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DM559 – Linear and integer programming

Sheet 5, Spring 2018 [pdf format]

Exercise 1*

Which of the following sets are subspaces of $\mathbb{R}^3$?

$$S_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\ z = y = 3x \right\} \quad S_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\ z + y = 3x \right\}$$

$$S_3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\ zy = 3x \right\} \qquad S_4 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\ xyz = 0 \right\}$$

Provide proofs or counterexamples to justify your answers.

Solution:

Exercise 5.1 The set S_1 is a subspace. We have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S_1 \iff z = y = 3x \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 3x \\ 3x \end{pmatrix} = x \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, x \in \mathbb{R}.$$

So, the set S_1 is the linear span of the vector $\mathbf{v} = (1, 3, 3)^T$ and is therefore a subspace of $\mathbb{R}^3$. (This is the line through the origin in the direction of the vector $\mathbf{v} = (1, 3, 3)^T$.)

The set S_2 is a subspace. Since

$$S_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| z + y = 3x \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| z + y - 3x = 0 \right\},$$

it is a plane through the origin in $\mathbb{R}^3$, and you have shown that a plane through the origin is a subspace (see Activity 5.38). You can also show directly that the set is non-empty and closed under addition and scalar multiplication.

The set S_3 is not a subspace. $\mathbf{0} \in S_3$, but S_3 is not closed under addition. For example,

$$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \in S_3, \quad \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \in S_3, \quad \text{but} \quad \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \notin S_3$$

since it does not satisfy the condition $zy = 3x$.

The set S_4 is not a subspace because it is not closed under addition. For example,

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in S_4, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in S_4, \quad \text{but} \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \notin S_4.$$

What is S_4? For a vector $\mathbf{x}$ to be in S_4, either $x = 0$, $y = 0$ or $z = 0$. So this set consists of the xy-plane (if $z = 0$), the xz-plane, and the yz-plane. But any vector in $\mathbb{R}^3$ which is not on one of these planes is not in S_4.

Exercise 2*

Suppose A is an $n \times n$ matrix and $\lambda \in \mathbb{R}$ is a fixed constant. Show that the set

$$S = \{\mathbf{x} \mid A\mathbf{x} = \lambda\mathbf{x}\}$$

is a subspace of $\mathbb{R}^n$.

Solution:

Exercise 5.2 If A is an $n \times n$ matrix, then all vectors $\mathbf{x}$ for which $A\mathbf{x}$ is defined must be $n \times 1$ vectors, so the set

$$S = \{\mathbf{x} \mid A\mathbf{x} = \lambda\mathbf{x}\}, \qquad \text{some } \lambda \in \mathbb{R},$$

is a subset of $\mathbb{R}^n$. To show it is a subspace, you have to show it is non-empty and closed under addition and scalar multiplication.

Since $A\mathbf{0} = \lambda\mathbf{0} = \mathbf{0}$, the vector $\mathbf{0} \in S$, so S is non-empty. (In fact, depending on λ, S may well be the vector space which contains only the zero vector; more on this is found in Chapter 8.)

Let $\mathbf{u}, \mathbf{v} \in S$ and $a \in \mathbb{R}$. Then you know that $A\mathbf{u} = \lambda\mathbf{u}$ and $A\mathbf{v} = \lambda\mathbf{v}$. Therefore,

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \lambda\mathbf{u} + \lambda\mathbf{v} = \lambda(\mathbf{u} + \mathbf{v})$$

and

$$A(a\mathbf{u}) = a(A\mathbf{u}) = a(\lambda\mathbf{u}) = \lambda(a\mathbf{u})$$

so $\mathbf{u} + \mathbf{v} \in S$ and $a\mathbf{u} \in S$. Therefore, S is a subspace of $\mathbb{R}^n$.

Exercise 3*

Consider the following vectors

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

- Show that $\mathbf{u}$ can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, and write down the linear combination; but that $\mathbf{w}$ cannot be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
- What subspace of $\mathbb{R}^3$ is given by $\text{Lin}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}\}$? What subspace of $\mathbb{R}^3$ is given by $\text{Lin}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}\}$?
- Show that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}, \mathbf{w}\}$ spans $\mathbb{R}^3$. Show also that any vector $\mathbf{b} \in \mathbb{R}^3$ can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}, \mathbf{w}$ in infinitely many ways.

Solution:

Exercise 5.3 We are given the vectors

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \text{and} \quad \mathbf{u} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}.$$

(a) The vector $\mathbf{u}$ can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ if you can find constants s, t such that $\mathbf{u} = s\mathbf{v}_1 + t\mathbf{v}_2$. Now,

$$\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \iff \begin{cases} -1 = -s + t \\ 2 = 2t \\ 5 = s + 2t. \end{cases}$$

From the middle component equation, we find $t = 1$, and substituting this into the top equation yields $s = 2$. Substituting these values for s and t in the bottom component equation gives $5 = 2 + 3(1)$, which is correct, so $\mathbf{u} = 2\mathbf{v}_1 + \mathbf{v}_2$. You can check this using the vectors,

$$2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}.$$

Exercise 4*

Let

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Find a vector $\mathbf{x}_3$ such that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a linearly independent set of vectors.
Find a condition that a, b, c must satisfy for the set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}\}$ to be linearly dependent.

