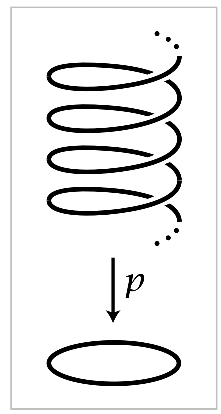
The Fundamental Group of the Circle

Our first real theorem will be the calculation $\pi_1(S^1) \approx \mathbb{Z}$. Besides its intrinsic interest, this basic result will have several immediate applications of some substance, and it will be the starting point for many more calculations in the next section. It should be no surprise then that the proof will involve some genuine work.

Theorem 1.7. *$\pi_1(S^1)$ is an infinite cyclic group generated by the homotopy class of the loop $\omega(s) = (\cos 2\pi s, \sin 2\pi s)$ based at $(1,0)$.*

Note that $[\omega]^n = [\omega_n]$ where $\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$ for $n \in \mathbb{Z}$. The theorem is therefore equivalent to the statement that every loop in S^1 based at $(1,0)$ is homotopic to ω_n for a unique $n \in \mathbb{Z}$. To prove this the idea will be to compare paths in S^1 with paths in $\mathbb{R}$ via the map $p:\mathbb{R}\to S^1$ given by $p(s) = (\cos 2\pi s, \sin 2\pi s)$. This map can be visualized geometrically by embedding $\mathbb{R}$ in $\mathbb{R}^3$ as the helix parametrized by $s \mapsto (\cos 2\pi s, \sin 2\pi s, s)$, and then p is the restriction to the helix of the projection of $\mathbb{R}^3$ onto $\mathbb{R}^2$, $(x,y,z) \mapsto (x,y)$. Observe that the loop ω_n is the composition $p\widetilde{\omega}_n$ where $\widetilde{\omega}_n : I \to \mathbb{R}$ is the path $\widetilde{\omega}_n(s) = ns$, starting at 0 and ending at n, winding around the helix $|n|$ times, upward if $n > 0$ and downward if $n < 0$. The relation $\omega_n = p\widetilde{\omega}_n$ is expressed by saying that $\widetilde{\omega}_n$ is a **lift** of ω_n.



We will prove the theorem by studying how paths in S^1 lift to paths in $\mathbb{R}$. Most of the arguments will apply in much greater generality, and it is both more efficient and more enlightening to give them in the general context. The first step will be to define this context.

Given a space X, a **covering space** of X consists of a space $\widetilde{X}$ and a map $p:\widetilde{X}\to X$ satisfying the following condition:

$(*)$ For each point $x \in X$ there is an open neighborhood U of x in X such that $p^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by p.

Such a U will be called **evenly covered.** For example, for the previously defined map $p:\mathbb{R}\to S^1$ any open arc in S^1 is evenly covered.

To prove the theorem we will need just the following two facts about covering spaces $p:\widetilde{X}\to X$.

(a) For each path $f:I\to X$ starting at a point $x_0 \in X$ and each $\widetilde{x}_0 \in p^{-1}(x_0)$ there is a unique lift $\widetilde{f}:I\to\widetilde{X}$ starting at $\widetilde{x}_0$.

(b) For each homotopy $f_t:I\to X$ of paths starting at x_0 and each $\widetilde{x}_0 \in p^{-1}(x_0)$ there is a unique lifted homotopy $\widetilde{f}_t:I\to\widetilde{X}$ of paths starting at $\widetilde{x}_0$.

Before proving these facts, let us see how they imply the theorem.

Proof of Theorem 1.7: Let $f:I\to S^1$ be a loop at the basepoint $x_0 = (1,0)$, representing a given element of $\pi_1(S^1,x_0)$. By (a) there is a lift $\widetilde{f}$ starting at 0. This path $\widetilde{f}$ ends at some integer n since $p\widetilde{f}(1) = f(1) = x_0$ and $p^{-1}(x_0) = \mathbb{Z} \subset \mathbb{R}$. nother path in $\mathbb{R}$ from 0 to n is $\widetilde{\omega}_n$, and $\widetilde{f} \simeq \widetilde{\omega}_n$ via the linear homotopy $(1-t)\widetilde{f} + t\widetilde{\omega}_n$. Composing this homotopy with p gives a homotopy $f \simeq \omega_n$ so $[f] = [\omega_n]$.

