

DETERMINING THE CHROMATIC NUMBER OF A GRAPH*

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Abstract. Certain branch-and-bound algorithms for determining the chromatic number of a graph are proved usually to take a number of steps which grows faster than exponentially with the number of vertices in the graph. A similar result holds for the number of steps in certain proofs of lower bounds for chromatic numbers.

Key words. graph, chromatic number, algorithm, proof

1. Introduction. Graph coloring problems arise in many practical situations, for example in various timetabling and scheduling problems (see for example [13], [14]). It would be very useful to be able to determine quickly the chromatic number of a graph. However, it is well known that this problem is *NP*-hard, and thus we do not expect to find good algorithms for the problem ([1], [9]). A class of branch-and-bound coloring algorithms, which we call “Zykov” algorithms (see [5]) has been proposed. We branch on whether or not two nonadjacent vertices will have the same color and bound by using the fact that the chromatic number is at least the size of any complete subgraph. Zykov algorithms always explore at least a “pruned Zykov tree” for a graph. We shall prove that for almost all graphs G_n on n vertices every pruned Zykov tree has size (number of vertices) at least $c^{n \log^{1/2} n}$, where c is a constant > 1 . It follows that for any Zykov algorithm the number of steps usually required grows faster than exponentially with the size of the graph.

E. L. Lawler [10] has recently noted that a simple algorithm involving the maximal stable sets of a graph requires a number of steps which grows only (!) exponentially with the size of the graph. The Lawler algorithm is then asymptotically faster than any Zykov algorithm. This result contrasts with the conclusions of D. G. Corneil and B. Graham [5].

In the next section we give some preliminary definitions, including those of Zykov trees and Zykov algorithms. In § 3 we investigate the size of (unpruned) Zykov trees. (The standard algorithm for determining the chromatic polynomial of a graph involves the exploration of a Zykov tree—see for example [2, chap. 15].) Then in § 4 we investigate the size of pruned Zykov trees and deduce that Zykov algorithms are slow. Finally in § 5 we give an interpretation of our earlier results in terms of the lengths of certain proofs concerning chromatic numbers. The results in this section are similar in spirit to some recent results of V. Chvátal [4], and indeed the research reported here was initially inspired by discussions with Chvátal concerning his results. He was interested in certain “recursive” proofs for establishing upper bounds for stability numbers of graphs, and showed that for almost all graphs with a (sufficiently large) linear number of edges, the number of steps in any such proof grows at least exponentially with the size of the graph. This result implies that for a certain (wide) class of algorithms which determine the stability number of a graph each member algorithm is “slow”.

Further related results are given in the forthcoming paper [12]. Both this paper and the paper [12] are based on the technical report [11].

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2. Preliminaries. We consider only graphs without loops or multiple edges. A (*proper*) *coloring* of a graph G is a coloring of the vertices of G so that no two adjacent vertices receive the same color; and the *chromatic number* $\chi(G)$ is the least number of colors in a proper coloring of G . A graph is *complete* if every two vertices are adjacent; and the *clique number* $\omega(G)$ is the greatest number of vertices in a complete subgraph of G . A set of vertices is *stable* if no two are adjacent; and the *stability number* $\alpha(G)$ is the greatest number of vertices in a stable set. A *proper partition* of G is a partition of the vertex set into stable sets. Thus proper partitions and proper colorings are closely related.

Let n be a positive integer. We denote by \mathcal{G}_n the set of all graphs with vertex set $\{1, 2, \dots, n\}$, and by \mathcal{G}_n^* the set of all graphs with vertex set the sets of a partition of $\{1, 2, \dots, n\}$. We may fail to distinguish between an integer k and the singleton set $\{k\}$ containing it, and for example consider that $\mathcal{G}_n \subseteq \mathcal{G}_n^*$. The use of sets to label vertices is simply a notational convenience.

We adopt in this paper a very simple probability model. (A more general model is considered in [12].) Throughout the paper p will be a real number with $0 < p < 1$ and q will be $1 - p$: usually p and q will be constants. We induce a probability distribution on the set \mathcal{G}_n of graphs by stipulating that each edge occurs independently with probability p . Thus for example the number of edges in a graph in \mathcal{G}_n is a binomial random variable $B = B\left(\binom{n}{2}, p\right)$ with parameters $\binom{n}{2}$ and p .

We now move on towards the definitions of Zykov trees and Zykov algorithms. Suppose that x and y are nonadjacent vertices in a graph H in \mathcal{G}_n^* . Following [5] we define the reduced graphs H'_{xy} and H''_{xy} (or simply H' and H''). The former, H'_{xy} , is obtained from H by simply adding an edge joining x and y ; and the latter, H''_{xy} , is obtained from H by replacing the vertices x and y by a single new vertex adjacent to each vertex to which x or y was adjacent. We say that H' and H'' are obtained from H by an “edge-addition” and a “vertex-contraction” respectively. In any proper coloring of H either x and y have different colors or they have same color. Thus we have the simple and well known result (see [15]) that

$$(2.1) \quad \chi(H) = \min \{\chi(H'), \chi(H'')\}.$$

Suppose that we have a graph H in \mathcal{G}_n^* which is itself a leaf in a binary tree. Then *branching* at H involves choosing nonadjacent vertices x and y in H and giving H the leftson H'_{xy} and the rightson H''_{xy} . Of course we cannot branch at H if H is complete. Now let G be a graph in \mathcal{G}_n . If we start with the single node G , the root of our binary tree, and branch repeatedly we obtain a *partial Zykov tree* for G . By (2.1) we know that $\chi(G)$ is the minimum value of $\chi(L)$ over all leaves L of any partial Zykov tree for G . A *Zykov tree* for G is a partial Zykov tree in which each leaf is a complete graph, that is in which we have branched until we can branch no more. We give below an example of a Zykov tree for a graph in \mathcal{G}_4 . (See also [2, chap. 15] and [5].)

