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LOWER BOUNDS FOR THE CLIQUE AND THE CHROMATIC NUMBERS OF A GRAPH

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G is any simple graph with m edges and n vertices where these vertices have vertex degrees $d(1) \geq d(2) \geq \cdots \geq d(n)$, respectively. General algorithms for the exact calculation of $\chi(G)$, the chromatic number of G, are almost always of 'branch and bound' type; this type of algorithm requires an easily constructed lower bound for $\chi(G)$. In this paper it is shown that, for a number of such bounds (many of which are described here for the first time), the bound does not exceed $\mathrm{cl}(G)$ where $\mathrm{cl}(G)$ is the clique number of G.

In 1972 Myers and Liu showed that $\mathrm{cl}(G) \geq n/(n-2m/n)$. Here we show that $\mathrm{cl}(G) \geq n/[n-(2m/n)(1+c_{\mathrm{v}}^2)^{1/2}]$, where c_{v} is the vertex degree coefficient of variation in G, that $\mathrm{cl}(G) \geq$ Bondy's constructive lower bound for $\chi(G)$, and that $\mathrm{cl}(G) \geq n/(n-W(G))$, where $W(G)$ is the largest positive integer such that $W(G) \leq d(W(G)+1)$ [$W(G)+1$ is the Welsh and Powell upper bound for $\chi(G)$]. It is also shown that $\mathrm{cl}(G)+\frac{1}{3} > n/(n-L(G)) \geq n/(n-\lambda_1)$ and that $\chi(G) \geq n/(n-\lambda_1)$; λ_1 is the largest eigenvalue of $\boldsymbol{A}$, the adjacency matrix of G, and $L(G)$ is a newly defined single-valued function of G similar to $W(G)$ in its construction from the vertex degree sequence of G [$L(G) \geq W(G)$].

Finally, a 'greedy' lower bound for $\mathrm{cl}(G)$ is constructed from $\boldsymbol{A}$ and it is shown that this lower bound is never less than Bondy's bound; thereafter, for 150 random graphs with varying numbers of vertices and edge densities, the values of lower bounds given in this paper are listed, thereby illustrating that this last greedily obtained lower bound is almost always better than each of the other bounds.

1. Introduction

Easily constructed lower bounds for $\chi(G)$, the chromatic number of a graph G, are of importance since they are required in branch and bound algorithms for the exact determination of $\chi(G)$. The simplest lower bound for $\chi(G)$ is the clique number, $\mathrm{cl}(G)$, but this bound is unsatisfactory for the following two reasons: first, the determination of $\mathrm{cl}(G)$ is itself an NP-hard problem; secondly, Tutte [9] has shown that there exists graphs of arbitrarily high chromatic number with $\mathrm{cl}(G)=2$. Thus, it is desirable to derive lower bounds for $\chi(G)$ which do not involve solutions of an NP-hard problem and which, ideally, are not necessarily also lower bounds for $\mathrm{cl}(G)$.

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The following notation will be employed: G denotes any simple graph with n vertices, m edges, chromatic number $\chi(G)=q$ and clique number $\mathrm{cl}(G)$. Let $d(1)\geq d(2)\geq\cdots\geq d(n)$ denote the vertex degree sequence of G, where $d(i)$ is the degree of the vertex v_i of G, and let i be called 'the vertex index' of v_i $(1\leq i\leq n)$. Let c_{v} be the non-negative number defined by

$$1+c_{\mathrm{v}}^2=\frac{n}{4m^2}\sum_{i=1}^{n}[d(i)]^2.$$

One of us [2] has suggested that c_{v} be called the 'vertex degree coefficient of variation' for G and has shown that $c_{\mathrm{v}}=0$ if and only if G is regular. G is called 'q-partite' since it is always possible to partition V, the vertex set of G, into q independent sets $V_1, V_2, \ldots, V_q$; let $\# V_i=n_i$ $(1\leq i\leq q)$, where $n_1\leq n_2\leq\cdots\leq n_q$. G is called 'complete-q-partite', (with colour classes $V_1, V_2, \ldots, V_q$), if, for every vertex v in V_i, v is adjacent to every vertex in $V\setminus V_i$ $(1\leq i\leq q)$. Finally, let $\boldsymbol{A}$ denote the adjacency matrix of G and let $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$ denote the eigenvalues of $\boldsymbol{A}$.

Section 2 contains results concerning lower bounds for $\mathrm{cl}(G)$ and $\chi(G)$ based on the degree sequence of G. In Section 3 several bounds based on the eigenvalues of a graph are discussed and in Section 4 a lower bound based on the adjacency matrix $\boldsymbol{A}$ is derived. The paper concludes with an experimental comparison of alternative lower bounds for $\mathrm{cl}(G)$ and $\chi(G)$ based on a study of 150 random graphs.

