Math 611 HW1

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Consider the queue model discussed in class (i.e. a single server model). As in class, assume that the interarrival times, T_i are iid as are the service times V_i. Assume further that each T_i is exponentially distributed with rate λ and each V_i is exponentially distributed with rate μ. Let $X(t)$ be the number of customers waiting in line at time t. Let W_i be the waiting time of the ith individual. Assume that initially the queue is empty, so $X(0) = 0$ and $W_1 = 0$.

a) Determine $P(W_2 \geq c)$ for c a positive number. (write down an integral and evaluate it, you won't need a computer).

$P(W_2 \geq c) = P(\max(0, D_1 - A_2) \geq c) = P((T_1 + V_1) - (T_1 + T_2) \geq c =$

$P(V_1 - T_1 \geq c) = P(V_1 \geq c + t_1) = \int_0^\infty \int_{c+t_1}^\infty \mu e^{-\mu v} \lambda e^{-\lambda t} dv dt = \int_0^\infty \lambda e^{-\lambda t} \left[-e^{-\mu v}\right]_{c+t_1}^\infty dt = \int_0^\infty \lambda e^{-\lambda t} e^{-\mu(c+t_1)} dt =$

$e^{-\mu c} \int_0^\infty \lambda e^{-\lambda t} e^{-\mu t} dt = \frac{e^{-\mu c} \lambda \left[e^{(-\lambda-\mu)t}\right]_0^\infty}{\lambda+\mu} = \frac{\lambda e^{-\mu c}}{\lambda+\mu}$

b) Write down an integral expression for $P(W_3 \geq c)$ (You don't need to evaluate the integral, unless you want to. Your answer may be the sum of two integrals.)

$P(W_3 > c) = P(\max(0, \hat{D}_2 - \hat{A}_3) > c) = P(D_2 - A_3 > c) = P((A_2 + W_2 + V_2) - (T_0 + T_1 + T_2) > x) =$

$P(W_2 + V_2 - T_2 > c) = P(W_2 > c - V_2 + T_2) = P(\max(0, V_1 - T_1) > c - V_2 + T_2)$

Two scenarios - either $c - V_2 + T_2 > 0$ or $c - V_2 + T_2 < 0$

If $c - V_2 + T_2 > 0$, then $P(\max(0, V_1 - T_1) > c - V_2 + T_2) = P(V_1 - T_1 > c - V_2 + T_2)$, $V_1 > T_1$

If $c - V_2 + T_2 < 0$, then $P(\max(0, V_1 - T_1) > c - V_2 + T_2) = 1$

$c - V_2 + T_2 > 0 \to c + T_2 > V_2$. If $c - V_2 + T_2 < 0 \to V_2 > c + T_2$

Add the two scenarios together:

$P(\max(0, V_1 - T_1) > c - V_2 + T_2) = \int_0^\infty dT_2 \int_0^\infty dT_1 \int_{T_1}^\infty dV_1 \int_0^{c+T_2} dV_2 \lambda e^{-\lambda t_1} \mu e^{-\mu v_1} \mu e^{-\mu v_2} \lambda e^{-\lambda t_2} + \int_0^\infty dT_2 \int_0^\infty dT_1 \int_{T_1}^\infty dV_1 \int_{c+T_2}^\infty dV_2 \lambda e^{-\lambda t_1} \mu e^{-\mu v_1} \mu e^{-\mu v_2} \lambda e^{-\lambda t_2}$

c) Write a function WaitingTimes(n λ, μ) that samples the waiting times of the first n customers. Your function should return a vector of length n with the sampled waiting time. Show the output of your function for $n = 10, \lambda = 1, \mu = 1$.