Example. See Fig. 2.1.

We have now described the “branching” process to be used in our branch-and-bound algorithms. The “bounding” process depends on the obvious result that for any graph G

$$(2.2) \quad \chi(G) \geq \omega(G).$$

A *Zykov algorithm* is a branch-and-bound algorithm for determining the chromatic number of a graph, using branch and bound processes as described above.

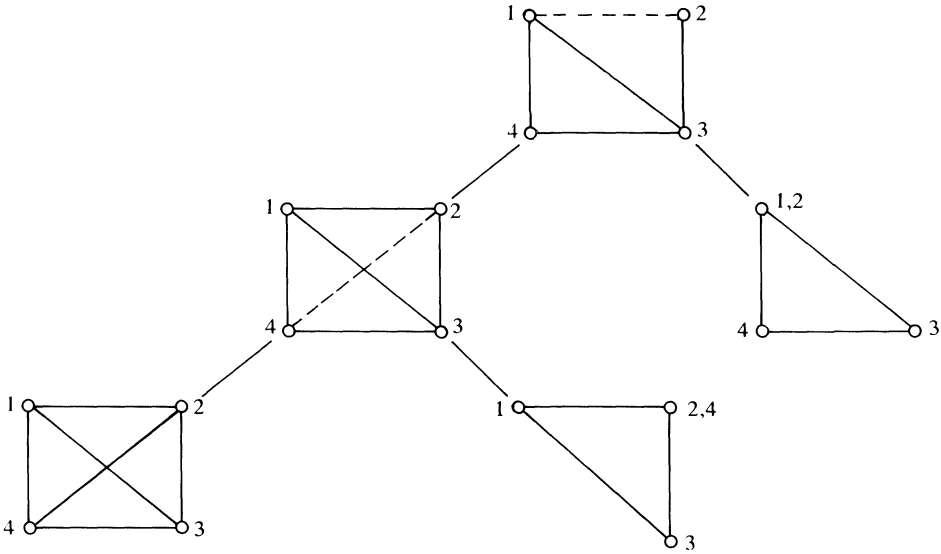


FIG. 2.1.

Such an algorithm has a subroutine for determining for each graph H a lower bound $\omega'(H)$ for the clique number $\omega(H)$ (for example by finding a complete subgraph of H). Also it maintains a current best upper bound for the chromatic number, which is always at most the number of vertices in any graph encountered. It operates as follows on a graph G . It begins to (construct and) explore a partial Zykov tree for G , starting with the root G . Suppose that at some stage we have explored a partial Zykov tree T for G and we have an upper bound b for $\chi(G)$. The algorithm chooses a leaf L of T with $\omega'(L) < b$ if there is such a leaf, then it branches at L and updates the upper bound: if there is no such leaf L the algorithm returns $\chi(G) = b$ and stops. Examples of Zykov algorithms are investigated in [5] and [12].

It is easy to see that after a finite number of steps a Zykov algorithm returns the correct value for the chromatic number and stops. Further if say it conducts a depth-first search of the partial Zykov tree the storage requirement need only be say $O(n^3)$. We shall see, however, that the number of steps required grows very quickly with n , even if we suppose that the subroutine can always determine $\omega(H)$ exactly and without cost, and that we can always start with the upper bound at the actual value of the chromatic number. (Both these suppositions are of course rather unlikely, since we would be solving NP -hard problems [1].)

Given a Zykov tree T for a graph G , and a positive integer k , the corresponding Zykov tree *pruned at k* consists simply of the root G if $\omega(G) \geq k$ and otherwise is the unique maximal rooted subtree of T containing as internal nodes precisely the nodes H of T with $\omega(H) < k$: the corresponding *pruned Zykov tree* T^* is the tree pruned at $k = \chi(G)$. Any Zykov algorithm for determining the chromatic number of a graph G must explore at least some pruned Zykov tree for G . We shall prove that pruned Zykov trees are usually very large and thus that Zykov algorithms are usually very slow.

Finally note that all logarithms are natural; and for any real number x we let $\lceil x \rceil$ denote the least integer not less than x and $\lfloor x \rfloor$ denote the greatest integer not more than x .

3. Zykov trees. In this section we investigate the sizes of Zykov trees. We have three main reasons for doing this. Firstly the sizes of Zykov trees are of interest if we wish for example to determine the chromatic polynomial of a graph ([2, chap. 15]); secondly some knowledge of the sizes of Zykov trees helps us to interpret results on the sizes of pruned Zykov trees; and thirdly some of the arguments brought up here will be useful later.

The first result in this section shows in particular that every Zykov tree for a given graph has the same size, that is the same number of nodes. Given a graph G let us denote by $C(G)$ the number of proper partitions of G (that is, the number of colorings with “color indifference”).

PROPOSITION 3.1. *Every Zykov tree T for a given graph G has $2C(G) - 1$ nodes.*

Proof. It is not hard to check that the vertex sets of the leaves of T are in 1–1 correspondence with the proper partitions of G . Alternatively we may use an easy inductive proof.

We are interested in the size of Zykov trees for a graph G_n on n vertices. By the above proposition we may state results in terms of the number $C(G_n)$ of proper partitions of G_n , and we choose to do so. The following proposition requires no proof.

