Two Graph Vertex Partitioning Algorithms for Part Consolidation in Axiomatic Design

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1 Graph Theory

This section presents the concepts, definitions, and theorems from the field of graph theory that are needed in the development of the two algorithms. This material is primarily taken from the textbooks used [1] and class notes compiled by the author during his undergraduate and graduate graph theory classes at SJSU.

1.1 Simple Graphs

The problem of part consolidation is best served by a class of graphs called *simple graphs*:

Definition: Simple Graph

A *simple graph* is a mathematical object represented by a tuple $G = (V, E, \ldots)$ consisting of a non-empty and finite set of *vertices* (also called *nodes*) $V(G)$, a finite and possibly empty set of edges $E(G)$, and zero of more relations. Each edge is represented by a two-element subset of $V(G)$ called the *endpoints* of the edge:

$$E(G) \subseteq \mathcal{P}_2\left(V(G)\right)$$

Each relation has $V(G)$ or $E(G)$ as its domain and is used to associated vertices or edges with problem-specific attributes.

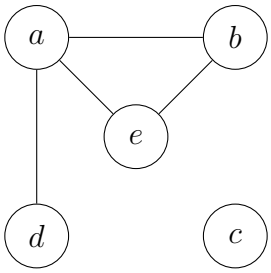
For the remainder of this work, the use of the term "graph" implies a "simple graph."

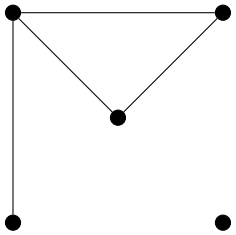
The choice of two-element subsets of $V(G)$ for the edges has certain ramifications that are indeed characteristics that differentiate a simple graph from other classes of graphs:

1. Every two vertices of a graph are the endpoints of at most one edge; there are no so-called *multiple* edges between two vertices.
2. The two endpoint vertices of an edge are always distinct; there are no so-called *loop* edges on a single vertex.
3. The two endpoint vertices are unordered, suggesting that an edge provides a bidirectional connection between its endpoint vertices.

A part consolidation problem can be represented by a graph whose vertices are the FRs and whose edges discourage combining their endpoint FRs into a single part: in the case of the first algorithm, each edge is given a numerical score (weight) indicating the magnitude of the desire to not combine the endpoint FRs into a single part, and in the case of the second algorithm, each edge indicates that the endpoint FRs should never be combined into a single part.

Graphs are often portrayed visually using labeled or filled circles for the vertices and lines for the edges such that each edge line is drawn between its two endpoint vertices. An example graph is shown in Figure 1.1.





$V = V(G) = \{a, b, c, d, e\}$

$E = E(G) = \big\{\{a, b\}, \{a, d\}, \{a, e\}, \{b, e\}\big\}$

Figure 1: An Example Graph (labeled and unlabeled)

When referring to the edges in a graph, the following common notation will be used:

Notation: Edge

The edge $\{u, v\}$ is represented by the simple juxtaposition uv or vu.

Note that there is no requirement that every vertex in a graph be an endpoint to some edge:

Definition: Isolated Vertex

Let G be a graph and let $u \in V(G)$. To say that u is an *isolated* vertex means that it is not an endpoint for any edge in $E(G)$:

$$\forall\, vw \in E(G), u \neq v \text{ and } u \neq w$$

In the example graph of Figure 1.1, notice that vertex c is an isolated vertex.

When two vertices are the endpoints of the same edge the vertices are said to be *adjacent* or are called *neighbors*:

Definition: Adjacent Vertices

Let G be a graph and let $u, v \in V(G)$. To say that u and v are *adjacent* vertices, also called *neighbors*, means that they are the endpoints of some edge $e \in E(G)$:

$$\exists\, e \in E(G), e = uv$$

The edge e is said to *join* its two endpoint vertices u and v. Furthermore, the edge e is said to be *incident* to its endpoint vertices u and v.

In the example graph of Figure 1.1, notice that vertex a is adjacent to vertices b, e, and d; and vertex b is adjacent to vertex e.

We can also speak of adjacent edges, which are edges that share an endpoint:

Definition: Adjacent Edges

Let G be a graph and left $e, f \in E(G)$. To say that e and f are *adjacent* edges means that they share some endpoint $v \in E(G)$:

$$\exists\, v \in V(G), e \cap f = \{v\}$$

or similarly:

$$|e \cap f| = 1$$

Note that two edges in a simple graph can only share one endpoint; otherwise, the two edges would be multiple edges, which are not allowed in simple graphs.

In the example graph of Figure 1.1, notice that ab is adjacent to ad, ae, and be; and ae is adjacent to be.

1.2 Order and Size

Two of the most important characteristics of a graph are the number of vertices in the graph, called the *order* of the graph, and the number of edges in the graph, called the *size* of the graph:

Definition: Order

Let G be a graph. The *order* of G, denoted by $n(G)$, is the number of vertices in G:

$$n = n(G) = |V(G)|$$

Definition: Size

Let G be a graph. The *size* of G, denoted by $m(G)$, is the number of edges in G:

$$m = m(G) = |E(G)|$$

In the example graph of Figure 1.1, notice that $n = 5$ and $m = 4$.

