DM559 – Linear and Integer Programming

Answers to Obligatory Assignment 0.1, Spring 2018

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Exercise 1

In order for the product between two matrices is valid, the number of coloumns in matrix A has to be equal to the number of rows in matrix B.

1: $\begin{bmatrix}1 & 2 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 1\\3 & 1 & 2\end{bmatrix} =$ NOT VALID.

2: $\begin{bmatrix}3 & 3 & 0\end{bmatrix}\begin{bmatrix}1 & 4 & 1\\3 & 1 & 2\end{bmatrix} =$ NOT VALID

3: $\begin{bmatrix}3 & 3 & 0\end{bmatrix}\begin{bmatrix}1 & 4 & 1\\3 & 1 & 2\end{bmatrix}^T \equiv \begin{bmatrix}3 & 3 & 0\end{bmatrix}\begin{bmatrix}1 & 3\\4 & 1\\1 & 2\end{bmatrix} =$ VALID. Resulting matrix: (1x2)

4: $\begin{bmatrix}1 & 4 & 1\\1 & 7 & 2\end{bmatrix}\begin{bmatrix}3 & 3 & 0\end{bmatrix}^T \equiv \begin{bmatrix}1 & 4 & 1\\1 & 7 & 2\end{bmatrix}\begin{bmatrix}3\\3\\0\end{bmatrix} =$ VALID. Resulting matrix: (2x1)

5: $\begin{bmatrix}1 & 4 & 1\\1 & 7 & 2\end{bmatrix}\begin{bmatrix}3 & 3 & 0\end{bmatrix} =$ NOT VALID

6: $\begin{bmatrix}2 & 1 & 5\end{bmatrix}\begin{bmatrix}1 & 6 & 2\end{bmatrix}^T \equiv \begin{bmatrix}2 & 1 & 5\end{bmatrix}\begin{bmatrix}1\\6\\2\end{bmatrix} =$ VALID. Resulting matrix: (1x1)

7: $\begin{bmatrix}2 & 1 & 5\end{bmatrix}^T\begin{bmatrix}1 & 6 & 2\end{bmatrix} \equiv \begin{bmatrix}2\\1\\5\end{bmatrix}\begin{bmatrix}1 & 6 & 2\end{bmatrix} =$ VALID. Resulting matrix: (3x3)

Exercise 2

$\mathrm{A} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ $\mathrm{B} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1x1 + ax0 & 1xb + ax1 \\ 0x1 + 1x0 & 0xb + 1x1 \end{bmatrix} = \begin{bmatrix} 1 & b+a \\ 0 & 1 \end{bmatrix}$$

Exercise 3

We use gauss-jordan elimination in order to try solving the three task a, b, c:

a)

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 3 & 1 \\ 0 & 0 & 5 & 3 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{R1 * } \frac{1}{2} \text{ -> R1}} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 5 & 3 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{R2 * } \frac{1}{5} \text{ -> R2}} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{5} & \frac{-1}{5} \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{R2 * } \frac{-1}{2} \text{ -> R1}}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{6}{5} & \frac{3}{5} \\ 0 & 0 & 1 & \frac{3}{5} & \frac{-1}{5} \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{R3 * } \frac{-6}{5} \text{ -> R1}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & \frac{3}{5} & \frac{-1}{5} \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{R3 * } \frac{-3}{5} \text{ ->R2}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] = \begin{bmatrix} x_1 = & -3 \\ x_3 = & -2 \\ x_4 = & 3 \end{bmatrix}$$

Reduced row echelon form is obtained and has a free variable x_2

b)

$$\left[\begin{array}{ccccc|c} 1 & 3 & -2 & 1 & 0 & 5 \\ 0 & 0 & 2 & -3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{array}\right] = \text{No solution}$$

Has no solution because no matter which number we multiply in row three the result will be zero, and can never have 2 as a solution.

c)

$$\left[\begin{array}{ccc|c} 10^{-20} & 0 & 1 & 1 \\ 1 & 10^{20} & 1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow{\text{Swap R2 and R3}} \left[\begin{array}{ccc|c} 1 & 10^{20} & 1 & 2 \\ 10^{-20} & 0 & 1 & 1 \\ 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow{\text{R2 - } 10^{-20} * \text{R1 -> R2}} \left[\begin{array}{ccc|c} 1 & 10^{20} & 1 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$$\xrightarrow{\text{R2 / -1 -> R2}} \left[\begin{array}{ccc|c} 1 & 10^{20} & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow{\text{R3 - 1 } * \text{ R2 -> R3}} \left[\begin{array}{ccc|c} 1 & 10^{20} & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -10^{-20} & 4 \end{array}\right] \xrightarrow{\text{R3 / } -10^{-20} \text{ -> R3}} \left[\begin{array}{ccc|c} 1 & 10^{20} & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -4 * 10^{20} \end{array}\right]$$

$$\xrightarrow{\text{R2 + 1 * R3 -> R2}} \left[\begin{array}{ccc|c} 1 & 10^{20} & 1 & 2 \\ 0 & 1 & 0 & -4 * 10^{20} \\ 0 & 0 & 1 & -4 * 10^{20} \end{array}\right] \xrightarrow{\text{R1 - 1 * R3 -> R1}} \left[\begin{array}{ccc|c} 1 & 10^{20} & 0 & 4 * 10^{20} \\ 0 & 1 & 0 & -4 * 10^{20} \\ 0 & 0 & 1 & -4 * 10^{20} \end{array}\right]$$

$$\xrightarrow{\text{R1 - } 10^{20} \text{ * R2 -> R1}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 * 10^{40} \\ 0 & 1 & 0 & -4 * 10^{20} \\ 0 & 0 & 1 & -4 * 10^{20} \end{array}\right] = \begin{bmatrix} x_1 = & 4 * 10^{40} \\ x_2 = & -4 * 10^{20} \\ x_3 = & -4 * 10^{20} \end{bmatrix}$$

