Binary Sequences with Structural Delays

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General Problem

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- We want to determine under which circumstances these sequences converge for all initial values.

Example 1

Let $\mathcal{F}_1 = \{1,5\}$ and $\mathcal{F}_2 = \{3,6\}$, i.e.

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Suppose the initial values in the sequence are 101101.

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Open Question

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For fixed $r > 1$, determine all sets $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_r$ such that all solutions $\{y_n\}$ converge.

Related Concepts

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- Fibonacci linear feedback shift register sequences (LFSRs; looking for long periods for all initial values, rather than short).
- Generalizations of Schur's Theorem (Coin Problem, Postage Stamp Problem).

Consecutive Delays

The case $\mathcal{F}_1 = \{a, a+1\}$ and $\mathcal{F}_2 = \{b, b+1\}$

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Conjecture

Suppose $a, b \geq 1$. *All solutions to* $\{y_n\}$, *where*

$$y_n = \begin{cases} 0 & \text{if } y_{n-a} = y_{n-a-1} = 0 \text{ or } y_{n-b} = y_{n-b-1} = 0 \\ 1 & \text{otherwise} \end{cases} \tag{3}$$

are convergent if and only if, for

$$g \stackrel{def}{=} \max\{\gcd(a, b+1), \gcd(a+1, b)\}, \tag{4}$$

we have $g \leq 2$.

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Isolated 0's

Conjecture

Suppose $\mathcal{F} = \{a, a+1\}$ *and* $\mathcal{F} = \{b, b+1\}$. *The set*

$$\mathcal{S} \stackrel{def}{=} \{n \geq 0 : y_{n-1} = y_{n+1} = 1 \text{ and } y_n = 0\}$$

is finite.

Theorem (Assuming no isolated 0's)

The case $\mathcal{F}_1 = \{a, a+1\}$ and $\mathcal{F}_2 = \{b, b+1\}$

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The case $\mathcal{F}_1 = \{a, a+1\}$ and $\mathcal{F}_2 = \{b, b+1\}$

Theorem

(Berenhaut and Patsolic) Suppose $a, b \geq 1$ and the set $\mathcal{S}$ is finite. Then all solutions to

$$y_n = \begin{cases} 0 & \text{if } y_{n-a} = y_{n-a-1} = 0 \text{ or } y_{n-b} = y_{n-b-1} = 0 \\ 1 & \text{otherwise} \end{cases} \tag{5}$$

are convergent if and only if $g \leq 2$, where

$$g \stackrel{def}{=} \max\{\gcd(a, b+1), \gcd(a+1, b)\}. \tag{6}$$

Question: Can there be infinitely many isolated 0's?

Example 1: $\mathcal{F}_1 = \{2, 3\}$ and $\mathcal{F}_2 = \{5, 6\}$

1. 010100

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1. 01010010110111111111111111
   11111111111111111111111111
   11111111111111111111111111
   11111111111111111111111111

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Example 2: $\mathcal{F}_1 = \{8, 9\}$ and $\mathcal{F}_2 = \{15, 16\}$

1. 0010001001001010

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1. 001000100100101001100110010101100
   11011100111111001111110011111100
   11111100111111001111110011111100
   11111100111111001111110011111100
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Example 3: $\mathcal{F}_1 = \{6, 7\}$ and $\mathcal{F}_2 = \{11, 12\}$

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1. 0100101010001101001110001111001110001111
   001110001111001110001111001110001111
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Related Theorem for Min-Max Sequences

It is not difficult to prove the following (employing the earlier result).

Theorem

Suppose $\mathcal{F}_1 = \{a, a+1\}$ and $\mathcal{F}_2 = \{b, b+1\}$, and the minimal value in the period is "non-isolated". Then all solutions to

$$y_n = \min_{1 \le i \le r} \left\{ \max_{j \in \mathcal{F}_i} \{y_{n-j}\} \right\}, \qquad n \ge 0 \tag{7}$$

are convergent if and only if $g \le 2$.

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Suppose $\mathcal{F}_1 = \{2, 3\}, \mathcal{F}_2 = \{4, 5\}$ and the initial values in the sequence are 23512.

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Generalized conjecture for the case $r = 2$

The case $\mathcal{F}_1 = \{a, a+k\}$ and $\mathcal{F}_2 = \{b, b+k\}$, with k odd.

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Conjecture

Suppose $a, b \geq 1$ and $k \geq 1$ is odd. Then all solutions to

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