DM559 – Linear and integer programming

Sheet 3, Spring 2018 [pdf format]

Exercise 1*

What is the change in the value of $|A|$ in the following circumstances:

- if R_2 is replaced by $R_2 - 3R_1$?
- if R_2 is replaced by $3R_1 - R_2$?

Solution:

In the first there is no change. In the second the determinant will change sign. This is because $3R_1 - R_2$ is actually two elementary row operations: first, we multiply row 2 by -1 and then we add three times row 1 to it. Hence, when performing row operation RO3, to leave the determinant unchanged, add a multiple of another row while leaving the row you are replacing unchanged.

Exercise 2*

Calculate the determinant of the following 4×4 matrix by using elementary row operations to speed up the calculation. [Hint: bring the first column in the form with a leading one and zeros below it. You are thus left with a 3×3 determinant to evaluate.]

$$|A| = \begin{vmatrix} 1 & 2 & -1 & 4 \\ -1 & 3 & 0 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 4 & 1 & 3 \end{vmatrix}$$

Solution:

$$|A| = \begin{vmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & -1 & 6 \\ 0 & -3 & 3 & -6 \\ 0 & 2 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 5 & -1 & 6 \\ -3 & 3 & -6 \\ 2 & 2 & -1 \end{vmatrix}.$$

At this stage you can expand the 3×3 matrix using a cofactor expansion, or continue a bit more with row operations:

$$|A| = 3\begin{vmatrix} 1 & -1 & 2 \\ 5 & -1 & 6 \\ 2 & 2 & -1 \end{vmatrix} = 3\begin{vmatrix} 1 & -1 & 2 \\ 0 & 4 & -4 \\ 0 & 4 & -5 \end{vmatrix} = 3\begin{vmatrix} 4 & -4 \\ 4 & -5 \end{vmatrix} = 3(-4) = -12.$$

In the last row we first multiplied the first row by $-1/3$, which implies multiplying $|A|$ by $-1/3$, and then permuted the first and the second row, which implies multiplying $|A|$ by -1. Hence, to undo the changes on $|A|$ and restore its correct value we multiply by 3. Then, linear combinations of rows are applied that do not affect the determinant calculation and, finally, the determinant is given as the cofactor expansion by the first column.

Exercise 3

Write the cofactor expansion by row one to determine the determinant of this matrix:

$$M = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$$

Use then an alternative way to calculate the determinant by row operations.
Finally, express M as a product of elementary matrices and evaluate $|M|$ as a product of elementary matrices.

Solution:

Cofactor expansion on row one:

$$|M| = -1(8-3) - 2(0-3) + 1(0-2) = -1$$

It is more convenient to expand by column 1 or row 2. Using column 1:

$$|M| = -1(8-3) + 1(6-2) = -1$$

Rest to do.

Exercise 4*

Find the inverse of the following matrices using: row operations and adjoint matrix. Which method resulted the fastest in terms of flops?

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Solution:

$$(A|I) = \left(\begin{array}{ccc|ccc} -2 & 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ -2 & 1 & 3 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_3+2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 5 & 3 & 1 & 0 & 2 \end{array}\right) \xrightarrow{(-1)R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 5 & 3 & 1 & 0 & 2 \end{array}\right)$$

$$\xrightarrow{R_3-5R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 8 & 1 & 5 & 2 \end{array}\right) \xrightarrow{\frac{1}{8}R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{8} & \frac{5}{8} & \frac{1}{4} \end{array}\right)$$

$$\xrightarrow{R_2+R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{8} & -\frac{3}{8} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{5}{8} & \frac{1}{4} \end{array}\right) \xrightarrow{R_1-2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{8} & \frac{6}{8} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{8} & -\frac{3}{8} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{5}{8} & \frac{1}{4} \end{array}\right).$$

So,

$$A^{-1} = \frac{1}{8}\begin{pmatrix} -2 & 6 & 4 \\ 1 & -3 & 2 \\ 1 & 5 & 2 \end{pmatrix}.$$

Now *check* that $AA^{-1} = I$.

When you carry out the row reduction, it is not necessary to always indicate the separation of the two matrices by a line as we have done so far. You just need to keep track of what you are doing.

