DISCRETE SERIES L-PACKETS FOR REAL REDUCTIVE GROUPS

1. Overview

Discrete series L-packets play an essential role within the Langlands program for both non-Archimedean and Archimedean local fields. These L-packets often from the base case for inductive style arguments. For real groups, these L-packets (along with the general case) were originally constructed by Langlands over 30 years ago. Langlands' approach was often motivated by certain endoscopic character identities for tempered L-packets (which were also studied by Shelstad).

However, 30 years ago, the ideas of endoscopy were still in their infancy. Many of the arguments were ad-hoc. Nowadays, the endoscopic character identities have been refined. In particular, Kottwitz and Shelstad defined a normalized transfer factor in broad generality. However, their definition has the disadvantage of requiring some choices, including an artificial object known as a z-extension. Adams and Vogan and more recently Kaletha have taken a different viewpoint on the endoscopic character identities. Instead of working with z-extensions, they work with certain covering groups. This viewpoint is more natural in the setting of the Langlands program and requires less artificial choices in defining a normalized transfer factor.

In this article, a modern construction of discrete series L-packets for real groups is given. In particular, the authors make use of covering groups, their normalized transfer factors, and the modern form of endoscopic character identities, all of which were not available to Langlands at the time. The article is well-written and will serve as an excellent entry point into the subject. Furthermore, the article also includes several new results regarding generic essentially discrete series representations.

As someone who largely works with these ideas in the p-adic setting, I have learned a great amount from this article. It has certainly achieved its goal of providing a modern entry point into how the situation work for real groups. I strongly recommend this article for publication in Essential Number Theory.

2. Comments

Here are several comments and suggestions.

(1) P. 1 In the table of contents, the title of §3.3 contains the term "eds" which is not clear at first. I recommend writing out essentially discrete series until eds is introduced.
(2) P. 2 In the last sentence of the third paragraph, the reference [Kal22] appears to reference the "technical work-arounds" rather than the "double covers of endoscopic groups". Maybe move the reference to that point.
(3) P. 3 In the second paragraph, I would also add that $W_{\mathbb{R}}$ denotes the Weil group.

(4) P 3 In the third paragraph, it might be useful to refer to readers to §2.8 for the notion of endoscopic datum.
(5) P. 3 In the third paragraph, I assume the subscript "sr" here means strongly regular.
(6) P. 5 In the third paragraph, I assume the $*$ in $\pi_0(S_\varphi^+)^*$ means representations of this group, correct?
(7) P. 8 In the first paragraph, why is the endoscopic datum "refined"?
(8) P. 8 In the third paragraph, "reps." should be "resp."
(9) P. 10 In the second paragraph, what does $\overline{z}_\sigma$ mean?
(10) P. 10 In the last paragraph, the notation in [Kal16b] seems to be different from the notation here. Could you explain the comparison between the different notations?
(11) P. 11 In the first paragraph, I think "$G_0(\mathbb{C})$" should be "$G(\mathbb{C})$".
(12) P. 11 In the fourth paragraph, "the equivalence classes in induces" should be "the equivalence classes it induces".
(13) P. 14 In the second paragraph, "More generally, $\mathrm{inv}((z_1,\delta_1),(z_2,\delta_2)) = 1, \ldots$ should maybe be "More generally, if $\mathrm{inv}((z_1,\delta_1),(z_2,\delta_2)) = 1, \ldots$ At first, I thought you were asserting that we always have $\mathrm{inv}((z_1,\delta_1),(z_2,\delta_2)) = 1$.
(14) P. 14 In the 5th paragraph, I assume that γ is a strongly regular semi-simple element. It would be helpful to write this somewhere.
(15) P. 14 In the final paragraph, we have a normalizing factor $D(g)$, but I do not think this is defined anywhere in the paper. It would be helpful to say some words about it.
(16) P. 16 In the second paragraph, should $\{\alpha, -\alpha\}$ be $\{\alpha, \alpha^{-1}\}$ since we are taking the roots for the Lie groups?
(17) P. 16 In the second paragraph, "Borel $\mathbb{C}$-subgroup V" should be "Borel $\mathbb{C}$-subgroup B".
(18) P 17 In the third paragraph, remind us that "eds" means essentially discrete series.
(19) P 17 In the first sentence of the proof of Lemma 2.7.1, "Condition 2." should be "Condition 2".
(20) P 19 In the proof of Lemma 2.7.5, there are two instances where ref appears. Presumably, this is the result of a missing reference.
(21) P 20. I think it would be useful to move Remark 2.7.8 to before Lemma 2.7.6.
(22) P. 20 Equation (2.6) is the value of $\Theta(\delta)$ where $\delta \in \mathcal{S}$. To help the reader, I suggest changing "Θ is given on $\mathcal{S}$ by..." to "Θ is given on $\delta \in \mathcal{S}$ by...".
(23) P. 23 For explication, I suggest changing "(4) an L-embedding $\mathcal{H} \to {}^L G$." to "(4) an L-embedding $\eta : \mathcal{H} \to {}^L G$."
(24) P. 24 In the fourth paragraph, I found the use of the notation + confusing. It is in the inverse image in $\widehat{\overline{H}}$. So $Z(\widehat{\overline{H}})^+$ should mean the inverse image of $Z(\widehat{\overline{H}})$ in $\widehat{\overline{H}}$, but this is already in $\widehat{\overline{H}}$, right?
(25) P. 28 In the first paragraph, "becaus e" should be "because".
(26) P. 29 I do not think that the notation $SJ(\gamma, f)$ is explained.
(27) P. 30 In Definition 2.11.1, it would be helpful to refer the reader to Theorem 2.9.1 for the transfers $f^{H_\pm}$ and f^{H_1}.
(28) P. 31 In the first paragraph, in the integral the term f^{H_1} should be $f^{H_\pm}$.

