Discrete series L-packets for real reductive groups

Contents

1 Recollections and preparations

1.1 The L-group

We review the L-group of a connected reductive group following [Vog93, §2].

Let F be a field. Assume first that F is separably closed. Let G be a connected reductive F-group. Given a Borel pair (T, B) of G one has the based root datum $\text{brd}(T, B, G) = (X^*(T), \Delta, X_*(T), \Delta^\vee)$, where $\Delta \subset X^*(T)$ is the set of B-simple roots for the adjoint action of T on $\text{Lie}(G)$, and $\Delta^\vee \subset X_*(T)$ are the corresponding coroots. For a second Borel pair (T', B'), there is a unique element of $T'(F)\backslash G(F)/T(F)$ that conjugates (T, B) to (T', B'). This element provides an isomorphism $\text{brd}(T, B, G) \to \text{brd}(T', B', G)$. This procedure leads to a system of based root data and isomorphisms, indexed by the set of Borel pairs of G. The limit of that system is the based root datum $\text{brd}(G)$ of G.

One can formalize the notion of a based root datum: we refer the reader to [Spr09, §7.4] for the formal notion of a root datum, to which one has to add a set of simple roots to obtain the formal notion of a based root datum. Based root data can be placed into a category, in which all morphisms are isomorphisms, for the evident notion of isomorphism of based root data. The classification of connected reductive F-groups [Spr09, Theorem 9.6.2, Theorem 10.1.1] can be stated as saying that $G \mapsto \mathrm{brd}(G)$ is a full essentially surjective functor from the category of connected reductive F-groups and isomorphisms to the category of based root data and isomorphisms. Moreover, two morphisms lie in the same fiber of this functor if and only if they differ by an inner automorphism.

Consider now a general field F, let F^s a separable closure, $\Gamma = \mathrm{Gal}(F^s/F)$ the Galois group. Given a connected reductive F-group G, there is a natural action of Γ on the set of Borel pairs of G_{F^s}, and this leads to a natural action of Γ on $\mathrm{brd}(G_{F^s})$. We denote by $\mathrm{brd}(G)$ the based root datum $\mathrm{brd}(G_{F^s})$ equipped with this Γ-action. Given two connected reductive F-groups G_1, G_2, an isomorphism $\xi : G_{1,F^s} \to G_{2,F^s}$ is called an *inner twist*, if $\xi^{-1} \circ \sigma \circ \xi \circ \sigma^{-1}$ is an inner automorphism of G_{1,F^s} for all $\sigma \in \Gamma$. The two groups G_1, G_2 are then called inner forms of each other. The functor $G \mapsto \mathrm{brd}(G)$ from the category of connected reductive F-groups to the category of based root data over F and isomorphisms is again essentially surjective. It maps inner twists to isomorphisms, and two inner twists map to the same isomorphism if they differ by an inner automorphism. The fiber over a given based root datum over F consists of all reductive groups that are inner forms of each other.

Given a based root datum $(X, \Delta, Y, \Delta^\vee)$ over F, its dual $(Y, \Delta^\vee, X, \Delta)$ is also a based rood datum over F. If G is a connected reductive F-group with based root datum $(X, \Delta, Y, \Delta^\vee)$, its dual $\widehat{G}$ is the unique split connected reductive group defined over a chosen base field (we will work with $\mathbb{C}$) with based root datum $(Y, \Delta^\vee, X, \Delta)$. Thus, given a Borel pair $(\widehat{T}, \widehat{B})$ of $\widehat{G}$ and a Borel pair (T, B) of G_{F^s}, one is given an identification $X_*(\widehat{T}) = X^*(T)$ that identifies the Weyl chambers associated to $\widehat{B}$ and B.

To form the L-group, one chooses a pinning $(\widehat{T}, \widehat{B}, \{Y_\alpha\})$ of $\widehat{G}$. The group of automorphisms of $\widehat{G}$ that preserve this pinning is in natural isomorphism with the group of automorphisms of $\mathrm{brd}(\widehat{G})$, hence with that of $\mathrm{brd}(G)$. The Γ-action on $\mathrm{brd}(G)$ then lifts to an action on $\widehat{G}$ by algebraic automorphisms, and ${}^L G = \widehat{G} \rtimes \Gamma$.

When G is quasi-split, (T, B) is an F-Borel pair, and $(\widehat{T}, \widehat{B})$ is a Γ-stable Borel pair of $\widehat{G}$, then the identification $X_*(T) = X^*(\widehat{T})$ is Γ-equivariant.

1.2 Inner forms

Let G be a connected reductive F-group. An *inner twist* of G is a pair (G_1, ξ) where G_1 is a connected reductive F-group and $\xi : G_{F^s} \to G_{1,F^s}$ is an isomorphism, such that for each $\sigma \in \Gamma$ the automorphism $\xi^{-1}\sigma(\xi) = \xi^{-1} \circ \sigma \circ \xi \circ \sigma^{-1}$ of G_{F^s} is inner. An isomorphism of inner twists $(G_1, \xi_1) \to (G_2, \xi_2)$ is a homomorphism $f : G_1 \to G_2$ of F-groups such that $\xi_2^{-1} \circ f \circ \xi_1$ is an inner automorphism of G_{F^s}. From an inner twist (G_1, ξ) we obtain the function $\Gamma \to G/Z(G)$ given by $\sigma \mapsto \xi^{-1}\sigma(x)$. It is an element of $Z^1(F, G/Z(G))$, whose cohomology class depends only on the isomorphism class of (G_1, ξ). In this way the set of isomorphism classes of inner twists is in bijection with $H^1(F, G/Z(G))$.

Following Vogan [Vog93], a *pure inner twist* of G is a triple (G_1, ξ, z), where G_1

is a connected reductive F-group, $\xi : G_{F^s} \to G_{1,F^s}$ is an isomorphism and $z \in Z^1(\Gamma, G)$, subject to $\xi^{-1}\sigma(\xi) = \mathrm{Ad}(\bar{z}_\sigma)$, where $\bar{z} \in Z^1(F, G/Z(G))$ is the image of z under the natural projection $G \to G/Z(G)$. An isomorphism of pure inner twists $(G_1, \xi_1, z_1) \to (G_2, \xi_2, z_2)$ is a pair (f, g) consisting of an isomorphism $f : G_1 \to G_2$ of F-groups and $g \in G_0(F^s)$ such that $\xi_2^{-1} \circ f \circ \xi_1 = \mathrm{Ad}(g)$ and $z_2(\sigma) = gz_1(\sigma)\sigma(g)^{-1}$ for all $\sigma \in \Gamma$. The map $(G_1, \xi, z) \mapsto z$ induces a bijection from the set of isomorphism classes of pure inner twists to $H^1(F, G)$.

There is a lighter notation in which inner twists and pure inner twists can be recorded. It is grounded on the fact that, if (G_1, ξ) is an inner twist of G, with $\bar{z} \in Z^1(F, G/Z(G))$ given by $\bar{z}(\sigma) = \xi^{-1}\sigma(\xi)$, and we let $G_{\bar{z}}$ be the algebraic F-group obtained from G by twisting the F-structure by $\bar{z}$, then ξ becomes an isomorphism of F-groups $G_{\bar{z}} \to G_1$, in fact an isomorphism of inner twists $(G_{\bar{z}}, \mathrm{id}) \to (G_1, \xi)$. Therefore the map $\bar{z} \mapsto (G_{\bar{z}}, \mathrm{id})$ induces an injection from $Z^1(F, G/Z(G))$ into the class of all inner twists of G which meets every isomorphism class. Two elements of $Z^1(F, G/Z(G))$ map into the same isomorphism class if and only if they lie in the same orbit for the action of $G(F^s)$ on $Z^1(F, G/Z(G))$ given by $(g \cdot z)(\sigma) = gz(\sigma)\sigma(g)^{-1}$. The orbits space for this action is $H^1(F, G/Z(G))$, and this gives the same identification between that orbit space and the set of isomorphism classes of inner twists as above.

The same simplification applies to the notion of pure inner twist. There we work with the set $Z^1(F, G)$ and obtain the embedding $z \mapsto (G_z, \mathrm{id}, z)$ from that set into the class of pure inner twists of G. Again the orbit space for the action of $G(F^s)$ on $Z^1(F, G)$ by $(g \cdot z)(\sigma) = gz(\sigma)\sigma(g)^{-1}$ equals $H^1(F, G)$ and is identified with the set of isomorphism classes of pure inner twists.

We now take $F = \mathbb{R}$, hence $F^s = \mathbb{C}$. Then $\Gamma = \{1, c\}$, where c is complex conjugation. An element of $Z^1(F, G)$ can be identified with the image of c, which is an element of $G(\mathbb{C})$. We can also think of this as an element of the coset $G(\mathbb{C}) \rtimes c \subset G(\mathbb{C}) \rtimes \Gamma$. The elements of $Z^1(F, G)$ correspond precisely to the elements of $G(\mathbb{C}) \rtimes c$ of order 2.

In [ABV92] the notion of strong real form was introduced. This is an element $\delta \in G(\mathbb{C}) \rtimes c$, such that δ^2 is a finite order element of $Z(G)(\mathbb{C})$. In [Kal16b] this notion was given a cohomological interpretation and was extended to non-archimedean local fields. Following the latter reference, a *rigid inner twist* of G is a triple (G_1, ξ, z), where G_1 is a connected reductive $\mathbb{R}$-group, $\xi : G_\mathbb{C} \to G_{1,\mathbb{C}}$ is an isomorphism and $z \in Z^1_{\text{bas}}(\mathcal{E}_R, G)$, subject to $\xi^{-1}\sigma(\xi) = \mathrm{Ad}(\bar{z}_\sigma)$. Here $1 \to u_\mathbb{R}(\mathbb{C}) \to \mathcal{E}_R \to \Gamma \to 1$ is a certain extension of the Galois group, $Z^1_{\text{bas}}(\mathcal{E}_\mathbb{R}, G)$ denotes the group of continuous 1-cocycles $\mathcal{E}_\mathbb{R} \to G(\mathbb{C})$ whose restriction to $u_\mathbb{R}(\mathbb{C})$ takes values in $Z(G)(\mathbb{C})$, and $\bar{z}$ is again the image of z under the natural projection map $G \to G/Z(G)$; it factors through the quotient $\mathcal{E}_\mathbb{R} \to \Gamma$. An isomorphism of pure inner twists $(G_1, \xi_1, z_1) \to (G_2, \xi_2, z_2)$ is a pair (f, g) consisting of an isomorphism $f : G_1 \to G_2$ of $\mathbb{R}$-groups and $g \in G_0(\mathbb{C})$ such that $\xi_2^{-1} \circ f \circ \xi_1 = \mathrm{Ad}(g)$ and $z_2(e) = gz_1(e)\sigma_e(g)^{-1}$ for all $e \in \mathcal{E}_\mathbb{R}$, where $\sigma_e \in \Gamma$ is the image of e. The set of isomorphism classes of rigid inner twists is in bijection with $H^1_{\text{bas}}(\mathcal{E}_\mathbb{R}, G)$. The equivalence between the cohomology classes in $H^1_{\text{bas}}(\mathcal{E}_\mathbb{R}, G)$ and the elements $G(\mathbb{C}) \rtimes c$ whose square is a finite order element of $Z(G)(\mathbb{C})$ is less obvious than in the Galois case, and is discussed in [Kal16b, §5.2].

Each of these three notions partitions the set of isomorphism classes of connected reductive $\mathbb{R}$-groups into equivalence classes. The notion of inner twist is the most classical, stemming from the classification of reductive groups, and each equivalence class has exactly one quasi-split member. Unfortunately, this notion is not rigid enough for representation theory – the group of automorphisms of an inner twist (G_1, ξ) is $(G_1/Z(G_1))(\mathbb{R})$, and the conjugation by such an element can be an outer automorphism of the Lie group $G_1(\mathbb{R})$.

The notion of pure inner twist is sufficiently rigid – the group of automorphisms of a pure inner twist (G_1, ξ, z) is $G_1(\mathbb{R})$, hence acts by inner automorphisms of $G_1(\mathbb{R})$. Unfortunately, the equivalence classes induced by this notion are generally smaller, and not all of them contain a quasi-split member.

The notion of a rigid inner twist is again sufficiently rigid, and in addition the equivalence classes in induces coincide with those induced by the notion of inner twist. This will be the notion that we will use.

For further discussion of this topic we refer the reader to [Vog93] and [Kal16a].

Given an inner twist $\xi : G \to G_1$, the isomorphism ξ induces an isomorphism $\mathrm{brd}(G) \to \mathrm{brd}(G_1)$, which is Γ-equivariant, even though ξ itself is not. This leads to an identification of dual groups $\widehat{G} = \widehat{G}_1$ and L-groups ${}^LG = {}^LG_1$.

1.3 Admissible embeddings of tori

Let F be a field and let G be a connected reductive F-group. Let $\widehat{G}$ be its dual group, defined over any base field, which we take to be $\mathbb{C}$, equipped with a Γ-action.

