Computing a Realistic Point Distribution of Hands for Cribbage

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A brute force method simulation was conducted to determine the realistic point distribution in the card game Cribbage. The expectation value for every possible hand is computed the score rounded and recorded. The results are compared against a Monte Carlo simulation of cribbage to produce a true, realistic distribution. A discussion of these results, including a discussion on the role of luck in cribbage are addressed here. All code written for simulation and analysis can be obtained by contacting the authors.

I. INTRODUCTION

Created in the late 17^{th} century by Sir John Suckling as a spin off from the card game "noddy", cribbage is one of the most popular card games in the English speaking world. The game is distinct from many other card games because it features two distinct phases in which players can gain points. In the first phase, "Pegging", players follow a rotation terminated by the dealer to lay cards in their hand on the table in an attempt to score points. This is the only phase of the game where players interact with each other. In the second phase, "Counting", players count their hands for points. The dealer counts a second hand called the "crib", the game's namesake. Score is kept on a cribbage board using pegs. Throughout the game, the goal of each player or team of players is to reach 121 points first. The aspect of the game that this paper seeks to analyze is the counting phase of the game.

Previous work on this subject was conducted by Porzio *et al* in 2014. That work found the possible point distribution in cribbage in order to definitively confirm or deny the existence of the "Big 19". However, the point distribution of *possible* hands in cribbage is not necessarily the same as the point distribution of *actual* hands in cribbage. This discrepancy between possible and actual hands arises from discarding cards to the crib, which happens immediately after dealing. This phase presents players with a choice, allowing them to preferentially select groups of four cards which are most likely to get them the most points in the counting phase when a fifth, communal card is drawn. Because of this choice, the distribution of possible hands is not representative of what players would experience during a real game of cribbage.

The experiment and arguments discussed in this paper seek to build upon the "The Big 19" paper's findings and to show the actual point distribution cribbage players would experience in a real game of cribbage. We do this by taking into account the choice players make during the discard phase. We then borrow ideas from quantum mechanics to find the expectation value of each subset of 4 cards, assuming the player will always choose the hand with the highest expectation value. Success will be characterized by being able to show the point distribution derived from first principles and brute force calculations. Since verification is impractical, results will be discussed from the mindset of an experienced cribbage player.

II. THEORY

The same Python classes and routines used in "The Big 19" are being expanded upon in this work. Past functionality allowed for the creation of cards, decks and hands. Methods for these classes allowed us to create every possible hand in cribbage as well as count those hands. A new method to find the expectation value of each possible hand is explained below.

In order to find a realistic point distribution in cribbage, we need to take into account the choice players make when they discard to the crib. The example we will explore will be 4-handed cribbage, where each player is dealt 5 cards and discards a single card to the crib. A fifth, common card is then "added" to each players hand from the deck during the counting phase. We will pretend that a single player is following this routine of receiving 5 cards and trying to select the set of 4 cards which are mostly likely to result in a high scoring hand once the common card is flipped. Although a real cribbage game would have three other players following the same routine and drawing from the deck's card pool, analyzing a single player game is much easier and still encompasses all of the information available to that single player in a real game.

There are $\binom{52}{5}$ (or $2,598,960$) possible combinations of 5 cards a player can be dealt. From each of these, the player must choose to discard one card and then receive a common card from the deck before counting. From the deck of 52 cards, 5 are given to the player, one of which is discarded to the crib. One of the remaining 47 cards is then chosen as the common card. This means the player gets to choose the set of four cards that are most likely to earn them the most points in the counting phase. For the purposes of this paper, we will assume the player makes the most statistically correct choice (i.e. they never gamble on the set of cards that has a 1% chance to get them 20 points, but a 99% chance to earn them 2 points over a set of cards that has an 80% chance to earn 10 points). To determine the most statistically correct choice, we will compute the expectation value for each subset of four cards, and use the highest value to determine which set of 4 cards to keep. This creates the need to count on the order of 611 million hands, about 47 times the computational cost of the last paper.

