UDC 519.14

THE CHROMATIC CLASS OF A MULTIGRAPH

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Kibernetika, Vol. 1, No. 3, pp. 29-39, 1965

1. Introduction

The problem of coloring the vertices of a graph in such a way that adjacent vertices have different colors has become a classical problem in the theory of graphs. A great many papers are concerned with the problem of coloring the vertices in this way with the minimum number of colors or, put another way, with finding the chromatic number of a graph. The problem of finding the chromatic class of a multigraph, i. e., the least number of colors needed to color all the rays of a multigraph in such a way that the incident rays of one vertex do not have the same color, has received much less attention.

The problem of the minimum coloring of the rays of a multigraph G reduces to the minimum coloring of the vertices of the graph H defined as follows: the vertices of H are the rays of G, the vertices u and u' of H being adjacent when, and only when, the rays u and u' of G are adjacent. It is apparently this that has, in some measure, caused the lack of interest in the study of chromatic class, although naturally it suggests that better results might be obtained in this way. The fact is that so far there has been no suitable practical algorithm for coloring the vertices of an arbitrary graph or the rays of an arbitrary multigraph with the minimum number of colors. It is therefore interesting to formulate relations between the chromatic number or the chromatic class and other structural properties of the graph or multigraph. For example, let m be the maximum power of a vertex of the multigraph G. Then the chromatic class q(G) of G is obviously not less than m and not greater than $2m - 1$.

The problem of the minimum coloring of the rays of a multigraph is not only of theoretical interest, but also has a practical application. To avoid errors in the assembly of certain forms of electrical circuits the wires are colored, those connected to the same device having different colors.

In connection with this technical problem C. E. Shannon [2] obtained an upper estimate of the chromatic class of a multigraph in terms of the maximum power of its vertex. He proved that if m is the maximum power of a vertex of the multigraph G, then its chromatic class $q(G) \leq [3m/2]$, where the square brackets denote the integral part of the number.

Before Shannon's work we had König's theorem [3], which states that if a graph does not contain any cycles of odd length, its chromatic class is equal to the maximum power of a vertex. König's theorem is generalized for a multigraph in [4]. In Section 4, using a lemma of Shannon, we give a proof of the generalized König theorem which we think is simpler than the other proofs known to us [1, 3, 4, 5, 6].

In [19] the author proves a theorem stating that the chromatic class of the p-graph G with maximum power of a vertex m does not exceed $m + p$. This theorem, Theorem 1 of this article, is an immediate corollary of Lemma 3 of Section 3. It is clear than when $p > [m/2]$ the estimate of Theorem 1 is worse than Shannon's, although, as we show in Section 4, it is easy to prove Shannon's estimate is exact for any m, it is nevertheless attained when $m \geq 4$ only on a narrow class of multigraphs.

In Section 5 we discuss the possibilities of Shannon's operation for coloring maximal two-color circuits. In Section 6 we examine the effect of certain structural and numerical characteristics of a graph on its chromatic class; Section 7 is concerned with the converse problem of determining the properties of a graph from its chromatic class. It should be noted that the questions dealt with in Sections 5, 6, and 7 have not by any means been exhaustively studied. In throwing out a number of hypotheses, the author in no way wishes to claim that he has defined the path for further research into chromatic class. In Section 8 we establish the relation between the chromatic class q(G) of the graph G and the chromatic class $q(\bar{G})$ of its complement.

Some of our results are given without proof in [20]. The author wishes to express his gratitude to A. A. Zykov for his thoughtful criticism and discussion of the results.

2. Basic Concepts. Two Lemmas of Shannon

In this article we mean by a multigraph a finite nonoriented multigraph without loops [1]. We shall denote multigraphs by capital latin letters: G, H, ..., sometimes together with a letter, the name of the multigraph, in brackets to indicate the set of vertices and rays. Thus, G(X, U) is the multigraph G with set of vertices X and set of rays U.

If A is a finite set, then $|A|$ denotes its power, i. e., the number of its elements. An empty set is denoted by $\emptyset$. Clearly, $|\emptyset| = 0$.

Let x and y be two different vertices of a multigraph. Then $E(x, y)$ denotes the set of all rays of the multigraph which are incident both at x and y.

If x and y are not adjacent, then, clearly $E(x, y) = \emptyset$. If $|E(x, y)| \geq 2$, then two different rays of $E(x, y)$ are called parallel. Often we use (x, y) to denote the ray with ends x and y, i.e., a certain definite ray from $E(x, y)$.

The multigraph $G(X, U)$ is called a p-graph (p is an integer ≥ 0) if $|E(x, y)| \leq p$ for any x and y from X. A 1-graph is called a graph.

The number of rays of a multigraph incident at the vertex x is called the degree of the vertex x and denoted by $\sigma(x)$. The vertex x is said to be isolated if $\sigma(x) = 0$. The maximum degree of a vertex of G is denoted by $\sigma(G)$. A multigraph is said to be homogeneous of degree m if the degrees of all its vertices are the same and equal to m.

A multigraph is said to be a complete p-graph if for any pair of different vertices x and y $|E(x, y)| = p$. If G is a graph, then $\overline{G}$ denotes a graph with the same set of vertices, two vertices of $\overline{G}$ being adjacent when, and only when, they are not adjacent in G. The graph $\overline{G}$ is called the complement of G.

The multigraph $H(X', U')$ is called a subgraph of $G(X, U)$ if $X' \subseteq X$, $U' \subseteq U$, and the ray $u \in U$ belongs to U' when, and only when, both ends of u belong to X'.

Any aggregate of mutually nonadjacent rays of a multigraph is called a pair-combination. The largest power of a pair-combination in G is denoted by $\pi(G)$.

The coloring of the rays of a multigraph is said to be regular if the rays colored the same form a pair-combination. The smallest number of colors required for a regular coloring of all the rays of G is called the chromatic class of G and denoted by $q(G)$.

