BITS, BYTES, AND INTEGERS

SYSTEMS I

Instructor:

-

Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
- Making ints from bytes
- Summary

Encoding Byte Values

- Byte = 8 bits
  - Binary 00000000_2 to 11111111_2
  - Decimal: 0_{10} to 255_{10}
  - Hexadecimal 00_{16} to FF_{16}
    - Base 16 number representation
    - Use characters ‘0’ to ‘9’ and ‘A’ to ‘F’
    - Write $FA1D37B_{16}$ in C as
      - 0xFA1D37B
      - 0xfa1d37b

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Boolean Algebra

- Developed by George Boole in 19th Century
  - Algebraic representation of logic
    - Encode "True" as 1 and "False" as 0

And

- **A&B = 1 when both A=1 and B=1**

&	0	1
0	0	0
1	0	1

Or

- **A|B = 1 when either A=1 or B=1**

|	0	1
0	0	1
1	1	1

Not

- **~A = 1 when A=0**

~	
0	1
1	0

Exclusive-Or (Xor)

- **A^B = 1 when either A=1 or B=1, but not both**

^	0	1
0	0	1
1	1	0

General Boolean Algebras

- Operate on Bit Vectors
  - Operations applied bitwise

```
  01101001     01101001     01101001
& 01010101   | 01010101   ^ 01010101   ~ 01010101
  01000001     01111101     00111100     10101010
```

- All of the Properties of Boolean Algebra Apply

Bit-Level Operations in C

- Operations &, |, ~, ^ Available in C
  - Apply to any “integral” data type
    - long, int, short, char, unsigned
  - View arguments as bit vectors
  - Arguments applied bit-wise
- Examples (Char data type [1 byte])
  - ~0x41 → 0xBE
    - $\sim 01000001_2 \rightarrow 10111110_2$
  - ~0x00 → 0xFF
    - $\sim 00000000_2 \rightarrow 11111111_2$
  - 0x69 & 0x55 → 0x41
    - $01101001_2 \,\&\, 01010101_2 \rightarrow 01000001_2$
  - 0x69 | 0x55 → 0x7D
    - $01101001_2 \mid 01010101_2 \rightarrow 01111101_2$

Representing & Manipulating Sets

- Representation
  - Width w bit vector represents subsets of {0, …, w–1}
  - $a_j = 1$ if $j \in A$
    - 01101001 { 0, 3, 5, 6 }
    - *76543210*
    - 01010101 { 0, 2, 4, 6 }
    - *76543210*
- Operations
  - & Intersection 01000001 { 0, 6 }
  - | Union 01111101 { 0, 2, 3, 4, 5, 6 }
  - ^ Symmetric difference 00111100 { 2, 3, 4, 5 }
  - ~ Complement 10101010 { 1, 3, 5, 7 }

Contrast: Logic Operations in C

- Contrast to Logical Operators
  - &&, ||, !
    - View 0 as "False"
    - Anything nonzero as "True"
    - Always return 0 or 1
    - Short circuit
- Examples (char data type)
  - !0x41 → 0x00
  - !0x00 → 0x01
  - !!0x41 → 0x01

  - 0x69 && 0x55 → 0x01
  - 0x69 || 0x55 → 0x01
  - p && *p (avoids null pointer access)

Shift Operations

- Left Shift: x << y
  - Shift bit-vector x left y positions
      - Throw away extra bits on left
    - Fill with 0's on right
- Right Shift: x >> y
  - Shift bit-vector x right y positions
    - Throw away extra bits on right
  - Logical shift
    - Fill with 0's on left
  - Arithmetic shift
    - Replicate most significant bit on left
- Undefined Behavior
  - Shift amount < 0 or ≥ word size

Argument x	01100010
<< 3	00010*000*
Log. >> 2	*00*011000
Arith. >> 2	*00*011000

Argument x	10100010
<< 3	00010*000*
Log. >> 2	*00*101000
Arith. >> 2	*11*101000

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Data Representations

C Data Type	Typical 32-bit	Intel IA32	x86-64
char	1	1	1
short	2	2	2
int	4	4	4
long	4	4	8
long long	8	8	8
float	4	4	4
double	8	8	8
long double	8	10/12	10/16
pointer	4	4	8

Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit

```
short int x =  15213;
short int y = -15213;
```

- C `short` 2 bytes long

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
y	-15213	C4 93	11000100 10010011

- Sign Bit
  - For 2's complement, most significant bit indicates sign
    - 0 for nonnegative
    - 1 for negative

Encoding Example (Cont.)

```
x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
```

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum		**15213**		**-15213**

Unsigned & Signed Numeric Values

X	**B2U(*X*)**	**B2T(*X*)**
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

- Equivalence
  - Same encodings for nonnegative values
- Uniqueness
  - Every bit pattern represents unique integer value
  - Each representable integer has unique bit encoding
- $\Rightarrow$ Can Invert Mappings
  - $\mathrm{U2B}(x) = \mathrm{B2U}^{-1}(x)$
    - Bit pattern for unsigned integer
  - $\mathrm{T2B}(x) = \mathrm{B2T}^{-1}(x)$
    - Bit pattern for two's comp integer

Numeric Ranges

- Unsigned Values
  - *UMin* = 0
    000…0
  - *UMax* = $2^w - 1$
    111…1
- Two's Complement Values
  - *TMin* = -2^{w-1}
    100…0
  - *TMax* = $2^{w-1} - 1$
    011…1
- Other Values
  - Minus 1
    111…1

Values for *W* = 16

	Decimal	Hex	Binary
UMax	**65535**	**FF FF**	11111111 11111111
TMax	**32767**	**7F FF**	01111111 11111111
TMin	**-32768**	80 00	10000000 00000000
-1	**-1**	**FF FF**	11111111 11111111
0	**0**	00 00	00000000 00000000

Values for Different Word Sizes

	W			
	8	**16**	**32**	**64**
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

- Observations
  - $| TMin | = TMax + 1$
    - Asymmetric range
  - $UMax = 2 * TMax + 1$

- **C Programming**
  - #include <limits.h>
  - Declares constants, e.g.,
    - ULONG_MAX
    - LONG_MAX
    - LONG_MIN
  - Values platform specific

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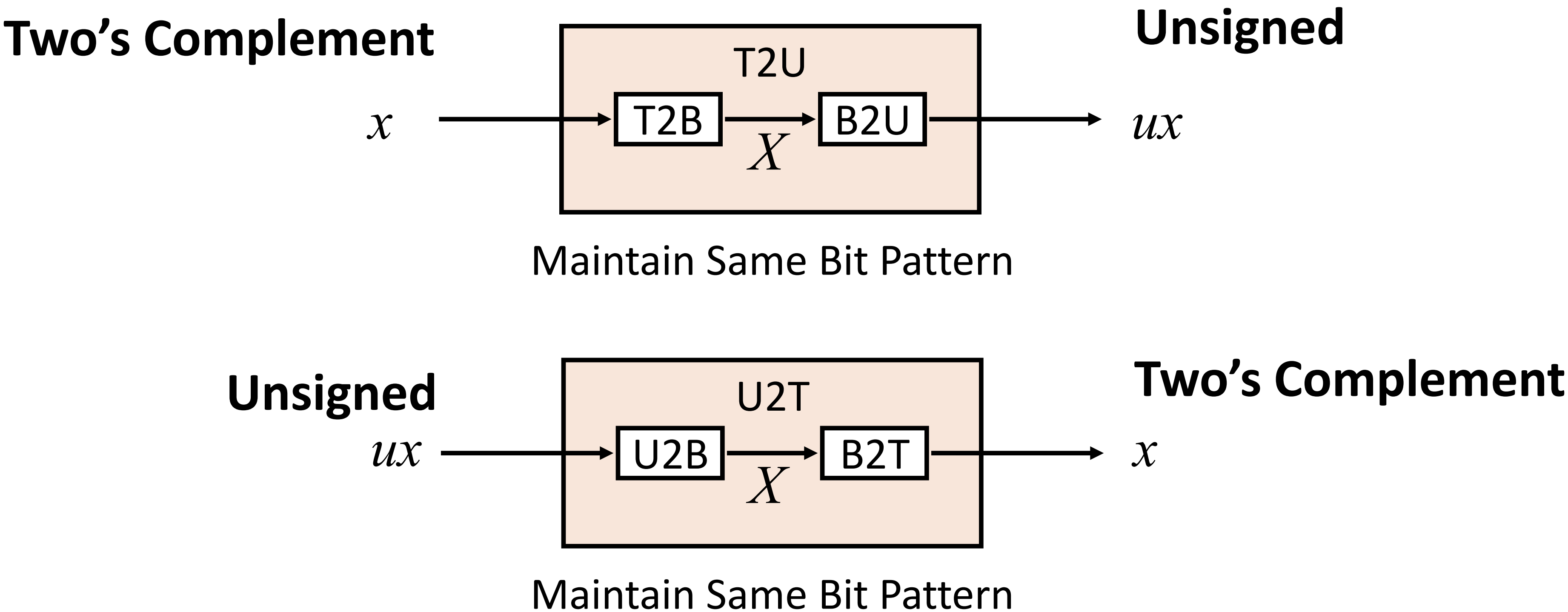
Mapping Between Signed & Unsigned