A. Expectation Value

To quantify the value of each set of 4 cards the player could choose to keep, we will measure each set's expectation value. An expectation value is a weighted average of the possible points you could score when you combine your 4 cards and the common card. The weights are the probability of getting each point value (i.e. how likely you are to draw a good common card). The expectation value is computed by

$$\langle S \rangle = \sum_{i=1}^{47} \mathcal{P}_i s_i \tag{1}$$

where $\langle S \rangle$ is the expectation value, s_i is the score of a hand given a common card, and $\mathcal{P}_i$ is the probability of getting score s_i. As an example, if you had a set of 4 cards that has a 80% chance to earn 10 points, and a 20% chance to earn 0 points, the expectation value for that set of 4 cards would be 8 points. It is important to note that the expectation value is not actually the number of points you could score. In the example, the player would receive either 0 or 10 points, never 8. This makes the expectation value a terrible choice for quantifying a single hand. However, on a large scale the expectation value is a useful tool to calculate the "value" of each choice the player makes and allows us to work in the random elements of cribbage into our calculations.

III. EXPERIMENT

To accomplish the counting of every possible hand in cribbage the authors implement the object oriented capabilities of the Python Programming Language. A Card and Hand class are constructed. The card class has members number, suit, and value. The cards number is between 1 and 13 to signify Ace through King. This quantity is used in constructing runs and pairs. The cards value is the value used when summing cards together for 15s (i.e. the value of all face cards is 10). The Hand class is simply a list of 5 cards, with the last card assumed to be the common card. The Hand class has a method countHand() which returns a value as per the counting rules. Additionally, the method getExpectationValue() has been added to perform the necessary countHand() operations to calculate the expectation values. This is done by calculating the expectation values for each set of five cards the player could be dealt. Each set of five dealt cards will be broken into the 5 sets of four cards (representing the 5 choices they could make when discarding). The expectation value for these sets of 4 will be computed and the highest value chosen (to represent the player making the most statistically correct choice). These highest expectation values are returned to the main program.

The expectation value returned by Hand.getExpectationValue() is rounded to the nearest point and then recorded in a dictionary whose keys are the score of each hand and whose values are the multiplicity of each expectation value. When all the hands are finished counting, the dictionary contains all of the realistic scores and the number of different hands that can produce each value. These are printed to the screen and recorded. This is a "continuous" distribution for a card game, where only integer values are technically possible. The expectation value gives us a better idea of what the odds are of obtaining a hand with each point value on the common card draw, like in a real game.

The original version of this code takes about 12 hours to run on an Intel i5 processor. Since the original conception of this project, performance enhancements have been made. Rather than count each hand 47 times, a pre-processing function is run which calculates the score of each hand and then stores the value of the hand in a hash map in which each hand has a unique hash string. This process takes about 3.5 minutes. Then, this hash map is used to perform a look up operation for each of the 47 times each of the 12,994,800 million hands. This brings the run time of the simulation down from 12 hours to about 1 hour and 45 minutes. In short, we calculated 12,994,800 hands and perform 611 million look ups rather than calculating 611 million hands.

A. Monte Carlo Simulation

In an attempt to justify using the expectation value as good metric for players to use in order to decide which 4 cards to keep and which card to discard and Monte Carlo simulation was performed. In this experiment a hand is generated by shuffling the deck (the deck.shuffle() method). This hand is used then used to select three different choices of 4 cards. The first set of cards, the "best" choice, is selected by keeping the 4 cards which have the highest expectation value. The second set of cards, the "random" cards, select a random set of 4 cards to keep. The final set of cards, the "worst" cards, are chosen by selecting the set of four cards with the lowest expectation value. Once these three decisions have been made, a single common card is drawn and given to each of the hands, which are then counted.

This is designed to remove the less than straight forward expectation value from the actual data. In this case actual hands are counted with actual (psuedo) randomness. In this way we can use the expectation value but still show a true representation of real scores, rather than fractional point values binned to the nearest full point. The experiment will include a data set constructed from 1 million hands counted in this way.

