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DM559 – Linear and integer programming

Sheet 6, Spring 2018 [pdf format]

Starred exercises are relevant for the exam.

Solution:
Included.

Exercise 1*

Suppose

$$M = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 7x - 2y \\ -x + 8y \end{bmatrix}.$$

[In the subtasks below, you can use Python to carry out matrix multiplications in fraction mode, but if you have to calculate the determinant and the inverse of a matrix then explain how you would carry out the calculations by hand.]
Show that M is a basis of $\mathbb{R}^2$. Write down the transition matrix from M coordinates to standard coordinates. Find $[\mathbf{v}]_M$, the M coordinates of the vector $\mathbf{v}$.

Solution:
To show that M is a basis of $\mathbb{R}$ it suffices to show that the two vectors are linearly independent. We can do this by calculating the determinant of a matrix made by those vectors as columns:

$$\det(M) = 2 + 1 = 3$$

The transition matrix is

$$B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

which is used to find

$$\mathbf{v} = B[\mathbf{v}]_M$$

Hence to calculate $[\mathbf{v}]_M$ we have to invert the matrix B. We use the matrix inverse by cofactor method, ie, we calculate the adjoint matrix and divide by the determinant:

$$B^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

Hence,

$$[\mathbf{v}]_M = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 5/3 \end{bmatrix}$$

Write down the matrix A of the linear transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

with respect to the standard basis.
Find the matrix of T in M coordinates. Call it D. Describe geometrically the effect of the transformation T as a map from $\mathbb{R}^2 \to \mathbb{R}^2$.

Solution:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 7x - 2y \\ -x + 8y \end{bmatrix} = x \begin{bmatrix} 7 \\ -1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 8 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Hence,

$$A = \begin{bmatrix} 7 & -2 \\ -1 & 8 \end{bmatrix}$$

To find the matrix D we need to calculate $B^{-1}AB$:

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 7 & -2 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix}$$

We used Python for the calculations:

```
np.dot(np.dot(np.array([[Fraction(1,3),Fraction(1,3)],,
                              [Fraction(-1,3),Fraction(2,3)]]), ,
                np.array([[7,-2],[-1,8]]) ), ,
        np.array([[2,-1],[1,1]]) )
```

```
array([[Fraction(6, 1), Fraction(0, 1)],,
                [Fraction(0, 1), Fraction(9, 1)]], dtype=object)
)
```

Geometrically, the effect of the transformation T is a stretch in the direction of the two axis represented by the vectors that make the two columns of B.
Find the image of $[\mathbf{v}]_M$ using the matrix D.
Show that your answer is correct using standard coordinates.

Solution:

$$T([\mathbf{v}]_M) = D[\mathbf{v}]_M = \begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \end{bmatrix}$$

In standard coordinates:

$$T(\mathbf{v}) = \begin{bmatrix} 7 & -2 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -11 \\ 17 \end{bmatrix}$$

which brought in M coordinates by

$$[\mathbf{v}]_M = B^{-1}\mathbf{v}$$

gives:

$$[\mathbf{v}]_M = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} -11 \\ 17 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \end{bmatrix}$$

Exercise 2*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{pmatrix} x_1 + x_2 + 2x_3 \\ x_1 + x_3 \\ 2x_1 + x_2 + 3x_3 \end{pmatrix}.$$

Is T invertible? If so how is it described?

Solution:

Determine the transformation matrix and then check if it is invertible, eg, the determinant is different from zero.

Exercise 3 Computing sparse matrix-vector product

If M is an $m \times n$ matrix and u is an n-vector, the ordinary definition of matrix-vector multiplication, $M \cdot u$, states that the result is the m-vector v such that, for each $i = 1..m$,

$$v[i] = \sum_{j=1}^{n} M[i, j]u[j],$$

foreach $i \in \{1, \ldots, m\}$ **do**
 $v[i] := \sum_{j=1}^{n} M[i,j]u[j]$

where we used the computer science notation [] to identify the elements of the matrix and of the vector. The most straightforward way to implement matrix-vector multiplication based on this definition is: Design an implementation that takes advantage of the fact that, if M is a sparse matrix, many entries of M are zero and do not even appear in the sparse representation of M.

