Binary Sequences with Structural Delays

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Abstract

In this poster we consider the propagation of values in recursive binary sequences. Such sequences have been heavily studied in the context of feedback shift registers. Here we discuss some recent results and conjectures on convergence (and periodicities) of sequences, in terms of number-theoretic properties of elements in underlying delay sets.

– The problem –

- Suppose $r \geq 1$ and $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are finite sets of positive integers.
- Let $s = \max\left\{\bigcup_{1\leq i\leq r} \mathcal{F}_i\right\}$. (Maximum delay)
- Consider a binary sequence $\{y_i\}$, with initial values $y_{-s}, y_{-s+1}, \ldots, y_{-1}$, satisfying

$$y_n = \begin{cases} 0 & \text{if there exists an } i \text{ s.t. } y_{n-j}=0, \text{ for all } j \in \mathcal{F}_i \\ 1 & \text{otherwise} \end{cases}, \qquad n \geq 0. \tag{1}$$

- We want to determine under which circumstances such sequences converge for all initial values.

Some simple examples

Example. Let $\mathcal{F}_1 = \{1,5\}$ and $\mathcal{F}_2 = \{3,6\}$, i.e.

$$y_n = \begin{cases} 0 & \text{if } y_{n-1}=y_{n-5}=0 \text{ or } y_{n-3}=y_{n-6}=0 \\ 1 & \text{otherwise} \end{cases}. \tag{2}$$

Suppose the initial values in the sequence are 101101 (i.e. $y_{-6}=1, y_{-5}=0, \ldots, y_{-1}=1$)

- Here $s=6$
- The sequence continues

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
y_n	1	0	1	1	0	1	1	0	1	1	0	1	1

Example. Let $\mathcal{F}_1 = \{2,3\}, \mathcal{F}_2 = \{4,5\}$ and suppose the initial values in the sequence are 11001.

- Here $s=5$
- The sequence continues:

n	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
y_n	1	1	0	0	1	0	1	0	1	1	1	1	1

– Main open question –

Open Question 1. *For fixed $r > 1$, determine all sets $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_r$ such that all solutions $\{y_n\}$ converge.*

Related Concepts

- Fibonacci linear feedback shift register sequences (LFSRs; looking for long periods for all initial values, rather than short).
- Generalizations of Schur's Theorem (Coin Problem, Postage Stamp Problem).

Periodicities

These sequences must, by necessity, be eventually periodic!

A case of two sets and consecutive delays, i.e. $r=2$, $\mathcal{F}_1=\{a,a+1\}$ and $\mathcal{F}_2=\{b,b+1\}$.

Conjecture 1. *Suppose $a,b \geq 1$. All solutions to $\{y_n\}$, where*

$$y_n = \begin{cases} 0 & \text{if } y_{n-a}=y_{n-a-1}=0 \text{ or } y_{n-b}=y_{n-b-1}=0 \\ 1 & \text{otherwise} \end{cases}, \tag{3}$$

for $n \geq 0$ are convergent if and only if, for

$$g_{a,b} \stackrel{def}{=} \max\{\gcd(a,b+1), \gcd(a+1,b)\}, \tag{4}$$

we have $g_{a,b} \leq 2$.

Maximal (minimal) period lengths

Maximal period lengths for $\mathcal{F}_1=\{a,a+1\}$ and $\mathcal{F}_2=\{b,b+1\}$

a/b	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2			3	1	1	3	1	1	3	1	1	3	1	1	3	1	1	3	1
3				4	3	1	1	8	1	1	3	12	1	3	1	16	3	1	1
4					5	1	8	1	1	10	12	1	1	1	16	1	1	1	20
5						6	1	1	10	1	1	12	1	15	3	1	1	18	20
6							7	3	1	1	12	1	1	14	1	1	18	1	1
7								8	1	1	1	12	14	1	1	16	1	1	1
8									9	1	12	3	1	1	16	1	1	18	20
9										10	3	1	1	3	30	1	18	1	1
10											11	1	1	30	1	1	1	1	20
11												12	1	1	3	16	1	36	1
12													13	3	16	1	36	1	20
13														14	1	1	1	1	1
14															15	1	1	3	1
15																16	3	1	60
16																	17	1	20
17																		18	1
18																			19

Example For $a=10$ and $b=14$, we have $(a,a+1)=(10,11)$, $(b,b+1)=(14,15)$, $\gcd(10,15)=5$ and $\gcd(11,14)=1$. Hence $g_{a,b}=5$. Here the minimal period lengths are 1, 5, 10, 15 and 30 with respective multiplicities 29533, 130, 254, 1362 and 1489.

For instance, for initial values 100001101010000, we have the minimal period (of length 30) $100011100010001100001100110000$.

- Understanding the possible periods (including the maximal period length) for given $\mathcal{F}$-sets in among the questions we are considering.

Isolated zeros

Conjecture 2. *Suppose $\mathcal{F}=\{a,a+1\}$, $\mathcal{F}=\{b,b+1\}$ and $\{y_n\}$ satisfies (3). The set*

$$\mathcal{S} \stackrel{def}{=} \{n \geq 0 : y_{n-1}=y_{n+1}=1 \text{ and } y_n=0\}$$

is finite (for all initial values).

Result (assuming no isolated 0's)

Theorem 1. *(Berenhaut and Patsolic) Suppose $\mathcal{F}=\{a,a+1\}$ and $\mathcal{F}=\{b,b+1\}$ with $a,b \geq 1$. If Conjecture 2 holds, then all solutions to (3) are convergent if and only if $g_{a,b} \leq 2$.*

Related result for min-max sequences

Theorem 2. *Suppose $\mathcal{F}_1=\{a,a+1\}$ and $\mathcal{F}_2=\{b,b+1\}$, and $\{z_n\}$ is defined via*

$$z_n = \min_{1\leq i\leq r}\left\{\max_{j\in\mathcal{F}_i}\{z_{n-j}\}\right\}, \qquad n \geq 0. \tag{5}$$

If the minimal value in the period of $\{z_n\}$ is always "non-isolated", then all solutions are convergent if and only if $g_{a,b} \leq 2$.

Example: Suppose $\mathcal{F}_1=\{2,3\}, \mathcal{F}_2=\{4,5\}$ and the initial values for the sequence are 23512.

- Continuation of the sequence:

n	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
y_n	2	3	5	1	2	3	2	3	2	3	3	3	3

Some conjectures for larger gaps

Conjecture 3. *Suppose $a,b \geq 1$ and $k \geq 1$ is odd. Then all solutions to*

$$y_n = \begin{cases} 0 & \text{if } y_{n-a}=y_{n-a-k}=0 \text{ or } y_{n-b}=y_{n-b-k}=0 \\ 1 & \text{otherwise} \end{cases} \tag{6}$$

are convergent if and only if $g_{a,b,k} \leq 2$, where

$$g_{a,b,k} \stackrel{def}{=} \max\{\gcd(a,b+k), \gcd(a+k,b)\}. \tag{7}$$

Conjecture 4. *Suppose $k \geq 1$, $r=2$ and $\mathcal{F}=\{a,a+k\}$ and $\mathcal{F}=\{b,b+k\}$. The set*

$$\mathcal{S} \stackrel{def}{=} \{n \geq 0 : y_{n-k}=y_{n+k}=1 \text{ and } y_n=0\}$$

is finite.