DETERMINING A GRAPH'S CHROMATIC NUMBER FOR PART CONSOLIDATION IN AXIOMATIC DESIGN

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Jeffery A. Cavallaro

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by

Jeffery A. Cavallaro

APPROVED FOR THE DEPARTMENT OF MATHEMATICS

SAN JOSÉ STATE UNIVERSITY

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Sogol Jahanbekam, Ph.D.   Department of Mathematics

Wasin So, Ph.D.   Department of Mathematics

Jordan Schettler, Ph.D.   Department of Mathematics

ABSTRACT

DETERMINING A GRAPH'S CHROMATIC NUMBER FOR PART CONSOLIDATION
IN AXIOMATIC DESIGN

by Jeffery A. Cavallaro

Mechanical engineering design practices are increasingly moving towards a
framework called *axiomatic design*, which starts with a set of independent *functional
requirements* (FRs) for a manufactured product. A key tenet of axiomatic design is to
decrease the *information content* of a design in order to increase the chance of
manufacturing success. One important way to decrease information content is to fulfill
multiple FRs by a single part: a process known as *part consolidation*. Thus, an important
parameter when comparing two candidate designs is the minimum number of parts
needed to satisfy all of the FRs. One possible method for determining the minimum
number of parts is to represent the problem by a graph, where the vertices are the FRs and
the edges represent the need to separate their endpoint FRs into separate parts. The
answer then becomes the solution to a vertex coloring problem: finding the chromatic
number of such a graph. Unfortunately, the chromatic number problem is known to be
NP-hard. This research investigates a new algorithm that determines the chromatic
number for a graph and compares the new algorithm's computer runtime performance to
existing Zykov branch and bound algorithms using random graph analysis.

ACKNOWLEDGMENTS

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1 AXIOMATIC DESIGN

The axiomatic design framework was developed in the late 20th century by Professor Nam P. Suh while at MIT and the NSF [1]. This was in response to concern in the engineering community that *design* was being practiced almost exclusively as an ad-hoc creative endeavor with very little in the way of scientific discipline. In the words of Professor Suh:

> It [design] might have preceding the development of natural sciences by scores of centuries. Yet, to this day, design is being done intuitively as an art. It is one of the few technical areas where experience is more important than formal education [1].

Professor Suh was not making these claims in an educational vacuum, but in the shadow of several recent major design failures such as the Union Carbide plant disaster in India, nuclear power plant accidents at Three Mile Island and Chernobyl, and the Challenger space shuttle O-ring failure. Furthermore, Professor Suh asserts that design-related issues resulting in production problems and operating failures were increasingly happening in everything from consumer products to big-ticket items. As a result, axiomatic design has been widely adopted by companies to promote efficiency and accuracy in the design process, resulting in more reliable products and reduced manufacturing costs [2].

The following sections provide an overview of axiomatic design as specified in detail by Professor Suh [1], [3], and summarized by Behdad, et al. [4], [5]. Following the overview is a description of how an algorithm proposed by this research can be a helpful tool to a designer using the axiomatic design framework.

1.1 Design

Design is defined as the process by which it is determined *what* needs to be achieved and then *how* to achieve it. Thus, the decisions on what to do are just as important as how to do it. *Creativity* is the process by which experience and intuition are used to generate solutions to perceived needs. This includes pattern matching to and adapting existing

solutions and synthesizing new solutions. Thus, creativity plays a vital role in design. Since different designers may approach the same problem differently, their level of creativity may lead to very different, yet plausible, solutions. Therefore, there needs to be a design-agnostic method for comparing different designs with the goal of selecting the best one.

This discussion will sound familiar to mathematicians, since creativity is a very important part of solving math problems, and in particular, writing proofs. Starting with the work of Peano in the 19th century, the field of mathematics has established various tests on what constitutes a good proof. For example:

- Does every conclusion result by proper implication from existing definitions, axioms, and previously proved conclusions?
- Is direct proof, contrapositive proof, proof by contradiction, or proof by induction the best approach for a particular problem?
- Do proofs by induction contain clear basic, assumptive, and inductive steps?
- Are all subset and equality relationships properly proved via membership implication?
- Are all necessary cases included and stated in a mutually exclusive manner?
- Are degenerate cases sufficiently highlighted?
- Are all equivalences proved in a proper circular fashion?
- Are key and reused conclusions highlighted in lemmas?

In short, Professor Suh was looking for a similar framework for the more general concept of design.

1.2   The Axiomatic Design Framework

The *best* design among a set of candidates is the design that exactly satisfies a clearly defined set of needs and has the greatest probability of success. In a desire not to hinder the creative element needed for design, yet provide some methodology to distinguish bad

designs from good designs from better designs, the diagram in Fig. 1 establishes the overall framework for axiomatic design.
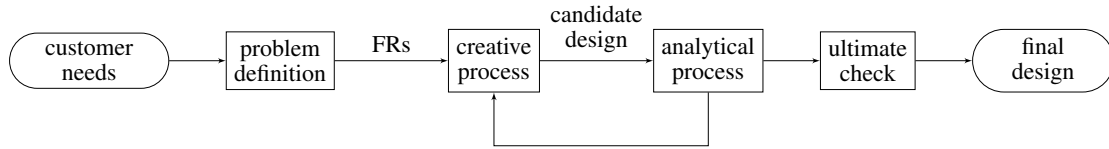
Fig. 1. The axiomatic design framework.

Design starts with the desire to satisfy a set of clear *customer needs*. The term *customer* refers to any entity that expresses needs, and can be as varied as individuals, organizations, or society. The designer, in the *problem definition* phase, determines how these customer needs will be met by generating a list of *functional requirements* (FRs). It is this list of FRs that determines exactly *what* is to be accomplished.

Once the set of FRs has been determined, the designer begins the *creative process* by mapping the FRs into solutions that are embodied in so-called *design parameters* (DPs). The DPs contain all of the information on *how* the various FRs are satisfied: parts lists, drawings, specifications, etc. The FRs exist in a design-agnostic *functional space* and the DPs exist in a solution-specific *physical space*. It is the designer's job to provide the most efficient mapping between the two spaces. This process is represented by Fig. 2.
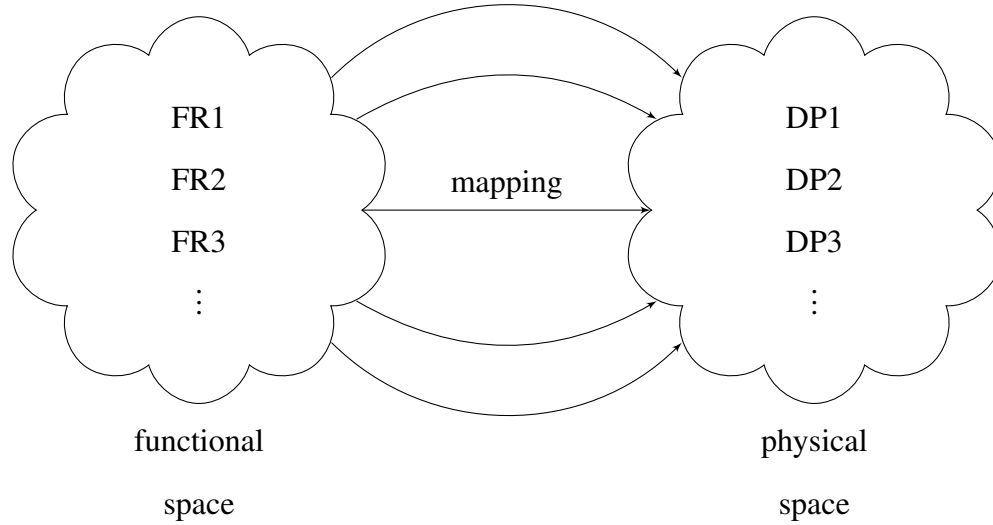
Fig. 2. Mapping FRs to DPs.

Simple problems may require only one level of FRs; however, more complicated designs may require a hierarchial structure of FRs from more general to more detailed requirements. This type of design is often referred to as *top-down* design. Each individual FR layer has its own DP mapping. In fact, the mapping process on one level should be completed prior to determining the FRs for the next level. This is because DP choices on one level may affect requirements on the next level. For example, consider a FR related to a moving part in a design. The DP for this FR could specify that the part be moved manually or automatically. Each choice would result in different FRs for the actual mechanism selected by the DP.

The FR/DP mapping at each level in the design hierarchy is described by the *design equation*, which is shown in Equation 1.

$$[\text{FR}] = [\text{A}][\text{DP}] \tag{1}$$

The design equation is a matrix equation that maps a vector of *m* FRs to a vector of *n* DPs via an $m \times n$ design matrix A. As will be shown, good designs require $m = n$. A full

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discussion of the design matrix element values is beyond the scope of this research. Instead, the following two values are used:

$$A_{ij} = \begin{cases} X, & \text{FR}_i \text{ depends on DP}_j \\ 0, & \text{FR}_i \text{ does not depend on DP}_j \end{cases}$$

Since the FR/DP mapping is non-unique, there needs to be a method to compare different plausible designs so that the best design can be selected as the final design. Thus, the framework in Fig. 1 includes an *analytical process*, where designs are judged by a set of axioms, corollaries, and theorems that specify the properties common to all good designs. Once the best design, according to this analysis, is selected, it undergoes an *ultimate check* to make sure that it sufficiently meets all of the customer's needs. If so, then that design is selected as the final design.

1.3 The Axioms

The analytical process is based on two main axioms: the independence axiom and the information axiom. This section describes these axioms and their related corollaries and theorems.

The independence axiom [1] imposes a restriction on the FR/DP mapping:

Axiom 1 (The Independence Axiom). *An optimal design always maintains the independence of the FRs. This means that the FRs and DPs are related in such a way that a specific DP can be adjusted to satisfy its corresponding FR without affecting other FRs.*

The ideal case is when the design matrix is a diagonal matrix, and so each FR is mapped to and is satisfied by exactly one DP. This is referred to as an *uncoupled* design, which is demonstrated in Fig. 3. Uncoupled designs completely adhere to the independence axiom.

$$\begin{bmatrix} \text{FR1} \\ \text{FR2} \\ \text{FR3} \end{bmatrix} = \begin{bmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X \end{bmatrix} \begin{bmatrix} \text{DP1} \\ \text{DP2} \\ \text{DP3} \end{bmatrix}$$

Fig. 3. An uncoupled design.

The next best situation is when the design matrix is a lower-triangular matrix. The idea is to finalize the first DPs before moving on to the later DPs. Thus, DP_i can be adjusted without affected FR_1 through FR_{i-1}. This is referred to as a *decoupled* design, which is demonstrated in Fig. 4. Although decoupled designs do not completely adhere to the independence axiom, they may be reasonable compromises in designs that address complex problems.

$$\begin{bmatrix} \text{FR1} \\ \text{FR2} \\ \text{FR3} \end{bmatrix} = \begin{bmatrix} X & 0 & 0 \\ X & X & 0 \\ X & X & X \end{bmatrix} \begin{bmatrix} \text{DP1} \\ \text{DP2} \\ \text{DP3} \end{bmatrix}$$

Fig. 4. A decoupled design.

The worst solution is a non-triangular matrix, where every change in a DP affects multiple FRs in an unconstrained fashion. This is referred to as a *coupled* design, which is demonstrated in Fig. 5. Coupled designs are in complete violation of the independence axiom and generally should be decoupled by reworking the FRs or by adding additional DPs.

$$\begin{bmatrix} \text{FR1} \\ \text{FR2} \\ \text{FR3} \end{bmatrix} = \begin{bmatrix} X & X & X \\ X & X & X \\ X & X & X \end{bmatrix} \begin{bmatrix} \text{DP1} \\ \text{DP2} \\ \text{DP3} \end{bmatrix}$$

Fig. 5. A coupled design.

Unfortunately, adding additional DPs runs counter to the second axiom: the information axiom [1].

Axiom 2 (The Information Axiom)**.** *The best design is a functionally uncoupled design that has the minimum information content.*

The amount of *Information* contained in a particular DP is inversely related to the probability that the DP can successfully satisfy its corresponding FR(s) by Equation 2.

$$I = \log_2\left(\frac{1}{p}\right) \tag{2}$$

where p is the probability of success and I is measured in bits. This probability must take into consideration such things as tolerances, ease of manufacture, failure rates, etc. The information content of a design is then the sum of the information content of its individual DPs.

From these two axioms come the following seven corollaries [1]:

Corollary 1. *Decouple or separate parts or aspects of a solution if the FRs are coupled or become interdependent in the designs proposed.*

Corollary 2. *Minimize the number of FRs.*

Corollary 3. *Integrate design features in a single physical part if FRs can be independently satisfied in the proposed solution.*

Corollary 4. *Use standardized or interchangeable parts if the use of these parts is consistent with the FRs.*

Corollary 5. *Use symmetrical shapes and/or arrangements if they are consistent with the FRs.*

Corollary 6. *Specify the largest allowable tolerance in stating FRs.*

Corollary 7. *Seek an uncoupled design that requires less information than coupled designs in satisfying a set of FRs.*

The theorems that arise from these axioms and corollaries are used to prove that an optimal design results from a square design matrix. In other words, the number of FRs should be equal to the number of DPs. First, consider the case where there are more FRs than DPs. This forces a single DP to be mapped to multiple FRs. Otherwise, some FRs cannot be satisfied by the DPs. This result is stated in Theorem 1 [1].

Theorem 1. *When the number of DPs is less that the number of FRs, either a coupled design results or the FRs cannot be satisfied.*

A possible solution to this problem is given by Theorem 2 [1].

Theorem 2. *A coupled design due to more FRs than DPs can be decoupled by adding new DPs if the additional DPs result in a lower triangular design matrix.*

An example is show in Fig. 6. Note that the addition of DP3 results in a decoupled design.

$$\begin{bmatrix} FR1 \\ FR2 \\ FR3 \end{bmatrix} = \begin{bmatrix} X & 0 \\ X & X \\ X & X \end{bmatrix} \begin{bmatrix} DP1 \\ DP2 \end{bmatrix} \implies \begin{bmatrix} FR1 \\ FR2 \\ FR3 \end{bmatrix} = \begin{bmatrix} X & 0 & 0 \\ X & X & 0 \\ X & X & X \end{bmatrix} \begin{bmatrix} DP1 \\ DP2 \\ DP3 \end{bmatrix}$$

Fig. 6. Decoupling a design by adding DPs.

Next, consider the case where the number of FRs is less than the number of DPs. Assuming that the design is not coupled, this means that either a DP exists that does not address any FRs or multiple DPs exist that address a single FR and hence can be integrated into a single DP. Such a design is called a *redundant* design. This is addressed by Theorem 3 [1].

Theorem 3. *When there are less FRs than DPs then the design is either coupled or redundant.*

Finally, the previous three theorems lead to the conclusion in Theorem 4 [1].

Theorem 4. *In an ideal design, the number of FRs is equal to the number of DPs.*

1.4   Part Consolidation

One particularly important design parameter is the number of parts in a product design. According to Professor Suh:

> Poorly designed products often cost more because they use more materials or parts than do well-designed products. They are often difficult to manufacture and maintain [1].

Decreasing the number of parts in a design while maintaining the independence of the FRs is consistent with Corollary 3 and lowers the information content of the design. In fact, Tang, et al. [6] describe how part consolidation reduces the weight and complexity of a final product while boosting reliability and reducing cost.

An informative example of part consolidation is the combination can/bottle opener shown in Fig. 7. The design of this handy utensil has two FRs, shown in Table 1. Both FRs are consolidated into a single part, yet remain independent as long as there is no desire to open a can and a bottle simultaneously (although that would be a popular trick around a campfire).
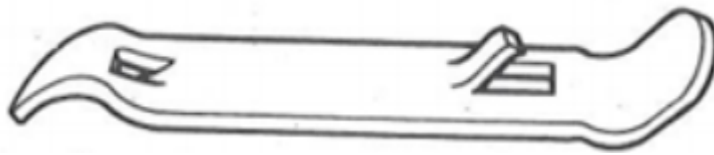
Fig. 7. A part consolidation example.

Table 1

Opener Functional Requirements

FR1	Open beverage cans
FR2	Open beverage bottles

The goal of this research is to provide designers with a tool that they can use to determine the minimum number of parts required to realize a particular design at a particular level in a FR/DP hierarchy. The designer is required to construct a graph whose vertices are the FRs and whose edges indicate that the endpoint FRs need to be realized by separate parts due to specific design constraints. How these edges are actually determined is beyond the scope of this research. Adjacent FRs are candidates for part consolidation. The goal is to find the chromatic number of the resulting graph, which corresponds to the minimum number of parts.

To be a viable tool, a computer program must be able to deliver an answer in a reasonable amount of time. Unfortunately, finding the chromatic number of a graph is known to be an NP-hard problem [7] and so the time required to find a solution grows exponentially with the number of FR vertices and edges in the graph. Thus, an algorithm is proposed with improved runtime complexity that can be run on a computer in order to provide a designer with an answer in a reasonable amount of time. Designs with different minimum part requirements can then be compared during the analytical process as part of the overall process of selecting the best design.

2 GRAPH THEORY

This section presents the concepts, definitions, and theorems from the field of graph theory that are needed in the development of the proposed algorithm. This material is primarily taken from the undergraduate graph theory text by Chartrand and Zhang (2012) [8] and the graduate graph theory text by West (2001) [9].

2.1 Simple Graphs

The problem of part consolidation is best served by a class of graphs called *simple graphs*. A *simple graph* is a mathematical object represented by an ordered pair $G = (V, E)$ consisting of a finite and non-empty set of *vertices* (also called *nodes*): $V(G)$, and a finite and possibly empty set of edges: $E(G)$. Each edge is represented by a two-element subset of $V(G)$ called the *endpoints* of the edge: $E(G) \subseteq \mathscr{P}_2(V(G))$. For the remainder of this work, the use of the term "graph" implies a "simple graph." Thus, a part consolidation problem can be represented by a graph whose vertices are the functional requirements (FRs) of the design and whose edges indicate which endpoint FRs should never be combined into a single part.

The choice of two-element subsets of $V(G)$ for the edges has certain ramifications that are indeed characteristics that differentiate a simple graph from other classes of graphs:

1) Every two vertices of a graph are the endpoints of at most one edge; there are no so-called *multiple* edges between two vertices.

2) The two endpoint vertices of an edge are always distinct; there are no so-called *loop* edges on a single vertex.

3) The two endpoint vertices are unordered, suggesting that an edge provides a bidirectional connection between its endpoint vertices.

When referring to the edges in a graph, the common notation of juxtaposition of the vertices will be used instead of the set syntax. Thus, edge $\{u, v\}$ is simply referred to as *uv* or *vu*.