- Mappings between unsigned and two's complement numbers: keep bit representations and reinterpret

Mapping Signed ↔ Unsigned

Bits	Signed	Unsigned
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	-8	8
1001	-7	9
1010	-6	10
1011	-5	11
1100	-4	12
1101	-3	13
1110	-2	14
1111	-1	15

T2U →

← U2T

Mapping Signed ↔ Unsigned

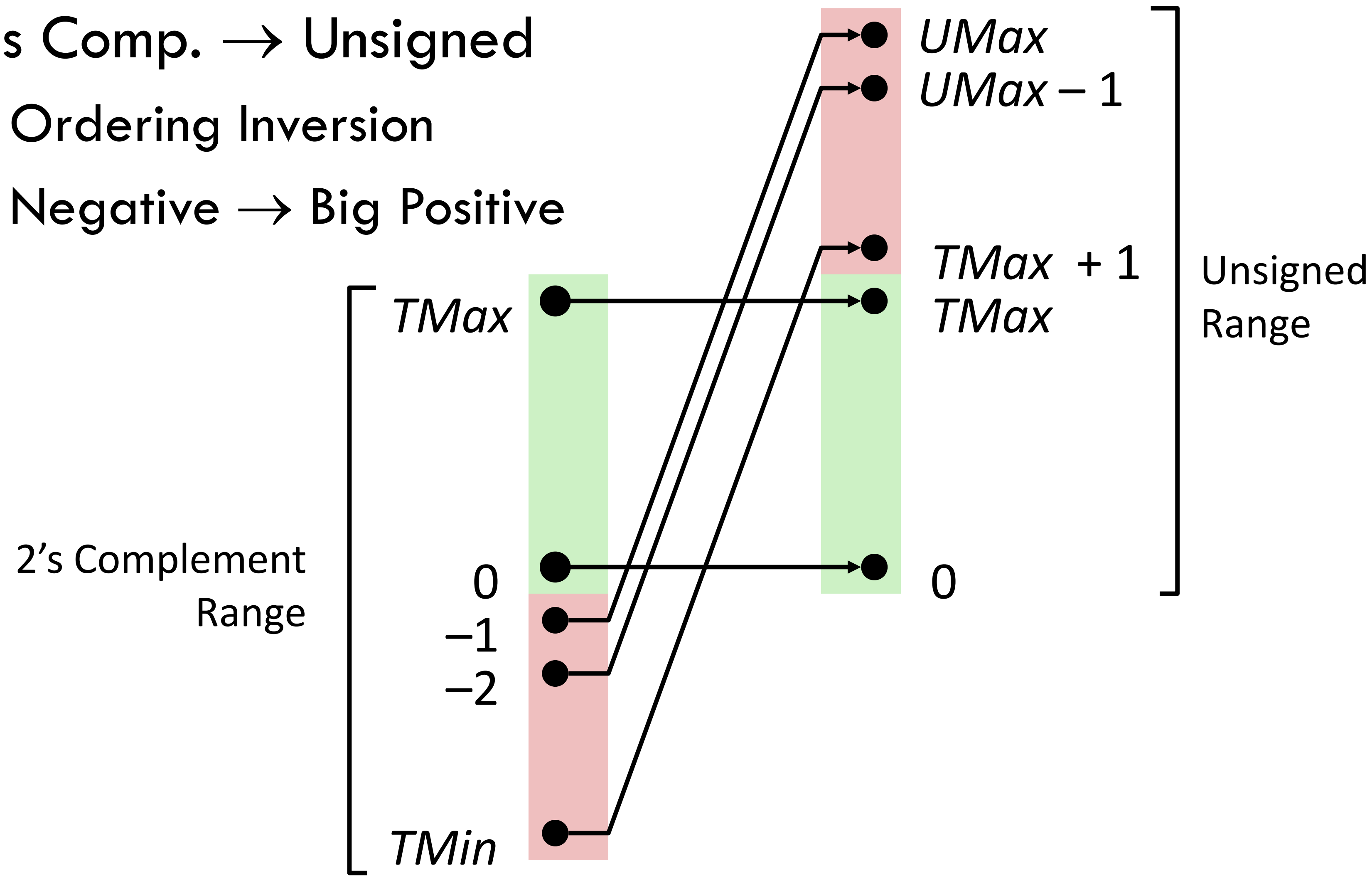
Bits	Signed	Unsigned
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	-8	8
1001	-7	9
1010	-6	10
1011	-5	11
1100	-4	12
1101	-3	13
1110	-2	14
1111	-1	15

=

+/- 16

Conversion Visualized

- 2’s Comp. → Unsigned
  - Ordering Inversion
  - Negative → Big Positive



Negation: Complement & Increment

- Claim: Following Holds for 2's Complement

  `~x + 1 == -x`

- Complement
  - Observation: `~x + x == 1111…111 == -1`

	x	1	0	0	1	1	1	0	1
+	~x	0	1	1	0	0	0	1	0
	-1	1	1	1	1	1	1	1	1

Complement & Increment Examples

x = 15213

	Decimal	Hex	Binary
`x`	15213	`3B 6D`	`00111011 01101101`
`~x`	-15214	`C4 92`	`11000100 10010010`
`~x+1`	-15213	`C4 93`	`11000100 10010011`
`y`	-15213	`C4 93`	`11000100 10010011`

x = 0

	Decimal	Hex	Binary
`0`	0	`00 00`	`00000000 00000000`
`~0`	-1	`FF FF`	`11111111 11111111`
`~0+1`	0	`00 00`	`00000000 00000000`

Signed vs. Unsigned in C

- Constants
  - By default are considered to be signed integers
  - Unsigned if have "U" as suffix

    `0U, 4294967259U`
- Casting
  - Explicit casting between signed & unsigned same as U2T and T2U

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

  - Implicit casting also occurs via assignments and procedure calls

```
tx = ux;
uy = ty;
```

Casting Surprises

- Expression Evaluation
  - If there is a mix of unsigned and signed in single expression, ***signed values implicitly cast to unsigned***
  - Including comparison operations <, >, ==, <=, >=
  - Examples for $W = 32$
    - **TMIN = -2,147,483,648, TMAX = 2,147,483,647**

Constant$_1$	Constant$_2$	Relation	Evaluation
0	0U	==	unsigned
-1	**0**	<	signed
-1	0U	>	unsigned
2147483647	-2147483647-1	>	signed
2147483647U	-2147483647-1	<	unsigned
-1	-2	>	signed
(unsigned)-1	-2	>	unsigned
2147483647	2147483648U	<	unsigned
2147483647	(int) 2147483648U	>	signed

Code Security Example

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void *user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

- Similar to code found in FreeBSD's implementation of getpeername
- There are legions of smart people trying to find vulnerabilities in programs

Typical Usage

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void *user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

```
#define MSIZE 528

void getstuff() {
    char mybuf[MSIZE];
    copy_from_kernel(mybuf, MSIZE);
    printf("%s\n", mybuf);
}
```

Malicious Usage

```
/* Declaration of library function memcpy */
void *memcpy(void *dest, void *src, size_t n);
```

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void *user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

```
#define MSIZE 528

void getstuff() {
    char mybuf[MSIZE];
    copy_from_kernel(mybuf, -MSIZE);
    . . .
}
```

Summary
Casting Signed ↔ Unsigned: Basic Rules

- Bit pattern is maintained
- But reinterpreted
- Can have unexpected effects: adding or subtracting 2^w

- Expression containing signed and unsigned int
  - `int` is cast to `unsigned`!!

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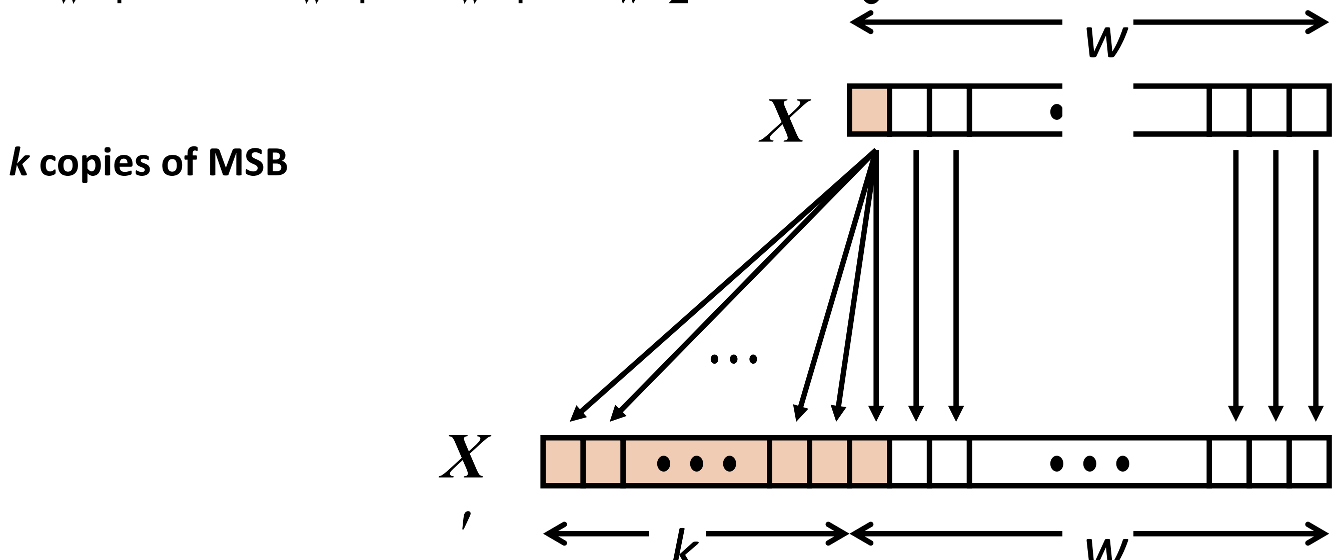
Sign Extension

