An algorithm for the chromatic number of a graph

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An algorithm is described for colouring the vertices of a graph using the minimum number of colours possible so that any two adjacent vertices are coloured differently. The algorithm can produce all the optimal independent ways of colouring the graph. In the process of deriving the algorithm the concept of the 'maximal' internally stable sets (Berge, 1962) is generalised to more than one group of sets.

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1. Introduction

Problems of production scheduling, construction of examination timetables (Welsh and Powell, 1967; Wood, 1969), storage of goods, etc., can generally be expressed in terms of graphs. In the timetable problem examinations are represented by the vertices of a graph with a link between two vertices indicating that two examinations cannot take place concurrently because some candidate is taking both. The problem is then to schedule the examinations in the smallest possible number of periods. In the storage problem the vertices of a graph represent goods (chemicals say), and a link between two vertices indicates that the two chemicals cannot, for safety reasons, be placed in the same compartment. The problem is then to store the chemicals in the smallest possible number of compartments.

Both of the above two problems are equivalent to colouring the vertices of a corresponding graph with the minimum number of colours, so that no two adjacent vertices are coloured with the same colour. This minimum number of colours is known as the chromatic number of the graph.

There are several heuristic procedures to find the chromatic number (Berge, 1962; Welsh and Powell, 1967), the one most widely used being as follows. The vertices are arranged in descending order of their degree. Colour the first vertex with colour α and scan the list of vertices downwards colouring with α any vertex which is not connected by a link to another vertex that has already been coloured with α. Starting from the top of the list colour the first uncoloured vertex by β and again scan the list downwards colouring with β any uncoloured vertex which is not connected by a link to another vertex that has already been coloured with β. Proceed in the same way with colours γ, δ, etc., until all the vertices have been coloured. The number of colours used is then taken as an approximation to the chromatic number of the graph. Refinements to the above method were proposed by Wood (1969) but the method is still only approximate.

2. Maximal subgraphs

The maximal internally stable set of graph $G = (X, \Gamma)$—where X is the set of vertices and Γ is a relation—is defined (Berge, 1962) as the set $S[G]$ where

$$S[G] \cap \Gamma(S[G]) = \phi \tag{1}$$

and there is no set $S'[G] \supset S[G]$ which satisfies (1).

Let us now define a maximal r-subgraph of a graph G as a graph $(S_r[G], \Gamma)$ which is r-chromatic (i.e. it can be coloured with r colours with no two adjacent vertices having the same colour) and there is no $S_r'[G] \supset S_r[G]$ which is r-chromatic. A subgraph having a maximal internally stable set as its vertices can then be called a maximal 1-subgraph, i.e. $S[G] = S_1[G]$.

The chromatic number of G is then obviously given by the smallest value of r so that any of the $S_r[G] = X$.

The following observation can now be made.

Theorem:
If a graph is r-chromatic it can be coloured with r colours colouring first with one colour a maximal internally stable set $S_1[G]$, next colouring with another colour a set $S_1[(X - S_1(G)), \Gamma]$ and so on, until all the vertices are coloured.

The fact that such a colouring using only r colours always exists can be shown as follows. Say a colouring with r colours exists so that one or more sets coloured with the same colour are not maximal internally stable sets in the above-mentioned sense. Numbering the colours in an arbitrary way we can always colour with colour 1 those vertices $\tilde{V}_1$ which are not coloured with colour 1 and which form an internally stable set with the vertices V_1 that are already of colour 1. This new colouring is possible because the set of vertices $\tilde{V}_1$ is not adjacent to the set of vertices V_1 and hence those vertices which are adjacent to $\tilde{V}_1$ have colours different from 1 and are therefore unaffected by the change of the colour of the vertices in $\tilde{V}_1$. Considering now the subgraph $((X - V_1 \cup \tilde{V}_1), \Gamma)$ the same procedure will lead to a recolouring containing a maximal internally stable set of colour 2 and so on.

Colourings of the type indicated in the theorem will be called optimal independent colourings.

A maximal r-subgraph with the vertex sets $S_r^i[G]$, say, can be obtained from a maximal $(r - 1)$-subgraph according to the recurrence relation:

$$S_r^i[G] = S_{r-1}^j[G] \cup S_1^k[(X - S_{r-1}^j[G], \Gamma)] \tag{2}$$

where $S_{r-1}^j[G]$ is any one of the family $\sum_{r-1}[G]$ of the vertex sets of the maximal $(r - 1)$-subgraphs of G, and $S_1^k[(X - S_{r-1}^j[G], \Gamma)]$ is any one of the vertex sets of the maximal 1-subgraphs (internally stable sets) of the subgraph formed by the vertices of G not included in the $(r - 1)$-subgraph $S_{r-1}^j[G]$.

To each $S_{r-1}^j[G]$. there correspond a number of such 1-subgraphs and the family of the vertex sets of the maximal r-subgraphs $\sum_r[G]$ is obtained by considering every union of any one $S_{r-1}^j[G]$ for $j = 1, \ldots, q_{r-1}$ (where q_{r-1} is the total number of maximal $r - 1$ sub-graphs of G) with every maximal 1-subgraph corresponding to the $S_{r-1}^j[G]$. If a set $S_r^{j1}[G] \supseteq S_r^{j2}[G]$, then $S_r^{j2}[G]$ must be removed from the family of sets $\sum_r[G]$ before continuing to the next stage.

It is tacitly assumed here that maximal 1-subgraphs can be obtained without any computational difficulty. This is in fact the case using either the method of Maghout (1963) or that of Bednarek and Taulbee (1966).

