Assignment 3

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Theoretical Exercises

1)

$$Y_i \overset{iid}{\sim} Gamma(\alpha, \beta)$$
$$P(\alpha, \beta) \propto \frac{\beta^{\alpha s}}{\Gamma(\alpha)^r} p^{\alpha-1} e^{-\beta q}$$

$$\mathcal{L}(\alpha, \beta | y_i, \ldots y_n) = \prod_{i=1}^{n} \frac{\beta^{\alpha}}{\Gamma(\alpha)} y_i^{\alpha-1} e^{-\beta y_i} = \frac{\beta^{n\alpha}}{\Gamma(\alpha)^n} \left(\prod_{i=1}^{n} y_i \right)^{\alpha-1} e^{-\beta \sum y_i}$$

$$\mathcal{L}(y|\alpha, \beta) p(\alpha, \beta) \propto \left(\frac{\beta^{n\alpha}}{\Gamma(\alpha)^n} \left(\prod_{i=1}^{n} y_i \right)^{\alpha-1} e^{-\beta \sum y_i} \right) \left(\frac{\beta^{\alpha s}}{\Gamma(\alpha)^r} p^{\alpha-1} e^{-\beta q} \right) =$$
$$\frac{\beta^{\alpha(n+s)}}{\Gamma(\alpha)^{n+r}} \left(p \prod_{i=1}^{n} y_i \right)^{\alpha-1} e^{-\beta(\sum y_i + q)}$$

Let: $p' = p \prod y_i$, $q' = q + \sum y_i$, $s' = s + n$, and $r' = n + r$

$$\frac{\beta^{\alpha(n+s)}}{\Gamma(\alpha)^{n+r}} \left(p \prod_{i=1}^{n} y_i \right)^{\alpha-1} e^{-\beta(\sum y_i + q)} = \frac{\beta^{\alpha s'}}{\Gamma(\alpha)^{r'}} (p')^{\alpha-1} e^{-\beta q'}$$

This takes the same form as the joint prior, therefore the prior is a conjugate prior for a Gamma distribution with unknown parameters α and β

2)

$y \sim MVN(X\beta, \lambda^{-1} I_{n\times n)}$
$\pi(\beta, \lambda) \propto \lambda^{-1}$

$$\mathcal{L}(Y|X, \beta, \lambda^{-1}) \propto |\lambda^{-1} I_{n\times n}|^{-1/2} \exp\left[-\frac{1}{2} (Y - X\beta)' (\lambda^[- 1]I)^{-1} (Y - X\beta) \right]$$
$$= \lambda^{n/2} \exp\left[-\frac{\lambda}{2} (Y - X\beta)'(Y - X\beta) \right]$$

$$
\begin{aligned}
P(\beta,\lambda) &\propto \pi(\beta,\lambda)\mathcal{L}(Y|X,\beta,\lambda^{-1}) \\
&\propto (\lambda^{-1})\lambda^{n/2}\exp\left[-\frac{\lambda}{2}(Y-X\beta)'(Y-X\beta)\right] \\
&= (\lambda)^{n/2-1}\exp\left[-\frac{\lambda}{2}(Y-X\beta)'(Y-X\beta)\right] \\
&= (\lambda)^{n/2-1}\exp\left[-\frac{\lambda}{2}\{(Y-X\hat{\beta})'(Y-X\hat{\beta})+(\hat{\beta}-\beta)'X'X(\hat{\beta}-\beta)\}\right] \\
&= (\lambda)^{n/2-1}\exp\left[-\frac{\lambda}{2}\{(Y-X\hat{\beta})'(Y-X\hat{\beta})\}\right]\exp\left[-\frac{\lambda}{2}\{(\hat{\beta}-\beta)'X'X(\hat{\beta}-\beta)\}\right]
\end{aligned}
$$

So for the marginal distribution of β:

$P(\beta|\lambda,Y,X) \propto \exp\left[-\frac{1}{2\lambda^{-1}}\{(\hat{\beta}-\beta)'X'X(\hat{\beta}-\beta)\}\right]$