- Task:
  - Given w-bit signed integer x
  - Convert it to $w+k$-bit integer with same value
- Rule:
  - Make k copies of sign bit:
  - $X' = x_{w-1}, \ldots, x_{w-1}, x_{w-1}, x_{w-2}, \ldots, x_0$



Sign Extension Example

```
short int x =  15213;
int       ix = (int) x;
short int y = -15213;
int       iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- Converting from smaller to larger integer data type
- C automatically performs sign extension

Summary:
Expanding, Truncating: Basic Rules

- Expanding (e.g., short int to int)
  - Unsigned: zeros added
  - Signed: sign extension
  - Both yield expected result

- Truncating (e.g., unsigned to unsigned short)
  - Unsigned/signed: bits are truncated
  - Result reinterpreted
  - Unsigned: mod operation
  - Signed: similar to mod
  - For small numbers yields expected behaviour

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  - Representation: unsigned and signed
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  - **Addition, negation, multiplication, shifting**
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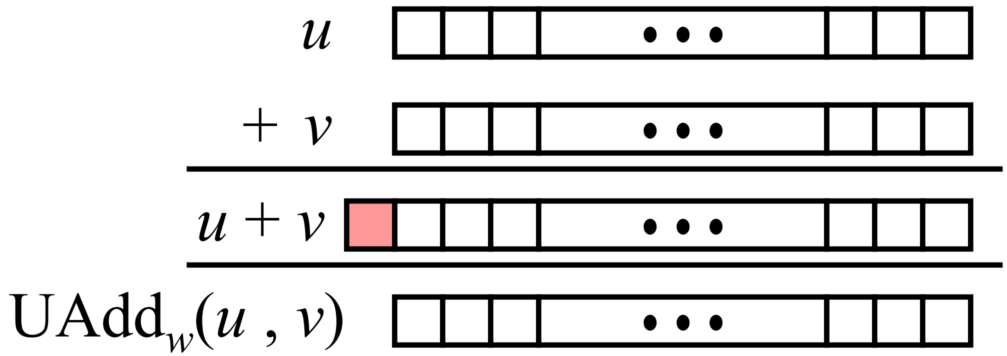
Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



- Standard Addition Function
  - Ignores carry output
- Implements Modular Arithmetic