Graphs are often portrayed visually using labeled or filled circles for the vertices and lines for the edges such that each edge line is drawn between its two endpoint vertices. An example graph is shown in Fig. 8.

$$V(G) = \{a,b,c,d,e\}$$

$$E(G) = \{ab,ad,ae,be\}$$

Fig. 8. An example graph (labeled and unlabeled).

When two vertices are the endpoints of the same edge, the vertices are said to be *adjacent* or are called *neighbors*, and the edge is said to *join* its two endpoint vertices. Furthermore, an edge is said to be *incident* to its endpoint vertices. In the example graph of Fig. 8, vertex *a* is adjacent to vertices *b*, *d*, and *e*; however, it is not adjacent to vertex *c*.

As demonstrated by vertex *c* in Fig. 8, there is no requirement that every vertex in a graph be an endpoint for some edge. In fact, a vertex that is not incident to any edge is called an *isolated* vertex.

We can also speak of adjacent edges, which are edges that share exactly one endpoint. Note that two edges cannot share both of their endpoints — otherwise they would be multiple edges, which are not allowed in simple graphs. In the example graph of Fig. 8, edge *ab* is adjacent to edges *ad* and *ae* via common vertex *a*, and *be* via common vertex *b*.

2.2 Order and Size

Two of the most important characteristics of a graph are its *order* and its *size*. The *order* of a graph G, denoted by $n(G)$ or just n when G is unambiguous, is the number of vertices in G: $n = |V(G)|$. The *size* of a graph G, denoted by $m(G)$ or just m when G is unambiguous, is the number of edges in G: $m = |E(G)|$. In the example graph of Fig. 8: $n = 5$ and $m = 4$.

Since every two vertices can have at most one edge between them, the number of edges has an upper bound:

Theorem 5. *Let G be a graph of order n and size m:*

$$m \leq \frac{n(n-1)}{2}$$

Proof. Since each pair of distinct vertices in $V(G)$ can have zero or one edges joining them, the maximum number of possible edges is $\binom{n}{2}$, and so:

$$m \leq \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

$\square$

Some choices of graph order and size lead to certain degenerate cases that serve as important termination cases for the the proposed algorithm:

- The *null* graph is the non-graph with no vertices ($n = m = 0$).
- The *trivial* graph is the graph with exactly one vertex and no edges ($n = 1, m = 0$). Otherwise, a graph is called *non-trivial*.
- An *empty* graph is a graph containing no edges ($m = 0$).

Thus, both the null and trivial graphs are empty.

2.3 Graph Relations

In addition to its vertices and edges, a graph may be associated with one or more relations. Each relation has $V(G)$ or $E(G)$ as its domain and is used to associate vertices or edges with problem-specific attributes such as labels or colors. Note that there are no particular limitations on the nature of such a relation — everything from a basic relation to a bijective function are possible. Some authors include these relations and their codomains as part of the graph tuple; however, since these extra tuple elements don't affect the structure of a graph, we will not do so.

In practice, when a graph theory problem requires a particular vertex or edge attribute, the presence of some corresponding relation $\mathscr{R}$ is assumed and we say something like, "vertex v has attribute a," instead of the more formal, "vertex v has attribute $\mathscr{R}(v)$."

The following sections describe the two relations used by the proposed algorithm.

2.3.1 Labels

One possible relation associated with a graph G is a bijective function $\ell : V(G) \to L$ that assigns to each vertex a unique identifying label. The codomain L is the set of available labels. When such a function is present, the graph is said to be a *labeled* graph and the vertices are considered to be distinct. Otherwise, a graph is said to be *unlabeled* and the vertices are considered to be identical (only the structure of the graph matters).

The vertices in a labeled graph are typically drawn as open circles containing the corresponding labels, whereas the vertices in an unlabeled graph are typically drawn as filled circles. This is demonstrated in the example graph of Fig. 8: the graph on the left is labeled and the graph on the right is unlabeled.

Since the labeling function ℓ is bijective, a vertex $v \in V(G)$ with label "a" can be identified by v or $\ell^{-1}(a)$. In practice, the presence of a labeling function is assumed for a labeled graph and so a vertex is freely identified by its label. This is important to note when a proof includes a phrase such as, "let $v \in V(G) \ldots$" since v may be a reference to

any vertex in $V(G)$ or may call out a specific vertex by its label; the intention is usually clear from the context.

The design graphs that act as the inputs to the proposed algorithm are labeled graphs, where the labels represent the various functional requirements: $FR_1, FR_2, FR_3, \ldots, FR_n$.

2.3.2   *Vertex Color*

Other graph theory problems require that a graph's vertices be distributed into some number of sets based on some problem-specific criteria. Usually, this distribution is a true partition (no empty sets), but this is not required depending on the problem. One popular method of performing this distribution on a graph G is by using a *coloring* function $c : V(G) \to C$, where C is a set of *colors*. Vertices with the same color are assigned to the same set in the distribution. Although the elements of C are usually actual colors (red, green, blue, etc.), a graph coloring problem is free to select any value type for the color attribute. Note that there is no assumption that c is surjective, so the codomain C may contain unused colors, which correspond to empty sets in the distribution.

A coloring $c : V(G) \to C$ on a graph G is called *proper* when no two adjacent vertices in G are assigned the same color: for all $u, v \in V(G)$, if $uv \in E(G)$ then $c(u) \neq c(v)$. Otherwise, c is called *improper*. A proper coloring with $|C| = k$ is called a *k-coloring* of G and G is said to be *k-colorable*, meaning the actual coloring (range of c) uses *at most k* colors.

An example of a 4-coloring is shown in Fig. 9.

$$C = \{\text{green}, \text{blue}, \text{red}, \text{orange}\}$$

$$c(a) = \text{green}$$

$$c(b) = \text{blue}$$

$$c(c) = \text{green}$$

$$c(d) = \text{orange}$$

$$c(e) = \text{red}$$
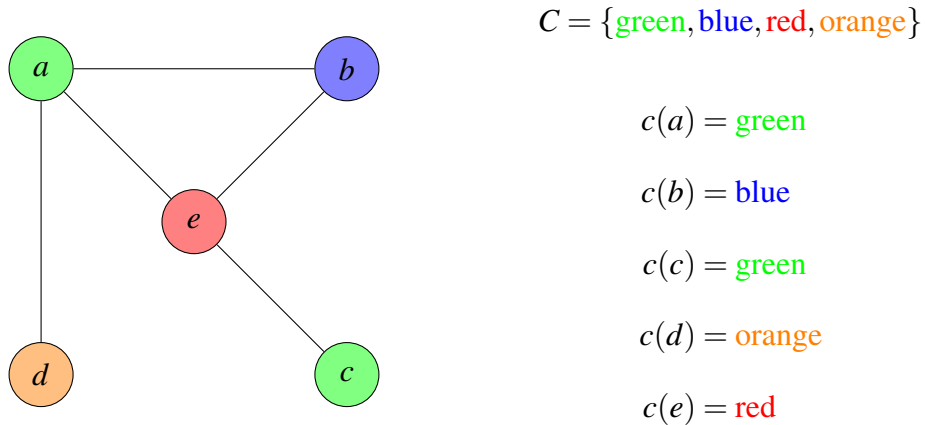
Fig. 9. A graph with a 4-coloring.

Since there is no requirement that a coloring c be surjective, the codomain C may contain unused colors. For example, the coloring shown in Fig. 9 is surjective, but we can add an unused color to C:

$$C = \{\text{green}, \text{blue}, \text{red}, \text{orange}, \text{brown}\}$$

Now, c is no longer surjective, and according to the definition, G is 5-colorable — the coloring c uses at most 5 colors (actually only 4), which is the cardinality of the codomain.

Thus, we can make the statement in Proposition 1.

Proposition 1. *Let G be a graph. If G is k-colorable then G is $(k+1)$-colorable*

By inductive application of Proposition 1, one can arrive at the conclusion in Proposition 2.

Proposition 2. *Let G be a graph and let $r \in \mathbb{N}$. If G is k-colorable then G is $(k+r)$-colorable.*

Furthermore, for a graph G of order n, if $n \leq k$ then we can conclude that G is k-colorable, since there are sufficient colors to assign each vertex its own unique color.

This is stated in Proposition 3, which will turn out to be an important termination case for the proposed algorithm.

Proposition 3. *Let G be a graph of order n and let $k \in \mathbb{N}$. If $n \leq k$ then G is k-colorable.*

Since $k \in \mathbb{N}$, by the well-ordering principle there exists some minimum k such that a graph G is k-colorable. This minimum k is called the *chromatic number* of G, denoted by $\chi(G)$. A k-coloring for a graph G where $k = \chi(G)$ is called a *k-chromatic* coloring of G.

Returning to the example 4-coloring of Fig. 9, note that vertex d can be colored blue and then orange can be excluded from the codomain, resulting in a 3-coloring. This is shown in Fig. 10. Since there is no way to use less than 3 colors to obtain a proper coloring of the graph, the coloring is 3-chromatic. Note that when a coloring is chromatic, there are no unused colors (empty sets) and hence the distribution is a true partition.

$$C = \{\text{green}, \text{blue}, \text{red}\}$$

$$c(a) = \text{green}$$
$$c(b) = \text{blue}$$
$$c(c) = \text{green}$$
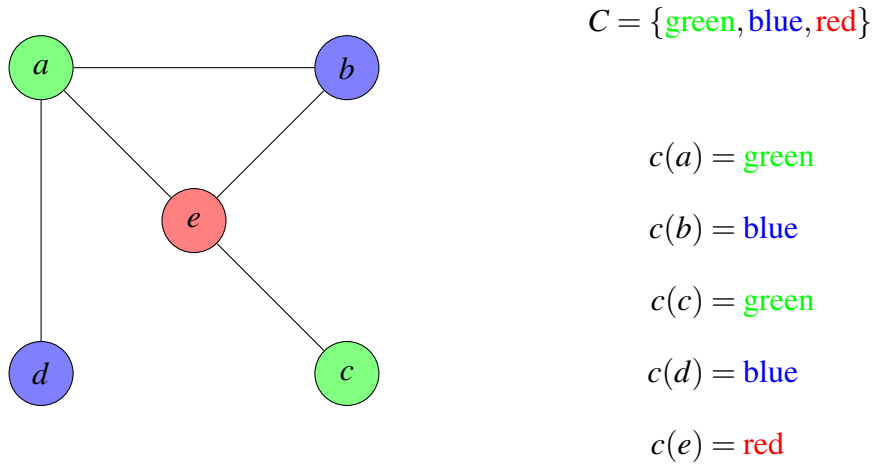$$c(d) = \text{blue}$$
$$c(e) = \text{red}$$

Fig. 10. A Graph with a 3-chromatic coloring.

2.4 Subgraphs

The basic strategy of the proposed algorithm is to arrive at a solution by mutating an input graph into simpler graphs such that a solution is more easily determined. The algorithm utilizes three particular mutators: vertex deletion, edge addition, and vertex

contraction. Before describing these mutators, it will be helpful to describe what is meant by graph equality and a *subgraph* of a graph.

To say that graph G is *equal* to graph H, denoted by $G = H$, means that the *exact same graph* is given two names: G and H. It is specifically *not* a comparison between two different graphs. Two different graphs that have the same structure, meaning there exists an adjacency-preserving bijection between the vertices of the two graphs, are referred to as being *isomorphic*, denoted by $G \cong H$, and are not considered to be equal. Of course, if $G = H$ then $G \cong H$; however, the converse is usually not true. In fact, $G = H$ if and only if $V(G) = V(H)$ and $E(G) = E(H)$.

To say that H is a *subgraph* of a graph G, denoted by $H \subseteq G$, means that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Thus, H can be achieved by removing zero or more vertices and/or edges from G, and G can be achieved by adding zero or more vertices and/or edges to H. Once again, H is not a different graph. If H is a different graph then one can say that it is isomorphic to a subgraph of G, but not a subgraph of G itself. By definition, $G \subseteq G$ and the null graph is a subgraph of every graph.

When G and H differ by at least one vertex or edge then H is called a *proper subgraph* of G, denoted by $H \subset G$. In fact, $H \subset G$ if and only if $H \subseteq G$ but $H \neq G$, meaning $V(H) \subset V(G)$ or $E(H) \subset E(G)$. When H and G differ by edges only: $V(H) = V(G)$ and $E(H) \subseteq E(G)$, then H is called a *spanning subgraph* of G.

The concept of subgraphs is demonstrated by graphs G, H, and F in Fig. 11. H is a proper subgraph of G by removing vertices c and d and edges ad and be. F is a proper spanning subgraph of G because F contains all of the vertices in G but is missing edges ab and be.
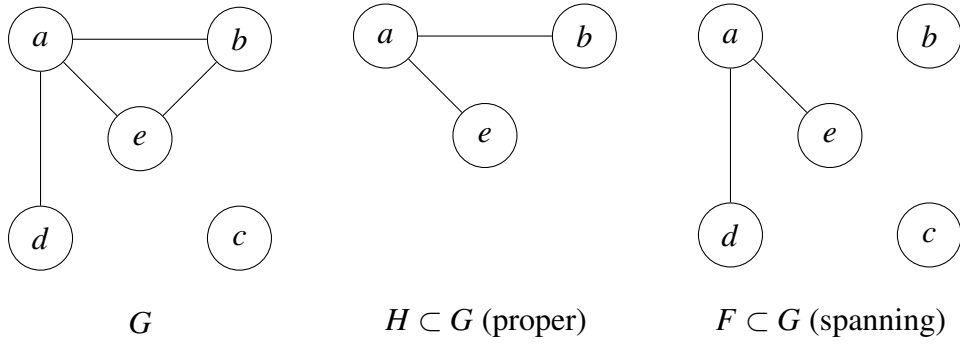
Fig. 11. Subgraph examples.

An *induced* subgraph is a special type of subgraph. Let G be a graph and let $S \subseteq V(G)$. The subgraph of G *induced* by S, denoted by $G[S]$, is a subgraph H such that $V(H) = S$ and for every $u, v \in S$, if u and v are adjacent in G then they are also adjacent in H. Such a subgraph H is called an *induced subgraph* of G.

In the examples of Fig. 11, H is not an induced subgraph of G because it is missing edge *be*. Likewise, a proper spanning subgraph like F can never be induced due to missing edges. In fact, the only induced spanning subgraph of a graph is the graph itself. Fig. 12 adds edge *be* so that H is now an induced subgraph of G.
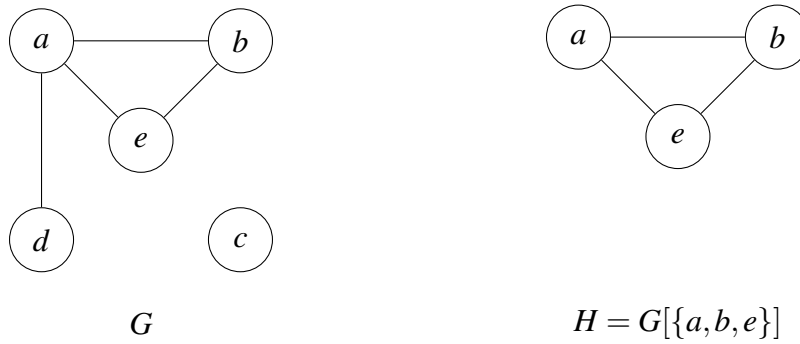
Fig. 12. An induced subgraph.

2.5   Mutators

The following sections describe the graph mutators used by the proposed algorithm.

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2.5.1 Vertex Removal

Let G be a graph and let $S \subseteq V(G)$. The induced subgraph obtained by removing all of the vertices in S (and their incident edges) is denoted by:

$$G - S = G[V(G) - S]$$

If $S \neq \emptyset$ then $G - S$ is a proper subgraph of G. If $S = V(G)$ then the result is the null graph.

Fig. 13 shows an example of vertex removal: vertices c and e are removed, along with their incident edges ae and be.

$$G \qquad\qquad G - \{c,e\}$$
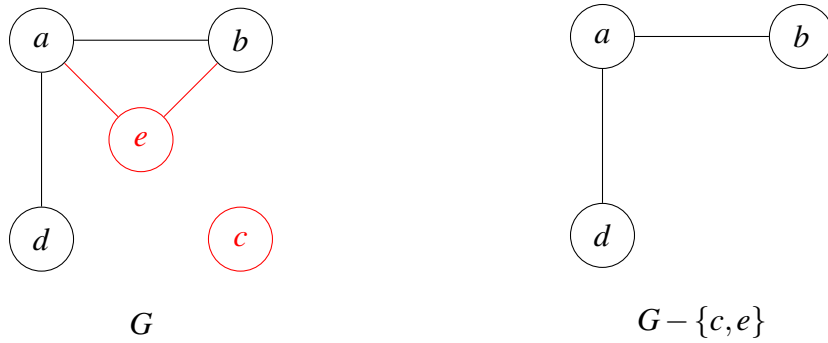
Fig. 13. Vertex removal.

If S consists of a single vertex v then the alternate syntax $G - v$ is used instead of $G - \{v\}$.

The proposed algorithm uses vertex removal to simplify a graph that is assumed to be k-colorable into a smaller graph that is also k-colorable.

2.5.2 Edge Addition

Let G be a graph and let $u, v \in V(G)$ such that $uv \notin E(G)$. The graph $G + uv$ is the graph with the same vertices as G and with edge set $E(G) \cup \{uv\}$. Note that G is a proper spanning subgraph of $G + uv$.

Fig. 14 shows an example of edge addition: edge cd is added.
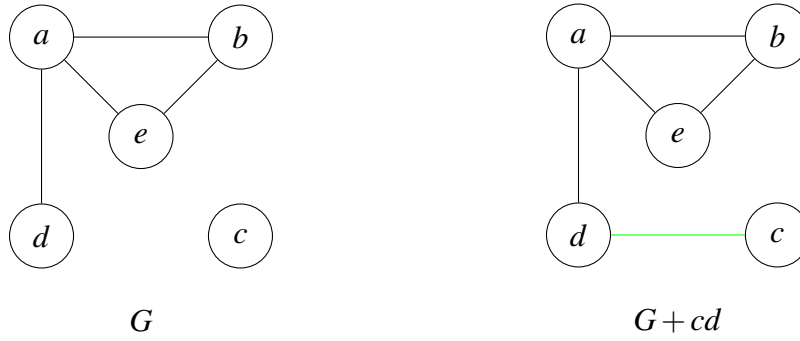
Fig. 14. Edge addition.

The proposed algorithm uses edge addition to prevent two non-adjacent FRs from being consolidated into the same part.

2.5.3 Edge Removal

The proposed algorithm does not use edge removal; however, a number of related algorithms do rely on this mutator so it is presented here. Let G be a graph and let $X \subseteq E(G)$. The spanning subgraph obtained by removing all of the edges in X is denoted by:

$$G - X = H\left(V(G), E(G) - X\right)$$

Thus, only edges are removed — no vertices are removed. If $X \neq \emptyset$ then $G - X$ is a proper subgraph of G. If $X = E(G)$ then the result is an empty graph.