IV. RESULTS AND ANALYSIS

The calculated distribution can be seen in Figure 2 and Figure 4, which show the multiplicities of the expectation values on a linear and logarithmic scale, respectively. It is important to remember that we are not looking at the distribution of possible point values here. Instead, we are looking at the expectation value, which can be thought of as the intrinsic value of the 4 cards a player chooses, taking into account the randomness of the 5th, common card that is drawn later for the counting phase. Also included are figures from previous work showing the same measurements for possible hands in Figure 1 and Figure 3. Do note that the total number of hands displayed in each distribution is not the same (there are 5x as many hands in the possible distribution), but comparing the ratios is valid.

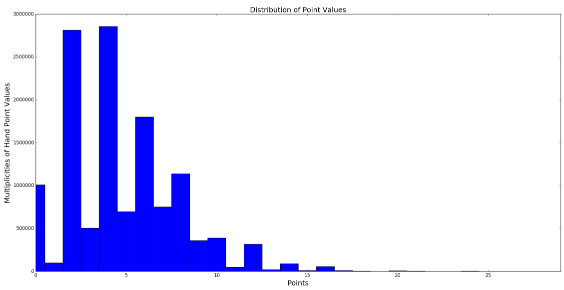

FIG. 1: The number of occurrences (multiplicity) for each expectation value binned to the nearest point. Some values are too small to be seen on this scale.

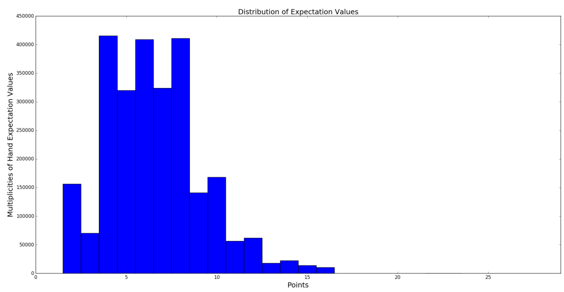

FIG. 2: The multiplicity for each expectation value binned to the nearest point shown on a logarithmic y-axis. Here, the values are squished enough to see those expectation values with very low multiplicities.

The distribution shows the behavior observed in previous work where adjacent point values alternated between

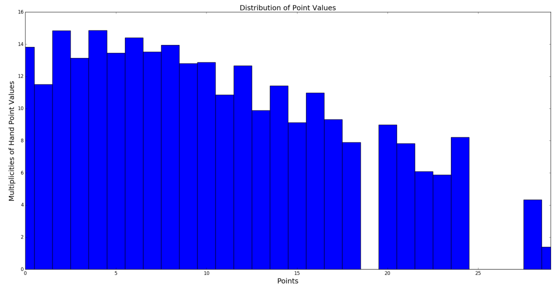

FIG. 3: The number of occurrences (multiplicity) for each possible hand binned to the nearest point. Some values are too small to be seen on this scale.

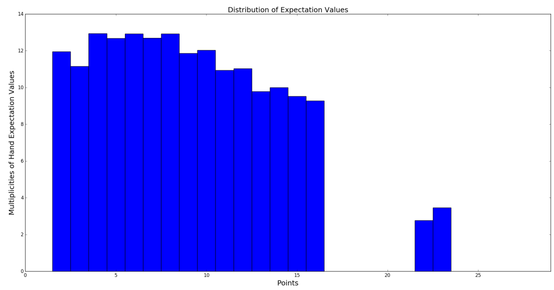

FIG. 4: The multiplicity for each possible point value binned to the nearest point shown on a logarithmic y-axis. Here, the values are squished enough to see those expectation values with very low multiplicities.

higher and lower values, each creating their own envelope. The average expectation value is 6.554 points with a standard deviation of 2.659 points. By comparing the distribution of expectation values to the distribution of possible values we can see how player choice affects the distribution of realistic hands as compared to possible hands.

