Image Reconstruction 1 – Planar reconstruction from projections

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Outline

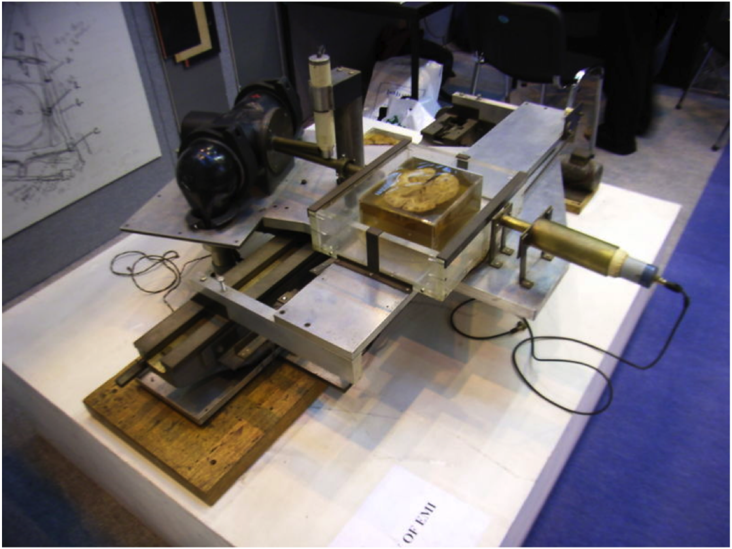
Invention of Computerized Tomography (CT)

Sir Godfrey N. Hounsfield
(Electrical Engineer)
EMI

Allan M. Cormack
(Physicist)
South Africa, Boston

Joint Nobel Prize for Physiology or Medicine, 1979

First CT scanner prototype (Hounsfield apparatus)

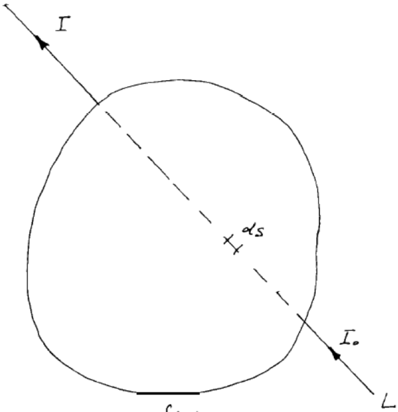


Outline

Projection

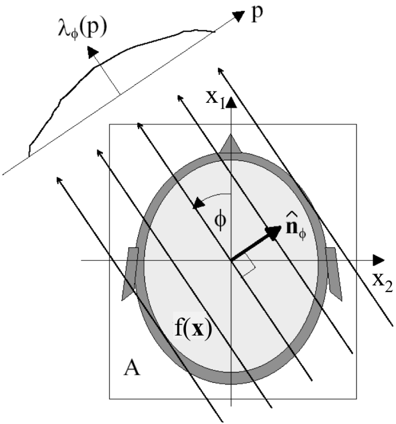
- Consider a function $f(\boldsymbol{x})$ of the variables $\boldsymbol{x} = (x_1, x_2)$ in the plane A.
- In CT, $f(\boldsymbol{x})$ stands for the distribution of attenuation coefficients in a planar cut through the patient's body.
- Let us assume that we know the "projections" (x-ray projections) of $f(\boldsymbol{x})$ for arbitrary projection angles.

From transmission to projection



$$I = I_0 \, e^{-\int_L f \, ds}$$

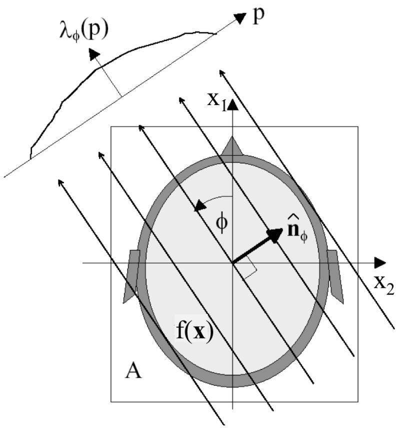
Projection



- Mathematically, the projection λ is the integral of $f(\boldsymbol{x})$ along a (parallel) set of projection lines:

$$\lambda_\phi(p) = \int_A f(\boldsymbol{x})\,\delta(p - \boldsymbol{x}\cdot\hat{\boldsymbol{n}}_\phi)\ \mathrm{d}^2 x$$

Projection



- Mathematically, the projection λ is the integral of $f(\boldsymbol{x})$ along a (parallel) set of projection lines:

$$\lambda_\phi(p) = \int_A f(\boldsymbol{x})\,\delta(p - \boldsymbol{x}\cdot\hat{\boldsymbol{n}}_\phi)\ \mathrm{d}^2x$$

1. Note: A projection line is described in the Hessian normal form by the equation $p = \boldsymbol{x}\cdot\hat{\boldsymbol{n}}_\phi$.
2. Note also: The δ-function “picks” those points $\boldsymbol{x}$ from the plane A that lie on the projection line.

