Binary Sequences with Structural Delays

Yujie Jiang



Department of Mathematics

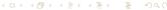
4th August 2016





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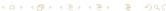




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 We want to determine under which circumstances these sequences converge for all initial values.





Let $\mathcal{F}_1=\{1,5\}$ and $\mathcal{F}_2=\{3,6\}$, i.e.

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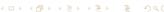
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Open Question

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For fixed r > 1, determine all sets $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_r$ such that all solutions $\{y_n\}$ converge.





Related Concepts





Related Concepts

- Fibonacci linear feedback shift register sequences (LFSRs; looking for long periods for all initial values, rather than short).
- Generalizations of Schur's Theorem (Coin Problem, Postage Stamp Problem).





Consecutive Delays

The case
$$\mathcal{F}_1 = \{a, a+1\}$$
 and $\mathcal{F}_2 = \{b, b+1\}$





Consecutive Delays

The case $\mathcal{F}_1 = \{a, a+1\}$ and $\mathcal{F}_2 = \{b, b+1\}$

Conjecture

Suppose $a, b \ge 1$. All solutions to $\{y_n\}$, where

$$y_n = \begin{cases} 0 & \text{if } y_{n-a} = y_{n-a-1} = 0 \text{ or } y_{n-b} = y_{n-b-1} = 0\\ 1 & \text{otherwise} \end{cases}$$
 (3)

are convergent if and only if, for

$$g \stackrel{\text{def}}{=} \max\{\gcd(a, b+1), \gcd(a+1, b)\},\tag{4}$$

we have $g \leq 2$.





Isolated 0's

Conjecture

Suppose
$$\mathcal{F}=\{a,a+1\}$$
 and $\mathcal{F}=\{b,b+1\}$. The set

$$S \stackrel{def}{=} \{ n \ge 0 : y_{n-1} = y_{n+1} = 1 \text{ and } y_n = 0 \}$$

is finite.

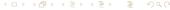




Theorem (Assuming no isolated 0's)

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Theorem (Assuming no isolated 0's)

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 and $\mathcal{F}_2 = \{b, b+1\}$

Theorem

(Berenhaut and Patsolic) Suppose $a,b \geq 1$ and the set $\mathcal S$ is finite. Then all solutions to

$$y_n = \begin{cases} 0 & \text{if } y_{n-a} = y_{n-a-1} = 0 \text{ or } y_{n-b} = y_{n-b-1} = 0\\ 1 & \text{otherwise} \end{cases}$$
 (5)

are convergent if and only if $g \leq 2$, where

$$g \stackrel{\text{def}}{=} \max\{\gcd(a, b+1), \gcd(a+1, b)\}. \tag{6}$$



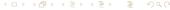


Question: Can there be infinitely many isolated 0's?

Example 1:
$$\mathcal{F}_1 = \{2,3\}$$
 and $\mathcal{F}_2 = \{5,6\}$

1 010100





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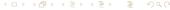




Example 2:
$$\mathcal{F}_1 = \{8,9\}$$
 and $\mathcal{F}_2 = \{15,16\}$

0 0010001001001010





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001000100100100100110011001010100
 1101110011111100111111100
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 11111100111111100111111100
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Example 3:
$$\mathcal{F}_1 = \{6,7\}$$
 and $\mathcal{F}_2 = \{11,12\}$

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Related Theorem for Min-Max Sequences

It is not difficult to prove the following (employing the earlier result).

$\mathsf{Theorem}$

Suppose $\mathcal{F}_1 = \{a, a+1\}$ and $\mathcal{F}_2 = \{b, b+1\}$, and the minimal value in the period is "non-isolated". Then all solutions to

$$y_n = \min_{1 \le i \le r} \left\{ \max_{j \in \mathcal{F}_i} \{ y_{n-j} \} \right\}, \qquad n \ge 0$$
 (7)

are convergent if and only if $g \le 2$.





Suppose $\mathcal{F}_1=\{2,3\}, \mathcal{F}_2=\{4,5\}$ and the initial values in the sequence are 23512.





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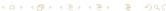




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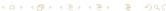




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The case $\mathcal{F}_1 = \{a, a+k\}$ and $\mathcal{F}_2 = \{b, b+k\}$, with k odd.





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The case $\mathcal{F}_1 = \{a, a+k\}$ and $\mathcal{F}_2 = \{b, b+k\}$, with k odd.

Conjecture

Suppose $a, b \ge 1$ and $k \ge 1$ is odd. Then all solutions to

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are convergent if and only if $g \leq 2$, where

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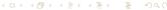
Professor Kenneth S. Berenhaut





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