function $f : \mathbb{R}^n \to \mathbb{R}^n$ defined as $f(t) = (1-t)a + tb$, $f(A) \subset A$.

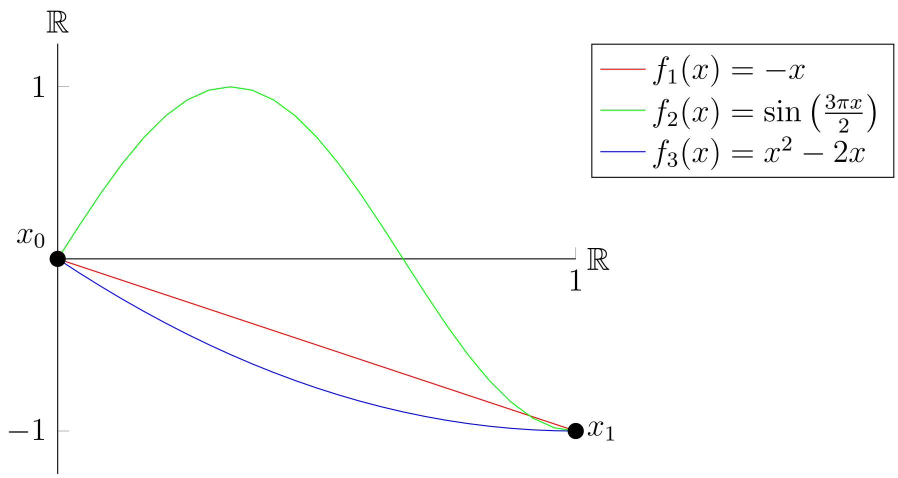
Thus, if $A \subset \mathbb{R}^n$ is convex then any two continuous functions in A are homotopic via the straight-line homotopy.

Definition: Path

Let X be a topological space and let $x_0, x_1 \in X$. To say that a continuous map $f : I \to X$ is a *path* from x_0 (the *initial* point) to x_1 (the *final* point) means that $f(0) = x_0$ and $f(1) = x_1$.

Example: Paths

Let $X = \mathbb{R}^2$ and let $x_0 = (0,0)$ and $x_1 = (1,-1)$:



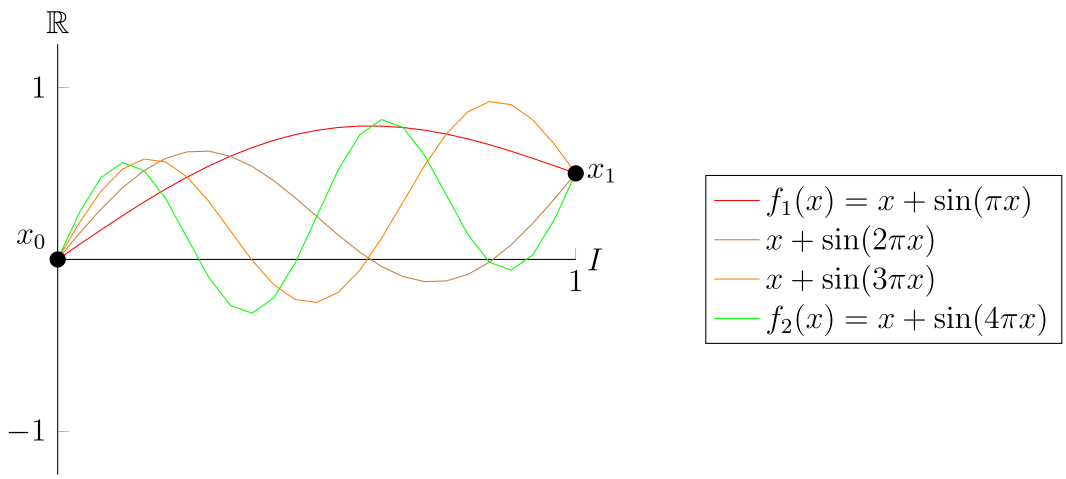
Definition: Path Homotopy

Let X be a topological space and let f_1 and f_2 be two paths in X. To say that f_1 and f_2 are *path homotopic* means that they have the same initial point x_0 and the same final point x_1 and there exists a continuous map $F : I \times I \to X$ such that:

1. $F(s,0) = f_1(s)$ and $F(s,1) = f_2(s)$
2. $F(0,t) = x_0$ and $F(1,t) = x_1$

Example: Path Homotopy

Let $X = \mathbb{R}^2$ and let $x_0 = (0,0)$ and $x_1 = (1,1)$:



Note that path homotopic is also an equivalence relation.

Definition

Let X be a topological space and let f_1 be a path in X from x_0 to x_1 and let f_2 be a path in X from x_1 to x_2. The *product* of f_1 and f_2 is given by:

$$f_1 * f_2 = \begin{cases} f_1(2k), & k \in \left[0, \frac{1}{2}\right] \\ f_2(2k-1), & k \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Note that f_1 and f_2 agree at $k = \frac{1}{2}$ $\left(f_1\left(\frac{1}{2}\right) = f_2\left(\frac{1}{2}\right) = x_1\right)$, so by the pasting lemma, $f_1 * f_2$ is continuous and thus a path in X from x_0 to x_2.