PROPOSITION 3.2. (a) *If G'_n is a subgraph of G_n then $C(G'_n) \leq C(G_n)$, with equality only if the graphs are the same.*

(b) *Let ϕ_n and K_n denote respectively the null (edge-less) and complete graphs on n vertices. Then $C(K_n) = 1$, and $C(\phi_n)$ is simply the number of partitions of a set of size n , so that (see for example [16])*

$$\log C(\phi_n) = n(\log n - \log \log n - 1 + o(1)).$$

The “extreme” properties of $C(G_n)$ are thus easily handled. We may also determine quite closely the “usual” properties.

THEOREM 3.3. (a) *The expected value $E(C_n)$ of $C(G_n)$ for graphs G_n in \mathcal{G}_n satisfies*

$$\log E(C_n) = n(\log n - (2 \log(1/q) \log n)^{1/2} - \frac{1}{2} \log \log n + O(1)).$$

(b) *With probability $1 - o(e^{-n})$*

$$n(\log n - 3(\frac{1}{4} \log(1/q) \log^2 n)^{1/3}) \leq \log C(G_n) \leq n(\log n - (2 \log(1/q) \log n)^{1/2}).$$

Proof. (a) We first show that $\log E(C_n)$ is at least the value given above. Let $d = d(n)$ be an integer-valued function such that $d(n) \rightarrow \infty$ as $n \rightarrow \infty$ but say $d(n) = O(n/\log n)$. We shall choose d below. Let \mathcal{R}_n be the set of partitions of $\{1, \dots, n\}$ into $k = \lfloor n/d \rfloor$ sets each of size d and (possibly) the $(n - kd)$ singleton sets $\{kd + 1\}, \dots, \{n\}$. Then the number of partitions in \mathcal{R}_n equals

$$\frac{(kd)!}{k!(d!)^k} \geq \frac{(n-d)!}{(n/d)!(d!)^{n/d}},$$

and the probability that a partition in \mathcal{R}_n is proper equals

$$q^{\binom{d}{2}k} \geq q^{(1/2)nd}.$$

Hence the logarithm of the expected number of proper partitions in \mathcal{R}_n is at least

$$\begin{aligned} (3.1) \quad & (n-d) \log(n-d) - (n/d) \log(n/d) - (n/d)(d \log d) - \frac{1}{2}nd \log(1/q) + O(n) \\ & = n(\log n - (\log n)/d - \log d - \frac{1}{2}d \log 1/q + O(1)). \end{aligned}$$

Now let

$$f_n(x) = (\log n)/x + \log x + \frac{1}{2} \log(1/q)$$

for $x > 0$. Then $f_n(x)$ achieves a unique minimum for $x > 0$ at $x = ((2 \log(1/q) \log n + 1)^{1/2} - 1) / \log(1/q)$ and this minimum equals

$$(3.2) \quad (2 \log(1/q) \log n)^{1/2} + \frac{1}{2} \log \log n + O(1).$$

We set $d(n) = \lfloor (2 \log(1/q) \log n)^{1/2} \rfloor$ and find that the right hand side in (3.1) equals the right hand side in the statement of (a).

We now prove the reverse inequality in (a). Let $k = k(n)$ be an integer i such that the expected number of proper partitions into i sets is a maximum. Then of course $E(C_n)$ is at most n times the expected number of proper partitions into k nonempty sets. Let $d = d(n) = n/k$. (Thus d is not necessarily an integer.)

Let Q be a partition of $\{1, \dots, n\}$ into k sets, of sizes s_1, \dots, s_k . Then as in [8] we see that the probability that Q is proper equals

$$\prod_{i=1}^k q^{(1/2)s_i(s_i-1)} = q^{(1/2)(\sum s_i^2 - n)} \leq q^{(1/2)(n^2/k - n)}.$$

Also the number of partitions of $\{1, \dots, n\}$ into k nonempty sets is at most $k^n/k!$. Hence

$$E(C_n) \leq n \frac{k^n}{k!} q^{(1/2)(n^2/k - n)},$$

and so

$$\begin{aligned} \log E(C_n) &\leq n \log k - k \log k - \frac{1}{2} (\log(1/q)) n^2/k + O(n) \\ &= n \log n - n \log d - (n/d) \log n - \frac{1}{2} (\log(1/q)) nd + O(n) \\ &= n(\log n - f_n(d) + O(1)). \end{aligned}$$

But by (3.2)

$$f_n(d) \geq (2 \log(1/q) \log n)^{1/2} + \frac{1}{2} \log \log n + O(1).$$

This completes the proof of part (a) of the theorem.

Proof of (b). The right hand inequality here follows from (a) and the standard result that for any nonnegative random variable X and any real number x ,

$$E(X) \geq x \text{Prob}\{X \geq x\}.$$

The left hand inequality follows from Lemma 3.4 below and the discussion preceding it.

Suppose that we have functions $l(n)$ and $r(n)$ with nonnegative integer values. For each positive integer n we let $T_n(l, r)$ be the set of graphs G_n in \mathcal{G}_n such that in some Zykov tree for G_n we may reach a leaf by starting at the root G_n and descending through the tree making at most $l(n)$ left turns and $r(n)$ right turns. If a graph G_n in \mathcal{G}_n is not in $T_n(l, r)$ then certainly every Zykov tree for G_n has at least $\binom{l+r}{r}$ nodes. We wish to find functions $l(n)$ and $r(n)$ so that $\text{Prob } T_n(l, r) \rightarrow 0$ as $n \rightarrow \infty$ and $\binom{l+r}{r}$ is as large as possible.