The easiest example illustrate this is by comparing the hands scoring 0 points in each distribution. The multiplicity of hands scoring 0 points is on the order of (after normalizing the two distributions) 0 as compared to 201800 in the possible distribution. This is because our "statistically correct" cribbage player used in this example would never elect to keep a hand scoring 0 points. This is the most telling flag that the expectation value is probably *not* a good predictor of a realistic point distribution – every cribbage player has had a 0 point hand before. It can be seen that the expectation values pull in the low and high end of the possible distribution, concentrating the hands with high multiplicities between 4 and 8 points.

The multiplicities for hands with expectation values scoring over 16 is extremely low – and more importantly lower than the corresponding possible distribution in this score range. This doesn't mean getting a hand scoring 16+ points is prohibitively unlikely (players might see two of more hands scoring over 16 points in a typical game to 121 points), but rather that there are very few sets of 4 cards a player can have where they could *expect* to get 16+ points, without getting lucky on the common card draw. It is regrettable to inform the cribbage community that getting high scoring hands is a stochastic process and depends more heavily on the common card than the four cards a player chooses to keep. More simply put: you're not a cribbage god, you're just really lucky.

This is not true for hands scoring 16 points or below. The more continuous distribution in this region and the closer match to the possible distribution suggests that the 4 cards a player chooses *does* play a large factor in determining the value of that hand once the common card is flipped. These sets of 4 cards do have an intrinsic value that is more likely to be realized in game. In other words, a player *can* expect to get value out of the set of 4 cards they choose. This is where the good cribbage players are able to separate themselves from the bad – by choosing cards that get reliably get them points instead of throwing a 'Hail Mary.'

A. Monte Carlo Simulation

To further illustrate this point and as an attempt to show that the expectation value is a good way to decide which 4 cards have value and which card to discard, another experiment was conducted. Rather than report the expectation value, we report the results of 1 million hands of cribbage in Figure 5. The "best" player scored an average of 6.65 ± 3.55; the "random" player scored an average of 4.77 ± 3.12 points; the "worst" player scored an average of 3.12 ± 2.44 points. To better show the distribution, see Figure 6 which has re-binned scores to remove the discrepancy between odd and even scoring hands. A χ^2 test shows that each distribution is significantly different from the others with a significance beyond the 6σ confidence interval. When the sample size is reduced to $N = 100$, it can be seen that there is less of a difference between "best" and "random" than between "random" and "worst" – suggesting you can do a lot more to tank your cribbage game by choosing bad cards than improve it by choosing good cards as compared to random chance.

One of the most telling differences between the three imaginary players is the number of hands scoring in the extreme low and extreme high point range. The "best" player had 2.29% of their hands score zero points, while the the "worst" player had 17.6% of their hands resulting in zero points. The "random" player had 7.74% of their hands with a point value of zero. At the other end of the distribution, the "best" player had 2.44% of their hands scoring above 16 points, the "worst" had 0.09%, and the "random" had 0.65%.

By selecting the cards with the best expectation value every time, the "best" player is able to shift their point distribution to higher point values compared to the "random" and "worst" players. Despite the differences between the "best" and "worst" players, it is worth mentioning that no player would willfully select the worst cards available to them – making the "worst" player's distribution more of a theoretical concern that something that might be seen in a real game of cribbage.

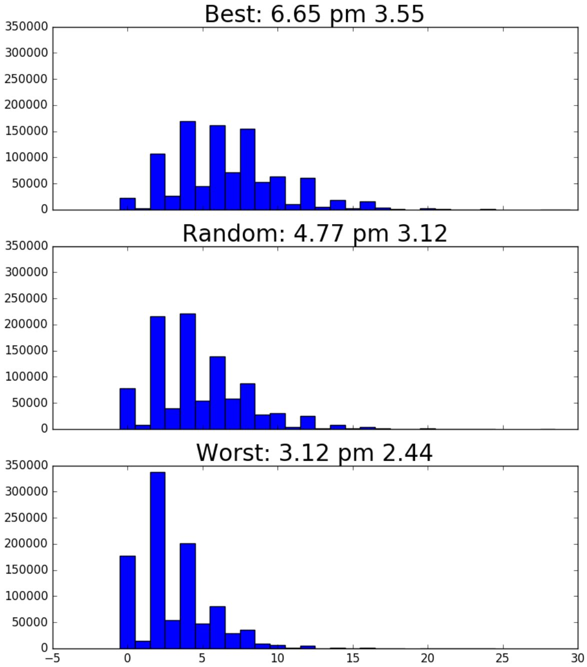

FIG. 5: Results of a Monte Carlo simulation of 1 million hands. Each hand of 5 cards was dealt to a "best", "random", and "worst" player who decided which 4 cards to keep. A random common card was then flipped and their score were tabulated.