Solution:

Exercise 5.3 We are given the vectors

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \text{and} \quad \mathbf{u} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}.$$

(a) The vector $\mathbf{u}$ can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ if you can find constants s, t such that $\mathbf{u} = s\mathbf{v}_1 + t\mathbf{v}_2$. Now,

$$\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \iff \begin{cases} -1 = -s + t \\ 2 = 2t \\ 5 = s + 2t. \end{cases}$$

From the middle component equation, we find $t = 1$, and substituting this into the top equation yields $s = 2$. Substituting these values for s and t in the bottom component equation gives $5 = 2 + 3(1)$, which is correct, so $\mathbf{u} = 2\mathbf{v}_1 + \mathbf{v}_2$. You can check this using the vectors,

$$2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}.$$

Attempting this for the vector $\mathbf{w}$,

$$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \iff \begin{cases} 1 = -s + t \\ 2 = 2t \\ 5 = s + 2t \ . \end{cases}$$

This time the top two component equations yield $t = 1$ and $s = 0$, and these values do not satisfy the bottom equation, $5 = s + 3t$, so no solution exists. The vector $\mathbf{w}$ cannot be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Since $\mathbf{u} = 2\mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{u} \in \text{Lin}\{\mathbf{v}_1, \mathbf{v}_2\}$. Therefore, $\text{Lin}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}\}$ and $\text{Lin}\{\mathbf{v}_1, \mathbf{v}_2\}$ are the same subspace. Any vector $\mathbf{x} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{u}$ can be expressed as a linear combination of just $\mathbf{v}_1$ and $\mathbf{v}_2$ by just substituting $2\mathbf{v}_1 + \mathbf{v}_2$ for $\mathbf{u}$. Therefore, this is the linear span of two non-parallel vectors in $\mathbb{R}^3$, so it is a plane in $\mathbb{R}^3$.

Since $\mathbf{w} \notin \text{Lin}\{\mathbf{v}_1, \mathbf{v}_2\}$, the subspace $\text{Lin}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}\}$ must be bigger than just the plane, so it must be all of $\mathbb{R}^3$. To show that $\text{Lin}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}\} = \mathbb{R}^3$, you can establish that any $\mathbf{b} \in \mathbb{R}^3$ can be expressed as a linear combination, $\mathbf{b} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{w}$, or equivalently that the system of equations $A\mathbf{x} = \mathbf{b}$ has a solution where A is the matrix whose columns are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{w}$. You can show this by reducing A to row echelon form, or by finding the determinant. Since

$$|A| = \begin{vmatrix} -1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 3 & 5 \end{vmatrix} = -4 \neq 0,$$

you know from the main theorem (Theorem 4.5) that $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^3$.

(c) You know from part (b) that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}\}$ spans $\mathbb{R}^3$, and therefore so does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}, \mathbf{w}\}$. But more efficiently, you can take the same approach as in part (b) to show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}, \mathbf{w}\}$ spans $\mathbb{R}^3$, and at the same time show that any vector $\mathbf{b} \in \mathbb{R}^3$ can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}, \mathbf{w}$ in infinitely many ways. If B is the matrix with these four vectors as its columns, then the solutions, $\mathbf{x}$, of $B\mathbf{x} = \mathbf{b}$ will determine the possible linear combinations of the vectors. We put the coefficient matrix B into row echelon form (steps not shown),

$$B = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 2 & 2 \\ 1 & 3 & 5 & 5 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since there is a leading one in every row, the system $B\mathbf{x} = \mathbf{b}$ is always consistent, so every vector $\mathbf{b} \in \mathbb{R}^3$ can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}, \mathbf{w}$. Since there is a free variable (in column three), there are infinitely many solutions to $B\mathbf{x} = \mathbf{b}$.

Exercise 5*

Let B be an $m \times k$ matrix whose row space $RS(B)$, is a plane in $\mathbb{R}^3$ with Cartesian equation $4x-5y+3z = 0$.
From the given information, can you determine either m or k for the matrix B? If it is possible, do so.
Can you determine the null space of B? If so, write down a general solution of $B\mathbf{x} = \mathbf{0}$.

Solution:

Exercise 6.14 From the given information, you can determine that $k = 3$, since the rows of B (written as vectors) must be in $\mathbb{R}^3$. You cannot determine m, but you can say that $m \geq 2$ because you know that B has rank 2, since its row space is a plane.