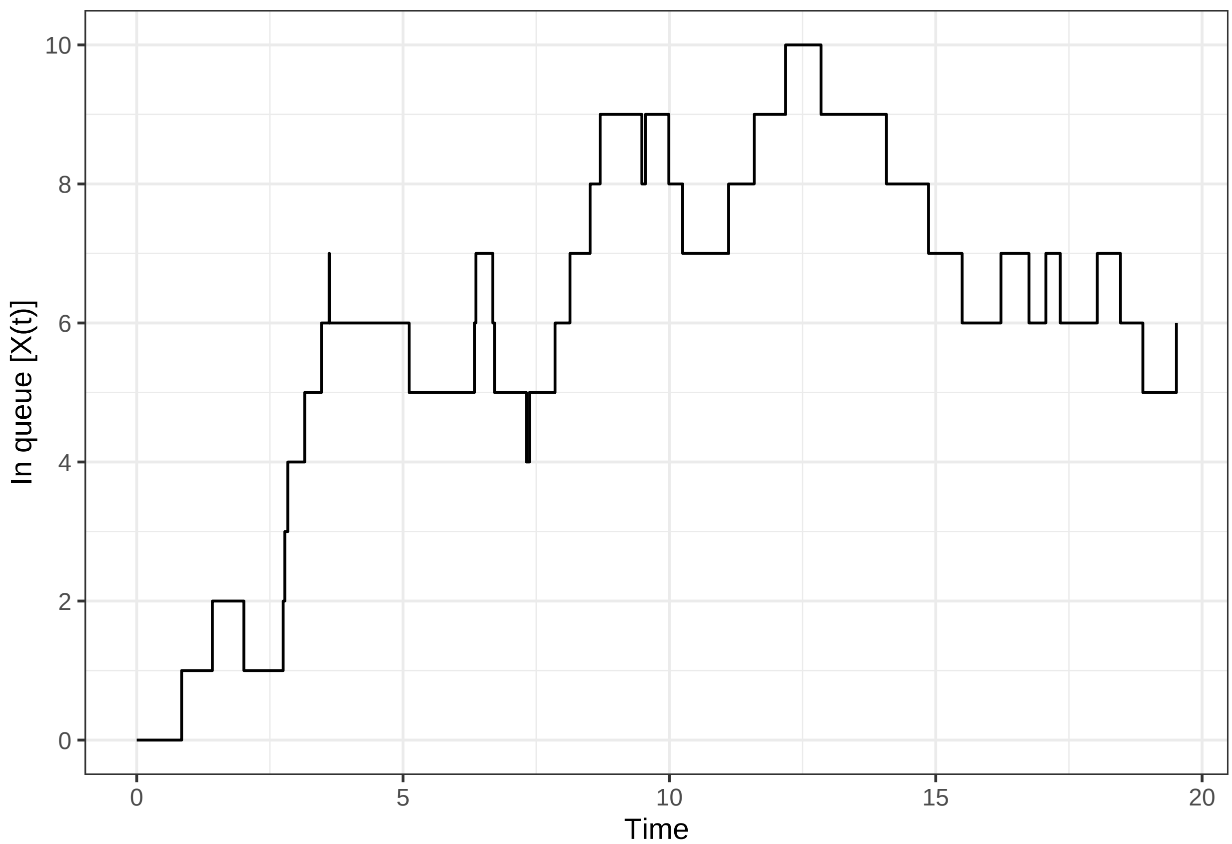
```r
waitingTimes <- function(n, lambda, mu){

  ### First, create a vector of length n for T_i's, the time between cutomer arrivals
  T_i <- rexp(n, rate = lambda)
  ### Turn T_i into a vector of arrival times, A
  A <- cumsum(T_i)

  ### Now create vector of service time for each person
  V <- rexp(n, rate = mu)

  ### Initialize W with W_1 = 0
  W <- c(0)

  ### The departure time of the first customer is simply their service time + arrival time
  D <- V[1] + A[1]

  for (i in 2:n) {
    ## The wait time of the ith customer is max(0, D_(i-1) - A_i)
    W[i] <- max(0, D[i - 1] - A[i])

    ## The departure time of the ith customer is their arrival time + their wait time + their service t
    D[i] <- A[i] + W[i] + V[i]
  }

  return(W)

}

set.seed(123)
waitingTimes(10,1,1)
```

```
##  [1] 0.0000000 0.4282198 0.0000000 0.2494363 0.5703431 0.4421259 0.9776848
##  [8] 2.3956215 0.1481455 0.7099269
```

