國立臺灣大學 109 學年度碩士班招生考試試題

※ 注意：請於試卷內之「非選擇題作答區」標明題號依序作答。

1. (5%) Consider

$$x_1 + x_2 + \cdots + x_n = r,$$

where $a \le x_i \le b$ for $1 \le i \le n$. What is the generating function for the number of integer solutions to the above equation (where the desired count appears as the coefficient of x^r, where $r = 0, 1, \ldots$)?

2. (5%) What is the number of functions from $\{1,2,\ldots,n\}^m$ to $\{1,2,\ldots,i\}^j$?

3. (10%) Consider

$$x_1 + x_2 + \cdots + x_n < r,$$

where $x_i \ge 0$ for $1 \le i \le n$. What is its number of nonnegative integer solutions when $n = 4$ and $r = 8$?

4. (10%) Derive the solution for a_n that satisfies the recurrence equation $a_n = 3a_{n-1} + n$ with $a_0 = 1$.

5. (10%) The generating function in partial fraction decomposition for the above recurrence equation is _________. (Note that expressions like

$$\frac{x-8}{(x-3)^2} - \frac{9}{x-1}$$

are *not* partial fraction decompositions.)

6. (10%) Prove the following inequality:

$$\binom{n}{\lfloor n/2 \rfloor} \ge \frac{2^n}{n}$$

where $2 \le n$.

7. (10%) If the polynomial function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ satisfies

$$\begin{aligned} f(-2) &= 150 \\ f(-1) &= 16 \\ f(0) &= 2 \\ f(1) &= 18 \\ f(2) &= 166, \end{aligned}$$

then a, b, c, d, e are ____, ____, ____, ____, ____, respectively.

8. (10%) The nullities of the matrices $BB^T - \lambda I$ for $\lambda = 0, 1, 2, 3, 4$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

are ____, ____, ____, ____, ____, respectively.

9. (10%) Let

$$A = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}.$$

Let

$$U = \left\{ a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

The numbers of elements -2, -1, 0, 1, 2 in a matrix B with

$$Bx = \begin{cases} Ax & \text{if } x \in U \\ 0 & \text{if } x \in U^{\perp}. \end{cases}$$

are ____, ____, ____, ____, ____, respectively.

10. (10%) If

$$A = \begin{bmatrix} 0 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

then the numbers of elements $-2, -1, 0, 1, 2$ in a matrix B with

$$\begin{aligned} ABA &= A \\ BAB &= B \\ (AB)^T &= AB \\ (BA)^T &= BA. \end{aligned}$$

are _____, _____, _____, _____, _____, respectively.

11. (10%) The numbers of elements 0, 1, 2, 3, 4 in a Jordan normal form of the matrix

$$A = \begin{bmatrix} 4 & 4 & 2 & 1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

are _____, _____, _____, _____, _____, respectively.