Since every two vertices can have at most one edge between them, the number of edges has an upper bound:

Theorem

Let G be a graph of order n and size m:

$$m \leq \frac{n(n-1)}{2}$$

Proof. Since each pair of distinct vertices in $V(G)$ can have zero or one edges joining them, the maximum number of possible edges is $\binom{n}{2}$, and so:

$$m \leq \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

■

Some choices of graph order and size lead to certain degenerate cases that serve as important termination cases for the two algorithms:

Definition: Degenerate Cases

- The *null* graph is the non-graph with no vertices $(n = m = 0)$.
- The *trivial* graph is the graph with exactly one vertex and no edges $(n = 1, m = 0)$. Otherwise, the graph is *non-trivial*.
- An *empty* graph is a graph with possibly some isolated vertices but with no edges $(m = 0)$.

Hence, both the null and trivial graphs are empty.

1.3 Graph Tuple Relations

Various problems in graph theory require that vertices and edges be assigned values of particular attributes. This is accomplished by adding relations to the graph tuple that map the vertices and/or edges to their attribute values. Note that there are no particular limitations on the nature of such a relation — everything from a basic relation to a bijective function is possible, depending on the problem.

In practice, when a graph theory problem requires a particular vertex or edge attribute, the presence of some corresponding relation $\mathscr{R}$ is assumed and we say something like, "vertex v has attribute a," instead of the more formal, "vertex v has attribute $\mathscr{R}(v)$."

The following sections describe the three relations used by the algorithms.

1.3.1 Labels

One of the possible relations in a graph tuple is a bijective function that assigns each vertex an identifying label. When such a function is present, the graph is said to be a *labeled* graph:

Definition: Labeled Graph

To say that a graph G is *labeled* means that its vertices are considered to be distinct and are assigned identifying names (labels) by adding a bijective labeling function to the graph tuple:

$$\ell : V(G) \to L$$

where L is a set of labels (names). Otherwise, the vertices are considered to be identical (only the structure of the graph matters) and the graph is *unlabeled*.

The vertices in a labeled graph are typically draw as open circles containing the corresponding labels, whereas the vertices in an unlabeled graph are typically drawn as filled circles. This is demonstrated in the example graph of Figure 1.1: the graph on the left is labeled and the graph on the right is unlabeled.

Since the labeling function ℓ is bijective, a vertex $v \in V(G)$ with label "a" can be identified by v or $\ell^{-1}(a)$. In practice, the presence of a labeling function is assumed for a labeled graph and so a vertex is freely identified by its label. This is important to note when a proof includes a phrase such as, "let $v \in V(G) \ldots$" since v may be a reference to any vertex in $V(G)$ or may call out a specific vertex by its label; the intention is usually clear from the context.

The FR graphs that act as the inputs to the two algorithms are labeled graphs, where the labels represent the various functional requirements:

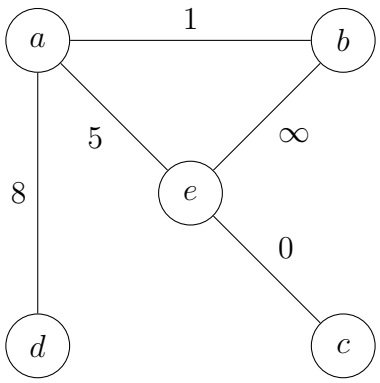
$$FR_1, FR_2, FR_3, \ldots, FR_n$$

1.3.2 Edge Weight

Certain graph theory problems require that each edge be assigned a *weight*. An edge weight suggests a certain cost associated with the use of the edge, usually interpreted as traversing the edge from one endpoint to the other. This requires the addition of a surjective weight function to the graph tuple:

$$w : E(G) \to W$$

where W is a set of edge weights, usually integers or rational numbers. A special edge weight of ∞ is used to indicate that an edge should never be used/traversed. Non-adjacent vertices can be viewed as being connected by edges with either zero or infinite weight, depending on the problem. In a drawn graph, edges are typically labeled with their weights, as shown in the example graph of Figure 1.3.2.



$$w(ab) = 1$$
$$w(ae) = 5$$
$$w(ad) = 8$$
$$w(be) = \infty$$
$$w(ce) = 0$$

Figure 2: A Graph with Edge Weights

The first algorithm uses edge weights to represent the cost or penalty for consolidating an edge's two FR endpoints into the same part. It is assumed that non-adjacent parts cannot be consolidated, and thus are considered to be joined by edges with infinite weight.

1.3.3 Vertex Color

Other graph theory problems require that the graph's vertex set be distributed into some number of sets based on some problem-specific criteria. Usually, this distribution is a true partition (no empty sets), but this is not required depending on the problem. One popular method of performing this distribution is by adding a *coloring* function to the graph tuple:

$$c : V(G) \to C$$

where C is a set of *colors*; vertices with the same color are assigned to the same set in the distribution. Although the elements of C are usually actual colors (red, green, blue, etc.), a graph coloring problem is free to select any value type for the color attribute. Note that there is no assumption that c is surjective, so the codomain C may contain unused colors, which corresponds to empty sets in the distribution.