To show that n is uniquely determined by $[f]$, suppose that $f \simeq \omega_n$ and $f \simeq \omega_m$, so $\omega_m \simeq \omega_n$. Let f_t be a homotopy from $\omega_m = f_0$ to $\omega_n = f_1$. By (b) this homotopy lifts to a homotopy $\widetilde{f}_t$ of paths starting at 0. The uniqueness part of (a) implies that $\widetilde{f}_0 = \widetilde{\omega}_m$ and $\widetilde{f}_1 = \widetilde{\omega}_n$. Since $\widetilde{f}_t$ is a homotopy of paths, the endpoint $\widetilde{f}_t(1)$ is independent of t. For $t = 0$ this endpoint is m and for $t = 1$ it is n, so $m = n$.

It remains to prove (a) and (b). Both statements can be deduced from a more general assertion about covering spaces $p:\widetilde{X}\to X$:

(c) Given a map $F:Y \quad I\to X$ and a map $\widetilde{F}:Y \quad \{0\}\to\widetilde{X}$ lifting $F|Y \quad \{0\}$, then there is a unique map $\widetilde{F}:Y \quad I\to\widetilde{X}$ lifting F and restricting to the given $\widetilde{F}$ on $Y \quad \{0\}$.

Statement (a) is the special case that Y is a point, and (b) is obtained by applying (c) with $Y = I$ in the following way. The homotopy f_t in (b) gives a map $F:I \quad I\to X$ by setting $F(s,t) = f_t(s)$ as usual. unique lift $\widetilde{F}:I \quad \{0\}\to\widetilde{X}$ is obtained by an application of (a). Then (c) gives a unique lift $\widetilde{F}:I \quad I\to\widetilde{X}$. The restrictions $\widetilde{F}|\{0\} \quad I$ and $\widetilde{F}|\{1\} \quad I$ are paths lifting constant paths, hence they must also be constant by the uniqueness part of (a). So $\widetilde{f}_t(s) = \widetilde{F}(s,t)$ is a homotopy of paths, and $\widetilde{f}_t$ lifts f_t since $p\widetilde{F} = F$.

To prove (c) we will first construct a lift $\widetilde{F}:N \quad I\to\widetilde{X}$ for N some neighborhood in Y of a given point $y_0 \in Y$. Since F is continuous, every point $(y_0,t) \in Y \quad I$ has a product neighborhood $N_t \quad (a_t,b_t)$ such that $F(N_t \quad (a_t,b_t))$ is contained in an evenly covered neighborhood of $F(y_0,t)$. By compactness of $\{y_0\} \quad I$, finitely many such products $N_t \quad (a_t,b_t)$ cover $\{y_0\} \quad I$. This implies that we can choose a single neighborhood N of y_0 and a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of I so that for each i, $F(N \quad [t_i,t_{i+1}])$ is contained in an evenly covered neighborhood U_i. ssume inductively that $\widetilde{F}$ has been constructed on $N \quad [0,t_i]$, starting with the given $\widetilde{F}$ on $N \quad \{0\}$. We have $F(N \quad [t_i,t_{i+1}]) \subset U_i$, so since U_i is evenly covered there is an open set $\widetilde{U}_i \subset \widetilde{X}$ projecting homeomorphically onto U_i by p and containing the point $\widetilde{F}(y_0,t_i)$. fter replacing N by a smaller neighborhood of y_0 we may assume that $\widetilde{F}(N \quad \{t_i\})$ is contained in $\widetilde{U}_i$, namely, replace $N \quad \{t_i\}$ by its intersection with $(\widetilde{F}|N \quad \{t_i\})^{-1}(\widetilde{U}_i)$. Now we can define $\widetilde{F}$ on $N \quad [t_i,t_{i+1}]$ to be the composition of F with the homeomorphism $p^{-1}:U_i\to\widetilde{U}_i$. fter a finite number of steps we eventually get a lift $\widetilde{F}:N \quad I\to\widetilde{X}$ for some neighborhood N of y_0.

