A Lower Bound on the Chromatic Number of a Graph

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1. INTRODUCTION.

D. P. Geller recently proposed the Problem [1]: For any graph G with p points, q lines and chromatic number χ, show

$$\chi \geq p^2/(p^2-2q) \tag{1}$$

In this paper we not only prove (1) but also, in the process, establish the stronger result that

$$\chi \geq P_0 \geq p^2/(p^2-2q) \tag{2}$$

where P_0 is the number of points in a largest complete subgraph G_0 of G. Since G_0 is complete,

$$2q_0 = P_0(P_0-1) \tag{3}$$

where q_0 is the number of lines in G_0. Combining (2) and (3) yields

$$q/q_0 \leq p^2/P_0^{\,2} \tag{4}$$

This inequality has obvious significance as a bound on the combinatorial complexity which can be realized with q lines on p points.

Another consequence of (2), and one which is of significance in relation to the Four Color Conjecture [2], is that when G is maximally planar,

$$q = 3(p-2) \tag{5}$$

$$3 \leq P_0 \leq 4 \tag{6}$$

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so that (2) becomes

$$\chi \geq p^2/(p^2 - 6p + 12) \tag{7}$$

The right-hand side of (7) achieves its maximum value, which is 4 (so that $\chi \geq 4$), when $p = 4 = P_0$; and has a value of 3 when $p = 3 = P_0$.

In the derivations which follow it is assumed that G is finite and nontrivial, there being no useful purpose served in assuming otherwise.

2. LOWER BOUND ON THE CHROMATIC NUMBER.

It is obvious that

$$\chi \geq P_0 \tag{8}$$

Since there can be at most p-1 lines incident with any one point in G, the total valence 2q of G is related to p as

$$2q \leq p(p-1) \tag{9}$$

where equality holds if and only if G is complete. Since G is finite, there exists a (not necessarily unique) finite set of nonempty subgraphs $G_0, G_1, \ldots, G_k$ of G which constitutes a line cover of G, such that G_j, $1 \leq j \leq k$, is a largest complete subgraph of $G-G_0-G_1- \ldots - G_{j-1}$. If p_i is the number of points and q_i is the number of lines in G_i,

$$p_i \geq p_j \tag{10}$$

and

$$2q_i = p_i(p_i-1) \geq 2q_j = p_j(p_j-1) \tag{11}$$

whenever $0 \leq i < j \leq k$. If $k = 0$, (2) is seen to be true by (8) and (3). We shall show that (2) is true for $k = 1$ and then show by induction that it is true for k arbitrary.

Suppose $k = 1$ and let $q_{0,1}$ be the number of lines in G each of which covers a point in G_0 and a point in G_1. Since G_0 is a largest complete subgraph in G,

$$q_{0,1} \leq p_1(p_0-1) \tag{12}$$

Thus, since $p_1 \leq p_0$, the number of lines q in G is bounded as

$$\begin{aligned} 2qp_0 &= 2p_0(q_o+q_1+q_{0,1}) \\ &\leq p_0^2\ (p_0-1) + p_0p_1(p_1-1) + 2p_0p_1(p_0-1) \\ &= p^2(p_0-1) - p_1(p_0-p_1) \\ &\leq p^2(p_0-1) \end{aligned} \tag{13}$$

This gives

$$p_0 \geq p^2/(p^2-2q) \tag{14}$$

so that (2) is true by (14) and (8) when k = 1.