2. Bounds based on the degree sequence

Theorem 1 is a result, due to Erdös [4], which can be used to derive lower bounds for $\mathrm{cl}(G)$ from lower bounds for $\chi(G)$; we shall use this theorem in the proofs of Theorem 3 and the Corollary to Theorem 5. Theorem 2 is the simplest degree-based lower bound for $\mathrm{cl}(G)$ and is due to Myers and Liu [6]; this bound is improved in Theorem 3 with the replacement of the mean vertex degree by a function of $\sum_{i=1}^{n}[d(i)]^2$. An alternative function of $\sum_{i=1}^{n}[d(i)]^2$ is used in the Corollary to Theorem 10 to provide a lower bound for $\chi(G)$ where we have not been able to show that this lower bound is also a lower bound for $\mathrm{cl}(G)$. A further lower bound for $\mathrm{cl}(G)$, based on the Welsh and Powell [11] upper bound for $\chi(G)$, is derived in Theorem 4; a related lower bound for $\mathrm{cl}(G)+\frac{1}{3}$ is given in Theorem 9. In the Corollary to Theorem 5 the Bondy [1] lower bound for $\chi(G)$ is shown also to be a lower bound for $\mathrm{cl}(G)$. We note that for a graph regular of degree d all of the lower bounds of Theorems 2 to 5 are equal to $n/(n-d)$.

Definition 1. If H is any graph of order n with degree sequence $d_H(1)\geq d_H(2)\geq\cdots\geq d_H(n)$, and if H^* is any graph of order n with degree sequence $d_{H^*}(1)\geq d_{H^*}(2)\geq\cdots\geq d_{H^*}(n)$, such that $d_H(i)\leq d_{H^*}(i)$ $(1\leq i\leq n)$, then H^* is said to 'dominate' H.

Theorem 1 (Erdös [4]). *If H is any graph of order n, then there exists a graph H^* of order n, where $\chi(H^*)=\mathrm{cl}(H)$, such that H^* dominates H.*

Theorem 2 (Myers and Liu [6]). $n/(n-2m/n) \le \mathrm{cl}(G)$.

Theorem 3. *If H is any graph of order n and degree sequence $d_H(1) \ge d_H(2) \ge \cdots \ge d_H(n)$, then*

$$n\Big/\left[n - \left(\frac{1}{n}\sum_{i=1}^{n} [d_H(i)]^2\right)^{1/2}\right] \le \mathrm{cl}(H).$$

Proof. Since G is any graph of order n with $\chi(G)=q$ and colour classes $V_1, V_2, \ldots, V_q$, we see that

$$\sum_{i=1}^{n} [d(i)]^2 \le \sum_{j=1}^{q} n_j(n-n_j)^2; \tag{1}$$

since $\sum_{j=1}^{q} n_j = n$, if $n_i \le n_j$, then it is easy to show that $n_i(n-n_i) \le n_j(n-n_j)$. Thus

$$\sum_{j=1}^{q} n_j(n-n_j)^2 = \sum_{j=1}^{q} n_j(n-n_j)(n-n_j) \le n(1-\frac{1}{q}) \sum_{j=1}^{q} n_j(n-n_j). \tag{2}$$

Next, since, by the Cauchy-Schwartz inequality,

$$\sum_{j=1}^{q} n_j^2 \ge \frac{1}{q}\left(\sum_{j=1}^{q} n_j\right)^2 = \frac{1}{q}n^2,$$

we see that

$$n\left(1-\frac{1}{q}\right)\sum_{j=1}^{q} n_j(n-n_j) = n\left(1-\frac{1}{q}\right)\left(n^2 - \sum_{j=1}^{q} n_j^2\right) \le n^3\left(1-\frac{1}{q}\right)^2.$$

Using (1) and (2), it follows that

$$\left(\frac{1}{n}\sum_{i=1}^{n} [d(i)]^2\right)^{1/2} \le n\left(1-\frac{1}{q}\right)$$

and, after re-arranging, we obtain

$$n\Big/\left[n - \left(\frac{1}{n}\sum_{i=1}^{n} [d(i)]^2\right)^{1/2}\right] \le q = \chi(G). \tag{3}$$

Finally, let H be any graph of order n with $\mathrm{cl}(H)=q$; it follows from Theorem 1 that there exists a graph H^* or order n and degree sequence $d_{H^*}(1) \ge d_{H^*}(2) \ge \cdots \ge d_{H^*}(n)$, where H^* dominates H, such that $\chi(H^*)=q$. Since we can put $H^*=G$ in (3), we see that

$$n\Big/\left[n - \left(\frac{1}{n}\sum_{i=1}^{n} [d_H(i)]^2\right)^{1/2}\right] \le n\Big/\left[n - \left(\frac{1}{n}\sum_{i=1}^{n} [d_{H^*}(i)]^2\right)^{1/2}\right] \le q = \mathrm{cl}(H).$$

This completes the proof.