d) Write a function plotQueue(t, λ, μ) that simulates (in other words samples) the queue and plots $X(t)$ up to a time t. Show a single simulation for $t = 20$, $\lambda = 1$, $\mu = 1$.

```r
set.seed(123)
plotQueue <- function(t, lambda, mu){

  T_i <- rexp(1, lambda)

  while (sum(T_i) <= t) {
    T_i <- c(T_i, rexp(1, lambda))
  }

  A <- cumsum(T_i)
  n <- length(T_i)
  ### Now create vector of service time for each person
  V <- rexp(n, rate = mu)

  ### Initialize W with W_1 = 0
```

```r
  W <- c(0)

  ### The departure time of the first customer is simply their service time + arrival time
  D <- V[1] + A[1]

  for (i in 2:n) {
    ## The wait time of the ith customer is max(0, D_(i-1) - A_i)
    W[i] <- max(0, D[i - 1] - A[i])

    ## The departure time of the ith customer is their arrival time + their wait time + their service t
    D[i] <- A[i] + W[i] + V[i]
  }

  X <- data.frame(
    steps = c(0,A,D),
    count = c(0, rep(1, length(A)), rep(-1, length(D)))  ### Add 1 if arrival, subtract 1 if departure
  ) %>%
    arrange(steps) %>% ## Put events in chronological order
    filter(steps <= t) %>% ### remove departures after t
    mutate(in_queue = cumsum(count))

  ggplot(X, aes(x = steps, y = in_queue)) +
    geom_step() +
    theme_bw() +
    labs(x = "Time", y = "In queue [X(t)]") +
    scale_y_continuous(breaks = seq(0,max(X$in_queue) + 2, 2))

}

plotQueue(20,1,1)
```



e) Using a Monte Carlo approach, estimate $P(W_2 \geq 1)$. Assume $\lambda = 1$, $\mu = 1$. Compare your estimate to the exact answer you derived in part (a). Repeat for $P(W_{100} \geq 1)$, except in this case you won't have the exact answer.

We can use our `waitingTimes` function created in 2a. Simply run the function N times, then calculate number of times $W_2 \geq 1/N$ to estimate the probabilty

Let $N = 100000$

```
set.seed(123)
N <- 100000
cl <- parallel::makeCluster(parallel::detectCores() - 1)

WaitTimes <- t(parallel::parSapply(cl, X = rep(2, N), FUN = waitingTimes, lambda = 1, mu = 1))

P <- mean(WaitTimes[,2] >=1)
```

Runing the above, we find that the probabilty the second customer has to wait in line more than 1 time unit is 0.18294

Compared with our answer from 2b: $\frac{\lambda e^{-\mu c}}{\lambda+\mu} = \frac{1e^{-1 \cdot 1}}{1+1} = 0.18394$

We follow the same process for W_{100}, simply expand `n` in `waitingTimes` to 100 instead of 2

```
set.seed(123)
N <- 100000
```

```r
WaitTimes <- t(parallel::parSapply(cl, X = rep(100, N), FUN = waitingTimes, lambda = 1, mu = 1))

P <- mean(WaitTimes[,100] >=1)
parallel::stopCluster(cl)
```

Runing the above, we find that the probabilty the 100^{th} customer has to wait in line more than 1 time unit is 0.8871

3 Let X be a multivariate normal, $X \sim \mathcal{N}(\mu, \Sigma)$.

a) Show that the coordinates of X are independent if and only if Σ is diagonal. (Hint: recall how the joint pdf and marginal pdfs relate for independent r.v.)

"$\Rightarrow$" Suppose the coordinates of X are independent, $x_i \sim N(\mu_i, \sigma_i^2)$. Then $f(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\ldots f(x_n)$

$$f(x_1)f(x_2)\ldots f(x_n) = \int\int\cdots\int f(x_1)f(x_2)\ldots f(x_n)dx_1dx_2\ldots dx_n = \int\int\cdots\int \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i}} e^{-\frac{(x_i-\mu_i)^2}{2\sigma_i^2}} dx_i =$$

$$\int\int\cdots\int \frac{1}{(2\pi)^{n/2}\prod_{i=1}^{n}\sqrt{\sigma_i}} e^{-\sum_{i=1}^{n}\frac{(x_i-\mu_i)^2}{2\sigma_i^2}} dx_1\ldots dx_n$$

