國立交通大學107學年度碩士班考試入學試題

科目：線性代數與離散數學(1102) 考試日期：107年2月2日 第2節

系所班別：資訊聯招 第1頁,共3頁

【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

First you need to write down your answers clearly and then explain how to compute the answers. You also need to answer the questions in order. Do not jump around.

1. **(5 points)** The notation 2^A denotes the collection of all subsets of the set A. The notation $A \times B$ denotes the Cartesian product of A and B. The notation $A \cap B$ denotes the intersection of A and B. How many elements are there in the following set?

$$(2^{\{a,b\}} \times 2^{\{b,c\}}) \cap (2^{\{a,b,c\}} \times 2^{\{\}})$$

2. **(5 points)** (subbags) We define a *bag* is a set that allows repetitions of elements. For example, {1, 2, 2, 2, 3, 3, 4} and {1, 2, 3} are (different) bags. Obviously, every set is also a bag.

   We define $\#(x|S)$ as the repetitions of element x in bag S. For example, $\#(2|\{1, 2, 2, 2, 3, 3, 4\}) = 3$, $\#(1|\{1, 2, 2, 2, 3, 3, 4\}) = 1$, $\#(5|\{1, 2, 2, 2, 3, 3, 4\}) = 0$, etc.

   A *subbag* S of a bag T satisfies the following conditions:

   (i) S is a subset of T when they are viewed as sets;

   (ii) for every element x, $\#(x|S) \leq \#(x|T)$.

   For example, {1, 2, 3, 3, 4} is a subbag of {1, 2, 2, 3, 3, 3, 4} while {1, 2, 2, 3, 3, 3, 4} is not a subbag of {1, 2, 2, 3, 3, 4}.

   How many subbags of {4, 5, 5, 6, 6, 6, 7, 7} are there?

3. **(5 points)** Assume we have already defined a predicate *prime*(x) which is true if and only if x is a prime number. Otherwise, the predicate is false. Also assume that we are concerned with natural numbers only. Please translate the following statement into a formula in predicate logic:

   "Every natural number that is greater than 6 can be written as the sum of three prime numbers."

4. **(10 points)** Let n be a natural number. Let $b = n \bmod 10$ and $a = (n-b)/10$, where a and b are nonnegative integers. Please find (negative or positive) integers x and y ($-9 \leq x \leq 9$ and $-9 \leq y \leq 9$) so that the following claim is true:

   "n is a multiple of 17 if and only if $xa + yb$ is a multiple of 17."

   You need to explain or prove your answer.

5. **(5 points)** Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$.

   (1) $a_n = 4n - 2$.

   (2) $a_n = n(n + 1)$.

6. **(5 points)** Find the flaw with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.

   Basis Step: We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.

   Inductive Step: Assume that we can form postage of j cents for all nonnegative integers j with $j \leq k$ using just three-cent and four-cent stamps. We can then form postage of k + 1 cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four-cent stamps by three three-cent stamps.

7. **(5 points)** What is the probability that when a coin is flipped six times in a row,

   (1) it lands heads up every time?

   (2) it lands heads up three times?

   (3) it lands heads up even number of times?

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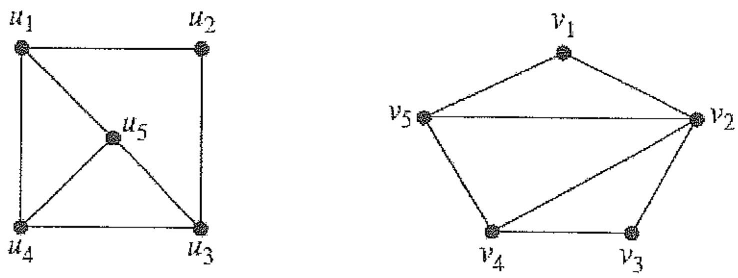
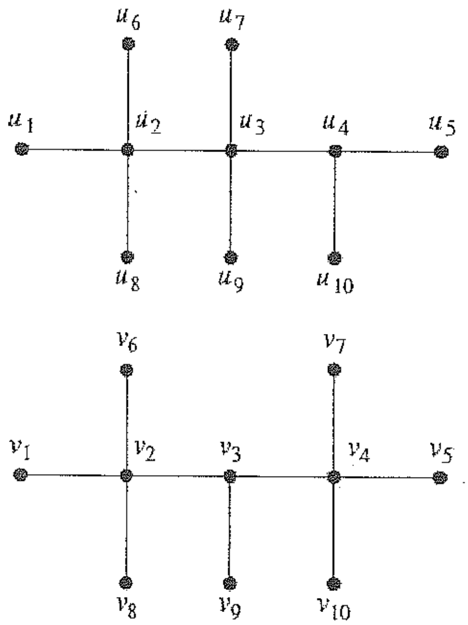
8. **(5 points)** (1) Find the smallest equivalence relation on the set {a, b, c, d, e} containing the relation {(a, b), (a, c), (d, e)}. (2) Identify the equivalent classes partitioned by this equivalent relation.