Corollary. $n/[n-(2m/n)(1+c_v^2)^{1/2}] \le \mathrm{cl}(G)$.

Proof. It may be readily verified that

$$\frac{2m}{n}(1+c_v^2)^{1/2} = \left(\frac{1}{n}\sum_{i=1}^{n} [d(i)]^2\right)^{1/2}.$$

Definition 2. $W(G) \triangleq \max_{1\le i\le n} \min[d(i), i-1]$.

Remark. $1+W(G)$ is the Welsh and Powell [11] upper bound for $\chi(G)$. $W(G)$ can be equivalently defined as the largest integer $i-1$ such that $i-1 \le d(i)$.

Theorem 4. $n/[n-W(G)] \le \mathrm{cl}(G)$.

Proof. Let $\bar{G}$ denote the complementary graph of G; then it is well known that $\chi(\bar{G}) \ge n/cl(G)$. Also, as remarked above, Welsh and Powell have shown that $\chi(G) \le 1+W(G)$, whilst Bondy [1] has shown that $W(G)+W(\bar{G}) \le n-1$. Thus,

$$\mathrm{cl}(G) \ge n/\chi(\bar{G}) \ge n/[1+W(\bar{G})] \ge n/[n-W(G)].$$

This completes the proof.

Theorem 5 (Bondy [1]). *If s_j is defined recursively by $s_j = n-d(1+\sum_{i=1}^{j-1} s_i)$, and if k is the largest integer such that $\sum_{j=1}^{k-1} s_j < n$, then $k \le \chi(G)$.*

Corollary. $k \le \mathrm{cl}(G)$.

Proof. Let H^* be any graph dominating G with $\chi(H^*) = \mathrm{cl}(G)$; (by Theorem 1, at least one such H^* exists). So, by this Theorem 5,

$$k(H^*) \le \mathrm{cl}(G), \tag{1}$$

where $k(H^*)$ is directly analogous to k.

By inspection of the forms of k and $k(H^*)$, we see first that $s_i \ge s_i^*$ $(i=1,2,\ldots)$, where s_i^* is the analogue for H^* of s_i for G, and so $k \le k(H^*)$. Using (1), we see that our result is obtained.

Remark. Where ambiguity otherwise could result we denote k by $k(G)$.

3. Bounds based on the largest eigenvalue

It is of interest to investigate eigenvalue based lower bounds for $\mathrm{cl}(G)$ and $\chi(G)$ since they can be used to deduce degree based bounds. For example, in Theorem 6 it will be shown that $n/(n-\lambda_1) \le \chi(G)$; since it is easy to show that $\lambda_1 \ge (2m/n) \times (1+c_v^2)^{1/2}$, using Theorem 1, we are able to deduce Theorem 3 trivially from

Theorem 6. It has not been possible to prove that $n/(n-\lambda_1)\le \mathrm{cl}(G)$ for all graphs, but Elphick [3] has proved this result for planar graphs and in Theorem 9 it is shown that $n/(n-\lambda_1)<\mathrm{cl}(G)+\frac{1}{3}$. For graphs regular of degree d the bounds of both Theorems 7 and 9 are equal to $n/(n-d)$; [in this context, we recall that Hoffmann [5] has shown that for graphs regular of degree d his own lower bound for $\chi(G)$, quoted in Theorem 11) is not less than $n/(n-d)$].

Definition 3. G^* denotes the complete q-partite graph with colour classes $V_1, V_2, \ldots, V_q$; let $d^*(1)\ge d^*(2)\ge\cdots\ge d^*(n)$ denote the vertex degree sequence of G^* and let m^* be the number of edges in G^*.

Lemma 1. *G is a subgraph of G^*.*

Proof. Obvious.

Corollary. *G^* dominates G.*

Proof. Immediate from the definitions in Section 1, Definitions 1 and 3.

Theorem 6. $n/(n-\lambda_1)\le\chi(G)$.

Proof. By Lemma 1, G is a subgraph of the complete q-partite graph G^*, also of order n. Schwenk and Wilson [8] have noted that, as a simple corollary of the Perron–Frobenius Theorem, $\lambda_1\le\lambda_1^*$, where λ_1^* is the largest eigenvalue of $\boldsymbol{A}^*$, the adjacency matrix of G^*. Therefore it is sufficient to prove our result with G^* replacing G.