In the calculation for the inverse of B, we have omitted the line but added a bit of space to make it easier for you to read.

$$(B|I) = \begin{pmatrix} 2 & 1 & 3 & \ 1 & 0 & 0 \\ 0 & -1 & 1 & \ 0 & 1 & 0 \\ 1 & 2 & 0 & \ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 0 & \ 0 & 0 & 1 \\ 0 & -1 & 1 & \ 0 & 1 & 0 \\ 2 & 1 & 3 & \ 1 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3-2R_1} \begin{pmatrix} 1 & 2 & 0 & \ 0 & 0 & 1 \\ 0 & -1 & 1 & \ 0 & 1 & 0 \\ 0 & -3 & 3 & \ 1 & 0 & -2 \end{pmatrix} \xrightarrow{(-1)R_2} \begin{pmatrix} 1 & 2 & 0 & \ 0 & 0 & 1 \\ 0 & 1 & -1 & \ 0 & -1 & 0 \\ 0 & -3 & 3 & \ 1 & 0 & -2 \end{pmatrix}$$

$$\xrightarrow{R_3+3R_2} \begin{pmatrix} 1 & 2 & 0 & \ 0 & 0 & 1 \\ 0 & 1 & -1 & \ 0 & -1 & 0 \\ 0 & 0 & 0 & \ 1 & -3 & -2 \end{pmatrix},$$

which indicates that the matrix B is not invertible; it is not row equivalent to the identity matrix.

The calculation by cofactor expansion by row requires $O(n!)$ calculations. The calculation by Gaussian Elimination requires instead $O(n^3)$, although care must be taken to avoid the cases where an exponential number of bits are needed to represent numbers. Gaussian elmination implies that we need to set to zero the $(n \cdot n - n)/2$ terms of the lower triangular part of the matrix and for each zero we may have to carry out at most two elementary row operations (each one takes $O(n)$).

Exercise 5*

Find the solution of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

using Cramer's rule.

Solution:

Exercise 3.9 The system of equations is

$$\begin{aligned} -x + 2y + z &= 1 \\ y + 2z &= 1 \\ 3x + y + 4z &= 5\,. \end{aligned}$$

This has as its coefficient matrix the matrix B of Exercise 3.1. We know from the previous exercise that $|B| = 7$. Since the determinant is non-zero, we can use Cramer's rule. Then

$$x = \frac{1}{|B|}\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 5 & 1 & 4 \end{vmatrix} = \frac{1(2) - 2(-6) + 1(-4)}{|B|} = \frac{10}{7},$$

$$y = \frac{1}{|B|}\begin{vmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 3 & 5 & 4 \end{vmatrix} = \frac{0 + 1(-7) - 2(-8)}{|B|} = \frac{9}{7},$$

$$z = \frac{1}{|B|}\begin{vmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 3 & 1 & 5 \end{vmatrix} = \frac{0 + 1(-8) - 1(-7)}{|B|} = -\frac{1}{7}.$$

which, of course, agrees with the result in Exercise 3.1.

Exercise 6*

Look at the reduced row echelon form of the matrix A of a system of linear equalities $A\mathbf{x} = \mathbf{b}$:

$$RREF(A) = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Explain why you can tell from this matrix that for all $\mathbf{b} \in \mathbb{R}^3$, the linear system $A\mathbf{x} = \mathbf{b}$ is consistent with infinitely many solutions.

Solution:

Activity 2.26 This is the reduced row echelon form of the coefficient matrix, A. The reduced row echelon form of any augmented matrix, $(A|\mathbf{b})$, will have as its first four columns the same four columns. As there is a leading one in every row, it is impossible to have a row of the form $(0\ 0\ \ldots\ 0\ 1)$, so the system will be consistent. There will be one free (non-leading) variable, (fourth column, say $x_4 = t$), so there will be infinitely many solutions.

Exercise 7*

Consider the following system of equations $A\mathbf{x} = \mathbf{b}$:

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 2x_2 & & & = & 2 \\ 3x_1 & + & 4x_2 & + & x_3 & = & 2 \end{array}$$

1. Solve the system of equations $A\mathbf{x} = \mathbf{b}$;
2. Solve the associated homogeneous system $A\mathbf{x} = \mathbf{0}$;
3. Describe the geometric configuration of intersecting planes for the system $A\mathbf{x} = \mathbf{b}$.
4. Describe the geometric configuration of intersecting planes for the system $A\mathbf{x} = \mathbf{0}$.