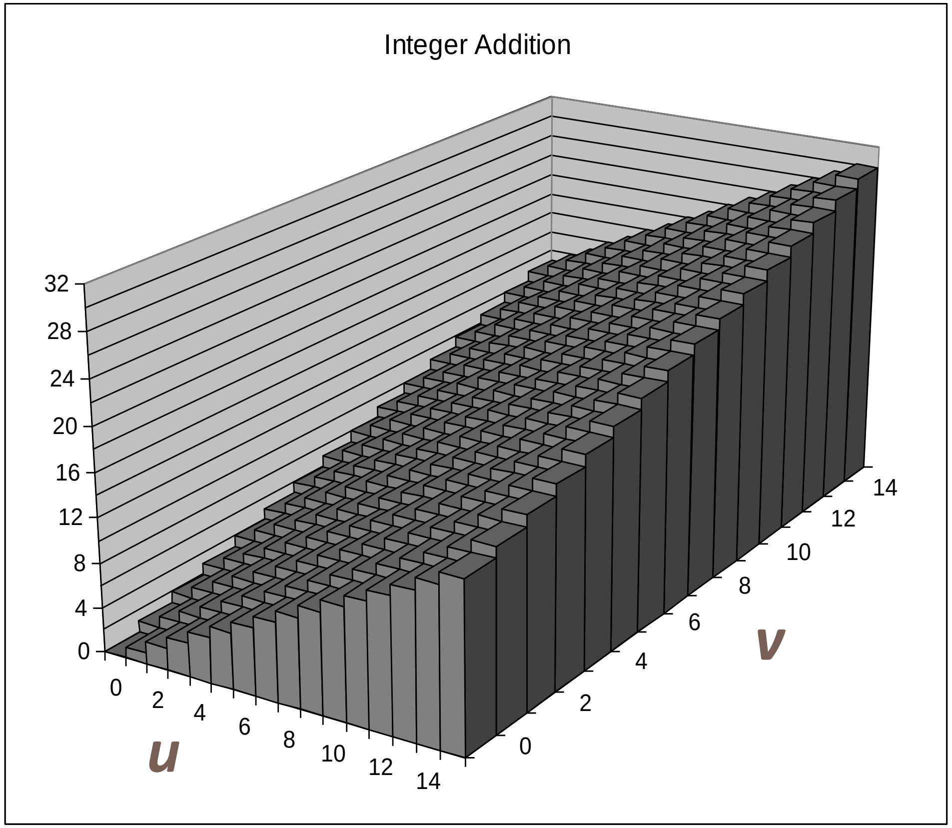
  $s \quad = \quad \mathrm{UAdd}_w(u\ ,\ v) \quad = \quad u + v \mod 2^w$

$$UAdd_w(u,v) = \begin{cases} u+v & u+v < 2^w \\ u+v-2^w & u+v \geq 2^w \end{cases}$$

Visualizing (Mathematical) Integer Addition

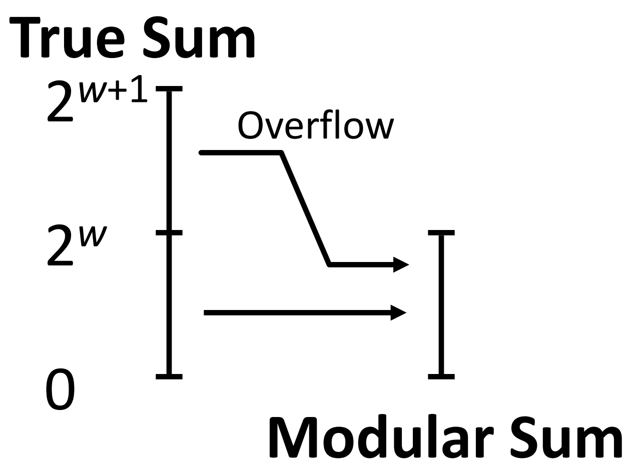
- Integer Addition
  - 4-bit integers *u*, *v*
  - Compute true sum $Add_4(u\ ,\ v)$
  - Values increase linearly with *u* and *v*
  - Forms planar surface

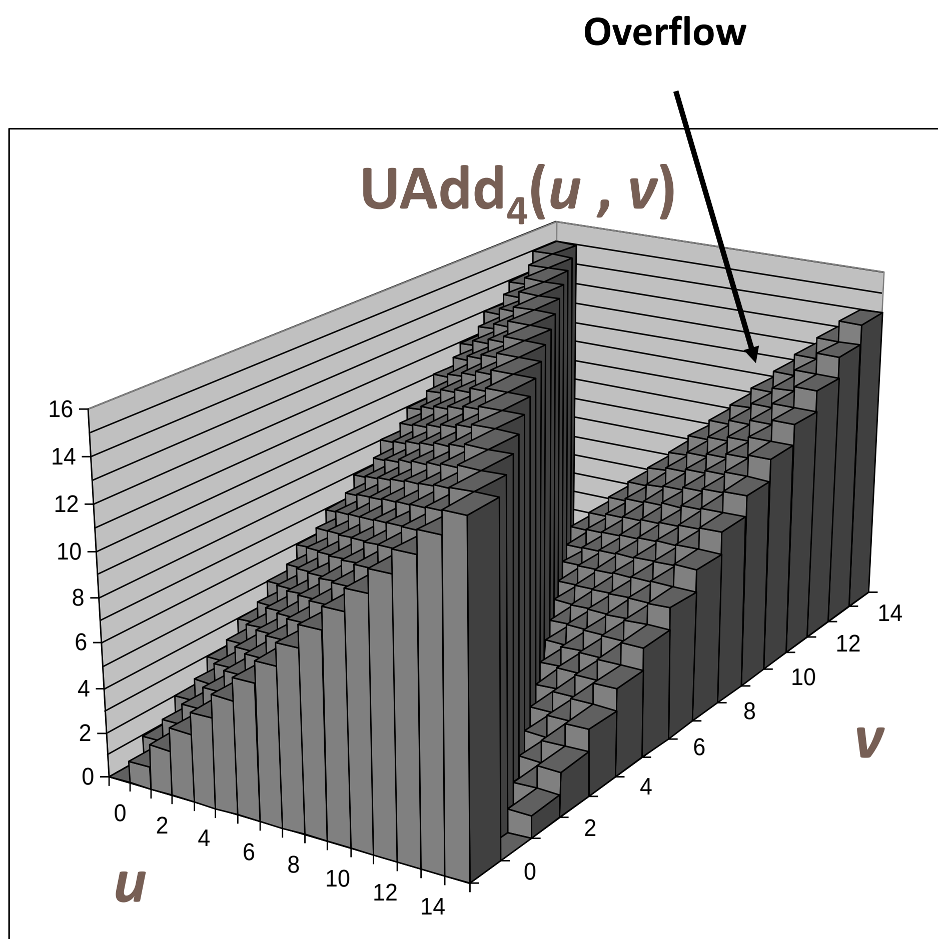
$Add_4(u\ ,\ v)$



Visualizing Unsigned Addition

- Wraps Around
  - If true sum $\geq 2^w$
  - At most once





Mathematical Properties

- Modular Addition Forms an *Abelian Group*
  - **Closed** under addition

    $0 \leq UAdd_w(u, v) \leq 2^w - 1$
  - **Commutative**

    $UAdd_w(u, v) = UAdd_w(v, u)$
  - **Associative**

    $UAdd_w(t, UAdd_w(u, v)) = UAdd_w(UAdd_w(t, u), v)$
  - **0** is additive identity

    $UAdd_w(u, 0) = u$
  - Every element has additive **inverse**
    - Let $UComp_w(u) = 2^w - u$

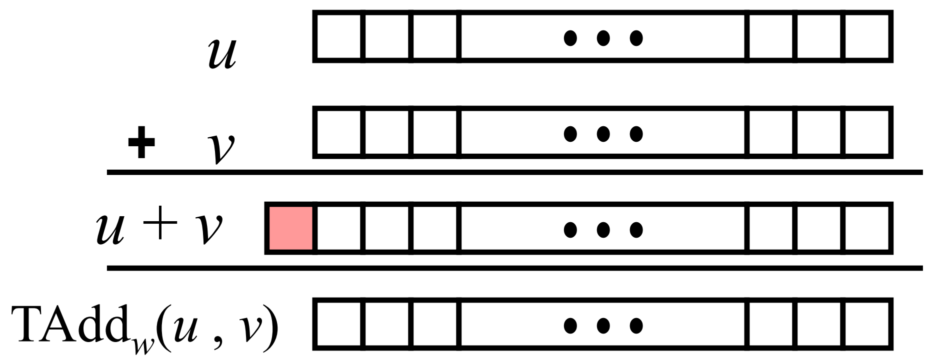
      $UAdd_w(u, UComp_w(u)) = 0$

Two's Complement Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



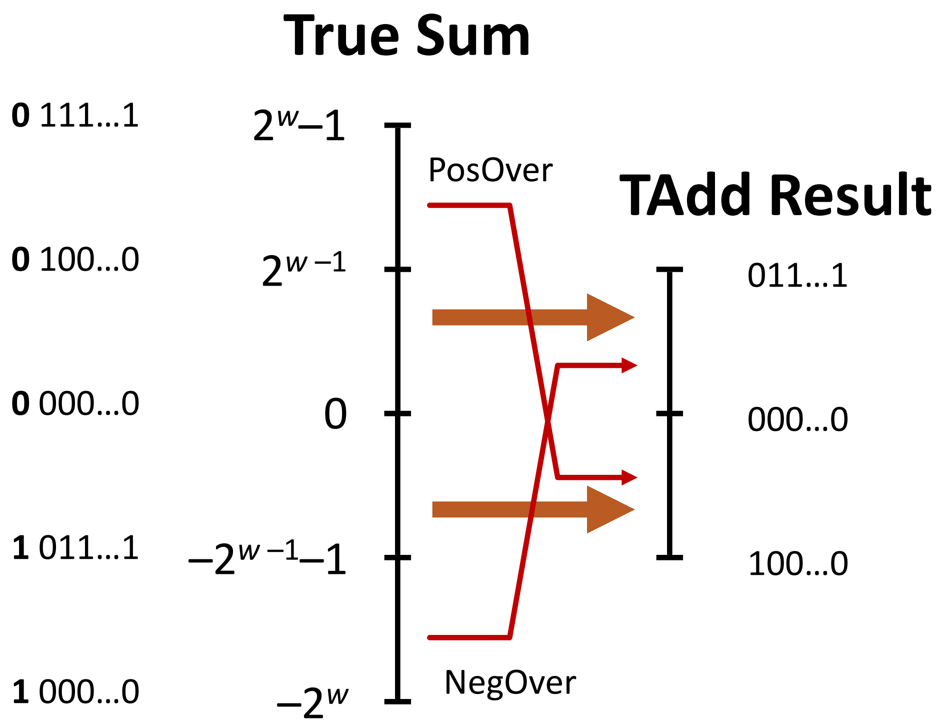
- TAdd and UAdd have Identical Bit-Level Behavior
  - Signed vs. unsigned addition in C:

