國立陽明交通大學 111 學年度碩士班考試入學試題

科目：線性代數與離散數學(1102)　　　　考試日期：111 年 2 月 9 日 第 2 節

系所班別：資訊聯招　　　　第 1 頁,共 3 頁

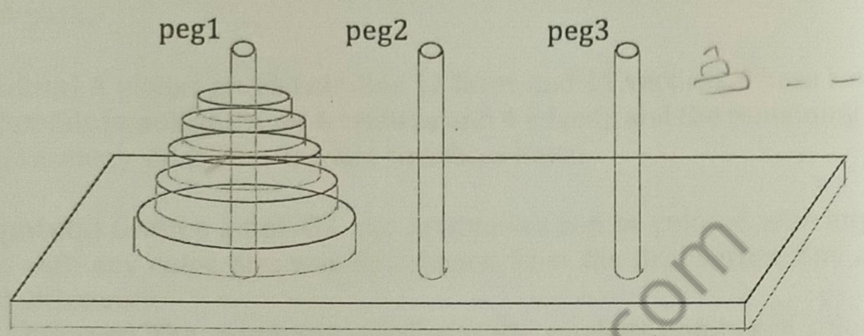
【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. **(25 points)** Consider the game of Hanoi-Tower, where on a board with three erected pegs pile of disks of different sizes are initially stacked at one of the pegs, in a size-ordered manner with the largest disk being at the bottom and on top the smallest.

The player is required to relocate the disk pile to any of the rest two pegs, in compliance with the following rules at any time during the play:

* moving one disk at a time from one peg to another;

* when moving a disk to a peg already piled with disks, the disk must be smaller than any in the pile (which would entail that, during the game, a disk pile at any peg will be size-ordered with smaller ones on top of larger ones);



3　　1 2 3 4
　　1 3 7

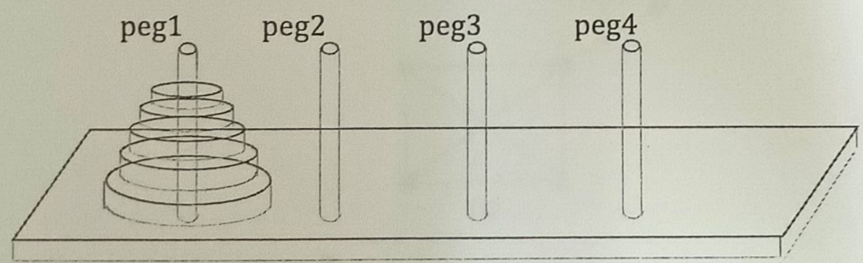
2^n-1

$a_n = 2a_{n-1}+1$

a. (2 points) Suppose that $H^3(n)$ is the number of moves required to relocate a pile of n disks at a peg to any of the rest two pegs. How would you formulate $H^3(n)$ in a recursive manner? And $H^3(n)$ =?

50

b. (3 points) The game can be extended to the case of 4-peg as shown below.



Suppose that $H^4(n)$ denotes the number of moves needed to relocate a pile of n disks at a peg to any of the rest three pegs. How would you formulate $H^4(n)$ in a recursive manner? And $H^4(n)$ =? Is there any connection that you see between $H^3(n)$ and $H^4(n)$?

c. (10 points) Let $H^5(n)$ be the number of moves for relocating a pile of n disks in 5-peg situation. Are there any connections that you see among $H^3(n)$, $H^4(n)$ and $H^5(n)$?

d. (10 points) What can you tell about $H^m(n)$ in the case of an m-peg Hanoi-Tower game? And the connections between $H^{m_1}(n)$ and $H^{m_2}(n)$?

2 8 26 80

4^n-1　　3 15 63

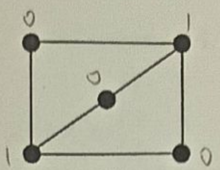
國立陽明交通大學 111 學年度碩士班考試入學試題

科目：線性代數與離散數學(1102)
系所班別：資訊聯招

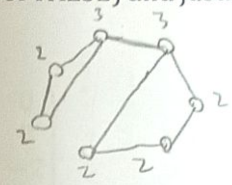
考試日期：111 年 2 月 9 日 第 2 節

【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

2. (**5 points**) Please indicate if the graph G1 is a bipartite graph (TRUE or FALSE) and justify your answer.



G1



33 22222

3. (**5 points**) Given a graph G2 that contains 7 vertices. The degree of two vertices in G2 is 3. The degree of the remaining vertices is 2. Please show if the G2 contains an Euler path. Construct the G2 if G2 exists an Euler path.

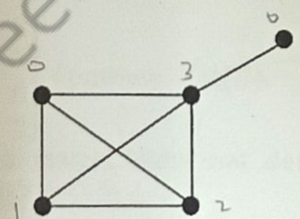
4. (**5 points**) A planar graph contains 12 faces and 11 vertices. These faces consist of six triangles, four quadrilaterals (a polygon with 4 vertices and 4 edges), and the remaining ones have the same number of sides. How many sides does the last two faces have?