Radon Transform

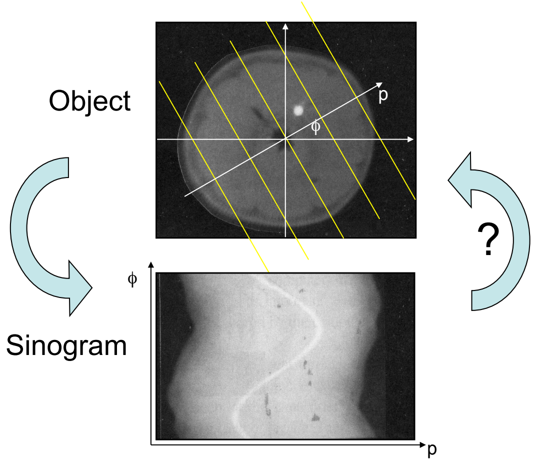
- We will consider all projections of f as a two-dimensional function with the arguments p and ϕ, and write it as $\lambda(p, \phi)$. The transform $f(x_1, x_2) \to \lambda(p, \phi)$ is called a **Radon transform**[1]
- In symbols:

$$\lambda(p, \phi) = \mathfrak{R}\{f(\boldsymbol{x})\}.$$

The problem of reconstructing $f(\boldsymbol{x})$ from the (known) projections $\lambda(p, \phi)$ is basically the determination of the inverse Radon transform, $\mathfrak{R}^{-1}$.

[1] After the mathematician Johann Radon, who described the first mathematical method for a reconstruction from projections as early as in 1917

The problem: inverting the Radon transform



Outline

Backprojection

By backprojection we mean "smearing out" of the values of $\lambda_\phi(p)$ along the projection lines, over the plane A, which results in a streak image. Mathematically, backprojection under an angle ϕ is simply given by:

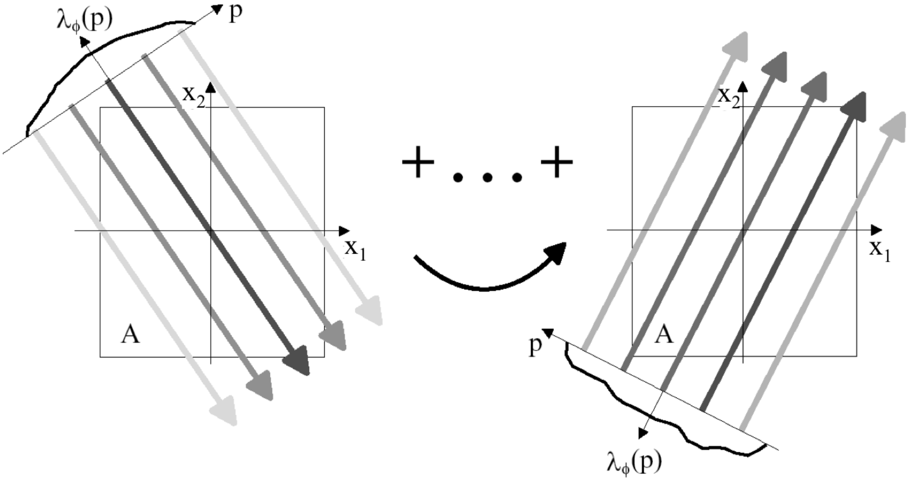
$$f_\phi(\boldsymbol{x}) = \lambda_\phi(\boldsymbol{x} \cdot \hat{\boldsymbol{n}}_\phi).$$

If we perform backprojections for all angles within the interval $[0, \pi)$ and integrate the results, we get