LEMMA 3.4. *There exist functions $l(n)$ and $r(n)$ such that*

$$\text{Prob } T_n(l, r) = o(e^{-n}) \quad \text{as } n \rightarrow \infty,$$

and

$$\log \binom{l+r}{r} \geq n(\log n - 3(\frac{1}{4} \log(1/q) \log^2 n)^{1/3})$$

for n sufficiently large.

Lemma 3.4 may be proved along the lines of the proof in the next section of the more important Lemma 4.7: we do not give a proof here. The lemma is in a sense best possible (see [12]).

4. Pruned Zykov trees. In this section we investigate the size of pruned Zykov trees. We do not manage to find out as much about the pruned trees as we found out about the unpruned trees in the last section, but we are able to prove a greater than exponential lower bound on their size. This result shows that Zykov algorithms for determining the chromatic number of a graph usually require more than exponential time.

We have seen that every Zykov tree for a given graph G_n has the same size (which is less than n). Thus certainly if we have to construct a Zykov tree there is no point in spending time choosing a “best” way of branching. The situation is quite different when we look at pruned Zykov trees.

PROPOSITION 4.1. *There is a sequence (G_n^*) of graphs on n vertices such that a smallest pruned Zykov tree for G_n^* has 3 nodes and a largest pruned Zykov tree for G_n^* has $n^{n(1+o(1))}$ nodes.*

Proof. For each integer $k \geq 5$ let H_k be the pentagon C_5 together with $(k-5)$ vertices adjacent to each other vertex. It is easy to check that $\omega(H_k) = k-3$ and $\chi(H_k) = k-2$; and that every pruned Zykov tree for H_k has exactly 3 nodes. Now for each positive integer n let

$$k = k(n) = \lfloor n(\log n)^{-1/2} \rfloor.$$

Then $k(n) \geq 5$ for $n \geq 7$, and we can let G_n^* be the graph H_k together with $(n-k)$ isolated vertices (see Fig. 4.1).

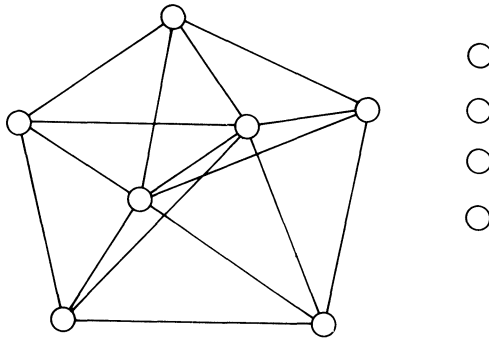


FIG. 4.1. G_{11}^* is H_7 plus 4 isolated vertices.

By branching within the H_k component of G_n^* we see that the size of a smallest pruned Zykov tree for G_n^* is 3. By branching first amongst the isolated vertices of G_n^* we see there is a pruned Zykov tree T_n for G_n^* “containing” as a subtree those nodes K of a Zykov tree for ϕ_{n-k} (the null graph on $n-k$ vertices) with $\omega(K) < \chi(G_n^*) = k-2$. Hence the number $|T_n|$ of nodes in T_n is at least the number of partitions of a set

of size $(n - k)$ into at most $(k - 3)$ sets. Let

$$d = d(n) = \lfloor (n - k)/(k - 3) \rfloor.$$

Then, much as in the proof of Theorem 3.3(a)

$$|T_n| \geq \frac{((k - 3)d)!}{(k - 3)!(d!)^{k-3}} \geq \frac{(n - 2k)!}{n^k} = n^{n(1+o(1))}.$$

We have now seen that the sizes of pruned Zykov trees for a given graph G may vary wildly. For lower bounds on running times of Zykov algorithms we are of course interested in the minimum size, say $Z^*(G)$, of a pruned Zykov tree for G . We investigate below both the “extremal” and the “usual” properties of $Z^*(G_n)$.

Consider first the extremal properties. It is clear that $Z^*(G) = 1$ if and only if $\omega(G) = \chi(G)$. However, it is not clear how large $Z^*(G_n)$ may be for graphs G_n with n vertices.

PROPOSITION 4.2. *There is a sequence (G_n^*) of graphs on n vertices such that*

$$Z^*(G_n^*) \geq n^{(1+o(1))n/10}.$$

To prove Proposition 4.2 we need one lemma. For each positive integer k we define a graph H_k on $5k$ vertices. The graph H_k is obtained from the pentagon C_5 by “expanding” each vertex into a complete graph on k vertices. More formally we let H_k have disjoint sets of vertices S_1, \dots, S_5 each of size k ; and let distinct vertices x in S_i and y in S_j be adjacent if $i - j \equiv 0, \pm 1 \pmod{5}$. (See Fig. 4.2.)

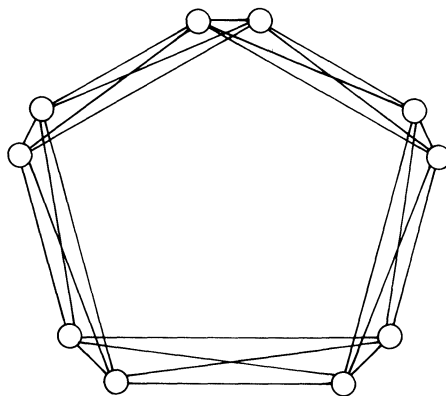


FIG. 4.2. H_2 .

LEMMA 4.3. *Let k and x be positive integers with $x \leq k$. Then every subgraph of H_k with $2k + x$ vertices misses at least kx edges.*

Proof. Let H be an (induced) subgraph of H_k with $2k + x$ vertices such that the number m of edges missing is a minimum. We must show that $m \geq kx$. For $i = 1, \dots, 5$ let n_i be the number of vertices of H in S_i . There are two cases to consider.

