DM559 – Linear and integer programming

Sheet 1, Spring 2018 [pdf format]

Solution:
Included.

Exercise 1* Modelling

Flying with the wind, an airplane traveled 1365 miles in 3 hours. The plane then turned against the wind and traveled another 870 miles in 2 hours. Find the speed of the airplane and the speed of the wind assuming they remained constant in the 5 hour time lapse. Recall that the distance traveled is equal to the average speed times the time traveled at that rate, $D = r \cdot t$.

Solution:
There is no obvious relationship between the speed of the plane and the speed of the wind. For this reason, use two variables as follows:
Let x be the speed of the airplane
Let w be the speed of the wind.
With the wind, the airplane's total speed is $x + w$. Flying against the wind, the total speed is $x - w$.

	Distance =	***Rate*** ×	***Time***
Flight with wind	1,365 mi	$x + w$	3 hrs
Flight against wind	870 mi	$x - w$	2 hrs
Total	2,235 mi		5 hrs

$$1{,}365 = (x + w) \cdot 3$$
$$870 = (x - w) \cdot 2$$

Thus we have a system of two linear equations in two variables.
We solve by Gaussian elimination or by Cramer's rule.
Answer: The speed of the airplane is 445 miles per hour and the speed of the wind is 10 miles per hour.

Exercise 2*

Use Gaussian elimination to solve the following system of equations

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 6 \\ 2x_1 & + & 4x_2 & + & x_3 & = & 5 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 6 \end{array}$$

Be sure to follow the algorithm to put the augmented matrix into row echelon form using row operations. Try then also the variant in which you put the augmented matrix in reduced row echelon form.

Solution:

Activity 2.13 Put the augmented matrix into reduced row echelon form. It should take five steps:

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & 5 \\ 2 & 3 & 1 & 6 \end{pmatrix} \longrightarrow (1) \longrightarrow (2) \longrightarrow (3) \longrightarrow (4)$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{pmatrix},$$

from which you can read the solution, $\mathbf{x} = (2, -1, 5)^{\mathrm{T}}$. We will state the row operations at each stage. To obtain (1), do $R_2 - 2R_1$ and $R_3 - 2R_1$; for (2) switch R_2 and R_3; for (3) do $R_3 - 2R_2$. The augmented matrix is now in row echelon form, so starting with the bottom row, for (4), do $R_2 + R_3$ and $R_1 - R_3$. The final operation, $R_1 - R_2$, will yield the matrix in reduced row echelon form.

Exercise 3

Do the True False exercises on page 8, 20 and 30 of the book [AR].

Solution:

On page 8, this is the sequence: F,F,F,T,F,F,T,F.
On page 20, this is the sequence: T,T,F,T,T,F,T,F,F.
On page 30, this is the sequence: T,F,F,T,T,F,F,T,T,T,T,T,T,T

a) T: the diagonal is defined only for square matrices

b) F: See definition of row and column vectors, partitioning and submatrices on page 25 of [AR]. We agreed to use the notation: $\mathbf{b}_{\cdot j}$ or $\mathbf{b}_j$ for column vectors and $\mathbf{a}_{i\cdot}$ for row vectors. At the blackboard the bold font is replaced by an arrow $\vec{a}$. Hence an $m \times n$ matrix has m row vectors and n column vectors.

c) F. Not necessarily. Write explicitly the element $(AB)_{ij}$ and element $(BA)_{ij}$ and argue that an example can be easily constructed in which each term is different.

d) F. Show instead how the matrix AB can be written as the product of A and column vectotrs of B or row vectors of A and B.

e) T

f) F

g) F

h) T

i) T

j) T

k) T

l) F

m) T

n) T

o) F

Exercise 4*

Do exercises 1, 2, 3, 15, 16, 17, 57, 58, 59, 71, 223, 230 from the Exercise set 1 on pages 75-87 of [AR]. Below is a set of exercises taken from [Le], in case someone does not have the book [AR] yet. You do not need to look at this if you have the book.

(a) The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ has no main diagonal.

(b) An $m \times n$ matrix has m column vectors and n row vectors.

(c) If A and B are 2×2 matrices, then $AB = BA$.

(d) The ith row vector of a matrix product AB can be computed by multiplying A by the ith row vector of B.

(e) For every matrix A, it is true that $(A^T)^T = A$.

(f) If A and B are square matrices of the same order, then
$$\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B)$$

(g) If A and B are square matrices of the same order, then
$$(AB)^T = A^T B^T$$

(h) For every square matrix A, it is true that $\mathrm{tr}(A^T) = \mathrm{tr}(A)$.

(i) If A is a 6×4 matrix and B is an $m \times n$ matrix such that $B^T A^T$ is a 2×6 matrix, then $m = 4$ and $n = 2$.

(j) If A is an $n \times n$ matrix and c is a scalar, then $\mathrm{tr}(cA) = c\,\mathrm{tr}(A)$.

(k) If A, B, and C are matrices of the same size such that $A - C = B - C$, then $A = B$.

(l) If A, B, and C are square matrices of the same order such that $AC = BC$, then $A = B$.

(m) If $AB + BA$ is defined, then A and B are square matrices of the same size.

(n) If B has a column of zeros, then so does AB if this product is defined.

(o) If B has a column of zeros, then so does BA if this product is defined.

