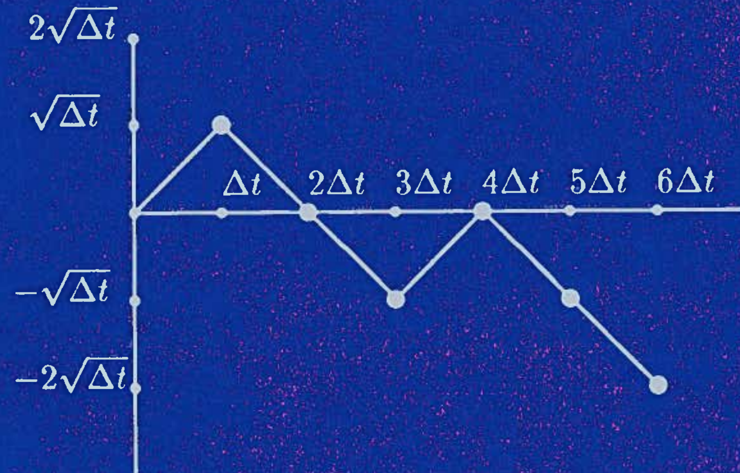
Introduction to Stochastic Processes

Second Edition



GREGORY F. LAWLER

Chapman & Hall/CRC
Taylor & Francis Group

Chapter 3

Continuous-Time Markov Chains

3.1 Poisson Process

Consider X_t the number of customers arriving at a store by time t. Time is now continuous so t takes values in the nonnegative real numbers. Suppose we make three assumptions about the rate at which customers arrive. Intuitively, they are as follows:

1. The number of customers arriving during one time interval does not affect the number arriving during a different time interval.

2. The "average" rate at which customers arrive remains constant.

3. Customers arrive one at a time.

We now make these assumptions mathematically precise. The first assumption is easy: for $s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_n \leq t_n$, the random variables $X_{t_1} - X_{s_1}, \ldots X_{t_n} - X_{s_n}$ are independent. For the second two assumptions, let λ be the rate at which customers arrive, i.e., on the average we expect λt customers in time t. In a small time interval $[t, t + \Delta t]$, we expect that a new customer arrives with probability about $\lambda \Delta t$. The third assumption states that the probability that more than one customer comes in during a small time interval is significantly smaller than this. Rigorously, this becomes

$$\mathbb{P}\{X_{t+\Delta t} = X_t\} = 1 - \lambda \Delta t + o(\Delta t), \tag{3.1}$$

$$\mathbb{P}\{X_{t+\Delta t} = X_t + 1\} = \lambda \Delta t + o(\Delta t), \tag{3.2}$$

$$\mathbb{P}\{X_{t+\Delta t} \geq X_t + 2\} = o(\Delta t). \tag{3.3}$$

Here $o(\Delta t)$ represents some function that is much smaller than Δt for Δt small, i.e.,

$$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0.$$

A stochastic process X_t with $X_0 = 0$ satisfying these assumptions is called a *Poisson process with rate parameter* λ.

We will now determine the distribution of X_t. We will actually derive the distribution in two different ways. First, consider a large number n and write

$$X_t = \sum_{j=1}^{n} [X_{jt/n} - X_{(j-1)t/n}]. \tag{3.4}$$

We have written X_t as the sum of n independent, identically distributed random variables. If n is large, the probability that any of these random variables is 2 or more is small; in fact,

$$\begin{aligned} &\mathbb{P}\{X_{jt/n} - X_{(j-1)t/n} \geq 2 \text{ for some } j \leq n\} \\ &\leq \sum_{j=1}^{n} \mathbb{P}\{X_{jt/n} - X_{(j-1)t/n} \geq 2\} \\ &= n\,\mathbb{P}\{X_{t/n} \geq 2\}. \end{aligned}$$

The last term goes to 0 as $n \to \infty$ by (3.3). Hence we can approximate the sum in (3.4) by a sum of independent random variables which equal 1 with probability $\lambda(t/n)$ and 0 with probability $1 - \lambda(t/n)$. By the formula for the binomial distribution,

$$\mathbb{P}\{X_t = k\} \approx \binom{n}{k} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k}.$$

Rigorously, we can then show:

$$\mathbb{P}\{X_t = k\} = \lim_{n \to \infty} \binom{n}{k} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k}.$$

To take this limit, note that

$$\lim_{n \to \infty} \binom{n}{k} n^{-k} = \lim_{n \to \infty} \frac{n(n-1)\cdots(n-k+1)}{k!\,n^k} = \frac{1}{k!},$$

and

$$\lim_{n \to \infty} \left(1 - \frac{\lambda t}{n}\right)^{n-k} = \lim_{n \to \infty} \left(1 - \frac{\lambda t}{n}\right)^{n} \lim_{n \to \infty} \left(1 - \frac{\lambda t}{n}\right)^{-k} = e^{-\lambda t}.$$

Hence,

$$\mathbb{P}\{X_t = k\} = e^{-\lambda t} \frac{(\lambda t)^k}{k!},$$

i.e., X_t has a Poisson distribution with parameter λt.

