

# DM559 – Linear and Integer Programming

## Answers to Obligatory Assignment 0.1, Spring 2018

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### Exercise 1

In order for the product between two matrices is valid, the number of columns in matrix  $A$  has to be equal to the number of rows in matrix  $B$ .

1:  $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix} = \text{NOT VALID.}$

2:  $\begin{bmatrix} 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 3 & 1 & 2 \end{bmatrix} = \text{NOT VALID}$

3:  $\begin{bmatrix} 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 3 & 1 & 2 \end{bmatrix}^T \equiv \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 1 & 2 \end{bmatrix} = \text{VALID. Resulting matrix: ( 1x2 )}$

4:  $\begin{bmatrix} 1 & 4 & 1 \\ 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix}^T \equiv \begin{bmatrix} 1 & 4 & 1 \\ 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = \text{VALID. Resulting matrix: ( 2x1 )}$

5:  $\begin{bmatrix} 1 & 4 & 1 \\ 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} = \text{NOT VALID}$

6:  $\begin{bmatrix} 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 6 & 2 \end{bmatrix}^T \equiv \begin{bmatrix} 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} = \text{VALID. Resulting matrix: ( 1x1 )}$

7:  $\begin{bmatrix} 2 & 1 & 5 \end{bmatrix}^T \begin{bmatrix} 1 & 6 & 2 \end{bmatrix} \equiv \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 6 & 2 \end{bmatrix} = \text{VALID. Resulting matrix: ( 3x3 )}$

## Exercise 2

$$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + a \times 0 & 1 \times b + a \times 1 \\ 0 \times 1 + 1 \times 0 & 0 \times b + 1 \times 1 \end{bmatrix} = \begin{bmatrix} 1 & b + a \\ 0 & 1 \end{bmatrix}$$

## Exercise 3

We use gauss-jordan elimination in order to try solving the three task a, b, c:

a)

$$\begin{aligned} \left[ \begin{array}{cccc|c} 2 & 0 & 1 & 3 & 1 \\ 0 & 0 & 5 & 3 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right] &\xrightarrow{R1 * \frac{1}{2} \rightarrow R1} \left[ \begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 5 & 3 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right] &\xrightarrow{R2 * \frac{1}{5} \rightarrow R2} \left[ \begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 1 & 3 \end{array} \right] &\xrightarrow{R2 * \frac{-1}{2} \rightarrow R1} \\ \left[ \begin{array}{cccc|c} 1 & 0 & 0 & \frac{6}{5} & \frac{3}{5} \\ 0 & 0 & 1 & \frac{3}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 1 & 3 \end{array} \right] &\xrightarrow{R3 * \frac{-6}{5} \rightarrow R1} \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & \frac{3}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 1 & 3 \end{array} \right] &\xrightarrow{R3 * \frac{-3}{5} \rightarrow R2} \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right] = \begin{bmatrix} x_1 = -3 \\ x_3 = -2 \\ x_4 = 3 \end{bmatrix} \end{aligned}$$

Reduced row echelon form is obtained and has a free variable  $x_2$

b)

$$\left[ \begin{array}{ccccc|c} 1 & 3 & -2 & 1 & 0 & 5 \\ 0 & 0 & 2 & -3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{array} \right] = \text{No solution}$$

Has no solution because no matter which number we multiply in row three the result will be zero, and can never have 2 as a solution.

c)

$$\begin{aligned} \left[ \begin{array}{ccc|c} 10^{-20} & 0 & 1 & 1 \\ 1 & 10^{20} & 1 & 2 \\ 0 & 1 & -1 & 3 \end{array} \right] &\xrightarrow{\text{Swap R2 and R3}} \left[ \begin{array}{ccc|c} 1 & 10^{20} & 1 & 2 \\ 10^{-20} & 0 & 1 & 1 \\ 0 & 1 & -1 & 3 \end{array} \right] &\xrightarrow{R2 - 10^{-20} * R1 \rightarrow R2} \left[ \begin{array}{ccc|c} 1 & 10^{20} & 1 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 3 \end{array} \right] \\ \xrightarrow{R2 / -1 \rightarrow R2} \left[ \begin{array}{ccc|c} 1 & 10^{20} & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 3 \end{array} \right] &\xrightarrow{R3 - 1 * R2 \rightarrow R3} \left[ \begin{array}{ccc|c} 1 & 10^{20} & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -10^{-20} & 4 \end{array} \right] &\xrightarrow{R3 / -10^{-20} \rightarrow R3} \left[ \begin{array}{ccc|c} 1 & 10^{20} & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -4 * 10^{20} \end{array} \right] \\ \xrightarrow{R2 + 1 * R3 \rightarrow R2} \left[ \begin{array}{ccc|c} 1 & 10^{20} & 1 & 2 \\ 0 & 1 & 0 & -4 * 10^{20} \\ 0 & 0 & 1 & -4 * 10^{20} \end{array} \right] &\xrightarrow{R1 - 1 * R3 \rightarrow R1} \left[ \begin{array}{ccc|c} 1 & 10^{20} & 0 & 4 * 10^{20} \\ 0 & 1 & 0 & -4 * 10^{20} \\ 0 & 0 & 1 & -4 * 10^{20} \end{array} \right] \\ \xrightarrow{R1 - 10^{20} * R2 \rightarrow R1} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 4 * 10^{40} \\ 0 & 1 & 0 & -4 * 10^{20} \\ 0 & 0 & 1 & -4 * 10^{20} \end{array} \right] = \begin{bmatrix} x_1 = 4 * 10^{40} \\ x_2 = -4 * 10^{20} \\ x_3 = -4 * 10^{20} \end{bmatrix} \end{aligned}$$