(a) At least two of the integers n_i are equal to k . We may suppose (without loss of generality) that $n_1 = k$. Then

$$m \geq k(n_3 + n_4)$$

and so we may suppose that n_3 and n_4 are less than k . But now we may assume that $n_2 = k$, and so

$$m \geq k(n_3 + n_4 + n_5) = kx.$$

(b) At most one of the n_i equals k . Note first that the n_i are not all equal, for then we would have

$$m = 5 \left(\frac{2k+x}{5} \right)^2 \geq \frac{3}{5}k^2 + kx > kx.$$

It follows that we may assume that $n_1 = \max(n_i)$ and $n_1 > n_3$. Note that $n_2, n_5 < k$. Suppose that $n_4 > 0$, and consider the graph H' obtained from H by removing a vertex in S_4 and adding one in S_5 . Then the number of edges missing on H' equals m less the positive number $n_1 - n_3$, which contradicts our choice of H . Thus $n_4 = 0$. But now $n_1 > n_4$ and so arguing as above we have $n_3 = 0$. Hence

$$n_2 + n_5 = 2k + x - n_1 \geq k + x,$$

and so

$$m = n_2 n_5 \geq kx.$$

This completes the proof of the lemma.

Proof of Proposition 4.2. Let n be an integer at least 5 and let $k = \lfloor n/5 \rfloor$. Let G_n^* be H_k together with $n - 5k$ vertices adjacent to each other vertex. Then $Z^*(G_n^*) = Z^*(H_k)$ and so it suffices to prove that

$$(4.1) \quad Z^*(H_k) = k^{k((1/2)+o(1))}.$$

Note first that every stable set in H_k contains at most two vertices, and so

$$\chi(H_k) \geq \lceil 5k/2 \rceil.$$

(In fact we have equality but this is not needed.)

For each k let

$$r(k) = \lceil k/2 \rceil - \lceil k/\log k \rceil$$

and

$$l(k) = \lceil k^2/\log k \rceil - 1.$$

Let T be any Zykov tree for H_k , and let K be any node in T which we may reach from the root by descending through the tree making at most $l(k)$ left turns and $r(k)$ right turns. If $\omega(K) \geq \chi(H_k)$ then H_k must have a subgraph on

$$\chi(H_k) - r(k) \geq 2k + \lceil k/\log k \rceil$$

vertices which misses at most $l(k)$ edges; and by Lemma 4.3 this is not possible. Thus the node K is in the pruned tree T^* corresponding to T . Hence

$$|T^*| \geq \binom{l+r}{r} = k^{k((1/2)+o(1))}.$$

This establishes (4.1) and so completes the proof of the proposition.

It is possible to prove that for the sequence (G_n^*) of graphs constructed above we actually have

$$Z^*(G_n^*) = n^{(1+o(1))n/10}.$$

Perhaps every such sequence (G_n^*) of graphs satisfies

$$Z^*(G_n^*) \leq n^{(1+o(1))n/10},$$

so that Proposition 4.2 is in a sense best possible?

We now move on towards our main result, which concerns the “usual” behavior of the minimum size $Z^*(G_n)$ of a pruned Zykov tree for graphs G_n . We need a number of lemmas. The first concerns the chromatic number of a random graph, and is taken essentially from [8].

LEMMA 4.4. $\text{Prob} \{\chi(G_n) \leq n \log(1/q)/(2 \log n)\} = o(n^{-k})$ for any k as $n \rightarrow \infty$.

Proof. Recall that the stability number $\alpha(G_n)$ satisfies $\alpha(G_n)\chi(G_n) \geq n$. Let

$$s = s(n) = \lceil 2 \log n / \log(1/q) \rceil.$$

Then

$$\begin{aligned} \text{Prob} \{\chi(G_n) \leq n \log(1/q)/(2 \log n)\} \\ &\leq \text{Prob} \{\alpha(G_n) \geq s\} \\ &\leq \binom{n}{s} q^{\binom{s}{2}} \leq \left(\frac{ne}{s}\right)^s q^{(1/2)(s-1)s} \\ &= \exp s(-\log s + O(1)) \\ &= o(n^{-k}) \quad \text{for any } k. \end{aligned}$$

We now look at the number of edges in a “contraction” of a random graph. Let Q be a family of disjoint subsets of the vertex set of a graph G . We say that Q is *proper* for G if each set in Q is stable. The “contracted” graph G_Q has vertices the sets in Q and an edge between two of these sets if there is an edge in G between some two vertices one from each set. Clearly G_Q may be formed from G by a sequence of vertex-contractions if and only if Q is proper for G . We are interested in the number of edges we are likely to have in G_Q .

Given two random variables X and Y we write $X \leq Y$ in distribution if $F_X(t) \geq F_Y(t)$ for each real number t , where F_X and F_Y are the distribution functions of X and Y respectively.

LEMMA 4.5. *Suppose that X, Y, Z are random variables, that $X \leq Y$ in distribution, and that the pairs X, Z and Y, Z are independent. Then $X + Z \leq Y + Z$ in distribution.*

Proof. For any real number t ,

$$\begin{aligned} F_{X+Z}(t) &= \int F_X(t-u) dF_Z(u) \\ &\geq \int F_Y(t-u) dF_Z(u) = F_{Y+Z}(t). \end{aligned}$$

Let m and n be positive integers, and let q be a real number with $0 < q < 1$. Let $B(q)$ be a binomial random variable with parameters $\binom{m}{2}$ and $1-q$, and for each partition Q of $\{1, \dots, n\}$ let the random variable $N(Q) = N(Q, n, 1-q)$ be the number of edges in the contracted graph G_Q , for graphs G in \mathcal{G}_n with edge-probability $1-q$.