Suppose we have a multigraph G with regularly colored rays and let s and t be two different colors. The circuit formed from the rays of G colored with s and t is called a two-color (s, t)-circuit. Note that a two-color circuit can be either an elementary circuit or an elementary cycle*; in particular, it may consist of one ray, or may contain no rays at all. We say that an (s, t)-circuit is maximal if it is not a singular part of any (s, t)-circuit.

We shall say that the color s is absent from the vertex x of G with regularly colored rays if not a single ray of G incident at x is colored with s.

The following two lemmas are given by Shannon [2].

Lemma 1. The regularity of the coloring of the rays of a multigraph is not destroyed by recoloring any maximal two-color circuit.

Lemma 2. Let x, y, z be three different vertices of the multigraph G with regularly colored rays, where at least one of the colors s and t is absent from x, y, and z. Then at least one of the vertices x, y, z is not connected to either of the other two by the (s, t) circuit.

Below we shall always refer to regularly colored rays. Let us denote by $\delta(x)$ the set of colors absent from the vertex x; we consider only those absent colors which belong to some previously fixed set of colors, which will be clear from the context.

3. On an Upper Estimate of the Chromatic Class of a p-graph

Lemma 3. Suppose that using the k colors $(1, 2, \ldots, k)$ we color some set M of the rays of $G(X, U)$ and suppose that the ray $(a, b) \in U$ (but $(a, b) \notin M$). Then if $|\delta(a)| \geqslant 1$, $|\delta(b)| \geqslant |E(a, b) \cap M| + 1$ and for all x different from a and b, $|\delta(x)| \geqslant |E(a, x) \cap M|$, using the same k colors we can color the set $M \cup (a, b)$ of rays of G.

Proof. If $\delta(a) \cap \delta(b) \neq \emptyset$, then (a, b) can be colored with the color which is absent both from a and from b, and this proves the lemma.

Now let $\delta(a) \cap \delta(b) = \emptyset$.

With each ray of $E(a, x) \cap M$, where $x \in X$, $x \neq a$, we associate one color absent from x, so that different rays of $E(a, x) \cap M$ are associated with different colors absent from x. This can always be done, since, from the conditions of the lemma, $|\delta(x)| \geqslant |E(a, x) \cap M|$ for all $x \neq a$. Since, moreover $|\delta(b)| \geqslant |E(a, b) \cap M| + 1$, there exists a color $s_1 \in \delta(b)$, which is not associated with any ray of $E(a, b) \cap M$.

Since $\delta(a) \cap \delta(b) = \emptyset$, we have $s_1 \notin \delta(a)$, i.e., the ray $(a, x_1) \in M (x_1 \neq b)$, colored with s_1 comes from the vertex a. Let s_2 be the color associated with the ray (a, x_1). Obviously, $s_2 \neq s_1$. If $s_2 \in \delta(a)$, then we color

* A circuit is said to be elementary if no vertex occurs in it twice. A cycle is said to be elementary if no vertex, apart from the coincident initial and final vertices, is encountered twice on going around it.

(a, x_1) with s_2; then (a, b) can be colored with s_1. Suppose that $s_2 \notin \delta(a)$; then the ray $(a, x_2) \in M$, colored with s_2 comes from a.

Suppose that $(a, x_1), (a, x_2), \ldots, (a, x_n)$ (where $n \geq 2$) are rays coming from a and belonging to the set M and colored with the colors $s_1, s_2, \ldots, s_n$, respectively, and that the colors s_j [and therefore also the rays (a, x_j)] are different in pairs $(j = 1, 2, \ldots, n)$*, the color $s_i \in \delta(x_{i-1})$ $(2 \leqslant i \leqslant n)$ being associated with the ray (a, x_i). Let the ray (a, x_n) be associated with the color $s_{n+1} \in \delta(x_n)$. There are two possibilities.

Case 1. $s_{n+1} \in \delta(a)$.

We recolor the ray (a, x_n) with the color s_{n+1} and (a, x_{n-1}) with s_n, (a, x_2) with s_3, (a, x_1) with s_2. The ray (a, b) can then be colored with s_1, which proves the lemma.

Case 2. $s_{n+1} \notin \delta(a)$.

The following possibilities arise:

a) $s_{n+1} = s_1$. Then $x_n \neq b$, since s_1 is not associated with any ray from $E(a, b) \cap M$.

Let us take the color $t \in \delta(a)$. We have $t \in \delta(a)$, $s_1 \in \delta(b)$, $s_1 \in \delta(x_n)$, and a, b and x_n are different vertices. We apply Lemma 2. If a and b are not connected by an (s_1, t)-circuit, then after recoloring of the maximal (s_1, t)-circuit starting from the vertex a the color s_1 will be absent from a and b and can be used to color (a, b). If x_n is not connected either to a or to b by an (s_1, t)-circuit, then after recoloring the maximal (s_1, t)-circuit starting at x_n the color $t \in \delta(a)$ will be absent from x_n. Associating it with the ray (a, x_n), we now have the conditions of Case 1 and can therefore also color the ray (a, b);

b) $s_{n+1} = s_i \, (2 \leqslant i \leqslant n-1)$. Note that we cannot have $s_{n+1} = s_n$ for the ray (a, x_n) is colored with s_n.

Then $x_{i-1} \neq x_n$, since otherwise the same color s_i would be associated with two parallel rays (a, x_{i-1}) and (a, x_n), which is impossible. Take the color $t \in \delta(a)$. We have $t \in \delta(a)$, $s_i \in \delta(x_{i-1})$ and $s_i \in \delta(x_n)$. If a is connected by an (s_i, t)-circuit with the vertex x_{i-1} then from Lemma 2, x_n is not connected by an (s_1, t)-circuit either to a or to x_{i-1}. After recoloring the maximal (s_1, t)-circuit starting at x_n the color $t \in \delta(a)$ will be absent from x_n; associating t with (a, x_n) we have the conditions of Case 1. But if a is not connected to x_{i-1} by an (s_i, t)-circuit, then, by recoloring the maximal (s_i, t)-circuit starting at x_{i-1} and then associating t with (a, x_{i-1}), considering the system of rays $(a, x_1), (a, x_2), \ldots, (a, x_{i-1})$, we have the conditions of Case 1, where (a, x_n) takes the place of (a, x_{i-1});

c) neither a) nor b) applies. This means that $s_{n+1} \neq s_j$ for any $j = 1, 2, \ldots, n$. Then the ray (a, x_{n+1}), colored with s_{n+1} and different from $(a, x_1), (a, x_2), \ldots, (a, x_n)$, comes from a and the whole argument can be repeated for the system of rays $(a, x_1), (a, x_2), \ldots, (a, x_n), (a, x_{n+1})$. Since G is finite this can happen only a finite number of times, i.e., after a finite number of steps we must arrive either at Case 1 or at a) or b) of Case 2.