Let S be an F-torus. We recall from [Kal19b, §5.1] that, given a Γ-stable $\widehat{G}$-conjugacy class $\widehat{J}$ of embeddings $\widehat{S} \to \widehat{G}$ whose images are maximal tori, there is an associated Γ-stable $G(F^s)$-conjugacy class J of embeddings $S \to G$. To obtain J, we first assume that G is quasi-splut. Fix Γ-stable Borel pairs $(\widehat{T}, \widehat{B})$ and (T, B) in $\widehat{G}$ and G, respectively. Thus we have the identifications $X_*(T) = X^*(\widehat{T})$ and $\Omega(T, G) = \Omega(\widehat{T}, \widehat{G})$ as Γ-modules. There is $\widehat{\jmath} \in \widehat{J}$ with image $\widehat{T}$ and we obtain the isomorphism $X_*(T) = X^*(\widehat{T}) \to X^*(\widehat{S}) = X_*(S)$, hence an isomorphism $j : S_{F^s} \to T_{F^s}$. We let J be the $G(F^s)$-conjugacy class of the composition of j with the inclusion $T \to G$.

For $\sigma \in \Gamma$ the embedding $\sigma(\widehat{\jmath}) = \sigma_{\widehat{G}} \circ \widehat{\jmath} \circ \sigma_{\widehat{S}}$ is $\widehat{G}$-conjugate to $\widehat{\jmath}$, but has the same image because $\widehat{T}$ is Γ-stable, so there exists $w \in \Omega(\widehat{T}, \widehat{G})$ such that $\sigma(\widehat{\jmath}) = w \circ \widehat{\jmath}$. By construction of j we have $\sigma(j) = w \circ j$, using $\Omega(T, G) = \Omega(\widehat{T}, \widehat{G})$, and conclude that J is Γ-stable. Since all Γ-stable Borel pairs of $\widehat{G}$ are conjugate under $\widehat{G}^\Gamma$, and all Γ-stable Borel pairs of G are conjugate under $G(F)$, the construction of J does not depend on the choices of Borel pairs.

Now drop the assumption that G is quasi-split. We consider an inner twist $\xi : G_0 \to G$ with G_0 quasi-split. It gives an identification $\widehat{G}_0 = \widehat{G}$ and we obtain from $\widehat{J}$ a Γ-stable $G_0(F^s)$-conjugacy class J_0 of embeddings $S \to G_0$. Composing with ξ we obtain the desired $G(F^s)$-conjugacy class J of embeddings $S \to G$. It is Γ-stable, because the $G(F^s)$-conjugacy class of ξ is.

This completes the construction of J. We will refer to it as the set of *admissible* embeddings $S \to G$. If we want to record the group G we will write J^G.

Write $J(F)$ for the set of Γ-fixed points in J, i.e. the set of embeddings $S \to G$ defined over F. When $G = G_0$ is quasi-split, a result of Kottwitz [Kot82, Corollary 2.2] guarantees that this set is non-empty. For a general G this set may be empty. The group $G(F)$ acts on $J(F)$ by conjugation and we will write $J(F)/G(F)$ for the set of orbits under this action.

Given any two elements $j_1, j_2 \in J(F)$ there exists $g \in G(F^s)$ such that $j_2 = \mathrm{Ad}(g) \circ j_1$. The map $\sigma \mapsto j_1^{-1}(g^{-1}\sigma(g))$ is a 1-cocycle of Γ valued in $S(F^s)$. Its class is independent of g and will be denoted by $\mathrm{inv}(j_1, j_2)$. For a fixed $j \in J(F)$ the map $j_2 \mapsto \mathrm{inv}(j_1, j_2)$ is a bijection between $J(F)/G(F)$ and $\ker(H^1(j_1) : H^1(F, S) \to H^1(F, G))$.

One can combine multiple inner forms in this discussion to obtain a more uniform picture. This requires the use of pure or rigid inner twists. Fix a quasi-split group G_0. We start with the case of pure inner twists and consider tuples (G, ξ, z, j), where (G, ξ, z) is a pure inner twist of G_0 and $j \in J^G(F)$. An isomorphism $(G_1, \xi_1, z_1, j_1) \to (G_2, \xi_2, z_2, j_2)$ between such tuples is an isomorphism $(f, g) : (G_1, \xi_1, z_1) \to (G_2, \xi_2, z_2)$ of inner twists that satisfies $j_2 = f \circ j_1$. Let $\mathcal{J}(F)$ be the category whose objects are these tuples and whose morphisms are these isomorphisms. Given two tuples as above there exists $g \in G_0(F^s)$ such that $j_2 = \xi_2 \circ \mathrm{Ad}(g) \circ \xi_1^{-1} \circ j_1$. The map $\sigma \mapsto j_1^{-1}(g^{-1}z_2(\sigma)\sigma(g)z_1(\sigma)^{-1})$ which side should z_1 be on? is a 1-cocycle $\Gamma \to S(F^s)$ whose class $\mathrm{inv}(j_2, j_1)$ depends only on the isomorphism classes of the tuples. Fixing the tuple (G_1, ξ_1, z_1, j_1), the map $(G_2, \xi_2, z_2, j_2) \mapsto \mathrm{inv}(j_1, j_2)$ induces a bijection from the set of isomorphism classes $[\mathcal{J}(F)]$ to $H^1(F, S)$.

The individual groups can be extracted from the combined picture as follows. For a fixed (G, ξ, z) we can view the set $J^G(F)$ as the set of objects in a category, with set of morphisms $j_1 \to j_2$ given by $\{g \in G(F) | j_2 = \mathrm{Ad}(g) \circ j_1\}$. Then we obtain an embedding (fully faithful functor) from $J^G(F)$ to $\mathcal{J}(F)$. The category $\mathcal{J}(F)$ decomposes into blocks, indexed by $H^1(F, G_0)$, and each block is equivalent to $J^G(F)$ for some (G, ξ, z). In particular, the set of isomorphism classes $[\mathcal{J}(F)]$ is the disjoint union $\bigcup_{H^1(F,G_0)}(J^G(F)/G(F))$. If we choose $j_0 \in J^{G_0}(F)$, then under the bijection $[\mathcal{J}(F)] \to H^1(F, S)$ given by $(G, \xi, z, j) \mapsto \mathrm{inv}(j, j_0)$, an individual $J^G(F)/G(F)$ coming from (G, ξ, z) is mapped bijectively onto the fiber of $H^1(j_0) : H^1(F, S) \to H^1(F, G)$ over the class of z.

The bijection $[\mathcal{J}(F)] \to H^1(F, S)$ coming from fixing (G, ξ, z, j) can be understood as the orbit map for a natural action of $H^1(F, S)$ on $[\mathcal{J}(F)]$ that does not depend on any choices. To see this action it is easier to consider the simplified notation, where a pure inner twist of G_0 is understood simply as an element $z \in Z^1(F, G_0)$, in the sense that it corresponds to (G_z, id, z), where G_z is the F-group obtained by twisting the rational structure of G_0 by z. Then $\mathcal{J}$ consists of pairs (z, j), where $z \in Z^1(F, G_0)$ and $j \in J^{G_0}$. Such a pair lies in $\mathcal{J}(F)$ if and only if $j \in J^{G_z}(F)$, which is explicitly given as $\mathrm{Ad}(z(\sigma))\sigma_{G_0} \circ j \circ \sigma_S^{-1} = j$. The group $G_0(F^s)$ acts on $\mathcal{J}$ by $g \cdot (z, j) = (gz(\sigma)\sigma(g)^{-1}, \mathrm{Ad}(g) \circ j)$. This action preserves $\mathcal{J}(F)$ and the orbit space is $[\mathcal{J}(F)]$. We introduce the action of $Z^1(F, S)$ on $\mathcal{J}(F)$ as $x \cdot (z, j) = (j(x) \cdot z, j)$ for $x \in Z^1(F, S)$ and $(z, j) \in \mathcal{J}(F)$. One checks directly that the actions of $Z^1(F, S)$ and $G(F^s)$ on $\mathcal{J}(F)$ commute and that the action of $S(F^s)$ on $Z^1(F, S)$ is compatible with the action of $Z^1(F, S)$ on $\mathcal{J}(F)$ in the sense that $(s \cdot x) \cdot (z, j) = j(s) \cdot (x \cdot (z, j))$. Thus we obtain an action of $H^1(F, S)$ on $[\mathcal{J}(F)]$.

Lemma 1.3.1. *The above action is simply transitive.*

Proof. For simplicity, let $x \in Z^1(F, S)$, $g \in G_0(F^s)$, and $(z, j) \in \mathcal{J}(F)$ be such that $g \cdot (z, j) = x \cdot (z, j)$. Then $j = \mathrm{Ad}(g) \circ j$ from which we see that $g = j(s)$

for some $s \in S(F^s)$. We also have $j(s)z(\sigma)\sigma_{G_0}(j(s))^{-1} = j(x(\sigma)) \cdot z(\sigma)$ and multiplying on the right by $z(\sigma)^{-1}$ and using that $(z, j) \in \mathcal{J}(F)$ we conclude $j(s \cdot \sigma_S(s)^{-1}) = j(x(\sigma))$, thus $[x] = 1$ in $H^1(F, S)$, as desired.

For transitivity, we need to show that any two elements of $[\mathcal{J}(F)]$ are in the same $H^1(F, S)$ orbit. Since the members of J^{G_0} are all conjugate under $G_0(F^s)$ we may represent the two elements of $[\mathcal{J}(F)]$ by (z_1, j) and (z_2, j) with the same j. The fact that both lie in $\mathcal{J}(F)$ implies that $\mathrm{Ad}(z_1(\sigma)) \circ \sigma_{G_0} \circ j = \mathrm{Ad}(z_2(\sigma)) \circ \sigma_{G_0} \circ j$. It follows that $\sigma_{G_0}^{-1}(z_2(\sigma)^{-1} z_1(\sigma))$ lies in the image of j, and we write it as $j(\sigma_S^{-1}(x(\sigma)))$ for some $x(\sigma) \in S(F^s)$. Using that $(z_1, j), (z_2, j) \in \mathcal{J}(F)$ we see that $j(x(\sigma)) = z_1(\sigma) z_2(\sigma)^{-1}$. From this one easily checks that $x \in Z^1(F, S)$ and concludes $(z_1, j) = x \cdot (z_2, j)$, as desired. □

The same discussion applies to the setting of rigid inner forms when the base field is local. One should only replace $H^1(F, -)$ with $H^1(u_F \to \mathcal{E}_F, Z(G) \to -)$.

1.4 Representations and elements in inner forms

In §1.3 we reviewed the idea of grouping the various admissible embeddings of a torus S into the different inner forms of a fixed group G_0, thereby obtaining a set $\mathcal{J}(F)$ with an action of $G_0(F^s)$ on it. Here we follow the same idea, but group elements or representations.

We focus on $F = \mathbb{R}$, although the same procedure applies for any F of interest. Let $\tilde{\Pi}$ be the set of pairs (z, π), where $z \in Z^1(F, G_0)$ and π is an isomorphism class of irreducible admissible representations of $G_z(F)$, where G_z is again the $\mathbb{R}$-group obtained by twisting the $\mathbb{R}$-structure of G_0 by z. The group $G_0(\mathbb{C})$ acts on this set by $g \cdot (z, \pi) = (gz(\sigma)\sigma(g)^{-1}, \pi \circ \mathrm{Ad}(g)^{-1})$. Note that $\mathrm{Ad}(g) : G_z \to G_{gz(\sigma)\sigma(g)^{-1}}$ is an isomorphism of algebraic $\mathbb{R}$-groups. Let $\Pi = \tilde{\Pi}/G_0(\mathbb{C})$. This is the set of isomorphism classes of irreducible admissible representations of pure inner forms of G_0. The pairs $(z, \pi) \in \tilde{\Pi}$ with a fixed z correspond to all isomorphism classes of representations of $G_z(\mathbb{R})$.

1.5 Weyl denominators

Let G be a connected reductive $\mathbb{R}$-group and $T \subset G$ a maximal $\mathbb{R}$-torus. One can consider the function $T(\mathbb{R}) \to \mathbb{R}$ defined as

$$D_G(t) = \prod_{\alpha \in R(T,G)} (1 - \alpha(t)).$$

In this paper we will normalize orbital integrals and characters by multiplying them by $|D_G(t)|^{1/2}$. Thus, for $t \in T(\mathbb{R})$ strongly regular and $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$ we set

$$J(t, f) = |D_G(t)| \int_{G(\mathbb{R})/T(\mathbb{R})} f(gtg^{-1}) dg,$$

while for π admissible representation of $G(\mathbb{R})$ we have

$$J(t, \pi) = |D_G(t)| \Theta_\pi(t),$$

where Θ_π is the character function of π. This has the advantage that the resulting functions remain bounded as t approaches singular elements in $T(\mathbb{R})$.