LEMMA 4.6. *For each partition Q of $\{1, \dots, n\}$ into m sets*

$$(4.2) \quad N(Q) \leq B(q^{\lceil n/m \rceil^2}) \quad \text{in distribution.}$$

Proof. We may of course assume that $m \geq 2$. We prove first that

$$(4.3) \quad N(Q) \leq B(q^{\lceil n/m \rceil^2}) \quad \text{in distribution.}$$

(The inequality (4.3) is in fact good enough for the purposes of this paper.) Let

$Q = (S_1, \dots, S_m)$ be a partition of $\{1, \dots, n\}$ into m sets, and suppose that $|S_1| + 1 \leq |S_2| - 1$. Let Q' be the partition obtained from Q by switching one element from S_2 to S_1 . In order to prove (4.3) it is sufficient to prove that

$$(4.4) \quad N(Q) \leq N(Q') \quad \text{in distribution.}$$

For $1 \leq i < j \leq m$ let $X_{ij} = 1$ if S_i and S_j are adjacent in G_Q and let $X_{ij} = 0$ otherwise. The random variables X_{ij} are of course all independent. Define independent random variables X'_{ij} from Q' in a similar way. Note that $X'_{ij} = X_{ij}$ for $i > 2$, and let the random variable Z be the sum of all such X_{ij} (or X'_{ij}). Then

$$N(Q) = X_{12} + \sum_{j=3}^m (X_{1j} + X_{2j}) + Z,$$

and

$$N(Q') = X'_{12} + \sum_{j=3}^m (X'_{1j} + X'_{2j}) + Z.$$

But it is straightforward to prove that

$$X_{12} \leq X'_{12} \quad \text{in distribution,}$$

and for $j = 3, \dots, m$

$$X_{1j} + X_{2j} \leq X'_{1j} + X'_{2j} \quad \text{in distribution.}$$

The result (4.3) now follows by repeated use of Lemma 4.5.

We now use (4.3) to prove (4.2). Given a set S of positive integers and a positive integer k let kS be the set of positive integers i such that $\lceil i/k \rceil$ is in S . Given a partition $Q = (S_1, \dots, S_m)$ of $\{1, \dots, n\}$ let kQ be the partition (kS_1, \dots, kS_m) of $\{1, \dots, kn\}$. (For example if Q is the partition $(\{1, 2\}, \{3\})$ of $\{1, 2, 3\}$ then $2Q$ is the partition $(\{1, 2, 3, 4\}, \{5, 6\})$ of $\{1, 2, \dots, 6\}$.)

It is easy to see that for each positive integer k

$$N(Q, n, 1 - q) = N(kQ, kn, 1 - q^{1/k^2}) \quad \text{in distribution.}$$

Hence by (4.3) for each k

$$N(Q, n, 1 - q) \leq B(q^{\lceil kn/m \rceil^2 / k^2}) \quad \text{in distribution.}$$

But $\lceil kn/m \rceil^2 / k^2 \rightarrow (n/m)^2$ as $k \rightarrow \infty$, and so (4.2) holds. This completes the proof of Lemma 4.6.

We need one more lemma in order to prove the main result. Suppose that we have a positive constant t and functions $l(n)$ and $r(n)$ with nonnegative integer values. For each positive integer n let $T'_n(l, r)$ be the set of graphs G_n in \mathcal{G}_n such that in some Zykov tree for G_n we may reach a leaf or node H with $\omega(H) \geq t\chi(G_n)$ by starting at the root and descending through the tree making at most $l(n)$ left turns and $r(n)$ right turns. (Compare with the definition of $T_n(l, r)$ preceding Lemma 3.4 in the last section.) If a graph G_n in \mathcal{G}_n is not in $T'_n(l, r)$ then certainly every Zykov tree for G_n has at least $\binom{l+r}{r}$ nodes H with $\omega(H) < t\chi(G_n)$. In the case $t = 1$ we see that if G_n is not in $T'_n(l, r)$ then every pruned Zykov tree for G_n has at least $\binom{l+r}{r}$ nodes. We wish to find functions $l(n)$ and $r(n)$ such that $\text{Prob } T'_n(l, r) \rightarrow 0$ as $n \rightarrow \infty$ and $\binom{l+r}{r}$ is as

large as possible.

LEMMA 4.7. *For any positive constant t there exist functions $l(n)$ and $r(n)$ such that as $n \rightarrow \infty$*

$$\text{Prob } T_n^t(l, r) = o(n^{-k}) \quad \text{for any } k$$

and

$$\log \binom{l+r}{r} \sim tn^{(1/27)} \log(1/q) \log n^{1/2}.$$

We may take l and r so that

$$l(n) = n^{5/3+o(1)}$$

and

$$r(n) = \lfloor tn(\log(1/q)/(12 \log n))^{1/2} \rfloor.$$

Lemma 4.7 above of course is similar to Lemma 3.4 in the last section, and we noted there that that lemma is in a sense best possible. Lemma 4.7 is also in a sense best possible [12].

