ECON 6300 - Math Methods for Economists

Answer Key.
Dec/13/2019
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Fall 2019 Final Exam

Answer all eight questions on the Bluebook. Show ALL work, including intermediate steps. Failure to show intermediate steps will result in zero credit. Please write your name on your exam.

1. Consider a function

$$f(x_1, x_2) = 2 \cdot \sqrt{x_1^3 \cdot x_2}$$
$$\text{s.t. } 6x_1 + x_2 = 80$$

(6pts) (a) Find out the optimal value(s) of this function under the constraint.

(6pts) (b) Are the optimal point(s) maximum or minimum? How do you know?

Solution:

Part (a)

$$\mathcal{L} = 2 \cdot \sqrt{x_1^3 \cdot x_2} - \lambda(6x_1 + x_2 - 80)$$

FOCs

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 \Rightarrow 2 \cdot \frac{3}{2} \cdot x_1^{1/2} x_2^{1/2} - 6\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 0 \Rightarrow 2x_1^{3/2} \frac{1}{2} \cdot x_2^{-1/2} - \lambda = 0$$

Organize

$$x_1^{1/2} \cdot x_2^{1/2} = 2\lambda \quad (1)$$
$$x_1^{3/2} \cdot x_2^{-1/2} = \lambda \quad (2)$$

Divide (1) by (2), we get

$$\frac{x_2}{x_1} = 2$$
$$x_2 = 2x_1$$

Plug in to the constriant, we get $x_1 = 10,\ x_2 = 20$

Then we need to check the Hessian Matrix.

$$H = \begin{bmatrix} \frac{3}{2} x_1^{-1/2} x_2^{1/2} & \frac{3}{2} x_1^{1/2} x_2^{-1/2} \\ \frac{3}{2} x_1^{1/2} x_2^{-1/2} & -\frac{1}{2} x_1^{3/2} x_2^{-3/2} \end{bmatrix}$$

$$H_1 > 0, H_2 < 0$$

H is a negative definitiness matrix at (10, 20), so (10, 20) is a maximum.

Note: use broader Hessian gives you the same results, Since the constraint is binding and solution is interior.

$$BH = \begin{bmatrix} 0 & -6 & -1 \\ -6 & & \\ -1 & & H_{2\times 2} \end{bmatrix}$$

$|BH| > 0$. max.

BH is a ND

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2. Evaluate the following double definite integral

$$\int_0^1 \int_{2y}^2 xy^2 dxdy$$

(12pts)

Solution:

$$\int_0^1 \int_{2y}^2 xy^2 dxdy$$

$$\Rightarrow \int_0^1 \frac{x^2}{2}y^2 \Big|_{x=2y}^{2} dy$$

$$\Rightarrow \int_0^1 2y^2 - 2y^4 dy$$

$$\Rightarrow 2\int_0^1 y^2 - y^4 dy$$

$$\Rightarrow 2[\frac{y^3}{3} - \frac{y^5}{5}]\Big|_{y=0}^{1} = \frac{4}{15}$$

3. Define the gradient of a function as the vector of the partial derivative to each variable. Consider a real value function $f(x, y, z)$ with three real number variables x, y, z, such that

$$f(x, y, z) = x^2 + 2xy + z^3 + e^{y+2y^2}$$

(6pts) (a) Find the gradient of $f(x, y, z)$, that is, find the vector $\nabla f = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}]'$

Solution:

$2x + (1+4y)e^{y+2y^2},$

$$\nabla f = [\ 2x + 2y,\ \cancel{2x + e^y},\ 3z^2]$$

(b) Evaluate the gradient of function, at the point $(x, y, z) = (1, 0, 2)$

(6pts)

Solution:

$$\nabla f\Big|_{x=1,y=0,z=2} = [\ 2x + 2y,\ 2x + (1 + 4y)e^{y+2y^2},\ 3z^2]\Big|_{x=1,y=0,z=2} = [2, 3, 12]$$

4. Consider a special case of Constant Elasticity of Substitution (CES) utility function with two goods x, y. Suppose U is the utility function of a representative consumer, where

(12pts)

$$U = (x^2 + y^2)^{\frac{1}{2}}$$

The budget constraint for this representative consumer is

$$p_x x + p_y y = I$$

and I denotes her total income. A rational representative consumer must maximize her utility and spend ALL her income I on goods x, y. Assume I, p_x, p_y are known. Set up the constrained optimization problem properly and find out the optimal consumption bundle (x^*, y^*)

Solution: Set up the constraint problem with Lagrangian multiplier.

$$x^* = \frac{p_x}{(p_x)^2 + (p_y)^2} I$$

$$y^* = \frac{p_y}{(p_x)^2 + (p_y)^2} I$$

5. In a four-dimensional space with variables w, x, y, z. Suppose w can be represented as a function of x, y, z. That is, $w = w(x, y, z)$. At the point of $(x, y, z) = (1, 2, 3)$, find the total differentiation of w.

(12Pts)

$$w = x^3yz + xy + 2z + 5$$

Solution:

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

$$dw = (3x^2yz + y)dx + (x^3z + x)dy + (x^3y + 2)dz$$

Plug in $x = 1, y = 2, z = 3$

$$\cancel{dw = 8dx + 4dy + 4dz}$$

$dw = 20dx + 4dy + 4dz$.

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6. Consider z as a positive integer and define a function

$$f(z+1) = \int_0^\infty x^z e^{-x} dx$$

(4pts) (a) Use the method of Integration By Part, show that

$$f(z+1) = z \cdot f(z)$$

(4pts) (b) Plug in $z = 1, 2,$ and 3. Show that $f(2) = 1$, $f(3) = 2 \cdot 1$, $f(4) = 3 \cdot 2 \cdot 1$.

