BS Formula : Theory and Application

一、Brownian Motion

(1)

```
randn('state',100)          % set the state of randn
T = 1; N = 500; dt = T/N;
dW = zeros(1,N);            % preallocate arrays for efficiency
W = zeros(1,N);
dW(1) = sqrt(dt)*randn;     % prepare the iteration W(j) = W(j-1) + dW(j)
W(1) = dW(1);
for j = 2:N                 % start the iteration
    dW(j) = sqrt(dt)*randn; % general increment
    W(j) = W(j-1) + dW(j);
end
plot([0:dt:T],[0,W],'r-')   % plot W against t
xlabel('t','FontSize',16)
ylabel('W(t)','FontSize',16,'Rotation',0)
```

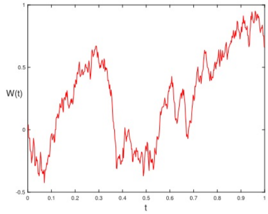


Figure 3.1: Discretized Brownian path

Time series

$B(t)$, $X(t)$, $W(t)$

B_t, X_t, W_t

The time series of BM is similar to stock price time series.

(2) def. $\{W_t, t \geq 0\}$

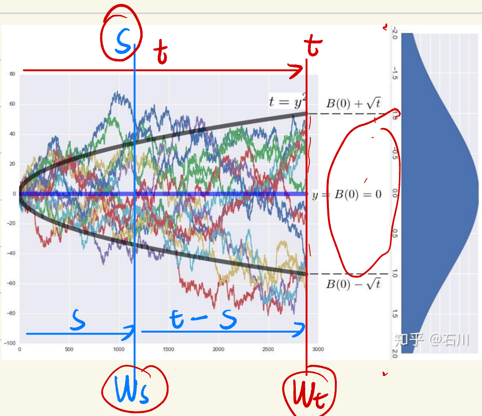
① $W_0 = 0$ (起点从 0 开始)

② 平稳性: $\forall\ 0 < s < t$ $\quad W_t - W_s \sim N(0, t-s)$

③ 独立增量 $\forall$ 区间 $[s_i, t_i]$, not overlapping

$W_{t_i} - W_{s_i}$ 之间相互独立

$\Rightarrow$ $W_{t_i} - W_{s_i}$, a series of independent normal distribution



W_t $\quad \underline{W_t - W_0} \sim N(0, t)$

$\quad W_0 = 0$

$W_s + (W_t - W_s)$

$N(0, s)$ $\quad N(0, t-s)$

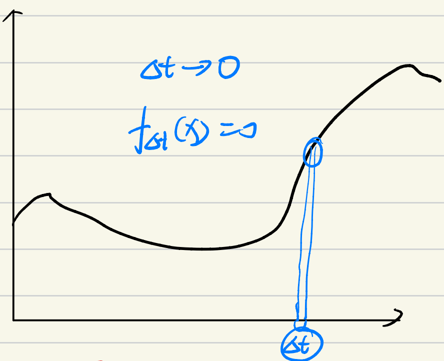
Think about it : $W_3 \sim N(0,3)$ $W_1 \sim N(0,1)$
What is the distribution of $W_1 + W_3$?

(3) quadratic variation

$[0,T]$ is divided into $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ $n \to \infty$

$$\lim_{|\Pi| \to 0} \sum_i (W_{t_{i+1}} - W_{t_i})^2 = T \qquad [0,T]$$

explaination: $|\Pi| = \max |t_{i+1} - t_i|$ $n \to \infty$



quadratic variation $\Rightarrow (dW_t)^2 = dt \Rightarrow$ Ito's lemma.