Proof. Let k be any positive integer. Let $l(n)$ and $r(n)$ be functions with nonnegative integer values, which we shall choose later. Let

$$b(n) = \lfloor tn \log(1/q)/(2 \log n) \rfloor$$

and let

$$B_n = \{G \in \mathcal{G}_n : \chi(G) < n \log(1/q)/(2 \log n)\}.$$

Let $C_n(l, r)$ be the set of graphs G in \mathcal{G}_n such that in some Zykov tree for G we may reach a leaf or node H with $\omega(H) \geq b(n)$ by starting at the root and descending through the tree making (as usual) at most $l(n)$ left turns and $r(n)$ right turns. Then

$$T_n^t(l, r) \subseteq B_n \cup C_n(l, r).$$

By Lemma 4.4 $\text{Prob } B_n = o(n^{-k})$ as $n \rightarrow \infty$. Thus we wish to choose functions $l(n)$ and $r(n)$ such that

$$(4.5) \quad \text{Prob } C_n(l, r) = o(n^{-k}) \quad \text{as } n \rightarrow \infty$$

and $\binom{l+r}{r}$ is as large as possible.

Let \mathcal{R} be the collection of all families of b disjoint subsets of $\{1, \dots, n\}$ with union containing at most $r+b$ elements. For each family Q in \mathcal{R} let T_Q be the set of graphs G in \mathcal{G}_n such that the “contracted” graph G_Q misses at most l edges. Now if G is a graph in $C_n(l, r)$ then some graph obtained from G by performing at most r vertex-contractions contains a subgraph on b vertices missing at most l edges; and so $G \in T_Q$ for some family Q (proper for G) in \mathcal{R} . Hence

$$(4.6) \quad C_n(l, r) \subseteq \bigcup \{T_Q : Q \in \mathcal{R}\}.$$

Next we find an upper bound for the $\text{Prob } T_Q$. Let N be a binomial random variable with parameters $\binom{b}{2}$ and q^{x^2} , where $x = r/b + 1$. By Lemma 4.6 for each Q in \mathcal{R}

$$(4.7) \quad \text{Prob } T_Q \leq \text{Prob } \{N \leq l\}.$$

Now let $l(n) = \lfloor \frac{1}{2}E(N) \rfloor$. Then by a standard inequality concerning the binomial distribution (see for example [6, pp. 17, 18])

$$(4.8) \quad \text{Prob} \{N \leq l\} \leq \exp \left(-\left(\frac{1}{8}\right)E(N) \right).$$

But \mathcal{R} of course contains at most n^n families Q . Hence by (4.6), (4.7) and (4.8)

$$(4.9) \quad \text{Prob } C_n(l, r) \leq n^n \exp \left(-\left(\frac{1}{8}\right)E(N) \right).$$

Let

$$r(n) = \lfloor un(\log(1/q)/\log n)^{1/2} \rfloor$$

for some constant u to be chosen with $0 < u < \frac{1}{2}t$. Then

$$x(n) = (1 + o(1))(2u/t)((\log n)/\log(1/q))^{1/2}$$

and so

$$\begin{aligned} E(N) &= \binom{b}{2} q^{x^2} \\ &= \exp(2 \log n - (4u^2/t^2) \log n + o(\log n)) \\ &= n^{(2-4u^2/t^2+o(1))}. \end{aligned}$$

But $4u^2/t^2 < 1$ and so (4.5) holds by (4.9).

It remains to choose u to maximize $\binom{l+r}{r}$. But

$$\begin{aligned} \log \binom{l+r}{r} &= r(\log l - \log r + O(1)) \\ &= un(\log(1/q)/\log n)^{1/2}(1 - 4u^2/t^2 + o(1)) \log n \\ &= (u - 4u^3/t^2 + o(1))(\log(1/q) \log n)^{1/2}n. \end{aligned}$$

The maximum value of $u - 4u^3/t^2$ for $u > 0$ is attained at $u = 12^{(-1/2)}t < \frac{1}{2}t$. Thus we give u this value, and find that $\log \binom{l+r}{r}$ is as in the statement of the lemma. The functions $l(n)$ and $r(n)$ are now also as in the lemma. This completes the proof.

Suppose that for each positive integer n we have a subset A_n of \mathcal{G}_n , the set of all graphs on $\{1, \dots, n\}$. We shall make statements like “the event A_n occurs for almost all G_n ” if the sum of the probabilities that each A_n fails to occur is convergent. (This definition corresponds to embedding all our probability spaces \mathcal{G}_n in a single space and using a Borel–Cantelli lemma—compare with [8].) Thus for example by Lemma 4.4, for almost all graphs G_n we have

$$\chi(G_n) \geq n \log(1/q)/(2 \log n).$$

From Lemma 4.7 and the discussion preceding it we may now deduce immediately our main result.

THEOREM 4.8. *If t is a positive constant then for almost all graphs G_n every Zykov tree for G_n contains at least*

$$\exp \left[(1 + o(1))tn \left(\frac{1}{27} \log(1/q) \log n \right)^{1/2} \right]$$

nodes H such that $\omega(H) < t\chi(G_n)$.

Recall that $Z^*(G)$ is the minimum size of a pruned Zykov tree for a graph G .

COROLLARY 4.9. *For almost all graphs G_n*

$$Z^*(G_n) \geq \exp \left[(1 + o(1)) n^{(1/27)} \log(1/q) \log n^{1/2} \right].$$

COROLLARY 4.10. *For almost all graphs G_n the number of steps needed by any Zykov algorithm to determine $\chi(G_n)$ is at least the quantity given above.*

In the case $p = q = \frac{1}{2}$ Corollary 4.9 yields

COROLLARY 4.11. *The proportion of graphs G_n on n vertices such that*

$$Z^*(G_n) \geq \exp(.157 n \log^{1/2} n)$$

tends to 1 as $n \rightarrow \infty$.

