DM559 – Linear and Integer Programming

Answers to Obligatory Assignment 0.2, Spring 2018

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Exercise 1

a)

To determine whether the column vectors of a matrix form a basis of R^4 the numpy function: numby_linalg_matrix_rank is used. As shown in the python code two matrices U and V is created with random values ranging from, respectively, -10 to 10 for U and 0 to 10 for V, both matrices has the a size of 4x4. As shown in the result matrix U and V both has the rank of 4. So the coloumn vectors of the matrix form a basis of R^4

Python code:

```
import numpy as np
U = np.random.randint(-10,10,size=(4,4))
V = np.random.randint(0,10,size=(4,4))
b = np.ones(4)

print("Rank of Matrix U: %d" %np.linalg.matrix_rank(U))
print("Rank of Matrix V: %d" %np.linalg.matrix_rank(V))
```

Result:

```
Rank of Matrix U: 4
Rank of Matrix V: 4
```

b)

Python code:

```
import numpy as np
U = np.random.randint(-10,10,size=(4,4))
V = np.random.randint(0,10,size=(4,4))
C = np.random.randint(0,10,size=(4,1))
b = np.ones(4)

u1, u2, u3, u4, c = ([] for i in range(5))
n = 0
while n < len(U):
  u1.append(U[n,0])
  u2.append(U[n,1])
  u3.append(U[n,2])
  u4.append(U[n,3])
  c.append(C[n,0])
  n+=1

Un_dot_Cn = np.dot(u1,C[0,0]) + \
            np.dot(u2,C[1,0]) + \
            np.dot(u3,C[2,0]) + \
            np.dot(u4,C[3,0])

U_dot_c = np.dot(U,c)

print("Vector b corrosponds to original?: %s" \
  %np.allclose(Un_dot_Cn, U_dot_c, rtol=1e-05, atol=1e-08))
```

Result

```
Vector b corrosponds to original?: True
```

c)

The previous task provided a random 4x4 matrix U and V, and 4x1 matrix C which will be denoted matrix d in this task. Below matrix V and d is illustrated.

$$V = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{bmatrix} 7 & 4 & 9 & 8 \\ 2 & 8 & 0 & 0 \\ 1 & 2 & 9 & 8 \\ 0 & 9 & 4 & 6 \end{bmatrix} \end{array}, \qquad \mathrm{d} = \begin{bmatrix} 3 \\ 6 \\ 7 \\ 0 \end{bmatrix}$$

To compute the transition matrix from the standard basis to the basis $F = v_1, v_2, v_3, v_4$ and use this matrix to find the coordinate vector b with respect to F is it required to verify:

$$\mathbf{b} = d_1v_1 + d_2v_2 + d_3v_3 + d_4v_4 = Vd$$

To verify this the $d_1v_1 + d_2v_2 + d_3v_3 + d_4v_4$ is computed first.

$$3\begin{bmatrix} 7 \\ 2 \\ 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 4 \\ 8 \\ 2 \\ 9 \end{bmatrix} + 7\begin{bmatrix} 9 \\ 0 \\ 9 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 8 \\ 0 \\ 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 108 \\ 54 \\ 78 \\ 82 \end{bmatrix}$$

Then the Vd-part is computed:

$$Vd = \begin{bmatrix} v_{11} \cdot d_1 & + & v_{21} \cdot d_2 & + & v_{31} \cdot d_3 & + & v_{41} \cdot d_4 \\ v_{12} \cdot d_1 & + & v_{22} \cdot d_2 & + & v_{32} \cdot d_3 & + & v_{42} \cdot d_4 \\ v_{13} \cdot d_1 & + & v_{23} \cdot d_2 & + & v_{33} \cdot d_3 & + & v_{43} \cdot d_4 \\ v_{14} \cdot d_1 & + & v_{24} \cdot d_2 & + & v_{34} \cdot d_3 & + & v_{44} \cdot d_4 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix}$$

$$Vd = \begin{bmatrix} 7 \cdot 3 & + & 4 \cdot 6 & + & 9 \cdot 7 & + & 8 \cdot 0 \\ 2 \cdot 3 & + & 8 \cdot 6 & + & 0 \cdot 7 & + & 0 \cdot 0 \\ 1 \cdot 3 & + & 2 \cdot 6 & + & 9 \cdot 7 & + & 8 \cdot 0 \\ 0 \cdot 3 & + & 9 \cdot 6 & + & 4 \cdot 7 & + & 6 \cdot 0 \end{bmatrix} = \begin{bmatrix} 108 \\ 54 \\ 78 \\ 82 \end{bmatrix}$$