Fig. 15 shows an example of edge removal: edges ae and be are removed.
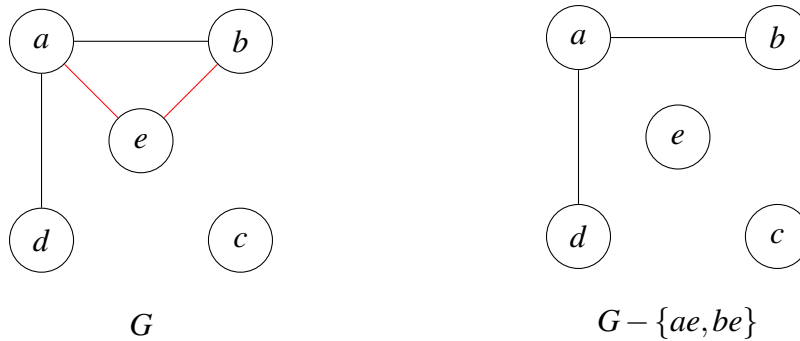
Fig. 15. Edge removal.

If *X* consists of a single edge *e* then the alternate syntax $G - e$ is used instead of $G - \{e\}$.

2.5.4 Vertex Contraction

Vertex contraction is a bit different because it does not involve subgraphs. Let *G* be a graph and let $u, v \in V(G)$. The graph $G \cdot uv$ is constructed by identifying *u* and *v* as one vertex (i.e., merging them). Any edge between the two vertices is discarded. Any other edges that were incident to the two vertices become incident to the new single vertex. Note that this may require supression of multiple edges to preserve the nature of a simple graph.

Fig. 16 shows an example of vertex contraction: vertices *a* and *b* are contracted into a single vertex. Since edges *ae* and *be* would result in multiple edges between *a* and *e*, one of the edges is discarded. Edges *bc* and *bd* also become incident to the single vertex.
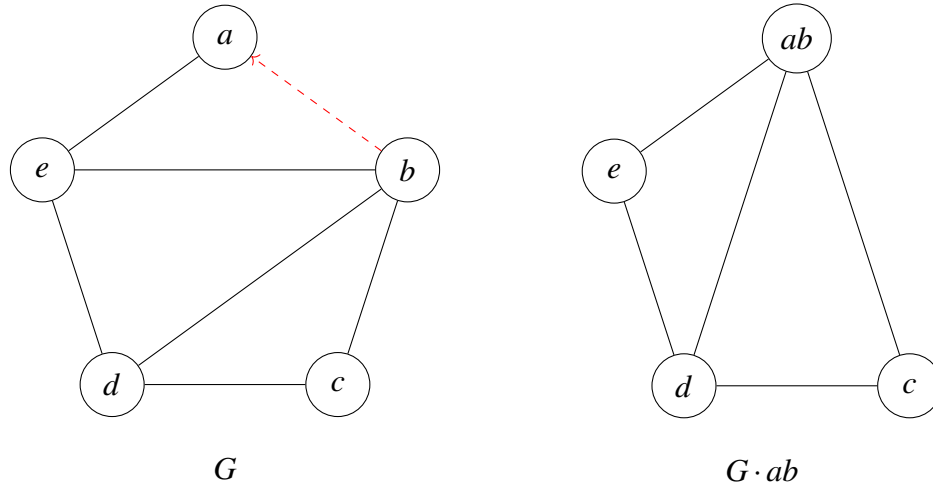
Fig. 16. Vertex contraction.

For the operation $G \cdot uv$, if $uv \in E(G)$ then the operation is also referred to as *edge contraction*. If $uv \notin E(G)$ then the operation is also referred to as *vertex identification*. The proposed algorithm uses vertex identification to consolidate two non-adjacent FRs into the same part.

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2.6   Independent Sets

The primary purpose of a k-coloring of a graph G is to distribute the vertices of G into k so-called *independent* (some possibly empty) sets. For a graph G, an *independent* set $S \subseteq V(G)$, sometimes called a *stable* set, is a set of pairwise non-adjacent vertices in G: for all $u, v \in S$, $uv \notin E(G)$. By definition, the empty set is an independent set. The cardinality of the largest possible independent set for a graph G, denoted by $\alpha(G)$, is called the *independence number* for G.

Since a k-chromatic coloring of a graph G is surjective, there are no unused colors (empty sets) and so the coloring partitions the vertices of G into exactly k non-empty independent sets. The goal of the proposed algorithm is to find a chromatic coloring of a design graph so that the resulting independent sets indicate how to consolidate the FRs into a minimum number of parts: one part per independent set.

A well-known algorithm for determining the chromatic number of a graph, introduced by Christofides (1971) [10], is based on finding the largest possible independent set S for a graph and then continually repeating the process on $G - S$ until $S = \emptyset$. The resulting partition represents a chromatic coloring of G. Part of this research compares the Christofides algorithm to the proposed algorithm.

2.7   Connected Graphs

The edges of a graph suggest the ability to "walk" from one vertex to another along the edges. A graph where this is possible for any two vertices is called a *connected* graph. The concept of connectedness is an important topic in graph theory; however, an ideal coloring algorithm should work regardless of the connected nature of an input graph. The concept of connectedness and how it impacts coloring is described in this section.

2.7.1   Walks

The undirected edges in a simple graph suggest bidirectional connectivity between their endpoint vertices. This leads to the idea of "traveling" between two vertices in a

graph by following the edges joining intermediate adjacent vertices. Such a journey is referred to as a *walk*.

A $u - v$ *walk* W in a graph G is a finite sequence of vertices $w_i \in V(G)$ starting with $u = w_0$ and ending with $v = w_k$:

$$W = (u = w_0, w_1, \ldots, w_k = v)$$

such that $w_i w_{i+1} \in E(G)$ for $0 \le i < k$. To say that W is *open* means that $u \ne v$. To say that W is *closed* means that $u = v$. The *length k* of W is the number of edges traversed: $k = |W|$. A *trivial* walk is a walk of zero length — i.e, a single vertex: $W = (u)$.

The bidirectional nature of the edges in a simple graph suggests the following proposition:

Proposition 4. *Let G be a graph and let $u - v$ be a walk of length k in G. G contains a $v - u$ walk of length k in G by traversing $u - v$ in the opposite direction.*

An example of two walks of length 4 is shown in Fig. 17. W_1 is an open walk because it starts and ends on distinct vertices, whereas W_2 is a closed walk because it starts and ends on the same vertex.
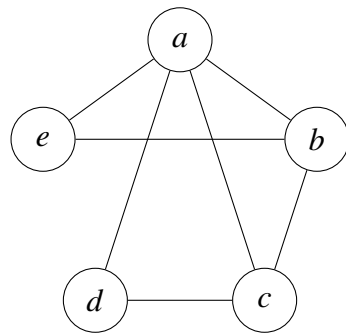
$W_1 = (a, b, e, a, c)$ is open

$W_2 = (a, e, b, c, a)$ is closed

$|W_1| = |W_2| = 4$

Fig. 17. Open and closed walks.

Vertices and edges are allowed to be repeated during a walk. Certain special walks can be defined by restricting such repeats:

trail	An open walk with no repeating edges	(a,b,c,a,e)
path	A trail with no repeating vertices	(a,e,b,c)
circuit	A closed trail	(a,b,e,a,c,d,a)
cycle	A closed path	(a,e,b,c,a)

The example special walks stated above refer to the graph in Fig. 17.

2.7.2 Paths

When discussing the connectedness of a graph, the main concern is the existence of paths between vertices. Let G be a graph and let $u, v \in V(G)$. To say that u and v are *connected* means that G contains a $u - v$ path.

But if there exists a $u - v$ walk in a graph G, does this also mean that there exists a $u - v$ path in G — i.e. a walk with no repeating edges or vertices? The answer is yes, as shown by the following theorem:

Theorem 6. *Let G be a graph and let $u, v \in V(G)$. If G contains a $u - v$ walk of length k then G contains a $u - v$ path of length $\ell \leq k$.*

Proof. Assume that G contains at least one $u - v$ walk of length k and consider the set of all possible $u - v$ walks in G; their lengths form a non-empty set of positive integers. By the well-ordering principle, there exists a $u - v$ walk P of minimum length $\ell \leq k$:

$$P = (u = w_0, \ldots, w_\ell = v)$$

We claim that P is a path.

Assume by way of contradiction that P is not a path, and thus P has at least one repeating vertex. Let $w_i = w_j$ for some $0 \leq i < j \leq \ell$ be such a repeating vertex. There are two possibilities:

Case 1: The walk ends on a repeated vertex ($j = \ell$). This is demonstrated in Fig. 18.

25
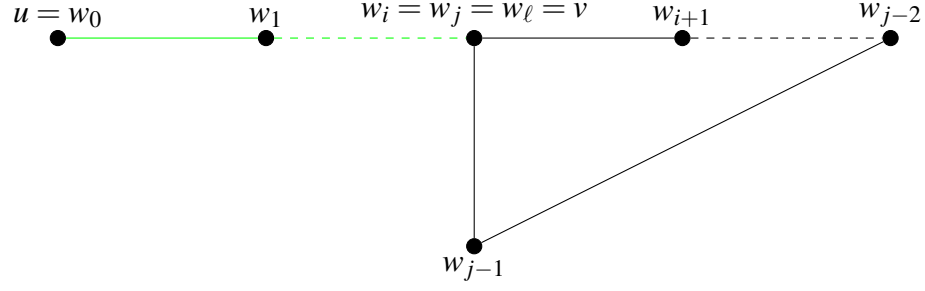
Fig. 18. Repeated vertex at end case.

Let $P' = (u = w_0, w_1, \ldots, w_i = v)$ be the walk shown in green in Fig. 18. P' is a $u - v$ walk of length $i < \ell$ in G.

Case 2: A repeated vertex occurs inside the walk ($j < \ell$). This is demonstrated in Fig. 19.
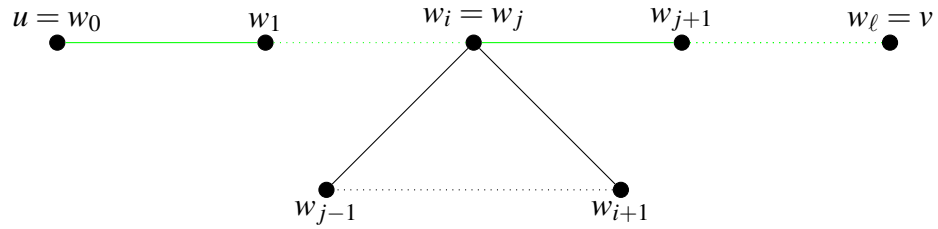
Fig. 19. Repeated vertex inside case.

Let $P' = (u = w_0, w_1, \ldots, w_i, w_{j+1}, \ldots, w_\ell = v)$ be the walk shown in green in the Fig. 19. P' is a $u - v$ walk of length $\ell - (j - i) < \ell$ in G.

Both cases contradict the minimality of the length of P.

$\therefore P$ is a $u - v$ path of length $\ell \leq k$ in G. $\qquad\square$

2.7.3 Connected

A *connected* graph G is a graph whose vertices are all connected: for all $u, v \in V(G)$ there exists a $u - v$ path. Otherwise, G is said to be *disconnected*. Examples of connected and disconnected graphs are shown in figure 20.

(a,b)
(a,c)
(a,d)
(b,a,c)
(b,a,d)
(c,a,d)

No path from any of a,b,c to any of d,e

DISCONNECTED
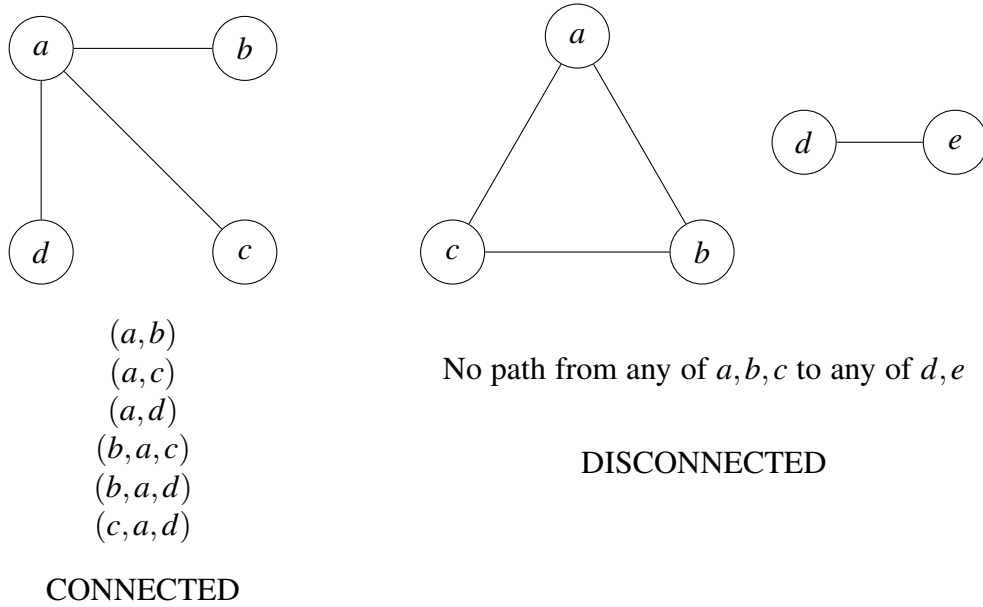
CONNECTED

Fig. 20. Connected and disconnected graphs.

By definition, the trivial graph is connected since the single vertex is connected to itself by a trivial path (of length 0).

2.7.4 Components

It would seem that a disconnected graph is composed of some number of connected subgraphs that partition the graph's vertex set under a connected equivalence relation. Each such subgraph is called a *component* of the graph.

Let G be a graph and let $\mathscr{G}$ be the set of all connected subgraphs of G. To say that a graph $H \in \mathscr{G}$ is a *component* of a G means that H is not a subgraph of any other connected subgraph of $\mathscr{G}$: for every $F \in \mathscr{G} - \{H\}$ it is the case that $H \not\subset F$. The number of distinct components in G is denoted by k(G), or just k if G is unambiguous. For a connected graph: $k(G) = 1$.

Each component of a graph G is denoted by G_i where $1 \leq i \leq k(G)$. We also use union notation to denote that G is composed of its component parts:

$$G = \bigcup_{0 \leq i \leq k(G)} G_i$$

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Furthermore the G_i are induced by the vertex equivalence classes of the connectedness relation:

Theorem 7. *Let G be a graph with component G_i. G_i is an induced subgraph of G.*

Proof. By definition, G_i is a maximal connected subgraph of G. So assume by way of contradiction that G_i is not an induced subgraph of G. Thus, G_i is missing some edges that when added would result in a connected induced subgraph H of G. But then $G_i \subset H$, contradicting the maximality of G_i.

$\therefore G_i$ is an induced subgraph of G. □

2.7.5 *Impact on Coloring*

The impact of disconnectedness on coloring depends on the selected algorithm. One might assume that the selected algorithm should be run on each component individually in order to determine each $\chi(G_i)$ and then, as pointed out by Zykov (1949) [11], conclude that the maximum such value is sufficient for $\chi(G)$:

$$\chi(G) = \max_{1 \leq i \leq k(G)} \chi(G_i)$$

For example, consider the disconnected graph in Fig. 20. The graph contains two components, so number the components from left-to-right:

$$\chi(G_1) = 3$$
$$\chi(G_2) = 2$$

$$\chi(G) = \max\{3, 2\} = 3$$

Using this technique requires application of an initial algorithm to partition the graph into components. Such an algorithm is well-known and is described by Hopcroft and Tarjan (1973) [12]. The algorithm is recursive. It starts by pushing a randomly selected

vertex on the stack and walking the vertex's incident edges, removing each edge as it is traversed. As each unmarked vertex is encountered, it is assigned to the current component. Vertices with incident edges are pushed onto the stack and newly isolated vertices are popped off the stack. Once the stack is empty, any previously unmarked vertex is selected to start the next component and the process continues until all vertices are marked. Given a graph G of order n and size m, this algorithm runs in $\max(n, m)$ steps.

Alternatively, an ideal coloring algorithm could be run on the entire graph at once regardless of the number of components in the graph. The proposed algorithm is such a solution, and therefore saves the needless work of partitioning the graph into components first.

2.8   Vertex Degree

Besides a graph's order and size, the next most important parameter is the so-called *degree* of each vertex. In order to define the degree of a vertex, we need to define what is meant by a vertex's *neighborhood* first. Let G be a graph and let $u \in V(G)$. If $v \in V(G)$ is adjacent to u then u and v are called *neighbors*. Note that for simple graphs, a vertex is never a neighbor of itself. The *neighborhood* of u, denoted by $N(u)$, is the set of all the neighbors of u in G:

$$N(u) = \{v \in V(G) \,|\, uv \in E(G)\}$$

The *degree* of u, denoted by $\deg_G(u)$ or just $\deg(u)$ if G is unambiguous, is then defined to be the cardinality of its neighborhood: $\deg(u) = |N(u)|$. Thus, the degree of a vertex can be viewed as the number of neighbor vertices or the number of incident edges.

When considering the degrees of all the vertices in a graph, the following limits are helpful:

$$\delta(G) = \min_{v \in V(G)} \deg(v)$$

$$\Delta(G) = \max_{v \in V(G)} \deg(v)$$

Therefore, we can state the conclusion of Proposition 5:

Proposition 5. *Let G be a graph of order n. For every vertex $v \in G$:*

$$0 \leq \delta(G) \leq \deg(v) \leq \Delta(G) \leq n - 1$$

Intuitively, as $\delta(G)$ increases, a graph becomes denser (more edges) resulting in more adjacencies, making it harder to find a proper coloring at lower values of k.

Vertices can be classified based on their degree, as shown in Table 2.

Table 2

Classifying Vertex u in Graph G

$\deg(u)$	TYPE
0	isolated
1	pendant, end, leaf
$n-1$	universal
even	even
odd	odd

Isolated vertices have degree 0; they are not adjacent to any other vertex in G. *Pendant* (also called *end* or *leaf*) vertices have degree 1; they are adjacent to exactly one other vertex in G. *Universal* vertices are adjacent to every other vertex in G. *Even* vertices are adjacent to an even number of vertices in G and *odd* vertices are adjacent to an odd number of vertices in G. Note that if G has a universal vertex then it cannot have an isolated vertex, and vice-versa.