Solution:

initialize v to zero vector;
foreach *pair* (i,j) *such that the sparse representation specifies* $M[i,j]$ **do**
 $v[i] = v[i] + M[i,j]u[j]$

Exercise 4 Linear span

Let $V = \text{Lin}\{[0,1,0,1], [0,0,1,0], [1,0,0,1], [1,1,1,1]\}$ where the vectors are over the Galois Field 2, $GF(2)$. For both the following vectors over $GF(2)$, show that it belongs to V by writing it as a linear combination of the vectors that generate V:

a) $[1,1,0,0]$

b) $[1,0,1,0]$

Solution:

We need to find $\alpha, \beta, \gamma, \delta \in \{0,1\}$ such that

$$\alpha \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \delta \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Hence,

$$\begin{cases} \gamma + \delta = 1 \\ \alpha + \delta = 1 \\ \beta + \delta = 0 \\ \alpha + \gamma + \delta = 0 \end{cases}$$

Trying both values for δ we see that only $\delta = 0$ leads to a solution: $\alpha = 1, \beta = 0, \gamma = 1$. Consequently, $[1,0,1,0]$ is the coordinate representation of the first vector with respect to V.
Similarly for the second vector:

$$\alpha \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \delta \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{cases} \gamma + \delta = 1 \\ \alpha + \delta = 0 \\ \beta + \delta = 1 \\ \alpha + \gamma + \delta = 0 \end{cases}$$

We see that $\delta = 1$ and $\alpha = 1, \beta = 0, \gamma = 0$ and $[1,0,0,1]$ is the coordinate representation of the second vector with respect to V.

Exercise 5 Basis and dimensions

Given the matrix:

$$A = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

a) Give a basis for the space determined by the row vectors of the matrix (row space)

b) Give a basis for the space determined by the column vectors of the matrix (ie, column space or range)

c) Show that the dimensions of the row and columns spaces are the same and argue that this is always the case. This dimension of a matrix is called rank.

Solution:

The row space is given by

$$\operatorname{Lin}\left(\left\{\begin{bmatrix}1\\4\\0\\0\end{bmatrix},\begin{bmatrix}0\\2\\2\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}\right)$$

Performing elementary row operations in the original matrix to put it in reduced row echelon form, does not change the row space. Indeed elementary row operations correspond to linear combinations of the three row vectors, and therefore are only generating vectors that belong to the row space and, being reversible, do not loose any member of the span. The RREF of the matrix calculated with sympy in Python is:

$$\begin{bmatrix}1 & 0 & 0 & 4\\0 & 1 & 0 & -1\\0 & 0 & 1 & 1\end{bmatrix}$$

From this form we can write a basis for th erow space, it is the set of non-zero rows (written as vectors). These are the rows with a leading one.

The column space is given by

$$\operatorname{Lin}\left(\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}4\\2\\0\end{bmatrix},\begin{bmatrix}0\\2\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\right)$$

To find a basis for this space we can also use the RREF of the original matrix. The columns in the RREF have been obtained by operations which are NOT linear combinations of the column vectors defining the column space. However, the columns of the RREF with leading ones provide a standard basis for $\mathbb{R}^4$ and we can use this information to deduce which column vectors of A are linearly independent, and by finding a linearly independent subset of the column vectors we indeed find a basis for the column space. Thus:

$$\begin{bmatrix}1 & 4 & 0\\0 & 2 & 2\\0 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Hence, the first three columns are linearly independent ($Ax = 0$ has only the trivial solution) and constitue a basis.

Since we showed above that to find a basis we need to look for both the row space and the column space at the leading ones in the RREF of the matrix A, then we showed that the dimensions of the two spaces are the same.

```
M = sy.Matrix([[1,4,0,0],
[0,2,2,0],
[0,0,1,1]])
M.rref()
```

Exercise 6 Determinant of orthogonal matrices

An $n \times n$ matrix Q is said to be an *orthogonal matrix* if the column vectors of Q form an *orthonormal set* in $\mathbb{R}^n$ (recall that a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is orthonormal if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$ and all vectors are unit vectors.) It follows from the definition of Q that $Q^T Q = I$ and hence that $Q^{-1} = Q^T$. Show that $|\det(Q)| = 1$

Solution:

Since

$$Q^T Q = I$$

then

$$\det(Q^T Q) = \det(I)$$

and using the product rule for determinants:

$$\det(Q^T)\det(Q) = (\det(Q))^2 = 1$$

Hence: $\det(Q) = \pm 1$

Exercise 7*

Diagonalise the matrix

$$A = \begin{bmatrix} 4 & 5 \\ -1 & -2 \end{bmatrix}$$

that is, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Carry out the calculations by hand and check your answer.
In python, the function `numpy.linalg.eig` can be used to find the eigenvalues and the eigenvectors of a matrix. Try to repeat the diagonalization procedure above on the following matrix:

