REPORT ON DISCRETE SERIES L-PACKETS FOR REAL REDUCTIVE GROUPS

The character identity is an important tool in the endoscopy theory of the Langlands correspondence and also has other applications. For real reductive groups, it was developed by several work of Shelsted, Kowittz, Langlands and others some time ago. The main result of this article is to give a conceptual proof on the character identity for discrete series L-packets (Theorem 5.2.1) using some modern developments such as pure and right inner forms. I believe the approach explained in the topic is largely new and uses some insights from some recent work of the first-named author such as rigid inner forms and double covers.

I personally very much like the way of writings and approaches in that article. I believe it fits the aims of Essential Number Theory. On the other hand, it also needs quite a number of prerequisites. Since the journal aims for wide audiences including graduate students, I suggest the authors may include what kind of background knowledge is needed for reading that article.

The main concern is on the structure of the article. It looks slightly awkward for a large portion of the article to introduce preliminaries. It is also not very meaningful for me to read the whole section 2 before going to the main context. For example, Lemma 2.11.2 is used to translate two languages of the character identities in Section 5. Probably it makes sense to put Section 2.11 to Section 5 (but then it will add some pages to that 11 pages) to shorten the 'looking-long' preliminaries. Section 2.10 to 2.12 are designed to prepare Section 5 and it may even make sense to be separated into another recollections for the proofs in Section 5.

I believe the article is not ready for publication because there are several typos. Here are some comments for authors' reference:

(1) Page 3 before (1.1): unique up to?
(2) Page 10 2nd paragraph: should $\overline{z}_\sigma$ be $\overline{z}$?
(3) Page 14 line -16: is γ equal to t?
(4) Page 15 line 6: there are two 'we'
(5) Page 15 line -6: maybe 'denotes'?
(6) Page 16: 'natural bijection', maybe elaborate more since it goes to dual side.
(7) Page 16: 'no canonical L-embedding': maybe add a reference, or maybe give an example if ones to understand what is happening there?
(8) Page 18: arises?
(9) Page 19 line -5: the subscript $\pi|_{\mathcal{H}}$?
(10) Page 19 line -4: ref?
(11) Page 19 line -2: allows?

(12) Page 19 line -1: ref?
(13) Theorem 2.7.7 line 2: add the word 'and' before 'is a compact...'?
(14) Theorem 2.7.7: it is an important theorem and it may be good to recall some notions: τ' and d'_τ or even $q(G_{sc})$
(15) Page 20 line -1: 'is' should be 'if'?
(16) Theorem 2.7.7 The notions could be more self-contained in the theorem, or recall suitable places? τ' and $d_{\tau'}$ are also defined before Definition 2.7.9 and after the theorem.
(17) Page 22 line -16: the formula needs Condition 4 of $\mathcal{G}$
(18) Page 23 (a): but you do not necessarily assume $\mathcal{H}$ is product of the two groups, maybe good to emphasis and elaborate a bit more
(19) Definition 2.7.3: there are two 'in the' in 2.
(20) Lemma 2.7.5: If and only if
(21) Lemma 2.7.5: In Proof of Claim, $\pi|_{\mathcal{H}}$ rather than $\pi|\mathcal{H}$.
(22) Page 26: stably conjugacy is explained in the forth paragraph
(23) Page 26, 5th paragraph: this formula means (2.7)? The explanation is quite concise and I cannot see where the conclusion is obtained at the end.
(24) Page 27 under the diagram: triangle
(25) Page 27 paragraph 5: provisions?
(26) After Theorem 2.10, do you want to specialize to $F = \mathbb{R}$?
(27) Lemma 2.11.2. It looks γ_1 is used for H_1. Do you want add resp. something in Lemma 2.11.?
(28) Page 33: D_T is defined in Section 2.5. In Proposition 2.12.1, maybe one also need to restrict the factor $|D_T|$ to T'?
(29) Few lines before (2.15): $|\nu_a|(b)$ rahter than $|\nu_a(h)|$
(30) Section 3.1: there are two constructions of Whittaker dutum: does one follow [Kos78] while does another one follow from [Vog78]? It also has a formulation related to be defined of the dual of the lie algebra. Maybe explain more on why need different constructions more.
(31) Proposition 3.3.3: The subscript for lim should be n rather than i.
(32) Lemma 3.3.6: $\mathcal{O}_\pi$ is defined in the middle of page 40. $\mathcal{O}_{d\tau}$ seems to be the same as $\mathcal{O}_\pi$. Proposition 3.3.3 seems to be used to conclude the first statement of Lemma 3.3.6 and it may be good to mention explicitly.
(33) Reference AA24: Is the reference the one in arXiv titled 'nilpotent invariants...'. The Appendix in that article has detailed computations for related concepts (wavefront sets and associated varieties).
(34) Page 42 last line of proof of lemma 3.3.6: there are two 'see's
(35) Page 49: good to say explicitly how the character depends on $\mathfrak{m}$, and also please recall the notion $e(G)$
(36) Definition 4.2.1 Recall that $\mathcal{J}(F)$ is defined in Section 2.3.1.

(37) Section 4.4: There are two $\mathcal{J}(F)$, and maybe $\mathcal{J}(\mathbb{R})$ since it is working on a specific case?
(38) Section 4.5, 2nd paragraph: the argument in Section 4.1. Is that the one proving lemma 4.1? Maybe good to be more precise.
(39) Section 4.5, second paragraph: $\mathcal{J}(\mathbb{R})$ rather than $\mathcal{J}(F)$?
(40) Section 4.5: maybe elaborate more on what 'non-covers' are working? In Kal22 Theorem 2.6.2, it uses z-extension. Is it talking the same thing?
(41) Sectin 4.5: does $G_x(\mathbb{R})$ mean the same as $G(\mathbb{R})_x$?
(42) Remark 5.1.2 line 5: there is extra 'are'
(43) Section 5.2 line 3: 'measures'
(44) Page 54 line 7: it is good to point out Lemma 4.3.1 is used
(45) Page 54 line 59: maybe good to recall where J_ζ is defined
(46) Under (5.8): the font for z-extension
(47) Page 59 line -4: in one angle bracket, it is $s_{\mathfrak{m}}$ rather than s?
(48) Why Lemma 5.5.1 completes the proof? If some previous result is used, maybe good to mention explicitly?