This proves Lemma 3.

Lemma 3 gives the theorem proved by the author in [19].

Theorem 1. Let m be the maximum degree of a vertex of the p-graph G. Then $q(G) \leq m + p$.

It is clear that if $p < [(\sigma(G))/2]$, then the estimate of the chromatic class of G, $q(G) \leq \sigma(G) + p$, is better than Shannon's estimate $q(G) \leq [(3\sigma(G))/2]$. On the other hand, if G is a p-graph and $q(G) = \sigma(G) + p$, then $p \leq [(\sigma(G))/2]$. The question that naturally arises is whether for any positive integers m and p satisfying the relation $p \leq [m/2]$ we can construct a p-graph G with $\sigma(G) = m$ and $q(G) = m + p$. As was shown in [19], when $p = 1$ this question can be answered in the affirmative. We have the following corollary.

Corollary 1. If m is the maximum degree of a vertex of the graph G, then $m \leq q(G) \leq m + 1$. For an integer $m \geq 2$ we can construct a graph G with $\sigma(G) = m$ and $q(G) = m + 1$. However, when $p \geq 2$ the estimate of Theorem 1 is not accurate in the sense indicated above.

Proposition. For any integer $p \geq 2$ the chromatic class of the p-graph G with $\sigma(G) = m = 2p + 1$ does not exceed $m + p - 1$.

Proof. Assume the contrary. Suppose that for some $p \geq 2$ there exists a p-graph G with $\sigma(G) = m = 2p + 1$ and $q(G) = m + p = 3p + 1$. Without loss of generality, we can assume that G possesses the following properties:

1) it contains the smallest number of vertices of all p-graphs with maximum degree of a vertex $2p + 1$ and chromatic class $3p + 1$;

2) if any ray is removed from G, the chromatic class of the resulting multigraph is equal to $3p = m + p - 1$.

It follows from properties 1) and 2) that for any vertex x of G we can find two different vertices y and y' such that $|E(x, y)| = |E(x, y')| = p$; in other words, two pencils of parallel rays, each of which has power p, come from

* The vertices x_j cannot be different in pairs. Therefore if $x_l = x_r$ $(l \neq r; 1 \leqslant l, r \leqslant n)$, then (a, x_l) and (a, x_r) are parallel rays.

each vertex of G. For suppose that the vertex x does not possess this property and that E(x, y) is the maximum power of pencils coming from x. From property 1), G has no isolated vertices; therefore $|E(x, y)| \geq 1$. We remove from G a ray belonging to E(x, y) and color all the rays of the resulting multigraph with $m + p - 1$ colors, this being possible because of property 2). We then restore the ray we removed. Then, from Lemma 3, whose conditions are easily verified, all the rays of G can be colored with $m + p - 1$ colors, and this contradicts $q(G) = m + p$.

Since, as we have shown, two pencils of parallel rays of power p come from each vertex of G, the rays which are not parallel form a (nonempty) pair-combination in G and can therefore be colored with one color.

After removing these rays we have the p-graph H with $\sigma(H) = 2p$ and $q(H) = 3p$. The multigraph H contains as a subgraph the complete three-vertex p-graph F, since otherwise, as it is not difficult to verify, we would have $q(H) \leq \leq 3p$. Obviously, F is a subgraph of G, not coincident with G. Let x_1, x_2, x_3 be vertices of F. We remove all the rays from F and make all the vertices x_1, x_2, x_3 identical; the p-graph thus obtained from G has k fewer vertices than G. Since, in addition, $\sigma(k) \leq \sigma(G)$, according to property 1) of G, $q(k) \leq 3p$. It follows that if we remove from G all the rays of F, the rays of the resulting multigraph L can be colored using 3p colors in such a way that all the rays of L incident at x_1, x_2, and x_3 have different colors. Let us color the rays of L in this way. Suppose that n rays ($n \leq 3$) of L are incident at x_1, x_2, x_3. Then we color n rays of F as follows: if a ray of L colored with s_i is incident at x_i ($1 \leq i \leq 3$), we color one of the rays of F that is not incident at the vertex x_i with s_i. Coloring the remaining rays of F using $3p - n$ different colors we obtain a coloring of all the rays of G with 3p colors, which contradicts the assumption that $q(G) = = 3p + 1$.

This proves the proposition.

4. König's and Shannon's Theorems

Theorem 2 (König). Let G be a multigraph without cycles of odd length. Then $q(G) = \sigma(G)$.

Proof. Let $\sigma(G) = m$. We number the colors $1, 2, \ldots, m$. Let us show, using the numbered colors, how we can color all the rays of G. Suppose that the ray (a, b) is still not colored. Since not more than $(m - 1)$ colored rays come from a and from b, $\delta(a) \neq \varnothing$, $\delta(b) \neq \varnothing$. If $\delta(a) \cap \delta(b) \neq \varnothing$, then we can color (a, b) with the color which is absent both from a and from b. But if $\delta(a) \cap \delta(b) = \varnothing$ and $s \in \delta(a)$, $t \in \delta(b)$, then a and b are not connected by an (s, t)-circuit, since otherwise G would contain a cycle of odd length. Therefore, recoloring the maximum (s, t)-circuit starting at b, we can color the ray (a, b) with the color s.