```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
t = u + v
```

  - Will give `s == t`

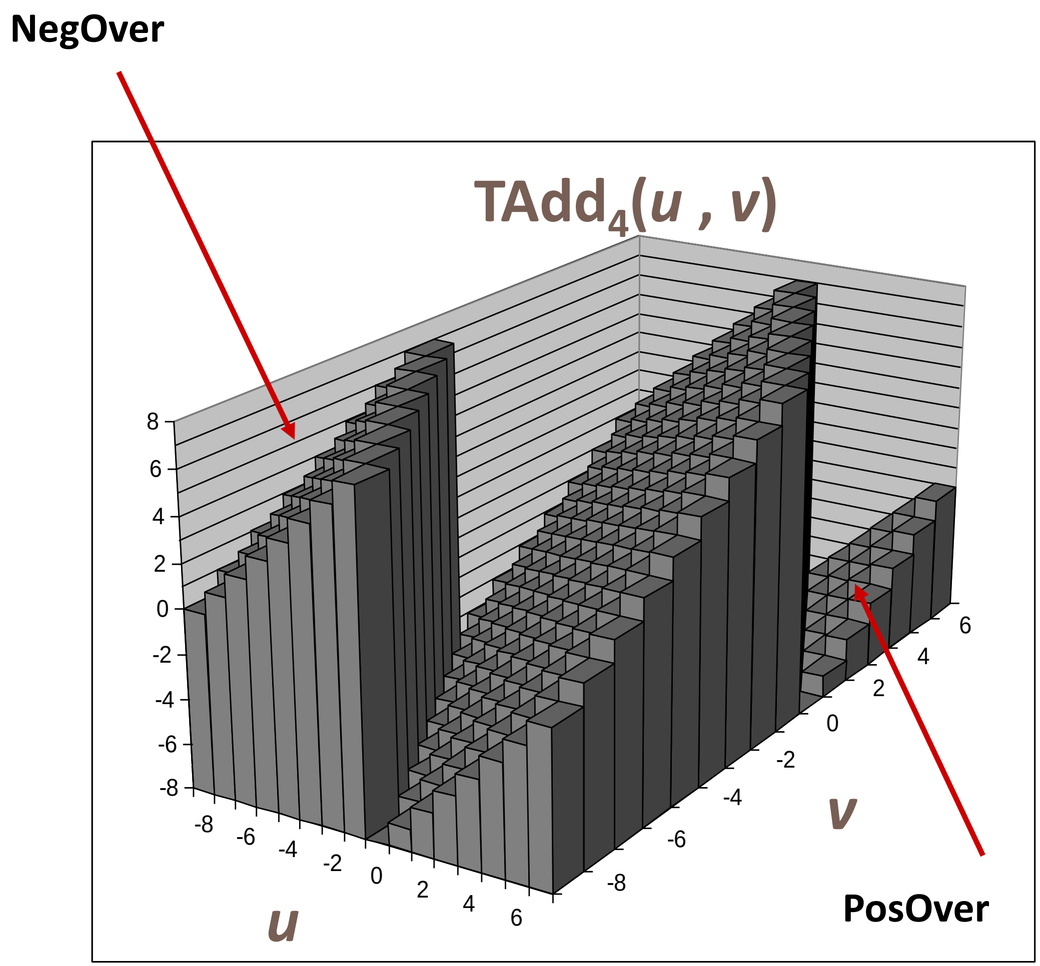
TAdd Overflow

- Functionality
  - True sum requires w+1 bits
  - Drop off MSB
  - Treat remaining bits as 2's comp. integer



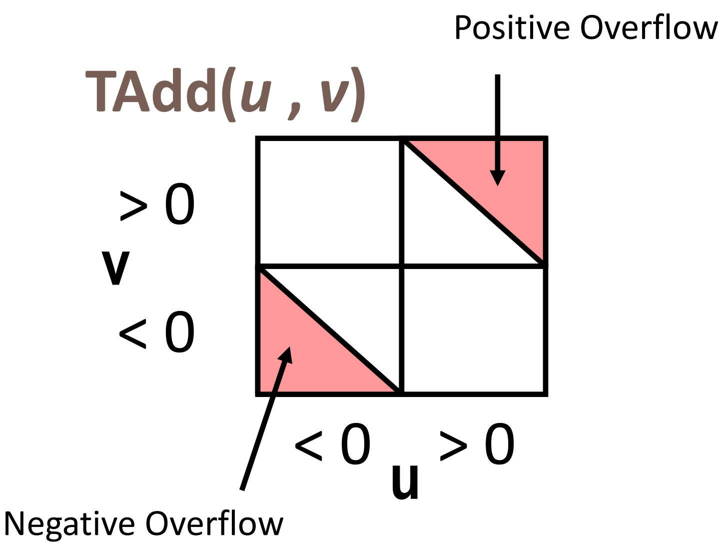
Visualizing 2's Complement Addition

- Values
  - 4-bit two's comp.
  - Range from -8 to +7
- Wraps Around
  - If sum $\geq 2^{w-1}$
    - Becomes negative
    - At most once
  - If sum $< -2^{w-1}$
    - Becomes positive
    - At most once



Characterizing TAdd

- Functionality
  - True sum requires w+1 bits
  - Drop off MSB
  - Treat remaining bits as 2's comp. integer



$$TAdd_w(u,v) = \begin{cases} u+v+2^w & u+v < TMin_w \quad \textbf{(NegOver)} \\ u+v & TMin_w \le u+v \le TMax_w \\ u+v-2^w & TMax_w < u+v \quad \textbf{(PosOver)} \end{cases}$$

Multiplication

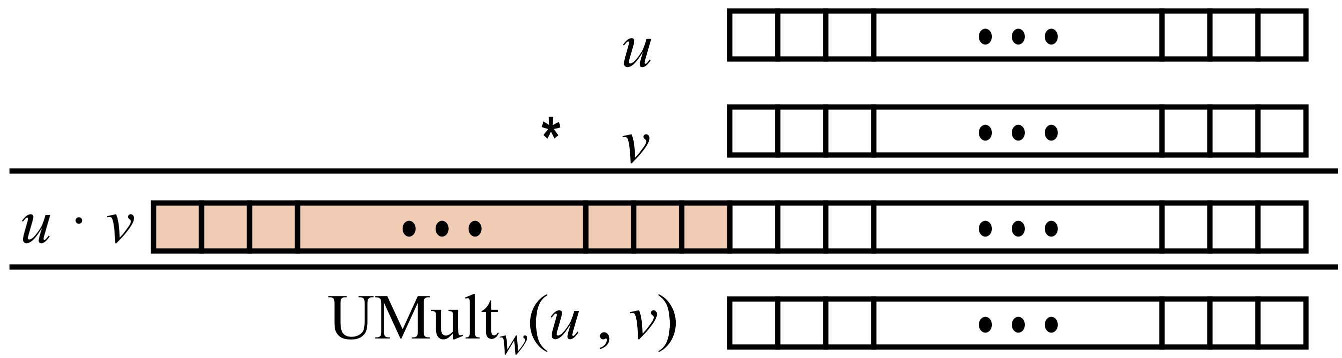
- Computing Exact Product of w-bit numbers x, y
  - Either signed or unsigned
- Ranges
  - Unsigned: $0 \leq x * y \leq (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1$
    - Up to $2w$ bits
  - Two's complement min: $x * y \geq (-2^{w-1})*(2^{w-1}-1) = -2^{2w-2} + 2^{w-1}$
    - Up to $2w-1$ bits
  - Two's complement max: $x * y \leq (-2^{w-1})^2 = 2^{2w-2}$
    - Up to $2w$ bits, but only for $(TMin_w)^2$
- Maintaining Exact Results
  - Would need to keep expanding word size with each product computed
  - Done in software by "arbitrary precision" arithmetic packages

Unsigned Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits



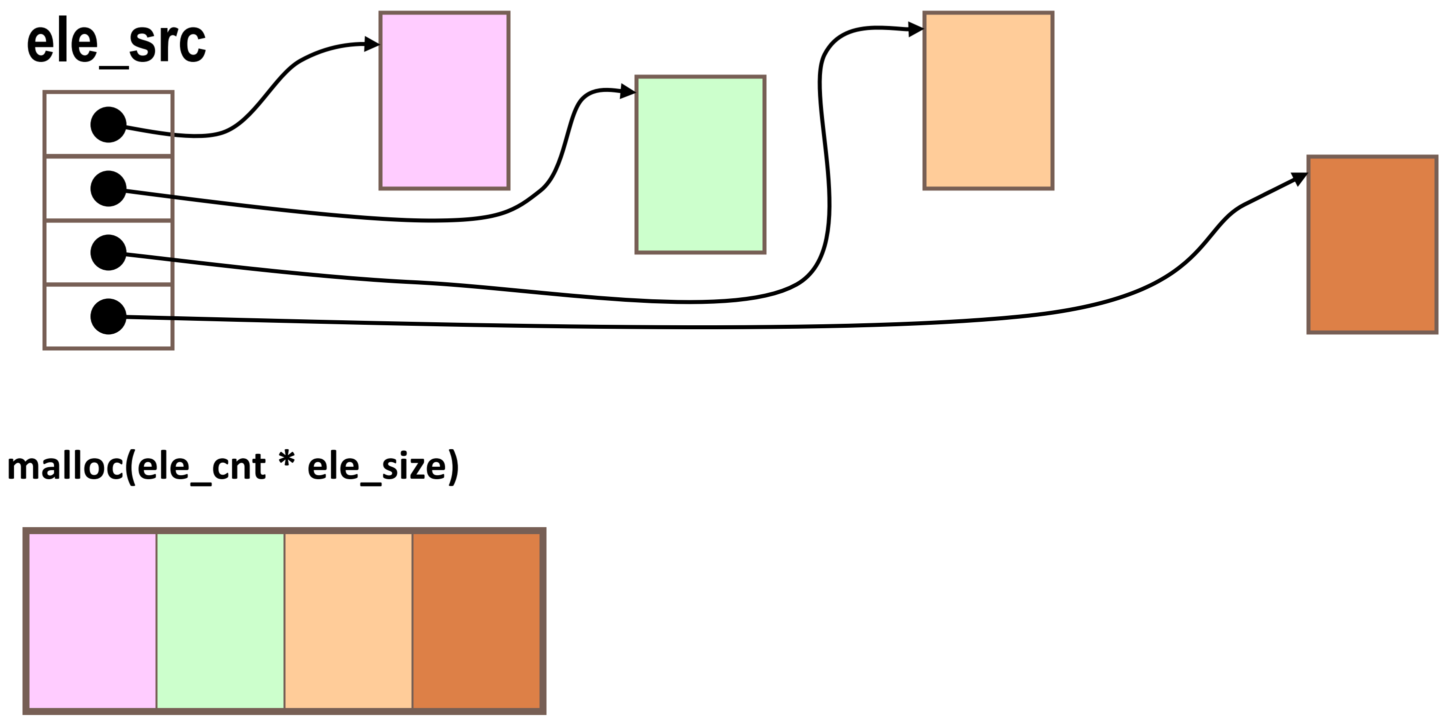
- Standard Multiplication Function
  - Ignores high order *w* bits
- Implements Modular Arithmetic

$$\text{UMult}_w(u, v) = u \cdot v \mod 2^w$$

Code Security Example #2

- SUN XDR library
  - Widely used library for transferring data between