9. **(5 points)** determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

a)



b)



Question # 10-14: This part consists of true-or-false questions. In each case, please answer true if the statement is always true and false otherwise.

10. **(5 points)** A matrix is said to be in reduced row echelon form, if
   (1) the nonzero entry in each nonzero row is 1.
   (2) row m does not consist entirely of zeros, the number of leading zero entries in row $m+1$ is greater than the number of leading zero entries in row m.
   (3) there are rows whose entries are all zero, they are below the rows having nonzero entries.
   (4) the first nonzero entry in each row is the only nonzero entry in its column.
   (5) its augmented matrix is treated as a homogeneous system.

11. **(5 points)** A nxn matrix A is said to be nonsingular, if
   (1) A is non-invertible.

(2) there exists a multiplicative inverse matrix **B** such that **A** is row equivalent to **B**.
(3) the determinate of **A** is nonzero.
(4) **Ax**=**0** where **x** is nonzero vector.
(5) **A** is row equivalent to **I**.

12. **(5 points)** There are many types of elementary matrices. The properties are described in elementary matrices if
(1) a *nxn* matrix is obtained by interchanging two rows of **I**.
(2) a *nxn* matrix is obtained by multiplying a column of **I** by a nonzero constant.
(3) a *nxn* matrix is obtained from **I** by adding a multiple of one row to another row.
(4) a *nxn* permutation matrix is formed from the **I** by reordering its columns.
(5) the determinate of *nxn* matrix is always 1.

13. **(5 points)** If there are two *nxn* matrices **A**, **B** and one scalar *r*,
(1) det(**A**) = det(**B**) implies **A** = **B**.
(2) det(r**A**+**B**) = r det(**A**)+det(**B**).
(3) det($(\mathbf{AB})^T$) = det(**A**)det(**B**).
(4) det(**A**) = det(**B**) where **A** and **B** are row equivalent matrices.
(5) det($(\mathbf{AB})^{-1}$) = det($\mathbf{A}^{-1}$)det($\mathbf{B}^{-1}$).

14. **(5 points)** Defining the $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ being n column vectors in $\mathbb{R}^n$ and let $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n]$,
(1) the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ will be linearly independent if and only if **A** is nonsingular.
(2) the determinate of **A** is nonzero.
(3) the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ will be linearly independent if and only if the null space $N(\mathbf{A}) = \{\mathbf{0}\}$.
(4) the rank of **A** plus the nullity of **A** equals n.
(5) the rank of $\mathbf{A}^T$ equals the rank of **A**.

15. Calculation

(1) **(6 points)** Given a matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 1 & 2 & 1 & 12 \\ 0 & 3 & -3 & 0 \end{bmatrix}$. What is the projection of $x = \begin{bmatrix} -3 \\ 5 \\ 8 \\ 5 \end{bmatrix}$ onto the row space of **A**.

(2) **(6 points)** Find orthogonal vectors that form an ordered basis B_1, where $B_1 = \left\{ \begin{bmatrix} b_{11} \\ b_{21} \\ 0 \end{bmatrix}, \begin{bmatrix} b_{12} \\ b_{22} \\ 3 \end{bmatrix}, \begin{bmatrix} b_{13} \\ b_{23} \\ -3 \end{bmatrix} \right\}$, of $\mathbb{R}^3$ by Gram-Schmidt from the columns of **A**.

(3) **(6 points)** Find the eigenvalues and the eigenvectors that form an ordered basis B_2, where $B_2 = \left\{ \begin{bmatrix} c_{11} \\ 1 \\ c_{31} \end{bmatrix}, \begin{bmatrix} c_{12} \\ -1 \\ c_{32} \end{bmatrix}, \begin{bmatrix} c_{13} \\ c_{23} \\ 1 \end{bmatrix} \right\}$, of $\mathbb{R}^3$ of the matrix $\begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

(4) **(7 points)** Find the coordinate transition matrix from the basis B_1 to the basis B_2.