The most popular coloring scheme for a graph requires that adjacent nodes be assigned different colors:

Definition: Proper Coloring

A coloring c on a graph G is called *proper* when no two adjacent nodes are assigned the same color:

$$\forall\, u, v \in V(G), uv \in E(G) \implies c(u) \neq c(v)$$

A proper coloring c with $|C| = k$ is called a k-*coloring* of G and G is said to be k-*colorable*, meaning the actual coloring (range of c) uses *at most* k colors.

An example of a 4-coloring is shown in Figure 1.3.3.

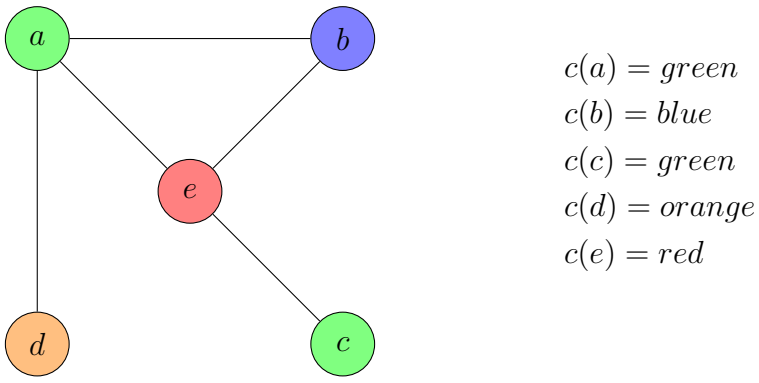


Figure 3: A Graph with a 4-coloring

Since there is no requirement that a coloring c be surjective, the codomain C may contain unused colors. For example, the codomain of the coloring shown in Figure 1.3.3 might be:

$$C = \{green, blue, red, orange\}$$

and hence c is surjective and G is 4-colorable. But we can always add an unused color to C:

$$C = \{green, blue, red, orange, brown\}$$

Now, c is no longer surjective, and according to the definition: G is 5-colorable — the coloring c uses at most 5 colors (actually only 4), which is the cardinality of the codomain.

Thus, we can make the following statement:

Proposition 1

Let G be a graph:

$$G \text{ is } k\text{-colorable} \implies G \text{ is } (k+1)\text{-colorable}$$

By inductive application of Proposition 1, one can arrive at the following conclusion:

Proposition 2

Let G be a graph:

$$G \text{ is } k\text{-colorable} \implies G \text{ is } (k+r)\text{-colorable for some } r \in \mathbb{N}.$$

1.3.4 Chromatic Coloring

Since $k \in \mathbb{N}$, by the well-ordering principle, there exists some minimum k such that a graph G is k-colorable:

Definition: Chromatic Coloring

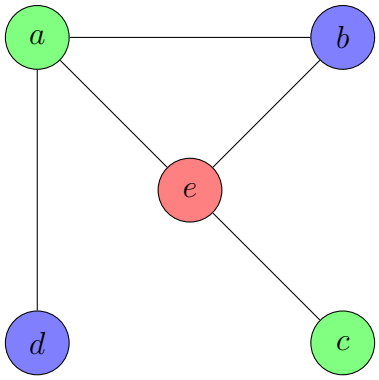
The minimum k such that a graph G is k-colorable is called the *chromatic number* of G, denoted by $\chi(G)$. A k-coloring for a graph G where $k = \chi(G)$ is called a k-*chromatic* coloring.

Returning to the example 4-coloring of Figure 1.3.3, note that vertex d can be colored blue and then orange can be excluded from the codomain, resulting in a 3-coloring. This is shown in Figure 1.3.4. Since there is no way to use less than 3 colors to obtain a proper coloring of the graph, the coloring is 3-chromatic. Note that when a coloring is chromatic, there are no unused colors (empty sets) and hence the distribution is a true partition.

1.3.5 Independent Sets

The primary purpose of a k-coloring of a graph G is to distribute the vertices of G into k so-called *independent* (some possibly empty) sets:

Definition: Independent Set



$$c(a) = green$$
$$c(b) = blue$$
$$c(c) = green$$
$$c(d) = blue$$
$$c(e) = red$$

Figure 4: A Graph with a Chromatic 3-coloring

Let G be a graph and let $S \subseteq V(G)$. To say that S is an *independent* set means that all of the vertices in S are non-adjacent in G:

$$\forall\, u, v \in S, uv \notin E(G)$$

Since a k-chromatic coloring of a graph G is surjective, there are no unused colors (empty sets) and so the coloring partitions the vertices of G into exactly k independent sets. The goal of the second algorithm is to find a chromatic coloring of an FR graph so that the resulting independent sets indicate how to consolidate the FRs into a minimum number of parts: one part per independent set.

References

[1] G. Chartrand and P. Zhang. *A First Course in Graph Theory*. Dover Publications, Mineola, New York, 2012.