## Exercise 4

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$$

By use of laplace expansion (cofactor expansion) we calculate the determinant by expanding along the first row:

$$\begin{aligned} \det(A) &= 1 \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} \\ &= (1 * (1 * (-1) * 2) - 3 * (2 * (-1) - 1 * (-2)) + 1 * (2 * 2 - 1 * (-2))) \underline{\underline{= 3}} \end{aligned}$$

The adjunct of a matrix is the transpose of its cofactors. So first step is to find the cofactor of each point in matrix A:

$$C = \begin{bmatrix} + A_{11} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3 & - A_{12} = \begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} = 0 & + A_{13} = \begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} = 6 \\ - A_{21} = \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = 5 & + A_{22} = \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} = 1 & - A_{23} = \begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix} = -8 \\ + A_{31} = \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2 & - A_{32} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 & + A_{33} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5 \end{bmatrix}$$

Next step is to create a matrix of the cofactors of matrix A and transpose it based on the cofactor-rule:

$$\text{adj}(A) = \begin{bmatrix} -3 & 0 & 6 \\ 5 & 1 & -8 \\ 2 & 1 & -5 \end{bmatrix}^T = \begin{bmatrix} -3 & 5 & 2 \\ 0 & 1 & 1 \\ 6 & -8 & -5 \end{bmatrix}$$

From  $\det(A) = 3$  we know that matrix A has an inverse. By augmenting an identity matrix to matrix A and use of gauss-jordan elimination we calculate the inverse of matrix A:

$$\begin{aligned} A^{-1} &= \left[ \begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ -2 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_2 - 2 * R_1 \rightarrow R_2} \left[ \begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -5 & -1 & -2 & 1 & 0 \\ -2 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_3 - (-2) * R_1 \rightarrow R_3} \left[ \begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -5 & -1 & -2 & 1 & 0 \\ 0 & 8 & 1 & 2 & 0 & 1 \end{array} \right] \\ &\xrightarrow{R_2 / (-5) \rightarrow R_2} \left[ \begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 8 & 1 & 2 & 0 & 1 \end{array} \right] \xrightarrow{R_3 - 8 * R_2 \rightarrow R_3} \left[ \begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & -\frac{3}{5} & -\frac{6}{5} & \frac{8}{5} & 1 \end{array} \right] \xrightarrow{R_3 / (-\frac{3}{5}) \rightarrow R_3} \left[ \begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & 2 & -\frac{8}{3} & -\frac{5}{3} \end{array} \right] \\ &\xrightarrow{R_2 - (\frac{1}{5}) * R_3 \rightarrow R_2} \left[ \begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 2 & -\frac{8}{3} & -\frac{5}{3} \end{array} \right] \xrightarrow{R_1 - 1 * R_3 \rightarrow R_1} \left[ \begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & \frac{8}{3} & \frac{5}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 2 & -\frac{8}{3} & -\frac{5}{3} \end{array} \right] \xrightarrow{R_1 - 3 * R_2 \rightarrow R_1} \\ &\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & \frac{5}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 2 & -\frac{8}{3} & -\frac{5}{3} \end{array} \right] \end{aligned}$$

Matrix A has the inverse:

$$A^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & -1 \\ -2 & 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & \frac{5}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 2 & -\frac{8}{3} & -\frac{5}{3} \end{bmatrix}$$

## Exercise 4 continued

To verify that the determinant of matrix  $A$  and matrix  $A^{-1}$  is correct, the matrix has been coded in python:

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```
import numpy as np
A=([[ 1., 3., 1.],
    [ 2., 1., 1.],
    [-2., 2., -1.]])

print(np.linalg.det(A))
print(np.linalg.inv(A))
```

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As shown below, the python program outputs the determinant of matrix  $A$  and the inverse matrix of matrix  $A$  which is consistent with our own results.

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```
3.0
[[-1.          1.66666667  0.66666667]
 [ 0.          0.33333333  0.33333333]
 [ 2.         -2.66666667 -1.66666667]]
```

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## Exercise 5

We have a line going through the points  $A = (-\frac{3}{2}, 2)$  and  $B = (3, 0)$ . The cartesian equation of a line has the form:  $y = ax + b$

To calculate  $a$  we simply:

$$a = \left( \frac{y_2 - y_1}{x_2 - x_1} \right) = \left( \frac{0 - 2}{3 - (-\frac{3}{2})} \right) = -\frac{4}{9}$$

Now our line is represented as:

$$y = -\frac{4}{9}x + b$$

We insert  $x_1$  and  $y_1$ , and isolate  $b$ :

$$2 = -\frac{4}{9} * -\frac{3}{2} + b$$

$$2 = b + \frac{6}{5}$$

$$b = \frac{4}{5}$$

The vector form of the equation is given as:  $x = p + tv$  where  $x$  is the position vector,  $p$  is any particular point and  $v$  is the direction of the line<sup>1</sup>.

$$\begin{aligned} \overrightarrow{AB} &= \begin{bmatrix} -\frac{3}{2} \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} \\ 2 \end{bmatrix} \\ x &= \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ 2 \end{bmatrix} + t \begin{bmatrix} -\frac{9}{2} \\ 2 \end{bmatrix} \end{aligned}$$

<sup>1</sup>According to slide on Geomtric Insight. Page 9: URL: <http://www.imada.sdu.dk/marco/DM559/Slides/dm559-lec5.pdf>