【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. (5 points) Consider the following real number:

$$0.02468101214161820\ldots$$

The 1st digit after the decimal point is 0; the second digit is 2; the 13th digit is 6; etc. What is the 1,000th digit after the decimal point?

2. (15 points)
(a) (5 points)
Let $P = \{2, \{(2)\}, \{(2), (2)\}, (2), (2, 2), \{2, (2), (2), 2\}, \{(2, 2), (2, 2), ((2), 2)\}\}$. How many elements are there in 2^P (which is the set of all subsets of P)?
(b) (5 points) What is the definition of an infinite set W?
(c) (5 points) Let N be the set of natural numbers. Assume that we already know N is an infinite set. Let $S = \{k/2^n \mid n = 0, 1, 2, \ldots; k \in N\}$. Note that / in this question is the division of real numbers. For example, $5/2 = 2.5$. Prove that S is an infinite set strictly according to your definition of an infinite set.

3. (5 points) Assume $n \geq 2$. Let $A = \{x_1, x_2, \ldots, x_n\}$ be a set of (not necessarily distinct) natural numbers. Let $g = gcd(x_1, x_2, \ldots, x_n)$ and $l = lcm(x_1, x_2, \ldots, x_n)$. Assume n, g, and l are appropriate fixed integers. Let $p = x_1 \cdot x_2 \cdot \ldots \cdot x_n$, the product of x_i's. What is the largest possible value of p in terms of n, g, and l? What is the smallest possible value of p in terms of n, g, and l?

4. (5 points) How many integers between 1000 and 5000 are not divisible by 4, 5, or 6?
5. (5 points) Let a_k be the number of integer solutions of $x_1 + x_2 + x_3 = k$ where $x_1 \geq 2$, $1 \leq x_2 \leq 5$, and $0 \leq x_3 \leq 3$. Show the generating function for $\{a_k\}$.
6. (5 points) Let R_1 and R_2 be equivalence relations on the set S. Prove that $R_1 \cap R_2$ is an equivalence relation.
7. Please answer TRUE or FALSE for the following statements and give CONCISE explanations.
   (a) (2 points) A simple graph is called regular if every vertex of this graph has the same degree. An n-cube, denoted by Q_n, is a graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position. Q_n is regular.
   (b) (2 points) Let V_n and E_n respectively be the number of vertices and edges of Q_n, the n-cube. Then, $E_1 = 1$, $E_2 = 4$, and $E_{n+1} = 2V_n + E_n$.
   (c) (2 points) The complete bipartite graphs $K_{2,3}$ and $K_{3,3}$ are planar.
   (d) (2 points) There are exactly 2 shortest paths between v_1 and v_2 in the graph represented by the adjacency matrix

$$\begin{array}{c c} & \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

   with $|e_1| = 1,\ |e_2| = 2,\ |e_3| = 1,\ |e_4| = 4,\ |e_5| = 1,\ |e_6| = 4,\ |e_7| = 5$.
   (e) (2 points) In the graph given by the previous question, there exists an Euler path from v_5 to v_1.

8. (10 pt) True or false: NO need to justify your answer. No credit for unanswered questions. Incorrect answers will deduct the credit until zero point is earned in this section.

   (a) (**correct: 1; incorrect: -1; unanswered: 0**) For any square real matrix A, $A^T + A$ is always symmetric.

   (b) (**correct: 1; incorrect: -1; unanswered: 0**) For a square real matrix A, if $A\mathbf{x} = \mathbf{0}$ has a non-zero solution, then A is invertible.

   (c) (**correct: 1; incorrect: -1; unanswered: 0**) $\{(0,0,0)\}$ is the orthogonal complement of $\mathbb{R}^3$ in $\mathbb{R}^3$.

   (d) (**correct: 1; incorrect: -1; unanswered: 0**) Let A be a $m \times n$ real matrix, where $m < n$, with full row rank. $A\mathbf{x} = \mathbf{b}$ has either 0 or 1 solution.

   (e) (**correct: 1; incorrect: -1; unanswered: 0**) Let $\mathbf{v}$ be a vector in S and $\mathbf{u}$ be a vector such that $< \mathbf{u}, \mathbf{v} >= 0$, then $\mathbf{u} \in \mathsf{S}^{\perp}$.

   (f) (**correct: 1; incorrect: -1; unanswered: 0**) Let A be a symmetric matrix. $C(A^T)$ and $N(A^T)$ are orthogonal complement.

   (g) (**correct: 1; incorrect: -1; unanswered: 0**) Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be linearly dependent. Then $\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}$ are linearly dependent.

   (h) (**correct: 1; incorrect: -1; unanswered: 0**) Let U be a subspace of a vector space V. Then $\left(\mathsf{U}^{\perp}\right)^{\perp} = \mathsf{U}$.

   (i) (**correct: 1; incorrect: -1; unanswered: 0**) The null space of A is equal to the column space of $A^T A$.