By Definition 3, the vertices of G^*, like those of G, are partitioned into the q independent vertex sets $V_1, \ldots, V_q$ [where, as we recall, $\# V_i=n_i$ $(1\le i\le q)$ and $n_1\le n_2\le\cdots\le n_q$]. Schwenk [7] and Waller [10] have proved that the largest eigenvalue λ_1^* of G^* is given by the matrix equation $\boldsymbol{S}\boldsymbol{y}=\lambda_1^*\boldsymbol{y}$, where the positive eigenvector $\boldsymbol{y}=(y_1,\ldots,y_q)$ and where

$$\boldsymbol{S}=\begin{bmatrix} 0 & n_2 & \cdots & n_q \\ n_1 & 0 & \cdots & n_q \\ \vdots & \vdots & & \vdots \\ n_1 & n_2 & \cdots & 0 \end{bmatrix}.$$

Let $c=\sum_{j=1}^{q} n_j y_j$; then the matrix equation may be equivalently rewritten as

$$c-n_j y_j=\lambda_1^* y_j \quad (1\le j\le q). \tag{1}$$

Therefore $y_j=c/(\lambda_1^*+n_j)$. Since each element y_j of $\boldsymbol{y}$ (which, of course, corresponds to λ_1^*), is positive, since $n_1\le n_2\le\cdots\le n_q$ and since $c=\sum_{j=1}^{q} n_j y_j>0$, it follows that $y_1\ge y_2\ge\cdots\ge y_q$; adding the q equations in (1), we see that

$$\lambda_1^* \sum_{j=1}^{q} y_j = (q-1) \sum_{j=1}^{q} n_j y_j \le n\left(1 - \frac{1}{q}\right) \sum_{j=1}^{q} y_j.$$

It follows that $\lambda_1^* \le n(1 - 1/q)$. Since $\lambda_1 \le \lambda_1^*$, this completes the proof.

Definition 4. $D(x) \triangleq d(\{x\})$ for each real x $(0 < x \le n)$ [where, as usual, $\{y\}$ denotes the smallest integer $\ge y$ for all real y]. $D(0) \triangleq d(1)$. $D^*(x) \triangleq d^*(\{x\})$ for each real x $(0 < x \le n)$, and $D^*(0) \triangleq d^*(1)$ (see Definition 3).

Lemma 2. *Each of $D(x)$ and $D^*(x)$ is a non-negative and non-increasing function of x, and $D(x) \le D^*(x)$ $(0 \le x \le n)$.*

Proof. Immediate from the Corollary to Lemma 1 and Definition 4.

Definition 5. $y(G)$ is any positive real defined by the equation:

$$y(G)[y(G) - 1] = \int_0^{y(G)} D(x)\,\mathrm{d}x;$$

$y(G^*)$ is analogously defined. Where no ambiguity results, $y(G)$ and $y(G^*)$ will be denoted by y and y^*, respectively.

Lemma 3. *y is uniquely determined, and $1 \le y \le n$, with $y = 1$ iff $m = 0$ and with $y = n$ iff G is complete.*

Proof. Since, by definition, $y > 0$, it follows from Definition 4 that $\int_0^y D(x)\,\mathrm{d}x \ge 0$. Again using Definition 5, since $d(1) \ge d(2) \ge \cdots \ge d(n) \ge 0$, it follows that $y \ge 1$, with equality iff $d(1) = 0$ and then $y = 1$ iff $m = 0$.

Since $y > 0$, and since, by Lemma 2, $D(x) \le n - 1$ is a non-increasing function of x, it follows that $(1/y)\int_0^y D(x)\,\mathrm{d}x$ is defined, does not exceed $n - 1$ and is a non-negative non-increasing continuous function of y, whilst $y - 1$ is a non-negative strictly increasing function of y continuous in the interval $0 \le y - 1 \le n - 1$. Thus, both y is uniquely determined by the equation $y(y-1) = \int_0^y D(x)\,\mathrm{d}x$ and also $y \le n$.

Finally, using Definition 4 we easily see that $\int_0^n D(x)\,\mathrm{d}x \le n(n-1)$, with equality iff $G = K_n$, the complete graph; since $y \ge 1$ is uniquely determined, it follows not only that $y \le n$, but also $y = n$ iff $G = K_n$.

This completes the proof.

Lemma 4. *$y \le y^*$.*

Proof. By Lemmas 2, 3 and Definition 5,

$$y - 1 \le \frac{1}{y} \int_0^y D^*(x)\,\mathrm{d}x. \tag{1}$$

Now, by the proof of Lemma 3 we see that $(1/y)\int_0^y D^*(x)\,\mathrm{d}x$ is a non-negative non-increasing continuous function $\le n-1$ of $y \ge 1$, whilst $y-1$ is an increasing function of y. Since y^* exists, since $y^* \ge 1$ (by Lemma 3), and since $y^*-1=(1/y^*)\int_0^{y^*} D^*(x)\,\mathrm{d}x$, using (1), our result is obtained.

Lemma 5. *If G is a subgraph of G^+, then $y(G) \le y(G^+)$.*

Proof. Considering any n largest vertex degrees of G^+, the proof becomes almost identical to that of Lemma 4.

Definition 6. $L(G) \triangleq y(G)-1$; where no ambiguity results, $L(G)$ and $L(G^*)$ will be denoted by L and L^*, respectively.

Lemma 6. $L(G) \ge W(G)$.