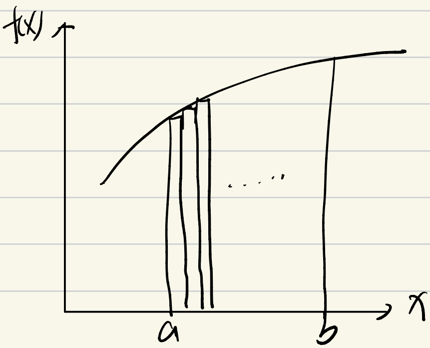
$\int f(t)\,dt = F(t) + C$ ✓ $\int f(W_t)\,dW_t$? (带 W_t 的函数，如何求导求积分?)

$F(W_t) + C$ ✗.

二、Ito integral

(1) How to do intergration on dW_t

Key point : Return to the definition of Riemann Integral



$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i)$$

where $x_0 = a$, $x_n = b$

$$S_t = \int_0^t dW_u \quad dx$$

Therefore : $\int_0^T f(W_t)\,dW_{t_i} = \sum_{i=0}^{n-1} f(W_{t_i})(W_{t_{i+1}} - W_{t_i})$

$N(0, t_{i+1} - t_i)$

Example 1

Guess : $E\left(\int_0^T f(W_t)\,dW_t \right) = \,?\, 0$

$f(s, W_t)$

$f(s)$

$X_t = \int_0^t f(W_s)\,dW_s$ $\qquad E(X_t)$

Example 2

$$\int_0^t W_s\,dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t$$

定积分 : $\int_0^t t_s\,dt_s = \frac{1}{2}t_s^2 \Big|_0^t$

$$\int_0^t W_s\,dW_s = \lim_{\delta t \to 0} \sum W_{t_i}(W_{t_{i+1}} - W_{t_i})$$

$$= \lim_{\delta t \to 0} \left(\sum_i \frac{1}{2}(W_{t_{i+1}}^2 - W_{t_i}^2) - \sum_i (W_{t_{i+1}} - W_{t_i})^2 \right)$$

quadratic variation

$$= \frac{1}{2}W_t^2 - \frac{1}{2}t$$

(2) Ito Isometry :

$$E\left[\int_0^T f(t,\omega)\,dW_t \int_0^T g(t,\omega)\,dW_t \right] = E\left(\int_0^T f(t,\omega)\,g(t,\omega)\,dt \right)$$

proof: $E\left(\sum_{i=0}^{n-1} f_{t_i}(W_{t_{i+1}} - W_{t_i}) \cdot \sum_{i=0}^{n-1} g_{t_i}(W_{t_{i+1}} - W_{t_i}) \right)$

Note : $E\left((W_{t_{j+1}} - W_{t_j})(W_{t_{i+1}} - W_{t_i}) \right) = 0$ for any $i \neq j$

iid normal distribution

$i \neq j$

$(W_{t_{j+1}} - W_{t_j})$ $\quad$ $(W_{t_{i+1}} - W_{t_i})$

$$= E\left(\sum_{i=1}^{n-1} f_{t_i} g_{t_i} (W_{t_{i+1}} - W_{t_i})^2 \right)$$

$$= E\left[\int_0^T f(t,w)\, g(t,w)\, \overset{dt}{(dW_t)^2}\right]$$

$$df(t,s) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial s}ds$$

$$df(t,W_t) = $$

三、 Ito lemma

Target: how to solve $df(t,s)$

We have learned chain rule: $df(t,s) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial s}ds$

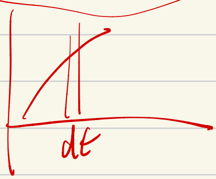
And what about: $df(t,W_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial W_t}dW_t$ TRUE or False?