```
void* copy_elements(void *ele_src[], int ele_cnt, size_t ele_size);
```



XDR Code

```
void* copy_elements(void *ele_src[], int ele_cnt, size_t ele_size) {
    /*
     * Allocate buffer for ele_cnt objects, each of ele_size bytes
     * and copy from locations designated by ele_src
     */
    void *result = malloc(ele_cnt * ele_size);
    if (result == NULL)
        /* malloc failed */
        return NULL;
    void *next = result;
    int i;
    for (i = 0; i < ele_cnt; i++) {
        /* Copy object i to destination */
        memcpy(next, ele_src[i], ele_size);
        /* Move pointer to next memory region */
        next += ele_size;
    }
    return result;
}
```

XDR Vulnerability

malloc(ele_cnt * ele_size)

- What if:
  - `ele_cnt` = $2^{20} + 1$
  - `ele_size` = 4096 = 2^{12}
  - Allocation = ??

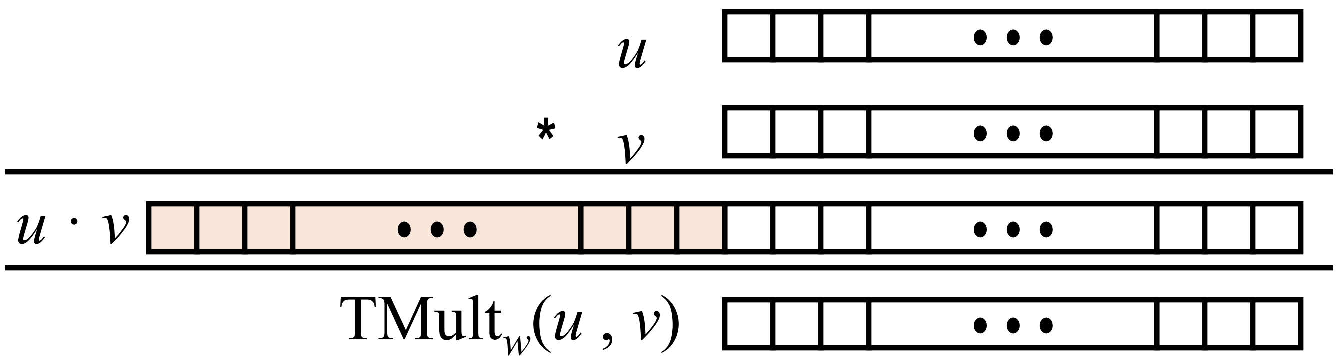
- How can I make this function secure?

Signed Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

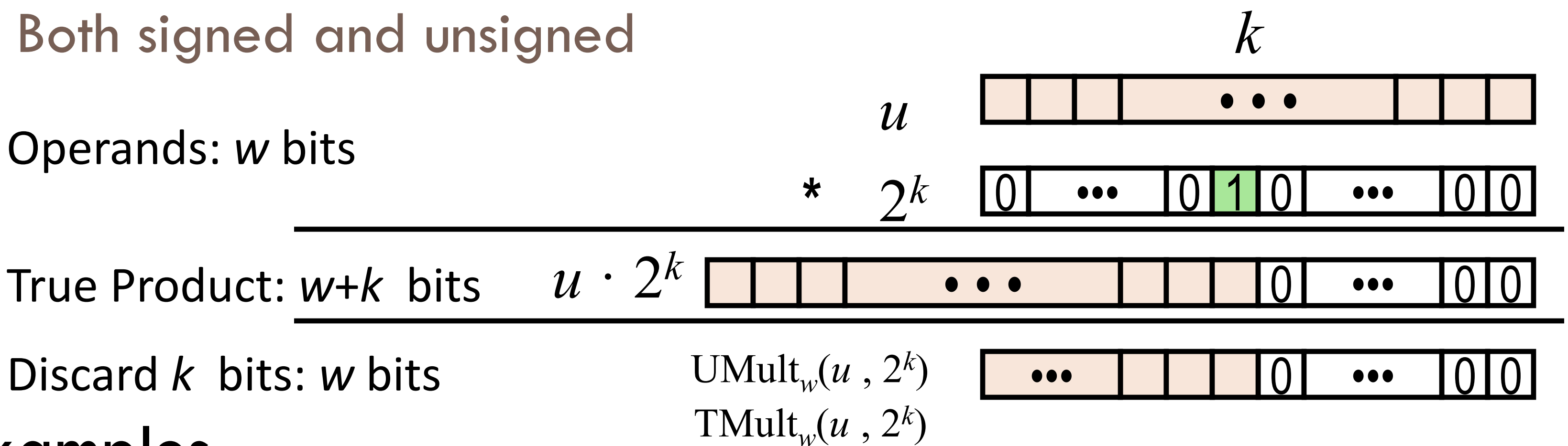
Discard *w* bits: *w* bits



- Standard Multiplication Function
  - Ignores high order *w* bits
  - Some of which are different for signed vs. unsigned multiplication
  - Lower bits are the same

Power-of-2 Multiply with Shift

- Operation
  - `u << k` gives `u * 2`^{k}
  - Both signed and unsigned



- Examples
  - `u << 3          ==  u * 8`
  - `u << 5 - u << 3  ==   u * 24`
  - Most machines shift and add faster than multiply
    - Compiler generates this code automatically

Compiled Multiplication Code

C Function

```
int mul12(int x)
{
  return x*12;
}
```

Compiled Arithmetic Operations

```
leal (%eax,%eax,2), %eax
sall $2, %eax
```

Explanation

```
t <- x+x*2
return t << 2;
```

- C compiler automatically generates shift/add code when multiplying by constant

Division

C Function

```
int mul12(int x)
{
  return x*12;
}
```

Compiled Arithmetic Operations

```
leal (%eax,%eax,2), %eax
sall $2, %eax
```

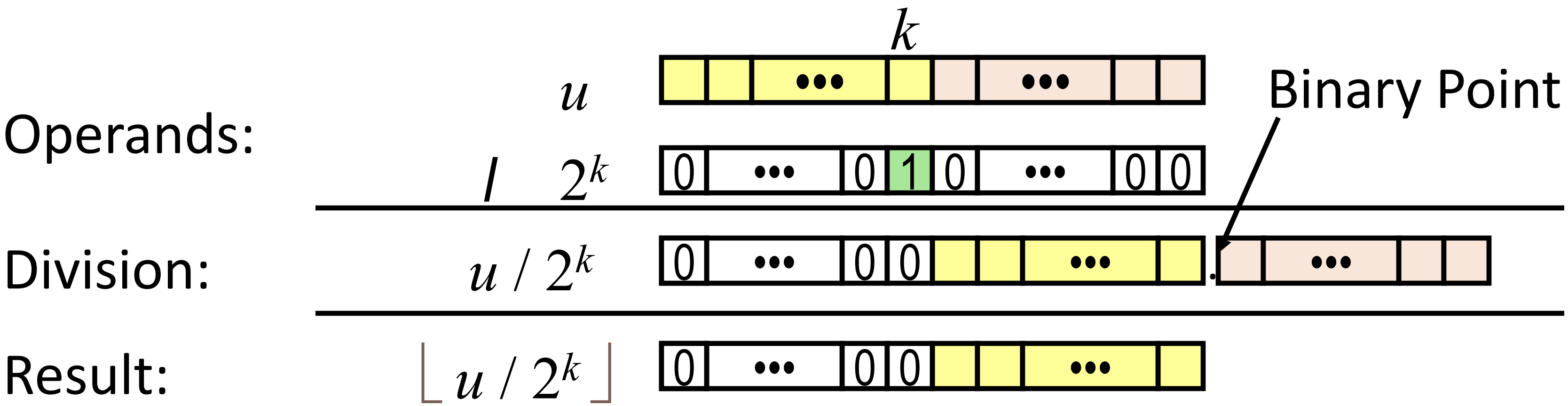
Explanation

```
t <- x+x*2
return t << 2;
```

- XXX implement divide with shift!

Unsigned Power-of-2 Divide with Shift

- Quotient of Unsigned by Power of 2
  - `u >> k` gives $\lfloor$ `u / 2`^{k} $\rfloor$
  - Uses logical shift



	Division	Computed	Hex	Binary
`x`	15213	15213	`3B 6D`	`00111011 01101101`
`x >> 1`	7606.5	7606	`1D B6`	`00011101 10110110`
`x >> 4`	950.8125	950	`03 B6`	`00000011 10110110`
`x >> 8`	59.4257813	59	`00 3B`	`00000000 00111011`

Compiled Unsigned Division Code

C Function

```
unsigned udiv8(unsigned x)
{
  return x/8;
}
```

Compiled Arithmetic Operations