This proves the theorem.

Theorem 3 (Shannon). For any multigraph G, $q(G) \leq [(3\sigma(G))/2]$.

Proof. Suppose that the contrary is true. Without loss of generality we can assume that there exists a multigraph G with $\sigma(G) \leq m$, $q(G) = [3m/2] + 1$ and when any ray is removed from G the resulting multigraph has a chromatic class equal to $[3m/2]$. According to Theorem 1 there exists in G a pair of vertices a and b with $|E(a, b)| \geq [m/2] + 1$. Let us color all the rays of G except for any ray from $E(a, b)$ with $[3m/2]$ colors. It is sufficient to show that $\delta(a) \cap \delta(b) \neq \neq \varnothing$. We have, clearly, $|\delta(a) \cup \delta(b)| \leq [3m/2] - |E(a, b)| + 1 \leq m$. On the other hand, $|\delta(a)| \geq [m/2] + 1$ and $|\delta(b)| \geq [m/2] + 1$. Hence, $|\delta(a)| + |\delta(b)| > m \geq |\delta(a) \cup \delta(b)|$. Therefore, $\delta(a) \cap \delta(b) \neq \varnothing$.

This proves the theorem.

Thus, although Theorem 1 is not a direct generalization of Shannon's theorem, the latter can very easily be proved using it. Moreover, using the proof of Theorem 1, we can easily construct an algorithm for coloring the rays of the arbitrary p-graph G using not more than $\min\{\delta(G) + p, [(3\sigma(G))/2]\}$ colors.

Let us call the three-vertex multigraph H with $[3m/2]$ rays and with $\sigma(H) = m$ a Shannon multigraph of degree m. It is clear that for fixed m all Shannon multigraphs of degree m are isomorphic to the multigraph with vertices x, y, and z and with $|E(x, y)| = [m/2]$; $|E(x, z)| = [m/2]$; $|E(y, z)| = [(m + 1)/2]$. It is also clear that the chromatic class of a Shannon multigraph of degree m is equal to $[3m/2]$.

Theorem 4. Let G be a multigraph with $\sigma(G) = m \geq 4$ and $q(G) = [3m/2]$. Then G contains a Shannon multigraph of degree m as a subgraph.

Proof. Obviously it is sufficient to prove that if H is a connected multigraph with $\sigma(H) = m \geq 4$, $q(H) = [3m/2]$ such that, when any ray of H is removed, the resulting multigraph has chromatic class less than $[3m/2]$, then H is a Shannon multigraph of degree m.

Let us consider two cases.

Case 1. m even, i.e., $\sigma(H) = zk$, where $k \geq 2$ and $q(H) = 3k$.

We will show that if the vertices x and y of H are adjacent, then $|E(x, y)| = k$. For we cannot have $|E(x, y)| \geq$ $\geq k + 1$, since by coloring all the rays of H except for one ray of E(x, y) using $3k - 1$ colors it is easy to calculate that we would have $\delta(x) \cap \delta(y) \neq \emptyset$ and this contradicts $q(H) = 3k$.

On the other hand, if $0 < |E(x, y)| \leq k - 1$, then, by removing from H the ray in the pencil of parallel rays of maximum power coming from x and coloring all the rays of the remaining multigraph with $3k - 1$ colors, we could, from Lemma 3, color all the rays of H with $3k - 1$ colors, and this again contradicts $q(H) = 3k$.

It is now clear that two pencils of parallel rays of power k come from each vertex of H. Since $q(H) = 3k$ and H is a connected multigraph, H can only be a complete three-vertex k graph, which is what we were required to prove.

Case 2. m odd, i.e., $\sigma(H) = 2k + 1$, where $k \geq 2$ and $q(H) = 3k + 1$.

According to the proposition proved in Section 3, there exists in H a pair of vertices x and y with $|E(x, y)| \geq$ $\geq k + 1$. We cannot have $|E(x, y)| \geq k + 2$, since otherwise, coloring all the rays of H except for one ray from E(x, y) with 3k colors, we would have $\delta(x) \cap \delta(y) \neq \emptyset$, i.e., $q(H) = 3k$. It can be shown similarly that if $|E(x, y)| = k + 1$, then $\sigma(x) = \sigma(y) = 2k + 1$.

Further, using Lemma 3, it can easily be shown that if a pencil of parallel rays of power $\leq k$ is incident at some vertex, then at least one more pencil of parallel rays of power $\geq k$ is incident at this vertex.

Considering the comments we have just made about the structure of H and using arguments similar to those made at the end of Section 3 in the proof of our proposition, it is not difficult to see that in order to complete the proof of Theorem 4 it is sufficient to show that, if x and y are vertices of H with $|E(x, y)| = k + 1$, then x (like y) is adjacent to only two vertices of H. We shall prove this statement for the case $k \geq 3$ only, leaving the case $k \geq 2$ to the reader.

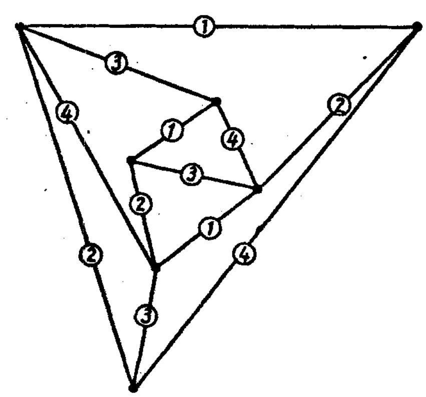
Thus, let $k \geq 3$, $|E(x, y)| \geq k + 1$ and suppose, in spite of our assertion, that the vertex x is adjacent to not less than three vertices of H. We remove from E(x, y) one ray u and color the remaining rays of H with 3k colors. If $\sigma(x) \cap$ $\cap \sigma(y) \neq \emptyset$, the proof is complete. Let $\delta(x) \cap \delta(y) = \emptyset$. Since $|\delta(y)| \geq k$, each of the k rays coming from x and not belonging to E(x, y) is colored with a color belonging to $\delta(y)$. Let x_1 and x_2 be two different vertices, adjacent to x and distinct from y. Since $\sigma(x_1) \leq 2k + 1$ and $\sigma(x_2) \leq 2k + 1$, we have $|\delta(x_1)| \geq k - 1$ and $|\delta(x_2)| \geq k - 1$. We cannot have $\delta(x_i) \cap \delta(x) \neq \emptyset$ $(i = 1, 2)$, since, if $s \in \delta(x_i) \cap \delta(x)$, by recoloring the ray $u' \in E(x, x_i)$ with the color s we could color the ray u with the color of the ray u'. We also have $\delta(x_i) \cap \delta(y) = \emptyset$ $(i = 1, 2)$. For if $s_1 \in \delta(x_1) \cap \delta(y)$ we can take the color $t \in \delta(x)$ and recolor the maximum (s_1, t)-circuit starting at the vertex x. If x and y are not connected by an (s_1, t)-circuit, then we will be able to color u with s_1; if they are connected, then after recoloring the (s_1, t)-circuit we obtain $\delta(x_1) \cap \delta(x) \neq \emptyset$, which is impossible.