Solution:

Activity 2.27 Using row operations to reduce the augmented matrix to echelon form, we obtain

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & 2 \\ 3 & 4 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

There is no reason to reduce the matrix further, for we can now conclude that the original system of equations is inconsistent: there is no solution. For the homogeneous system, $A\mathbf{x} = \mathbf{0}$, the row echelon form of A consists of the first three columns of the echelon form of the augmented matrix. So starting from these and continuing to reduced row echelon form, we obtain

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Setting the non-leading variable x_3 to $x_3 = t$, we find that the null space of A consists of all vectors, $\mathbf{x}$, of the following form:

$$\mathbf{x} = t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

The system of equations $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

Geometrically, the associated homogeneous system represents the equations of three planes, all of which pass through the origin. These planes intersect in a line through the origin. The equation of this line is given by the solution we found.

The original system represents three planes with no common points of intersection. No two of the planes in either system are parallel. Why? Look at the normals to the planes: no two of these are parallel, so no two planes are parallel. These planes intersect to form a kind of triangular prism; any two planes intersect in a line, and the three lines of intersection are parallel, but there are no points which lie on all three planes. (If you have trouble visualising this, take three cards, place one flat on the table, and then get the other two to balance on top, forming a triangle when viewed from the side.)

Exercise 8*

Consider the system of linear equations $A\mathbf{x} = \mathbf{b}$, where λ and μ are constants and

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 5 & 1 & \lambda \\ 1 & -1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 7 \\ \mu \end{bmatrix}$$

Compute the determinant of A, $|A|$.
Determine for which values of λ and μ this system has

(a) a unique solution

(b) no solution

(c) infinitely many solutions

In case (a), use Cramer's rule to find the value of z in terms of λ and μ. In case (c), solve the system using row operations and express the solution in vector form, $\mathbf{x} = \mathbf{p} + t\mathbf{v}$.

Solution:

Exercise 4.3 $|A| = 3\lambda - 9$.
(a) If $|A| \neq 0$, that is if $\lambda \neq 3$, then the system will have a unique solution. In this case, using Cramer's rule, $z = (3 - 3\mu)/(\lambda - 3)$.

To answer (b) and (c), reduce the augmented matrix to echelon form with $\lambda = 3$

$$(A|\mathbf{b}) = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 5 & 1 & 3 & 7 \\ 1 & -1 & 1 & \mu \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & -9 & 3 & -3 \\ 0 & -3 & 1 & \mu - 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 3 & -1 & 1 \\ 0 & -3 & 1 & \mu - 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & \mu - 1 \end{pmatrix}.$$

So if $\lambda = 3$, this system will be inconsistent if $\mu \neq 1$, which answers part (b).

If $\lambda = 3$ and $\mu = 1$, we have (c) infinitely many solutions. Setting $\mu = 1$ and continuing to reduced echelon form,

$$(A|\mathbf{b}) \longrightarrow \cdots \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \frac{2}{3} & \frac{4}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The solution can now be read from the matrix. Setting the non-leading variable z equal to t,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{4}{3} - \frac{2}{3}t \\ \frac{1}{3} + \frac{1}{3}t \\ t \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} = \mathbf{p} + t\mathbf{v}, \qquad t \in \mathbb{R}.$$

Exercise 9

Consider the matrices:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 2 & -1 & 8 \\ 3 & 1 & 7 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 3 & -2 & 5 \\ 3 & -6 & 9 & -6 \\ -2 & 9 & -1 & 9 \\ 5 & -6 & 9 & -4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ a \\ b \end{bmatrix}$$

For both systems $A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} = \mathbf{b}$:

- write in vector form the composition of the null space $N(A)$.
- write the general solution of the system.
- Find all real numbers a and b such that $\mathbf{b} \in R(A)$, where $\mathbf{b}$ is the vector given above and $R(A)$ is the range of A, that is, the vector space made of all linear combinations of the column vectors of A.
- Write a non-trivial linear combination of the column vectors of A which is equal to the zero vector $\mathbf{0}$, or justify why this is not possible.

Solution:

Exercise 4.6 You need to put the matrix into row echelon form to answer the first question, and into reduced row echelon form for the second,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 2 & -1 & 8 \\ 3 & 1 & 7 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rank of A is 2. There is one non-leading variable. If you write $\mathbf{x} = (x, y, z)^{\mathrm{T}}$, then setting $z = t$, you will obtain the solution

$$\mathbf{x} = t \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Since there are non-trivial solutions of $A\mathbf{x} = \mathbf{0}$, it is possible to express $\mathbf{0}$ as a linear combination of the columns of A with non-zero coefficients. A non-trivial linear combination of the column vectors which is equal to the zero vector is given by any non-zero vector in the null space. For example, using $t = 1$, the product $A\mathbf{x}$ yields

$$-3\mathbf{c}_1 + 2\mathbf{c}_2 + \mathbf{c}_3 = -3 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 8 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}.$$