3. Description of the algorithm

The following is a simplified description of an algorithm to find the chromatic number of a graph and the colouring realising this number.

1. Set $r = 1$, $i = 0$. Find the vertex sets $S_r^j[G]$, $(j = 1, 2, \ldots, q_r)$, of the maximal r-subgraphs of G (say there are q_r such sets). Set $j = 1$.
2. Find the vertex sets of a maximal 1-subgraph of the graph $G_j = (X - S_r^j[G], \Gamma)$. If one exists go to step 4. If all of them have already been found go to step 3.
3. If $j \neq q_r$ set $j = j + 1$. If $j = q_r$ set $r = r + 1$, $i = 0$ and $j = 1$. Go to step 2.
4. Set $i = i + 1$.
Calculate $S_{r+1}^i[G] = S_r^j[G] \cup S_1[(X - S_r^j[G], \Gamma)]$ as indicated by equation (2).
5. If $S_{r+1}^i[G] = X$ stop. The number $(r + 1)$ is the chromatic number. The subsets that were introduced into the set $S_{r+1}^i[G]$ according to (2) give the actual colouring. (These subsets can be kept separate with markers as they are introduced.) If $S_{r+1}^i[G] \neq X$, go to step 6.
6. If $S_{r+1}^i[G] \subseteq S_{r+1}^k[G]$, for any $k = 1, 2, \ldots, (i - 1)$; set $S_{r+1}^i[G] = \phi$ and $i = i - 1$. If $S_{r+1}^i[G] \supseteq S_{r+1}^k[G]$ set $S_{r+1}^k[G] = S_{r+1}^i[G]$ and $i = i - 1$. If neither, set $q_{r+1} = i$. Go to step 2.

If the algorithm is not stepped when the first $S_{r+1}^i[G] = X$, it will continue to produce an alternative colouring with $r + 1$ colours, if such a colouring exists. One should note however that the algorithm will not give a complete enumeration of all possible colourings with $r + 1$ colours but will only produce the optimal independent colours. Such colourings may be only a small fraction of the total possible number of colourings using $r + 1$ colours.

4. Example

Consider the 7-vertex graph G shown in **Fig. 1**.

Step 1: The vertex sets of the maximal 1-subgraphs are:

$S_1^1[G] = \{1, 4, 6\}$; $S_1^2[G] = \{2, 3, 5\}$; $S_1^3[G] = \{2, 7, 5\}$; $S_1^4[G] = \{2, 6\}$. Hence $q_1 = 4$.

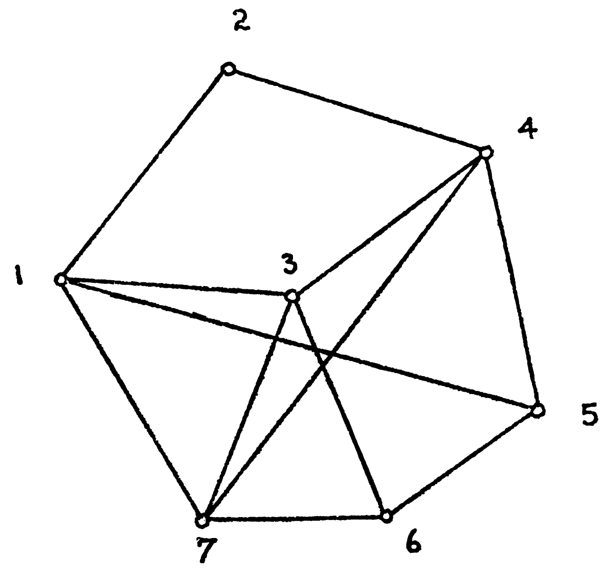

Fig. 1. Graph for the example

Iterative application of steps 2–6 produces:

Step 2: $G_1 = (X - S_1^1[G], \Gamma) = (\{2, 3, 5, 7\}, \Gamma)$; $S_1[G_1] = \{2, 5, 3\}$

Step 4: $S_2^1[G] = \{1, 4, 6 \uparrow 2, 5, 3\}$

Step 2: $S_1[G_1] = \{2, 5, 7\}$

Step 4: $S_2^2[G] = \{1, 4, 6 \uparrow 2, 5, 7\}$

Similarly $S_2^3[G] = \{2, 3, 5 \uparrow 7\}$; $S_2^4[G] = \{2, 6 \uparrow 1, 4\}$

$S_2^5[G] = \{2, 6 \uparrow 3, 5\}$; $S_2^6[G] = \{2, 6 \uparrow 5, 7\}$

(Note: three sets $S_2^i[G]$ were eliminated by step 6)

Step 3: $r = 2$, $i = 0$, $j = 1$

Step 2: $G_1 = (X - S_2^1[G], \Gamma) = (\{7\}, \Gamma)$; $S_1[G_1] = \{7\}$

Step 4: $S_3^1[G] = \{1, 4, 6 \uparrow 2, 5, 3 \uparrow 7\}$

Step 5: $S_3^1[G] = X$ and hence the chromatic number is 3 and the corresponding colouring is $\{1, 4, 6\}$, $\{2, 5, 3\}$, $\{7\}$

If one decides to continue, another possible colouring is found as:

$$S_3^2[G] = \{1, 4, 6 \uparrow 2, 5, 7 \uparrow 3\} = X$$

all other $S_r^j[G]$ are either contained in the previous two sets or do not contain all the vertices in X.

Notice that colours such as $\{5, 3\}$, $\{1, 4, 6\}$, $\{2, 7\}$ which are possible but are not independent are not produced.

References

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