The product operator is not a binary operator because it is only defined on paths where $f_1(1) = f_2(0)$. Thus, the product operator creates a *groupoid*.

Lemma

Let X and Y be topological spaces and let f_1 and f_2 be two paths in X such that $f_1(1) = f_2(0)$, and let $g : X \to Y$ be continuous:

$$g \circ (f_1 * f_2) = (g \circ f_1) * (g \circ f_2)$$

Proof. Let f_1 is a path from x_0 to x_1 and let f_2 be a path from x_1 to x_2.

case 1: $k \in \left[0, \frac{1}{2}\right]$

$$(g \circ (f_1 * f_2))(k) = g((f_1 * f_2)(k)) = g(f_1(k)) = (g \circ f_1)(k) = ((g \circ f_1) * (g \circ f_2))(k)$$

case 2: $k \in \left[\frac{1}{2}, 1\right]$

$$(g \circ (f_1 * f_2))(k) = g((f_1 * f_2)(k)) = g(f_2(k)) = (g \circ f_2)(k) = ((g \circ f_1) * (g \circ f_2))(k)$$

At $k = \frac{1}{2}$:

$$(g \circ f_1)\left(\frac{1}{2}\right) = g(f_1(1)) = x_1$$

$$(g \circ f_2)\left(\frac{1}{2}\right) = g(f_2(0)) = x_1$$

So the function agree on the intersection. Furthermore:

$$((g \circ f_1) * (g \circ f_2))(0) = g(f_1(0)) = x_0$$

$$((g \circ f_1) * (g \circ f_2))(1) = g(f_2(1)) = x_2$$

And so the initial and final points are correct. ∎

Notation

Let X be a topological space and let $x \in X$. The constant path that maps all of I to x is denoted by e_x.

Lemma

Let X be topological space and let f be a path from x_0 to x_1 in X:

1. $[f] \circ e_0 = [e_{x_0}]$
2. $[f] \circ e_1 = [e_{x_1}]$

Proof. Assume that f is a path from x_0 to x_1 in X:

1. $(f \circ e_0)(j) = f(e_0(j)) = f(0) = x_0$
2. $(f \circ e_1)(j) = f(e_1(j)) = f(1) = x_1$

∎

Definition: Reverse

Let X be a topological space and let f be a path from x_0 to x_1 in X. The *reverse* of f is the path from x_1 to x_0 in X, denoted by $\bar{f}$, is given by:

$$\bar{f}(k) = f(1-k)$$

Lemma

Let X be a topological space. For all paths f in X:

1. $[f] \circ \iota_I = [f]$
2. $[f] \circ \bar{\iota}_I = [\bar{f}]$

Proof. Assume that f is a path in X.

1. $(f \circ \iota_I)(k) = f(\iota_I(k)) = f(k)$
2. $(f \circ \bar{\iota_I})(k) = f(\bar{\iota_I}(k)) = f(1-k) = \bar{f}(k)$

■

Theorem

Let X be a topological space and assume that f, g. and h be paths in X where f is a path from x_0 to x_1. The product operator on a topological space X has the following properties:

1. Associative: $([f] * [g]) * [h]$ is defined if and only if $[f] * ([g] * [h])$ is defined and if defined then they are equal.
2. Identity: $[e_{x_0}] * [f] = [f]$ and $[f] * [e_{x_1}] = [f]$.
3. Inverse: $[f] * [\bar{f}] = [e_{x_0}]$ and $[\bar{f}] * [f] = [e_{x_1}]$.

Proof.

1. Associative:
2. Identity:

   Consider the path $e_0 * \iota_I$ as a path in I from 0 to 1. Since I is convex, there exists a path homotopy F in I between ι_I and $e_0 * \iota_I$. This means that $f \circ F$ is also a path homotopy in X between the paths $f \circ \iota_I = f$ and $f \circ (e_0 * \iota_I) = (f \circ e_0) * (f \circ \iota_I) = e_{x_0} * f$.

   Therefore $[f] = [e_{x_0}] * [f]$

   Similarly, consider the path $\iota_I * e_1$ as a path in I from 0 to 1. Since I is convex, there exists a path homotopy F in I between ι_I and $\iota_I * e_1$. This means that $f \circ F$ is also a path homotopy in X between the paths $f \circ \iota_I = f$ and $f \circ (\iota_I * e_1) = (f \circ \iota_I) * (f \circ e_1) = f * e_{x_1}$.

   Therefore $[f] = [f] * [e_{x_1}]$

3. Inverse:

   Consider the path $\iota_I * \bar{\iota}_I$ as a path in I from 0 to 0. Since I is convex, there exists a path homotopy F in I between e_0 and $\iota_I * \bar{\iota}_I$. This means that $f \circ F$ is also a path homotopy in X between the paths $f \circ e_0 = e_{x_0}$ and $f \circ (\iota_I * \bar{\iota}_I) = (f \circ \iota_I) * (f \circ \bar{\iota}_I) = f * \bar{f}$.