The vector $\mathbf{b}$ is in $R(A)$ if $\mathbf{b}$ is a linear combination of the column vectors of A, which is exactly when $A\mathbf{x}=\mathbf{b}$ is consistent. Notice that the matrix A has rank 2, so the augmented matrix must also have rank 2. Reducing $(A|\mathbf{b})$ using row operations,

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -2 & 1 \\ 2 & -1 & 8 & a \\ 3 & 1 & 7 & b \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & -3 & 6 & a-8 \\ 0 & -2 & 4 & b-12 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & a-5 \\ 0 & 0 & 0 & b-10 \end{pmatrix}.$$

Therefore, $A\mathbf{x}=\mathbf{b}$ is consistent if and only if $a=5$ and $b=10$. In that case, continuing to reduced echelon form,

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, a general solution is

$$\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \qquad t \in \mathbb{R}.$$

(b) Using row operations,

$$|B| = \begin{vmatrix} -2 & 3 & -2 & 5 \\ 3 & -6 & 9 & -6 \\ -2 & 9 & -1 & 9 \\ 5 & -6 & 9 & -4 \end{vmatrix} = (-3) \begin{vmatrix} 1 & -2 & 3 & -2 \\ -2 & 3 & -2 & 5 \\ -2 & 9 & -1 & 9 \\ 5 & -6 & 9 & -4 \end{vmatrix}$$

$$= (-3) \begin{vmatrix} 1 & -2 & 3 & -2 \\ 0 & -1 & 4 & 1 \\ 0 & 5 & 5 & 5 \\ 0 & 4 & -6 & 6 \end{vmatrix}$$

$$= (30) \begin{vmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 4 & 1 \\ 0 & 2 & -3 & 3 \end{vmatrix} = (30) \begin{vmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & -5 & 1 \end{vmatrix} = 450.$$

Since $\det(B) \neq 0$, the rank of B is 4. Therefore, the main theorem (Theorem 4.1.2) tells us that $B\mathbf{x}=\mathbf{0}$ has only the trivial solution. Therefore, there is no way to write $\mathbf{0}$ as a linear combination of the column vectors of B except the trivial way, with all coefficients equal to 0.

Also, using this theorem, $B\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^4$. Therefore, $R(B) = \mathbb{R}^4$. That is, a and b can be any real numbers, the system $B\mathbf{x} = \mathbf{b}$ is always consistent.

Exercise 10*

Plot the line $y = 2x + 1$ and the position vector $\mathbf{q}$ of the point $(3, 7)$ which is on this line. Express $\mathbf{q}$ as the sum of two vectors $\mathbf{q} = \mathbf{p} + t\mathbf{v}$ for some $t \in \mathbb{R}$ in both cases in which $\mathbf{p} = (0, 1)^T$ and $\mathbf{p} = (1, 3)^T$.

Solution:

We have $\mathbf{v} = (1, 2)$ (why?) hence for $\mathbf{p} = (0, 1)^T$ we have to set $t = 3$ and for $\mathbf{p} = (1, 3)^T$ $t = 2$. Note that the vector $\mathbf{q}$ does not lie on the line.

Exercise 11*

Write a vector equation of the line through the points $P = (-1, 1)$ and $Q = (3, 2)$. What is the direction of this line? What is the Cartesian equation of it?

Exercise 12*

The lines ℓ_1 and ℓ_2 are given by the following equations (for $t \in \mathbb{R}$)

$$\ell_1 : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\ell_2 : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Find the intersection point and the angle between them.

Solution:

Example 1.53 The lines ℓ_1 and ℓ_2 are given by the following equations (for $t \in \mathbb{R}$)

$$\ell_1 : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$
$$\ell_2 : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

These lines are not parallel, since their direction vectors are not scalar multiples of one another. Therefore, they intersect in a unique point. We can find this point either by finding the Cartesian equation of each line and solving the equations simultaneously, or using the vector equations. We will do the latter. We are looking for a point (x, y) on both lines, so its position vector will satisfy

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

for some $t \in \mathbb{R}$ and for some $s \in \mathbb{R}$. We need to use different symbols (s and t) in the equations because they are unlikely to be the same number for each line. We are looking for values of s and t which will give us the same point. Equating components of the position vectors of points on the lines, we have

$$\left.\begin{aligned} 1 + t &= 5 - 2s \\ 3 + 2t &= 6 + s \end{aligned}\right\} \Rightarrow \begin{aligned} 2s + t &= 4 \\ -s + 2t &= 3 \end{aligned} \Rightarrow \begin{aligned} 2s + t &= 4 \\ -2s + 4t &= 6 \end{aligned}.$$

Adding these last two equations, we obtain $t = 2$, and therefore $s = 1$. Therefore, the point of intersection is $(3, 7)$:

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

What is the angle of intersection of these two lines? Since

$$\left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\rangle = 0,$$

the lines are perpendicular.