Next we show the uniqueness part of (c) in the special case that Y is a point. In this case we can omit Y from the notation. So suppose $\widetilde{F}$ and $\widetilde{F}'$ are two lifts of $F:I\to X$

such that $\widetilde{F}(0) = \widetilde{F}'(0)$. s before, choose a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of I so that for each i, $F([t_i, t_{i+1}])$ is contained in some evenly covered neighborhood U_i. ssume inductively that $\widetilde{F} = \widetilde{F}'$ on $[0, t_i]$. Since $[t_i, t_{i+1}]$ is connected, so is $\widetilde{F}([t_i, t_{i+1}])$, which must therefore lie in a single one of the disjoint open sets $\widetilde{U}_i$ projecting homeomorphically to U_i as in $(*)$. By the same token, $\widetilde{F}'([t_i, t_{i+1}])$ lies in a single $\widetilde{U}_i$, in fact in the same one that contains $\widetilde{F}([t_i, t_{i+1}])$ since $\widetilde{F}'(t_i) = \widetilde{F}(t_i)$. Because p is injective on $\widetilde{U}_i$ and $p\widetilde{F} = p\widetilde{F}'$, it follows that $\widetilde{F} = \widetilde{F}'$ on $[t_i, t_{i+1}]$, and the induction step is finished.

The last step in the proof of (c) is to observe that since the $\widetilde{F}$'s constructed above on sets of the form $N \quad I$ are unique when restricted to each segment $\{y\} \quad I$, they must agree whenever two such sets $N \quad I$ overlap. So we obtain a well-defined lift $\widetilde{F}$ on all of $Y \quad I$. This $\widetilde{F}$ is continuous since it is continuous on each $N \quad I$. nd $\widetilde{F}$ is unique since it is unique on each segment $\{y\} \quad I$. $\square$

Now we turn to some applications of the calculation of $\pi_1(S^1)$, beginning with a proof of the Fundamental Theorem of lgebra.

Theorem 1.8. *Every nonconstant polynomial with coefficients in has a root in .*

Proof: We may assume the polynomial is of the form $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. If $p(z)$ has no roots in , then for each real number $r \geq 0$ the formula

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$$

defines a loop in the unit circle $S^1 \subset$ based at 1. s r varies, f_r is a homotopy of loops based at 1. Since f_0 is the trivial loop, we deduce that the class $[f_r] \in \pi_1(S^1)$ is zero for all r. Now fix a large value of r, bigger than $|a_1| + \cdots + |a_n|$ and bigger than 1. Then for $|z| = r$ we have

$$|z^n| > (|a_1| + \cdots + |a_n|)|z^{n-1}| > |a_1 z^{n-1}| + \cdots + |a_n| \geq |a_1 z^{n-1} + \cdots + a_n|$$

From the inequality $|z^n| > |a_1 z^{n-1} + \cdots + a_n|$ it follows that the polynomial $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$ has no roots on the circle $|z| = r$ when $0 \leq t \leq 1$. Replacing p by p_t in the formula for f_r above and letting t go from 1 to 0, we obtain a homotopy from the loop f_r to the loop $\omega_n(s) = e^{2\pi i n s}$. By Theorem 1.7, ω_n represents n times a generator of the infinite cyclic group $\pi_1(S^1)$. Since we have shown that $[\omega_n] = [f_r] = 0$, we conclude that $n = 0$. Thus the only polynomials without roots in are constants. $\square$

Our next application is the Brouwer fixed point theorem in dimension 2.

Theorem 1.9. *Every continuous map $h : D^2 \to D^2$ has a fixed point, that is, a point $x \in D^2$ with $h(x) = x$.*

Here we are using the standard notation D^n for the closed unit disk in $\mathbb{R}^n$, all vectors x of length $|x| \leq 1$. Thus the boundary of D^n is the unit sphere S^{n-1}.