Proof. Since $W(G)$ is an integer, using Definitions 4, 5, 6, Lemmas 2, 3, and the alternative definition of $W(G)$ as the largest integer $i-1$ such that $i-1 \le d(i)$, our result is immediately obtained.

Lemma 7. *If G is a subgraph of G^+, then $L(G) \le L(G^+)$.*

Proof. Immediate from Lemma 5 and Definition 6.

Corollary. $L \le L^*$.

Lemma 8. $L^* < n(1-1/(q+\frac{1}{3}))$.

Proof. Suppose that $L^* > n(1-1/q)$ or, equivalently, that

$$y^*-1 > n\left(1-\frac{1}{q}\right). \tag{1}$$

Now, $n_1 \le n_2 \le \cdots \le n_q$ where $n_i = \# V_i$ $(1 \le i \le q)$ [in the present context, we recall Definition 3]. Using (1), it follows that

$$\begin{aligned} y^*(y^*-1)+(n-y^*)(n-n_q) &= \int_0^{y^*} D(x)\,\mathrm{d}x + (n-y^*)(n-n_q) = 2m^* \\ &\le (n-n_q)^2\left(1-\frac{1}{q-1}\right) + 2n_q(n-n_q), \end{aligned}$$

i.e.

$$\begin{aligned} y^*(y^*-1) &\le (n-n_q)\left[(n-n_q)\left(1-\frac{1}{q-1}\right)+2n_q-n+y^*\right] \\ &= (n-n_q)\left[n_q+y^*-\frac{1}{q-1}(n-n_q)\right] \end{aligned}$$

$$= (n-n_q)\left[n + y^* - \frac{q}{q-1}(n-n_q)\right]$$

$$= \frac{q}{q-1}(n-n_q)\left[\left(1-\frac{1}{q}\right)(n+y^*) - (n-n_q)\right]$$

$$\leq \frac{q}{q-1}\left[\tfrac{1}{2}\left(1-\frac{1}{q}\right)(n+y^*)\right]^2$$

$$= \left(1-\frac{1}{q}\right)[\tfrac{1}{2}(n+y^*)]^2,$$

and so

$$L^*(L^*+1) \leq \left(1-\frac{1}{q}\right)[\tfrac{1}{4}n^2 + \tfrac{1}{2}n(L^*+1) + \tfrac{1}{4}(L^*+1)^2],$$

i.e.

$$4qL^{*2} + 4qL^* \leq (q-1)(n^2 + 2nL^* + 2n + L^{*2} + 2L^* + 1)$$

i.e.

$$(3q+1)L^{*2} - 2[(q-1)(n+1) - 2q]L^* \leq (q-1)(n^2+2n+1),$$

i.e.

$$(3q+1)L^{*2} - 2[(q-1)(n-1) - 2]L^* \leq (q-1)(n+1)^2.$$

Thus,

$$\left[L^* - \frac{(q-1)(n-1)-2}{3q+1}\right]^2 (3q+1)^2$$
$$\leq (q-1)^2(n-1)^2 - 4(q-1)(n-1) + 4 + (q-1)(3q+1)(n+1)^2$$
$$= (q-1)(qn^2 - 2qn + q - n^2 + 2n - 1 - 4n + 4$$
$$+ 3qn^2 + 6qn + 3q + n^2 + 2n + 1) + 4$$
$$= (q-1)(4qn^2 + 4qn + 4q + 4) + 4$$
$$= 4[q(q-1)n(n+1) + q^2]$$
$$= 4[(q-\tfrac{1}{2})^2(n+\tfrac{1}{2})^2 - \tfrac{1}{4}q(q-1) - \tfrac{1}{4}n(n+1) - \tfrac{1}{16} + q^2]$$
$$= 4[(q-\tfrac{1}{2})^2(n+\tfrac{1}{2})^2 - \tfrac{1}{4}(q-\tfrac{1}{2})^2 - \tfrac{1}{4}(n+\tfrac{1}{2})^2 + \tfrac{1}{16} + q^2].$$

However, since $(q-\frac{1}{2})^2 + (n+\frac{1}{2})^2 \geq 4(q-\frac{1}{2})(n+\frac{1}{2})$, it follows that

$$(q-\tfrac{1}{2})^2(n+\tfrac{1}{2})^2 - \tfrac{1}{4}(q-\tfrac{1}{2})^2 - \tfrac{1}{4}(n+\tfrac{1}{2})^2 + \tfrac{1}{16} + q^2$$
$$\leq (q-\tfrac{1}{2})^2(n+\tfrac{1}{2})^2 - (q-\tfrac{1}{2})(n+\tfrac{1}{2}) + \tfrac{1}{16} + q^2,$$

and so,

$$\left[L^* - \frac{(q-1)(n-1)-2}{3q+1}\right]^2 (3q+1)^2$$
$$\leq 4[(q-\tfrac{1}{2})^2(n+\tfrac{1}{2})^2 - (q-\tfrac{1}{2})(n+\tfrac{1}{2}) + \tfrac{1}{16} + q^2]. \tag{2}$$