Exercise 13 Cross product

The *vector product* or *cross product* of two vectors is defined in $\mathbb{R}^3$ as follows (see also [AR sec. 3.5 or

Le, sec. 2.3]). If

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

then $\mathbf{a} \times \mathbf{b}$ is the vector given by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{e}_1 - (a_1b_3 - a_3b_1)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3$$

The vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

a) Calculate $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ for the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}$. Check that $\mathbf{w}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

b) Show that for the general vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ the scalar triple product, $\langle \mathbf{a}, \mathbf{b} \times \mathbf{c} \rangle$ is given by the following determinant:

$$\langle \mathbf{a}, \mathbf{b} \times \mathbf{c} \rangle = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Use this and properties of the determinant to show that the vector $\mathbf{b} \times \mathbf{c}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$.

c) Show that the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar if and only if the determinant at point (b) is equal to 0.

Find the constant t if the vectors

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} t \\ 5 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

are coplanar.

Solution:

Exercise 3.11 (a) The cross product $\mathbf{u} \times \mathbf{v}$ is

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 2 & 3 \\ 2 & -5 & 4 \end{vmatrix} = 23\mathbf{e}_1 + 2\mathbf{e}_2 - 9\mathbf{e}_3 = \begin{pmatrix} 23 \\ 2 \\ -9 \end{pmatrix}.$$

This vector is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ since

$$\left\langle \begin{pmatrix} 23 \\ 2 \\ -9 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\rangle = 23 + 4 - 27 = 0,$$
$$\left\langle \begin{pmatrix} 23 \\ 2 \\ -9 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} \right\rangle = 46 - 10 - 36 = 0.$$

(b) You are being asked to show that the inner product of a vector $\mathbf{a} \in \mathbb{R}^3$ with the cross product of two vectors $\mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ is given by the determinant with these three vectors as its rows. To show this, start with an expression for $\mathbf{b} \times \mathbf{c}$

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{e}_3$$

and then take the inner product with $\mathbf{a}$:

$$\begin{aligned} \langle \mathbf{a}, \mathbf{b} \times \mathbf{c} \rangle &= \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} a_1 - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} a_2 + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} a_3 \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \end{aligned}$$

This shows that the inner product is equal to the given 3×3 determinant (as it is equal to its expansion by row 1).

To show that $\mathbf{b} \times \mathbf{c}$ is orthogonal to both $\mathbf{b}$ and $\mathbf{c}$, we just calculate the inner products $\langle \mathbf{b}, \mathbf{b} \times \mathbf{c} \rangle$ and $\langle \mathbf{c}, \mathbf{b} \times \mathbf{c} \rangle$ using the above determinant expression and show that each is equal to 0:

$$\langle \mathbf{b}, \mathbf{b} \times \mathbf{c} \rangle = \begin{vmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \quad \text{and}$$

$$\langle \mathbf{c}, \mathbf{b} \times \mathbf{c} \rangle = \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0,$$

since each of these determinants has two equal rows. Hence the vector $\mathbf{b} \times \mathbf{c}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$.

(c) Therefore, the vector $\mathbf{b} \times \mathbf{c} = \mathbf{n}$ is orthogonal to all linear combinations of the vectors $\mathbf{b}$ and $\mathbf{c}$ and so to the plane determined by these vectors; that is, $\mathbf{b} \times \mathbf{c}$ is a normal vector to the plane containing the vectors $\mathbf{b}$ and $\mathbf{c}$.

If $\langle \mathbf{a}, \mathbf{b} \times \mathbf{c} \rangle = \langle \mathbf{a}, \mathbf{n} \rangle = 0$, then $\mathbf{a}$ must be in this plane; the three vectors are coplanar.

All these statements are reversible.

The given vectors are coplanar when

$$\begin{vmatrix} 3 & -1 & 2 \\ t & 5 & 1 \\ -2 & 3 & 1 \end{vmatrix} = 0,$$

so if and only if $t = -4$.