B. Reporting a Realistic Distribution

When starting this experiment, it was thought the distribution of expectation values would be a good predictor of the distribution of realistic values. It turns out that the expectation value removes too much of the element of chance in order to be accurate. However, where the expectation value distribution and the "best" player distribution differ tells us about to what degree chance plays a role in cribbage. This also reinforces the claim that hands scoring over 16 points are due to luck more so than skilled decision making – because the expectation value diminishes their value because they are so unlikely.

Lastly, the expectation value distribution and "best" player distribution have similar means, but the "best" player (who is actually subjected to chance) as a higher standard deviation. Because of this, the "best" player scores more zero point hands and more high scoring point

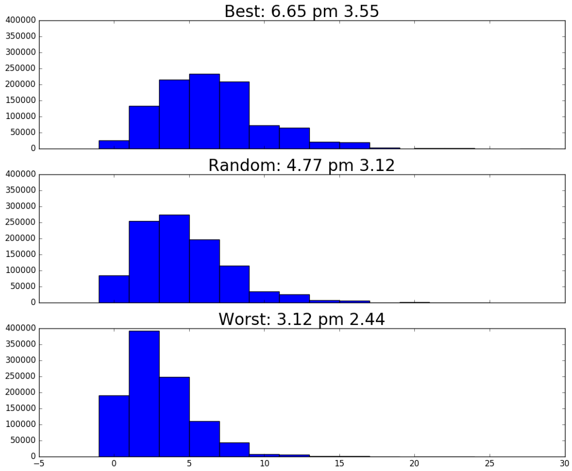

FIG. 6: Results of a Monte Carlo simulation of 1 million hands re-binned to smoother the distribution. Each hand of 5 cards was dealt to a "best", "random", and "worst" player who decided which 4 cards to keep. A random common card was then flipped and their score were tabulated.

hands than the expectation value would suggest. Including chance by way of the Monte Carlo simulation spreads the "best" player's distribution out from that of the more narrow expectation value distribution.

V. CONCLUSIONS

The purpose of this paper was to determine to what degree player agency during the discard phase could affect the distribution of points in a "real" game of cribbage. To do this we implemented the expectation value approach to look at how many points players could expect to earn for every set of 4 cards they start each round with. From the data, we were able to determine that the higher scoring hands are more dependent on luck, while the low to middle scoring hands are more dependent on player choice.

However, due to the differences in the expectation value distribution and the "best" player distribution (Figure 2 and 5, respectively), we can see that the expectation value distribution is *not* a good predictor of what a realistic point distribution could look like. Although the expectation value is a useful predictor of a hand's potential point value, it does too much to remove the elements of luck from the calculation to provide an accurate distribution of real hands.

As far as the role that player choice plays in shifting the distribution of realistic point values, we can report that the average point value of all possible hands is 4.769 ± 3.125 points, as shown in previous work, while the average of the expectation values is 6.554 ± 2.659 points ad the value of the "best" player from the Monte Carlo simulation was 6.65 ± 3.55 points. The differences in these values is attributed to player choice, while the difference in their standard deviations is attributed to the role that chance plays in cribbage.

From the work shown here, we now know that the choice players make when they choose their set of 4 cards is enough to skew the average point values of hands as compared to the average of all the possible hands. We also can say that hands scoring higher than 16 points have their value determined primarily by luck, while hands scoring below 16 points have their value determined primarily by the choice the player makes in selecting the 4 cards they would like to keep. Cribbage players everywhere can now be assured that methodically making correct decisions during the discard phase will net them more points during the counting phase in the long run.