Referee report on
Discrete series L-packets for real reductive groups
by Jeffrey Adams and Tasho Kaletha

The concept of discrete series representations comes from the theory of harmonic analysis on semisimple Lie groups. Given such a group G, one is interested in analyzing the space $L^2(G)$ on which G acts by translations: the goal is to write this representation as a direct integral of irreducibles. To do this, one needs to describe the Plancherel measure on the set of unitary representations of G. This was achieved by Harish-Chandra, whose approach — called the philosophy of cusp forms — was to reduce the problem to the case of discrete series representations, i.e. those representations that appear discretely in the support of the Plancherel measure.

These ideas played a major role in shaping what is known today as the Langlands program. In its local version, Langlands reciprocity asserts that the representations of a reductive group G over a local field should be organized into finite packets, which (roughly speaking) correspond to homomorphisms from the absolute Galois group to the dual group ${}^LG$. Shelstad elucidated the structure of these packets by showing that they satisfy certain character relations coming from endoscopic groups. Unsurprisingly, and true to Harish-Chandra's ideas, the case of discrete series (and, more generally, tempered) representations plays a prominent role in this theory.

The goal of the paper under review is to give a modern treatment of the construction, parametrization, and endoscopic character identities for discrete series L-packets of real reductive groups. Although the main results have been known for a while, there are good reasons for writing such an article at this point in time:

First, because of their importance in representation theory and number theory, the topics discussed in this paper have been in a state of almost constant development for the past fifty years. Because of this, there are not many sources in the literature that offer a state-of-the-art overview of these results. This paper will certainly bridge the existing gap.

Secondly, in addition to providing a modern account of the existing theory, this paper incorporates new developments not present in the work of Langlands and Shelstad. The most important of these is the use of double covers of tori and endoscopic groups (as developed recently by the second author) which allows one to obtain a much cleaner and simpler framework for the relevant results. In particular, the authors prove a new form of endoscopic character identities, which in this work become free of arbitrary choices. The paper also includes new results as yet unavailable in the literature, such as a criterion for detecting generic essentially discrete series representations for a fixed Whittaker datum.

This paper does a great job of striking a balance between providing ample background material and getting to the main results quickly and efficiently. The large preliminary section collects various results that are otherwise hard to find in a single publication. The approach taken here is meant to be precise, but not necessarily encyclopedic; where details are omitted to keep the paper in manageable in length, precise references are provided instead.

Throughout the paper, the authors choose a less-is-more approach, for example by restricting their attention to discrete series (as opposed to tempered) representations and focusing on pure (as opposed to rigid) inner twists when explaining the main arguments. In doing so, they successfully convey the main ideas without obscuring them with technical details, and present this highly technical topic in a breathable and readable way.

Because of all this, the paper under review represents a valuable addition to the literature devoted to this subject. I expect it to be highly useful to a very wide audience interested in this topic, and I am therefore happy to recommend it for publication in ENT.

Below I list some questions and comments, mostly of minor/typographical nature, which should be considered before publication:

1) page 3, line 16: "to an equivalence class [...] Langlands constructs" $\to$ "to an equivalence class [...] Langlands attaches"
2) page 5, line 5 of third paragraph: "(1.3) hold" $\to$ "(1.3) holds"
3) page 10, line 7: $\sigma \mapsto \xi^{-1}\sigma(x)$. I assume this should be $\sigma \mapsto \xi^{-1}\sigma(\xi)$, where $\xi^{-1}\sigma(\xi)$ is to be interpreted as the element x such that $\xi^{-1}\sigma(\xi)$ is equal to conjugation by x?
4) page 10, line 12: To be consistent with notation used elsewhere, I suggest using $Z^1(F, G)$ instead of $Z^1(\Gamma, G)$.
5) page 10, line 15 (and page 11, line 3): The group fixed in §2.2 is denoted G, so $G_0(F_s)$ (resp. $G_0(\mathbb{C})$) should be $G(F_s)$ (resp. $G(\mathbb{C})$).
6) page 11, §2.3: This is the first time the notation $\Omega(T, G)$ appears; I would suggest reminding the readers that it denotes the Weyl group.
7) page 11, line -3: $\sigma(\hat{\jmath}) = \sigma_{\widehat{G}} \circ \hat{\jmath} \circ \sigma_{\widehat{S}}$. Since we're not assuming $F = \mathbb{R}$ in this section, should $\sigma_{\widehat{S}}$ be $\sigma_{\widehat{S}}^{-1}$?
8) page 12, line 18: "for a fixed j" $\to$ "for a fixed j_1"
9) page 12, lines -16, -15: "Pick a Γ-stable Borel pairs" $\to$ "Pick Γ-stable Borel pairs" (or "Pick a Γ-stable Borel pair")
10) page 13, line 7: It is clear that $gz(\sigma)\sigma(g)^{-1}$ means "the cocycle sending σ to this expression", but strictly speaking, this notation is not valid. One solution is perhaps to say something like "${}^g z$, where ${}^g z(\sigma) = gz(\sigma)\sigma(g)^{-1}$". (Alternatively, if we're working with $\mathbb{C}/\mathbb{R}$, one could simply identify the cocycle z with $z(\sigma)$.) The same comment applies to page 14, line 3.
11) page 13, line 9: Similarly, it's probably better to write $(j \circ x) \cdot z$ than $j(x) \cdot z$
12) page 14, line 18: "more generally, inv$= 1$" $\to$ "more generally, if inv$= 1$"
13) page 14, lines -6 and -8: the symbols t and γ appearing here should be the same
14) page 15, lines 6 and 7: "we we" $\to$ "we"

15) page 17, line -7: $[\mathcal{G}^\natural \cdot Z(\mathcal{G}) : \mathcal{G}]$ should be $[\mathcal{G} : \mathcal{G}^\natural \cdot Z(\mathcal{G})]$

16) page 19, lines -1 and -4: There is a missing reference.

