A RESULT OF HARISH CHANDRA

Let G be a connected real reductive group. We let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. Let θ be a Cartan involution of G.

Let G_r be the open set of regular semisimple elements if G. Suppose H is a θ-stable Cartan subgroup of G, and set $\mathfrak{h} = \mathrm{Lie}(H) \otimes \mathbb{C}$ and $H_r = H \cap G_r$. Define

$$\Delta(h) = (\det(\mathrm{Ad}(h)^{-1} - 1)|_{\mathfrak{g}/\mathfrak{h}} \quad (h \in H_r).$$

We identify the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ with the algebra of left-invariant differential operators on G. Let $\mathfrak{Z}$ be the center of $\mathfrak{U}(\mathfrak{g})$. Similarly we identify $S(\mathfrak{h})$ with the algebra of differential operators on H. Let W be the Weyl group of H in G, and let $\gamma : \mathfrak{Z} \to S(\mathfrak{h})^W$ be the Harish Chandra isomorphism. Let G_r be the open set of regular semisimple elements of G, and let C^∞ be the smooth function on G_r.

Lemma 0.0.1. *Suppose $f \in C^\infty(G_r)$ is a class function. Then*

$$(zf)(h) = |\Delta(h)|^{-\frac{1}{2}}\gamma(z)(|\Delta|^{\frac{1}{2}} f)(h) \quad (h \in H_r).$$

Proof. This statement can be found in the proof of [HC65, Lemma 24]. To be precise, write $h = \exp(H)$ with $H \in \mathfrak{g}$. Then there is a neighborhood U of H such that the the map $\exp : U \to U_G = \exp(H)$ is an analytic diffeomorphism [HC65, Section 10]. Let ξ be the analytic, non-zero function on U of [HC65, Section 10]. If ϕ is a locally invariant C^∞ function on U (see [HC64, Section 8]) let $f_\phi(\exp X) = \xi(X)^{-1}\phi(X)$. The map $\phi \to f_\phi$ is a bijection between the smooth locally invariant functions on U and U_G.

According to the proof of [HC65, Lemma 24] it follows from [HC65, Lemma 17] and [HC56, Theorem 2] that

$$(zf_\phi)(h) = |\Delta(h)|^{-\frac{1}{2}}\gamma(z)(|\Delta|^{\frac{1}{2}} f_\phi)(h) \quad (h \in H_r). \tag{0.1}$$

By the preceding discussion $f|_{U_G}$ is a locally invariant function, so $f_{U_G} = f_\phi$ for some locally invariant function ϕ on U. The result follows. ☐

1 DISCUSSION

Surely this result is known, but I could not find a clear statement and proof. In particular I do not see how (0.1) follows from the cited results. Here are a few comments. All references are to [HC65] unless otherwise noted.

The references for the proof of (0.1) are Lemma 17 and [HC56, Theorem 2]. I don't see how Lemma 17 applies, except as a step in the proof of Lemma 18, which *is* relevant. However Lemma 18 refers to the locally defined differential operator $\delta_a(z)$ (see Section 5). On the other hand [HC56, Theorem 2] refers to the differential operator $\beta(z)$ which is defined on all of H_r. I *think* that the proof is using the fact that if we write $h = ay$ with y near the identity then $\beta(z)(f)(ay) = \delta_a(z)(f)(ay)$. However I was unable to convince myself this is true.

Note that Lemma 13 says that

$$\delta_a(z)(f)(h) = |\Delta_a(h)|^{-\frac{1}{2}}\gamma(z)(|\Delta_a|^{\frac{1}{2}}f)(h)$$

where

$$\Delta_a(h) = \Delta(ah).$$

This is very close to the statement needed for the result, except that it involves the function Δ_a, not Δ. This has to do with the issue of defining the differential operator on H_r using the local expression at each point. It is worth pointing out that there are some fraught technical details here. For example see the proof of Lemma 23 and [HC56, Lemma 25]. Finally, note that Harish Chandra does *not* cite Lemma 13 in his proof, but rather the more distantly related [HC56, Theorem 2].

The conclusion is that I don't know how to fill in the details of the proof.

References

[HC56] Harish-Chandra, *The characters of semisimple Lie groups*, Trans. Amer. Math. Soc. **83** (1956), 98–163. MR 80875

[HC64] ———, *Invariant distributions on Lie algebras*, Amer. J. Math. **86** (1964), 271–309. MR 161940

[HC65] ———, *Invariant eigendistributions on a semisimple Lie algebra*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 27, 5–54. MR 180630