5. (**10 points**) Given a graph G3, the first vertex can be colored with any color. The second one can be colored with any color that was not chosen from the first vertex. The adjacent vertices of the G3 are colored differently.

a. (2 points) The chromatic number is the smallest number of colors needed to produce a proper coloring of a graph. What is the chromatic number of G, denoted by $_\chi(G)$?

b. (3 points) The chromatic polynomial is a polynomial that represents the number of distinct ways to color the vertices of a graph. What is the chromatic polynomial of G3, P_G?

c. (5 points) Let $P_G(n)$ be the number of different ways to color the vertices of G3 using n colors, where $n \geq 0$ is an integer. What is minimum number of $P_G(n)$, where $P_G(n) > 0$?



G3

$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-1)$

6. **(25 points)**

✓ a. (6 points) Project the vector $\boldsymbol{b}$ onto the nullspace of $\boldsymbol{A}$, where

$$\boldsymbol{b}=\begin{bmatrix}-3\\-2\\1\\2\\1\end{bmatrix} \text{ and } \boldsymbol{A}=\begin{bmatrix}1&2&1&0&0\\2&5&1&1&0\\3&7&2&2&-2\\4&9&3&-1&4\end{bmatrix}.$$

-3 4
1 -2
1 0
0 2
0 1

1 0 3 0 -4
0 1 -1 0 2
0 0 0 1 -2
0 0 0 0 0

$x_1=-3x_3+4x_5$
$x_2=x_3-2x_5$
$x_4=2x_5$

$= \frac{2}{11}\ -\frac{8}{11}\ \frac{14}{11}\ \frac{22}{11}\ \frac{11}{11}$

(2, -8, 14, 22, 11)

b. Orthogonal Bases.

✓ (i) (8 points) Apply the Gram-Schmidt process (**Requirement: you must process following the column order**, i.e. first column first, then second column, etc. **or you will get zero point**) to obtain orthonormal vectors from the columns of

$$A=\begin{bmatrix}1&4&-2&2\\2&1&3&2\\-1&0&-2&0\\0&1&-1&8\end{bmatrix}$$

$4\ {-2}\ 0\ 2\ 1 - \frac{\langle -3\ 1\ 1\ 0\ 0,\ 4\ {-2}\ 0\ 2\ 1\rangle}{11}$

$4\ {-2}\ 0\ 2\ 1 + \frac{14}{11}(-3\ 1\ 1\ 0\ 0)$

✓ (ii) (3 points) As we known, the orthonormal vectors obtained from (i) cannot span $\mathbb{R}^4$. We can add some additional orthonormal vectors to those obtained from (i) and the new set of orthonormal vectors will span $\mathbb{R}^4$, and what are the additional orthonormal vectors?

c. (8 points) Let $\boldsymbol{A}$ be an m by n matrix. Show that nullspace $N(\mathbf{A}^T\mathbf{A})=N(\mathbf{A})$. (Please leave it blank if you don't know the correct answer, or you will get **at most minus 5 points** for the wrong answer. 不會寫請留白，答錯最多倒扣 5 分，扣至本題組 0 分為止。)

7. **(6 points)**

a. (3 points) Let $B=\begin{bmatrix}1&-1&0&2\\0&0&3&3\\2&1&0&0\\1&1&2&2\end{bmatrix}$. Compute $det(B)$.

1 -1 0 2
0 0 3 3
2 1 0 0
1 1 2 2

1 -1 0 2
0 0 3 3
0 3 0 0
0 2 2 2

b. (3 points) Let Q be an orthogonal matrix. Show that $det(Q)$ is either 1 or -1. （不會寫請留白，答錯最多倒扣 2 分，扣至本題組 0 分為止。）

8. **(12 points)**

$x=1, 1, \frac{3}{2}$

0 1 1
0 0 0
0 ½ ½

Let $A=\begin{bmatrix}4&-1\\6&-1\end{bmatrix}$, $B=\begin{bmatrix}1&1&1\\0&1&0\\0&0.5&1.5\end{bmatrix}$, and $C=\begin{bmatrix}4&-4.5\\2&-2\end{bmatrix}$.

a. (6 points) Check the diagonalizability of each of the above matrices. If it is diagonalizable, please diagonalize it; otherwise, clearly explain why it is not.

b. (3 points) Find A^8 and A^∞.

✓ c. (3 points) Find $det(B+6I_3)$ using eigenvalues.

$77\frac{15}{2}$

9. **(7 points)**

Let $A=\begin{bmatrix}6&2&2\\-2&6&2\end{bmatrix}$, and consider the singular value decomposition (SVD) $A=U\Sigma V^T$.

a. (3 points) Find $U, \Sigma, and\ V$.

b. (4 points) Find orthonormal bases for $C(A^T)$, $N(A)$, $C(A)$, and $N(A^T)$, respectively, from the results obtained from SVD.