The degrees of the vertices in a graph and the number of edges in the graph are related by the so-called First Theorem of Graph Theory:

Theorem 8 (First Theorem of Graph Theory). *Let G be a graph of size m:*

$$\sum_{v \in V(G)} \deg(v) = 2m$$

Proof. When summing all the degrees, each edge is counted twice: once for each endpoint. ☐

These concepts are demonstrated by the graph in Fig. 21.

vertex	degree	type
v_1	7	universal,odd
v_2	4	even
v_3	4	even
v_4	3	odd
v_5	4	even
v_6	4	even
v_7	3	odd
v_8	1	pendant,odd
total	30	

$n = 8$ $\quad m = 15 = \frac{30}{2}$

$\delta(G) = 1$ $\quad \Delta(G) = 7$
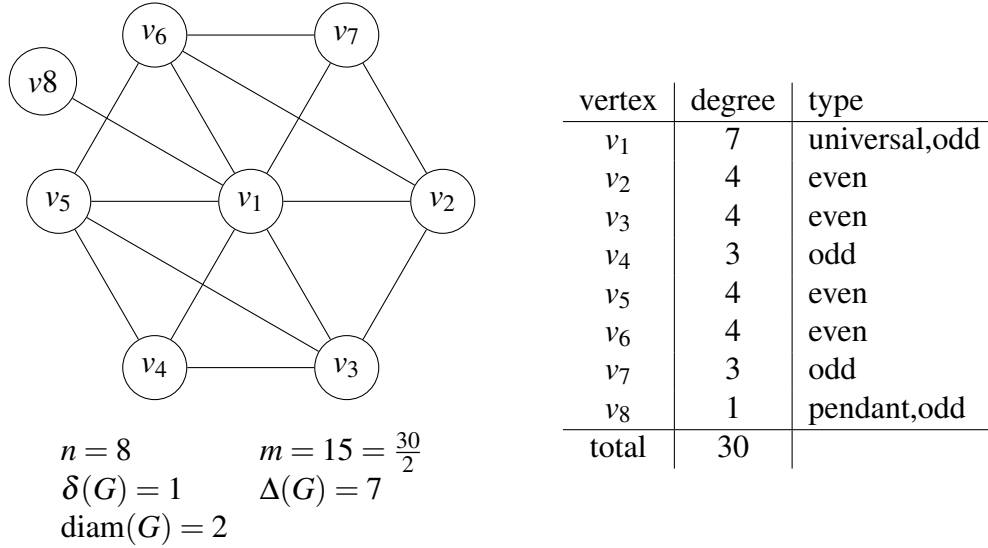
$\text{diam}(G) = 2$

Fig. 21. Vertex degrees and the first theorem of graph theory.

2.9 Special Graphs

The following sections described some special classes of graphs that are important to the execution of the proposed algorithm.

2.9.1 Empty Graphs

An *empty* graph of order n, denoted by E_n, is a graph with one or more vertices ($n > 1$) and no edges ($m = 0$). An empty graph is connected if and only if $n = 1$. Examples of empty graphs are shown in Fig. 22.

E_1 $\qquad$ E_4

E_9
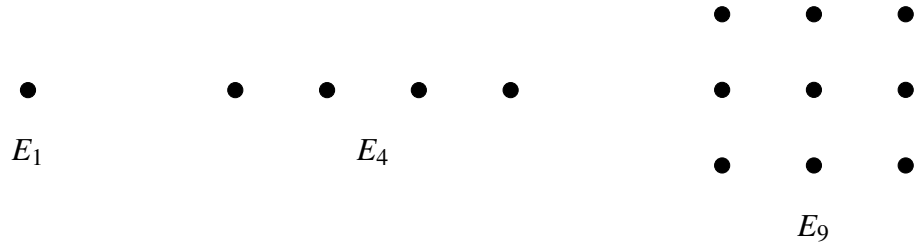
Fig. 22. Empty graphs.

The null graph ($n = 0$) is denoted by E_0 and is defined to be 0-chromatic. All other empty graphs are 1-chromatic and thus are important termination conditions for the proposed algorithm.

2.9.2 Paths

A *path* graph of order n and length $n - 1$, denoted by P_n, is a connected graph consisting of a single open path. Examples of path graphs are shown in Fig. 23.

P_1 $\qquad$ P_4

P_9

Fig. 23. Path graphs.

Note that $P_1 = E_1$ is 1-chromatic, whereas $P_{n>1}$ is 2-chromatic.

Paths are not particularly important to the proposed algorithm; however, they are used in the definition of cycles.

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2.9.3 Cycles

A *cycle* graph of order n and length n for $n \geq 3$, denoted by C_n, is a connected graph consisting of a single closed path. When n is odd then C_n is called an *odd* cycle and when n is even then C_n is called an *even* cycle.

Examples of cycle graphs are shown in Fig. 24.
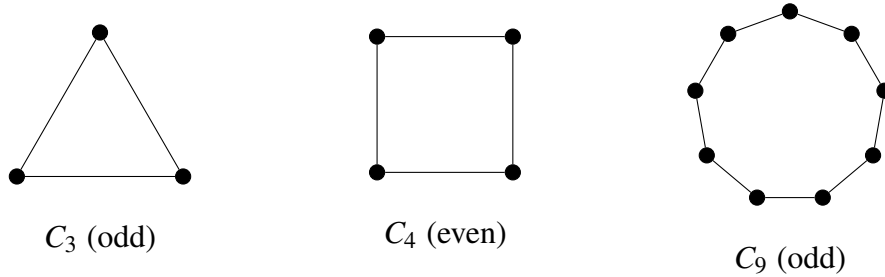
Fig. 24. Cycle graphs.

Note that even cycles are 2-chromatic; however, odd cycles are 3-chromatic.
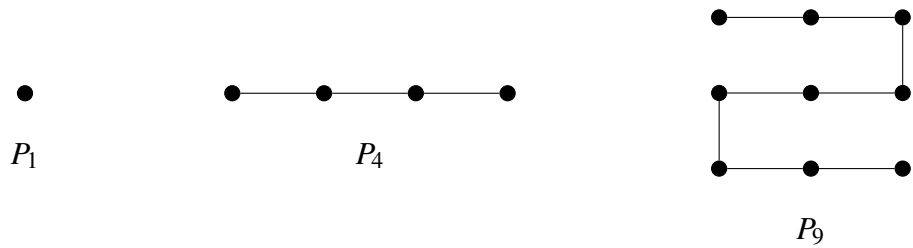
Cycles are not particularly important to the proposed algorithm; however, they are used in the definition of trees, which are important to the later analysis of coloring algorithms.

2.9.4 Complete Graphs

A *complete* graph of order n and size $\frac{n(n-1)}{2}$, denoted by K_n, is a connected graph that contains every possible edge: $E(G) = \mathscr{P}_2(V(G))$. Thus, all of the vertices in a complete graph are universal.

Examples of complete graphs are shown in Fig. 25.
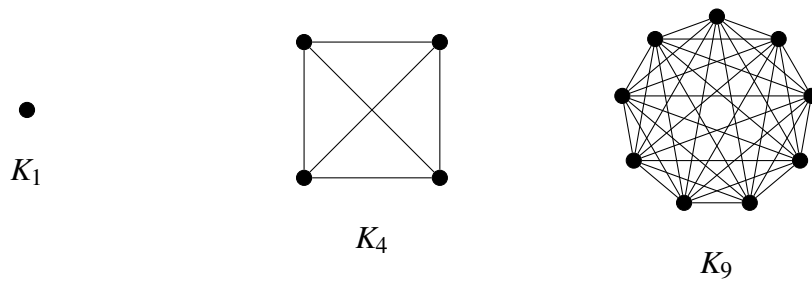
Fig. 25. Complete graphs.

Note that $K_1 = P_1 = E_1$.

Since all of the vertices in a complete graph are adjacent to each other, each vertex requires a separate color in order to achieve a proper coloring. Thus, K_n is n-chromatic and is also an important termination condition for the proposed algorithm.

2.9.5 Trees

A *tree* is a connected graph that contains no cycles as subgraphs. Typically, one vertex of the tree is selected as the *root* vertex and then the tree is depicted in layers that contain vertices that are equidistant from the root vertex. Thus, the bottom layer is composed entirely of pendant vertices, but pendant vertices can exist in the other layers as well. Such pendant vertices are usually referred to as *leaves* in this context.

An example tree is shown in Fig. 26. The root vertex r is shown in red and the leaf vertices b, e, g, h, i, j, k are shown in green.

Fig. 26. A tree organized from root to leaves.

Trees are important because they can be used to track so-called "branch-and-bound" algorithms; each vertex represents a branch choice of the algorithm and thus a particular

state of the problem. All such states can be visited using a so-called *depth-first* walk. In the example in Fig. 26, such a depth-first walk would be:

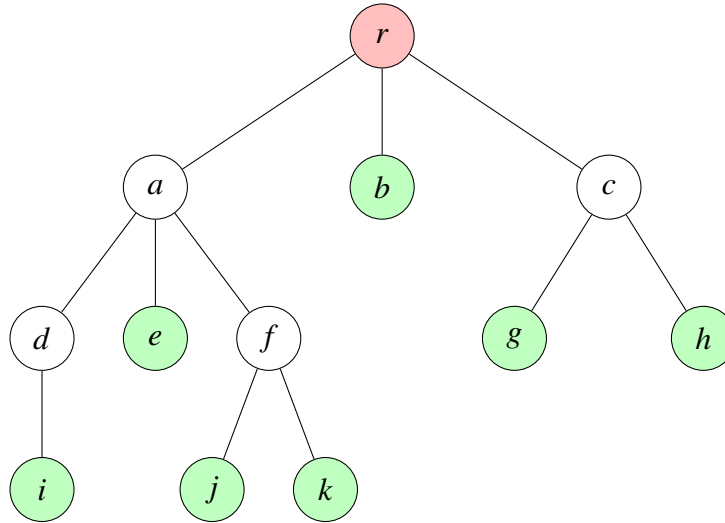$$(r,a,d,i,d,a,e,a,f,j,f,k,f,a,r,b,r,c,g,c,h,c,r)$$

Note that this walk guarantees that each vertex is visited at least once.

When such a tree is applied to the problem of exhaustively finding the chromatic number of a graph via a sequence of vertex contraction and edge addition choices, the tree is called a *Zykov* tree and the algorithm is called a *Zykov* algorithm [7]. Zykov algorithms are described in detail in the following sections. In fact, the proposed algorithm is a variation of a standard Zykov algorithm.

2.10   Cliques

A *clique* is a complete subgraph of a graph. A *maximal* clique is a clique that cannot be extended by an additional vertex. The *maximum* clique is the maximal clique of highest order. The *clique number* of a graph G, denoted by $\omega(G)$, is the order of the maximum clique in G. Consider the example in Fig. 27. G has many 2-cliques; however, none of them are maximal. G has several maximal 3-cliques — for example: $G[\{b,e,g\}]$. There is only one maximal 4-clique: $G[\{a,b,c,d\}]$ and no clique of higher order. Thus, $\omega(G) = 4$.

G
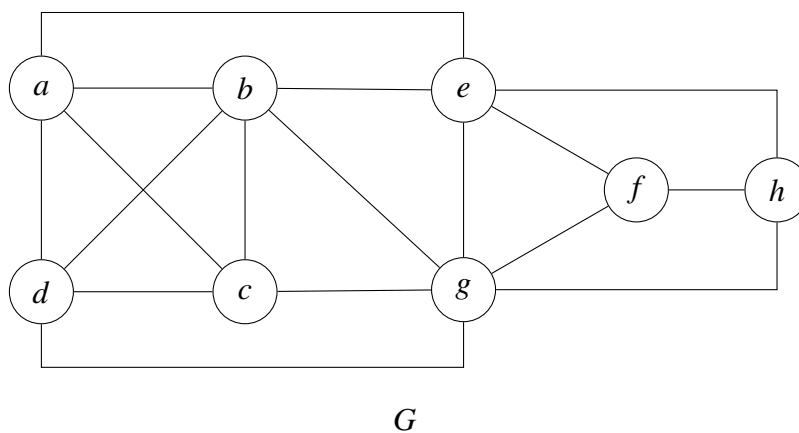
Fig. 27. A graph with cliques.

Since a *k*-clique of a graph *G* needs at least *k* colors in any proper coloring of *G*, the clique number of *G* provides a nice lower bound for the chromatic number of *G*. Unfortunately, the clique number problem is known to be NP-hard as well [13]. Thus, there are many attempts in the literature to find a good lower bound for the clique number of a graph *G*, usually denoted by $\omega'(G)$. If such a lower bound is known then the conclusion of Proposition 6 holds:

Proposition 6. *Let G be a graph with clique number lower bound* $\omega'(G)$*:*

$$\omega'(G) \leq \omega(G) \leq \chi(G)$$

Examples of such algorithms are discussed in detail in later sections, since they are typically used to "bound" Zykov branch-and-bound algorithms.

3 ALGORITHMS AND COMPUTATIONAL COMPLEXITY

The chromatic number problem is *inherently intractable*. Informally, this means that finding a solution for a given input graph, regardless of the means, can take a very, very long time in the worst cases. In fact, the chromatic number problem is a member of a class of problems referred to as being *NP-hard*, meaning that the time to find a solution increases exponentially with the number of vertices. Thus, trying to find an efficient but exact method of solution that satisfies all cases is a fool's errand; about the best that can be done is to perform better than existing well-known methods in most cases [14]. In order to better understand why the proposed algorithm is inherently intractable and how to compare it with existing well-known chromatic number algorithms, this section provides an overview of the computational complexity of algorithms.

3.1 Problems

The various questions asked and answered in this section can be a bit confusing, so it is essential to clearly define the various terms that will be used. A *problem* consists of three parts:

1) A specific question to be answered.
2) A description of zero or more input *parameters*.
3) A statement of the properties that the *solution* is required to satisfy.

An *instance* of a problem is constructed by specifying particular values for each input parameter. The step-by-step procedure that translates the input parameters to a corresponding well-defined solution is called an *algorithm*. To say that an algorithm *solves* a problem Π means that the algorithm produces a valid solution for every possible instance I of the problem Π [14].

Thus, the chromatic number problem accepts a graph G and uses an algorithm to obtain a number $k \in \mathbb{N}$ where k is the minimum value such that G is k-colorable. The proposed algorithm is one such algorithm that can be used to solve the problem. Other

examples are the Christofides (1971) [10] and the Corneil/Graham (1973) [13] algorithms, which are compared to the proposed algorithm in later sections.

3.2   Enigma

A good example of an intractable problem that involved most of the personalities who would make important contributions to the development of a theory for computation occurred in the field of cryptography during WWII. Although cryptography had been important in WWI, the horrors of the new era of warfare instigated by Hitler's Third Reich brought the importance of codebreaking to the forefront. The story of the Polish code breakers and their *bombe* device and the efforts of the British at Bletchley Park using the bombes are now legendary. The problem was to break the German Enigma code: given a set of ciphertext messages, convert them to intelligible cleartext messages. It is in the nature of this problem that the inherent intractability of some problems becomes evident [15].

The Enigma code was a letter-swapping code; however, the swap table changed with each letter. Cleartext messages were encoded using a typewriter-like device with the following characteristics:

1) Three slightly different 26-position (one for each letter) rotors that selected the swap for the current letter in the cleartext. As each character was added to the ciphertext, the rotors would rotate based on their gear mechanisms to select the next swap, thus changing the swap table each time.

2) The ability to initialize the position of each rotor at the start of the message, which acted as a seed for the current cipher.

3) A plugboard where ten pairs of bidirectional manual letter swaps could be specified.

Each of the twenty letters that made up the ten pairs on the plugboard were distinct. Furthermore, the ordering of the pairs was insignificant. Thus, the number of possible

configurations N of the plugboard is given by:

$$
\begin{aligned}
N &= \frac{\binom{26}{2}\binom{24}{2}\binom{22}{2}\binom{20}{2}\binom{18}{2}\binom{16}{2}\binom{14}{2}\binom{12}{2}\binom{10}{2}\binom{8}{2}}{10!} \\
&= \frac{26!24!22!20!18!16!14!12!10!8!}{2^{10}24!22!20!18!16!14!12!10!8!6!10!} \\
&= \frac{26!}{2^{10}6!10!}
\end{aligned}
$$

The German army used a set of five rotors, any three of which could be selected for a day's messages. Thus, the number of possible initial configurations for the army Enigma machine M was:

$$
M = 5 \cdot 4 \cdot 3 \cdot 26^3 \cdot \frac{26!}{2^{10}6!10!} = 158.9 \times 10^{18}
$$

Given a ciphertext message, the bombes would (mechanically) try all possible Enigma initial configurations in an attempt to convert the ciphertext into an intelligible cleartext message (in German). To put this in perspective, an Intel i7 has a speed of about $50,000$ MIPS. Assuming that each one of Enigma's initial configurations could be tried in one instruction means that it would take about 100 years to try all possible combinations. Since it would actually take several thousand instructions to test each possible configuration, the real answer is upwards of 100,000 years — intractable indeed. Furthermore, although the German army used a set of five rotors, the German navy used a set of eight rotors. And in 1942, both changed from a three rotor system to a four rotor system.

Clearly, a day's Enigma configuration needed to be determined within a couple hours in order to make it useful to decode the day's messages. It is at this point that the art of crypography comes into play. The goal was to use techniques like letter frequency and candidate cleartext segments to eliminate enough of the possible initial configurations so that bombes could find the correct configuration in a reasonable amount of time. Thus, there was no "aha" moment where the Enigma code was broken once and for all. Each

new day was a new instance of the problem that required a tremendous manual effort to eliminate possible configurations so that the bombes could try a minimum number of combinations to arrive at the current day's solution.

3.3   Algorithms

The etymology for the word *algorithm* is the Latinized name *Algorithmi* for Muḥammad ibn Mūsā al-Khwārizmī of algebra fame. In Europe, the term algorithmi became synonymous with any well-defined procedure used in mathematical problem solving, such as the division algorithm of the Babylonians, al-Khwārizmī's completing the square, the sieve of Eratosthenes for finding primes, and Euclid's algorithm for find the GCD of two numbers [16].

In today's modern computer age, the term *algorithm* is used without much thought to represent a well-defined and finite (however large) sequence of steps to solve some problem, usually within the context of a computer program. However, practical programmable computers didn't exist until well into the 1950s. In fact, the formal study of algorithms actually began in the mid to late 19th century as part of the effort to formalize mathematical systems. In order to understand why the chromatic number problem is so intractable and how to measure that intractability, it will be helpful to investigate this fairly recent history of algorithms.

In the mid 19th century, the field of mathematics found itself in a bit of a crisis. Although logic was already the foundation of mathematics, there was a feeling that, like Euclid, many of the accepted axioms were making too many assumptions about the involved non-defined terms, ultimately resulting in paradoxes. One of the most famous is the Russell paradox, communicated by British mathematician Bertrand Russell to German mathematician Gottlob Frege circa 1902, just as Frege was about to publish a major work on the set theory foundations of mathematical systems. Frege had used naïve set theory and had made too many assumptions about the nature of collections, an idea that was left

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to intuition instead of definition. This lead to paradoxes involving sets that were members of themselves [16].