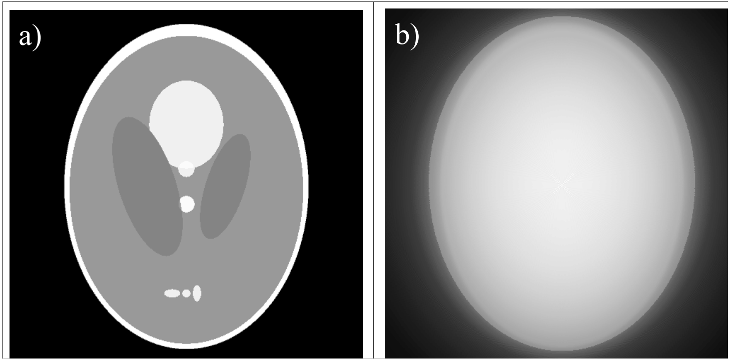
$$f_b(\boldsymbol{x}) = \int_0^\pi \lambda_\phi(\boldsymbol{x} \cdot \hat{\boldsymbol{n}}_\phi)\, \mathrm{d}\phi.$$

$$f_b(\boldsymbol{x}) = \mathfrak{B}\left\{\lambda(p, \phi)\right\} = \mathfrak{B}\,\mathfrak{R}\left\{f(\boldsymbol{x})\right\}.$$

Backprojection



Backprojection alone does not reconstruct the object!



(a) Shepp and Logan phantom

(b) "Reconstruction" of (a) with backprojection

Backprojection alone does not reconstruct the object!

The **Central Slice Theorem** provides the relationship between the one-dimensional (1-D) FT of a projection $\lambda_\phi(p) = \mathfrak{R}_\phi\{f(\boldsymbol{x})\}$ and the 2-D FT of $f(\boldsymbol{x})$:

$$
\begin{aligned}
\Lambda_\phi(\nu) &= \mathfrak{F}_1\left\{\mathfrak{R}_\phi\{f(\boldsymbol{x})\}\right\} \\
&= \int_{-\infty}^{\infty} \left[\int_A f(\boldsymbol{x})\,\delta(p - \boldsymbol{x}\cdot\hat{\boldsymbol{n}}_\phi)\ \mathrm{d}^2 r\right] \exp(-2\pi i\,\nu p)\ \mathrm{d}p \\
&= \int_A f(\boldsymbol{x}) \left[\int_{-\infty}^{\infty} \delta(p - \boldsymbol{x}\cdot\hat{\boldsymbol{n}}_\phi) \exp(-2\pi i\,\nu p)\ \mathrm{d}p\right] \mathrm{d}^2 x \\
&= \int_A f(\boldsymbol{x})\ \exp(-2\pi i\,\nu \boldsymbol{x}\cdot\hat{\boldsymbol{n}}_\phi)\ \mathrm{d}^2 x.
\end{aligned}
$$

The last integral is the 2-D Fourier transform $F(\boldsymbol{\rho})$ of the function $f(\boldsymbol{x})$ along the line $\boldsymbol{\rho} = \nu\hat{\boldsymbol{n}}_\phi$.

Central Slice Theorem

Theorem (Central Slice Theorem)

The 1-D FT of the projection of a 2-D function yields the 2-D FT of the function along a line through the origin of the frequency domain.

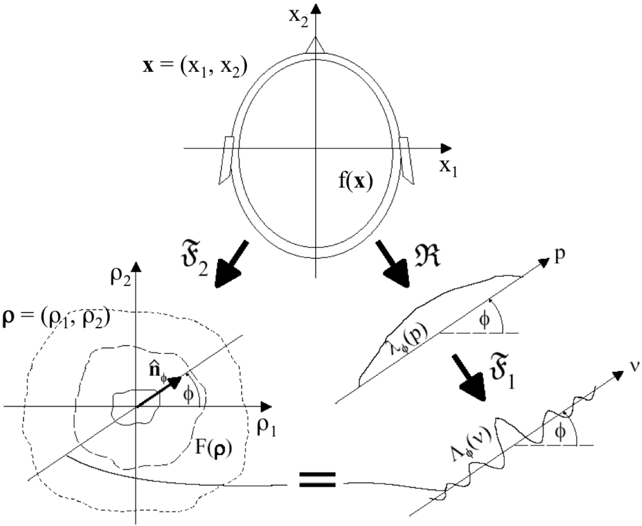
Using operator notation we can write this as:

$$\mathfrak{F}_1\left\{\mathfrak{R}_\phi\left\{f(\boldsymbol{x})\right\}\right\}(\nu) = \mathfrak{F}_2\left\{f(\boldsymbol{x})\right\}(\boldsymbol{\rho} = \nu\hat{\boldsymbol{n}}_\phi)$$

or just

$$\mathfrak{F}_1\mathfrak{R} = \mathfrak{F}_2.$$

Central Slice Theorem



Filtered Backprojection: formal derivation

Write $f(\boldsymbol{x})$ as the inverse Fourier transform of $F(\boldsymbol{\rho})$, in polar coordinates:

$$f(\boldsymbol{x}) = \int_{\infty} F(\boldsymbol{\rho}) \exp(2\pi i \, \boldsymbol{x} \cdot \boldsymbol{\rho}) \, \mathrm{d}^2\rho$$

$$= \int_0^{2\pi} \int_0^{\infty} \nu F(\nu \hat{\boldsymbol{n}}_\phi) \exp(2\pi i \, \nu \boldsymbol{x} \cdot \hat{\boldsymbol{n}}_\phi) \, \mathrm{d}\nu \, \mathrm{d}\phi$$