```
  shrl $3, %eax
```

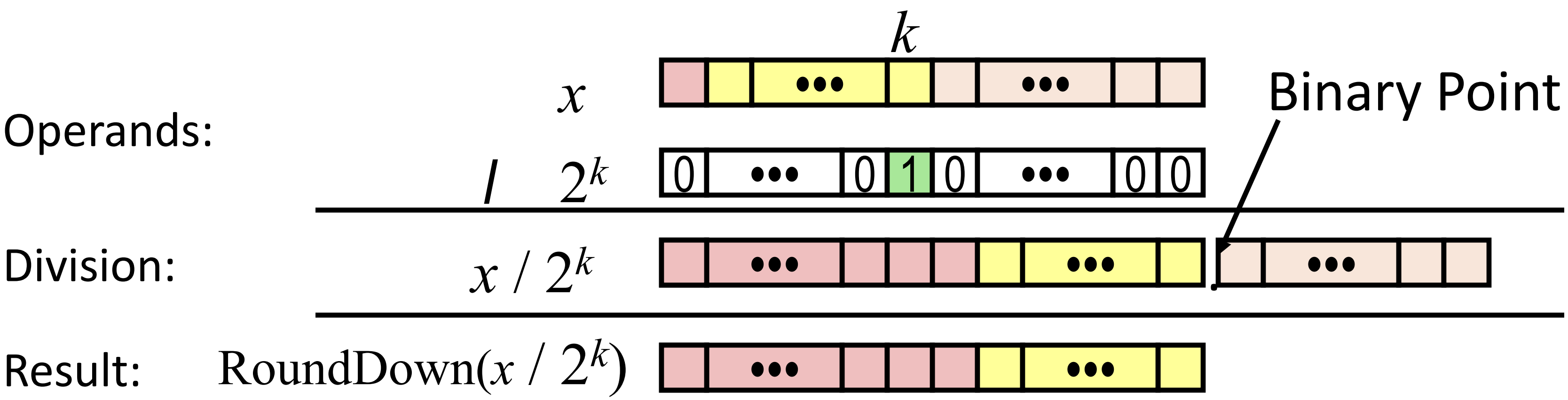
Explanation

```
  # Logical shift
  return x >> 3;
```

- Uses logical shift for unsigned
- For Java Users
  - Logical shift written as >>>

Signed Power-of-2 Divide with Shift

- Quotient of Signed by Power of 2
  - **x >> k** gives $\lfloor \mathbf{x} / 2^k \rfloor$
  - Uses arithmetic shift
  - Rounds wrong direction when **u < 0**



	Division	Computed	Hex	Binary
`y`	-15213	-15213	`C4 93`	`11000100 10010011`
`y >> 1`	-7606.5	-7607	`E2 49`	`11100010 01001001`
`y >> 4`	-950.8125	-951	`FC 49`	`11111100 01001001`
`y >> 8`	-59.4257813	-60	`FF C4`	`11111111 11000100`

Arithmetic: Basic Rules

- Addition:
  - Unsigned/signed: Normal addition followed by truncate, same operation on bit level
  - Unsigned: addition mod 2^w
    - Mathematical addition + possible subtraction of 2w
  - Signed: modified addition mod 2^w (result in proper range)
    - Mathematical addition + possible addition or subtraction of 2w

- Multiplication:
  - Unsigned/signed: Normal multiplication followed by truncate, same operation on bit level
  - Unsigned: multiplication mod 2^w
  - Signed: modified multiplication mod 2^w (result in proper range)

Arithmetic: Basic Rules

- Unsigned ints, 2's complement ints are isomorphic rings: isomorphism = casting

- Left shift
  - Unsigned/signed: multiplication by 2^k
  - Always logical shift

- Right shift
  - Unsigned: logical shift, div (division + round to zero) by 2^k
  - Signed: arithmetic shift
    - Positive numbers: div (division + round to zero) by 2^k
    - Negative numbers: div (division + round away from zero) by 2^k
      Use biasing to fix

Today: Integers

- Representing information as bits
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - **Summary**
- Making ints from bytes
- Summary

Properties of Unsigned Arithmetic

- Unsigned Multiplication with Addition Forms Commutative Ring
  - Addition is commutative group
  - Closed under multiplication

    $0 \leq \mathrm{UMult}_w(u\,,\,v) \leq 2^w - 1$
  - Multiplication Commutative

    $\mathrm{UMult}_w(u\,,\,v) = \mathrm{UMult}_w(v\,,\,u)$
  - Multiplication is Associative

    $\mathrm{UMult}_w(t, \mathrm{UMult}_w(u\,,\,v)) = \mathrm{UMult}_w(\mathrm{UMult}_w(t, u\,), v)$
  - 1 is multiplicative identity

    $\mathrm{UMult}_w(u\,,\,1) = u$
  - Multiplication distributes over addtion

    $\mathrm{UMult}_w(t, \mathrm{UAdd}_w(u\,,\,v)) = \mathrm{UAdd}_w(\mathrm{UMult}_w(t, u\,), \mathrm{UMult}_w(t, v))$

Properties of Two's Comp. Arithmetic

- Isomorphic Algebras
  - Unsigned multiplication and addition
    - Truncating to *w* bits
  - Two's complement multiplication and addition
    - Truncating to *w* bits
- Both Form Rings
  - Isomorphic to ring of integers mod 2^w
- Comparison to (Mathematical) Integer Arithmetic
  - Both are rings
  - Integers obey ordering properties, e.g.,

    $u > 0 \quad \Rightarrow \quad u + v > v$

    $u > 0, v > 0 \quad \Rightarrow \quad u \cdot v > 0$
  - These properties are not obeyed by two's comp. arithmetic

    TMax `+ 1` `==` *TMin*

    `15213 * 30426 == -10030` (16-bit words)

Why Should I Use Unsigned?

- *Don't* Use Just Because Number Nonnegative
  - Easy to make mistakes

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

  - Can be very subtle

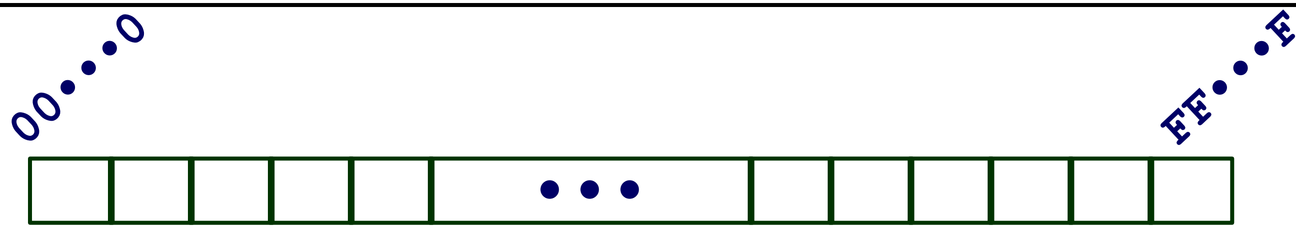
```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
  . . .
```

- *Do* Use When Performing Modular Arithmetic
  - Multiprecision arithmetic
- *Do* Use When Using Bits to Represent Sets
  - Logical right shift, no sign extension

Today: Integers

- Representing information as bits
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - Summary
- Making ints from bytes
- Summary

Byte-Oriented Memory Organization



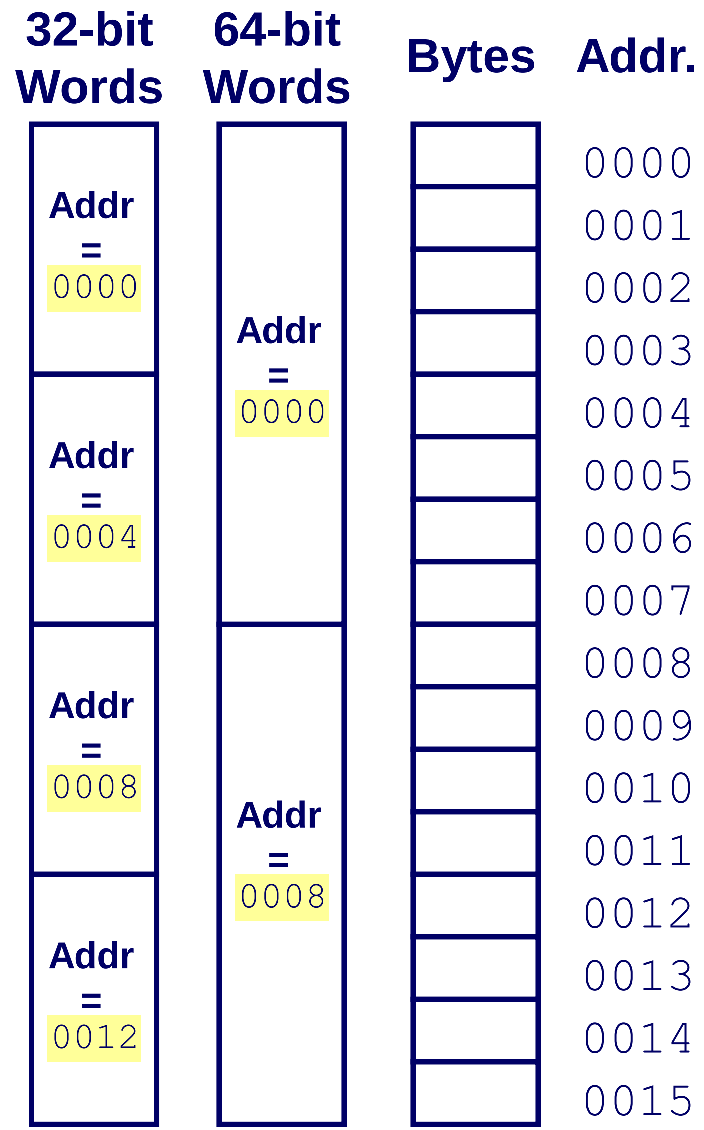
- Programs Refer to Virtual Addresses
  - Conceptually very large array of bytes
  - Actually implemented with hierarchy of different memory types
  - System provides address space private to particular "process"
    - Program being executed
    - Program can clobber its own data, but not that of others
- Compiler + Run-Time System Control Allocation
  - Where different program objects should be stored
  - All allocation within single virtual address space

Machine Words

- Machine Has “Word Size”
  - Nominal size of integer-valued data
    - Including addresses
  - Most current machines use 32 bits (4 bytes) words
    - Limits addresses to 4GB
    - Becoming too small for memory-intensive applications
  - High-end systems use 64 bits (8 bytes) words
    - Potential address space ≈ 1.8×10^{19} bytes
    - x86-64 machines support 48-bit addresses: 256 Terabytes
  - Machines support multiple data formats
    - Fractions or multiples of word size
    - Always integral number of bytes

Word-Oriented Memory Organization

- Addresses Specify Byte Locations
  - Address of first byte in word
  - Addresses of successive words differ by 4 (32-bit) or 8 (64-bit)



Byte Ordering

- How should bytes within a multi-byte word be ordered in memory?
- Conventions
  - Big Endian: Sun, PPC Mac, Internet
    - Least significant byte has highest address
  - Little Endian: x86
    - Least significant byte has lowest address

Byte Ordering Example

- Big Endian
  - Least significant byte has highest address
- Little Endian
  - Least significant byte has lowest address
- Example
  - Variable x has 4-byte representation 0x01234567
  - Address given by &x is 0x100

Big Endian

		0x100	0x101	0x102	0x103		
		01	23	45	67		

Little Endian

		0x100	0x101	0x102	0x103		
		67	45	23	01		

Reading Byte-Reversed Listings

- Disassembly
  - Text representation of binary machine code
  - Generated by program that reads the machine code
- Example Fragment

Address	Instruction Code	Assembly Rendition
`8048365:`	`5b`	`pop %ebx`
`8048366:`	`81 c3 ab 12 00 00`	`add $0x12ab,%ebx`
`804836c:`	`83 bb 28 00 00 00 00`	`cmpl $0x0,0x28(%ebx)`

- Deciphering Numbers
  - Value: 0x12ab
  - Pad to 32 bits: 0x000012ab
  - Split into bytes: 00 00 12 ab
  - Reverse: ab 12 00 00

Examining Data Representations

- Code to Print Byte Representation of Data
  - Casting pointer to unsigned char * creates byte array