   (j) (**correct: 1; incorrect: -1; unanswered: 0**) Let A and B be $n \times n$ real matrices. Then, $rank(AB) \leq \min(rank(A), rank(B))$

9. (15 pt) Multiple choice questions and fill-in-the-blank questions. NO need to justify your answer. No credit for unanswered questions. Incorrect answers will deduct the credit until zero point is earned in this section.

   (a) (**correct: 3; incorrect: -1; unanswered: 0**) Let W be a subspace of $\mathbb{R}^n$ and $\mathsf{W}^{\perp}$ denotes its orthogonal complement. If W_1 is a subspace of $\mathbb{R}^n$ such that $\mathbf{x} \in \mathsf{W}_1$, then $\mathbf{x}^T \mathbf{u} = 0$ for all $\mathbf{u} \in \mathsf{W}^{\perp}$. Justify whether the following statements are true or false.

      (i) $\dim(\mathsf{W}_1^{\perp}) \leq \dim(\mathsf{W}^{\perp})$

      (ii) $\dim(\mathsf{W}_1^{\perp}) \leq \dim(\mathsf{W})$

      (iii) $\dim(\mathsf{W}_1^{\perp}) \geq \dim(\mathsf{W})$

      (iv) $\dim(\mathsf{W}_1^{\perp}) \geq \dim(\mathsf{W}^{\perp})$

   (b) (**correct: 3; incorrect: -1; unanswered: 0**) Let A be a 7×5 matrix with $rank(A) = 5$. Justify whether the following statements are true or false.

      (i) There exists at least one $\mathbf{b} \in \mathbb{R}^7$ such that $A\mathbf{x} = \mathbf{b}$ has infinite number of least square solutions.

      (ii) For any $\mathbf{b} \in \mathbb{R}^7$, $A\mathbf{x} = \mathbf{b}$ has infinite number of solution.

      (iii) There exists at least one $\mathbf{b} \in \mathbb{R}^7$ such that $A\mathbf{x} = \mathbf{b}$ has a unique least square solution.

      (iv) For any $\mathbf{b} \in \mathbb{R}^7$, $A\mathbf{x} = \mathbf{b}$ has a unique solution.

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(c) **(correct: 3; incorrect: -1; unanswered: 0)** Let $\hat{\mathbf{B}} = \begin{bmatrix} (i) \\ (ii) \end{bmatrix} \in \mathbb{R}^2$ be the least-squares solution that minimize $\|\mathbf{Y} - \mathbf{XB}\|^2$ where

$$\mathbf{X} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{Y} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

Find the value for (i) and (ii).

(d) **(correct: 3; incorrect: -1; unanswered: 0)** Let $\begin{bmatrix} (i) \\ (ii) \end{bmatrix}$ be the orthogonal projection of $\mathbf{b}$ onto $\mathbf{u}$, where

$$\mathbf{b} = \begin{bmatrix} -24 & 2 \\ -10 & -10 \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 2 \\ -10 \end{bmatrix},$$

Find the value for (i) and (ii).

(e) **(correct: 3; incorrect: -1; unanswered: 0)** Let $\begin{bmatrix} (i) \\ (ii) \\ iii \end{bmatrix}$ be the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$, where

$$\mathbf{y} = \begin{bmatrix} 12 \\ -1 \\ 2 \end{bmatrix} \quad , \quad \mathbf{u}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad , \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Find the value for (i), (ii), and (iii).

10. **(25 points)**

a. Determine.

(i) (1 points) True (T) or false (F):

If $\boldsymbol{A}$ is an invertible matrix, then $\det(\boldsymbol{A}^{-1}) = [\det(\boldsymbol{A})]^{-1}$.

(ii) (1 points) True (T) or false (F):

If $\boldsymbol{M}$ is an $n\times n$ matrix and can be written in the form

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{O} & \boldsymbol{C} \end{pmatrix},$$

where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{O}$ are square matrices, and $\boldsymbol{O}$ is all zero matrix. Then $\det(\boldsymbol{M}) = \det(\boldsymbol{A}) \cdot \det(\boldsymbol{C})$.

(iii) (3 points) Find the determinant of the following matrix:

$$\begin{bmatrix} 2 & 0 & -1 & 1 & 3 \\ 0 & -1 & 0 & -2 & 0 \\ 2 & 1 & 4 & 0 & 3 \\ 2 & -1 & -1 & -3 & 3 \\ 0 & 2 & 0 & 3 & 1 \end{bmatrix}.$$

b. (10 points) Given a sequence 7, -6, 20, -24, 64, -96,... comes from $G_{k+3} = -2G_{k+2} + 2G_{k+1} + 4G_k$ for $k \geq 0$, with $G_0 = 7$, $G_1 = -6$ and $G_2 = 20$. Please find the number G_{100}.

c. (10 points) Prove **generally** that eigenvectors of an $n\times n$ matrix corresponding to distinct eigenvalues are linearly independent. Notice that the number of distinct eigenvalues k is not necessarily equal to n (i.e. a general case with $k \leq n$). (Please leave it blank if you don't know the correct answer, or you will get **at most minus 5 points** for the wrong answer. 不會寫請留白，答錯最多倒扣 5 分，扣至本題組 0 分為止。)