國立交通大學 109 學年度碩士班考試入學招生試題

科目：線性代數與離散數學(1102)　　考試日期：109 年 2 月 4 日 第 2 節

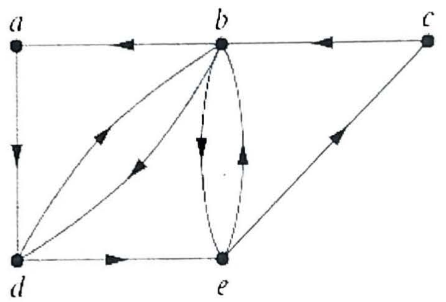
系所班別：資訊聯招　　

【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. Which of these relations on {0, 1, 2, 3} are partial orderings? Note that a partial ordering relation is reflexive, antisymmetric, and transitive. If not a partial ordering relation, explain why it is not.
   [3 points] (a) {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}
   [3 points] (b) {(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)} ✓
   [3 points] (c) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

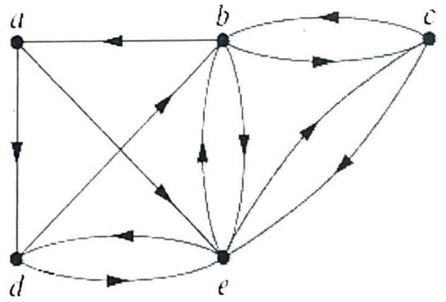
2. Give a recursive definition of the sequence $\{a_n\}$, n = 1, 2, 3, . . . if
   [4 points] (a) $a_n = 1 + (-1)^n$.　$a_n = a_{n-1} + 2(-1)^n$　$a_0 = 2$
   [4 points] (b) $a_n = n(n + 1)$.　$a_1 = 2$　$a_2 = 6$　$a_3 = 12$　$a_1 = 0$　$a_2 = 2$

3. Determine whether the directed graph shown has an Euler circuit starting from vertex a. Construct an Euler circuit if one exists. If no Euler circuit exists, determine whether the directed graph has an Euler path also starting from vertex a. Construct a Euler path if one exists.
   [4 points] (a)



$a_0 = 2$
$a_1 =$
$a_1 = a_0 + (-2) = 0$
$a_2 = a_1 + 2 = 2$

   [4 points] (b)



4. Consider a finite set N. Let $f\colon P(N) \to \mathbb{R}$ be a function from $P(N)$ to $\mathbb{R}$, where $P(N)$ is the power set of N and $\mathbb{R}$ is the set of all real numbers.
   [4 points] (a) What is the maximum cardinality of the range of f?
   [4 points] (b) Suppose that for all $S_1, S_2 \subseteq N$ we have $f(S_1) + f(S_2) \le f(S_1 \cup S_2) + f(S_1 \cap S_2)$. Prove that for all finite sets C and T such that $C \subseteq T \subset N$ and $\forall i \in N - T$, we have $f(C \cup \{i\}) - f(C) \le f(T \cup \{i\}) - f(T)$.

5. [5 points] (a) It is known that there are infinitely many prime numbers. We can prove this by contradiction as outlined below. Assume that $P = \{p_1, p_2, \cdots, p_n\}$ is the set of all prime numbers, where $|P| = n$ and n is a finite number. Consider a number q. If q is a prime number, then we have a prime number not in P, a contradiction. If q is not a prime number, then none of

$p_1, p_2, \cdots, p_n$ divides q, which is another contradiction. We thus have the proof. What is the value of q to make the proof valid?

[5 points] (b) Let domain U be all positive integers in the range $[2, 10]$. Let propositional function $P(x)$ denote "x is a prime number," and $Q(x)$ denote "$x \equiv 3 \pmod 5$." Find a set S with the maximum cardinality that makes "$\forall x \in S\ (P(x) \to Q(x))$" true and "$\exists x \in S\ (P(x) \wedge Q(x))$" false at the same time.

6. [7 points] If a and b are positive integers, then there exist integers s and t such that $\gcd(a,b) = sa - tb$. Let $a = 277$ and $b = 91$ and find the values of s and t such that $\gcd(a,b) = sa - tb$.

7. Let E_1, E_2, and E_3 be 3x3 elementary matrices of type I (interchanging two rows of I), II (multiplying a row of I by a nonzero constant), and III (adding a multiple of one row to another row), respectively. Also let A be a 3x3 matrix with $\det(A)$=6. Assume, additionaly, that E_2 was formed from identity matrix by multiplying its second row by 3. Find the value of each of the following:

[1 point] (a) $\det(E_1 A)$

[1 point] (b) $\det(E_2 A)$

[1 point] (c) $\det(A E_3)$

[1 point] (d) $\det(E_1^2)$

[1 point] (e) $\det(E_3 E_2 E_1)$

8. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix}$,

[5 points] (a) Find elementary matrices F_1, F_2, and F_3 such that $F_3 F_2 F_1 A = U$, where U is an upper triangular matrix.