Now, if $n=q$, then it is easy to see that $G=K_n$ and then, from the definition of L^*, we see that

$$L^*=n-1=n\left(1-\frac{1}{q}\right)\leq n\left(1-\frac{1}{q+\frac{1}{3}}\right)$$

and our result is obtained for $q=n$. Thus, since $q\geq 1$, we can suppose that $1\leq q\leq n-1$, and it follows that

$$q^2-(q-\tfrac{1}{2})(n+\tfrac{1}{2})=q^2-nq-\tfrac{1}{2}q+\tfrac{1}{2}n=q(q-\tfrac{1}{2})-n(q-\tfrac{1}{2})$$
$$=(q-n)(q-\tfrac{1}{2})\leq -(q-\tfrac{1}{2})<-\tfrac{1}{16}.$$

Using (2), it follows that if $q<n$, then

$$\left[L^*-\frac{(q-1)(n-1)-2}{3q+1}\right]^2(3q+1)^2<4(q-\tfrac{1}{2})^2(n+\tfrac{1}{2})^2. \tag{3}$$

Since, by (1), $L^*>n(1-1/q)$, and since

$$n\left(1-\frac{1}{q}\right)>\frac{(q-1)(n-1)-2}{3q+1}$$

whenever $q\geq 1$, using (3), it follows that

$$L^*<\frac{(q-1)(n-1)-2+(2q-1)(n+\frac{1}{2})}{3q+1}$$
$$=\frac{qn-n-q+1-2+2qn-n+q-\frac{1}{2}}{3q+1}$$
$$=\frac{(3q-2)n-\frac{3}{2}}{3q+1}<n\left(1-\frac{1}{q+\frac{1}{3}}\right)$$

Since $n(1-1/(q+\frac{1}{3}))>n(1-1/q)$, we see that the proof of our lemma is complete.

Corollary. $L<n(1-1/(q+\frac{1}{3}))$.

Proof. Immediate, using the Corollary to Lemma 7.

Theorem 7. $n/(n-L(G))<\mathrm{cl}(G)+\frac{1}{3}$.

Proof. By the Corollary to Lemma 8,

$$\frac{n}{n-L(G)}<\chi(G)+\tfrac{1}{3}. \tag{1}$$

Let H^* be any graph of order n, where $\chi(H^*)=\mathrm{cl}(G)$, such that H^* dominates G; (by Theorem 1, at least one such graph H^* exists). Writing H^* for G in (1), we see that

$$\frac{n}{n-L(H^*)} < \mathrm{cl}(G) + \tfrac{1}{3}. \tag{2}$$

However, from Definitions 4, 5 and 6, $L(G) \le L(H^*)$ and so, by (2),

$$\frac{n}{n-L(G)} \le \frac{n}{n-L(H^*)} < \mathrm{cl}(G) + \tfrac{1}{3}.$$

This completes the proof.

Theorem 8. *If G is connected, then* $L(G) \ge \lambda_1$.

Proof. Evidently, if G is connected, then $\boldsymbol{A}$ is irreducible and, by the Perron-Frobenius Theorem,

$\boldsymbol{A}$ has a unique (to a constant multiplier) positive eigenvector $\boldsymbol{x}$ corresponding to the largest eigenvalue $\lambda_1 > 0$. (1)

Let $x_{i(1)} \ge x_{i(2)} \ge \cdots \ge x_{i(n)}$, be the elements of x, where $i(k)$ is the vertex index, (see Introduction), corresponding to the row (and column) $i(k)$ of $\boldsymbol{A}$ $(1 \le k \le n)$. Thus,

$$\lambda_1 x_{i(k)} = \sum_{j=1}^{n} a_{i(k),j} x_j \quad (1 \le k \le n). \tag{2}$$

Recalling Definition 5 and Lemma 3, if $[y]$ here denotes the largest integer $\le y$, then

$$x^* \triangleq \frac{1}{y}\left(\sum_{k=1}^{[y]} x_{i(k)} + (y-[y])x_{i(\{y\})}\right); \tag{3}$$

using (2), we see that

$$\lambda_1 y x^* = \sum_{k=1}^{[y]} \sum_{j=1}^{n} a_{i(k),j} x_j + (y-[y]) \sum_{j=1}^{n} a_{i(\{y\}),j} x_j. \tag{4}$$