Return to its definition ☆ Taylor.

$$f(t,s) = f(0,0) + \frac{\partial f}{\partial t}t + \frac{\partial f}{\partial s}s + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}t^2 + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}s^2 + \frac{1}{2}\frac{\partial f}{\partial t}\frac{\partial f}{\partial s}ts + \dots$$

$$\Downarrow$$

$$df(t,s) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial s}ds + \frac{1}{2}\frac{\partial^2 f}{\partial^2 t}(dt)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial^2 s}(ds)^2 + \frac{1}{2}\frac{\partial f}{\partial t}\frac{\partial f}{\partial s}dt \cdot ds + \dots$$



$$df(t,W_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial W_t}dW_t + \cancel{\frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2} + \frac{1}{2}\frac{\partial^2 f}{\partial W_t}\overset{dt}{(dW_t)^2} + \frac{1}{2}\frac{\partial f}{\partial t}\frac{\partial f}{\partial W_t}dt\,dW_t \to 0 + 0$$

$$= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W_t}\right)dt + \frac{\partial f}{\partial W_t}dW_t$$

四. BS formula

W_t

(1) 刻画标的资产 S_t

Use Geometric Brownian Motion

$$dS_t = uS_t\,dt + \sigma S_t\,dW_t$$

$\ln S_t = f(S_t, t)$, Itô lemma

对数

$$d\ln S_t = (u-\frac{\sigma^2}{2})dt + \sigma\,dW_t$$

$$\Downarrow$$

$$S_t = S_0 \exp\left((u-\frac{\sigma^2}{2})t + \sigma W_t\right)$$

$$S_{t+\Delta t} = S_t \exp\left((u-\frac{\sigma^2}{2})\Delta t + \sigma W_{\Delta t}\right)$$

$W_{\Delta t} \sim N(0, \Delta t)$

σ'

(2) BS assumptions

1. 期权的行权方式为欧式，即只有到期日才可以行权。 T
2. 股票的价格符合几何布朗运动，即股票的不确定性满足对数正态分布。
3. 可以做空证券，且证券可以被分割（如可以买卖半手股票）。
4. 市场无摩擦，即不存在交易费用和税收。
5. 在期权期限内，标的股票不支付股息。
6. 在期权期限内，标的股票年收益率的标准差 σ 已知且保持不变。
7. 市场不存在无风险套利机会。
8. 标的资产交易是连续的（如股票市场始终开市）。
9. 短期无风险利率（由 r 表示）为常数并已知。

Suppose call option price function: $C(S_t, t)$

$$dC(S_t,t) = \frac{\partial C}{\partial S_t}dS_t + \frac{\partial C}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}\,\underbrace{d\langle S,S\rangle_t}_{(dS_t)^2} \quad \text{Itô lemma}$$

代入 $dS_t = uS_t\,dt + \sigma S_t\,dW_t$

$$= [\cdots]\,dt + [\cdots]\,dW_t$$

$$\left[\frac{\partial C}{\partial t} + uS_t\frac{\partial C}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S_t^2}\right]dt + \sigma S_t\frac{\partial C}{\partial S_t}\,dW_t$$

空一份 option, call $\quad -C(S_t, t)$

多 $\frac{\partial C}{\partial S_t}$ 的 stock $\quad + \frac{\partial C}{\partial S_t} \quad \frac{\partial C}{\partial S_t} \quad \Delta$

$-dC(S_t,t) + \frac{\partial C}{\partial S_t}dS_t \rightarrow dW_t$ term 消去

对冲风险

Thus construct a portfolio: $d\Pi_t(S_t,t) = dC(S_t,t) - d\frac{\partial C}{\partial S_t} S_t$ (0 above the last term)

$\Downarrow$

$d\Pi_t(S_t,t) = dC(S_t,t) - \frac{\partial C}{\partial S_t} dS_t$

risk neutral $\quad d\Pi_t(S_t,t) = r\Pi(S_t,t)\,dt$

$\Rightarrow$ BS 偏微分方程

$$\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S_t^2} = rC$$

解偏微分方程:

☆ BS Formula:

$$\begin{cases} C = S_0 N(d_1) - K e^{-r(T-t)} N(d_2) \\ P = K e^{-rt} N(-d_2) - S N(-d_1) \\ C + K e^{-rt} = P + S \\ d_1 = \dfrac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = d_1 - \sigma\sqrt{T-t} \end{cases}$$

Call

put