As shown $d_1 v_1 + d_2 v_2 + d_3 v_3 + d_4 v_4 = \begin{bmatrix} 108 \\ 54 \\ 78 \\ 82 \end{bmatrix}$

And $Vd = \begin{bmatrix} 108 \\ 54 \\ 78 \\ 82 \end{bmatrix}$

d)
n/a

Exercise 2

a)

Given are the matrices:

$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Find eigenvalues and eigenvectors for A:

$\det\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - \gamma \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - \gamma \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix}\right) = \det \begin{bmatrix} 3-\gamma & -1 \\ 1 & 1-\gamma \end{bmatrix}$
$= (3-\gamma)(1-\gamma) - (-1)1$
$= 3 - 4\gamma + \gamma^2 + 1$
$= \underline{\underline{\gamma^2 - 4\gamma + 4}}$

$\gamma^2 - 4\gamma = (\gamma^2 - 2\gamma) \cdot (-2\gamma + 4) = 0$
hence 2 and 2 is the only eigenvalue of matrix A

We solve for:

$A - 2I = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

Reduce to row echolon form

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 - 1 \cdot \mathrm{R}_1 \text{ -> } R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$A - 2I) \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 - y_1 = 0$

$x = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix}$

$let\ \mathrm{x}_2 = 1$

Eigenvector for A: $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Find eigenvalues and eigenvectors for matrix B with python:

Python code

```
import numpy as np

B = ([[ 1, 1, 1 ],
      [ 0, 2, 1 ],
      [ 0, 0, 1 ]])

print(np.linalg.eig(B))
```

Result:

```
(array([ 1.,  2.,  1.]), array([[ 1.        ,  0.70710678,  0.        ],
       [ 0.        ,  0.70710678, -0.70710678],
       [ 0.        ,  0.        ,  0.70710678]]))
```

b)

Matrix A has the eigenvalue 2 and 2, and does not have eigenvectors that are equal to its matrix dimensions. Matric A has One eigenvector that are less than 2, and therefor the matrix cannot be diagonalized.

Matrix B has the eigenvalues 1, 1 and 2, and the eigenvectors: $1 : \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, 1 : \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, 2 : \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 + 1 \cdot \mathrm{R}_3 \text{ -> } R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1 \cdot \mathrm{R}_2 \text{ -> } R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PDP^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

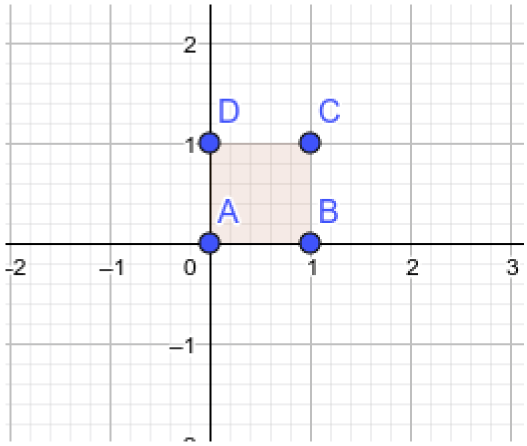
As shown matrix B can be diagonalized.

Exercise 3

a)

$R = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

The coloumns of R represents the coordinats of points in the plane, and forms a square figure.



b)

i)

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

A $= \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

ii)

Reflection of matrix A about x-axis:

reflection matrix of x-axis is $= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{bmatrix}$

$iii)$

Rotate around origin counter clockwise (90 degress)

A $= \begin{bmatrix} Cos(90) & Sin(90) \\ -Sin(90) & Cos(90) \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

$iv)$

$A = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} =$