Can you determine the null space of B? Yes, because the row space and the null space are orthogonal subspaces of $\mathbb{R}^3$, or simply because you know that the null space consists of all vectors for which $B\mathbf{x} = \mathbf{0}$, so all vectors such that $\langle \mathbf{r}_i, \mathbf{x} \rangle = 0$ for each row, $\mathbf{r}_i$ of B. Therefore, the null space must consist of all vectors on the line through the origin in the direction of the normal vector to the plane. So a basis of this space is given by $\mathbf{n} = (4, -5, 3)^{\mathrm{T}}$ and a general solution of $B\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = t \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Exercise 6*

Consider the sets

$$U = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}, \quad W = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \right\}$$

What subspace of $\mathbb{R}^3$ is $\mathrm{Lin}(U)$? And $\mathrm{Lin}(W)$? Find a basis for each subspace and show that one of them is a plane in $\mathbb{R}^3$. Find a Cartesian equation for the plane.

Solution:

Exercise 6.8 Observe that each set of vectors contains at least two linearly independent vectors since no vector in either set is a scalar multiple of another vector in the set. Write the vectors of each set as the columns of a matrix:

$$B = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & 3 & 5 \end{pmatrix}, \qquad A = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 3 & 5 \end{pmatrix}.$$

$|A| \neq 0$, so W is a basis of $\mathbb{R}^3$ and $\text{Lin}(W) = \mathbb{R}^3$. (Therefore, another basis of $\text{Lin}(W)$ is the standard basis, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.)

$|B| = 0$, so the set U is linearly dependent and one of the vectors is a linear combination of the other two. Since any two vectors of U are linearly independent, we know that we will need two vectors for a basis and $\text{Lin}(U)$ is a two-dimensional subspace of $\mathbb{R}^3$, which is a plane. So we can take the first two vectors in U to be a basis of $\text{Lin}(U)$.

There are two ways you can find the Cartesian equation of the plane. A vector equation is given by

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad s, t \in \mathbb{R},$$

and you can find the Cartesian equation by equating components to obtain three equations in the two unknowns s and t. Eliminating s and t between the three equations, you will obtain a single equation relating x, y, and z. Explicitly, we have $x = -s + t$, $y = 2t$, $z = s + 3t$, so

$$t = \frac{y}{2}, \quad s = t - x = \frac{y}{2} - x \quad \text{and so} \quad z = s + 3t = \left(\frac{y}{2} - x\right) + \frac{3}{2}y.$$

Therefore, $x - 2y + z = 0$ is a Cartesian equation of the plane.

Alternatively, you could write the two basis vectors and the vector $\mathbf{x}$ as the columns of a matrix M and, using the fact that $|M| = 0$ if and only if the columns of M are linearly dependent, you have the equation

$$\begin{vmatrix} -1 & 1 & x \\ 0 & 2 & y \\ 1 & 3 & z \end{vmatrix} = -2x + 4y - 2z = 0.$$

Exercise 7*

Write down a basis for the xz-plane in $\mathbb{R}^3$.

Solution:

The vectors in xz are all vectors in the form $[x, 0, z]^T$, so the set of vectors $\{e_1, e_3\}$ is a basis.

Exercise 8*

Let B be the set of vectors $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = [1, 1, 0]^T$, $\mathbf{v}_2 = [-4, 0, 3]^T$, $\mathbf{v}_3 = (3, 5, 1)^T$. Show that B is a basis of $\mathbb{R}^3$.

Let $\mathbf{w} = [-1, 7, 5]^T$ and $e_1 = [1, 0, 0]^T$. Find the coordinates of $\mathbf{w}$ and $\mathbf{e}_1$ with respect to the basis B.

Exercise 9*

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 0 & 1 \\ -4 & -5 & -2 & 4 \end{bmatrix}$$

Find a basis of the range of A, $R(A) = CS(A)$. State why $R(A)$ is a plane in $\mathbb{R}^3$ and find a Cartesian equation of this plane.
For what real values of a is the vector

$$\mathbf{b} = \begin{bmatrix} -1 \\ a \\ a^2 \end{bmatrix}$$

in the range of A? Write down any vectors in $R(A)$ of this form.
Determine the dimension of the null space, $N(A)$.

Exercise 10*

Determine for what values of the constant λ, the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ \lambda \end{bmatrix}$$

form a basis of $\mathbb{R}^3$.
Let $\mathbf{b} = [2, 0, 1]^T$ and $\mathbf{s} = [2, 0, 3]^T$. Deduce that each of the sets

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{b}\} \text{ and } S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{s}\}$$

is a basis of $\mathbb{R}^3$. Find the transition matrix P from S coordinates to B coordinates.
If

$$[\mathbf{w}]_S = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}_S$$

find $[\mathbf{w}]_B$.