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For symmetry reasons:

$$f(\boldsymbol{x}) = \int_0^{\pi} \int_{-\infty}^{\infty} |\nu| F(\nu \hat{\boldsymbol{n}}_\phi) \exp(2\pi i \, \nu \boldsymbol{x} \cdot \hat{\boldsymbol{n}}_\phi) \, \mathrm{d}\nu \, \mathrm{d}\phi$$

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With the Central Slice Theorem we obtain finally:

$$f(\boldsymbol{x}) = \int_0^{\pi} \int_{-\infty}^{\infty} |\nu| \Lambda_\phi(\nu) \exp(2\pi i\, \nu \boldsymbol{x} \cdot \hat{\boldsymbol{n}}_\phi)\ \mathrm{d}\nu\, \mathrm{d}\phi$$

Filtered Backprojection: algorithm

The function $f(\boldsymbol{x})$ can be reconstructed from the projection profiles $\lambda_\phi(p)$ using the following steps:

1. *Fourier transform* of $\lambda_\phi(p) \rightarrow \Lambda_\phi(\nu)$;
2. *multiplication* of $\Lambda_\phi(\nu)$ with $|\nu| \rightarrow \Lambda^*_\phi(\nu)$;
3. *inverse Fourier transform* of $\Lambda^*_\phi(\nu) \rightarrow \lambda^*_\phi(p')$;
4. *backprojection* of $\lambda^*_\phi(p')$ *and integration* over $\phi \rightarrow f(\boldsymbol{x})$.

The first three steps are a filtering (convolution) of the projection profiles with the filter $h^{-1}(p)$, which is the inverse FT of $H^{-1}(\nu) = |\nu|$.

Filtered Backprojection: intuitive explanation

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3. This results in a low-pass filtering (blurring) of the image.
4. Can be corrected with $|\nu|$ filter.

Outline

Discrete projection data (sinogram):

- We know $\lambda_{m\cdot\Delta\phi}(n\cdot\Delta p)$ for $n=-N,\ldots,N$, and $m=1,\ldots,M$ with $M=\pi/\Delta\phi$.
- Assume that the sampling interval, Δp, satisfies the Nyquist sampling condition. This means, we assume that projection profiles in the Fourier domain, $\Lambda_\phi(\nu)$, are bandlimited within $-\frac{1}{2\Delta p}<\nu<\frac{1}{2\Delta p}$.
- Then the inverse transfer function $H^{-1}(\nu)=|\nu|$ can be restricted to the same interval, $\left[-\frac{1}{2\Delta p},\frac{1}{2\Delta p}\right]$.
- The modified function

$$H_r^{-1}(\nu)=\begin{cases}|\nu| & \text{for} \quad |\nu|\leq\frac{1}{2\Delta p}\\ 0 & \text{otherwise}\end{cases}$$

is called **“ramp filter”**.

To determine the filter $h_r^{-1}(p)$ in the **spatial domain** we have to do an inverse Fourier transform of $H_r^{-1}(\nu)$m which yields:

$$\begin{aligned} h_r^{-1}(p) &= \mathfrak{F}_1^{-1}\left\{H_r^{-1}(\nu)\right\} \\ &= \frac{1}{4\Delta p^2}\left(2\ \mathrm{sinc}\left(\frac{p}{\Delta p}\right) - \mathrm{sinc}^2\left(\frac{p}{2\Delta p}\right)\right), \end{aligned}$$

where $\mathrm{sinc}(x)$ stands for $\sin(\pi x)/(\pi x)$.