Exercise 4

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$$

By use of laplace expansion (cofactor expansion) we calculate the determinant by expanding along the first row:

$$\det(A) = 1\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} + 1\begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix}$$

$$= (1 * (1 * (-1) * 2) - 3 * (2 * (-1) - 1 * (-2)) + 1 * (2 * 2 - 1 * (-2)) \underline{\underline{= 3}}$$

The adjunct of a matrix is the transpose of its cofacters. So first step is to find the cofacter of each point in matrix A:

$$C = \begin{bmatrix} + A_{11} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \text{-3} & - A_{12} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} = 0 & + A_{13} = \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix} = 6 \\ - A_{21} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} = 5 & + A_{22} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = 1 & - A_{23} = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} = \text{-8} \\ + A_{31} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} = 2 & - A_{32} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = 1 & + A_{33} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = \text{-5} \end{bmatrix}$$

Next step is to create a matrix of the cofacters of matrix A and transpose it based on the cofacter-rule:

$$\text{adj}(A) = \begin{bmatrix} -3 & 0 & 6 \\ 5 & 1 & -8 \\ 2 & 1 & -5 \end{bmatrix}^T = \begin{bmatrix} -3 & 5 & 2 \\ 0 & 1 & 1 \\ 6 & -8 & -5 \end{bmatrix}$$

From $det(A) = 3$ we know that matrix A has an inverse. By augmenting an identity matrix to matrix A and use of gauss-jordan elimination we calculate the inverse of matrix A:

$$A^{-1} = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ -2 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2\text{-}2^*R_1\ \text{->}\ R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -5 & -1 & -2 & 1 & 0 \\ -2 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3\text{-(-2)}^*R_1\ \text{->}\ R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -5 & -1 & -2 & 1 & 0 \\ 0 & 8 & 1 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2/(-5)->R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 8 & 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - 8 * R_2 -> R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & -\frac{3}{5} & -\frac{6}{5} & \frac{8}{5} & 1 \end{array}\right] \xrightarrow{R_3/(-\frac{3}{5})->R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & 2 & -\frac{8}{3} & -\frac{5}{3} \end{array}\right]$$

$$\xrightarrow{R_2 - (\frac{1}{5}) * R_3 -> R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 2 & -\frac{8}{3} & -\frac{5}{3} \end{array}\right] \xrightarrow{R_1 - 1 * R_3 -> R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & \frac{8}{3} & \frac{5}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 2 & -\frac{8}{3} & -\frac{5}{3} \end{array}\right] \xrightarrow{R_1 - 3 * R_2 -> R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & \frac{5}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 2 & -\frac{8}{3} & -\frac{5}{3} \end{array}\right]$$

Matrix A has the inverse:

$$A^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & -1 \\ -2 & 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & \frac{5}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 2 & -\frac{8}{3} & -\frac{5}{3} \end{bmatrix}$$

Exercise 4 continued

To verify that the determinant of matrix A and matrix A^{-1} is correct, the matrix has been coded in python:

```
import numpy as np
A=([[ 1., 3., 1.],
    [ 2., 1., 1.],
    [-2., 2.,-1.],])

print(np.linalg.det(A))
print(np.linalg.inv(A))
```

As shown below, the python program outputs the determinant of matrix A and the inverse matrix of matrix A which is consistent with our own results.

```
3.0
[[-1.          1.66666667  0.66666667]
 [ 0.          0.33333333  0.33333333]
 [ 2.         -2.66666667 -1.66666667]]
```

Exercise 5

We have a line going through the points A = (-$\frac{3}{2}$, 2) and B = (3, 0). The cartesian equation of a line has the form: $y = ax + b$

To calculate a we simply:
a = $(\frac{y_2-y_1}{x_2-x_1}) = (\frac{0-2}{3-(-\frac{3}{2})}) = -\frac{4}{9}$

Now our line is represented as:
$y = -\frac{4}{9}x + b$

We insert x_1 and y_1, and isolate b:
$2 = -\frac{4}{9} * -\frac{3}{2} + b$
$2 = b + \frac{6}{5}$
$b = \frac{4}{5}$

The vector form of the equation is given as: $x = p + tv$ where x is the position vector, p is any particular point and v is the direction of the line[1].

$\overrightarrow{AB} = \begin{bmatrix} -\frac{3}{2} \\ \\ 2 \end{bmatrix}$ - $\begin{bmatrix} 3 \\ \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} \\ \\ 2 \end{bmatrix}$
x = $\begin{bmatrix} x \\ \\ y \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ \\ 2 \end{bmatrix}$ + t $\begin{bmatrix} -\frac{9}{2} \\ \\ 2 \end{bmatrix}$

[1] According to slide on Geomtric Insight. Page 9: URL: http://www.imada.sdu.dk/ marco/DM559/Slides/dm559-lec5.pdf