   Therefore $[f] * [\bar{f}] = [e_{x_0}]$.

   Similarly, consider the path $\bar{\iota_I} * \iota$ as a path in I from 1 to 1. Since I is convex, there exists a path homotopy F in I between e_1 and $\bar{\iota}_I * \iota_I$. This means that $f \circ F$ is also a path homotopy in X between the paths $f \circ e_1 = e_{x_1}$ and $f \circ (\bar{\iota}_I * \iota_I) = (f \circ \bar{\iota}_I) * (f \circ \iota_I) = \bar{f} * f$.

Therefore $[\bar{f}] * [f] = [e_{x_1}]$.

■

Definition: Loop

Let X be a topological space and let $x_0 \in X$. A path in X whose initial and final point are both x_0 is called a *loop* in X based at x_0.

Definition: Fundamental Group

Let X be a topological space and let $x_0 \in X$. The set of path homotopy classes of loops based at x_0 along with the $*$ operator is called the *fundamental group* of X relative to the *base point* x_0 and is denoted by $\pi_1(X, x_0)$.

Note that $[e_{x_0}]$ is the identity element in the the fundamental group relative to x_0.

Theorem

Let A be a subspace of a topological space X and let $x_0 \in A$. If A is convex then

$$\pi_1(A, x_0) = \{e_{x_0}\}$$

Proof. Assume that A is convex and assume that f is a loop based at x_0. Since A is convex, $f \simeq (e_{x_0}$ by the straight-line homotopy. Therefore $f \in [e_{x_0}]$ and hence the fundamental group is trivial. ■

Corollary

All open balls in $\mathbb{R}^n$ have the trivial fundamental group.

Definition

Let X be a topological space and let α be a path in X between x_0 and x_1. Define the function $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ as follows:

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$$

Theorem

Let X be a topological space and let α be a path in X between x_0 and x_1. $\hat{(\alpha)}$ is a group isomorphism.

Proof. Assume that $f, g \in \pi_1(X, x_0)$:

$$\begin{aligned}
\hat{\alpha}([f]) * \hat{\alpha}(g) &= ([\bar{\alpha}] * [f] * [\alpha]) * ([\bar{\alpha}] * [g] * [\alpha]) \\
&= [\bar{\alpha}] * [f] * [e_{x_1}] * [g] * [\alpha] \\
&= [\bar{\alpha}] * [f] * [g] * [\alpha] \\
&= \hat{\alpha}([f] * [g])
\end{aligned}$$

Therefore $\hat{\alpha}$ is a homomorphism.

Assume $h \in \pi_1(X, x_0)$ and consider $\hat{\alpha}^{-1} = \hat{\bar{\alpha}}$:

$$\begin{aligned}
(\hat{\bar{\alpha}} \circ \hat{\alpha})([f]) &= \hat{\bar{\alpha}}(\hat{\alpha}([f])) \\
&= \hat{\bar{\alpha}}([\bar{\alpha}] * [f] * [\alpha]) \\
&= [\alpha] * ([\bar{\alpha}] * [f] * [\alpha]) * [\bar{\alpha}] \\
&= [e_{x_0}] * [f] * [e_{x_0}] \\
&= [f]
\end{aligned}$$

$$\begin{aligned}
(\hat{\alpha} \circ \hat{\bar{\alpha}})([h]) &= \hat{\alpha}(\hat{\bar{\alpha}}([h])) \\
&= \hat{\alpha}([\alpha] * [h] * [\bar{\alpha}]) \\
&= [\bar{\alpha}] * ([\alpha] * [h] * [\bar{[}\alpha]) * [\alpha] \\
&= [e_{x_1}] * [h] * [e_{x_1}] \\
&= [h]
\end{aligned}$$

Therefore $\hat{\alpha}$ is invertible and hence bijective.

Therefore $\hat{\alpha}$ is an isomorphism. ■

Corollary

Let X be a path-connected topological space. For all $x_0, x_1 \in X$, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

Note that for topological spaces that are not path connected, the fundamental group relative to some base point only describes the path component containing the base point; nothing else is known about the other path components. Thus, fundamental groups are usually applied to path connected spaces only.

Definition: Simply Connected

Let X be a path connected topological space. To say that X is *simply connected* means that for every $x_0 \in X$, $\pi_1(X, x_0) = \{[e_{x_0}]\}$. This is often denoted by $\pi_1(X, x_0) = 0$.

Theorem

Let X be a simply connected topological space. Any two paths in X having the same initial and final points are path homotopic.

Proof. Assume that f and g are two paths in X with initial point x_0 and final point x_1. This means that $f * \bar{g}$ is a loop in X based at x_0. Furthermore, since X is simply connected, $f * \bar{g} \in [e_{x_0}]$. ■

The fundamental group is a topological invariant property.

$\pi_1(S^1, x_0) \sim \mathbb{Z}$

$\pi_1(S^n, x_0) = 0$

$\pi_1(T = S^1 \times S^1) \sim \mathbb{Z} \times \mathbb{Z}$

$\pi_1(T \# T)$ is not abelian.