The first order of business was to formalize the rules for the algorithms that would be used to solve problems, and this was done by English mathematician George Boole in 1847 and 1856 with his papers that formalized the so-called zero-order logic, which includes: and, or, not, and implication. This work was supplemented by the aforementioned Frege in 1879 and independently by American mathematician Charles Peirce in 1885 when they added quantifiers to Boole's work, resulting in the so-called first-order logic [16].

The modern understanding of algorithms has its roots in attempts by mathematicians of the 1920s and 1930s to formalize mathematical systems. German mathematician David Hilbert begins in 1920 by stating what he felt were requirements for a proper mathematical system based on some language or set of rules:

Completeness. Every proposition that is true can be proven so using the rules of the system.

Consistency. No contradictions arise when following the rules.

Decidability. There exists a method for deciding whether something is true or false.

It is probably true that Hilbert had in mind mathematical systems based on first-order logic, which was the modus operandi of mathematicians of the time. However, Austrian mathematician Kurt Gödel upset the apple cart in 1931 by showing that mathematical systems cannot be proved to be both complete and consistent. Thus, only the third requirement of decideability remained, what was commonly referred to as the *Entscheidunsproblem* [15].

In 1935, British mathematician and early pioneer of programmable computing Max Newman was giving a lecture on the Entscheidunsproblem at Cambridge. Newman put forth of the question of whether there exists a *mechanical* process that would meet the

requirements of decideability. In the audience was a young Alan Turing who would take the challenge for a mechical process to heart. In 1936, Turing would submit his famous paper, "On Computable Numbers, with an Application to the Entscheidungsproblem [17]," which provided a definition of computable functions and a theoretical general-purpose machine to compute such functions: the A-machine, what we know today as the Turing machine. Turing showed that there are some problems that appear to be true; however, there is no way to design a process to prove that they are actually true - or that the original process even terminates with an answer (the halting problem). Most of these types of problems involve inherent contractions such as the Epimenides paradox: Epimenides the Cretan says that all Cretans are liars. [15]. Thus, Hilbert's third pillar was demolished. [15].

It should be noted that American mathematician Alan Church arrived at the same conclusions as Turing a bit earlier with his 1935 paper on lambda calculus. Unlike Newton and Leibniz, Turing and Church were willing collaborators, with Church eventually becoming Turing's PhD advisor at Princeton. Indeed, Turing adopted the use of the lambda calculus in his work [15].

4 THE PROPOSED ALGORITHM

The major advantages of the Zykov and Christofides algorithms described in the previous section are that they don't depend on the connectedness of a graph, an example of a chromatic coloring is readily available, and the fact that the algorithms can be coded rather easily to run on a computer. Their major disadvantage is its their high runtime complexity, which as was shown is inherent to the chromatic number problem.

Thus, the goals of the proposed algorithm are as follows:

1) It should not depend on whether the graph is connected or not.

2) An example of a chromatic coloring should be available.

3) It can be easily coded for execution on a computer.

4) It has better runtime performance than the existing algorithms.

To accomplish these goals, the proposed algorithm loops on successively higher values of k. For each candidate k value, a graph is assumed to be k-colorable and a modified version of a Zykov algorithm is executed to either prove or disprove this assumption. Since a candidate k value is known, certain reversible steps can be applied to mutate G into simpler graphs with equivalent colorability and test for early termination of the current Zykov tree. The first k for which G (or one of its simplifications) is found to be k-colorable is the chromatic number of G.

One slight disadvantage of the proposed algorithm is that whereas the other algorithms readily provide examples of actual chromatic colorings, the proposed algorithm requires a reverse traversal of its reversible steps in order to construct such a coloring. However, as was stated earlier, it is more important during the axiomatic design analytical process to know the minimum number of parts as opposed to an actual FR allocation to those parts.

This algorithm was first proposed by the author and his advisor in collaboration with a team of mechanical engineering researchers from SUNY Buffalo [4]. It accepts a graph G as input, provides $\chi(G)$ as output, and is composed of an outer loop on values of k and a

subroutine called by the outer loop to determine if G is k-colorable. The outer loop and called subroutine are summarized in the following sections. A complete description of the theorems that support the various steps in algorithm and the application of the algorithm to a sample graph then follow.

4.1 Outer Loop

The outer loop accepts a graph G as input and returns $\chi(G)$. It initially checks for some degenerate cases and then loops on increasing values of k. For each value of k, the called subroutine executes a modified Zykov algorithm to determine if G is k-colorable. The first such successful return identifies $\chi(G)$.

The steps of the outer loop are as follows:

1) If $n = 0$ then return 0, thus handling the degenerate case of a null graph.

2) If $m = 0$ then return 1, thus handling the degenerate case of an empty graph.

3) Initialize k to 2.

4) Call the subroutine to determine if G is k-colorable. The subroutine returns a possibly simplified G called G' and a boolean value R that reports the result of the test.

5) If G is k-colorable ($R = $ true) then return k.

6) Replace G with G'. As will be seen, doing this avoids needless reapplication of certain steps in the called subroutine.

7) Increment k.

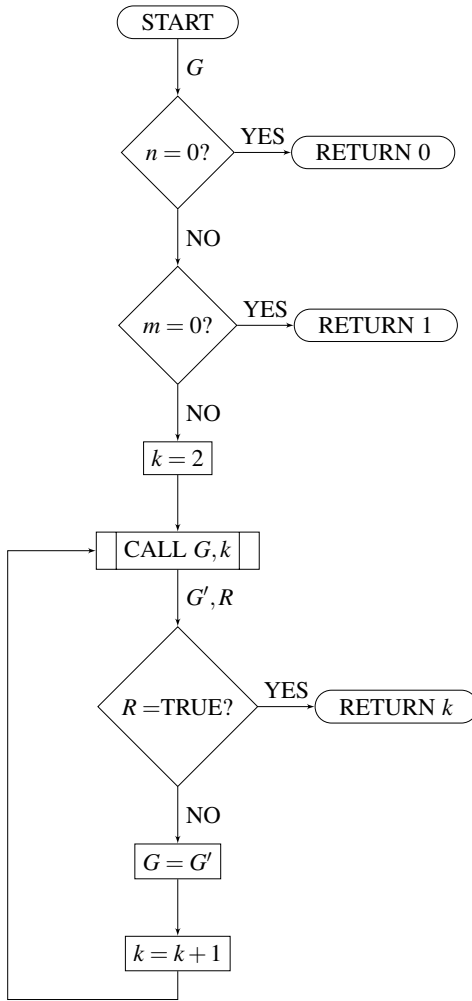8) Go to step 4.

A flowchart of these steps is shown in Fig. 28.

Fig. 28. Proposed algorithm outer loop.

The outer loop is guaranteed to terminate because k will eventually be greater than or equal to n. Thus, by Proposition 3, the current state of G is k-colorable, causing the called subroutine to return true.

4.2 Called Subroutine

The called subroutine executes a modified version of a Zykov algorithm that determines whether a graph is k-colorable. It accepts the current state of G of order n and size m and the current value of $k \geq 2$ as inputs. It returns a possibly simplified version of

G and a boolean value indicating whether or not G is k-colorable. Internally, various tests are applied to trim the corresponding Zykov tree or abandon it all together based on the current value of k.

The steps of the called subroutine and references to their associated theorems are as follows:

1) If $n \leq k$ then return true (Proposition 3).
2) Calculate a maximum edge threshold:
$$a = \frac{n^2(k-1)}{2k}$$
3) If $m > a$ then return false (Corollary 8).
4) Construct the set X of all vertices with degree less than k:
$$X = \{v \in V(G) \,|\, \deg(v) < k\}$$
5) If $X \neq \emptyset$ then replace G with $G - X$ and go to step 1 (Corollary 9).
6) Calculate the common number of neighbors between each pair of vertices in G, stopping if one vertex's neighborhood is found to be a subset of another.
7) If G has vertices u and v such that $N(u) \subseteq N(v)$ then replace G with $G - u$ and go to step 1 (Theorem 11).
8) Let b be the smallest number of common neighbors between any pair of vertices in G as found in step 6:
$$b = \min_{u,v \in V(G)} |N(u) \cap N(v)|$$
9) Calculate an upper bound for the minimum number of common neighbors between any pair of vertices in G:
$$c = n - 2 - \frac{n-2}{k-1}$$
10) If $b > c$ then return false (Corollary 10).

11) Select two non-adjacent vertices $u, v \in V(G)$ with the smallest number of common neighbors as found in step 6. It will be shown below that such a pair of vertices is guaranteed to exist in the current state of G.

12) Assume that u and v are assigned the same color by letting $G' = G \cdot uv$. Recursively call this routine to see if G' is k-colorable. If so, then return true (Theorem 13).

13) Assume that u and v are assigned different colors by letting $G' = G + uv$. Recursively call this subroutine to see if G' is k-colorable. If so, then return true (Theorem 13).

14) Since neither of the assumptions in steps 12 nor 13 hold, conclude that G is not k-colorable and return false.
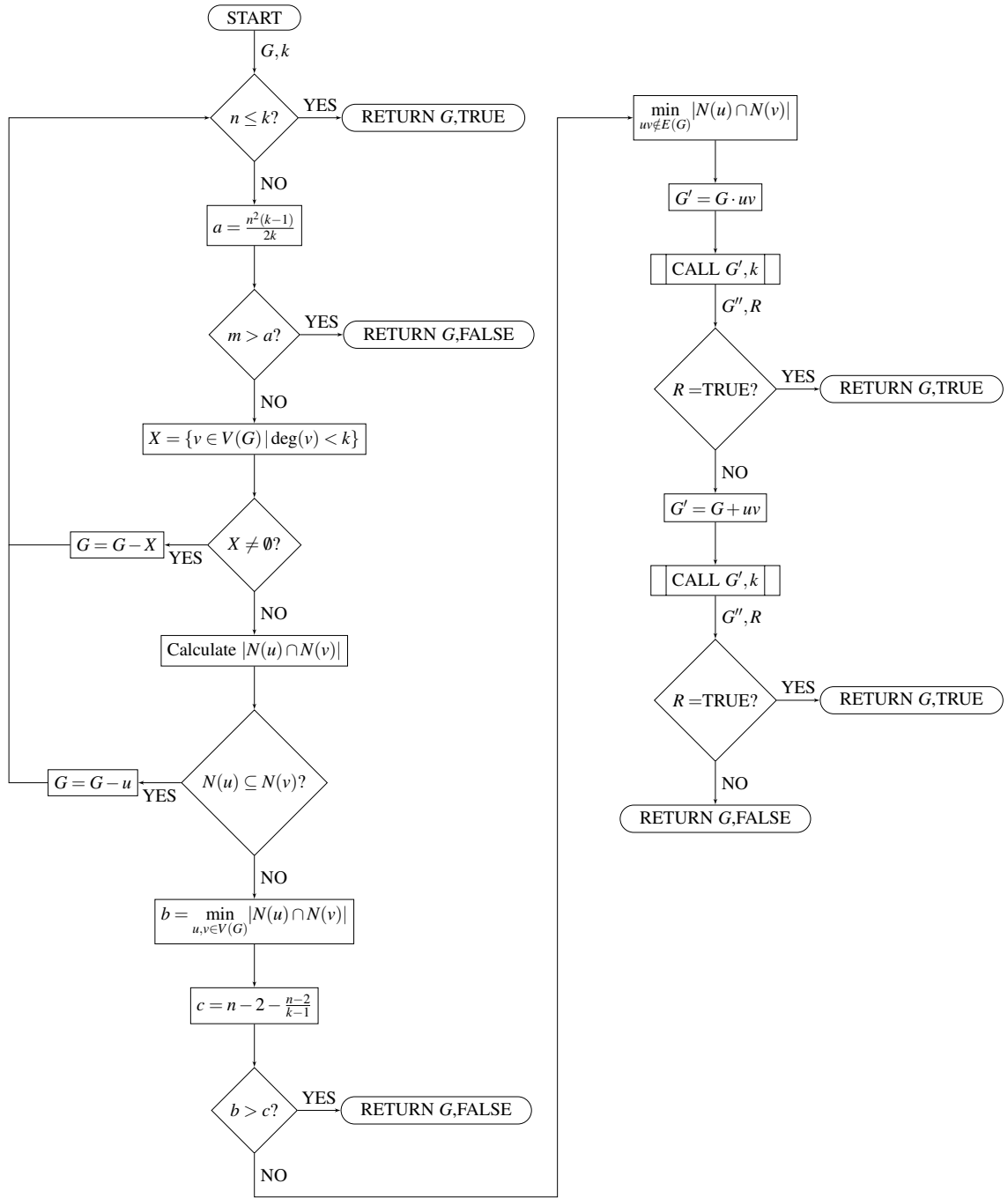
A flowchart of these steps is shown in Fig. 29.

START

G, k

$n \leq k$? —YES→ RETURN G,TRUE

NO

$a = \frac{n^2(k-1)}{2k}$

$m > a$? —YES→ RETURN G,FALSE

NO

$X = \{v \in V(G) \,|\, \deg(v) < k\}$

$X \neq \emptyset$? —YES→ $G = G - X$

NO

Calculate $|N(u) \cap N(v)|$

$N(u) \subseteq N(v)$? —YES→ $G = G - u$

NO

$b = \min_{u,v \in V(G)} |N(u) \cap N(v)|$

$c = n - 2 - \frac{n-2}{k-1}$

$b > c$? —YES→ RETURN G,FALSE

NO

$\min_{uv \notin E(G)} |N(u) \cap N(v)|$

$G' = G \cdot uv$

CALL G', k

G'', R

$R =$ TRUE? —YES→ RETURN G,TRUE

NO

$G' = G + uv$

CALL G', k

G'', R

$R =$ TRUE? —YES→ RETURN G,TRUE

NO

RETURN G,FALSE

Fig. 29. Proposed algorithm called subroutine.

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Step 1 is the success termination condition. Success occurs when G is simplified by removing suffcient vertices (steps 4–7) or when the outer loop has sufficiently incremented k (step 7) such that $n \leq k$.

Steps 4–7 attempt to remove vertices to achieve a simpler graph that is equivalently k-colorable. Each time a vertex is removed, the Zykov branches associated with that vertex are skipped. Since these same steps would just be repeated for $k+1$, the subroutine returns the current state of the possibly simplified G to the outer loop as a starting point for the next candidate value of k.

Steps 2–3 and 8–10 apply tests that attempt to disprove that the current state of G is k-colorable for the current value of k. If so, then the current Zykov tree is abandoned and the subroutine returns false. This allows the outer loop to continue with $k+1$.

The remaining steps of the called subroutine, steps 11–14, constitute the recursive portion of the modified Zykov algorithm. The recursive calls are guaranteed to terminate because either there will be sufficient vertex contractions such that $n \leq k$, resulting in a true return, or sufficient edge additions such that the graph becomes complete and (as will be shown) is rejected by step 3, resulting in a false return. Note that in the event of a false return, any modifications to the current state of G resulting from the recursive calls are not returned to the outer loop.

4.3 Supporting Theorems

This section contains the theorems that support the steps in the called subroutine. Remember that the success check of step 1 is already supported by Proposition 3.

4.3.1 Maximum Edge Threshold

The maximum edge threshold test of steps 2 and 3 is supported by Theorem 9.

Theorem 9 (Maximum Edge Threshold). *Let G be a graph of order n and size m and let $k \in \mathbb{N}$. If G is k-colorable then:*

$$m \leq \frac{n^2(k-1)}{2k}$$

Proof. Assume that G is k-colorable. This means that $V(G)$ can be distributed into k independent (some possibly empty) subsets. Call these subsets $A_1, \ldots A_k$ and let $a_i = |A_i|$. Thus, each $v \in A_i$ can be adjacent to at most $n - a_i$ other vertices in G, and hence the maximum number of edges incident to vertices in A_i is given by: $a_i(n - a_i) = na_i - a_i^2$. Now, using Theorem 8, the maximum number of edges in G is given by:

$$m \leq \frac{1}{2} \sum_{i=1}^{k} (na_i - a_i^2)$$

with the constraint:

$$\sum_{i=1}^{k} a_i = n$$

This problem can be solved using the Lagrange multiplier technique. We start by defining:

$$F(a_1, \ldots, a_k) = f(a_1, \ldots, a_k) - \lambda g(a_1, \ldots, a_k)$$
$$= \frac{1}{2} \sum_{i=1}^{k} (na_i - a_i^2) - \lambda \sum_{i=1}^{k} a_i$$
$$= \sum_{i=1}^{k} \left(\frac{1}{2} na_i - \frac{1}{2} a_i^2 - \lambda a_i \right)$$

Now, optimize by taking the gradient and setting the resulting vector equation equal to the zero vector:

$$\nabla F = \sum_{i=1}^{k} (\frac{n}{2} - a_i - \lambda) \hat{a}_i = \mathbf{0}$$

This results in a system of k equations of the form:

$$\frac{n}{2} - a_i - \lambda = 0$$

And so:

$$a_i = \frac{n}{2} - \lambda$$

Plugging this result back into the contraint:

$$\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} \left(\frac{n}{2} - \lambda \right) = k \left(\frac{n}{2} - \lambda \right) = n$$

Solving for λ yields:

$$\lambda = \frac{n}{2} - \frac{n}{k}$$

And finally, to get a_i in terms of n and k:

$$a_i = \frac{n}{2} - \left(\frac{n}{2} - \frac{n}{k} \right) = \frac{n}{k}$$

Therefore:

$$m \leq \frac{1}{2} \sum_{i=1}^{k} \left[n \left(\frac{n}{k} \right) - \left(\frac{n}{k} \right)^2 \right] = \frac{k}{2} \left(\frac{n^2 k - n^2}{k^2} \right) = \frac{n^2(k-1)}{2k}$$

$\square$

The called subroutine actually uses the contrapositive of this result, as stated in Corollary 8.

Corollary 8. *Let G be a graph of order n and size m and let $k \in \mathbb{N}$. If:*

$$m > \frac{n^2(k-1)}{2k}$$

then G is not k-colorable.

Corollary 8 is demonstrated by Fig. 30. The shown graph G has $n = 4$, $m = 5$, and $\chi(G) = 3$. Testing for $k = 2$:

$$a = \frac{4^2(2-1)}{2 \cdot 2} = 4$$

But $m = 5 > 4 = a$ and so we can conclude that G is not 2-colorable. However, testing for $k = 3$;

$$a = \frac{4^2(3-1)}{2 \cdot 3} = 5.3$$

So $m = 5 \ngtr 5.3 = a$ and thus *G may* be 3-colorable, since this test only provides a necessary and not a sufficient condition.
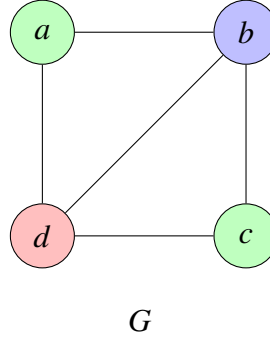

G

Fig. 30. Corollary 8 example.