Since $|\delta(x)| + |\delta(y)| + |\delta(x_1)| + |\delta(x_2)| \geq 4k - 2 > 3k$ for $k \geq 3$, we have $\delta(x_1) \cap \delta(x_2) \neq \emptyset$. Let $s_0 \in \delta(x_1) \cap \delta(x_2)$. We take the color $t \in$ $\in \delta(x)$. Then x is not connected by an (s_0, t)-circuit to at least one of the vertices x_1 and x_2. Suppose, for example, that x is not connected by an (s_0, t) circuit to x_1. Recoloring the maximum (s_0, t)-circuit starting at x_1 we shall have $\delta(x_1) \cap \delta(x) \neq \emptyset$, which, as we showed above, contradicts $q(H) = 3k + 1$.

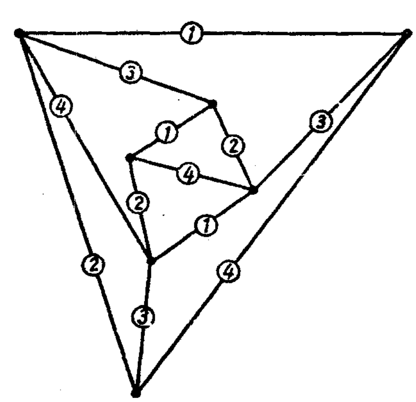
This proves the theorem.

5. On Recoloring Maximum Two-Color Circuits

On the basis of the proofs of the preceding sections we can draw the conclusion that the operation of recoloring maximum two-color circuits enables us to "economize" on colors in coloring the rays of a multigraph. Thus, for example, if more than $\sigma(G) + p$ colors are used to color the rays of the p-graph G, then (without introducing new colors) by recoloring certain maximum two-color circuits we can obtain a coloring of the rays of G with $\sigma(G) + p$ colors. The question that naturally arises is whether we can in this way obtain a minimum coloring of the rays of a multigraph? The author has not answered this question. Obviously, the answer would be in the affirmative if the following were true: for any two different colorings of the rays of a multigraph with the same number of the same colors there always exists a finite sequence for recoloring maximum two-color circuits which translates one coloring into another; of course, it is taken for granted here that it is impossible to resort to colors that do not appear in the original coloring of the rays. However, the statement is not true, as the example to the right shows.



I



II

In the diagram we show two different colorings of the rays of the same graph using the same four colors – the numbers of the colors are in the circles. It is impossible to go from coloring I to coloring II (and therefore from II to I) by recoloring the maximum two-color circuits, since any maximum two-color (i, j)-circuit ($1 \le i,\ j \le 4,\ i \ne j$) in I consists of all rays colored with i and j and so the recoloring of any such circuit would simply lead to the renumbering of colors or to an isomorphic coloring. The colorings I and II are not isomorphic.

The colorings I and II are not isomorphic.

6. Chromatic Class and Structure of a Graph

Corollary 1 of Section 3 states that the chromatic class of a graph with maximum degree of a vertex m is equal either to m or to m + 1. It seems to be a very difficult problem to give criteria enabling one to use several easily recognized parameters of an arbitrary graph to determine its chromatic class exactly.

In this section we examine the effect of certain numerical and structural characteristics of a graph on its chromatic class. In particular, we show which of the two possible values the chromatic class of several narrow classes of graphs has.

Theorem 5. If G is a homogeneous graph of (even) degree m > 0 with an odd number of vertices, then q(G) = m + 1.

Proof. Let $n \ge 3$ be the number of vertices of G. Then it has $n \cdot m/2$ rays. Since n is odd by hypothesis, the largest power of a pair-combination in G is $\pi(G) \leqslant \frac{n-1}{2}$. Hence $q(G) \geqslant \frac{n \cdot m}{2} \Big/ \frac{n-1}{2} = m + \frac{m}{n-1} > m$. Therefore, from Corollary 1, q(G) = m + 1. This proves the theorem.

This proves the theorem.

Corollary 2. If G is a complete (2k + 1)-vertex graph (the integer $k \ge 1$), then q(G) = 2k + 1.

Corollary 3. If G is a complete 2k-vertex graph (the integer $k \ge 1$), then q(G) = 2k – 1.

Proof. When k = 1 the statement is trivial. Let $k \ge 2$ and let $x_1, x_2, \dots, x_{2k-1}, x_{2k}$ be the vertices of G. Let H denote the subgraph of G generated by the vertices $x_1, x_2, \dots, x_{2k-1}$. From Corollary 2, q(H) = 2k – 1. Let us color all the rays of H with 2k – 1 colors. Clearly, we have $|\delta(x_l)| = 1$ for all $l = 1, 2, \dots, 2k-1$. $\delta(x_i) \cap \delta(x_j) = \varnothing$, where $i \ne j$ $(1 \leqslant i, j \leqslant 2k-1)$. Therefore the ray (x_{2k}, x_l) of G $(1 \leqslant l \leqslant 2k-1)$ can be colored in a color that is absent from x_l.