(4pts) (c) Then, conclude that $f(z+1) = z!$ or $f(z+1) = z \cdot (z-1) \cdot (z-2) \cdot \ldots \cdot 2 \cdot 1$, for z is a positive integer.

Solution: This is the definition of Gamma function. Using Integration by Parts (I.B.P), we can simply shows the result. Let

$$u = x^z, dv = e^{-x} dx$$

$$du = z \cdot x^{(z-1)}, v = -e^{-x}$$

$$\int u dv = uv - \int v du$$

$$f(z+1) = \int_0^\infty x^z e^{-x} dx = -e^{-x} x^z \Big|_{x=0}^{\infty} - \int_0^\infty z \cdot x^{(z-1)} - e^{-x} dx$$

$$= 0 + z \int_0^\infty x^{(z-1)} e^{-x} dx = z \cdot f(z)$$

Then the original equation is shown.

Let's check if $z = 1$,

$$f(2) = \int_0^\infty x e^{-x} dx = 1$$

If $z = 2$,

$$f(3) = \int_0^\infty x^2 e^{-x} dx = 2 \cdot f(2) = 2 \cdot 1 = 2!$$

Once again, check if $z = 3$

$$f(4) = \int_0^\infty x^3 e^{-x} dx = 3 \cdot f(3) = 3 \cdot 21 = 3!$$

By the induction, we could know

$$f(z) = z!$$

holds when z is a positive integer.

7. Consider the problem

$$\min_{x,y,z} x^2 + y^2 + z^2$$

$$\text{subject to} \quad x + 2y = 1, \quad 2x - z = 4.$$

12pts. Find the stationary point of (x, y, z)

4pts 4pts 4pts

8. Consider the following system F of two equations $f_1(x, y), f_2(x, y)$.

$$f_1(x, y) = 2x - y$$

$$f_2(x, y) = 3x + y$$

3pts (a) Find the Jacobian matrix J_F and its determinant $|J_F|$.

3pts (b) Let $f_1(x, y) = u$, $f_2(x, y) = v$. Solve x, y as the functions of u, v.

3pts (c) Let $x(u, v) = g_1(u, v)$ and $y(u, v) = g_2(u, v)$ as a new system of equations G. Find the Jacobian matrix J_G and its determinant $|J_G|$.

3pts (d) What is the relationship between J_F and J_G? What about $|J_F|$ and $|J_G|$? (Hint : $|X^{-1}| = \frac{1}{|X|}$ for all non-singular square matrix X)

End

Q7. Answer. Dec 13 2019 Je-Wei Kuo

$$\min \quad x^2+y^2+z^2$$

subject to $\quad x+2y=1, \quad 2x-z=4.$

$$\mathcal{L} = (x^2+y^2+z^2) - \lambda_1(x+2y-1) - \lambda_2(2x-z-4)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow 2x - \lambda_1 - \lambda_2 \cdot 2 = 0. \quad — ①$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0 \Rightarrow 2y - 2\lambda_1 = 0 \Rightarrow \lambda_1 = y \quad — ②$$

$$\frac{\partial \mathcal{L}}{\partial z} = 0 \Rightarrow 2z + \lambda_2 = 0. \Rightarrow \lambda_2 = -2z \quad — ③$$

plug ②, ③ into ①

$$2x - y + 4z = 0 \quad — ④$$

two constraints

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = 0 \Rightarrow x+2y-1=0. \quad — ⑤$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = 0 \Rightarrow 2x-z-4=0. \quad — ⑥$$

solve out system of ④, ⑤, ⑥

$$④ \times 2 + ⑤ \Rightarrow 5x+8z=1 \quad — ⑦$$

$$⑥ \Rightarrow 2x - z = 4.$$

$$⑥ \times 8 \Rightarrow 16x - 8z = 32.$$

$$+⑦ \quad 5x + 8z = 1$$

$$21x = 33$$

$$x = \frac{11}{7}$$

$$z = 2x-4 = \frac{22}{7} - \frac{28}{7} = \frac{-6}{7}$$

$$y = (1-x)\frac{1}{2} = \left(1-\frac{11}{7}\right)\left(\frac{1}{2}\right) = -\frac{2}{7}$$

$$(x^*, y^*, z^*) = \left(\frac{11}{7}, -\frac{2}{7}, \frac{-6}{7}\right)$$

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Q8. Answer This is an application of inverse function theorem.

(a) $$J_F = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \#$$

$$|J_F| = 5. \#$$

(b) $$u = 2x - y$$
$$v = 3x + y$$

$$u + v = 5x \Rightarrow x = \frac{1}{5}(u+v).$$

$$y = 2x - u \Rightarrow y = \frac{2}{5}(u+v) - u.$$

$$y = \frac{-3u+2v}{5}$$

$$x = \frac{1}{5}u + \frac{1}{5}v$$
$$y = -\frac{3}{5}u + \frac{2}{5}v. \#$$

(c) $$g_1 = \frac{1}{5}u + \frac{1}{5}v$$
$$g_2 = -\frac{3}{5}u + \frac{2}{5}v$$

$$J_G = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix} \qquad |J_G| = \frac{2}{25} - \frac{-3}{25} = \frac{5}{25} = \frac{1}{5}. \#$$

(d) $$J_F = J_G^{-1} \text{ or } J_F^{-1} = J_G \text{ or } J_F \cdot J_G = I_2$$

$$|J_F| = |J_G|^{-1} \text{ or } |J_F| = \frac{1}{|J_G|}$$

$$\#$$