In fact, the the test of Corollary 8 will always fail for a complete graph when $k < n$. Since $k, n > 0$:

$$
\begin{aligned}
\frac{n(n-1)}{2} - \frac{n^2(k-1)}{2k} &= \frac{kn(n-1) - n^2(k-1)}{2k} \\
&= \frac{kn^2 - kn - kn^2 + n^2}{2k} \\
&= \frac{n^2 - kn}{2k} \\
&= \frac{n(n-k)}{2k} \\
&> 0 \qquad (n > k)
\end{aligned}
$$

4.3.2 Vertex Removal

The theorems that support vertex removal make use of Lemma 1.

Lemma 1. *Let G be a graph and let $v \in V(G)$. If G is k-colorable then $G - v$ is also k-colorable.*

Proof. Assume that G is k-colorable. Let $c : V(G) \to C$ be such a coloring, and so $|C| = k$. Intuitively, removing v from G should not affect the proper coloring of the

remaining vertices. Thus, we should be able to construct a proper coloring for $G - v$ based upon c. So consider the restricted coloring function $c' = c|_{V(G-v)}$ and assume $uw \in E(G - v)$. Since c is proper:

$$c'(u) = c(u) \neq c(w) = c'(w)$$

Thus, c' is a proper coloring of $G - v$ using at most k colors.

Therefore $G - v$ is k-colorable. ☐

Lemma 1 is demonstrated in Fig. 31. No matter which vertex is removed, the resulting subgraph is still properly colored using at most four (in fact, three) colors.
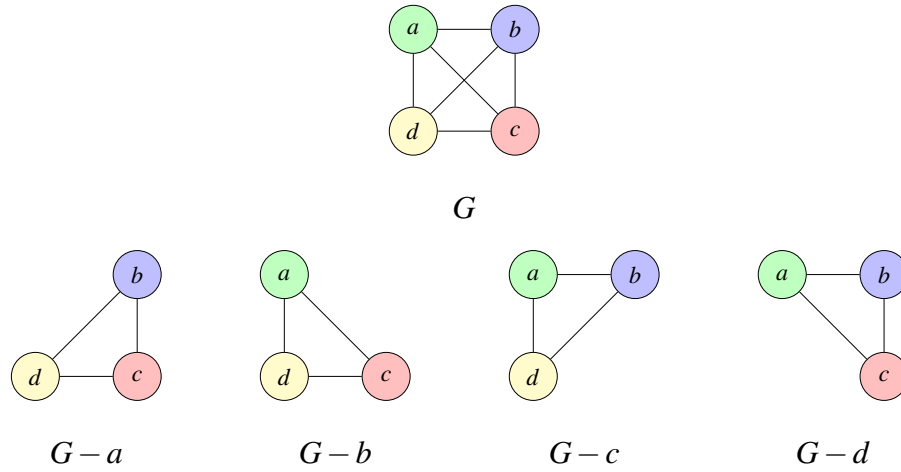
Fig. 31. Lemma 1 example.

Steps 4 and 5 remove vertices with degrees less than k. This is supported by Theorem 10.

Theorem 10. *Let G be a graph and let $v \in V(G)$ such that $\deg(v) < k$ for some $k \in \mathbb{N}$. G is k-colorable if and only if $G - v$ is k-colorable.*

Proof. Assume that G is k-colorable. Therefore, by Lemma 1, $G - v$ is also k-colorable.

For the converse, assume that $G - v$ is k-colorable. Let $c : V(G - v) \rightarrow C$ be such a coloring, and so $|C| = k$. By assumption, $\deg(v) < k$, so v has at most $k - 1$ neighbors in

G, using at most $k-1$ colors. This means that there should be an additional color that can be assigned to v in G such that the coloring remains proper. So let $N(v) = \{v_1, \ldots, v_r\} \subseteq V(G-v)$ for some $r < k$, and let $c[N(v)] = \{c_1, \ldots, c_s\} \subset C$ for some $s \le r < k$. Since $c[N(v)]$ is a proper subset of C, select $c_k \in C - c[N(v)]$ and define $c' : V(G) \to C$ as follows:

$$c'(u) = \begin{cases} c(u), & u \ne v \\ c_k, & u = v \end{cases}$$

Now, assume that $uw \in E(G)$ and consider the following two cases:

Case 1: $v \notin uw$

Since c is proper:

$$c'(u) = c(u) \ne c(w) = c'(w)$$

Case 2: $v \in uw$

Assume without loss of generality (AWLOG) that $u = v$. This means that $c'(v) = c_k$ and $c'(w) = c(w) \in c[N(v)]$. But $c_k \notin c[N(v)]$ and so $c'(v) \ne c'(w)$.

Thus, c' is a proper coloring of G using at most k colors.

Therefore G is k-colorable. $\qquad\square$

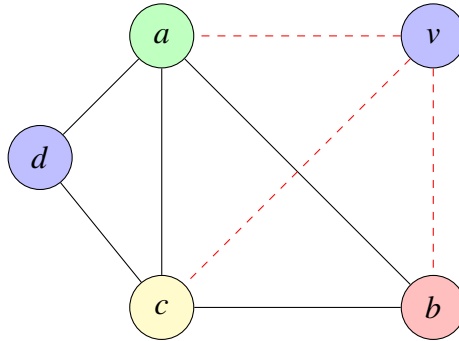Theorem 10 is demonstrated in Fig. 32 for $k = 4$ and $\deg(v) = 3$.

Fig. 32. Theorem 10 example.

The called subroutine actually removes all such vertices at once, which is supported by the inductive proof in Corollary 9.

Corollary 9. *Let G be a graph of order n and let $X = \{v \in V(G) \,|\, \deg(v) < k\}$ for some $k \in \mathbb{N}$. G is k-colorable if and only if $G - X$ is k-colorable.*

Proof. (by induction on $|X|$)

Base Case: Let $|X| = 0$.

But $G - X = G$ (trivial case).

Inductive Assumption: Let $|X| = r$.

Assume that G is k-colorable if and only if $G - X$ is k-colorable.

Inductive Step: Consider $|X| = r + 1$.

Since $|X| = r + 1 > 0$, there exists $v \in X$ such that $\deg(v) < k$. Let $Y = X - \{v\}$ and note that $|Y| = |X| - 1 = (r+1) - 1 = r$. So, G is k-colorable if and only if $G - v$ is k-colorable (Theorem 10) if and only if $(G - v) - Y$ is k-colorable (inductive assumption).

Therefore, by the principle of induction, G is k-colorable if and only if $G - X$ is k-colorable. $\qquad\square$

Returning to the example in Fig. 32, note that $X = \{v, b\}$ is the set of all vertices with degree less than 4 and so both could be removed at once in accordance with Corollary 9. Furthermore, after these vertices are removed, the remaining vertices a, b, and c will all have degree 2. Since $2 < 4$, the remaining vertices are subsequently removed, leaving $n = 0 < 4$, indicating that the graph is indeed 4-colorable. This iterative collapsing of a graph is an ideal situation.

Step 7 removes vertices whose neighborhoods are subsets of other vertices. This is supported by Theorem 11.

Theorem 11. *Let G be a graph and let $u, v \in V(G)$ such that $N(u) \subseteq N(v)$. G is k-colorable if and only if $G - u$ is k-colorable.*

Proof. Assume that G is k-colorable. Therefore, by Lemma 1, $G - u$ is also k-colorable.

For the converse, assume that $G - u$ is k-colorable. Let $c : V(G - u) \to C$ be such a coloring, and so $|C| = k$. By definition, $u \notin N(u)$, and since, by assumption, $N(u) \subseteq N(v)$, it is also the case that $u \notin N(v)$. Hence, u is not adjacent to v in G. But everything that is adjacent to u in G is also adjacent to v in both $G - u$ and G. Since everything adjacent to v has a different color than v, we should be able to assign u the same color as v in G in order to get a proper coloring of G.

Let $S = N(u) \cap N(v) = \{w_1, \ldots w_r\} \subset V(G - u)$ for some $r = \deg(u) \leq \deg(v)$. For all $w \in S$, $w \in N(v)$ meaning $vw \in E(G - u)$. And, because c is proper, it must be the case that $c(w) \neq c(v)$. So define $c' : V(G) \to C$ as follows:

$$
c'(w) = \begin{cases} c(w), & w \neq u \\ c(v), & w = u \end{cases}
$$

Now, assume that $wz \in E(G)$ and consider the following two cases:

Case 1: $u \notin wz$

Since c is proper:

$$c'(w) = c(w) \neq c(z) = c'(z)$$

Case 2: $u \in wz$

Assume without loss of generality (AWLOG) that $u = w$. Since $vz \in E(G)$ and c is proper:

$$c'(u) = c(v) \neq c(z) = c'(z)$$

Thus, c' is a proper coloring of G using at most k colors.

Therefore G is k-colorable. □

Theorem 11 is demonstrated in Fig. 33. Since $N(u) \subseteq N(v)$, G and $G - u$ are equivalently colorable. Futhermore, once u is removed, the degrees of vertices a and c will have degree 2. So if $k = 3$, those two vertices are subsequently removed by step 5. The remaining graph is of order 2, which then passes the success check of step 1 because $2 < 3$, so indeed the graph is 3-colorable. Once again, this iterative removal of vertices is very powerful.
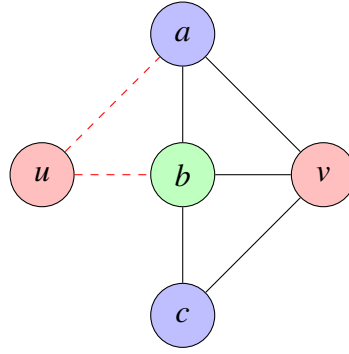

Fig. 33. Theorem 11 example.

4.3.3 Minimum Common Neighbor Upper Bound

Steps 8–10 establish an upper bound for the minimum common neighbor count between any two vertices in a graph that is assumed to be k-colorable. This limit is dependent on the following facts that are guaranteed by previous steps:

1) $2 \leq k < n$

2) There are no $u, v \in V(G)$ such that $N(u) \subseteq N(v)$

The supporting theorem uses these facts along with Lemma 2 in its proof.

Lemma 2. *Let G be a graph and let S be a non-empty independent subset of $V(G)$. If there exists a vertex $v \in S$ such that v is adjacent to all vertices in $V(G) - S$ (i.e., $N(v) = V(G) - S$) then for all vertices $u \in S$ it is the case that $N(u) \subseteq N(v)$.*

Proof. Assume that such a v exists and then assume that $u \in S$. If $u = v$ then (trivially) $N(v) = N(v)$, so assume $u \neq v$. Furthermore, since $u, v \in S$ and S is independent (by assumption), it must be the case that u and v are not neighbors.

Case 1: $N(u) = \emptyset$.

Therefore, by definition, $N(u) = \emptyset \subseteq N(v)$.

Case 2: $N(u) \neq \emptyset$.

Assume that $w \in N(u)$. This means that w is adjacent to u and hence $w \notin S$, since S is an independent set. So $w \in V(G) - S$ and thus, by assumption, v is adjacent to w and we can conclude that $w \in N(v)$. Therefore $N(u) \subseteq N(v)$.

Therefore, for all $u \in S$, $N(u) \subseteq N(v)$. $\qquad\qquad\square$

Lemma 2 is demonstrated in Fig. 34. Note that since $v \in S$ is adjacent to every vertex in $V(G) - S$, vertex $u \in S$ can't help but be adjacent to some subset of $N(v)$.
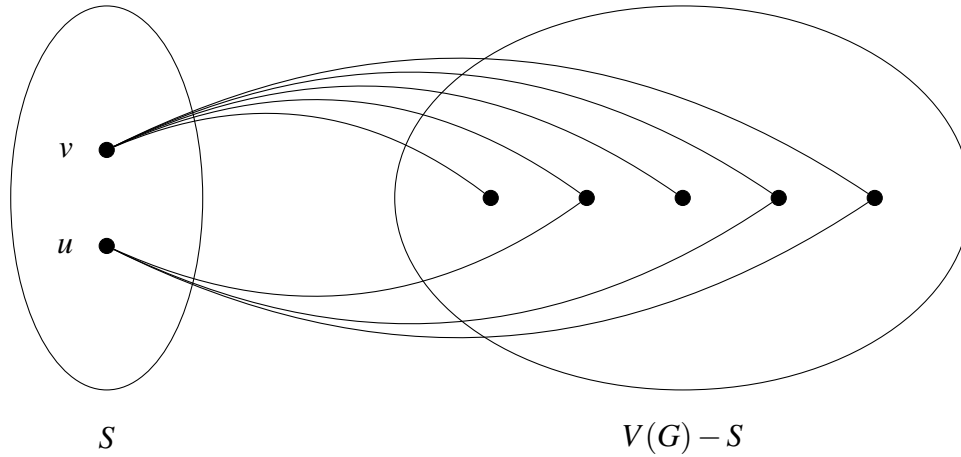

Fig. 34. Lemma 2 example.

Theorem 12 establishes the desired upper bound.

Theorem 12. *Let G be a graph of order n and size m such that there are no $u, v \in V(G)$ where $N(u) \subseteq N(v)$, and let $k \in \mathbb{N}$ such that $2 \leq k < n$. If G is k-colorable then there*

exists two vertices $w, z \in V(G)$ *such that:*

$$|N(w) \cap N(z)| \leq n - 2 - \frac{n-2}{k-1}$$

Proof. Assume that G is k-colorable. This means that $V(G)$ can be distributed into k independent (some possibly empty) subsets $A_1, \ldots, A_k$ such that $a_i = |A_i|$ and $a_1 \geq a_2 \geq \cdots \geq a_k$. Since $n > k$, by the pigeonhole principle, it must be the case that $a_1 \geq 2$. Assume that $v \in A_1$.

First, assume by way of contradiction (ABC) that v is adjacent to all other vertices in $V(G) - A_1$. Since $a_1 \geq 2$, there exists $u \in A_1$ such that $u \neq v$ and u is not adjacent to v. Thus, by Lemma 2, $N(u) \subseteq N(v)$, contradicting the assumption. Note that this contradiction also eliminates the degenerate case where $A_1 = V(G)$; however, this case does not occur here because the graph would be an empty graph and would have been eliminated by previous steps. Therefore, there exists some $v' \in V(G) - A_1$ such that v is not adjacent to v'. Assume that $v' \in A_i$ for some i such that $1 < i \leq k$:

Case 1: $a_i = 1$

By the pigeonhole principle:

$$a_1 \geq \left\lceil \frac{n-1}{k-1} \right\rceil \geq \frac{n-1}{k-1}$$

Now, assume by way of contradiction (ABC) that v' is adjacent to all vertices in $V(G) - A_1 - A_i$ and assume $u \in N(v)$. Then it must be the case that $u \in V(G) - A_1 - A_i$, and so u is adjacent to v', and thus $u \in N(v')$. Therefore $N(v) \subseteq N(v')$, which contradicts the assumption. This situation is demonstrated by Fig. 35.
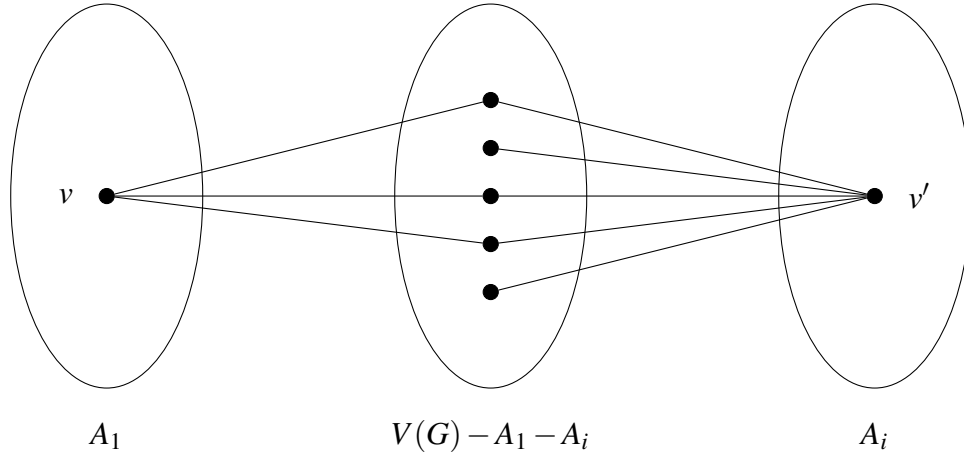
A_1 $V(G) - A_1 - A_i$ A_i

Fig. 35. Case $a_i = 1$ contradiction.

So there must exist some $u \in V(G) - A_1 - A_i$ such that u is not adjacent to v'. This results in the upper bound:

$$|N(v) \cap N(v')| \leq n - |\{u, v'\}| - a_1 \leq n - 2 - \frac{n-1}{k-1}$$

Note that since $v \in A_1$, it is already counted in a_1. Comparing this bound to the desired bound:

$$\left(n - 2 - \frac{n-2}{k-1}\right) - \left(n - 2 - \frac{n-1}{k-1}\right) = \frac{(n-1) - (n-2)}{k-1} = \frac{1}{k-1} > 0$$

for $k \geq 2$. Thus the new bound is tighter and so:

$$|N(v) \cap N(v')| \leq n - 2 - \frac{n-1}{k-1} \leq n - 2 - \frac{n-2}{k-1}$$

Case 2: $a_i = 2$

By the pigeonhole principle:

$$a_1 \geq \left\lceil \frac{n-2}{k-1} \right\rceil \geq \frac{n-2}{k-1}$$

This results in the upper bound:

$$|N(v) \cap N(v')| \le n - a_i - a_1 \le n - 2 - \frac{n-2}{k-1}$$

Case 3: $a_i \ge 3$

By the pigeonhole principle:

$$a_1 \ge \left\lceil \frac{n-3}{k-1} \right\rceil \ge \frac{n-3}{k-1}$$

This results in the upper bound:

$$|N(v) \cap N(v')| \le n - a_i - a_1 \le n - 3 - \frac{n-3}{k-1}$$

Comparing this to the desired bound:

$$\left(n - 2 - \frac{n-2}{k-1}\right) - \left(n - 3 - \frac{n-3}{k-1}\right) = 1 - \frac{(n-3)-(n-2)}{k-1} = 1 - \frac{1}{k-1} > 0$$

for $k \ge 2$. Thus the new bound is tighter and so:

$$|N(v) \cap N(v')| \le n - 3 - \frac{n-3}{k-1} \le n - 2 - \frac{n-2}{k-1}$$

Therefore, there exists $w, v \in V(G)$ such that:

$$|N(w) \cap N(z)| \le n - 2 - \frac{n-2}{k-1}$$

$\square$

The called subroutine actually uses the contrapositive of this result, as stated in Corollary 10.