Remark. Corollaries 2 and 3 have a curious interpretation with reference to games. Suppose that the conditions of the contest require that each of n participants shall compete once with each of the other n – 1. What is the minimum number of contests required in the competition if in any one contest each participant can meet not more than one opponent? Corollaries 2 and 3 give the answer to this question: if the number of participants n is odd, then n contests are required, but if n is even, then n – 1 contests are required.

Berger's well-known tournament tables are schedules for the meetings of contestants in competitions such as these.

Theorem 6. If the connected homogeneous graph G of degree $m \ge 2$ has a coupling point* then q(G) = m + 1.

Proof. The theorem follows from Theorem 5 if the number of vertices of G is odd. Now suppose that there is an even number of vertices in G and let x be a coupling point. Then at least one of the connected components of the subgraph obtained from G by removing x has an odd number of vertices. Let this component be denoted by H. Suppose that H has n vertices of which k (k < m) have degree m – 1 and n – k degree m. Since n is odd, $\pi(H) \le (n-1)/2$. Therefore

$$q(G) \geqslant \frac{k \cdot (m-1) + (n-k) \cdot m}{2} \Big/ \frac{n-1}{2} = m + \frac{m-k}{n-1} > m.$$

This proves the theorem.

Let us introduce some new notation. Let the graph G contain at least one cycle different from a single-vertex cycle. We shall denote the smallest length of such a cycle by t(G).

Theorem 7. For any integers $m \ge 2$ and $k \ge 3$ there exists a graph G with $\sigma(G) = m$, q(G) = m + 1 and $t(G) \ge k$.

Proof. Let $m \ge 2$ and $k \ge 3$ be arbitrary integers. Then, as is shown, for example, in [7], we can construct a homogeneous graph H of degree m with $t(H) \ge k$. Let n the number of vertices of H. If n is odd, then, according to Theorem 5, q(H) = m + 1. If n is even, we take the vertex z outside H and remove from H any ray (x, y) and connect z to x and y. The resulting graph G has, clearly, $n \cdot m/2 + 1$ rays, $\sigma(G) = m$ and $t(G) \ge k$. Since $\pi(G) \le n/2$, $q(G) \geqslant \frac{n \cdot m}{2} + 1 \Big/ \frac{n}{2} = m + \frac{2}{n} > m$, i.e., q(G) = m + 1.

This proves the theorem.

*The vertex x of a connected graph is a coupling point if the subgraph obtained by removing x is unconnected.

Lemma 4. When all the rays of a complete 2k-vertex graph ($k \geq 1$) are colored there exists a pair-combination consisting of $k-1$ rays colored with different colors.

Proof. When $k = 1$ and $k = 2$ the lemma is trivial. Suppose that for all $k < k_0$ ($k_0 \geq 3$) the lemma is proved and let G be a $2k_0$-vertex complete graph, all of whose rays are colored; according to the induction hypothesis there exists a pair-combination of $k_0 - 2$ rays in G: (x_1, y_1), $(x_2, y_2), \ldots, (x_{k_0-2}, y_{k_0-2})$, colored with the different colors: $s_1, s_2, \ldots, s_{k_0-2}$ respectively. Let z_1, z_2, z_3, z_4 be vertices of G different from $x_1, x_2, \ldots, x_{k_0-2}, y_1, y_2, \ldots, y_{k_0-2}$. Let $U(z_j)$ ($1 \leq j \leq 4$) denote the set of rays incident at the vertex z_j and colored with colors different from $s_1, s_2, \ldots, s_{k_0-2}$. Clearly, $|U(z_j)| \geqslant k_0 + 1$ for any $j = 1, 2, 3, 4$. If for some $j_1 \neq j_2$ ($1 \leqslant j_1, j_2 \leqslant 4$) $U(z_{j_1}) \cap U(z_{j_2}) \neq \varnothing$, then the lemma is proved, since then the rays $(x_1, y_1), (x_2, y_2), \ldots, (x_{k_0-2}, y_{k_0-2}), (z_{j_1}, z_{j_2})$ form the required pair-combination. Now let $U(z_{j_1}) \cap U(z_{j_2}) = \varnothing$ for any $j_1 \neq j_2$; $1 \leqslant j_1, j_2 \leqslant 4$. Let $U_z(x_i)$ ($1 \leq i \leq k_0 - 2$) denote the subset of rays of $\bigcup_{j=1}^{4} U(z_j)$, incident at x_i. We define $U_z(y_i)$ similarly. Since $\left|\bigcup_{i=1}^{k_0-2} U_z(x_i) \cup \bigcup_{i=1}^{k_0-2} U_z(y_i)\right| = \sum_{i=1}^{k_0-2} |U_z(x_i)| + \sum_{i=1}^{k_0-2} |U_z(y_i)| = \sum_{j=1}^{4} |U(z_j)| \geqslant 4(k_0+1)$, for some i_0 ($1 \leqslant i_0 \leqslant k_0 - 2$) $|U_z(x_{i_0})| + |U_z(y_{i_0})| \geqslant 5$. We shall assume that $|U_z(x_{i_0})| \geqslant |U_z(y_{i_0})|$ (this obviously does not destroy the generality of our reasoning). Then $|U_z(x_{i_0})| \geqslant 3$, and since $|U_z(x_{i_0})| \leqslant 4$, we have $|U_z(y_{i_0})| \geqslant 1$. Let $u_1 \in U_z(y_{i_0})$. Since $|U_z(x_{i_0})| \geqslant 3$, there exists a ray $u_2 \in U_z(x_{i_0})$, that is not adjacent to the ray u_1 and is colored with a color different from that of u_1. Therefore the rays $u_1, u_2, (x_1, y_1), \ldots, (x_{i_0-1}, y_{i_0-1}), (x_{i_0+1}, y_{i_0+1}), \ldots, (x_{k_0-2}, y_{k_0-2})$ form the required pair-combination.