A key role in this paper will be played by a function D_B which has the property that $|D_B| = |D_G|^{1/2}$. To see this function, let us interpret D_G as an element of the group ring $\mathbb{Z}[Q]$, where $Q \subset X^*(T) \otimes \mathbb{Q}$ is the root lattice, we we write the group operation on Q multiplicatively. Given a Borel $\mathbb{C}$-subgroup $B \subset G$ containing T we write $\alpha > 0$ when α is a B-positive root, and define

$$D'_B = \prod_{\alpha>0}(1-\alpha^{-1}) \in \mathbb{Q}[Q], \qquad D_B = \prod_{\alpha>0}(\alpha^{1/2}-\alpha^{-1/2}) \in \mathbb{Q}[Q]. \tag{1.1}$$

In $\mathbb{Q}[Q]$ we have the identity

$$D_B = \rho \cdot D'_B, \tag{1.2}$$

where $\rho = \prod_{\alpha>0} \alpha^{1/2} \in Q$. This implies

$$D_G = D'_B \cdot D'_{\bar{B}} = D_B \cdot D_{\bar{B}},$$

where $\bar{B}$ is the Borel subgroup opposite to B. Moreover, for $w \in \Omega(T, G)$ we have

$$wD_B = D_{w^{-1}Bw} = \operatorname{sgn}(w) D_B. \tag{1.3}$$

In particular, $|D_B|$ is independent of the choice of B and hence $|D_G|^{1/2} = |D_B|$, provided we can interpret D_B as a function on $T(\mathbb{R})$.

It is clear that D'_B is a function on $T(\mathbb{R})$. If we want to interpret D_B as a function of $T(\mathbb{R})$, the occurrence of $\alpha^{1/2}$ in the formula causes a problem. From (1.2) we see that D_B will be a function of $T(\mathbb{R})$ if and only if ρ is, which is equivalent to the element ρ lying in $X^*(T)$. This is always the case when G is semi-simple and simply connected, but can fail in general. To remedy this situation, one can introduce a double cover of $T(\mathbb{R})$, which will be discussed in the next section.

1.6 Double covers of tori and L-embeddings

Let G be a connected reductive $\mathbb{R}$-group and let $T \subset G$ be a maximal torus. An obstruction to the element D_B defining a function on $T(\mathbb{R})$ is the fact that $\rho \in \frac{1}{2}X^*(T)$ may not lie in $X^*(T)$. To remedy this, Adams–Vogan introduce in [AV92], [AV16] the ρ-double cover $T(\mathbb{R})_\rho$ as the pull-back of the diagram

$$T(\mathbb{R}) \xrightarrow{\rho^2} \mathbb{C}^\times \xleftarrow{(-)^2} \mathbb{C}^\times,$$

which comes equipped with a natural character $\rho : T(\mathbb{R})_\rho \to \mathbb{C}^\times$, namely the projection onto the right factor $\mathbb{C}^\times$. By construction we have an exact sequence

$$1 \to \{\pm 1\} \to T(\mathbb{R})_\rho \to T(\mathbb{R}) \to 1$$

and ρ is a genuine character, i.e. $\rho(-x) = -\rho(x)$ for $x \in T(\mathbb{R})_\rho$, where $-x$ denote the product of x and the element -1.

While the double cover $T(\mathbb{R})_\rho$ appears to depend on ρ, this is actually not so. Indeed, for any other Borel $\mathbb{C}$-subgroup B' we have $\rho'/\rho \in X^*(T)$, which allows us to define the genuine character $\rho' : T(\mathbb{R})_\rho \to \mathbb{C}^\times$ as $\rho \cdot (\rho'/\rho)$. Combining this character with the natural projection $T(\mathbb{R})_\rho \to T(\mathbb{R})$ gives a map from $T(\mathbb{R})_\rho$ to the diagram defining $T(\mathbb{R})_{\rho'}$, hence an isomorphism $T(\mathbb{R})_\rho \to T(\mathbb{R})_{\rho'}$.

To emphasize the independence of $T(\mathbb{R})_\rho$ on ρ, and emphasize the dependence on the ambient group G, we will write $T(\mathbb{R})_G$ for this cover. For each Borel $\mathbb{C}$-subgroup B we have the genuine character $\rho_B : T(\mathbb{R})_G \to \mathbb{C}^\times$.

In this paper we will be particularly interested in the case when T is elliptic. Then there is a different way to obtain the ρ-cover that generalizes to all local fields, as discussed in [Kal19a]. One first defines the "big cover" of $T(\mathbb{R})$ as follows. Each root provides a homomorphism $\alpha : T(\mathbb{R}) \to \mathbb{S}^1$. Combining these homomorphisms for a pair $A = \{\alpha, -\alpha\}$ provides a homomorphism $A : T(\mathbb{R}) \to \mathbb{S}^1_-$, where $\mathbb{S}^1_- \subset \mathbb{S}^1 \times \mathbb{S}^1$ is the kernel of the product map $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$. Define the "big cover" as the pull-back of

$$T(\mathbb{R}) \xrightarrow{(\alpha)} \prod \mathbb{S}^1_- \xleftarrow{z/\bar{z}} \prod (\mathbb{C}^\times/\mathbb{R}_{>0})_- \tag{1.4}$$

where the products run over the set of pairs $A = \{\alpha, -\alpha\}$ consisting of a root and its negative, and $(\mathbb{C}^\times/\mathbb{R}_{>0})_-$ denotes analogously the anti-diagonal in $(\mathbb{C}^\times/\mathbb{R}_{>0}) \times (\mathbb{C}^\times/\mathbb{R}_{>0})$. The result is an extension

$$1 \to \prod \{\pm 1\} \to T(\mathbb{R})_{GG} \to T(\mathbb{R}) \to 1.$$

Under the isomorphism $\mathbb{S}^1 \to \mathbb{C}^\times/\mathbb{R}_{>0}$ the map $z/\bar{z}$ becomes the squaring map, and we see that $T(\mathbb{R})_{GG}$ is equipped with a character $\alpha^{1/2} : T(\mathbb{R})_{GG} \to \mathbb{S}^1$ for each root α, and that $\beta^{1/2} = (\alpha^{1/2})^{-1}$ whenever $\beta = \alpha^{-1}$. There is an obvious surjective homomorphism $T(\mathbb{R})_{GG} \to T(\mathbb{R})_G$ whose kernel is the kernel of the multiplication map $\prod\{\pm 1\} \to \{\pm 1\}$. The function $\alpha^{1/2} - \alpha^{-1/2}$ is well-defined on the big cover, while for any choice Borel $\mathbb{C}$-subgroup V the function

$$D_B := \prod_{\alpha > 0} (\alpha^{1/2} - \alpha^{-1/2})$$

descends to the double cover $T(\mathbb{R})_G$.

The action of the Weyl group $\Omega_G(T)(\mathbb{R})$ lifts naturally to an action on $T(\mathbb{R})_G$, even on $T(\mathbb{R})_{GG}$, because $\Omega_G(T)(\mathbb{R})$ acts naturally on each term in (1.4). The identity (1.3) holds for this function.

What makes the double cover $T(\mathbb{R})_G$ very useful in the setting of the Langlands program is the fact that there is an associated L-group ${}^LT_G$, as well as a *canonical* $\widehat{G}$-conjugacy class of L-embeddings ${}^LT_G \to {}^LG$, cf. [Kal19a, §4.1]. The property of the L-group ${}^LT_G$ is that the set of $\widehat{T}$-conjugacy classes of L-homomorphisms $W_\mathbb{R} \to {}^LT_G$ is in natural bijection with the set of genuine characters of $T(\mathbb{R})_G$. Therefore, any L-parameter for G that factors through the image of the embedding of ${}^LT_G$ provides in a canonical way an $\Omega_G(T)(\mathbb{R})$-orbit of genuine characters of $T(\mathbb{R})_G$.

In contrast to ${}^LT_G$, there is generally no canonical L-embedding ${}^LT \to {}^LG$. In fact, if the Galois form of ${}^LT$ is used, there is generally no L-embedding ${}^LT \to {}^LG$ at all, let alone a canonical one. If the Weil form of ${}^LT$ is used, then there always do exist L-embeddings ${}^LT \to {}^LG$, but there is generally no canonical choice. If one chooses a genuine character of $T(\mathbb{R})_G$, then the pointwise product of its L-parameter $W_\mathbb{R} \to {}^LT_G$ with the natural inclusion $\widehat{T} \to {}^LT_G$ does lead to an L-isomorphism ${}^LT \to {}^LT_G$ between the Weil forms of the L-groups for $T(\mathbb{R})$ and $T(\mathbb{R})_G$. Composing this isomorphism with the canonical L-embedding ${}^LT_G \to {}^LG$ provides an L-embedding ${}^LT \to {}^LG$, and every L-embedding arises from this construction. A convenient choice for a genuine character on $T(\mathbb{R})_G$ is the character ρ associated to some Borel $\mathbb{C}$-subgroup of G containing T.

It is worth pointing out that all L-embeddings ${}^LT \to {}^LG$, as well as all L-embeddings ${}^LT_G \to {}^LG$, that extend a fixed embedding $\widehat{\jmath} : \widehat{T} \to \widehat{G}$, have the same image, namely

$$\{x \in N_{{}^LG}(\widehat{T}) \mid x \cdot \widehat{\jmath}(t) \cdot x^{-1} = \widehat{\jmath}(\sigma_x(t))\ \forall t \in \widehat{T}\}, \tag{1.5}$$

where $\sigma_x \in \Gamma$ is the image of x under the natural projection ${}^LG \to \Gamma$.

1.7 Essentially square-integrable representations

Let G be a connected reductive $\mathbb{R}$-group. An essentially square integrable representation of $G(\mathbb{R})$ is one which, after possibly tensoring with a continuous character $G(\mathbb{R}) \to \mathbb{C}^\times$, has a unitary central character, and such that every matrix coefficient is square-integrable on $G(\mathbb{R})/Z_G(\mathbb{R})$ (since the central character is unitary, the absolute value of a matrix coefficient is trivial on $Z_G(\mathbb{R})$).

Harish-Chandra has shown that the set of isomorphism classes of essentially square-integrable representations is in bijection with the set of $G(\mathbb{R})$-conjugacy classes of pairs (S, τ), where $S \subset G$ is an elliptic maximal torus, and τ is a genuine character of the double cover $S(\mathbb{R})_G$ whose differential is regular. This bijection is characterized by the fact that the character function of the representation corresponding to (S, τ), evaluated at a regular element $\delta \in S(\mathbb{R})$, is given by

$$(-1)^{q(G)} \sum_{w \in N(S,G)(\mathbb{R})/S(\mathbb{R})} \frac{\tau}{d_\tau}(w\delta) = (-1)^{q(G)} \sum_{w \in N(S,G)(\mathbb{R})/S(\mathbb{R})} \frac{\tau'}{d'_\tau}(w\dot{\delta}). \tag{1.6}$$

We explain the notation. Pull back τ to a character of $S_{\text{sc}}(\mathbb{R})_G$, where S_{sc} is the preimage of S in the universal cover G_{sc} of the derived subgroup of G. The cover $S_{\text{sc}}(\mathbb{R})_G$ splits canonically, because ρ is divisible by 2 in $X^*(S_{\text{sc}})$. Therefore τ provides a character τ_{sc} of $S_{\text{sc}}(\mathbb{R})$. This being a compact torus, τ_{sc} is an algebraic character, i.e. an element of $X^*(S_{\text{sc}})$, and coincides with its differential, which is still regular. Thus τ specifies a choice of positive roots, i.e. a Borel $\mathbb{C}$-subgroup B containing S. Write D_τ in place of D_B for the Weyl denominator (1.1). Since our convention (cf. §1.5) is to normalize orbital integrals and characters by the absolute value of this denominator, we will only need $d_\tau = \arg D_\tau$. Both τ and d_τ are genuine functions of $S(\mathbb{R})_G$, so their quotient $\Theta := \tau/d_\tau$ descends to $S(\mathbb{R})$.

In the second sum we have set $\tau' = \tau \cdot \rho_B^{-1}$, and $d'_\tau = d_\tau \cdot \rho_B^{-1}$, cf. (1.2). In this way, both numerator and denominator are functions of $S(\mathbb{R})$. Note that ρ_B takes values in $\mathbb{S}^1$ because S is elliptic.

1.8 Endoscopic groups and double covers

The notion of endoscopic data is introduced in [LS87, §1.2], and is a variation of the notion of endoscopic pairs or endoscopic triples discussed in [Kot84] and [Kot86].

It can be described equivalently as follows. An endoscopic datum for G is a tuple $(H, s, \mathcal{H}, \eta)$ consisting of

(1) a quasi-split connected reductive group H,

(2) an extension $1 \to \widehat{H} \to \mathcal{H} \to \Gamma \to 1$ of topological groups,

(3) a semi-simple element $s \in Z(\widehat{H})$, and

(4) an L-embedding $\mathcal{H} \to {}^L G$.

It is required that

(a) the extension $\mathcal{H}$ admits a splitting by a continuous group homomorphism $\Gamma \to \mathcal{H}$,

(b) the homomorphism $\Gamma \to \mathrm{Out}(\widehat{H})$ provided by $\mathcal{H}$ coincides with the one provided by the extension ${}^L H$,

(c) η identifies $\widehat{H}$ with the identity component of the centralizer of $\eta(s)$ in $\widehat{G}$,

(d) there exists $z \in Z(\widehat{G})$ such that $s\eta^{-1}(z) \in Z(\widehat{H})^\Gamma$.

The map η produces a Γ-equivariant embedding $Z(\widehat{G}) \to Z(\widehat{H})$, that we will use without explicit notation.

