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Bounds for the Chromatic Number of a Graph*

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ABSTRACT. A lower bound is obtained for the chromatic number $\chi(G)$ of a graph G in terms of its vertex degrees. A short proof of a known upper bound for $\chi(G)$, again in terms of vertex degrees, is also given.

1. INTRODUCTION

In this note we present lower and upper bounds for the chromatic number $\chi(G)$ of a graph G in terms of its vertex degrees. The upper bound is not new. It was originally obtained by Welsh and Powell [3], and is also a consequence of a theorem of Szekeres and Wilf [2]. However we give here a simple proof of the result. In what follows, G is a finite undirected graph with no loops or multiple edges. G has order N and vertex degrees $\{d(i)\}_1^N$, where $d(1) \geqslant d(2) \geqslant \cdots \geqslant d(N)$. We shall adopt the convention that

$$\sum_j^k a_i = 0 \qquad \text{when} \quad k < j.$$

2. LOWER BOUND

THEOREM 1. *Let σ_j be defined recursively by*

$$\sigma_j = N - d\left(\sum_1^{j-1} \sigma_i + 1\right).$$

Suppose that k is some integer satisfying

$$\sum_1^{k-1} \sigma_j < N. \tag{A}$$

Then $\chi(G) \geqslant k$.

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PROOF: G has chromatic number $\chi(G)$ and therefore the set $V(G)$ of vertices of G can be partitioned into $\chi(G)$ subsets $\{V_i\}_1^{\chi(G)}$, where each V_i is an independent set of vertices in G. Suppose V_i has cardinal number n_i and that $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_{\chi(G)}$. Each vertex in V_i has degree at most $N - n_i$ since it is not joined to any of the other $n_i - 1$ vertices of V_i. Hence

$$d\left(\sum_1^{j-1} n_i + 1\right) \leqslant N - n_j \qquad (1 \leqslant j \leqslant \chi(G)). \tag{1}$$

We show, by induction, that $n_i \leqslant \sigma_i$ $(1 \leqslant i \leqslant \chi(G))$. This is so for $i = 1$ since, by (1), $n_1 \leqslant N - d(1) = \sigma_1$. Assume the result true for all $i < j \leqslant \chi(G)$. Then, again by (1) and the assumption that $d(\cdot)$ is a decreasing function of i,

$$n_j \leqslant N - d\left(\sum_1^{j-1} n_i + 1\right) \leqslant N - d\left(\sum_1^{j-1} \sigma_i + 1\right) = \sigma_j .$$

Therefore

$$N = \sum_1^{\chi(G)} n_i \leqslant \sum_1^{\chi(G)} \sigma_i .$$

By (A) this means that $\chi(G) > k - 1$, that is, $\chi(G) \geqslant k$.

COROLLARY 1.1. *If G is regular of degree d then $\chi(G) \geqslant N/(N - d)$.*

3. UPPER BOUND

THEOREM 2.

$$\chi(G) \leqslant \max_{1 \leqslant i \leqslant N} \min\{d(i) + 1, i\}.$$

PROOF: Let G' be a critical subgraph of G. Then each vertex of G' has degree at least $\chi(G) - 1$. Therefore G' has at least $\chi(G)$ vertices of degree at least $\chi(G) - 1$ and *a fortiori* the same is true of G. Hence $d(\chi(G)) \geqslant \chi(G) - 1$. Therefore

$$\max_{1 \leqslant i \leqslant N} \min\{d(i) + 1, i\} \geqslant \min\{d(\chi(G)) + 1, \chi(G)\} = \chi(G).$$

COROLLARY 2.1 (Nordhaus and Gaddum [1]). *Let $\bar{G}$ be the complement of G. Then*

$$\chi(G) + \chi(\bar{G}) \leqslant N + 1.$$

PROOF: $\bar{G}$ has degree sequence $\{\bar{d}(i)\}_1^N$, where

$$\bar{d}(i) = N - 1 - d(N - i + 1).$$

Therefore, by Theorem 2,

$$\begin{aligned}
\chi(G) + \chi(\bar{G}) &\leqslant \max_{1 \leqslant i \leqslant N} \min\{d(i) + 1, i\} + \max_{1 \leqslant i \leqslant N} \min\{N - d(N - i + 1), i\} \\
&= \max_{1 \leqslant i \leqslant N} \min\{d(i) + 1, i\} + N + 1 \\
&\quad - \min_{1 \leqslant i \leqslant N} \max\{d(N - i + 1) + 1, N - i + 1\} \\
&\leqslant \max_{1 \leqslant i \leqslant N} \min\{d(i) + 1, i\} + N + 1 \\
&\quad - \max_{1 \leqslant i \leqslant N} \min\{d(N - i + 1) + 1, N - i + 1\} \\
&= N + 1.
\end{aligned}$$

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