[5 points] (b) Determine the inverses of F_1, F_2, and F_3 and set $L = F_1^{-1} F_2^{-1} F_3^{-1}$. What type of matrix is L? Verify that $A = LU$.

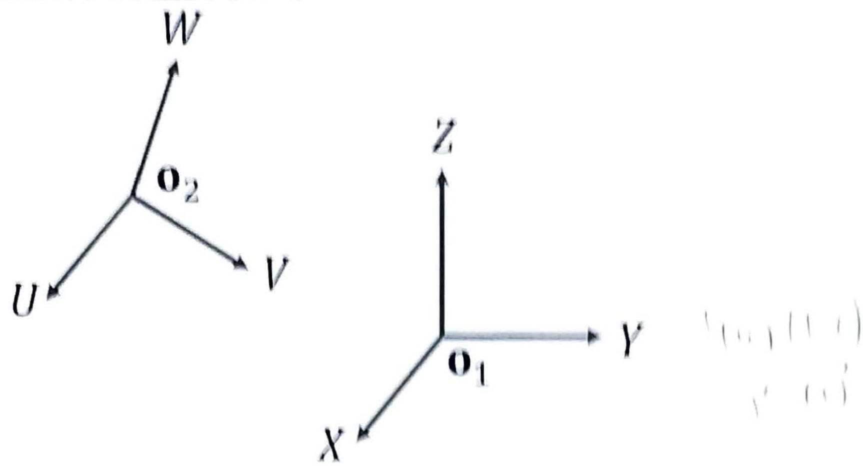
9. Let $A = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3 \quad \boldsymbol{a}_4]$ be a 4x4 matrix with reduced row echelon form given by $U = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. If $\boldsymbol{a}_1 = \begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{a}_2 = \begin{bmatrix} 4 \\ -3 \\ 7 \\ -1 \end{bmatrix}$,

[2 points] (a) Find the rank of A.

[4 points] (b) Find a basis of null space of A.

[4 points] (c) Find $\boldsymbol{a}_3$ and $\boldsymbol{a}_4$.

10. The homogeneous coordinates map $[x \quad y \quad z]^T$ to $[x \quad y \quad z \quad 1]^T$, that can help us format translation in the form of linear transformations. Consider the figure below.



There are two coordinate systems. One is with basis $B_1 = \{\boldsymbol{e}_X, \boldsymbol{e}_Y, \boldsymbol{e}_Z\}$ and origin $\boldsymbol{o}_1$, and the other is with basis $B_2 = \{\boldsymbol{e}_U, \boldsymbol{e}_V, \boldsymbol{e}_W\}$ and origin $\boldsymbol{o}_2$. Both B_1 and B_2 are orthonormal bases and obey the right-handed convention. The relation between B_1 and B_2 is a rotation operation.

[3 points] (a) If $[\boldsymbol{e}_X]_{B_2} = [1 \quad 0 \quad 0]^T$ and $[\boldsymbol{e}_Y]_{B_2} = \left[0 \quad \frac{1}{2} \quad \frac{\sqrt{3}}{2}\right]^T$, please find $[\boldsymbol{e}_Z]_{B_2}$.

[4 points] (b) Without considering translation, i.e., $\boldsymbol{o}_1 = \boldsymbol{o}_2$, please find a 3×3 rotation matrix that can transform B_2 coordinates to B_1 coordinates based on (a).

[3 points] (c) If $[\boldsymbol{o}_1 - \boldsymbol{o}_2]_{B_2} = [1 \quad 2 \quad 1]^T$, please find $[\boldsymbol{o}_1 - \boldsymbol{o}_2]_{B_1}$ based on (a).

[5 points] (d) Find the linear transformation in the homogeneous coordinate system to transform $[u \quad v \quad w \quad 1]^T$ to $[x \quad y \quad z \quad 1]^T$ based on (a) and (c). In details, please find the 4×4 matrix in the following equation.

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ 1 \end{bmatrix}$$

11. Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & i \\ -i & 1 \end{bmatrix}$, $D = \begin{bmatrix} i & 1 \\ 1 & 1 \end{bmatrix}$, and $E = \begin{bmatrix} i & 0 \\ 0 & 2i \end{bmatrix}$.

[2 points] (a) A matrix M is symmetric if $M^T = M$. List matrices that are symmetric.

[2 points] (b) A matrix M is Hermitian if $M^H = M$. List matrices that are Hermitian.

12. [6 points] Let $A = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$. Find a unitary matrix U to diagonalize A, i.e., $D = U^{-1}AU$. Please must arrange the eigenvalues in descending order in the matrix D.