An isomorphism $(H_1, s_1, \mathcal{H}_1, \eta_1) \to (H_2, s_2, \mathcal{H}_2, \eta_2)$ is an element $g \in \widehat{G}$ that satisfies the following properties. First, $\mathrm{Ad}(g)\eta_1(\mathcal{H}_1) = \eta_2(\mathcal{H}_2)$. In particular, $\eta_2^{-1} \circ \mathrm{Ad}(g) \circ \eta_1$ is an L-isomorphism $\mathcal{H}_1 \to \mathcal{H}_2$, and restricts to a Γ-equivariant isomorphism $Z(\widehat{H}_1) \to Z(\widehat{H}_2)$. The second condition is that the resulting isomorphism $\pi_0(Z(\widehat{H}_1)/Z(\widehat{G})) \to \pi_0(Z(\widehat{H}_2)/Z(\widehat{G}))$ maps the coset of s_1 to the coset of s_2.

In this paper we are working with pure (resp. rigid) inner twists, and this necessitates a slight refinement of the notion of endoscopic datum. A *pure refined* endoscopic datum is one in which it is required $s \in Z(\widehat{H})^\Gamma$ in point (3), and this eliminates the need for condition (d). An isomorphism of such data is required to map the coset of s_1 to the coset of s_2 under $\pi_0(Z(\widehat{H}_1)^\Gamma) \to \pi_0(Z(\widehat{H}_2)^\Gamma)$, without dividing by $Z(\widehat{G})$. A rigid refined endoscopic datum replaces $s \in Z(\widehat{H})^\Gamma$ by $\dot{s} \in Z(\widehat{\bar{H}})^+$. An isomorphism of such data is required to map the coset of $\dot{s}_1$ to the coset of $\dot{s}_2$ under $\pi_0(Z(\widehat{\bar{H}_1})^+) \to \pi_0(Z(\widehat{\bar{H}_2})^+)$.

Given an L-parameter $\varphi : W_\mathbb{R} \to {}^L G$ and a semi-simple element $s \in S_\varphi$, where $S_\varphi = \mathrm{Cent}(\varphi, \widehat{G})$, one obtains a pure refined endoscopic datum as follows. Set $\widehat{H} = \mathrm{Cent}(s, \widehat{G})^\circ$. The homomorphism $\varphi : W_\mathbb{R} \to \mathrm{Cent}(s, \widehat{G}) \to \mathrm{Out}(\widehat{H})$ factors through the projection $W_\mathbb{R} \to \Gamma$. There is a unique (up to isomorphism) quasi-split connected reductive $\mathbb{R}$-group H with dual group $\widehat{H}$ such that the homomorphism $\Gamma \to \mathrm{Out}(H) = \mathrm{Out}(\widehat{H})$ induced by the $\mathbb{R}$-structure of H matches the one induced by φ. Set $\mathcal{H} = \widehat{H} \cdot \varphi(W_\mathbb{R})$, and let η be the tautological inclusion $\mathcal{H} \to {}^L G$. In the rigid setting, the same construction works starting with $\dot{s} \in S_\varphi^+$, where S_φ^+ is the preimage in $\widehat{\bar{G}}$ of S_φ.

By construction the parameter φ takes values in $\mathcal{H}$. However, the extensions $\mathcal{H}$ and ${}^L H$ of Γ by $\widehat{H}$ need not be isomorphic, and even if they are, there is no natural isomorphism between them. Therefore, φ is *not* a parameter for H in any natural way. There are two ways to remedy this situation.

The classical approach is to choose a z-extension $H_1 \to H$ and an L-embedding $\mathcal{H} \to {}^L H_1$ that extends the natural embedding $\widehat{H} \to \widehat{H}_1$. These choices (which always exist, cf. [KS99, §2.2]) are called a z-pair. They provide a parameter φ_1 for H_1.

An approach introduced in [Kal22] is to extend the theory of double covers of tori from [Kal19a] to the setting of quasi-split connected reductive groups. The datum $\mathcal{H}$ then leads to a *canonical* double cover $H(F)_\pm$ of $H(F)$ and a *canonical* isomorphism ${}^LH_\pm \to \mathcal{H}$. In this way, φ naturally becomes a parameter for $H(F)_\pm$.

The transfer of orbital integrals and characters between G and H is governed by the transfer factor. In the classical case, it is a function

$$\Delta : H_1(\mathbb{R})^{\mathrm{rs}} \times G(\mathbb{R})^{\mathrm{rs}} \to \mathbb{C}$$

that depends on the z-pair datum, while in the setting of covers it is a function

$$\Delta : H(R)^{\mathrm{rs}}_\pm \times G(\mathbb{R})^{\mathrm{rs}} \to \mathbb{C}$$

that is genuine in the first argument. In the classical case, it is given as the product

$$\epsilon \cdot \Delta_I^{-1} \Delta_{II} \Delta_{III_1}^{-1} \Delta_{III_2}.$$

The individual factors are defined in [LS87], except for ϵ, which is defined in the more general twisted setting in [KS99, §5.3], and Δ_{III_1}, whose relative definition is given in [LS87], but whose absolute definition is given in [Kal11] in the setting of pure inner forms, and in [Kal16b] in the setting of rigid inner forms. The inverses appear due to the conventions of [KS, (1.0.4)], which we will use in this paper. The term Δ_{IV} is missing because we have normalized orbital integrals and characters by the Weyl denominator. We will not review the construction of the individual pieces here, as it has been reviewed in various other places, such as [Kal, §3.5,§4.2,§4.3]. The individual factors depend on auxiliary data, known as a-data and χ-data. The total factor depends on a choice of Whittaker datum, and z-datum.

In the case of covers, the transfer factor becomes the product

$$\epsilon \cdot \Delta_I^{-1} \cdot \Delta_{III}.$$

The terms Δ_I and Δ_{III} are slightly different from the original ones, and are defined in [Kal22, §4.3]. Neither of them depends on auxiliary data, although they are defined on certain covers of tori, and one could argue that the elements of those covers count as auxiliary data.

We now state the theorem asserting transfer of orbital integrals. It is a fundamental result of Shelstad, [She82], [She08a]. We state two versions, one using the cover $H(\mathbb{R})_\pm$ and one using a z-pair (H_1, η_1).

Theorem 1.8.1. *Let $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$.*

1. *There exists a genuine function $f^{H_\pm} \in \mathcal{C}_c^\infty(H(\mathbb{R})_\pm)$ such that for all $\dot\gamma \in H(F)^{rs}_\pm$*
$$SO_{\dot\gamma}(f^{H_\pm}) = \sum_\delta \Delta(\dot\gamma, \delta) O_\delta(f).$$

2. *Assume chosen a z-pair (H_1, η_1). There exists a genuine function $f^{H_1} \in \mathcal{C}_c^\infty(H_1(\mathbb{R}))$ such that for all $\gamma_1 \in H_1(F)^{rs}$*
$$SO_{\gamma_1}(f^{H_1}) = \sum_\delta \Delta(\gamma_1, \delta) O_\delta(f).$$

In both cases δ runs over the set of $G(\mathbb{R})$-conjugacy classes in $G(\mathbb{R})^{rs}$.

Definition 1.8.2. The functions f and $f^{H_\pm}$ are called *matching*. The functions f and f^{H_1} are called *matching*.

For the computations of this paper it would be useful to review the factor ϵ and compute it in the case of the base field $\mathbb{R}$, where the computation is rather straightforward. Consider the universal maximal torus T_0^G of G and T_0^H of H. The complexified character modules $V_G := X^*(T_0^G)\otimes_{\mathbb{Z}}\mathbb{C}$ and $V_H := X^*(T_0^H)\otimes_{\mathbb{Z}}$ $\mathbb{C}$ are self-dual Artin representations of the same dimension. Choose an $\mathbb{R}$-pinning of the quasi-split form G_0 of G and a non-trivial additive character $\Lambda : \mathbb{R} \to \mathbb{C}$, which combine to the Whittaker datum fixed for G_0. Then

$$\epsilon = \epsilon(1/2, V_G - V_H, \Lambda),$$

where we have use Langlands' convention [Tat79, (3.6.4)] for the ϵ-factor. The pinning is used in the construction of G_0. [May be better to review the construction of Δ_I and Δ_{III} here.]

Lemma 1.8.3. *Let $\Lambda(x) = e^{irx}$ with $r > 0$. Then*

$$\epsilon = (-1)^{q(H)-q(G_0)} i^{r_G/2-r_H/2},$$

where r_G is the number of roots in the absolute root system of G, and r_H is the analogous number for H.

Proof. Let $A_0^G \subset T_0^G$ be the maximal split torus ($X_*(A_0^G) = X_*(T_0^G)^\Gamma$), and $S_0^G \subset T_0^G$ the maximal anisotropic torus ($X^*(S_0^G) = X^*(T_0^G)/X^*(T_0^G)^\Gamma$). Then $X^*(T_0^G)_{\mathbb{C}} = X^*(A_0^G)_{\mathbb{C}} \oplus X^*(S_0^G)_{\mathbb{C}}$, where we have abbreviated $\otimes_{\mathbb{Z}}\mathbb{C}$ by the subscript $\mathbb{C}$. One has $\epsilon(1/2, \mathbf{1}, \Lambda) = 1$ and $\epsilon(1/2, \mathrm{sgn}, \Lambda) = i$ according to [Tat79, (3.2.4)], hence

$$\epsilon(1/2, V_G, \Lambda) = i^{d-\dim(A_0^G)},$$

where $d = \dim(T_0^G)$. We use the same computation for H and conclude

$$\epsilon = \frac{\epsilon(1/2, X^*(T_0^G)_{\mathbb{C}}, \Lambda)}{\epsilon(1/2, X^*(T_0^H)_{\mathbb{C}}, \Lambda)} = \frac{i^{d-\dim(A_0^G)}}{i^{d-\dim(A_0^H)}} = i^{\dim(A_0^H)-\dim(A_0^G)}.$$

The Iwasawa decomposition $\mathrm{Lie}(G_0) = \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{k}$ shows $2q(G_0) = \dim(\mathfrak{a}) + \dim(\mathfrak{n}) = \dim(A_0^G) + r_G/2$. We note that $q(G_0)$ is an integer, becaus e G_0 has an elliptic maximal torus, and $q(G_0)$ equals the number of positive non-compact roots with respect to any Weyl chamber. Therefore

$$\dim(A_0^H) - \dim(A_0^G) = 2(q(H) - q(G_0)) + (r_G/2 - r_H/2). \qquad \square$$

1.9 Whittaker data and Kostant sections

We assume that G is a quasi-split connected reductive $\mathbb{R}$-group. Suppose B is a Borel $\mathbb{R}$-subgroup of G, N its unipotent radical, and $\mathfrak{n} = \mathrm{Lie}(N)$. Since $N(\mathbb{R})$ is a connected Lie group, any character $\eta : N(\mathbb{R}) \to \mathbb{C}^\times$ is determined by its differential $d\eta : \mathfrak{n}(\mathbb{R}) \to \mathbb{C}$, which is an $\mathbb{R}$-linear form. We have $\eta(\exp(Y)) = e^{\langle d\eta, Y\rangle}$ for $Y \in \mathfrak{n}(\mathbb{R})$. In fact, the real Lie group $N(\mathbb{R})$ is connected, nilpotent, and simply connected (see [Kna02, Theorem 6.46], where the assumption that

G is semi-simple is unnecessary) and hence the exponential map $\exp : \mathfrak{n}(\mathbb{R}) \to N(\mathbb{R})$ is a diffeomorphism by [Kna02, Theorem 1.127].

An $\mathbb{R}$-linear form $\mathfrak{n}(\mathbb{R}) \to \mathbb{C}$ can be obtained as the restriction of a $\mathbb{C}$-linear form $\mathfrak{n}(\mathbb{C}) \to \mathbb{C}$. Let $\bar{N}$ be the unipotent radical of B that is T-opposite to N. Since the Killing form κ induces a non-degenerate $\mathbb{C}$-linear pairing $\mathfrak{n} \times \bar{\mathfrak{n}} \to \mathbb{C}$, such forms are in 1-1 correspondence with elements of $\bar{\mathfrak{n}}(\mathbb{C})$. Given $X \in \bar{\mathfrak{n}}(\mathbb{C})$ we thus obtain the character $\eta_X(\exp(Y)) = e^{\kappa(X,Y)}$. It is unitary if and only if $X \in i\bar{\mathfrak{n}}(\mathbb{R})$.

Given a maximal torus $T \subset B$ defined over $\mathbb{R}$, the space $\mathfrak{n}(\mathbb{R})$ decomposes as the direct sum of relative root spaces, with respect to the action of the split part of T. We say a character $\eta : N(\mathbb{R}) \to \mathbb{C}^\times$ is *non-degenerate* if its differential $d\eta$ restricts non-trivially to any relative root space in $\mathfrak{n}(\mathbb{R})$. Since all choices of T are conjugate under $N(\mathbb{R})$, this condition is independent of T. For η_X with $X \in i\bar{\mathfrak{n}}(\mathbb{R})$ we can write $X = \sum_\alpha X_{-\alpha}$ where the sum runs over the set of absolute positive roots and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$. Then η_X is non-degenerat if and only if $X_{-\alpha} \neq 0$ for all simple roots α, i.e. if and only if X is a regular nilpotent element.

By a *Whittaker datum* we mean a $G(\mathbb{R})$-conjugacy class of pairs (B, η) where B is a Borel subgroup defined over $\mathbb{R}$ and η is a non-degenerate unitary character of $N(\mathbb{R})$. We write $\mathfrak{w} = [(B, \eta)]$ for it $G(\mathbb{R})$-conjugacy class of (B, η).

