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Reallocation of Housing by use of Network Analysis

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A Housing Authority owns a number of houses which are let to tenants whose needs change over a period of time so that some of the houses no longer meet the requirements of their tenants. Reallocation of the houses can result in a series of moves forming a chain, cyclic in character. By use of a network best path algorithm these chains of varying length can be ascertained and facilitate the reallocation of the housing.

INTRODUCTION

A HOUSING AUTHORITY has a number of houses at its disposal which are let to tenants. Each house is characterized by the possession or otherwise of certain attributes. For example, a house might have a garage, four bedrooms and a rent within a specified range. It is clear that if there are n such attributes which may or may not be present there will be 2^n possible combinations or categories into which a house might fall. However, many of the attributes are mutually exclusive and the number of categories will in practice be smaller but might well be large enough to give rise to a problem when reallocation becomes necessary.

Over a period of time a number of tenants will surrender their tenancies as they move away or choose to live in alternative accommodation. No doubt in the various housing situations existing in many countries there will be more than an equal supply of prospective tenants. Furthermore, the requirements of tenants will change with time, families arrive and grow up, incomes and jobs change and so on. The Housing Authority will wish to match the requirements of the tenants, prospective or existing, to their houses; and, whilst much can be done by simple exchanges there will be cases where one tenant moving to another house is replaced by another tenant from a house in a different category who is replaced by another and so on, so that a chain of exchanges will be created. In graph theory terms a path progression (hereafter in this paper referred to as a path) is produced which consists of a sequence of arcs. The number of arcs in a path will be referred to as the length of the path. It will be seen that if conservation of supply and demand is to be preserved the paths must be cyclic forming a circuit.

The isolation of these circuits, where the housing pool is large, can be an onerous task. In the following section a simple method using a network best path algorithm is presented which enables this to be done.

SOLUTION BY NETWORK BEST PATH ALGORITHM

A requirements matrix $A = a_{ij}$ is prepared first showing the number of tenants in row i occupying a house with attributes in category i who wish to move to a house in column j which has attributes in category j. An associated "distance" matrix, $D = d_{ij}$ is generated such that for all i,j: $d_{ij} = 1$ if $a_{ij} \neq 0$, otherwise $d_{ij} = \infty$.

The matrix D may now be entered into a network best path routine whereby D is replaced by a new matrix $D' = d'_{ij}$. A suitable algorithm for this purpose appears in Floyd.[1] Each element d'_{ij} will give the minimum distance between i and j. This distance will equal the sum of the values given to each member of the path and, where the sum $\sim\infty$ these values will each be equal to unity and d'_{ij} in this case will be equal to the number of arcs on the path, i.e. length of the path. Down the diagonal of D' the values d'_{ss} will indicate the length of the minimum circuit, where one exists, beginning and starting at s.

Each value d_{ij} of matrix D having a value of unity is associated with a corresponding value in A having a value $a_{ij} \neq 0$. Those values in D having a value of unity will remain unchanged in D'. The routine can therefore be used to find minimum length paths in A each element of which has a value $a_{ij} \neq 0$.

FINDING THE MEMBERS OF THE CIRCUIT

The existence or otherwise of a circuit in A starting at a particular value i can therefore be determined by examining the diagonal d'_{ii}. If a circuit is selected for processing it is necessary to find which other elements in A are also members of the circuit by means of a suitable decoding routine. Such a routine relies on the principle of optimality familiar in dynamic programming.

If l_{ss} is the length of a circuit starting and finishing at s and t is some element on the circuit and l_{st} and l_{ts} are the lengths of the paths from s to t and t to s respectively such that $l_{ss} = l_{st} + l_{ts}$, then, if l_{ss} is optimal, so are l_{st} and l_{ts}. The minimum values of l_{ss}, l_{st} and l_{ts} all appear in D'. Thus any element x can be a member of a minimum circuit if $d'_{sx} + d'_{xs} = d'_{ss}$ and not otherwise. There can of course be alternative circuits of the same minimum length.