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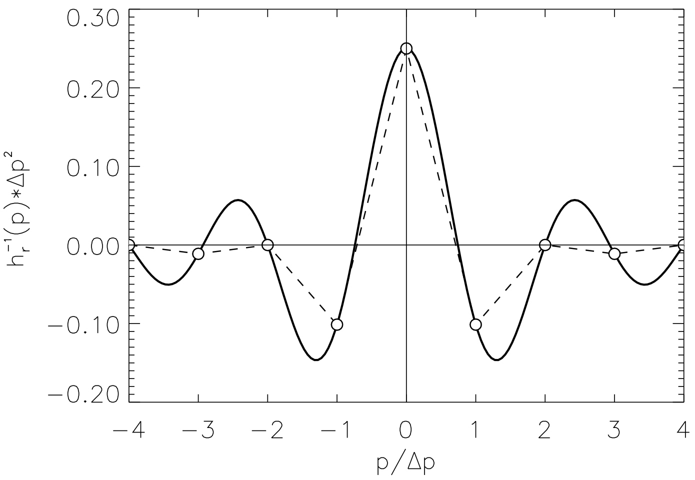
where $\mathrm{sinc}(x)$ stands for $\sin(\pi x)/(\pi x)$.
A sampling at discrete positions $p = n\Delta p$ yields the discrete version:

$$h_r^{-1}(n\Delta p) = \begin{cases} \dfrac{1}{4\Delta p^2} & \text{for} \quad n = 0 \\ 0 & \text{for} \quad n \ \text{ even}, \ \neq 0 \\ -\dfrac{1}{n^2\pi^2\Delta p^2} & \text{for} \quad n \ \text{ odd} \ . \end{cases}$$

This filter goes back to Ramachandran and Lakshminarayanan.
It is known as **"Ram-Lak"** filter.

Ram-Lak filter



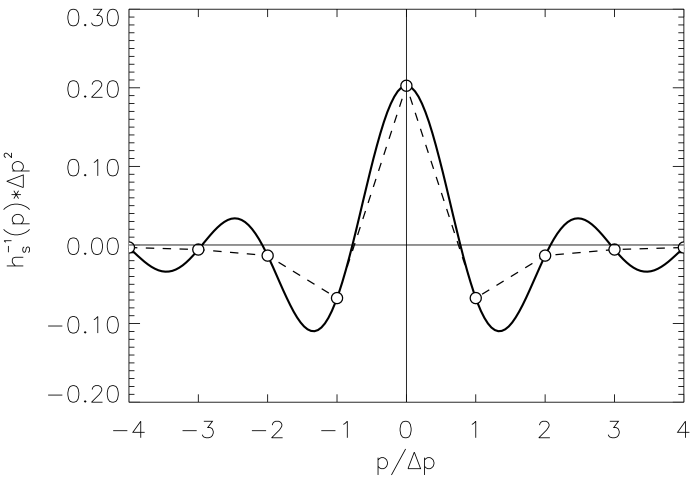
Another commonly used filter is the so-called "Shepp and Logan" filter, which results from averaging (smoothing) of the Ram-Lak filter over intervals of the width Δp (or in the frequency domain from the ramp filter $|\nu|$ by multiplication with $\mathrm{sinc}(\nu \Delta p)$):

$$h_s^{-1}(p) = -\frac{2}{\pi^2 \Delta p^2} \frac{1 - 2(p/\Delta p)\sin(\pi p/\Delta p)}{4(p/\Delta p)^2 - 1}.$$

The discrete version of this filter is very simple:

$$h_s^{-1}(n\Delta p) = -\frac{2}{\pi^2 \Delta p^2 (4n^2 - 1)} .$$

Shepp-Logan filter



Simple backprojection

Filtered backprojection

Homework

Homework 1: reconstruct yourself!

a) Take a picture of yourself and convert it to a grayscale 100 x 100 pixel square image.
b) Create your sinogram space for 100 projection angles.
c) Reconstruct your image by filtered backprojection using (i) the Ram-Lak filter, and (ii) the Shepp-Logan filter. Do the filtering in the **spatial domain** using filters h_r^{-1} (Ram-Lak) and h_s^{-1} (Shepp-Logan).

Further Reading

- **A.C. Kak, M. Slaney**: *Principles of Computerized Tomographic Imaging*. Reprint: SIAM Classics in Applied Mathematics, 2001. PDF available: http://www.slaney.org/pct/pct-toc.html
- **F. Natterer**: *The Mathematics of Computerized Tomography*. Reprint: SIAM Classics in Applied Mathematics, 2001.
- **R.N. Bracewell**: *The Fourier Transform and its Applications*. McGraw-Hill, New York, 3rd edition, revised, 1999.
- **T. Bortfeld**: *Röntgencomputertomographie: Mathematische Grundlagen*. In: Schlegel W, Bille J, eds. Medizinische Physik 2 (Medizinische Strahlenphysik). Heidelberg: Springer; 2002: 229-245. English translation available from author.
- **J. Radon**: *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten*. Berichte der Sächsischen Akademie der Wissenschaften – Math.-Phys. Klasse, 69:262–277, 1917.