Corollary 10. *Let G be a graph of order n and size m such that there are no $u, v \in V(G)$ where $N(u) \subseteq N(v)$, and let $k \in \mathbb{N}$ such that $2 \le k < n$. If for all $w, z \in V(G)$ it is the case*

that:

$$|N(w) \cap N(z)| > n - 2 - \frac{n-2}{k-1}$$

then G is not k-colorable.

Corollary 10 is demonstrated in Fig. 36. The shown graph has $n = 5$, is 3-chromatic, and has:

$$\min_{u,v \in V(G)} |N(u) \cap N(v)| = 1$$

Testing for $k = 2$:

$$5 - 2 - \frac{5-2}{2-1} = 0$$

But $1 > 0$ and so we can conclude that G is not 2-colorable. However, testing for $k = 3$:

$$5 - 2 - \frac{5-2}{3-1} = \frac{3}{2}$$

So $1 \not> \frac{3}{2}$ and thus G *may* be 3-colorable, since this test only provides a necessary and not a sufficient condition.

G
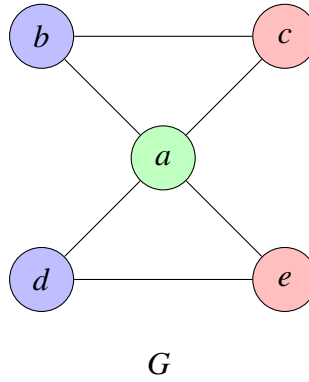
Fig. 36. Corollary 10 example.

4.3.4 Recursive Steps

If nothing more can be done in the preceding steps then steps 11–14 revert to branching. Step 11 selects two non-adjacent vertices with the smallest number of common neighbors. Such a pair must exist. Otherwise, the current state of G is complete, which would have been eliminated by step 3. The first recursive call (step 12) assumes that the two selected vertices have the same color, so they are contracted. The second recursive call (step 13) assumes that the two selected vertices have different colors, so they are joined by an added edge. Each call starts a new branch of the Zykov tree corresponding to the current value of k. If either call returns true then it can be concluded that the input graph was indeed k-colorable. Otherwise, it can be concluded that the input graph is not k-colorable and the called subroutine returns the state of G prior to the recursive calls to the outer loop.

These steps are supported by Theorem 13.

Theorem 13. *Let G be a graph of order $n \geq 2$ and let $u, v \in G$ such that u and v are not adjacent. G is k-colorable if and only if $G \cdot uv$ or $G + uv$ is k-colorable.*

Proof. Assume that G is k-colorable. Let $c : V(G) \to C$ be such a coloring, and so $|C| = k$. There are two possibilities, corresponding to the two recursive choices:

Case 1: *u and v have the same color: $c(u) = c(v)$.*

Let $c_{uv} = c(u) = c(v) \in C$. For all $w \in N(u) \cup N(v)$, since w is adjacent to u and v and because c is proper, it must be the case that w is a different color than the color of u and v: $c(w) \neq c_{uv}$. Let v' be the contracted vertex, so that $N(v') = N(u) \cup N(v)$ and assign color c_{uv} to v'. Define $c' : V(G \cdot uv) \to C$ as follows:

$$c'(w) = \begin{cases} c(w), & w \neq v' \\ c_{uv}, & w = v' \end{cases}$$

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Now, assume that $wz \in E(G \cdot uv)$ and consider the following two cases:

Case a: $v' \notin wz$

Since c is proper:

$$c'(w) = c(w) \neq c(z) = c'(z)$$

Case b: $v' \in wz$

Assume without loss of generality that $v' = w$. Since $z \in N(v')$:

$$c'(v') = c_{uv} \neq c(z) = c'(z)$$

Thus, c' is a proper coloring of $G \cdot uv$ using at most k colors.

Therefore $G \cdot uv$ is k-colorable.

Case 2: u and v have the different colors: $c(u) \neq c(v)$.

By adding edge uv, u and v become adjacent and thus must have different colors.

Thus, u and v can retain their same colors. So define $c' : V(G + uv) \to C$ as follows:

$$c'(w) = c(w)$$

Now, assume $wz \in E(G)$. Since c is proper:

$$c'(w) = c(w) \neq c(z) = c'(z)$$

Thus, c' is a proper coloring of $G + uv$ using at most k colors.

Therefore $G + uv$ is k-colorable.

Therefore $G \cdot uv$ or $G + uv$ is k-colorable.

For the converse, assume $G \cdot uv$ or $G + uv$ is k-colorable. Again, there are two cases:

Case 1: $G \cdot uv$ is k-colorable.

Let $c : V(G \cdot uv) \to C$ be such a coloring, and so $|C| = k$. Let v' be the contracted vertex, and so $N(v') = N(u) \cup N(v)$. Let $c(v') = c_{uv} \in C$. For all $w \in N(v')$ in $G \cdot uv$, since w and v' are adjacent, they must have different colors: $c(w) \neq c_{uv}$. Consider u and v in G. Since they are not adjacent, they can be assigned the same color. So, defined $c' : V(G) \to C$ as follows:

$$c'(w) = \begin{cases} c(w), & w \neq u, v \\ c_{uv}, & w = u \\ c_{uv}, & w = v \end{cases}$$

Now, assume $wz \in E(G)$ and consider the following cases:

Case a: $uv = wz$

This is not possible since, by assumption, u is not adjacent to v.

Case b: $u \in wz$ and $v \notin wz$

Assume without loss of generality (AWLOG) that $u = w$. Since $z \in N(u)$ in G, it must be the case that $z \in N(v')$ in $G \cdot uv$. And so:

$$c'(u) = c_{uv} \neq c(w) = c'(w)$$

Case c: $u \notin wz$ and $v \in wz$

Assume without loss of generality (AWLOG) that $v = w$. Since $z \in N(v)$ in G, it must be the case that $z \in N(v')$ in $G \cdot uv$. And so:

$$c'(v) = c_{uv} \neq c(w) = c'(w)$$

Case d: $u, v \notin wz$

Since c is proper:

$$c'(w) = c(w) \neq c(z) = c'(z)$$

Thus, c' is a proper coloring of G using at most k colors.

Case 2: $G + uv$ is k-colorable.

Let $c : V(G + uv) \to C$ be such a coloring, and so $|C| = k$. Since u and v are adjacent in $G + uv$, they must have different colors. Once uv is removed in G, u and v are no longer adjacent and so there are no requirements on their colors. Thus, they can retain their original colors from $G + uv$. So define $c' : V(G) \to C$ as follows:

$$c'(w) = c(w)$$

Now assume $wz \in E(G)$. Since c is proper:

$$c'(w) = c(w) \neq c(z) = c'(z)$$

Thus, c' is a proper coloring of G using at most k colors.

Therefore G is k-colorable. $\square$

4.4 An Example

In this section, the proposed algorithm will be applied to the example graph G of order $n = 8$ and size $m = 12$ shown in Fig. 37 in order to determine $\chi(G)$. The steps of the algorithm are then traversed in reverse order to determine a chromatic coloring.
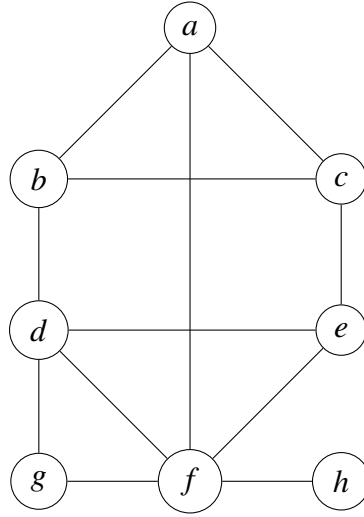
Fig. 37. An example graph.

The algorithm steps are as follows. Outer loop steps are marked by "O#" and called subroutine steps are marked by "I#.". Recursive subroutine calls add a call level: "I#–#."

1) (O1) Since $n = 8 > 0$, G is not the null graph, so continue.

2) (O2) Since $n = 8 > 1$, G is not an empty graph, so continue.

3) (O3) Initialize k to 2.

4) (O4) Call the subroutine with G and $k = 2$.

5) (I1) $n = 8 \not\leq k = 2$, so continue.

6) (I2) Calculate the maximum edge threshold for $n = 8$ and $k = 2$:

$$a = \frac{n^2(k-1)}{2k} = \frac{8^2(2-1)}{2 \cdot 2} = \frac{64}{4} = 16$$

7) (I3) Since $m = 12 \not> 16 = a$, continue.

8) (I4) Since $\deg(h) = 1 < 2$, set $X = \{h\}$.

9) (I5) Since $X \neq \emptyset$, replace G with $G - X$. The result is shown in Fig. 38. Now $n = 7$ and $m = 11$.
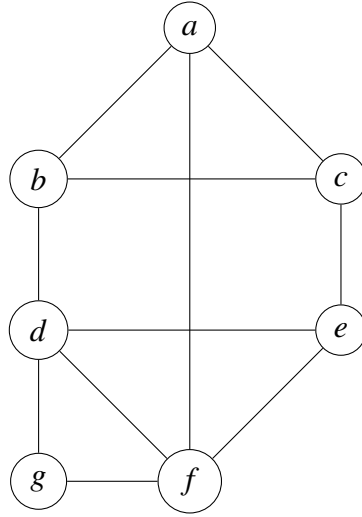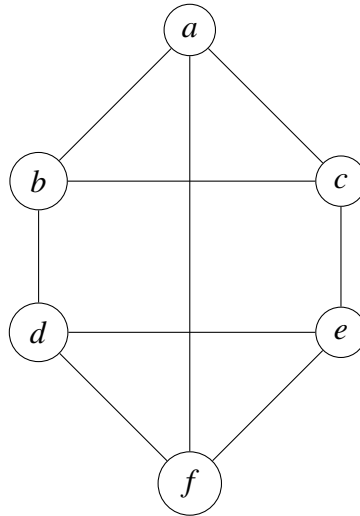
Fig. 38. Step: $G - \{h\}$.

10) (I1) $n = 7 \nleq k = 2$, so continue.

11) (I2) Calculate the maximum edge threshold for $n = 7$ and $k = 2$:

$$a = \frac{n^2(k-1)}{2k} = \frac{7^2(2-1)}{2 \cdot 2} = \frac{49}{4} \approx 12.3$$

12) (I3) Since $m = 12 \ngtr 12.3 = a$, continue.

13) (I4) Since $(G) = 2 \geq 2 = k$, set $X = \emptyset$.

14) (I5) Since $X = \emptyset$, continue.

15) (I6) Calculate the number of common neighbors, looking for any subsets. The results are shown in Table 3.

Table 3

Calculating Common Neighbors 1

u	v	$|N(u) \cap N(v)|$	subset?	adjacent?
a	b	1	N	Y
a	c	1	N	Y
a	d	2	N	N
a	e	2	N	N
a	f	0	N	Y
a	g	1	N	N
b	c	1	N	Y
b	d	0	N	Y
b	e	2	N	N
b	f	2	N	N
b	g	1	N	N
c	d	2	N	N
c	e	0	N	Y
c	f	2	N	N
c	g	0	N	N
d	e	1	N	Y
d	f	2	N	Y
d	g	1	N	Y
e	f	1	N	Y
e	g	2	Y	N
f	g	1	N	Y

16) (I7) Note that $N(g) = \{d, f\}$ and $N(e) = \{c, d, f\}$. Since $N(g) \subseteq N(e)$, replace G with $G - g$. The result is shown in Fig. 39. Now $n = 6$ and $m = 9$.

Fig. 39. Step: $G - g$.

17) (I1) $n = 6 \nleq k = 2$, so continue.

18) (I2) Calculate the maximum edge threshold for $n = 6$ and $k = 2$:

$$a = \frac{n^2(k-1)}{2k} = \frac{6^2(2-1)}{2 \cdot 2} = \frac{36}{4} = 9$$

19) (I3) Since $m = 9 \ngtr 9 = a$, continue.

20) (I4) Since $(G) = 2 \geq 2 = k$, set $X = \emptyset$.

21) (I5) Since $X = \emptyset$, continue.

22) (I6) Calculate the number of common neighbors, looking for any subsets. The results are shown in Table 4.

Table 4

Calculating Common Neighbors 2

u	v	$|N(u) \cap N(v)|$	subset?	adjacent?
a	b	1	N	Y
a	c	1	N	Y
a	d	2	N	N
a	e	2	N	N
a	f	0	N	Y
b	c	1	N	Y
b	d	0	N	Y
b	e	2	N	N
b	f	2	N	N
c	d	2	N	N
c	e	0	N	Y
c	f	2	N	N
d	e	1	N	Y
d	f	1	N	Y
e	f	1	N	Y

23) (I7) Since there are no $u, v \in V(G)$ such that $N(u) \subseteq N(v)$, continue.

24) (I8) From Table 4, the minimum number of common neighbors is 0 (e.g., a and f). So set $b = 0$.

25) (I9) Calculate the upper bound for minimum number of common neighbors for $n = 6$ and $k = 2$:

$$c = n - 2 - \frac{n-2}{k-1} = 6 - 2 - \frac{6-2}{2-1} = 4 - 4 = 0$$

26) (I10) Since $b = 0 \not> 0 = c$, conclude that G may be 2-colorable and continue.

27) (I11) Based on the results in Table 4, all of the nonadjacent vertices share two common neighbors, so select a and d.

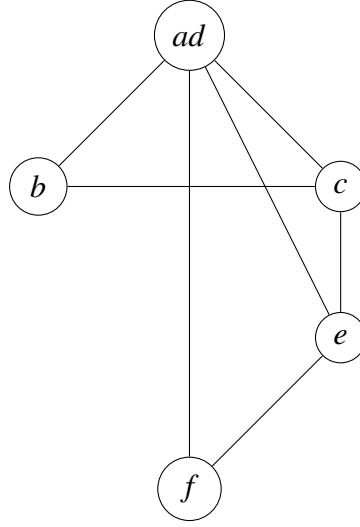28) (I12) Recursively call the subroutine with $G \cdot ad$, as shown in Fig. 40. Now, $n = 5$ and $m = 7$.

Fig. 40. Step: $G \cdot ad$.

29) (I1-1) $n = 5 \not\leq k = 2$, so continue.

30) (I2-1) Calculate the maximum edge threshold for $n = 5$ and $k = 2$:

$$a = \frac{n^2(k-1)}{2k} = \frac{5^2(2-1)}{2 \cdot 2} = \frac{25}{4} = 6.25$$

31) (I3-1) Since $m = 7 > 6.25 = a$, conclude that G is not 2-colorable and return false.

32) (I13) Returning to the G of Fig. 39, recursively call the subroutine with $G + ad$, as shown in Fig. 41. Now, $n = 6$ and $m = 10$.
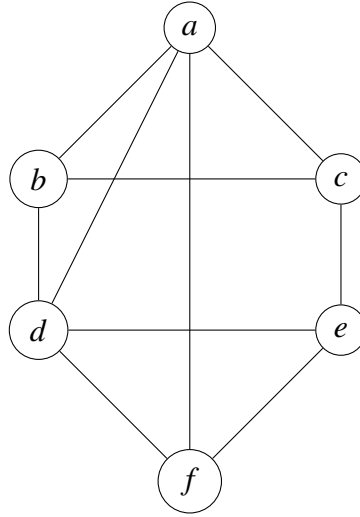
Fig. 41. Step: $G + ad$.

33) (I1-1) $n = 6 \not\leq k = 2$, so continue.

34) (I2-1) Calculate the maximum edge threshold for $n = 6$ and $k = 2$:

$$a = \frac{n^2(k-1)}{2k} = \frac{6^2(2-1)}{2 \cdot 2} = \frac{36}{4} = 9$$

35) (I3-1) Since $m = 10 > 9 = a$, conclude that G is not 2-colorable and return false.

36) (I14) Conclude that the G of Fig. 39 is not 2-colorable and return false with this G.

37) (O5) Since the called subroutine returned false, G is not 2-colorable, so continue.

38) (O6) Replace G with the graph returned by the previous call (Fig. 39).

39) (O7) Increment k to 3.

40) (O4) Call the subroutine with the new G and $k = 3$.

41) (I1) $n = 6 \not\leq k = 3$, so continue.

42) (I2) Calculate the maximum edge threshold for $n = 6$ and $k = 3$:

$$a = \frac{n^2(k-1)}{2k} = \frac{6^2(3-1)}{2 \cdot 3} = \frac{72}{6} = 12$$

43) (I3) Since $m = 9 \not> 11 = a$, continue.

44) (I4) Since $\delta(G) = 3 \geq 3 = k$, set $X = \emptyset$.

45) (I5) Since $X = \emptyset$, continue.

46) (I6) Calculate the number of common neighbors, looking for any subsets. Note that the results are the same as those shown in Table 4.

47) (I7) Since there are no $u, v \in V(G)$ such that $N(u) \subseteq N(v)$, continue.

48) (I8) From Table 4, the minimum number of common neighbors is 0 (e.g., a and f). So set $b = 0$.

49) (I9) Calculate the upper bound for minimum number of common neighbors for $n = 6$ and $k = 3$:

$$c = n - 2 - \frac{n-2}{k-1} = 6 - 3 - \frac{6-2}{3-1} = 3 - 2 = 1$$

50) (I10) Since $b = 0 \ngtr 1 = c$, conclude that G may be 3-colorable and continue.

51) (I11) Based on the results in Table 4, all of the nonadjacent vertices share two common neighbors, so select a and d.

52) (I12) Recursively call the subroutine with $G \cdot ad$, as shown in Fig. 40. Now, $n = 5$ and $m = 7$.

53) (I1-1) $n = 5 \nleq k = 2$, so continue.

54) (I2-1) Calculate the maximum edge threshold for $n = 5$ and $k = 3$:

$$a = \frac{n^2(k-1)}{2k} = \frac{5^2(3-1)}{2 \cdot 3} = \frac{25}{3} = 8.3$$

55) (I3-1) Since $m = 7 \ngtr 8.3 = a$, conclude that G may be 3-colorable and continue.

56) (I4-1) Since $\deg(b) = \deg(f) = 2 < 3 = k$, set $X = \{b, f\}$.

57) (I1-5-1) Since $X \neq \emptyset$, replace G with $G - X$. The result is shown in Figure 42. Now $n = 3$ and $m = 3$.
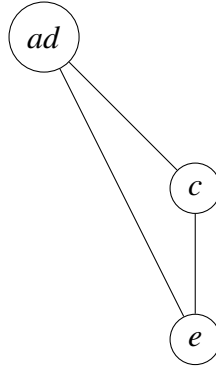
Fig. 42. Step: $G - \{b, f\}$

58) (I1-1) $n = 3 \le k = 3$, so conclude that the G of Fig. 42 is 3-colorable and return true and this G.

59) (I12) The recursive call returned true, so conclude that the returned G of Fig. 42 is 3-colorable and return true with this G.

60) (O4) The called subroutine returned true, so conclude that the returned G of Fig. **??** is 3-chromatic and return $\chi(G) = 3$.