Suppose $X \in i\mathfrak{g}(\mathbb{R})$ is a regular nilpotent element. Let $\overline{B}$ be the unique Borel subgroup containing X. Then $\overline{B}$ is defined over $\mathbb{R}$. Let B be an $\mathbb{R}$-Borel subgroup that is opposite to $\bar{B}$, i.e. such that $B \cap \bar{B}$ is a maximal torus. The Whittaker datum defined by X is $\mathfrak{w}_X = [(B, \eta_X)]$.

Lemma 1.9.1. *The Whittaker datum $\mathfrak{w}_X$ depends only on the $G(\mathbb{R})$-conjugacy class of X.*

Proof. The choice of B is equivalent to a choice of a maximal torus $T \subset \bar{B}$ defined over $\mathbb{R}$, because T is determined by B as $T = B \cap \bar{B}$ and B is determined by T as the unique T-opposite of $\bar{B}$. But all maximal $\mathbb{R}$-tori in $\bar{B}$ are conjugate under $\bar{B}(\mathbb{R})$ by [Bor91, Theorem 19.2]. □

There is a similar but slightly different way to obtain a Whittaker datum. Let $(T, B, \{X_\alpha\})$ be an $\mathbb{R}$-pinning of G and let $\Lambda : \mathbb{R} \to \mathbb{C}^\times$ be a non-trivial unitary character. The element X_α specifies an isomorphism u_α from $\mathbb{G}_a$ to the root group U_α of G associated to the absolute root α, namely the unique isomorphism whose differential maps 1 to X_α. If N is the unipotent radical of B then the composition $\prod_{\alpha\in\Delta} U_\alpha \to N \to N/[N,N]$ is an isomorphism of complex algebraic groups, which we compose with (u_α). The inverse of the result, composed with the product map $\prod_\alpha \mathbb{G}_a \to \mathbb{G}_a$, becomes a homomorphism of algebraic groups $N/[N,N] \to \mathbb{G}_a$ defined over $\mathbb{R}$, which leads to a homomorphism $N(\mathbb{R}) \to \mathbb{R}$. Composing that with the character Λ provides a non-degenerate unitary character $\psi : N(\mathbb{R}) \to \mathbb{C}^\times$. The $G(\mathbb{R})$-conjugacy class of the pair (B, ψ) is a Whittaker datum and depends only on the $G(\mathbb{R})$-conjugacy class of the pinning $(T, B, \{X_\alpha\})$.

Lemma 1.9.2. *Let $(T, B, \{X_\alpha\})$ be an $\mathbb{R}$-pinning of G and let $\Lambda(x) = e^{2\pi i x}$. Then the Whittaker datum associated to this pinning and character by the above procedure is equal to the Whittaker datum $\mathfrak{w}_{\bar{X}}$, where $\bar{X} = 2\pi i \sum_{\alpha\in\Delta} X_{-\alpha}$, where $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ is determined by the rule $[X_\alpha, X_{-\alpha}] = H_\alpha$.*

Proof. Set $X_- = \sum_{\alpha\in\Delta} X_{-\alpha}$. Then X_- is a regular nilpotent element of the Lie algebra of G and is fixed by the Galois action, hence an $\mathbb{R}$-point. It lies in $\bar{\mathfrak{n}}$, the Lie algebra of the unipotent radical of the Borel subgroup T-opposite to B. Thus $\bar{X} \in i\bar{\mathfrak{n}}(\mathbb{R})$ is also regular nilpotent.

Given $Y \in \mathfrak{n}(\mathbb{R})$ write it as $\sum_{\alpha\in\Delta} y_\alpha \cdot X_\alpha + Y'$ with $Y' \in [\mathfrak{n},\mathfrak{n}](\mathbb{R})$. Then we have

$$\kappa(Y, X_-) = \sum_{\alpha\in\Delta} y_\alpha \kappa(X_\alpha, X_{-\alpha}) = \sum_{\alpha\in\Delta} y_\alpha,$$

therefore

$$\psi_{\bar{X}}(\exp(Y)) = e^{\kappa(Y,\bar{X})} = e^{2\pi i \kappa(Y,X_-)} = \Lambda(\sum_{\alpha\in\Delta} y_\alpha) = \psi(\exp(Y)),$$

where $\psi : N(\mathbb{R}) \to \mathbb{C}^\times$ is constructed from the pinning and Λ as above. □

Suppose $X \in \mathfrak{g}$ is a regular nilpotent element. Choose an $SL(2)$-triple $[X, H, Y]$ [reference]. The *Kostant Section* $\mathcal{K}(X)$ of X is the affine space $X + \mathrm{Cent}_{\mathfrak{g}}(Y)$. Kostant showed [Kos63] that the Kostant section meets every regular orbit in a unique point. If the tripple lies in $\mathfrak{g}(\mathbb{R})$, then $\mathcal{K}(X)$ is Galois stable. The Kostant section $\mathcal{K}(X)$ depends on a choice of triple, but any two such choices that lie in $\mathfrak{g}(\mathbb{R})$ are $G(\mathbb{R})$-conjugate, and the $G(\mathbb{R})$-conjugacy class of $\mathcal{K}(X)$ only depends on the $G(\mathbb{R})$-conjugacy class of X.

Suppose $X \in \mathfrak{g}(\mathbb{R})$ and $\mathcal{O}$ is a regular orbit which is defined over $\mathbb{R}$. Then, since X is also defined over $\mathbb{R}$, the unique point in $\mathcal{K}(X) \cap \mathcal{O}$ is contained in $\mathfrak{g}(\mathbb{R})$. The $G(\mathbb{R})$-orbit of $\mathcal{K}(X) \cap \mathcal{O}$ depends only on the $G(\mathbb{R})$-orbit of X.

1.10 Generic discrete series representations

Definition 1.10.1. Suppose $\mathfrak{w} = [(B, \eta)]$ is a Whittaker datum. We say that a representation π of G is $\mathfrak{w}$-generic if there is a non-zero smooth vector v in the space of π such that $\pi(X)(v) = \eta(v)$ for all $x \in \mathfrak{n}(\mathbb{R})$. We say π is generic if it is $\mathfrak{w}$-generic for some $\mathfrak{w}$. Jeff, do you want $\pi(x)v = \eta(x)$ for all $x \in N(\mathbb{R})$, or $d\pi(X)v = d\eta(X)$ for all $X \in \mathfrak{n}(\mathbb{R})$?

Suppose π is an irreducible essentially discrete series representation and write $\pi = \pi(S, \tau)$ as in Section 1.7. Let H_π be the element of $i\mathfrak{s}(\mathbb{R})$ corresponding to $d\tau$ via κ. The $G(\mathbb{R})$-conjugacy class of H_π is well defined. The purpose of this subsection is to prove the following result.

Proposition 1.10.2. *Suppose π is a generic discrete series representation. Then π is $\mathfrak{w}$-generic for a unique Whittaker datum $\mathfrak{w}$. Write $\mathfrak{w} = \mathfrak{w}_X$ for some regular nilpotent element $X \in i\mathfrak{g}(\mathbb{R})$. Then $H_\pi \in i\mathfrak{s}(\mathbb{R})$ is $G(\mathbb{R})$-conjugate to an element of the Kostant section of X.*

We begin with some preparations. If W is a subset of a real or complex vector space V, define $\mathrm{AC}(W)$, the *asymptotic cone* of W as in [BV80, Proposition 3.7], [AV21, Definition 2.9]:

$$\mathrm{AC}(W) = \{v \in V \mid \exists t_i \in \mathbb{R}_{>0}, t_i \to 0, w_i \in W, \lim_{i\to\infty} t_i w_i = v\}$$

This is a closed cone.

Lemma 1.10.3. *Let $\mathcal{O}_{\mathbb{R}} \subset \mathfrak{g}(\mathbb{R})$ be a regular $G(\mathbb{R})$-orbit. Assume there exists a regular nilpotent element $X \in \mathrm{AC}(\mathcal{O}_{\mathbb{R}})$. Then $\mathcal{K}(X)$ meets $\mathcal{O}_{\mathbb{R}}$.*

Proof. Let $\mathcal{O}_{\mathbb{C}}$ be the $G(\mathbb{C})$-orbit of $\mathcal{O}_{\mathbb{R}}$. It is a regular orbit defined over $\mathbb{R}$ and hence meets $\mathcal{K}(X)$ in a unique point, which lies in $\mathcal{O}_{\mathbb{C}}(\mathbb{R})$.

By definition of $AC(\mathcal{O}_{\mathbb{R}})$, the distance (in a Euclidean metric on $\mathfrak{g}(\mathbb{R})$) between $t\mathcal{O}_{\mathbb{R}}$ and X goes to zero as $t \to 0$.

Let $\mathcal{O}_X$ be the $G(\mathbb{C})$-orbit of X. By [Kos63] Can you give a more precise reference? $\mathcal{K}(Y)$ is transverse to $\mathcal{O}_X$ for any $Y \in \mathcal{O}_X$. Since $\mathcal{O}_{\mathbb{R}}$ and $\mathcal{K}(Y)$ are defined over $\mathbb{R}$, we also have $\mathcal{K}(Y)(\mathbb{R})$ is transverse to $\mathcal{O}_X(\mathbb{R})$.

By the definition of $\mathrm{AC}(\mathcal{O}_{\mathbb{R}})$, tX approaches $\mathcal{O}_{\mathbb{R}}$ as $t \to \infty$. By transversality this says $\mathcal{O}_{\mathbb{R}} \cap \mathcal{K}(tX)(\mathbb{R}) \neq \emptyset$ for $t >> 0$. But tX is $G(\mathbb{R})$-conjugate to X so $\mathcal{O}_{\mathbb{R}} \cap \mathcal{K}(X)$ is also non-empty. ☐

Proof of Proposition 1.10.2. Suppose π is $\mathfrak{w}_X$-generic. By [Mat92, Theorem A] $X \in \mathrm{WF}(\pi)$. By [HHO16, Theorem 1.2] $\mathrm{WF}(\pi) = \mathrm{AC}(G(\mathbb{R}) \cdot H_\pi)$. The proposition then follows from the Lemma (applied with $i\mathfrak{g}(\mathbb{R})$ in place of $\mathfrak{g}(\mathbb{R})$). ☐

Remark 1.10.4. The maps $\pi \to H_\pi$ and $\mathfrak{w} \to X$ both depend on the choice of κ. This dependence is very minor (only on the center), and in any event these two choices cancel so the statement of the Proposition is independent of the choice of κ.

Corollary 1.10.5. *Let $S \subset G_0$ be an elliptic maximal torus, ρ a Weyl chamber in $X^*(S/Z_{G_0})$, and τ_0 a character of $S(\mathbb{R})$ whose differential is ρ-dominant and ρ-integral. Let $(T_0, B_0, \{X_\alpha\})$ be a pinning of G_0 and $\Lambda(x) = e^{2\pi i x}$. If the discrete series representation associated to (S, ρ, τ_0) is generic with respect to the Whittaker datum associated to the pinning and Λ, then the element $\rho^\vee(-i) \in Lie(S_{sc})(\mathbb{R})$ is $G_0(\mathbb{R})$-conjugate to the Kostant section associated to the pinning.*

Proof. Let π be the essentially discrete series representation associated to (S, ρ, τ_0) and let $H_\pi \in i\mathrm{Lie}(S)(\mathbb{R})$ be the associated element, determined by $d\tau(Z) = \kappa(H_\pi, Z)$.

Let $X_- = \sum_{\alpha \in \Delta} X_{-\alpha}$ and let $\bar{X} = 2\pi i X_-$. According to Lemma 1.9.2, being generic with respect to the pinning and Λ is equivalent to being generic with respect to $\mathfrak{w}_{\bar{X}}$, and Proposition 1.10.2 implies that H_π is $G(\mathbb{R})$-conjugate to the Kostant section $K(\bar{X})$.

Let $H = \rho^\vee(i) \in \mathrm{Lie}(S_{\mathrm{sc}})$. This element is Galois-fixed and thus lies in $\mathrm{Lie}(S_{\mathrm{sc}})(\mathbb{R})$. Argue from above statement to the fact that H is $G(\mathbb{R})$-conjugate to the Kostant section $K(i\bar{X})$. But $K(i\bar{X}) = K(-2\pi X_-)$ and we conclude that $\rho^\vee(-i)$ is $G(\mathbb{R})$-conjugate to $K(2\pi X_-)$. ☐

Example 1.10.6. Set $G = \mathrm{SL}(2, \mathbb{R})$. Let $\mathfrak{s}(\mathbb{C}) = \{t_z \mid z \in \mathbb{C}\}$ where

$$t_z = \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}$$

Then $\mathfrak{s}(\mathbb{R}) = \{t_x \mid x \in \mathbb{R}\}$.