Now,

$$\sum_{k=1}^{r} \sum_{j=1}^{n} a_{i(k),j} = \sum_{k=1}^{r} d[i(k)] \le \sum_{k=1}^{r} d(k) \quad (1 \le r \le n);$$

thus, slightly extending the previous argument, using Definition 5, we see that

$$\sum_{k=1}^{[y]} \sum_{j=1}^{n} a_{i(k),j} + (y-[y]) \sum_{j=1}^{n} a_{i(\{y\}),j}$$
$$\le \sum_{k=1}^{[y]} d(k) + (y-[y])d(\{y\}) = \int_0^y D(z)\,\mathrm{d}z = y(y-1),$$

and so

$$\sum_{k=1}^{[y]} \sum_{j=1}^{n} a_{i(k),j} + (y-[y]) \sum_{j=1}^{n} a_{i(\{y\})} \leq y(y-1). \tag{5}$$

Also, since each principal diagonal element of $\boldsymbol{A}$ is zero,

$$\text{within } \sum_{k=1}^{[y]} \sum_{j=1}^{n} a_{i(k),j}x_j + (y-[y]) \sum_{j=1}^{n} a_{i(\{y\}),j}x_j, \\ \text{each of } x_{i(1)}, x_{i(2)}, \ldots, x_{i([y])}, \text{ has sum of coefficients } \leq y-1; \tag{6}$$

since, by (5), the sum of all coefficients of $x_1, x_2, \ldots, x_n$, within

$$\sum_{k=1}^{[y]} \sum_{j=1}^{n} a_{i(k),j}x_j + (y-[y]) \sum_{j=1}^{n} a_{i(\{y\}),j}x_j$$

does not exceed $y(y-1)$, since $x_{i(1)} \geq x_{i(2)} \geq \cdots \geq x_{i(n)}$, since, by (3), $x^* \geq x_{i(\{y\})} \geq x_{i(\{y\}+1)} \geq \cdots \geq x_{i(n)}$, and since, using (6), we see that $(y-1)[y] \geq$ the total of all coefficients of all $x_{i(k)}$, $(k < 1 + [y])$, within

$$\sum_{k=1}^{[y]} \sum_{j=1}^{n} a_{i(k),j}x_j + (y-[y]) \sum_{j=1}^{n} a_{i(\{y\}),j}x_j,$$

then, again using both (3) and (6), it follows that

$$\sum_{k=1}^{[y]} \sum_{j=1}^{n} a_{i(k),j}x_j + (y-[y]) \sum_{j=1}^{n} a_{i(\{y\}),j}x_j \leq y(y-1)x^*.$$

Using (4), it follows that

$$\lambda_1 y x^* \leq y(y-1)x^*.$$

Since, by Lemma 3, $y \geq 1$, and so, using (1) and (3), $x^* > 0$, by Definition 6, it follows that $L(G) \geq \lambda_1$, (whenever G is connected), as required.

Corollary. $L(G) \geq \lambda_1$.

Proof. Let G' be any connected component of G, where λ_1 is the largest eigenvalue both of $\boldsymbol{A}$ and also of $\boldsymbol{A}'$, the adjacency matrix of G'; (evidently, G contains at least one such component G'). Thus, $L(G') \geq \lambda_1$.

However, by Lemma 7, $L(G) \geq L(G')$ and so $L(G) \geq \lambda_1$, as required.

Theorem 9.

$$\frac{n}{n-\lambda_1} \leq \frac{n}{n-L(G)} < \mathrm{cl}(G) + \tfrac{1}{3}.$$

Proof. Immediate, using Theorem 7 and the Corollary to Theorem 8.

Theorem 10 (Elphick [3]). $2m/(2m-\lambda_1^2)\le\chi(G)$.

Corollary. $n/[n-(2m/n)(1+c_v^2)]\le\chi(G)$.

Proof. Since, as stated earlier, it is easy to show that $\lambda_1\ge(2m/n)(1+c_v^2)^{1/2}$ or, equivalently that $\lambda_1^2\ge(2m/n)^2(1+c_v^2)$, dividing both numerator and denominator of the left hand side fraction, in the statement of the theorem, by $2m/n$, we see that our result is obtained immediately.

Remark. As indicated earlier, it is not possible, apparently, to substitute $\mathrm{cl}(G)$ for $\chi(G)$ in the statement of this corollary with the use of Theorem 1; indeed, we have been unable to prove the result that arises from this substitution, (cf. Theorems 2 and 3 and see the beginning of Section 2).

Theorem 11 (Hoffman [5]). *If* $m>0$, *then* $1-\lambda_1/\lambda_n\le\chi(G)$.

4. A bound based on the adjacency matrix

In this section we describe a constructive lower bound for $\mathrm{cl}(G)$. In Theorem 12 it is shown that this bound is always at least as good as the Bondy bound given in Theorem 5 and its Corollary.