This proves the lemma.

Theorem 8. If the n-vertex graph G has at least one vertex whose degree does not exceed $[n/2]$, then $q(G) \leq n - 1$.

Proof. When n is even, the theorem follows from Corollary 3.

Now suppose that $n = 2k + 1$ ($k \geq 1$) and that j rays $(x, x_1), (x, x_2), \ldots, (x, x_j)$, come from the vertex x, where $j \leq k$. Without loss of generality we shall assume that the subgraph H obtained from G by removing x is a complete 2k-vertex graph. From Lemma 4, when all the rays of H are colored with $2k - 1$ colors (that this is possible follows from Corollary 3) there exists a pair-combination of the $k - 1$ rays $u_1, u_2, \ldots, u_{k-1}$, colored with the different colors $s_1, s_2, \ldots, s_{k-1}$ respectively. Again without loss of generality, we can assume that u_1 is incident at x_1, u_2 to x_2, u_{j-1} to x_{j-1} and that not one of the rays $u_1, u_2, \ldots, u_{j-1}$ is incident at the vertex x_j. We now introduce a new, 2k-th color and recolor the rays $u_1, u_2, \ldots, u_{j-1}$, with it, and also use it to color the ray (x, x_j) of G. Then the rays $(x, x_1), (x, x_2), \ldots, (x, x_{j-1})$ can be colored with the colors $s_1, s_2, \ldots, s_{j-1}$ respectively.

This proves the theorem.

The following theorem is proved in [8]: if in the n-vertex graph G the sum of the degrees of any two nonadjacent vertices is not less than n, then the graph G has a Hamilton circuit. From this theorem of Ore's and Theorem 8 we get the following corollary.

Corollary 4. If G is an n-vertex graph with $q(G) = n$, then G has a Hamilton circuit.

7. Critical Graphs

The graph G is said to be $(m + 1)$-critical, where m is an integer ≥ 2, if:

a) G is connected;

b) $\delta(G) = m$;

c) $q(G) = m + 1$;

d) the chromatic class of the graph obtained from G by removing any ray is equal to m.

The study of $(m + 1)$-critical graphs is of interest primarily for the following two reasons. First, critical graphs can be used in inductive proofs; therefore a knowledge of the basic properties of $(m + 1)$-critical graphs can be very useful. Second, in studying $(m + 1)$-critical graphs we may discover the structural and numerical features of graphs that determine which of two possible values the chromatic class takes (Corollary 1)*.

We give only a few simple properties of $(m + 1)$-critical graphs and produce certain hypotheses.

Property 1. An $(m + 1)$-critical graph cannot have a coupling point.

Proof. Assume the contrary. Let $G(X, U)$ be an $(m + 1)$-critical graph; x a coupling point of G; $G_1(X_1, U_1)$; $G_2(X_2, \ldots, U_2)$; ... ; $G_k(X_k, U_k)$ the connected components of the subgraph generated by removing x. Let H_i ($i = 1, 2, \ldots, k$) denote the subgraphs of G generated by the vertices $X_i \cup x$, respectively. Obviously, $q(H_i) \leq m$ for all

* Dirac and other authors [9-16] have studied critical graphs associated with the chromatic number (i.e., graphs such that when any vertex is removed the chromatic number is reduced).

$i = 1, 2, \ldots, k$. On the other hand, it is clear that all the rays of H_i can be (regularly) colored with m colors in such a way that the rays of different subgraphs incident at x all have different colors. But then all the rays of G could also be colored using m colors, which contradicts the fact that $q(G) = m + 1$.

Property II. If G is an $(m + 1)$-critical graph and a and b are any two adjacent vertices of G, then $\sigma(a) + \sigma(b) \geqslant m + 2$.

Proof. Let us color all the rays of G with the exception of (a, b) with m colors. Since $q(G) = m + 1$, we have $\delta(a) \cap \delta(b) = \varnothing$; therefore all m colors are used to color rays coming from a and b, i.e., $\sigma(a) - 1 + \sigma(b) - 1 \geqslant m$, and so $\sigma(a) + \sigma(b) \geqslant m + 2$.

Property III. In an $(m + 1)$-critical graph each vertex is adjacent to at least two vertices of degree m.

Proof. Assume the contrary. Let a be a vertex of the $(m + 1)$ critical graph G which is adjacent to less than two vertices of degree m and let b be the vertex of highest degree adjacent to a. Then the degrees of all the other vertices of G adjacent to a do not exceed $m - 1$. Since we can color all the rays of G except (a, b) using m colors, then, according to Lemma 3, the conditions of which, as it is easy to verify, are satisfied, we can color the ray (a, b) also using the same m colors, and this contradicts $q(G) = m + 1$.

Further investigation of $(m + 1)$-critical graphs is possible in other directions. Thus, it would be interesting to estimate the number of rays in an $(m + 1)$-critical graph or the length of the maximum elementary circuit or the maximum elementary cycle.

Hypothesis 1. (Generalization of Theorem 8).

The number of rays in an $(m + 1)$-critical graph $> m^2/2$.

Hypothesis 2. (Generalization of Corollary 4.)

An $(m + 1)$-critical graph has an elementary cycle whose length is $\geq m + 1$.

More essential properties of $(m + 1)$-critical graphs are reflected in the following hypothesis.

Hypothesis 3. An $(m + 1)$-critical graph possesses a factoroid*.

The following hypothesis follows from Hypothesis 3.

Hypothesis 4. If G is an $(m + 1)$-critical graph its internal stability number** $\leq n/2$.

To conclude this section, we use $(m + 1)$-critical graphs to prove Theorem 9, which should belong to Section 6.

Theorem 9. If G is a plane graph with $\sigma(G) \geq 10$, then $q(G) = \sigma(G)$.