The search for such a path is facilitated by the fact that the value of d'_{sx} for an arc adjacent to s will have a value of unity. Thus if x is adjacent to s then it is a member of a minimum circuit if $d'_{xs} = d'_{ss} - 1$. Using a stepwise procedure the members of a circuit from s to s can be found.

THE BALANCED MATRIX

Each element a_{ij} of matrix A specifies the number of tenants occupying a house with attributes in category i who wish to move to a house with attributes in category j. Consider a matrix in which the sum of the entries in each row is equal to the sum of the entries in each column where the row index equals the column index, that is

$$\sum_{i=1}^{n} is = \sum_{i=1}^{n} si \quad \text{for all } s = 1, 2, \ldots, n.$$

In this case the number of paths leading to each category will equal the number of paths leaving that category and it will be possible to find a single circuit, possibly incorporating several sub-circuits, which will accommodate all exchanges. Indeed such a circuit could be established in the manner in which a Euler circuit is found although in large problems this could be a tedious task. Again such a circuit would be inconveniently long except in very small problems. In practice, Housing Authorities prefer to use short circuits since they are easier to handle administratively than longer ones.

ATTEMPTS AT THE OPTIMAL SOLUTION

Using this criteria a solution may be regarded as optimal if it employs the least number of arcs to effect all exchanges. Unfortunately this solution proves difficult to find in sizeable matrices. Some sort of iterative procedure is required, but it will be apparent that an arc, allocated to a circuit in an early iteration and thus precluded from a later iteration, might well have been better employed in the later iteration.

One procedure aiming to find the largest number of short circuits might take the following form. As many circuits of length 2 are processed, the adjustment to matrix A being made as each circuit is extracted. Matrix D after all such circuits have been dealt with is re-established and processed by the network best path routine to form matrix D' when circuits of length 3 are extracted and so on.

A drawback to this method lies in the fact that isolated arcs tend to be left towards the end of the processing and they then have to be joined to form circuits which may become unduly lengthy. An alternative method, illustrated in the example following, processes all circuits starting at a particular value of i before proceeding to the next. Systematically proceeding line by line is suggested although strict sequence is not necessary. Since the length of a circuit cannot exceed the number of lines waiting to be processed (otherwise a sub-circuit would emerge), during the early iterations when more alternatives are available, circuits produced tend to be of short length and during the later stages constrained within bounds.

Furthermore, it becomes unnecessary to prepare the whole of the matrix D'. Only the line and column under consideration need attention. They can be dropped from matrices A and D as they become exhausted. Matrix D' can be superimposed on D and D itself need not be reconstituted between iterations since apart from the line and column being processed it will contain only information about the existence of adjacent arcs, which is all that is required.

A number of methods of finding the shortest path from a single location are described in the literature on graph theory, e.g. Moore.[2]

A WORKED EXAMPLE

The following matrix A represents a small example deliberately made sparse so that the method can be clearly demonstrated. In practice larger matrices are encountered.

Examples of category descriptions 1–10 are

1 3 beds, 1 garage, medium rent, outer zone C
2 4 beds, 2 garages, high rent, inner zone A
⋮
5 3 beds, no garage, low rent, middle zone F
6 2 beds, 1 garage, medium rent, middle zone F
⋮
10 5 beds, 1 garage, garden, high rent, central heating, zone G

The rent ranges would be normally quantified.

Matrix A:

		Desired categories									
		1	2	3	4	5	6	7	8	9	10
Occupied categories	1		6		7		3		6	5	
	2				3	17	12			9	7
	3	8	4				8	10			5
	4			8		3		7			
	5	6		4			10	6	5	10	
	6		22		8			3	4	6	
	7	6	7	13						8	
	8		9	5			6		11	6	
	9			5		16		19			4
	10	7			5	4					

The following matrix D' includes the shortest distance from line $i = 1$ and to column $j = 1$,

	1	2	3	4	5	6	7	8	9	10
1	3	1	2	1	2	1	2	1	1	2
2	2			1	1	1			1	1
3	1	1				1	1			1
4	2		1		1		1	1		
5	1		1			1	1	1	1	1
6	2	1		1			1		1	
7	1	1	1					1	1	
8	2	1	1			1			1	
9	2		1		1		1		1	
10	1			1	1					