1. In each part, determine whether the equation is linear in x_1, x_2, and x_3.

(a) $x_1 + 5x_2 - \sqrt{2}x_3 = 1$ (b) $x_1 + 3x_2 + x_1x_3 = 2$

(c) $x_1 = -7x_2 + 3x_3$ (d) $x_1^{-2} + x_2 + 8x_3 = 5$

(e) $x_1^{3/5} - 2x_2 + x_3 = 4$ (f) $\pi x_1 - \sqrt{2}x_2 = 7^{1/3}$

2. In each part, determine whether the equation is linear in x and y.

(a) $2^{1/3}x + \sqrt{3}y = 1$ (b) $2x^{1/3} + 3\sqrt{y} = 1$

(c) $\cos\left(\frac{\pi}{7}\right)x - 4y = \log 3$ (d) $\frac{\pi}{7}\cos x - 4y = 0$

(e) $xy = 1$ (f) $y + 7 = x$

3. Using the notation of Formula (7), write down a general linear system of

(a) two equations in two unknowns.

(b) three equations in three unknowns.

(c) two equations in four unknowns.

4. Write down the augmented matrix for each of the linear systems in Exercise 3.

▶ In each part of Exercises 5–6, find a linear system in the unknowns $x_1, x_2, x_3, \dots,$ that corresponds to the given augmented matrix. ◀

5. (a) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & -2 & 5 \\ 7 & 1 & 4 & -3 \\ 0 & -2 & 1 & 7 \end{bmatrix}$

6. (a) $\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 0 & 1 & -4 & 3 \\ -4 & 0 & 4 & 1 & -3 \\ -1 & 3 & 0 & -2 & -9 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$

▶ In each part of Exercises 7–8, find the augmented matrix for the linear system. ◀

7. (a) $\begin{aligned} -2x_1 &= 6 \\ 3x_1 &= 8 \\ 9x_1 &= -3 \end{aligned}$ (b) $\begin{aligned} 6x_1 - x_2 + 3x_3 &= 4 \\ 5x_2 - x_3 &= 1 \end{aligned}$

(c) $$\begin{aligned} 2x_2 \qquad - 3x_4 + x_5 &= 0 \\ -3x_1 - x_2 + x_3 \qquad\qquad &= -1 \\ 6x_1 + 2x_2 - x_3 + 2x_4 - 3x_5 &= 6 \end{aligned}$$

8. (a) $\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \\ 7x_1 + 3x_2 &= 2 \end{aligned}$ (b) $\begin{aligned} 2x_1 \qquad + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{aligned}$

(c) $$\begin{aligned} x_1 \qquad\qquad &= 1 \\ x_2 \qquad &= 2 \\ x_3 &= 3 \end{aligned}$$

9. In each part, determine whether the given 3-tuple is a solution of the linear system

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= 1 \\ x_1 - 3x_2 + x_3 &= 1 \\ 3x_1 - 5x_2 - 3x_3 &= 1 \end{aligned}$$

(a) $(3, 1, 1)$ (b) $(3, -1, 1)$ (c) $(13, 5, 2)$

(d) $\left(\frac{13}{2}, \frac{5}{2}, 2\right)$ (e) $(17, 7, 5)$

In Exercises 17–18, find a single elementary row operation that will create a 1 in the upper left corner of the given augmented matrix and will not create any fractions in its first row.

17. (a) $\begin{bmatrix} -3 & -1 & 2 & 4 \\ 2 & -3 & 3 & 2 \\ 0 & 2 & -3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 & -5 & 0 \\ 2 & -9 & 3 & 2 \\ 1 & 4 & -3 & 3 \end{bmatrix}$

18. (a) $\begin{bmatrix} 2 & 4 & -6 & 8 \\ 7 & 1 & 4 & 3 \\ -5 & 4 & 2 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & -4 & -2 & 2 \\ 3 & -1 & 8 & 1 \\ -6 & 3 & -1 & 4 \end{bmatrix}$

21. The curve $y = ax^2 + bx + c$ shown in the accompanying figure passes through the points (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that the coefficients a, b, and c form a solution of the system of linear equations whose augmented matrix is

$$\begin{bmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{bmatrix}$$

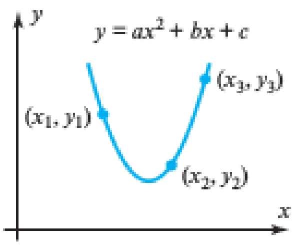

Figure Ex-21

Exercise 5*

34. The accompanying table shows a record of May and June unit sales for a clothing store. Let M denote the 4×3 matrix of May sales and J the 4×3 matrix of June sales.

(a) What does the matrix $M + J$ represent?

(b) What does the matrix $M - J$ represent?

(c) Find a column vector $\mathbf{x}$ for which $M\mathbf{x}$ provides a list of the number of shirts, jeans, suits, and raincoats sold in May.

(d) Find a row vector $\mathbf{y}$ for which $\mathbf{y}M$ provides a list of the number of small, medium, and large items sold in May.

(e) Using the matrices $\mathbf{x}$ and $\mathbf{y}$ that you found in parts (c) and (d), what does $\mathbf{y}M\mathbf{x}$ represent?

Table Ex-34

May Sales

	Small	Medium	Large
Shirts	45	60	75
Jeans	30	30	40
Suits	12	65	45
Raincoats	15	40	35

June Sales

	Small	Medium	Large
Shirts	30	33	40
Jeans	21	23	25
Suits	9	12	11
Raincoats	8	10	9