Assuming (2) true for k=n, we shall show that it is true for k=n+1 and hence, since true for k=0,1, that it is true for k arbitrary. Let k=n+1 and let $q_{j,n+1}$, where $0 \leq j \leq n$, be the number of lines in G each of which covers a point in G_{n+1} and a point in G_j, so that

$$q_{j,n+1} \leq p_{n+1}(p_j-1) \tag{15}$$

Under the induction hypothesis that (2) is assumed true for $p-p_{n+1}$ points in G,

$$\begin{aligned} 2qp_0 &\leq (p-p_{n+1})^2(p_0-1) + p_0p_{n+1}(p_{n+1}-1) \\ &\quad + 2p_0p_{n+1}(p_0-1 + p_1-1 +\ldots+ p_n-1) \\ &= p^2(p_0-1) + p_{n+1}\{2p-(2n+3)p_0-p_{n+1}\} \end{aligned} \tag{16}$$

Since

$$\begin{aligned} 2p &= 2(p_0+p_1 +\ldots+ p_{n+1}) \\ &\leq 2(n+1)p_0 + 2p_{n+1} \\ &= (2n+3)p_0 + p_{n+1}-(p_0-p_{n+1}) \end{aligned} \tag{17}$$

and since $p_{n+1} \leq p_0$, then

$$2p-(2n+3)\ p_0-p_{n+1} \leq p_{n+1}-p_0 \leq 0 \tag{18}$$

From (16) and (18),

$$2qp_0 \le p^2(p_0-1) \tag{19}$$

for k = n+1. Thus, from (19) and (8)

$$\chi \ge p_0 \ge p^2/(p^2-2q)$$

is true for k arbitrary, and is the central result of this paper.

In those cases when (as is more often the case than not) it is not convenient to identify a largest complete subgraph of a given graph, and hence to determine p_0, a lower bound on χ can be computed directly from p and q in accord with (2)

3. BOUND ON THE MAXIMUM NUMBER OF LINES.

As noted in the Introduction, (2) and (3) combined give

$$q/q_0 \le p^2/p_0^2 \tag{4}$$

Thus, (4) provides an upper bound on the maximum number of lines $q = q_{max}$ which G can contain when a largest complete subgraph G_0 in G has q_0 lines on p_0 points (so that $2q_0=p_0(p_0-1)$). A lower bound on q_{max} is obtained when G contains a maximal number of largest complete subgraphs each isomorphic to G_0, as follows.

For n any real number denote by [n] the smallest integer no less than n. Let

$$k = [p_0/p] \tag{20}$$

so that $k \ge 1$ and

$$p/p_0 = k-\varepsilon \tag{21}$$

where $0 \le \varepsilon < 1$. The maximal number of largest complete subgraphs each with q_0 lines on p_0 points is thus k when $\varepsilon = 0$, and k-1 otherwise. In either case, when $q = q_{max}$, G contains a set of k point-disjoint (and hence line-disjoint) complete subgraphs $G_0, G_1, \ldots, G_{k-1}$ which constitutes a line cover [2] of G with $p_i = p_0$ the number of points in G_i, $i = 0,1,\ldots, k-2$, and $p_{k-1} = p_0(1-\varepsilon)$ the number of points in G_{k-1}. For maximal connectedness, it is easily shown that

$$2q_{max} = (2q_0p^2/p_0^2) - p_0\varepsilon(1-\varepsilon) \qquad (22)$$

it follows from (22) and (4) that

$$(q_0p^2/p_0^2) - p_0\varepsilon(1-\varepsilon)/2 \leq q_{max} \leq q_0p^2/p_0^2 \qquad (23)$$

Since $0 \leq \varepsilon<1$, so that $\varepsilon(1-\varepsilon) \leq 1/4$,

$$(q_0p^2/p_0^2) - p_0/8 \leq q_{max} \leq q_0p^2/p_0^2 \qquad (24)$$

The tightness of these bounds on q_{max} is made evident by noting the smallness of the ratio of $p_0\varepsilon(1-\varepsilon)/2$ to q_0p^2/p_0^2. Since $2q_0 = p_0(p_0-1)$, and to the extent that $p_0(p_0-1) \cong p_0^2$ is a valid approximation, this ratio is of the order of 1 in $4p^2/p_0$ lines in the "worst case" when $\varepsilon = 1/2$.

REFERENCES

1. Problem 5713 *Amer. Math. Monthly* 77 (1970), 85.

2. Frank Harary, *Graph Theory*, Addison-Wesley Publishing Co., Reading Mass., 1969.

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