Blank entries in this matrix represent values of ∞. $d_{11} = 3$ shows that the shortest circuit from $i = 1$ includes 3 arcs. The first adjacent arc from node 1 is 1–2. Since $d'_{12}+d'_{21}$ gives $1+2 = 3$ node 2 will appear on a minimum circuit. The first adjacent arc from node 2 is 2–4 but $d'_{24}+d'_{41}$ gives $1+2 = 3$ which is greater than d'_{21} so that node 4 is not on this minimum circuit, although it can of course appear on another circuit. The next adjacent arc from node 2 is 2–5. $d'_{25}+d'_{51}$ gives $1+1 = 2$ so that node 5 is on this circuit giving the complete circuit:

1–2–5–1.

The minimum value of a_{12}, a_{25} and a_{51} is 6 so that the occupants of 6 houses may be moved from houses in category 1 to houses in category 2, 6 from 2 to 5 and 6 from 5 to 1. a_{12} and a_{51} become zero and a_{25} becomes 11. d'_{12} and d'_{51} are made equal to ∞ and the process repeated.

Thus 6 families in category description 1 (3 beds, 1 garage, medium rent, outer zone) would move to houses in category description 2 (4 beds, 2 garages, high rent, inner zone A) who would move to houses in category description 6 (2 beds, 1 garage, medium rent). The occupants in 6 houses in category 6 would replace those moving out of category 1, thus completing the cycle. The remaining 5 families in category 2 who wish to move to category 5 will be accommodated in other circuits, the complete list of which is given below. The following circuits are extracted:

1–4–3–1	7 sets	1–6–7–1	3 sets	1–8–3–1	1 set
1–9–7–1	3 sets	1–9–10–1	2 sets		

No more circuits of length 3 being left the network algorithm is reapplied to matrix D and a circuit of length 4 is disclosed. Matrix D' at this stage is as follows:

	1	2	3	4	5	6	7	8	9	10
1	4	2	2	3	3	2	3	1	2	3
2	2			1	1	1			1	1
3	2	1				1	1			1
4	3		1		1		1	1		
5	3		1			1	1	1	1	
6	3	1		1					1	
7	3	1	1					1	1	
8	3	1	1		1				1	
9	2		1		1		1			1
10	1			1	1					

The next circuit extracted is

1–8–2–10–1 5 sets

Row 1 and column 1 will now contain no adjacent arcs (d'_{11} will become ∞) and this row and column in both A and D can be dropped. Continuing the process the complete list of circuits becomes:

1–2–5–1	6 sets	2–5–6–2	8 sets	4–5–6–4	2 sets
1–4–3–1	7	2–9–7–2	5	4–5–8–6–4	1
1–6–7–1	3	2–9–3–6–2	2	4–7–8–6–4	2
1–8–3–1	1	2–9–5–8–2	2	5–9–5	10
1–9–7–1	3	2–10–4–8–2	2	5–7–9–5	4
1–9–10–1	2	3–7–3	10	5–7–9–10–5	2
1–8–2–10–1	5	3–6–9–3	3	6–9–7–8–6	3
2–6–2	12	3–10–5–3	1	7–8–9–7	6
2–4–3–2	1	3–6–4–7–3	3	7–9–7	2
2–4–7–2	2	3–10–4–8–3	3		
2–5–3–2	3	3–10–5–8–3	1		

All possible exchanges are now accounted for. It should be noted that it is not necessary to stipulate that the sum of each row should match the sum of the corresponding column. If they do not tally the algorithm will, after extracting all possible routes, fail to find a circuit ($d_{ss} = \infty$ for one or more s).

The method can be extended to deal with new housing hitherto unoccupied, and housing taken out of the pool by the creation of a dummy row and column to represent houses with no attributes. The method so described whilst not necessarily finding the optimum solution nevertheless will find a good solution, with relatively little computing effort and affords an example of the use of network analysis in the solution of a practical problem.

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[2] W. Hoffman and R. Payley (1959) A method for the solution of *N*th best path problem. *JACM* **6**, 506.