This gives us $\prod_{i=1}^{n}\sqrt{\sigma_i} = |\det\Sigma|^{1/2}$. Since we already know that each $\Sigma_{ii} = \sigma_i^2$, then the fact that the determinant of Σ is the product of it's diagonal entries means Σ is at least a triangular matrix, ie the lower half of Σ is all 0's. Since Σ is also symmetric, this means that Σ is diagonal

"$\Leftarrow$" Now suppose Σ is a diagonal matrix. Then each $\Sigma_{ij} = 0 \ \forall i \neq j$, and $\Sigma_{ii} = \sigma_i^2$

$f(x_1, x_2, \ldots, x_n) = \int\int\cdots\int \frac{1}{(2\pi)^{n/2}|\det(\Sigma)|^{1/2}} e^{-(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)/2} dx_1dx_2\ldots dx_n$

Then $\det(\Sigma) = \prod \sigma_i$, as Σ is diagonal, and $(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu) = \sum_{i=1}^{n}(x_i-\mu_i)\sigma_i^{-1}(x_i-\mu_i) = \sum_{i=1}^{n}(x_i-\mu_i)^2/\sigma_i^2$

So, $\int\int\cdots\int \frac{1}{(2\pi)^{n/2}|\det(\Sigma)|^{1/2}} e^{-(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)/2} = \int\int\cdots\int \frac{1}{(2\pi)^{n/2}\prod_{i=1}^{n}\sigma_i^{1/2}} e^{-\sum_{i=1}^{n}(x_i-\mu_i)^2/2\sigma_i^2} = \int\int\cdots\int \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i}} e^{-\frac{x_i-\mu_i}{2\sigma_i^2}} dx_1dx_2\ldots dx_n =$

$\int\int\cdots\int \prod_{i=1}^{n} f(x_i) = f(x_1)f(x_2)\ldots f(x_n)$ therefore the coordinates of X are independent.

b) Suppose that $X \in \mathbb{R}^2$ and that Σ is diagonal. Show that the level curves of the pdf are either ellipses or lines. If the level curves are lines, what can you say about Σ?

$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \to \lambda_1 = \sigma_1^2, \lambda_2 = \sigma_2^2, Q = I$

$\mu = (\mu_1 \ \ \mu_2)^T$

Let $c \in \mathbb{R}$, then the points $\{(x_1, x_2) \in \mathbb{R}^2 : f(x_1, x_2) = c\}$ is the level curve of X at c

The ellipse has axes defined by the eigenvectors of Σ, $q^{(1)}, q^{(2)}$ and the length of the axes by the corresponding eigenvalues, $\lambda_1 = \sigma_1^2$ and $\lambda_2 = \sigma_2^2$

Define a level curve at c as $c^2 = (\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)$

If $\sigma_1, \sigma_2 \neq 0$, then $c^2 = (\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu) = \frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}$ is the formula for an ellipse.

However, if $\sigma_1 = 0$ or $\sigma_2 = 0$, then Σ is not an invertible matrix.

Consider the case when $\sigma_2^2 = 0$. We will sub in ϵ and take the limit as $\epsilon \to 0$

The length of the axes for $q^{(2)}$ is defined by λ_2. As $\lim_{\epsilon\to 0} \lambda_2 \Rightarrow \lambda_2 \to 0$, since $\lambda_2 = \sigma_2$. So there is no width to the $q^{(2)}$ axis, thus we end up with a line on $q^{(1)}$. We can show the opposite for if $\sigma_1^2 = 0$.

c) Suppose $Y \sim \mathcal{N}(\mu', \Sigma')$ and that X and Y are independent. $X+Y$ will also be normal. Compute the mean and covariance matrix of $X+Y$. (You will find that the means and covariances sum. Don't just quote a result, show that this is the case.) Suppose $X \in \mathbb{R}^n$ and M a $n \times n$ matrix. Show that $MX \sim \mathcal{N}(M\mu, M\Sigma M^T)$

