6 Brownian Motion

6.1. Basic Definitions

Brownian motion is the most important example of a process with continuous time and continuous paths. Mathematically, it lies in the intersection of three classes of stochastic processes: it is a Gaussian process, a Markov process with continuous paths, and has independent increments. Because of this, it is possible to use techniques from all three fields, so there is a rich and detailed theory of its properties. From a practical view, Brownian motion is often used as a component of models of physical, biological, and economic phenomena.

(1.1) **Definition.** $B(t)$ is a (one-dimensional) Brownian motion with variance σ^2 if it satisfies the following conditions:

(a) $B(0) = 0$, a convenient normalization.

(b) *Independent increments.* Whenever $0 = t_0 < t_1 < \ldots < t_n$

$$B(t_1) - B(t_0), \ldots, B(t_n) - B(t_{n-1}) \quad \text{are independent.}$$

(c) *Stationary increments.* The distribution of $B_t - B_s$ only depends on $t - s$.

(d) $B(t)$ is normal$(0, \sigma^2 t)$.

(e) $t \to B_t$ is continuous.

To explain the definition we begin by discussing the situation that gives the process its name. Historically, Brownian motion originates from the fact observed by Robert Brown (and others) in the 1800s that pollen grains under a microscope perform a "continuous swarming motion." To explain the source of this motion, we imagine a spherical pollen grain that collides with water molecules at times of a Poisson process with a large rate λ. The displacement of the pollen grain between time 0 and time 1 is the sum of a large number of small independent random effects. To see that this implies the change in the x coordinate, $X(1) - X(0)$, will be approximately normal$(0, \sigma^2)$, we recall:

(1.2) **Central Limit Theorem.** *Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 0$ and $\text{var}(X_i) = \sigma^2$, and let $S_n = X_1 + \cdots + X_n$. Then for all x we have*

$$P\left(\frac{S_n}{\sigma\sqrt{n}} \le x\right) \to P(\chi \le x)$$

where χ has a standard normal distribution. That is,

$$P(\chi \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

Formally, (1.2) says that $S_n/\sigma\sqrt{n}$ **converges in distribution** to χ, a conclusion we will write as $S_n/\sigma\sqrt{n} \Rightarrow \chi$.

Since there are a huge number of water molecules in the system ($> 10^{20}$), it is reasonable to think that after a collision the molecule disappears into the crowd never to be seen again, and with a small leap of faith we can assert that $X(t)$ will have independent increments. The movement of the pollen grain is clearly continuous in space, so if we impose the condition $X(0) = 0$ by measuring displacements from the initial position, then the first coordinate of the pollen grain will be a Brownian motion with variance σ^2.

Does Brownian motion exist? Is there a process with the properties given in (1.1)? The answer is yes, of course, or this chapter wouldn't exist. The technical problem is to define the uncountably many random variables B_t for each nonnegative real number t on the same space so that $t \to B_t$ is continuous with probability one. One solution to this problem is provided in Exercise 6.11. However, the details of the proof are not important for understanding how B_t behaves, so we suggest that the reader take the existence of Brownian motion on faith.

Multidimensional Brownian motion. If we are looking at the pollen grain through a microscope then we can observe two coordinates of its motion $(X^1(t), X^2(t))$. Again we can argue that the displacement of the pollen grain between time 0 and time t will be the sum of a large number of small independent random effects, so if we suppose that $(X^1(0), X^2(0)) = (0, 0)$, then the position at time t, $(X^1(t), X^2(t))$, will have a two-dimensional normal distribution.

Multidimensional normal distributions $(Z^1, \ldots, Z^n)$ are characterized by giving the vector of means $\mu_i = EZ^i$ and the covariance matrix:

$$\Gamma_{ij} = E\{(Z^i - \mu_i)(Z^j - \mu_j)\}$$

In the case under consideration $\mu_1 = \mu_2 = 0$. The spherical symmetry of the pollen grain and of the physical system implies that

$$\Gamma_{ij} = \sigma^2 I_{ij}$$

where I is the identity matrix: $I_{ii} = 1$ and $I_{ij} = 0$ for $i \neq j$. Recalling that:

(1.3) **Lemma.** *Two components Z_i and Z_j of a multidimensional normal are independent if and only if their covariance $E(Z_iZ_j) - EZ_iEZ_j = 0$*

we see the two components of the movement of the pollen grain are independent. Generalizing (and specializing), we state:

(1.4) **Definition.** $(B^1(t), \dots B^n(t))$, $t \geq 0$ *is a standard n-dimensional Brownian motion if each of its components are independent one-dimensional Brownian motions with $\sigma^2 = 1$.*

We have removed the parameter σ^2 since it is trivial to reintroduce it. If B_t is a standard Brownian motion, then σB_t gives a Brownian motion with variance σ^2. A slight variation on the last calculation gives the following useful result:

(1.5) **Scaling relation.** *The processes $\{B_{ct}, t \geq 0\}$ and $\{c^{1/2}B_t, t \geq 0\}$ have the same distribution.*

Proof. Speeding up time by a factor of c does not change properties (a), (b), (c), or (e), but in (d) it makes the variance $c(t-s)$. By the previous observation this can be achieved by multiplying a standard Brownian motion by $c^{1/2}$. □

A second basic property of Brownian motion is:

(1.6) **Covariance formula.** $E(B_sB_t) = s \wedge t$.

Proof. If $s = t$, this says $EB_s^2 = s$, which is correct since B_s is normal$(0, s)$, so suppose now that $s < t$. Writing $B_t = B_s + (B_t - B_s)$ and using the fact that B_s and $B_t - B_s$ are independent with mean 0, we have

$$E(B_sB_t) = E(B_s^2) + E(B_s(B_t - B_s)) = s + 0 = s = s \wedge t \qquad \square$$

It follows easily from the definition given in (1.1) that Brownian motion is a **Gaussian process**. That is, for any $t_1 < t_2 \dots < t_n$ the vector

$$(B(t_1), B(t_2), \dots B(t_n))$$

has a multivariate normal distribution. Since multivariate normal distributions are characterized by giving their means and covariances, it follows that Brownian motion is the only Gaussian process with continuous paths that has $EB_s = 0$ and $E(B_s B_t) = s \wedge t$.

We will not use the Gaussian viewpoint here, except in a couple of the exercises. However, to complete the story we should mention two important Gaussian relatives of Brownian motion:

Example 1.1. Brownian bridge. Let $B_t^0 = B_t - tB_1$. By definition $B_1^0 = 0$. The basic fact about this process is the following:

(1.7) **Theorem.** $\{B_t^0, 0 \le t \le 1\}$ *and* $(\{B_t, 0 \le t \le 1\}|B_1 = 0)$ *have the same distribution. Each is a Gaussian process with mean 0 and covariance* $s(1-t)$ *if* $s \le t$.

Why is this true? Since each of the processes is Gaussian and has mean 0, it suffices to show that their covariances are equal to $s(1-t)$. If $s < t$,

$$\begin{aligned} E(B_s^0 B_t^0) &= E((B_s - sB_1)(B_t - tB_1)) = EB_sB_t - sEB_1B_t - tEB_sB_1 + stB_1^2 \\ &= s - st - st + st = s(1-t) \end{aligned}$$

With considerably more effort (see Exercise 6.2) one can compute the covariance function for $(B_t|B_1 = 0)$ and show that it is also $s(1-t)$, so the two processes have the same distribution. □

Example 1.2. Ornstein–Uhlenbeck process. Let

$$Z(t) = e^{-t} B(e^{2t}) \qquad \text{for } -\infty < t < \infty$$

The definition may look strange, but note the very special property that for each t, $Z(t)$ has a normal(0,1) distribution. To compute the joint distribution of $Z(t)$ and $Z(t+s)$ we note that

$$\begin{aligned} Z(s+t) &= e^{-(s+t)} B(e^{2(s+t)}) \\ &= e^{-(s+t)} B(e^{2t}) + e^{-(s+t)}(B(e^{2(s+t)}) - B(e^{2t})) \end{aligned}$$

The first term is equal to $e^{-s}Z(t)$. The second is independent of $Z(t)$ and has a normal distribution with variance $e^{-2(s+t)}(e^{2(s+t)} - e^{2t}) = 1 - e^{-2s}$. Thus introducing an independent standard normal χ we can write

$$Z(s+t) = e^{-s}Z(t) + \chi\sqrt{1 - e^{-2s}} \tag{$\star$}$$