Thus, the algorithm determines that the example G of Figure 37 is 3-chromatic. In order to determine a 3-chromatic coloring for G, first construct a set C of three colors:

$$C = \{\text{green}, \text{blue}, \text{red}\}$$

Next, follow the algorithm's modification steps in reverse order and color the corresponding vertices accordingly. The steps are as follows:

1) First, color the vertices that are present in the final simplified graph. Assign each vertex its own color, except assign contracted vertices the same color. This is shown in Fig. 43.

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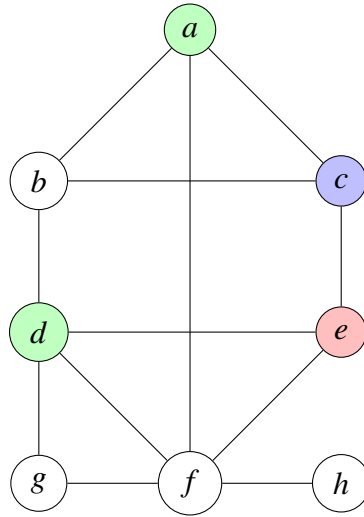
Fig. 43. Initial coloring.

2) The last modification was the removal of vertices b and f. Note that color selection needs to honor the vertices already colored. This is shown in Fig. 44.
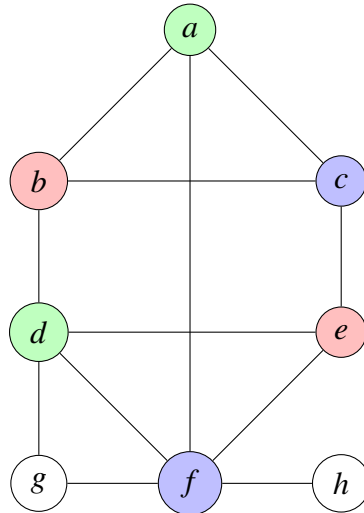

Fig. 44. Coloring b and f.

3) The contracted vertices have already been colored, so the next simplification to address is the removal of g due to the neighborhood subset test. Since $N(g) \subseteq N(e)$,

color *g* and *e* the same color. Since *e* has already been assign a color, use it for *g*.
The result is shown in Fig. 45.
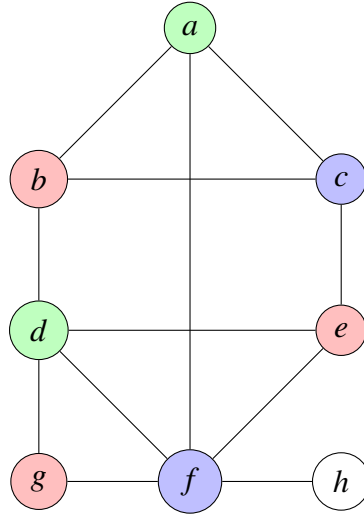
Fig. 45. Coloring *g*.

4) Finally, color the first removed vertex *h* with any appropriate color. The final result is shown in Fig. 46.
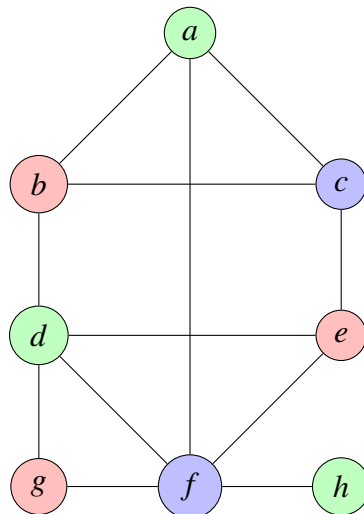
Fig. 46. Final chromatic coloring.

At first glance, one might wonder what advantage this method of coloring has over other sequential methods. The difference is that the other sequential methods only have knowledge about what has already been colored, whereas this method uses assumptions that are made during the simplifications and recursive calls on the entire graph. Thus, the sequential algorithms are heuristic, but this method selects a particular workable coloring path through the resulting Zykov tree.

5 TOASTER DESIGN CASE STUDY

This section contains a simplistic case study of how the proposed algorithm could be used to compare the FR graphs resulting from two slightly different designs of the basic kitchen toaster shown in Fig. 47.

Fig. 47. An example toaster.

The FRs are shown in Table 5. It is assumed that the design is either uncoupled or decoupled and hence the independence of the FRs is as strong as possible.

Table 5

Toaster Functional Requirements

FR_1	Body contains all parts
FR_2	Can be safely moved while hot
FR_3	Can hold two slices of bread
FR_4	Heats each slice of bread on both sides
FR_5	Toasting is manually started
FR_6	Toasting is automatically or can be manually stopped
FR_7	Heat level can be controlled

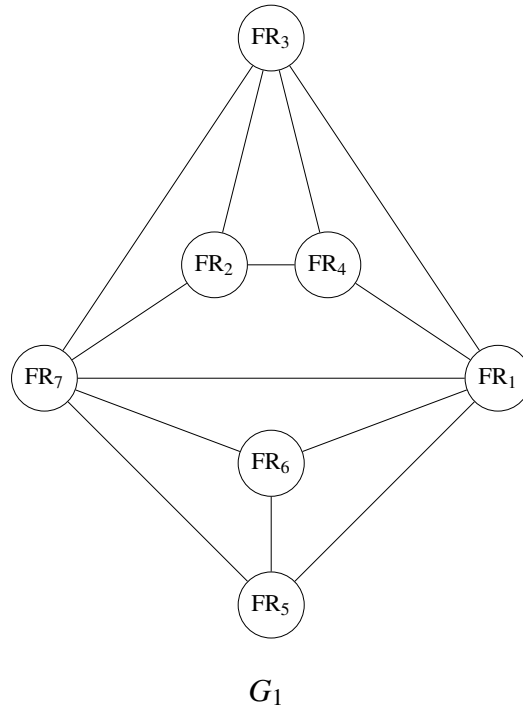The graph G_1 for the first candidate design with $n = 7$ and $m = 13$ is shown in Fig. 48.

G_1

Fig. 48. First candidate design.
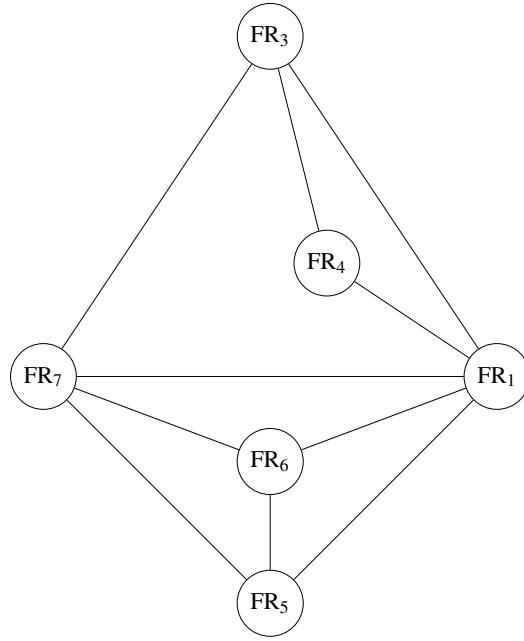
For $k = 2$, the maximum edge threshold test fails:

$$\frac{n^2(k-1)}{2k} = \frac{7^2(2-1)}{2 \cdot 2} = \frac{49}{4} \approx 12.3 < 13 = m$$

so G_1 is not 2-colorable.

For $k = 3$, the maximum edge threshold test passes:

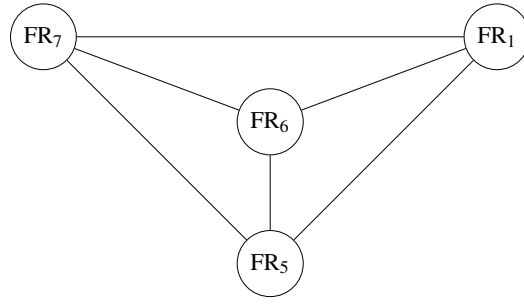$$\frac{n^2(k-1)}{2k} = \frac{7^2(3-1)}{2 \cdot 3} = \frac{98}{6} \approx 16.3 > 13 = m$$

There are no vertices with degrees less than $k = 3$; however, note that $N(\text{FR}_2) \subseteq N(\text{FR}_1)$, so remove vertex FR_2. The result is shown in Fig. 49.

$$G_1 - \text{FR2}$$

Fig. 49. Design 1—remove FR_2.

Next, since $\deg(\text{FR}_4) = 2 < 3 = k$, remove FR_4. As a result of removing FR_4, $\deg(\text{FR}_3) = 2 < 3 = k$, so remove FR_3 as well. The result is show in Fig. 50.

$$G_1 - \text{FR}_4 - \text{FR}_3$$

Fig. 50. Design 1—Remove FR_4 and FR_3.

At this point, $G_1 = K_4$, so it is guaranteed to fail the maximum edge threshold check for $k = 3$ and thus not be 3-colorable. At $k = 4$ we have $n = 4 \leq 4 = k$. Thus, we can conclude that G_1 is 4-chromatic. An example chromatic coloring with:

$$C = \{\text{green}, \text{blue}, \text{red}, \text{orange}\}$$
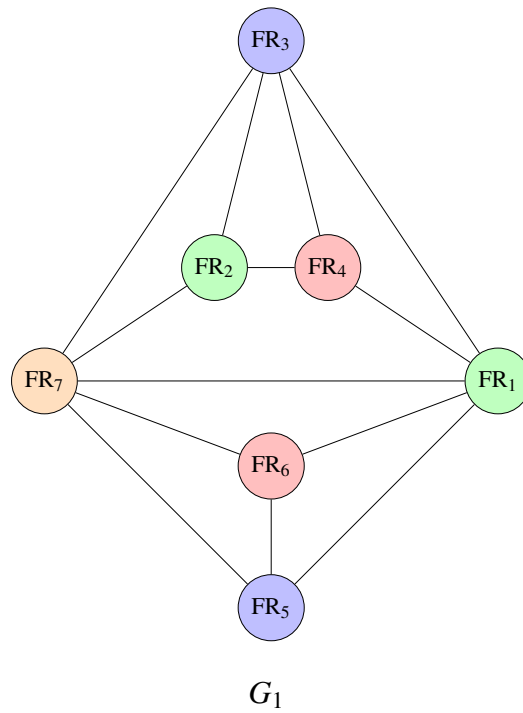
is show in Fig. 51.

G_1

Fig. 51. Design 1—chromatic coloring.

Notice in the design that FR_5 (start toasting) and FR_6 (stop toasting) have been forced into separate parts, conceivably to accommodate the separate "cancel" button shown in Fig. 47. But what if the designer decides to eliminate the cancel button and allow manual cancellation via the lever? Thus, FR_5 and FR_6 no longer need to be separated, so the edge between their vertices can be eliminated. The result is shown in Fig. 52.
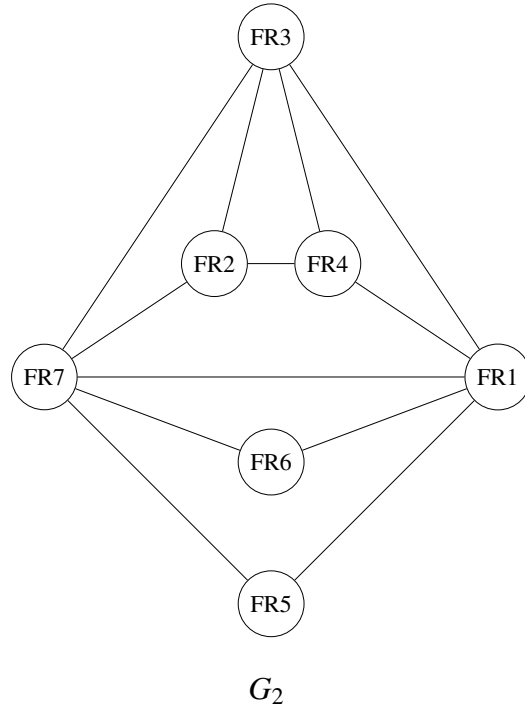
G_2

Fig. 52. Second candidate design.

Rerun the algorithm on G_2 for $n = 7$ and $m = 12$. For $k = 2$, the maximum edge threshold test passes:

$$\frac{n^2(k-1)}{2k} = \frac{7^2(2-1)}{2 \cdot 2} = \frac{49}{4} \approx 12.3 > 12 = m$$

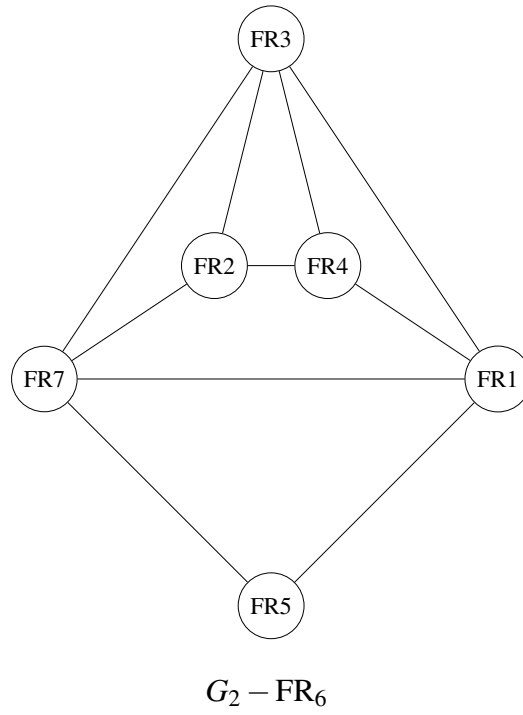Note that $N(\text{FR}_6) \subseteq N(\text{FR}_5)$, so remove FR_6. The result is shown in Fig. 53.

$$G_2 - \text{FR}_6$$

Fig. 53. Design 2—remove FR_6.

Now, with $n = 6$ and $m = 10$, G_2 fails the maximum edge threshold test for $k = 2$:

$$\frac{n^2(k-1)}{2k} = \frac{6^2(2-1)}{2 \cdot 2} = \frac{36}{4} = 9 < 10 = m$$

so G_2 is not 2-colorable.

For $k = 3$, note that $\deg(\text{FR}_5) = 2 < 3 = k$, so remove FR_5. The result is shown in Fig. 54.
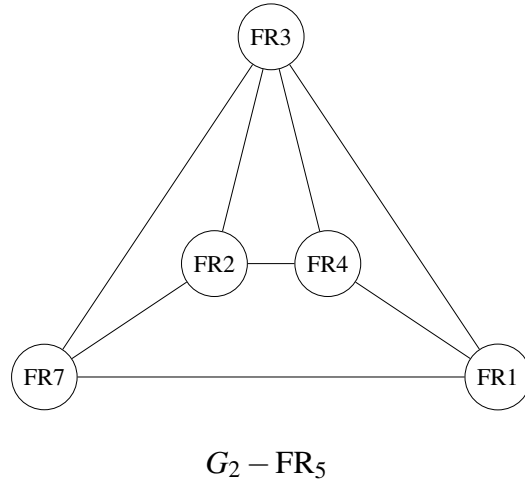
$$G_2 - FR_5$$

Fig. 54. Design 2—remove FR_5.

Now, with $n = 5$ and $m = 8$, G_2 passes the maximum edge threshold test for $k = 3$:

$$\frac{n^2(k-1)}{2k} = \frac{5^2(3-1)}{2 \cdot 3} = \frac{50}{6} \approx 8.3 > 8 = m$$

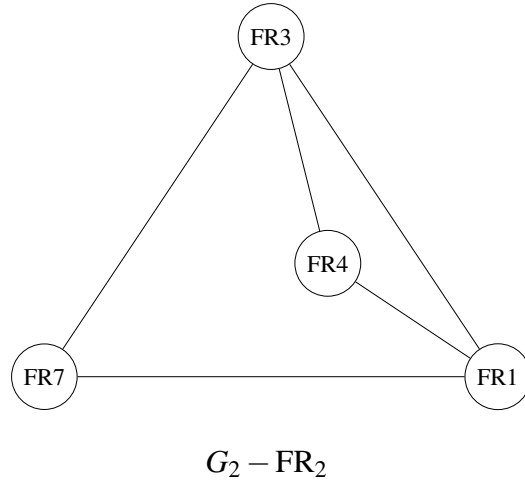Since $N(\text{FR}_2) \subseteq N(\text{FR}_1)$, remove FR_2. The result is shown in Fig. 55.

$$G_2 - FR_2$$

Fig. 55. Design 2—remove FR_2.

Finally, $\deg(\text{FR}_4) = \deg(\text{FR}_7) = 2 < 3 = k$, so remove FR_4 and FR_7. The result is shown in Fig. 56.
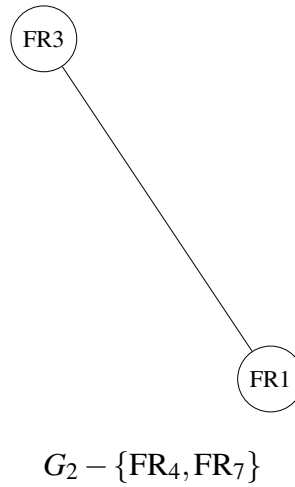
$$G_2 - \{\text{FR}_4, \text{FR}_7\}$$

Fig. 56. Design 2—remove FR_4 and FR_7.

This final state of G_2 has $n = 2 \leq 3 = k$, so we can conclude that G_2 is 3-chromatic. An example chromatic coloring with:

$$C = \{\text{green}, \text{blue}, \text{red}\}$$
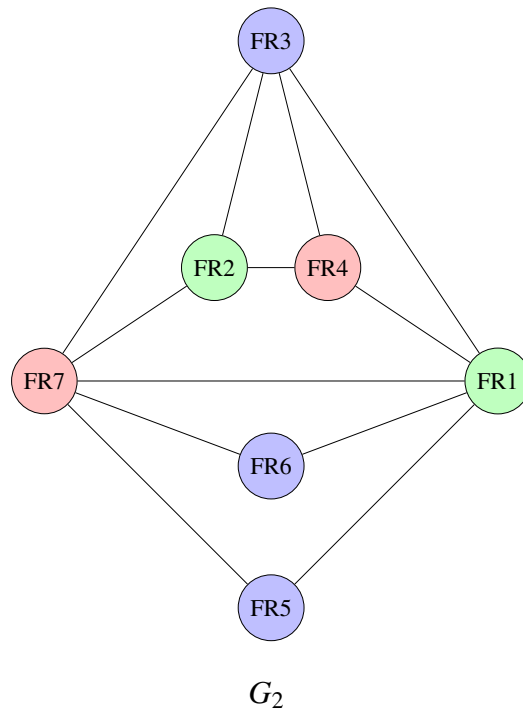
is show in Fig. 57.

G_2

Fig. 57. Design 2—chromatic coloring.

This process gives the designer the feedback that the second design requires only three parts instead of four, and thus has less information content and hence a higher chance of success than the first design. It will be up to the designer to weigh this result against other aspects of the design.

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