Which is the kernel for a multivariate normal distribution with mean = $\hat{\beta}$ and a Covariance matrix of $\Sigma = \lambda^{-1}(X'X)^{-1}$

3)

$P(\theta|W) \sim \mathcal{N}(\hat{\theta},[I(\hat{\theta})]^{-1})$

$I[\theta] = -\frac{d^2}{d\theta^2}\log[P(\theta|W)]$

$$
\begin{aligned}
P(\theta|W) &\propto \pi(\theta)\mathcal{L}(W|\theta) \\
&\propto (\tau^2)^{-1}\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\tau^2}}\exp[-\frac{1}{2}\frac{(w_i-\mu)^2}{\tau^2}] \\
&\propto (\tau^2)^{-(n/2+1)}\exp\left[-\frac{1}{2\tau^2}\sum(w_i-\mu)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
\log P(\theta|W) &= -(n/2+1)\log(\tau^2) - \frac{1}{2\tau^2}\sum(w_i-\mu)^2 \\
\frac{\partial}{\partial\mu}\log P &= \frac{1}{\tau^2}\sum(w_i-\mu) = 0 \rightarrow \\
\sum(w_i-\mu) &= 0 \rightarrow \\
\hat{\mu} &= \frac{1}{n}\sum w_i = \bar{w}
\end{aligned}
$$

$$
\frac{\partial^2}{\partial\mu^2}\log P = \frac{\partial}{\partial\mu}\frac{1}{\tau^2}\sum(w_i-\mu) = \frac{-n}{\tau^2}
$$

$$
\begin{aligned}
\frac{\partial}{\partial\tau^2}\log P &= \frac{-(n/2+1)}{\tau^2} + \frac{1}{2(\tau^2)^2\sum(w_i-\mu)^2} = 0 \rightarrow \\
\frac{1}{2(\tau^2)^2}\sum(w_i-\bar{w})^2 &= \frac{(n/2+1)}{\tau^2} \rightarrow \frac{1}{2}\sum(w_i-\bar{w})^2 = \tau^2(n/2+1) \rightarrow \\
\hat{\tau}^2 &= \frac{1}{n+2}\sum(w_i-\bar{w})^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial(\tau^2)^2}\log P &= \frac{\partial^2}{\partial(\tau^2)}\left[\frac{-(n/2+1)}{\tau^2} + \frac{1}{2(\tau^2)^2\sum(w_i-\mu)^2}\right] \\
&= \frac{(n/2+1)}{(\tau^2)^2} - (\tau^2)^{-3}\sum(w_i-\mu)^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial\mu\partial(\tau^2)}\log P &= \frac{\partial}{\partial\tau^2}\left[\frac{1}{\tau^2}\sum(w_i-\mu)\right] \\
&= -(\tau^2)^{-2}\sum(w_i-\mu)
\end{aligned}
$$

$$
\begin{aligned}
I &= -\begin{bmatrix} \frac{\partial^2}{\partial\mu^2} & \frac{\partial^2}{\partial\mu\partial\tau^2} \\ \frac{\partial^2}{\partial\mu\partial\tau^2} & \frac{\partial^2}{\partial(\tau^2)^2} \end{bmatrix} \\
&= -\begin{bmatrix} \frac{-n}{\tau^2} & -(\tau^2)^{-2}\sum(w_i-\mu) \\ -(\tau^2)^{-2}\sum(w_i-\mu) & \frac{(n/2+1)}{(\tau^2)^2} - (\tau^2)^{-3}\sum(w_i-\mu)^2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{n}{\tau^2} & (\tau^2)^{-2}\sum(w_i-\mu) \\ (\tau^2)^{-2}\sum(w_i-\mu) & -\frac{(n/2+1)}{(\tau^2)^2} + (\tau^2)^{-3}\sum(w_i-\mu)^2 \end{bmatrix}
\end{aligned}
$$