Let $Z = X + Y$

Then $E[Z] = E[X+Y] = E[X] + E[Y] = \mu + \mu'$

Let Σ^* be the covariance matrix for Z. Then $\Sigma^*_{ij} = E[(x_i + y_i)(x_j + y_j)] - E[x_i + y_i]E[x_j + y_j] = E[x_i x_j] + E[y_i x_j] + E[x_i y_j] + E[x_i x_j] - E[x_i][x_j] - E[y_i]E[x_j] - E[x_i]E[y_j] - E[y_i]E[z_j] = (E[x_i x_j] - E[x_i]E[x_j]) + (E[y_i y_j] - E[y_i]E[y_j]) + (E[y_i x_j] - E[y_i]E[x_j]) + (E[x_i y_j] - E[x_i]E[y_j])$

Since X and Y are independent, $E[x_i y_j] = E[x_i]E[y_j]$ and $E[x_j y_i] = E[x_j]E[y_i]$. So the last equation becomes: $E[(x_i + y_i)(x_j + z_j)] - E[x_i + y_i]E[x_j + y_j] = (E[x_i x_j] - E[x_i]E[x_j]) + (E[y_j y_j] - E[y_i]E[y_j]) = \Sigma_{ij} + \Sigma'_{ij} \to var(Z) = \Sigma + \Sigma'$

Thus, combined with the property that the sum of normal r.v.'s is also a normal r.v., $Z = X + Y \sim \mathcal{N}(\mu + \mu', \Sigma + \Sigma'$

- Suppose $X \in \mathbb{R}^n$ and M a $n \times n$ matrix. Show that $MX \sim \mathcal{N}(M\mu, M\Sigma M^T)$

Define Y as the random variable generated by MX. $E[Y] = E[MX] = E[M]E[X] = ME[X] = M\mu$

$Var[Y] = E[(y - \bar{y})(y - \bar{y})^T] = E[(Mx - M\mu)(Mx - M\mu)^T] = E[(M(x-\mu)(M(x-\mu))^T] = E[M(x-\mu)(x-\mu)^T M^T] = ME[(x-\mu)(x-\mu)^T]M^T = M\Sigma M^T$

$Var(Y) = Var(MX) = M\Sigma M^T$

$Y = MX \sim \mathcal{N}(M\mu, M\Sigma M^T)$

(another way to look at 3c.1?)

$f_Z(z) = \int \cdots \int f_Y(z-x) f_X(x) dx_1 \ldots dx_n =$
$\int \cdots \int \frac{1}{\sqrt{2\pi}^n \det \Sigma'^{1/2}} \exp\left[-(z - x - \mu')^T \Sigma'^{-1}(z - x - \mu')/2\right] \frac{1}{\sqrt{2\pi}^n \det \Sigma^{1/2}} \exp\left[-(x-\mu)^T \Sigma^{-1}(x-\mu)/2\right] =$
$\int \cdots \int \frac{1}{\sqrt{2\pi}^n \det \Sigma'^{1/2}} \frac{1}{\sqrt{2\pi}^n \det \Sigma^{1/2}} \exp\left[-((z - x - \mu')^T \Sigma'^{-1}(z - x - \mu') + -(x-\mu)^T \Sigma^{-1}(x-\mu))/2\right]$

$(-(z - x - \mu')^T \Sigma'^{-1}(z - x - \mu') + -(x-\mu)^T \Sigma^{-1}(x-\mu)) =$
$-(z - x - \mu') \cdot \Sigma'^{-1}(z - x - \mu') + -(x-\mu) \cdot \Sigma^{-1}(x-\mu) =$
$(-z + x + \mu' - x + \mu) \cdot (\Sigma'^{-1}(z - x - \mu') + \Sigma^{-1}(x-\mu)) = -(z - (\mu' + \mu)) \cdot (\Sigma + \Sigma')^{-1}(z - (\mu' + \mu))$

$\frac{1}{\sqrt{2\pi}^n \det \Sigma'^{1/2}} \frac{1}{\sqrt{2\pi}^n \det \Sigma^{1/2}} = \frac{1}{\sqrt{2^n \pi^n \det \Sigma' \det \Sigma}}$