```
typedef unsigned char *pointer;

void show_bytes(pointer start, int len){
  int i;
  for (i = 0; i < len; i++)
    printf(”%p\t0x%.2x\n",start+i, start[i]);
  printf("\n");
}
```

Printf directives:
%p**:** Print pointer
%x**:** Print Hexadecimal

`show_bytes` Execution Example

```
int a = 15213;
printf("int a = 15213;\n");
show_bytes((pointer) &a, sizeof(int));
```

Result (Linux):

```
int a = 15213;
0x11ffffcb8 0x6d
0x11ffffcb9 0x3b
0x11ffffcba 0x00
0x11ffffcbb 0x00
```

Data alignment

- A memory address a, is said to be n-byte aligned when a is a multiple of n bytes.
  - n is a power of two in all interesting cases
  - Every byte address is aligned
  - A 4-byte quantity is aligned at addresses 0, 4, 8,…
- Some architectures require alignment (e.g., MIPS)
- Some architectures tolerate misalignment at performance penalty (e.g., x86)

Data alignment in C structs

- Struct members are never reordered in C & C++
- Compiler adds padding so each member is aligned
  - struct {char a; char b;} no padding
  - struct {char a; short b;} one byte pad after a
- Last member is padded so the total size of the structure is a multiple of the largest alignment of any structure member (so struct can go in array)
  - struct containing int requires 4-byte alignment
  - struct containing long requires 8-byte (on 64-bit arch)

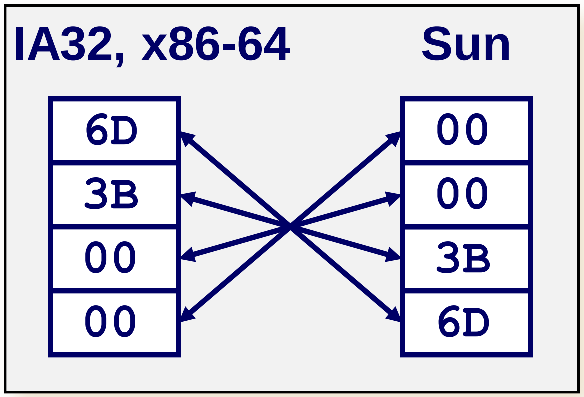
Data alignment malloc

- malloc(1)
  - 16-byte aligned results on 32-bit
  - 32-byte aligned results on 64-bit
- int posix_memalign(void **memptr, size_t alignment, size_t size);
  - Allocates size bytes
  - Places the address of the allocated memory in *memptr
  - Address will be a multiple of alignment, which must be a power of two and a multiple of sizeof(void *)

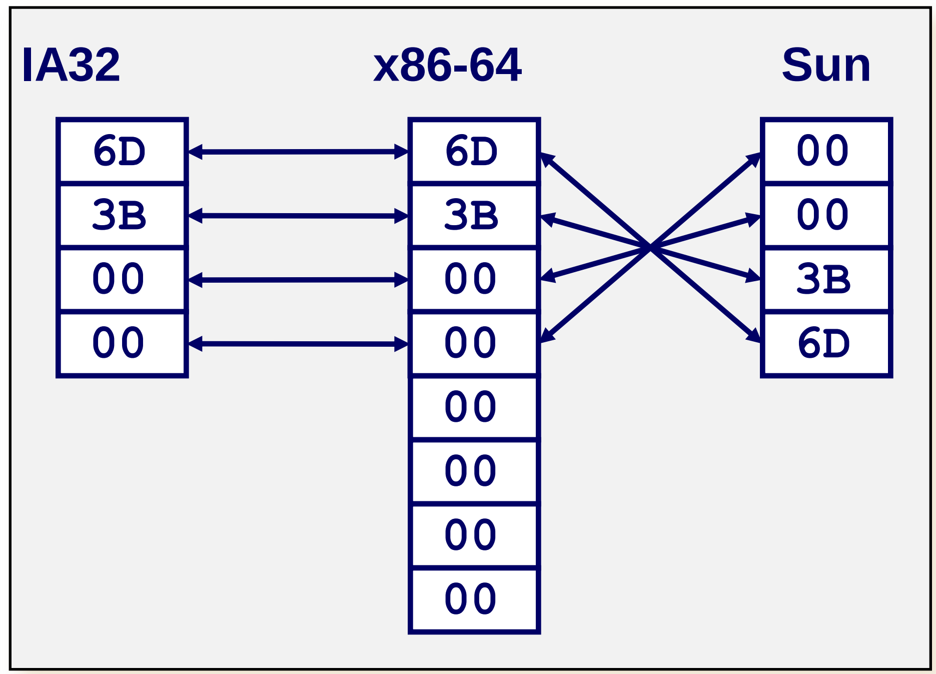
Representing Intege

Decimal:	15213
Binary:	0011 1011 0110 1101
Hex:	3 B 6 D

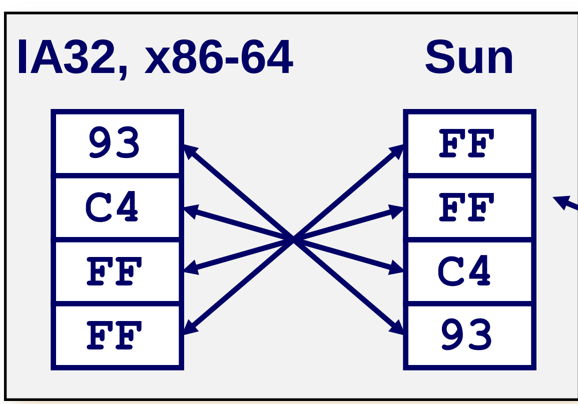
int A = 15213;



long int C = 15213;



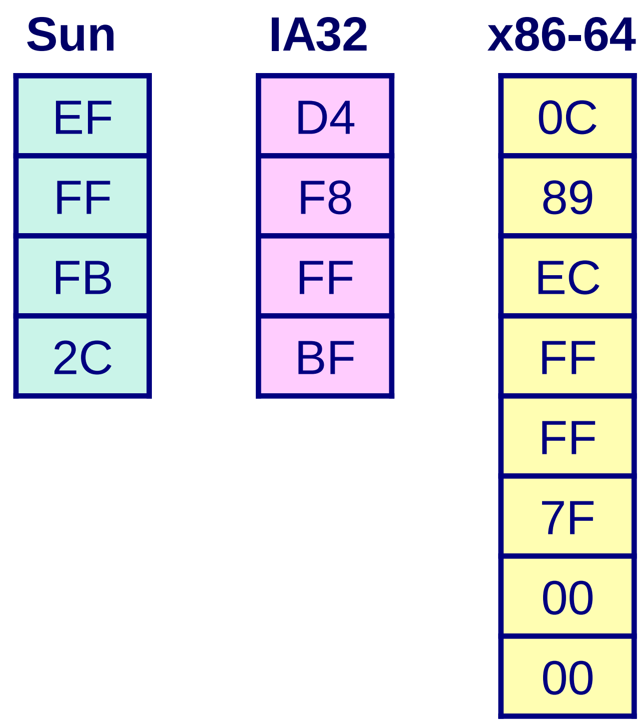
int B = -15213;



Two’s complement representation
(Covered later)

Representing Pointers

```
int B = -15213;
int *P = &B;
```

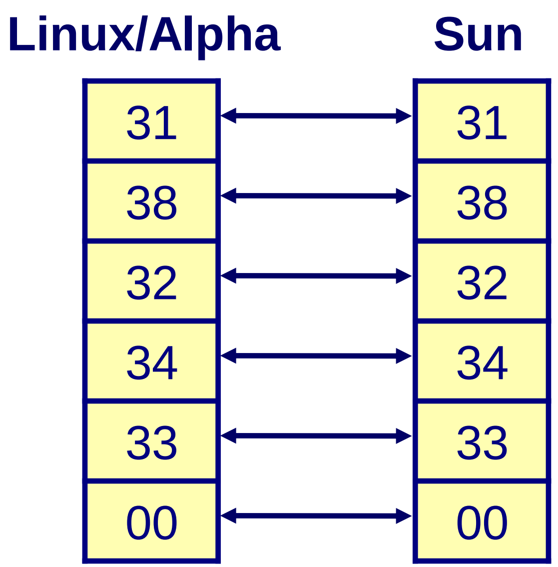


Different compilers & machines assign different locations to objects

Representing Strings

```
char S[6] = "18243";
```

- Strings in C
  - Represented by array of characters
  - Each character encoded in ASCII format
    - Standard 7-bit encoding of character set
    - Character “0” has code 0x30
      - Digit *i* has code 0x30+*i*
  - String should be null-terminated
    - Final character = 0
- Compatibility
  - Byte ordering not an issue



Integer C Puzzles

- x < 0 ⇒ ((x*2) < 0)
- ux >= 0
- x & 7 == 7 ⇒ (x<<30) < 0
- ux > -1
- x > y ⇒ -x < -y
- x * x >= 0
- x > 0 && y > 0 ⇒ x + y > 0
- x >= 0 ⇒ -x <= 0
- x <= 0 ⇒ -x >= 0
- (x|-x)>>31 == -1
- ux >> 3 == ux/8
- x >> 3 == x/8
- x & (x-1) != 0

Initialization

```
int x = foo();

int y = bar();

unsigned ux = x;

unsigned uy = y;
```