For $z \in \mathbb{C}$ define $\lambda_z \in \mathfrak{s}(\mathbb{C})^*$ by $\lambda_z(t_x) = xz$. The positive root is $\alpha(t_z) = 2iz$, i.e.

$$\alpha = \lambda_{2i}, \quad \rho = \lambda_i.$$

and

$$^\vee\alpha = t_{-i}, \quad ^\vee\rho = t_{-i/2}$$

In particular

$$^\vee\rho(-i) = t_{-\frac{1}{2}} = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

Define $H_\lambda \in \mathfrak{s}(\mathbb{C})$ so that $\kappa(H_\lambda, t_x) = \lambda(t_x)$. Then $H_{\lambda_z} = t_{-\frac{z}{8}}$:

$$H_{\lambda_z} = t_{\frac{-z}{8}} = \begin{pmatrix} 0 & -\frac{z}{8} \\ \frac{z}{8} & 0 \end{pmatrix}$$

Take $z = ik$, so

$$H_{\lambda_{ik}} = \begin{pmatrix} 0 & -\frac{ik}{8} \\ \frac{-ik}{8} & 0 \end{pmatrix}$$

Let $\pi(\lambda_{ik})$ be the discrete series representation with Harish-Chandra parameter λ_{ik} ($k \in \mathbb{Z}_{\neq 0}$). If $\pi = \pi(\lambda)$ let $H_\pi = H_\lambda \in i\mathfrak{s}(\mathbb{R})^*$.

For an $SL(2)$-triple we can take $\{X_\alpha, X_{-\alpha}, t_{-i}\}$ where

$$X_\alpha = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad X_{-\alpha} = \overline{X_\alpha}.$$

The Killing form satisfies

$$\kappa(t_x, t_y) = -8xy.$$

Conjugating this by $\mathrm{diag}(x, \frac{1}{x})$ takes it to $\begin{pmatrix} 0 & -x^2\frac{ik}{8} \\ \frac{ik}{x^2} & 0 \end{pmatrix}$, and taking the limit we see

$$\mathrm{AC}(G(\mathbb{R}) \cdot H_{\lambda_{ik}}) = \begin{cases} \mathbb{R}^+ * \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} & k > 0 \\ \mathbb{R}^+ * \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} & k < 0 \end{cases}$$

Therefore

$$\mathrm{WF}(\pi(\lambda_{ik})) = \begin{cases} \mathbb{R}^+ * \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} & k > 0 \\ \mathbb{R}^+ * \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} & k < 0 \end{cases}$$

Note that

$$\mathcal{K}\begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} = \{\begin{pmatrix} 0 & -i \\ z & 0 \end{pmatrix} \mid z \in \mathbb{C}\}$$

and

$$\mathcal{K} \cap i\mathfrak{g}(\mathbb{R}) = \{\begin{pmatrix} 0 & -i \\ iy & 0 \end{pmatrix} \mid y \in \mathbb{R}\}$$

In particular

$$\mathcal{K}\begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} \cap i\mathfrak{g}(\mathbb{R}) \ni H_\pi = \begin{pmatrix} 0 & -\frac{ik}{8} \\ \frac{ik}{8} & 0 \end{pmatrix} \quad (k > 0)$$

as required by Proposition 1.10.2. Similarly

$$\mathcal{K}\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \cap i\mathfrak{g}(\mathbb{R}) \ni H_\pi = \begin{pmatrix} 0 & -\frac{ik}{8} \\ \frac{ik}{8} & 0 \end{pmatrix} \quad (k < 0).$$

In this case Proposition **??** amounts to

$$\mathcal{K}(-iX) = \mathcal{K}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \ni {}^\vee\!\rho(-i) = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad (k > 0)$$

and

$$\mathcal{K}(-iX) = \mathcal{K}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ni {}^\vee\!\rho(-i) = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \quad (k < 0)$$

Note that the coroot ${}^\vee\alpha : \mathbb{C}^\times \to S$ is given by

$${}^\vee\alpha(e^z) = \begin{pmatrix} \cos(z) & \sin(z) \\ -\sin(z) & \cos(z) \end{pmatrix}$$

or more algebraically

$${}^\vee\alpha(z) = \begin{pmatrix} \frac{z+\frac{1}{z}}{2} & \frac{z-\frac{1}{z}}{2i} \\ -\frac{z-\frac{1}{z}}{2i} & \frac{z+\frac{1}{z}}{2} \end{pmatrix}.$$

2 Construction of L-packet and internal structure

Let G_0 be a quasi-split connected reductive $\mathbb{R}$-group with dual group $\widehat{G}$ and L-group ${}^LG$. Let $\varphi : W_\mathbb{R} \to {}^LG$ be a discrete Langlands parameter, given up to conjugation by $\widehat{G}$.

2.1 Factorization of a parameter

We choose a Γ-invariant Borel pair $(\widehat{T}, \widehat{B})$ of $\widehat{G}$. Conjugating by $\widehat{G}$ we arrange that $\varphi(z) \in \widehat{T}$ for all $z \in \mathbb{C}^\times$. Thus, $\varphi|_{\mathbb{C}^\times}$ is a continuous group homomorphism $\mathbb{C}^\times \to \widehat{T}$. Every continuous group homomorphism $\mathbb{C}^\times \to \mathbb{C}^\times$ is of the form $z^a\bar{z}^b = |z|^{a+b}\arg(z)^{a-b}$ for some $a, b \in \mathbb{C}$ with $a - b \in \mathbb{Z}$. Thus there exist $\lambda, \mu \in X_*(\widehat{T}) \otimes_\mathbb{Z} \mathbb{C}$ with $\lambda - \mu \in X_*(\widehat{T})$ such that

$$\varphi(z) = \lambda(z) \cdot \mu(\bar{z}), \qquad \forall z \in \mathbb{C}^\times.$$

Lemma 2.1.1. *Assume that G_0 is semi-simple and simply connected.*

1. $\lambda, \mu \in X_*(\widehat{T})$
2. $\langle \lambda, \alpha \rangle \neq 0$ *for all* $\alpha \in R(\widehat{T}, \widehat{G})$.
3. $\mu = -\lambda$ *in* $X_*(\widehat{T}/Z(\widehat{G}))$

Proof. [TODO.] ☐

We can apply Lemma 2.1.1 to the composition of φ with the projection ${}^LG \to {}^LG/\widehat{Z}$, where $\widehat{Z}$ is the center of $\widehat{G}$, noting that $\widehat{G}/\widehat{Z}$ is the dual group of G_{sc}. It implies that the centralizer of $\varphi|_{\mathbb{C}^\times}$ in $\widehat{G}$ equals $\widehat{T}$. Since $\mathbb{C}^\times$ is normal in $W_\mathbb{R}$, the image of φ lies in $N(\widehat{T}, \widehat{G})$. Its projection to $\Omega(\widehat{T}, \widehat{G})$ factors through a homomorphism $\xi : \Gamma \to \Omega(\widehat{T}, \widehat{G}) \rtimes \Gamma$. Let $\widehat{S}$ denote the Γ-module with underlying abelian group $\widehat{T}$ and Γ-structure given by $\mathrm{Ad} \circ \xi$. Let S be the

$\mathbb{R}$-torus whose dual is $\widehat{S}$, i.e. the $\mathbb{R}$-torus determined by $X^*(S) = X_*(\widehat{S})$ as Γ-modules.

By construction we have $R(\widehat{T}, \widehat{G}) \subset X^*(\widehat{T}) = X^*(\widehat{S}) = X_*(S)$, and we write $R^\vee(S, G)$ for this set. Analogously we have a subset $R(S, G) \subset X^*(S)$. Both of these subsets are Γ-stable. Moreover $\mu = \sigma\lambda$ in $X_*(\widehat{S})$ and according to Lemma 2.1.1 the action of σ on $R(S, G)$ is by negation.

Let $S(\mathbb{R})_G$ be the double cover of $S(\mathbb{R})$ reviewed in §1.6, associated to the subset $R(S, G) \subset X^*(S)$. As discussed there, there is a canonical $\widehat{G}$-conjugacy class of L-embeddings ${}^LS_G \to {}^LG$. Inside of this class, there is a unique $\widehat{S}$-conjugacy class, call it ${}^Lj$, whose restriction to $\widehat{S}$ is the tautological embedding $\widehat{S} \to \widehat{G}$. The image of this L-embedding is described in (1.5), and contains the image of φ by construction. Thus $\varphi = {}^Lj \circ \varphi_S$ for a unique $\widehat{S}$-conjugacy class of L-homomorphisms $\varphi_S : W_\mathbb{R} \to {}^LS_G$. According to [Kal19a, Theorem 3.15], φ_S corresponds to a genuine character $\tau : S(\mathbb{R})_G \to \mathbb{C}^\times$.

The construction of (S, τ) depended on the choice of $\widehat{T}$.

Lemma 2.1.2. *If (S_1, τ_1) and (S_2, τ_2) are two pairs obtained from two different choices of $\widehat{T}$, there exists $g \in \widehat{G}^\Gamma$ such that $Ad(g)\widehat{S}_1 = \widehat{S}_2$, $Ad(g) : \widehat{S}_1 \to \widehat{S}_2$ is Γ-equivariant, and its dual isomorphism $S_2 \to S_1$ identifies τ_2 with τ_1.*

Proof. [TODO] □

2.2 Construction of the L-packet

The natural embedding $\widehat{S} \to \widehat{G}$ is not Γ-equivariant, but its $\widehat{G}$-conjugacy class is. From §1.3 we obtain the category $\mathcal{J}$ of embeddings of S into all pure (or rigid) inner forms of G_0.

Consider $(G, \xi, z, j) \in \mathcal{J}(F)$. As discussed in §1.7, there exists a unique essentially discrete series representation π_j of $G(\mathbb{R})$ associated to the pair (S, τ), transported to G via j.

Definition 2.2.1. We define the pure (resp rigid) compound L-packet

$$\tilde{\Pi}_\varphi = \{(G, \xi, z, \pi_j) | (G, \xi, z, j) \in \mathcal{J}(F)\} \subset \tilde{\Pi},$$

$$\Pi_\varphi = \tilde{\Pi}_\varphi / G_0(\mathbb{C}) \subset \Pi.$$

For each pure (or rigid) inner twist (G, ξ, z) of G_0, we define

$$\Pi_\varphi(G, \xi, z) = \{\pi | (G, \xi, z, \pi) \in \tilde{\Pi}_\varphi\}.$$

Lemma 2.2.2. *The set of representations $\Pi_\varphi((G, \xi, z))$ equals $\{\pi_j | j \in J^G(\mathbb{R})/G(\mathbb{R}))\}$. In particular, it is independent of z. It coincides with the set $\Pi_\varphi(G)$ constructed by Langlands in [Lan89, §3].*

Proof. As discussed in §1.3, the set of $(G, \xi, z, j) \in \mathcal{J}(\mathbb{R})$ with fixed triple (G, ξ, z) corresponds to $J^G(\mathbb{R})$, in particular is independent of z. Since the images of the members of $J^G(\mathbb{R})$ are elliptic maximal tori, and all such are conjugate under $G(\mathbb{R})$, we can choose representatives of $J^G(\mathbb{R})/G(\mathbb{R})$ that all have the same image, call it $S' \subset G$. Then these representatives are a single orbit under $\Omega(S', G)$.

In other words, if τ' is the transport of τ under one admissible embedding $j : S \to G$ with image S', then all others make out the $\Omega(S', G)$-orbit of τ'. Since we have specified the representations π_j by their character values on regular elements of $S'(\mathbb{R})$ in the same way as in [Lan89, §3] or [AV16, §4]. $\square$

2.3 Internal structure of the compound packet

Recall from Lemma 1.3.1 that the abelian group $H^1(\Gamma, S)$ in the pure case (resp. $H^1(u \to W, Z(G_0) \to S)$ in the rigid case) acts simply transitively on the set $\mathcal{J}(\mathbb{R})/G_0(\mathbb{C})$. By construction we have a bijection $\mathcal{J}(\mathbb{R})/G_0(\mathbb{C}) \to \Pi_\varphi$. At the same time, Lemma 2.1.1 provides an identification $S_\varphi = \widehat{S}^\Gamma$, hence by Tate-Nakayama duality $\pi_0(S_\varphi)^* = \pi_0(\widehat{S}^\Gamma)^* = H^1(\Gamma, S)$. Analogously, in the rigid setting we obtain $\pi_0(S_\varphi^+)^* = H^1(u \to W, Z(G_0) \to S)$. This provides a simply transitive action of the abelian group $\pi_0(S_\varphi)^*$ in the pure setting, and $\pi_0(S_\varphi^+)^*$ in the rigid setting, on the set Π_φ.

Lemma 2.3.1. *The set $\Pi_\varphi((G_0, 1, 1))$ contains a unique $\mathfrak{w}$-generic member.*

Proof. Choose a G_0-equivariant extension κ to $\mathfrak{g}$ of the Killing form on $[\mathfrak{g}, \mathfrak{g}]$. Its pull-back to $\mathfrak{s}$ along any embedding $j \in J^{G_0}$ is the same, since all these embeddings are conjugate under G_0.