Proof. Suppose that there exists a graph $G(X, U)$ with $\sigma(G) = m \geq 10$ and $q(G) = m + 1$. Obviously, we can assume that G is an $(m + 1)$-critical graph. Let S denote the subset of those vertices, and only those, of G whose degree does not exceed 5. The set S is nonempty, for G is a plane graph (see [1]). Since the subgraph generated by the set of vertices $X \setminus S$, is also plane, there is a vertex $x \in X \setminus S$, in G which is adjacent to not more than five vertices of the set $X \setminus S$. But since $x \notin S$, there exists a vertex $y \in S$ adjacent to x. Let us use m colors to color all the rays of G apart from (x, y); this is possible, for G is an $(m + 1)$-critical graph. We have:

$$|\delta(y)| \geqslant m - 4; \qquad |\delta(x)| \geqslant 1,$$

and since $q(G) = m + 1$, $\delta(x) \cap \delta(y) = \varnothing$. Hence not fewer than $m - 4$ rays come from x, colored in colors absent from y. Let these rays be (x, x_1), (x, x_2), $\ldots$, (x, x_k), where $k \geq m - 4$. Since $k \geq m - 4 \geq 6$, at least one of the vertices $x_j (j = 1, 2, \ldots, k)$ belongs to S. Suppose, for definiteness, than $x_1 \in S$ and let the ray (x, x_1) be colored in the color $s_1 \in \delta(y)$. Since $m \geq 10$; $|\delta(x_1)| \geqslant m - 5$ and $|\delta(y)| \geqslant m - 4$, we have $\delta(x_1) \cap \delta(y) \neq \varnothing$, i.e., there is a color $s \in \delta(x_1) \cap \delta(y)$. Let us take the color $t \in \delta(x)$ and consider the three different vertices x_1, y, and x. We have $s \in \delta(x_1)$; $s \in \delta(y)$; $t \in \delta(x)$, and $s \neq t$, since $\delta(x) \cap \delta(y) = \varnothing$.

The vertices x and y of G must be connected by an (s, t)-circuit for otherwise, by recoloring the maximum (s, t)-circuit starting at x, we could color (x, y) with s, which contradicts $q(G) = m + 1$. Then, from Lemma 2, x_1 is not connected to either x or y by an (s, t)-circuit. Therefore, after recoloring the maximum (s, t)-circuit starting at x_1, we would be able, by recoloring (x, x_1) in t, to color (x, y) of G with the color s_1, which contradicts $q(G) = m + 1$.

This proves the theorem.

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* The graph G possesses a factoroid if, by removing certain rays from G, we obtain a homogeneous graph of degree 2 with the same set of vertices.

** The internal stability number of a graph is the maximum number of mutually nonadjacent vertices.

8. On the Chromatic Class of a Complementary Graph

In this section we study the relation between the chromatic class $q(G)$ of the graph G and the chromatic class $q(\bar{G})$ of its complement. Similar studies were made in [17] and [18].

Theorem 10. If G is an n-vertex graph, then

$$n-1 \leqslant q(G)+q(\bar{G}) \leqslant 2(n-1),$$
$$0 \leqslant q(G)\cdot q(\bar{G}) \leqslant (n-1)^2.$$

When n is even the theorem follows from Corollary 3. Obviously, for any even $n \geq 4$ the estimates of Theorem 10 cannot be improved. When n is odd we have the following stronger theorem.

Theorem 10'. If G is a graph with an odd number $n \geq 3$ of vertices, then

$$n \leqslant q(G)+q(\bar{G}) \leqslant 2n-3,$$
$$0 \leqslant q(G)\cdot q(\bar{G}) \leqslant (n-1)\cdot(n-2).$$

Proof. The inequalities $n \leqslant q(G)+q(\bar{G})$ and $0 \leqslant q(G)\cdot q(\bar{G})$ are obvious. The inequality $q(G)\times q(\bar{G}) \leqslant (n-1)\cdot(n-2)$ follows from $q(G)+q(\bar{G}) \leqslant 2n-3$. We have therefore only to prove $q(G)+q(\bar{G}) \leqslant 2n-3$.

When $n = 3$ this is trivial. Now let $n = 2k + 1$, where $k \geq 2$.

If $\max[\sigma(G), \sigma(\bar{G})] \leqslant n-3$, then $q(G) \leqslant n-2$ and $q(\bar{G}) \leqslant n-2$, and, therefore $q(G)+q(\bar{G}) < 2n-3$. There are two other possibilities: either $\max[\sigma(G), \sigma(\bar{G})] = n-2$, or $\max[\sigma(G), \sigma(\bar{G})] = n-1$.

Case 1. Let $\max[\sigma(G), \sigma(\bar{G})] = \sigma(G) = n-2$. Then $q(G) \leqslant n-1$. Let us show that $q(\bar{G}) \leqslant n-2$. Let x be a vertex which has degree $n-2$ in G; and y the only vertex in G which is not adjacent to x. Let H denote the subgraph of $\bar{G}$ obtained from $\bar{G}$ by removing x. According to Corollary 3 we can color all the rays of H using not more than $n-2$ colors. Since $\sigma(\bar{G}) \leq n-2$ the degree of y in H does not exceed $n-3$. Therefore one of the $n-2$ colors is absent from y in H. We can use this color to color the ray (x, y) of $\bar{G}$, i.e., $q(\bar{G}) \leq n-2$, which is what we were required to prove.

Case 2. Let $\max[\sigma(G), \sigma(\bar{G})] = \sigma(G) = n-1$. Then, from Corollary 3, $q(\bar{G}) \leqslant n-2$. If $q(G) = n-1$, then $q(G)+q(\bar{G}) \leqslant 2n-3$. Let $q(G) = n = 2k+1$. Then, from Theorem 8, the minimum degree of a vertex of G is not less than $k + 1$; hence $\sigma(\bar{G}) \leq k-1$. Therefore $q(\bar{G}) \leq k$. Then $q(G)+q(\bar{G}) \leqslant 3k+1 = 2n-(k+1)$, and so for $k \geqslant 2$, $q(G)+q(\bar{G}) \leqslant 2n-3$.

This proves the theorem.

It is clear that the estimates of Theorem 10 are exact for any odd $n \geq 3$.

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