We now derive this formula in a different way. Let

$$P_k(t) = \mathbb{P}\{X_t = k\}.$$

Note that $P_0(0) = 1$ and $P_k(0) = 0,\ k > 0$. Equations (3.1) through (3.3) can be used to give a system of differential equations for $P_k(t)$. The definition of the derivative gives

$$P_k'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}(\mathbb{P}\{X_{t+\Delta t} = k\} - \mathbb{P}\{X_t = k\}).$$

Note that

$$\begin{aligned}\mathbb{P}\{X_{t+\Delta t} = k\} &= \mathbb{P}\{X_t = k\}\,\mathbb{P}\{X_{t+\Delta t} = k \mid X_t = k\} \\ &\quad + \mathbb{P}\{X_t = k-1\}\,\mathbb{P}\{X_{t+\Delta t} = k \mid X_t = k-1\} \\ &\quad + \mathbb{P}\{X_t \le k-2\}\,\mathbb{P}\{X_{t+\Delta t} = k \mid X_t \le k-2\} \\ &= P_k(t)\,(1 - \lambda \Delta t) + P_{k-1}(t)\,\lambda \Delta t + o(\Delta t).\end{aligned}$$

Therefore,

$$P_k'(t) = \lambda P_{k-1}(t) - \lambda P_k(t).$$

We can solve these equations recursively. For $k = 0$, the differential equation

$$P_0'(t) = -\lambda P_0(t), \quad P_0(0) = 1$$

has the solution

$$P_0(t) = e^{-\lambda t}.$$

To solve for $k > 0$ it is convenient to consider

$$f_k(t) = e^{\lambda t} P_k(t).$$

Then $f_0(t) = 1$ and the differential equation becomes

$$f_k'(t) = \lambda f_{k-1}(t), \quad f_k(0) = 0.$$

It is then easy to check inductively that the solution is

$$f_k(t) = \lambda^k t^k / k!,$$

and hence

$$P_k(t) = e^{-\lambda t} \frac{(\lambda t)^k}{k!},$$

which is what we derived previously.

Another way to view the Poisson process is to consider the waiting times between customers. Let $T_n, n = 1, 2, \ldots$ be the time between the arrivals of the $(n-1)$st and nth customers. Let $Y_n = T_1 + \cdots + T_n$ be the total amount of time until n customers arrive. We can write

$$Y_n = \inf\{t : X_t = n\},$$

$$T_n = Y_n - Y_{n-1}.$$

Here inf stands for "infimum" or least upper bound which is the generalization of minimum for infinite sets; e.g., the infimum of the set of positive numbers is 0. The T_i should be independent, identically distributed random variables. One property that the T_i should satisfy is the loss of memory property: if we have waited s time units for a customer and no one has arrived, the chance that a customer will come in the next t time units is exactly the same as if there had been some customers before. Mathematically, this property is written

$$\mathbb{P}\{T_i \geq s+t \mid T_i \geq s\} = \mathbb{P}\{T_i \geq t\}.$$

The only real-valued functions satisfying $f(s+t) = f(s)f(t)$ are of the form $f(t) = e^{-bt}$. Hence the distribution of T_i must be an exponential distribution with parameter b. [Recall that a random variable Z has an exponential distribution with rate parameter b if it has density

$$f(z) = be^{-bz}, \quad 0 < z < \infty,$$

or equivalently, if it has distribution function

$$F(z) = \mathbb{P}\{Z \leq z\} = 1 - e^{-bz}, \quad z \geq 0.$$

An easy calculation gives $\mathbb{E}(Z) = 1/b$.] It is easy to see what b should be. For large t values we expect for there to be about λt customers. Hence, $Y_{\lambda t} \approx t$. But $Y_n \approx n\mathbb{E}(T_i) = n/b$. Hence $\lambda = b$. This gives a means of constructing a Poisson process: take independent random variables $T_1, T_2, \ldots,$ each exponential with rate λ, and define

$$Y_n = T_1 + \cdots + T_n,$$

$$X_t = n, \quad \text{if } Y_n \leq t < Y_{n+1}.$$

From this we could then conclude in a third way that the random variables X_t have a Poisson distribution. Conversely, given that we already have the Poisson process, it is easy to compute the distribution of T_i since

$$\mathbb{P}\{T_1 > t\} = \mathbb{P}\{X_t = 0\} = e^{-\lambda t}.$$

3.2 Finite State Space

In this section we discuss continuous-time Markov chains on a finite state space. We start by discussing some facts about exponential random variables.