Let $H \in i\mathfrak{s}(\mathbb{R})$ be the element such that $\kappa(H, Y) = d\tau(Y)$ for all $Y \in \mathfrak{s}(\mathbb{R})$. For $j \in J^{G_0}(\mathbb{R})$, the element $dj(H) \in i\mathfrak{g}(\mathbb{R})$ is associated to the representation π_j as in Proposition 1.10.2, which implies that π_j is $\mathfrak{w}$-generic if and only if $dj(H)$ meets the associated Kostant section. But as j varies over $J^{G_0}(\mathbb{R})/G_0(\mathbb{R})$, the element $dj(H)$ varies over the $G_0(\mathbb{R})$-classes in a fixed stable class. Therefore, $dj(H)$ meets any Kostant section for precisely one $j \in J^{G_0}(\mathbb{R})/G_0(\mathbb{R})$. At the same time, $j \mapsto \pi_j$ is a bijection from $j \in J^{G_0}(\mathbb{R})/G_0(\mathbb{R})$ to $\Pi_\varphi((G_0, 1, 1))$ by construction of the latter. $\square$

Taking the unique $\mathfrak{w}$-generic member of $\Pi((G_0, 1, 1)) \subset \Pi_\varphi$, provided by Lemma 2.3.1, as a base-point, the simply-transitive action turns into the desired bijection from $\pi_0(S_\varphi)^*$ (resp. $\pi_0(S_\varphi^+)^*$) to Π_φ.

2.4 The case of a cover of G

[TODO]

2.5 Dependence on the choice of Whittaker datum

[TODO]

3 Endoscopic character identities

Let $\varphi : W_\mathbb{R} \to {}^L G$ be a discrete parameter. Let $s \in S_\varphi$ (resp. $s \in S_\varphi^+$) be a semi-simple element. Let $(H, s, \mathcal{H}, \eta)$ be the (pure or rigid) refined endoscopic datum associated to the pair (φ, s), whose construction was reviewed in §1.8.

3.1 Statement of the main theorem

As discussed in §2.4 there is an associated compound L-packet Π_{φ_H}. We will be only interested in the contribution of the trivial twist $(H,1,1)$, and we write $\Pi_\varphi(H) \subset \Pi_{\varphi_H}$ for it. Consider the virtual character

$$S\Theta_{\varphi_H} := \sum_{\sigma\in\Pi_\varphi(H)} \langle\sigma, s\rangle\Theta_\sigma = \sum_{\sigma\in\Pi_\varphi(H)} \langle\sigma, 1\rangle\Theta_\sigma = \sum_{\sigma\in\Pi_\varphi(H)} \Theta_\sigma$$

on $H(\mathbb{R})_\pm$, where $\langle\sigma,-\rangle$ is the character of the irreducible representation of $\pi_0(S_\varphi)$ (resp. $\pi_0(S_\varphi^+)$) associated to σ by the bijection of §2.3. Let us argue the two equalities. Since $Z(\widehat{H})^\Gamma$ (resp. $Z(\widehat{\bar{H}})^+$) acts trivially on this irreducible representation, and s belongs by construction to this group, we see $\langle\sigma, s\rangle = \langle\sigma, 1\rangle$, hence the first equality. The second comes from the fact that S_φ is abelian, because it lies in $\widehat{S}$ (and S_φ^+ lies in $\widehat{\bar{S}}$), where $\widehat{S}$ is the torus involved in the construction of the L-packet on H. Note that, while the bijection of §2.3 depends on the choice of a Whittaker datum, the argument of §1.9 shows that the value $\langle\sigma,1\rangle$ does not depend on this choice.

Let (G,ξ,z) be a pure (resp. rigid) inner twist of G_0. We have the virtual character on $G(\mathbb{R})$ given by

$$\Theta_\varphi^{\mathfrak{w},s} := e(G) \sum_{\pi\in\Pi_\varphi((G,\xi,z))} \langle\pi, s\rangle\Theta_\pi.$$

This virtual character does depend on $\mathfrak{w}$.

The following is the main theorem of this article. It is a fundamental result of Shelstad [She82], [She10], [She08b].

Theorem 3.1.1. *Let $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$ be a test function.*

1. *If $f^{H_\pm} \in \mathcal{C}_c^\infty(H(\mathbb{R})_\pm)$ matches f as in Definition 1.8.2, then*

$$\Theta_\varphi^{\mathfrak{w},s}(f) = S\Theta_{\varphi_H}(f^{H_\pm}).$$

2. *If $f^{H_1} \in \mathcal{C}_c^\infty(H_1(\mathbb{R}))$ matches f as in Definition 1.8.2, then*

$$\Theta_\varphi^{\mathfrak{w},s}(f) = S\Theta_{\varphi_{H_1}}(f^{H_1}).$$

3.2 Reduction to the elliptic set

Theorem 3.2.1. 1. *For every strongly regular semi-simple element $\delta \in G(\mathbb{R})$ the following identity holds*

$$\Theta_\varphi^{\mathfrak{w},s}(\delta) = \sum_{\gamma\in H(\mathbb{R})/st} \Delta[\mathfrak{w},\mathfrak{e},z](\dot\gamma,\delta) S\Theta_{\varphi_H}(\dot\gamma).$$

2. *For every strongly regular semi-simple element $\delta \in G(\mathbb{R})$ the following identity holds*

$$\Theta_\varphi^{\mathfrak{w},s}(\delta) = \sum_{\gamma\in H(\mathbb{R})/st} \Delta[\mathfrak{w},\mathfrak{e},\mathfrak{z},z](\gamma_1,\delta) S\Theta_{\varphi_{H_1}}(\gamma_1).$$

Lemma 3.2.2. *Theorem 3.1.1 is equivalent to Theorem 3.2.1.*

Proof. [TODO] $\square$

Lemma 3.2.3. *If Theorem 3.2.1 holds for all elliptic δ, then it holds for all δ.*

Proof. [TODO] $\square$

3.3 The left hand side

In this subsection we will provide a formula for the left hand side of the identity in Theorem 3.2.1, i.e. $\Theta^{\mathfrak{w}}_{\varphi,s}(\delta)$, for strongly regular semi-simple elliptic $\delta \in G(\mathbb{R})$. The end result is (3.1).

The members of $\Pi_\varphi((G,\xi,z))$ are parameterized by the set of $G(\mathbb{R})$-conjugacy classes of admissible embedding $j : S \to G$. Given such an embedding let π_j be the corresponding representation. Let $j_{\mathfrak{w}} : S \to G_0$ be the unique embedding for which $\pi_{j_{\mathfrak{w}}}$ is the unique $\mathfrak{w}$-generic member of $\Pi_\varphi((G_0,1,1))$. Then $\mathrm{inv}(j_{\mathfrak{w}}, j) \in H^1(\Gamma, S) = \pi_0(\widehat{S}^\Gamma)^* = \pi_0(S_\varphi)^*$ equals ρ_{π_j}. Therefore the left hand side becomes

$$\Theta^{\mathfrak{w}}_{\varphi,s}(\delta) = e(G)\sum_j \langle s, \mathrm{inv}(j_{\mathfrak{w}}, j)\rangle \Theta_{\pi_j}(\delta),$$

where j runs over (a set of representatives for) the set of $G(\mathbb{R})$-conjugacy classes in J_ξ. Harish-Chandra's character formula (1.6) states

$$\Theta_{\pi_j}(\delta) = (-1)^{q(G)} \sum_{w \in W_{\mathbb{R}}(G,jS)} \frac{\tau'}{d'_\tau}(j^{-1}w^{-1}\delta),$$

where we have conjugated δ within $G(\mathbb{R})$ to land in $jS(\mathbb{R})$. Combining the two formulas and using $e(G) = (-1)^{q(G_0)-q(G)}$, we obtain

$$\Theta^{\mathfrak{w}}_{\varphi,s}(\delta) = (-1)^{q(G_0)} \sum_j \langle s, \mathrm{inv}(j_{\mathfrak{w}}, j)\rangle \sum_{w \in W_{\mathbb{R}}(G,jS)} \frac{\tau'}{d'_\tau}(j^{-1}w^{-1}\delta).$$

Instead of conjugating δ to land in jS, we can conjugate j by $G(\mathbb{R})$ to achieve this, without changing π_j. With this shift in point of view we can combine the two sums and arrive at

$$\Theta^{\mathfrak{w}}_{\varphi,s}(\delta) = (-1)^{q(G_0)} \sum_j \langle s, \mathrm{inv}(j_{\mathfrak{w}}, j)\rangle \frac{\tau'}{d'_\tau}(j^{-1}w^{-1}\delta),$$

where now the sum runs over the set of those $j \in J_\xi$ whose image contains δ.

As j runs over this set, $j_{\mathfrak{w}} j^{-1}(\delta)$ runs over the set of elements $\delta_0 \in S_{\mathfrak{w}}(\mathbb{R})$ that are stably conjugate to δ, where $S_{\mathfrak{w}} \subset G_0$ is the image of $j_{\mathfrak{w}}$, an elliptic maximal torus of G_0. Moreover, $j_{\mathfrak{w}}$ transports $\mathrm{inv}(j_{\mathfrak{w}}, j) \in H^1(\mathbb{R}, S)$ to $\mathrm{inv}(\delta_0, \delta) \in H^1(\mathbb{R}, S_{\mathfrak{w}})$. So we arrive at

$$\Theta^{\mathfrak{w}}_{\varphi,s}(\delta) = (-1)^{q(G_0)} \sum_{\delta_0} \langle s_{\mathfrak{w}}, \mathrm{inv}(\delta_0, \delta)\rangle \frac{\tau'_{\mathfrak{w}}}{d'_{\mathfrak{w}}}(\delta_0), \tag{3.1}$$

where the sum runs over the set of elements $\delta_0 \in S_{\mathfrak{w}}(\mathbb{R})$ that are stably conjugate to δ, and we have used the subscript $\mathfrak{w}$ to indicate various transports under $j_{\mathfrak{w}} : S \to S_{\mathfrak{w}}$.

3.4 The right hand side: covers

In this subsection we will show that the right hand side of the identity of Theorem 3.2.1(1) is also equal to (3.1).

We begin by applying (3.1) to the group H, the parameter φ_H, and the trivial endoscopic element, and obtain

$$S\Theta_{\varphi_H}(\dot{\gamma}) = \sum_{\dot{\gamma}_0} \frac{\tau_H}{d_H}(\dot{\gamma}_0),$$

where we have fixed an embedding $j_H : S \to H$ for which the corresponding discrete series representation of $H(\mathbb{R})_\pm$ is generic with respect to some Whittaker datum and denote by subscript H the various transports under j_H, $\dot{\gamma}_0$ runs over the elements of $S_H(\mathbb{R})_\pm$ that are H-stably conjugate to $\dot{\gamma}$, and we have used $\tau_H/d_H = \tau'_H/d'_H$.

The right hand side of Theorem 3.2.1(1) then becomes

$$\sum_{\gamma \in H(\mathbb{R})/\mathrm{st}} \Delta[\mathfrak{w}, \mathfrak{e}, z](\dot{\gamma}, \delta) \sum_{\dot{\gamma}_0} \frac{\tau_H}{d_H}(\dot{\gamma}_0).$$

The first sum runs over elements of $H(\mathbb{R})$ that are related to δ, up to stable conjugacy under H. Each such stable conjugacy class consists of regular semisimple elliptic elements (because δ is such), and hence intersects $S_H(\mathbb{R})$. The second sum runs over elements $\dot{\gamma}_0 \in S_H(\mathbb{R})_\pm$ that lie in the H-stable class of the lift $\dot{\gamma} \in S_H(\mathbb{R})_\pm$ of γ. Since Δ is H-stably invariant in the first factor, its values at $\dot{\gamma}$ and $\dot{\gamma}_0$ are the same. We can combine the two sums together and obtain

$$\sum_{\gamma_0 \in S_H(\mathbb{R})} \Delta[\mathfrak{w}, \mathfrak{e}, z](\dot{\gamma}_0, \delta) \frac{\tau_H}{d_H}(\dot{\gamma}_0),$$

where now γ_0 runs over all elements of $S_H(\mathbb{R})$, equivalently all those that are related to δ, since the transfer factor vanishes for the others.

Having fixed the embeddings $j_\mathfrak{w}$ and j_H, they provide an isomorphism $S_\mathfrak{w} \to S_H$, and this isomorphism induces a bijection

$$\delta_0 \leftrightarrow \gamma_0$$

between the set of elements of $S_\mathfrak{w}(\mathbb{R})$ that are stably conjugate to δ and the set of elements of $S_H(\mathbb{R})$ related to δ. Using the basic property of transfer factors ref we obtain

$$\sum_{\delta_0 \in S_\mathfrak{w}(\mathbb{R})} \Delta[\mathfrak{w}, \mathfrak{e}, z](\dot{\gamma}_0, \delta_0)\langle s_\mathfrak{w}, \mathrm{inv}(\delta_0, \delta)\rangle \frac{\tau_H}{d_H}(\dot{\gamma}_0), \tag{3.2}$$

where δ_0 runs over the elements of $S_\mathfrak{w}(\mathbb{R})$ that are stably conjugate to δ, $\gamma_0 \in S_H(\mathbb{R})$ denotes the element corresponding to δ_0 under above bijection, and $\dot{\gamma}_0 \in S_H(\mathbb{R})_\pm$ is an arbitrary lift of γ_0.

We now unpack the transfer factor. It is given as

$$\Delta(\dot{\gamma}_0, \delta_0) = \epsilon \Delta_I^{-1}(\dot{\gamma}_0, \dot{\delta}_0)\Delta_{III}(\dot{\gamma}_0, \dot{\delta}_0),$$

where we recall that the term Δ_{IV} is missing because we are working with normalized characters and orbital integrals, and $\dot{\delta}_0 \in S_\mathfrak{w}(\mathbb{R})_{G/H}$ is an arbitrary lift of δ_0.