17) page 19, line -2: "observation allow" $\to$ "observation allows" (or "These observations allow")

18) page 20, paragraph before Theorem 2.7.7: $S_{\mathrm{ad}}(\mathbb{R})_\pm$ is the first time the subscript $\pm$ appears; this notation is also used throughout the section. While this is compatible with the use of $\pm$ to denote covers of endoscopic groups, the notation in §2.6 (and later sections, like §4.1) involves G (the ambient group). A similar comment applies to the big cover, which is denoted using the subscript $\pm\pm$ in §5.4.

19) page 20, line 2 of Theorem 2.7.7: "is the preimage in $\mathcal{G}$ is" $\to$ "is the preimage in $\mathcal{G}$ of"

20) page 20, equation (2.6). Is there a reason for using $\dot{\delta}$ on the left-hand side and δ on the rhs? The notation $q(G_{sc})$ should be explained in the paragraph following Remark 2.7.8.

21) page 23, line 4: I suggest including η: "an L-embedding $\mathcal{H} \to {}^L G$" $\to$ "an L-embedding $\eta : \mathcal{H} \to {}^L G$"

22) page 24, line 8 of §2.9 "[...] setting order" $\to$ "[...] setting, in order"

23) page 26, line 14: "stably" $\to$ "stable"

24) page 28, line -4 of §2.9: "becaus e" $\to$ "because"

25) page 29, line -3 in proof of Corollary 2.10.2: SJ and J should be SO and O

26) page 30, lines 23/24: "ϕ is [...] stable". Do you mean "ϕ is [...] constant"?

27) page 31, line 1: $f^{H_1}(\dot{\gamma})$ should be $f^{H_\pm}(\dot{\gamma})$

28) page 31, line 2: "factors [...] depends" $\to$ "factors [...] depend"

29) page 33, line 5: although this is obvious from context, this is the first time $\mathcal{U}(\mathfrak{g})$ of appears, so I suggest adding something like "where $\mathcal{U}(\mathfrak{g})$ is the u.e.a. of $\mathfrak{g}$".

30) page 34, line 8: "aide" $\to$ "aid"

31) page 40, displayed equation defining $AC(W)$: $\lim_{i\to\infty}$ should be $\lim_{n\to\infty}$

32) page 41, line 14: $\mathcal{O}$ should be $\mathcal{O}_{\mathbb{R}}$

33) page 43, line 11: "and consists" $\to$ "consists"

34) page 44, line -4 in proof of (4): "conjutation" $\to$ "conjugation"

35) page 45, line 5: "elements if" $\to$ "elements of"

36) page 46, line 4: Although the notation $\mathcal{J}(F)$ makes sense, I suggest using $\mathcal{J}(\mathbb{R})$ to ensure compatibility with the rest of §4.2.

37) page 46, Lemma 4.2.2: here and in §4.3, double parentheses are used when writing $\Pi_\varphi((G, \xi, z))$, in contrast with $\Pi_\varphi(G, \xi, z)$ used in Definition 4.2.1.

38) page 46, proof of Lemma 4.2.2: The last sentence of the proof is unfinished.

39) page 46, lines 1 and 5 of §4.3: Like in 4), I suggest using $H^1(\mathbb{R}, S)$ instead of $H^1(\Gamma, S)$ here.

40) page 47, line 9: Should $x \cdot (z, j) = (x \cdot z, j)$ be $x \cdot (z, j) = (j(x) \cdot z, j)$? A similar question applies to line 11, as well as other instances of (x, j) appearing on this page.

41) page 47, line 12 of §4.4: $g(g^{-1}x\sigma(g), j)g^{-1}$ should be $g(g^{-1}x\sigma(g), j)$ (there is no right action).

42) page 47, line 13 of §4.4: $(x, \mathrm{Ad}(\overline{g}), j)$ should be $(x, \mathrm{Ad}(\overline{g}) \circ j)$

43) page 50, Theorem 5.2.1: The notation $\Delta[\mathfrak{w}, \mathfrak{e}, z]$ and $\Delta[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}, z]$ should be explained.

44) page 52, line -18: "is also" $\rightarrow$ "also"

45) page 53, line -10: "$d\tau$ With" $\rightarrow$ "$d\tau$. With"

46) page 54, line 8: Although it is clear what it means, I think this is the first time the notation ρ_{π_j} appears.

47) page 54, line 10: This is not crucial, but up until this point s always appears as the second of the two terms (as in $\langle \pi, s \rangle$) insted of the first (i.e. $\langle s, \pi \rangle$). The latter prevails in the remainder of §5.

48) page 56, line -4: Should pin be $\mathcal{P}$?

49) page 57, line 13: The reference [Kal19a] points to the arXiv version of the preprint, but this version does not include section 3.7 to which the inline citation refers. However, there is another version available on the second author's personal webpage, and that one does include §3.7, along with the relevant discussion of the exponential map. I suggest updating the arXiv entry.

50) page 57, line -14: $K(rX_\kappa)$ should be $\mathcal{K}(rX_\kappa)$. The same comment applies to line 12 of page 60.

51) page 59, line 17: The sentence continues, so there should be no period following the displayed equation.