(29) P. 32 In the second paragraph, "distributions that are representably by..." should be "distributions that are represented by...".
(30) P. 33 In the first paragraph, I assume that $\mathcal{U}(\mathfrak{g})$ refers to the universal enveloping algebra. I did not see this written anywhere so it may be useful to say so here.
(31) P. 34 In the last equation in the proof of Proposition 2.12.1, the term "$L_{a^{-1}}f$" should be "$L_{a^{-1}}(f)$".
(32) P. 35 In the third paragraph, $N(\mathbb{R})$ is unipotent, not nilpotent.
(33) P. 38 In the definition of $Q(G(\mathbb{R}))$, "$\mathrm{ad}(G(\mathbb{R}))$" should be "$\mathrm{Ad}(G(\mathbb{R}))$".
(34) P. 38 In the proof of Lemma 3.1.4, "this is a well-know fact" should be "this is a well-known fact".
(35) P. 39 I found it confusing that the Kostant section is denoted by $\mathcal{K}(X) \subseteq \mathfrak{g}$ in the first paragraph while it is also denoted by $\mathcal{K}(X) \subseteq \mathfrak{g}^*$ in the fourth paragraph. It is difficult to figure out later is we are working with the Kostant section in $\mathfrak{g}$ or $\mathfrak{g}^*$.
(36) P. 39 In the sixth paragraph, it would be helpful to remind us that $X = X_1 + \cdots + X_n$ is an element of $\mathfrak{g}^*$ here.
(37) P. 40 In the definition of $AC(W)$, the limit should be $n \to \infty$, not $t \to \infty$.
(38) P. 40 This is less of a comment and more of a personal confusion. In Lemma 3.3.4, why take $\epsilon \in \{1, -1, i, -i\}$ instead of $\epsilon \in \{1, i\}$?
(39) P. 41 In the third paragraph, E and F are used but it is unclear what they are. I think $E = X$ and $F = Y$.
(40) P. 42 It is probably useful to note that Corollary 3.3.5 is false p-adically. This is more for my benefit, but is it clear why Corollary 3.3.5 is expected for real groups? I suppose there are more discrete series for p-adic groups than real groups, but this alone does not account for the corollary.
(41) P. 42 In the proof of Corollary 3.3.5, "$\mathfrak{w}_2$.)" should be "$\mathfrak{w}_2$).".
(42) P. 43 In the proof of Lemma 4.1.1, A^2 denotes $\{x^2 \mid x \in A\}$ rather than $AA = \{xy \mid x, y \in A\}$, correct?
(43) P. 43 In the proof of Lemma 4.1.1, "The subgroup $\varphi(\mathbb{C}^\times)$ of $\widehat{G}$ and consists" should be "The subgroup $\varphi(\mathbb{C}^\times)$ of $\widehat{G}$ consists".
(44) P. 43 In the final paragraph, you say "B is normal in A of index 2". Is it not possible for $B = A$? The same reasoning probably still works in this case.
(45) P. 47 In the fifth paragraph, I think "$(x, \mathrm{Ad}(\overline{g}), j)$" should be "$(x, \mathrm{Ad}(\overline{g}) \circ j)$".
(46) P. 49 In the fourth paragraph, I assume that $e(G)$ is the Kottwitz sign, but I do not think this is written down.
(47) P. 49 In the fourth paragraph, the references "[She82], [She10], [She08b]." maybe should be reordered chronologically "[She82], [She08b], [She10]."
(48) P. 50 After Lemma 5.2.2, it would be helpful to write explicitly "Proof of Step 1" rather than proof. Also the underline of Step 1 can be confused with the closure of $Z_G(\mathbb{R})$.
(49) P. 52 In Step 2, I suggest replacing "Both sides are" with "Both sides of Theorem 5.2.1 are".
(50) P. 53 In the fourth paragraph, there is a missing period at "the isomorphism j_i produces $d\tau$".
(51) P. 56 In the first line, "idelized" should be "idealized".

(52) P. 56 In the first paragraph, “paremter” should be “parameter”.
(53) P. 62 “[Ka1]” maybe should be “[Kal1]”.
(54) P. 63 Maybe “[KS]” should be “[KS12]”.