$$B = \begin{bmatrix} 0 & 2 & 1 \\ 16 & 4 & -6 \\ -16 & 4 & 10 \end{bmatrix}.$$

Solution:
For P to exists we need to find a set of 2 linearly independent eigen vectors of A. The eigenvalues of A, that is, the valeus λ for which there exist vectors $\mathbf{x}$ that satisfy

$$A\mathbf{x} = \lambda\mathbf{x},$$

can be found by imposing that the homogeneous linear system $A - \lambda I$ admits non trivial solutions. This can be done by imposing:

$$|A - \lambda I| = 0$$

The corresponding characteristic equation for the given matrix A is:

$$\begin{vmatrix} 4-\lambda & 5 \\ -1 & -2-\lambda \end{vmatrix} = (4-\lambda)(-2-\lambda) + 5 = \lambda^2 - 2\lambda - 3 = 0$$

The polynomial of second degree has roots (ie, solutions of the quadratic equation):

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4+12}}{2} = 2 \pm 4$$

Hence the eigenvalues are -1 and 3. The corresponding eigenvectors can be found by solving:

$$(A + 1I)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 4+1 & 5 \\ -1 & -2+1 \end{bmatrix}\mathbf{x} = \mathbf{0} \implies \begin{bmatrix} 5 & 5 \\ -1 & -1 \end{bmatrix}\mathbf{x} = \mathbf{0} \rightarrow \overset{RREF}{\cdots} \rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \begin{bmatrix} -t \\ t \end{bmatrix}, t \in \mathbb{R}$$

$$\begin{bmatrix} 4-3 & 5 \\ -1 & -2-3 \end{bmatrix}\mathbf{x} = \mathbf{0} \implies \begin{bmatrix} 1 & 5 \\ -1 & -5 \end{bmatrix}\mathbf{x} = \mathbf{0} \rightarrow \overset{RREF}{\cdots} \rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 5 \end{bmatrix}\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \begin{bmatrix} -5t \\ t \end{bmatrix}, t \in \mathbb{R}$$

From theory we know that the eigenvectors are the columns of the matrix P, hence:

$$P = \begin{bmatrix} -1 & -5 \\ 1 & 1 \end{bmatrix}$$

We find $P-1$ in Python.

```
Asy = sy.Matrix(A-3*sc.eye(2))
Asy.rref()
Asy = sy.Matrix(A+1*sc.eye(2))
Asy.rref()

P = sc.array([[-1,-5],[1,1]])
P_1 = sl.inv(P)
sc.matrix(P_1)*sc.matrix(A)*sc.matrix(P)
```

```
matrix([[-1.,  0.],
        [ 0.,  3.]])
```

Exercise 8*

Diagonalize the matrix A:

$$\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Use Python `numpy.linalg.eig` for the calculations. Describe the eigenspace of each eigenvalue.

Solution:

```
A = sc.array([[0,0,-2],[1,2,1],[1,0,3]])
sl.eig(A)
```

which gives:

```
(array([ 2.+0.j,  1.+0.j,  2.+0.j]),
 array([[ 0.        , -0.81649658,  0.70710678],
        [ 1.        ,  0.40824829,  0.        ],
        [ 0.        ,  0.40824829, -0.70710678]]))
```

Hence the eigenvalues are 1 and 2. The eigenvalue 2 has algebraic multiplicity 2. The eigenspace is given by the eigenvectors. The eigenspace of the eigenvalue 1 is given by $\mathrm{Lin}([-0.82, 0.41, 0.41])$. The eigenspace of the eigenvalue 2 is $\mathrm{Lin} = ([0, 1, 0], [0.7, 0, -0.7])$ and since the two vectors are linear independent this eigenspace has dimension 2.

Exercise 9*

Sequences $\mathbf{x}_t = [x_t, y_t, z_t]$ are defined by $x_0 = -1$, $y_0 = 2$,$z_0 = 1$ and

$$\begin{aligned} x_{t+1} &= 7x_t - 3z_t \\ y_{t+1} &= x_t + 6y_t + 5z_t \\ z_{t+1} &= 5x_t - z_t \end{aligned}$$

Find a matrix expression for $\mathbf{x}_t$ and calculate the term $\mathbf{x}_5$. Do this in two ways: by implementing a loop in Python to calculate the powers of the matrix (power method) and by using the theory of diagonalization. Verify that you obtain the same solutions.