In G, let $v(t_1)=v_1$, an arbitrarily chosen vertex amongst those of maximum degree; let $v(t_2)$, of vertex degree $d(t_2)$, denote the vertex of smallest vertex index adjacent to $v(t_1)$ in G; in general let $v(t_j)$, of vertex degree $d(t_j)$, denote the vertex of smallest vertex index adjacency to each of $v(t_1), v(t_2), \dots, v(t_{j-1})$ $(1\le j\le k^*)$ [it is assumed that there exists in G no vertex $v(t_{k^*+1})$ which is adjacent to each of $v(t_1), v(t_2), \dots, v(t_{k^*})$]. Thus, the vertices $v(t_1), v(t_2), \dots, v(t_{k^*})$ form a complete subgraph K_{k^*} and so $k^*<\mathrm{cl}(G)$; hereafter, this lower bound k^* for $\mathrm{cl}(G)$ will be denoted either by $k^*(G)$ or, where no ambiguity results, by k^*. (We recall that the somewhat analogous Bondy lower bound for $\chi(G)$ and $\mathrm{cl}(G)$, given in Theorem 5 and its Corollary, is denoted either by $k(G)$ or, where no ambiguity results, by k.)

Theorem 12. $k^*(G)\ge k(G)$.

Proof. The notation used in Theorem 5 can be reformulated, equivalently, as follows: $s_j\triangleq n-d(r_j)$, where recursively

$$r_j\triangleq 1+\sum_{i=1}^{j-1}[n-d(r_i)] \quad (1\le j\le k), \tag{1}$$

as in Theorem 5, $k(G)$ and k are each defined as the largest integer such that

$$\sum_{i=1}^{k-1} s_i<n. \tag{2}$$

We now introduce the sequence of vertices $v(t_1), v(t_2), \ldots, v(t_{k^*})$, with corresponding vertex degrees $d(t_1), d(t_2), \ldots, d(t_{k^*})$, as defined in the introduction to this section; thereafter,

$$u_j \triangleq n - d(t_j) \quad (1 \le j \le k^*). \tag{3}$$

It follows that

$$\sum_{i=1}^{k^*} u_i = \sum_{i=1}^{k^*} [n - d(t_i)]$$

$$= \sum_{i=1}^{k^*} [\text{number of vertices not adjacent to } v(t_i)] \ge n;$$

(otherwise there exists a vertex $v(t_{k^*+1})$ adjacent to each of $v(t_1), v(t_2), \ldots, v(t_{k^*})$, contrary to assumption). Thus

$$\sum_{i=1}^{k^*} u_i \ge n. \tag{4}$$

We can now prove that $t_j \le r_j$ $(j = 1, 2, \ldots)$. Observing that $t_1 = r_1$, we proceed by induction on j, as follows:

Induction Hypothesis: $t_i \le r_i$ $(1 \le i \le j)$, for some $j \ge 1$.

Using the definition of r_{j+1} in (1), it follows that

$$r_{j+1} = 1 + \sum_{i=1}^{j} [n - d(r_i)] \ge 1 + \sum_{i=1}^{j} [n - d(t_i)]$$

$$= 1 + \sum_{i=1}^{j} [\text{the number of vertices not adjacent to } v(t_i)] \ge t_{j+1}$$

[by definition $v(t_{j+1})$ is the vertex of smallest vertex index adjacent to each of the vertices $v(t_1), v(t_2), \ldots, v(t_j)$]. Thus $t_{j+1} \le r_{j+1}$.

Since $t_1 = r_1$, we see that, by induction, we have shown that $t_j \le r_j$ for all positive integers j such that both t_j and r_j are defined. Using (1) and (3), it follows that

$$u_j \le s_j \text{ for all } j \text{ such that both } u_j \text{ and } s_j \text{ are defined.} \tag{5}$$

Next, suppose that $k \ge k^* + 1$. Using (2), (4) and (5), it follows that

$$\sum_{j=1}^{k-1} s_j \ge \sum_{j=1}^{k^*} s_j \ge \sum_{j=1}^{k^*} u_j \ge n,$$

and so $\sum_{j=1}^{k-1} s_j \ge n$, contrary to the definition of $k(G)$ in (2). This completes the proof.

5. An experimental comparison of lower bounds

The lower bounds for $\mathrm{cl}(G)$ and $\chi(G)$ discussed in this paper have been computed for 150 random graphs of type G_{np}, defined as follows:

A random graph G_{np} is a graph on n vertices in which each of the $\frac{1}{2}n(n-1)$ possible edges occurs with an independent probability p $(0<p<1)$. A summary of the results is given in Table 1, which lists the means over 50 graphs for three (n, p) pairs. The bounds are referenced by their theorem numbers.

The results provide strong evidence that the best available lower bound for $\chi(G)$ is the simple constructive lower bound for $\mathrm{cl}(G)$ described in Section 4. The best algebraic lower bound available is the Hoffman eigenvalue bound and the best degree sequence bound is Bondy's bound.

Table 1

n	p	2	3	10C	4	10	5	11	12
20	0.2	1.24	1.27	1.30	1.31	1.37	2.00	2.41	2.86
20	0.5	1.90	1.94	1.98	1.89	2.06	2.92	3.35	4.64
50	0.8	4.69	4.73	4.77	3.95	4.86	6.06	7.62	13.18

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