國立交通大學 108 學年度碩士班考試入學招生試題

科目：線性代數與離散數學(1102) 考試日期:108年2月13日 第 2 節

系所班別：資訊聯招

【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. **(8 points)** Let V be the vector space of 2×2 matrices with real entries, and P_3 the vector space of real polynomials of degree 3 or less. Define the linear transformation $T: V \to P_3$ by
$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 2a + (b-d)x - (a+c)x^2 + (a+b-c-d)x^3.$$
Find the rank and nullity of T.

2. **(5 points)** Let $\mathbf{A}$ be an $n \times n$ matrix with real entries and n is odd. Show that it is not possible for $\mathbf{A}^2 + \mathbf{I} = \mathbf{O}$, in which $\mathbf{I}$ is the identity matrix and $\mathbf{O}$ is the zero matrix.

3. Let $C[-1,1]$ be the vector space over R of all continuous functions defined on the interval $[-1,1]$. Let $V: \{f(x) \in C[-1,1] | f(x) = ae^x + be^{2x} + ce^{3x}, a, b, c \in R\}$.
   a. **(2 points)** Prove that V is a subspace of $C[-1,1]$.
   b. **(5 points)** Prove that $B = \{e^x, e^{2x}, e^{3x}\}$ is a basis of V.
   c. **(5 points)** Prove that $B' = \{e^x - 2e^{3x}, e^x + e^{2x} + 2e^{3x}, 3e^{2x} + e^{3x}\}$ is a basis of V.

4. Given $\mathbf{A} = \begin{bmatrix} 1 & -3 & -5 \\ 1 & 1 & -2 \\ 1 & -3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -6 \\ 1 \\ 1 \\ 6 \end{bmatrix}$.
If the Gram-Schmidt process is applied to determine an orthonormal basis for $R(\mathbf{A}) = \{\mathbf{b} \in R^m | \mathbf{b} = \mathbf{A}_{mn}\mathbf{x}\}$ and QR factorization of $\mathbf{A}$, then, after the first one orthonormal vector $\mathbf{q}_1$ and r_{11} are computed, we have
$$\mathbf{Q} = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_2] = \begin{bmatrix} 0.5 & - & - \\ 0.5 & - & - \\ 0.5 & - & - \\ 0.5 & - & - \end{bmatrix} \text{ and } \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} = \begin{bmatrix} 2 & - & - \\ 0 & - & - \\ 0 & - & - \end{bmatrix}.$$
   a. **(5 points)** Finish above process and determine $\mathbf{q}_2$ and $\mathbf{q}_3$, and fill in the columns of $\mathbf{Q}$.
   b. **(5 points)** Finish above process and determine $\mathbf{R}$.
   c. **(5 points)** Use the QR factorization to find the least squares solution to $\mathbf{Ax} = \mathbf{b}$.

5. Let $\mathbf{C} = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$ is a 3x3 matrix.
   a. **(5 points)** Find the value of $\lim_{n\to\infty} \mathbf{C}^n$.
   b. **(5 points)** Compute the value of $e^{\mathbf{C}}$.

6. Let $Q(x)$ be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are these truth values?
   a. **(2 points)** $\exists x\, Q(x)$
   b. **(2 points)** $\forall x\, Q(x)$
   c. **(2 points)** $\exists x\, \neg Q(x)$
   d. **(2 points)** $\forall x\, \neg Q(x)$

7. Determine whether each of the following conditional statements is a tautology or not. If yes, provide a proof. If no, provide a counter example.
   a. **(3 points)** $p \to (\neg q \vee r)$
   b. **(3 points)** $\neg p \to (q \to r)$
   c. **(3 points)** $(p \to q) \vee (\neg p \to r)$

8. Determine whether each of these functions is a bijection from R to R.
   a. **(2 points)** $f(x) = -3x + 4$
   b. **(2 points)** $f(x) = -3x^2 + 7$
   c. **(2 points)** $f(x) = (x+1)/(x+2)$
   d. **(2 points)** $f(x) = x^5 + 1$

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9. Let R and B be sets of red and blue balls, respectively, with $m = |R|$ and $n = |B|$. Suppose that $m < n$. Let $G = (V, E)$ be a undirected graph with vertex set V and edge set E such that $V = R \cup B$ and $E = \{(u, v) \mid u \in R \text{ and } v \in B\}$. Let E' be an edge cut of G with the minimum number of edges.

a. **(4 points)** What is the value of $|E'|$?

b. **(6 points)** Let $G' = G - E'$. What is the value of $\sum_{v \in G'} \deg(v)$, where $\deg(v)$ is the node degree of v?

10. **(5 points)** Let S be the set of all bit strings of length n. Let $\angle$ be a binary relation defined on S such that $a \angle b$ if and only if a and b differ in exactly k bit positions for any $a, b \in S$. Let $R = \{(a, b) \mid a \angle b, a, b \in S\}$. If we represent R using a zero-one matrix, how many 1's are there in the matrix?

11. A DNA sequence of length n is a sequence of n molecules, where each molecule is represented by either 'C', 'G', 'A', or 'T'. Consider a sequence $a_1, a_2, \ldots, a_n$, where a_i, $1 \le i \le n$, denotes the number of DNA sequences of length i that contain two consecutive 'G's. We may express a_n as a recurrence relation

$$a_n = s \times a_{n-1} + t \times a_{n-2} + f(n),$$

where s and t are positive integers and $f(n)$ is a function of n.

a. **(3 points)** $a_4 = ?$

b. **(3 points)** What is the value of $s + t$?

c. **(4 points)** What is the definition of $f(n)$?