Corollary 4.10 above shows that the time taken by Zykov algorithms for determining chromatic numbers grows faster than exponentially with the number of vertices; and thus that these algorithms are slower asymptotically than the algorithm considered by E. L. Lawler [10].

M. R. Garey and D. S. Johnston [7] have shown that the problem of determining the chromatic number of a graph to within a factor less than 2 is *NP*-hard. By analogy one might possibly have expected some effect in Theorem 4.8 at any $t = \frac{1}{2}$, but none is apparent (see also Corollary 5.1 below).

5. Lengths of proofs. The above results may be phrased in terms of the lengths of certain kinds of proof which determine chromatic numbers or which establish lower bounds for chromatic numbers. We then obtain results concerning chromatic numbers which are similar in spirit to recent results of V Chvátal [4] concerning stability numbers. Indeed this paper was initially motivated by discussions with Chvátal concerning his results.

If k is an integer at least as great as $\chi(G)$ then there is a short proof that $\chi(G) \leq k$ —namely we may exhibit a proper coloring of G using at most k colors. In general such a proof is hard to find but it must of course exist. However, if k is at most $\chi(G)$ then it is not clear if there is necessarily a short proof of this fact.

Consider the following proof system for establishing lower bounds for chromatic numbers. A *statement* is simply a pair (G, b) where G is a graph and b is a nonnegative integer. (Such a statement is to be interpreted as the inequality $\chi(G) \geq b$, which may of course be false.) A *recursive proof* of (G, b) is a sequence of statements (G_k, b_k) ($k = 1, \dots, m$) such that $(G_m, b_m) = (G, b)$ and for each $1 \leq k \leq m$ either $\omega(G_k) \geq b_k$ or there are integers $1 \leq i, j < k$ such that G_i and G_j are a pair of reduced graphs for G_k and $b_k \leq \min(b_i, b_j)$. We call the integer m the *length* of the proof. If there is a recursive proof of (G, b) then by (2.1) and (2.2) we have $\chi(G) \geq b$; and conversely if $\chi(G) \geq b$ then we can construct a recursive proof of (G, b) from any Zykov tree for G pruned at b . In fact if $\chi(G) \geq b$ there is close correspondence between recursive proofs of (G, b) and Zykov trees for G pruned at b . In particular the minimum length of a recursive proof of (G, b) equals the minimum size of a Zykov tree for G pruned at b .

From Theorem 4.8 we obtain

COROLLARY 5.1. *Let $0 < t \leq 1$. Then for almost all graphs G_n on n vertices, every recursive proof of $(G_n, t\chi(G_n))$ has length at least $c^{n^{(\log n)^{1/2}}}$, where c is a constant > 1 .*

From Corollary 4.11 we obtain

COROLLARY 5.2. *Consider the property for graphs G_n on n vertices that every recursive proof of $(G_n, \chi(G_n))$ has length at least*

$$\exp(.157 n \log^{1/2} n).$$

The proportion of graphs on n vertices with this property tends to 1 as $n \rightarrow \infty$.

The reader is reminded that further related results are given in [12].

REFERENCES

- [1] A. V. AHO, J. E. HOPCROFT AND J. D. ULLMAN, *The Design and Analysis of Computer Algorithms*, Addison-Wesley, Reading, MA, 1974.
- [2] C. BERGE, *Graphs and Hypergraphs*, North-Holland, London, 1973.
- [3] N. CHRISTOFIDES, *An algorithm for the chromatic number of a graph*, Comput. J., 14 (1971), pp. 38–39.
- [4] V. CHVÁTAL, *Determining the stability number of a graph*, this Journal, 6 (1977), pp. 643–662.
- [5] D. G. CORNEIL AND B. GRAHAM, *An algorithm for determining the chromatic number of a graph*, this Journal, 2 (1973), pp. 311–318.
- [6] P. ERDOS AND J. SPENCER, *Probabilistic Methods in Combinatorics*, Academic Press, New York and London, 1974.
- [7] M. R. GAREY AND D. S. JOHNSON, *The complexity of near-optimal graph coloring*, J. Assoc. Comput. Mach., 23 (1976), pp. 43–49.
- [8] G. R. GRIMMETT AND C. J. H. McDIARMID, *On coloring random graphs*, Math. Proc. Camb. Philos. Soc., 77 (1975), pp. 313–324.
- [9] R. M. KARP, *Reducibility among combinatorial problems*, Complexity of Computer Computations, R. E. Miller and J. W. Thatcher, eds., Plenum Press, New York, 1972, pp. 85–104.
- [10] E. L. LAWLER, *A note on the complexity of the chromatic number problem*, Information Processing Lett., 5 (1976), pp. 66–67.
- [11] C. J. H. McDIARMID, *Determining the chromatic number of a graph*, Technical Report STAN-CS-76-576, Stanford University, 1976.
- [12] ———, *Determining the chromatic number of a graph II*, to appear.
- [13] D. J. A. WELSH AND M. B. POWELL, *An upper bound to the chromatic number of graph and its application to time-tabling problems*, Comput. J., 10 (1967), pp. 85–86.
- [14] D. C. WOOD, *A technique for coloring a graph applicable to large scale time-tabling problems*, Ibid., 12 (1969), pp. 317–319.
- [15] A. A. ZYKOV, *On some properties of linear complexes*, Mat. Sb., 24 (1949), pp. 163–188; English translation, Amer. Math. Soc. Translation 79, 1952.
- [16] N. DE BRUIJN, *Asymptotic Methods in Analysis*, 2nd ed., North-Holland, 1961.