By construction, [ref]

$$\Delta_{III}(\dot\gamma_0,\dot\delta_0)=\frac{\tau_{\mathfrak{w}}}{\tau_H}(\dot\delta_0).$$

The following lemma completes the proof of Theorem 3.2.1(1).

Lemma 3.4.1. *For any $\dot\delta_0\in S_{\mathfrak{w}}(\mathbb{R})_{G/H}$, the following identity holds*

$$\Delta_I(\dot\gamma_0,\dot\delta_0)=\epsilon\cdot(-1)^{q(G_0)-q(H)}\cdot\prod_{\alpha\in R(S,G/H)^+}\arg(\alpha^{1/2}(\dot\delta_0)-\alpha^{-1/2}(\dot\delta_0)),$$

where $\dot\gamma_0\in S_H(\mathbb{R})_{G/H}$ is the image of $\dot\delta_0$ under the isomorphism $j_H\circ j_{\mathfrak{w}}^{-1}$.

Proof. We first investigate how both sides vary as functions of $\dot\delta_0$. Consider another strongly regular $\dot\delta_1$. Replacing $\dot\delta_0$ with $\dot\delta_1$ in the right hand side results in multiplication by $\prod_\alpha\arg(b_\alpha)$, where the product runs again over $R(S,G/H)^+$ and $b_\alpha=(\dot\delta_{1,\alpha}-\dot\delta_{1,-\alpha})/(\dot\delta_{0,\alpha}-\dot\delta_{0,-\alpha})$. By construction $\dot\delta_{0,-\alpha}=\sigma(\dot\delta_{0,\alpha})$, and the same holds for $\dot\delta_1$, from which follows $b_\alpha\in\mathbb{R}^\times$, and hence $\arg(b_\alpha)=\mathrm{sgn}(b_\alpha)$.

We now look at the left hand side. Replacing $\dot\delta_0$ by $\dot\delta_1$ multiplies $\mathrm{inv}(\dot\delta_0,\mathrm{pin})$ by $\prod_{\alpha>0,\sigma\alpha<0}\alpha^\vee(b_\alpha)$, and hence Δ_I by the Tate-Nakayama pairing of this 1-cocycle with the endoscopic element $s_{\mathfrak{w}}$. Since the torus S is elliptic, the conditions $\alpha>0$ and $\sigma\alpha<0$ are equivalent, and the value of the 1-cocycle at σ equals $\prod_{\alpha>0}\alpha^\vee(b_\alpha)$. This is the product over $\alpha>0$ of the images of the 1-cocycles $b_\alpha\in Z^1(\Gamma,R^1_{\mathbb{C}/R}\mathbb{G}_m)$ under the homomorphisms $\alpha^\vee:R^1_{\mathbb{C}/R}\mathbb{G}_m\to S$, so the change in $\Delta_{I,\pm}$ is given by $\prod_{\alpha>0}\langle b_\alpha,s_\alpha\rangle$, where s_α is the image of $s\in\widehat S$ under $\widehat\alpha:\widehat S\to\mathbb{C}^\times$. This is a Galois-equivariant homomorphism, with σ acting as inversion on $\mathbb{C}^\times$. Since s is σ-fixed, so is s_α, i.e. $s_\alpha\in\{\pm1\}\subset\mathbb{C}^\times$. By construction of the endoscopic group H, we have $s_\alpha=1$ precisely for $\alpha\in R(S,H)$. On the other hand, when $s_\alpha=-1$, then $\langle b_\alpha,s_\alpha\rangle=\mathrm{sgn}(b_\alpha)$.

We have thus shown that both sides of the identity multiply by the same factor upon replacing $\dot\delta_0$ by a different element $\dot\delta_1$. To establish the identity we may thus evaluate at an arbitrary element $\dot\delta_0$. For this we note that both sides descend to functions on $S_{\mathrm{ad}}(\mathbb{R})_{G/H}$, so we may assume that G is adjoint. Let $\rho^\vee\in X_*(S_{\mathfrak{w}})$ denote half the sum of the coroots that pair positively with $d\tau_{\mathfrak{w}}$. Since complex conjugation acts on $X_*(S_{\mathfrak{w}})$ by multiplication by -1, we have $X=\rho^\vee(-ir)\in\mathrm{Lie}(S_{\mathfrak{w}})(\mathbb{R})$ for any $r\in\mathbb{R}$. We will choose $r>0$ small enough and set $\ddot\delta=\exp(X)\in S_{\mathfrak{w}}(\mathbb{R})_{\pm\pm}$, where we are using the exponential map $\mathrm{Lie}(S_{\mathfrak{w}})(\mathbb{R})\to S_{\mathfrak{w}}(\mathbb{R})_{\pm\pm}$ discussed in [Kal19a, §3.7].

Considering the right hand side, we have for each $\alpha\in R(S,G)$

$$\alpha^{1/2}(\dot\delta_0)=\dot\delta_{0,\alpha}=\exp(d\alpha(X)/2)=e^{-ir\langle d\alpha,\rho^\vee\rangle/2}.$$

Then

$$\alpha^{1/2}(\dot\delta_0)-\alpha^{-1/2}(\dot\delta_0)=-2i\sin(r\langle d\alpha,\rho^\vee\rangle/2).$$

Choosing $r>0$ so that $r\langle d\alpha,\rho^\vee\rangle/2<\pi$ for all $\alpha\in R(S,G)^+$ we obtain

$$\arg(\alpha^{1/2}(\dot\delta_0)-\alpha^{-1/2}(\dot\delta_0))=(-i)^{\#R(S,G/H)^+}.$$

Choose a pinning $(T_0,B_0,\{X_\alpha\})$, which, together with $\Lambda(x)=e^{2\pi ix}$, produces the chosen Whittaker datum $\mathfrak{w}$. Corollary 1.10.5 then shows that $\rho^\vee(-i)$ is $G_0(\mathbb{R})$-conjugate lies in the Kostant section of that pinning. Thus X is $G_0(\mathbb{R})$-conjugate to the Kostant section of the pinning $(T_0,B_0,\{rX_\alpha\})$. If we construct

$\Delta_I(\dot\gamma_0, \dot\delta_0)$ with respect to this rescaled pinning, then [Kal22, Lemma 4.1.4] show that $\Delta_I(\dot\gamma_0, \dot\delta_0) = 1$. On the other hand, the rescaled pinning together with the character $\Lambda_r(x) = e^{r2\pi i x}$ also produces the Whittaker datum $\mathfrak{w}$. According to Lemma 1.8.3 the right hand side equals 1. $\square$

3.5 The right hand side: classical set-up

In this subsection we will show that the right hand side of the identity of Theorem 3.2.1(2) is also equal to (3.1). The initial arguments of §3.4, which applied to the right hand side of the identity in Theorem 3.2.1(1), have direct analogs in the setting of Therem 3.2.1(2) and show that the right hand side of that identity equals the analog of the expression (3.2), which is given by

$$\sum_{\delta_0 \in S_{\mathfrak{w}}(\mathbb{R})} \Delta[\mathfrak{w}, \mathfrak{e}, z](\gamma_1, \delta_0)\langle s_{\mathfrak{w}}, \mathrm{inv}(\delta_0, \delta)\rangle \frac{\tau'_{H_1}}{d'_{H_1}}(\gamma_1), \tag{3.3}$$

where we have used the parameter φ_{H_1} of the z-extension H_1 to obtain the character τ'_{H_1} of the torus $S_{H_1}(\mathbb{R})$.

It is the handling of the transfer factor that is slightly different. Indeed, in this setting without covers the transfer factor is given by

$$\Delta = \epsilon \Delta_I^{-1} \Delta_{II} \Delta_{III_2}.$$

The construction of the pieces involves a choice of an admissible isomorphism $S^H \to S_{\mathfrak{w}}$, which we take to be $j_{\mathfrak{w}} \circ j_H^{-1}$, as we did in §3.4. It further involves choices of χ-data and a-data for S. We take ρ-based χ-data, and $(-\rho)$-based a-data, so that $\chi_\alpha(x) = \arg(x)$ when $\alpha > 0$ and $a_\alpha = i$ when $\alpha < 0$.

We claim that

$$\Delta_{III_2}(\gamma_1, \delta_0) = \frac{\tau'_{\mathfrak{w}}(\delta_0)}{\tau'_{H_1}(\gamma_1)}.$$

We will explain this under the assumption that the z-pair is trivial, i.e. there exists an L-isomorphism ${}^L\eta : {}^L H \to \mathcal{H}$, the general case being entirely analogous by requiring more cumbersome notation. We then have the commutative diagram

$$\begin{tikzcd}
& {}^L S_H \arrow[r, "\rho^H"] \arrow[dd, "a"'] & {}^L H \arrow[dd, "{}^L\eta"] \\
W_{\mathbb{R}} \arrow[ur, "\varphi_S^H"] \arrow[dr, "\varphi_S"'] \arrow[urr, bend left, "\varphi^H"] \arrow[drr, bend right, "\varphi"'] & & \\
& {}^L S \arrow[r, "\rho"] & {}^L G,
\end{tikzcd}$$

where the horizontal arrows are the L-embeddings obtained via ρ-based and ρ^H-based χ-data, respectively. We have $\Delta_{III_2}(\gamma_1, \delta_0) = \langle a, \delta_0\rangle$, where $a \in Z^1(W_{\mathbb{R}}, \widehat{S})$ is the 1-cocycle that makes the above diagram commute, the pairing is the Langlands pairing, and $\delta_0 \in S(\mathbb{R})$ is the image of $\gamma_0 \in S^H(\mathbb{R})$ to $S(\mathbb{R})$ under the chosen fixed admissible isomorphism. The claim now follows from the above commutative diagram and the fact that τ and τ^H are the characters with parameters φ_S and φ_S^H.

Next we consider
$$\frac{\Delta_{II}(\gamma_0,\delta)}{d'_H(\gamma_0)}.$$
By definition,
$$\Delta_{II}(\gamma_0,\delta)=\frac{\Delta_{II}^G(\gamma_0,\delta)}{\Delta_{II}^H(\gamma_0,\delta)}.$$
With the chosen a-data and χ-data we have
$$\Delta_{II}^H(\gamma_0)=\prod_{\substack{\alpha\in R(S^H,H)\\ \langle\alpha,\rho^H\rangle>0}}\arg\Big(\frac{\alpha(\gamma_0)-1}{-i}\Big)=i^{\#R(S^H,H)/2}\cdot\prod_{\substack{\alpha\in R(S^H,H)\\ \langle\alpha,\rho^H\rangle>0}}\arg(\alpha(\gamma_0)-1).$$
On the other hand,
$$(\alpha(\gamma_0)-1)(1-\alpha(\gamma_0)^{-1})=\alpha(\gamma_0)+\alpha(\gamma_0)^{-1}-2=2(\mathrm{Re}(\alpha(\gamma_0)-1)<0.$$
Hence
$$\Delta_{II}^H(\gamma_0,\delta)d'_H(\gamma_0)=(-i)^{\#R(S^H,H)/2}.$$
In the same way one shows
$$\Delta_{II}^G(\gamma_0,\delta)d'_{G_0}(\delta_0))=(-i)^{\#R(S,G_0)/2}.$$
and we conclude
$$\frac{\Delta_{II}(\gamma_0,\delta)}{d'_H(\gamma_0)}=\frac{i^{\#R(S^H,H)/2-\#R(S,G_0)/2}}{d'_{G_0}(\delta_0)}.$$
With this (3.3) becomes
$$(-1)^{q(H)}i^{\#R(S^H,H)/2-\#R(S,G_0)/2}\epsilon\sum_{\delta_0\in S_{\mathfrak{w}}(\mathbb{R})}\Delta_I(\gamma_0,\delta)^{-1}\langle s_{\mathfrak{w}},\mathrm{inv}(\delta_0,\delta)\rangle\cdot\frac{\tau_{\mathfrak{w}}(\delta_0)}{d'_{G_0}(\delta_0)}.$$
We have
$$\Delta_I(\gamma_0,\delta)^{-1}=\langle s,\lambda\rangle^{-1}$$
where $\lambda\in H^1(\Gamma,S)$ is the splitting invariant of S relative to the chosen a-data. Using Lemma 1.8.3 we obtain
$$(-1)^{q(G_0)}\langle s,\lambda\rangle^{-1}\sum_{\delta_0}\langle s,\mathrm{inv}(\delta_0,\delta)\rangle\cdot\frac{\tau_{\mathfrak{w}}(\delta_0)}{d'_{G_0}(\delta_0)}.$$

The following lemma completes the proof.

Lemma 3.5.1. *The splitting invariant λ of S computed in terms of $(-\rho)$-based a-data is trivial.*

Proof. Let $X=\rho^\vee(-i)\in\mathrm{Lie}(S_{\mathrm{sc}})$. This element is Galois-fixed and thus lies in $\mathrm{Lie}(S_{\mathrm{sc}})(\mathbb{R})$. For every $\alpha>0$ the complex number $d\alpha(X)$ is a positive real multiple of $-i$. Therefore we can replace the $(-\rho)$-based a-data with the a-data $d\alpha(X)$ without changing the splitting invariant. According to [Kot99, Theorem 5.1] and Corollary 1.10.5, λ is trivial. $\square$

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