Solution:

Let $\mathbf{x}_t = [x_t, y_t, z_t]$. Then:

$$\mathbf{x}_{t+1} = A\mathbf{x}_t = \begin{bmatrix} 7 & 0 & -3 \\ 1 & 6 & 5 \\ 5 & 0 & -1 \end{bmatrix}$$

and from theory

$$\mathbf{x}_5 = A^5\mathbf{x}_0 = PD^5P^{-1}\mathbf{x}_0$$

Hence, we need first to diagonalize A.

```
A = sc.array([[7,0,-3],[1,6,5],[5,0,-1]])
val, vec = sl.eig(A)
```

```
(array([ 6.+0.j,  4.+0.j,  2.+0.j]),
 array([[ 0.        ,  0.30151134,  0.32929278],
        [ 1.        , -0.90453403, -0.76834982],
        [ 0.        ,  0.30151134,  0.5488213 ]]))
```

Hence, recalling that the eigenvalues form the diagonal of the matrix D:

$$\mathbf{x}_5 = PD^5P^{-1}\mathbf{x}_0 = \begin{bmatrix} 0. & 0.30151134 & 0.32929278 \\ 1. & -0.90453403 & -0.76834982 \\ 0. & 0.30151134 & 0.5488213 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}^5 \begin{bmatrix} 4. & 1. & -1. \\ 8.29156198 & 0. & -4.97493719 \\ -4.55521679 & 0. & 4.55521679 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Calculating the value in Python gives:

$$x_5 = \begin{bmatrix} 2. \\ -12. \\ 14. \end{bmatrix}$$

Exercise 10

Suppose that a car rental agency has three locations, numbered 1, 2, and 3. A customer may rent a car from any of the three locations and return it to any of the three locations. Records show that cars are rented and returned in accordance with the following probabilities:

		Rented from location		
		1	2	3
Rented to location	1	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{5}$
	2	$\frac{4}{5}$	$\frac{3}{10}$	$\frac{1}{5}$
	3	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{5}$

(a) Assuming that a car is rented from location 1, what is the probability that it will be at location 1 after two rentals?

(b) Assuming that this dynamical system can be modeled as a Markov chain, find the steady-state vector.

(c) If the rental agency owns 120 cars, how many parking spaces should it allocate at each location to be reasonably certain that it will have enough spaces for the cars over the long term? Explain your reasoning.

Solution:

Let $\mathbf{x}_t = [x_t, y_t, z_t]$. Then the distribution is given by:

Alternatively, modeling as a Markov chain:

$$\mathbf{x}_{t+1} = A\mathbf{x}_t = A^{t+1}\mathbf{x}_0 = PD^{t+1}P^{-1}\mathbf{x}_0$$

After two changes the probability distribution starting at $\mathbf{x}_0 = [1, 0, 0]$ is:

$$\ldots$$

$$\mathbf{x}_2 = A^2\mathbf{x}_0 = PD^2P^{-1}\mathbf{x}_0$$

The steady state can be calculated

Exercise 11

Physical traits are determined by the genes that an offspring receives from its parents. In the simplest case a trait in the offspring is determined by one pair of genes, one member of the pair inherited from the male parent and the other from the female parent. Typically, each gene in a pair can assume one of two forms, called *alleles*, denoted by A and a. This leads to three possible pairings:

$$AA, Aa, aa$$

called *genotypes* (the pairs Aa and aA determine the same trait and hence are not distinguished from one another). It is shown in the study of heredity that if a parent of known genotype is crossed with a random parent of unknown genotype, then the offspring will have the genotype probabilities given in the following table, which can be viewed as a transition matrix for a Markov process:

		Genotype of Parent		
		AA	Aa	aa
Genotype of Offspring	AA	$\frac{1}{2}$	$\frac{1}{4}$	0
	Aa	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	aa	0	$\frac{1}{4}$	$\frac{1}{2}$

Thus, for example, the offspring of a parent of genotype AA that is crossed at random with a parent of unknown genotype will have a 50% chance of being AA, a 50% chance of being Aa , and no chance of being aa.

(a) Show that the transition matrix is *regular*.

[A stochastic matrix P is said to be *regular* if P or some positive power of P has all (strictly) positive entries. A Markov chain whose transition matrix is regular has the following properties:

i. it has a unique vector $\mathbf{q}$ such that $P\mathbf{q} = \mathbf{q}$.

ii. for any initial probability vector $\mathbf{x}_0$, the sequence of state vectors:

$$\mathbf{x}_0, \quad P\mathbf{x}_0, \ldots, \quad P^k\mathbf{x}_0, \ldots$$

converges to $\mathbf{q}$.

Such vector $\mathbf{q}$ is called the *stady-state* vector of the Markov chain.]

(b) Find the steady-state vector, and discuss its physical interpretation.