Discrete series L-packets for real reductive groups

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Abstract

We give a modern exposition of the construction, parameterization, and character relations for discrete series L-packets of real reductive groups, which are fundamental results due to Langlands and Shelstad. This exposition incorporates normalized geometric transfer factors, the canonical double covers of tori and endoscopic groups. We also prove some new results, such as a simple criterion for detecting generic representations for a prescribed Whittaker datum, and an explicit formula for the term Δ_I in terms of covers of tori.

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1 Introduction

The construction of the discrete series of representations for a real reductive group was a fundamental achievement of Harish-Chandra and a key step towards understanding the tempered representations and obtaining an explicit Plancherel formula. Motivated by the emerging theory of the trace formula, Langlands organized the tempered representations into packets [Lan89] and Shelstad proved that these packets satisfy character relations with respect to endoscopic groups [She82]. These fundamental results lie at the core of the stabilization of the spectral side of the Arthur-Selberg trace formula and are the cornerstone of many applications of the Langlands program to arithmetic questions.

In this paper we give an exposition of these important results that is largely self-contained and incorporates a number of developments and insights that have occurred since the results were originally obtained. We focus on the case of discrete series representations. Our motivation to do so is two-fold. First, the available expositions of the internal structure and character identities treat the general case of tempered representations, which is more complicated due to the reducibility of parabolic induction and the theory of the Knapp-Stein R-group. The case of discrete series representations on the other hand is more direct and is a good setting in which the basic ideas become visible. At the same time, it is sufficient for many important applications.

Second, a number of developments over the ensuing years have enabled a simplified construction of the L-packets and their internal structure, a more natural statement of the character identities, and a more direct approach to their proof. These developments include the introduction of pure and rigid inner forms and the resulting normalizations of transfer factors [Vog93], [Kal11], [Kal16b], the introduction of double covers of tori and the recognition of the role they play in the classification of discrete series representations and the construction of their packets [AV92], [AV16], [Kal19a], and the introduction of double covers of endoscopic groups, which aleviate the need for ad-hoc constructions and technical work-arounds in the considerations of endoscopy [Kal22].

We utilize these developments to give a clean and direct account of the construction, internal structure, and character identities, of discrete series L-packets. This exposition is rather different from the classical literature and follows more closely the developments in the p-adic case such as [Kal19b], whose combination with double covers was hinted at in [Kal19a].

To say more precisely what we do and how it differs from classical expositions, we first review the classical results in the setting of discrete parameters. The notation used in this review will be explained in §2. Let G be a connected reductive $\mathbb{R}$-group, $\widehat{G}$ its complex dual group, and ${}^LG = \widehat{G} \rtimes \Gamma$ its L-group, where Γ is the Galois group of $\mathbb{C}/\mathbb{R}$. A discrete Langlands parameter is an L-homomorphism $W_\mathbb{R} \to {}^LG$ whose image is not contained in a proper parabolic subgroup of ${}^LG$, or equivalently for which the centralizer group $S_\varphi = \text{Cent}(\varphi, \widehat{G})$ contains $Z(\widehat{G})^\Gamma$ with finite index; the finite group $S_\varphi/Z(\widehat{G})^\Gamma$ turns out to be an abelian 2-group. Two such φ are considered equivalent if they are $\widehat{G}$-conjugate. To an equivalence class of φ Langlands constructs in [Lan89, §3] a finite set $\Pi_\varphi(G)$ of discrete series representations of $G(\mathbb{R})$. This construction is a bit complicated due to the fact that Langlands extracts by hand the explicit data contained in φ and uses it to specify the representations in $\Pi_\varphi(G)$.

In [She82], Shelstad establishes an injective map $\pi \mapsto \langle \pi, - \rangle$ from Π_φ to the set of characters of the finite abelian group $S_\varphi/Z(\widehat{G})^\Gamma$. This map depends on some auxiliary choices. Shelstad then proceeds to prove the following result. Let $(H, s, \mathcal{H}, \eta)$ be an endoscopic datum with $\eta(s) \in S_\varphi$. Assume that there exists, and fix, an L-isomorphism ${}^LH \to \mathcal{H}$ and let ${}^L\eta : {}^LH \to {}^LG$ denote its composition with $\eta : \mathcal{H} \to {}^LG$. We note that such an L-isomorphism need not exist in general. By construction φ factors through ${}^L\eta$ as $\varphi = {}^L\eta \circ \varphi'$ for a necessarily discrete parameter $\varphi' : W_\mathbb{R} \to {}^LH$. Let $\Pi_{\varphi'}(H)$ be the associated L-packet on $H(\mathbb{R})$. Shelstad proves the existence of a function $\Delta : H(\mathbb{R})_{\mathrm{sr}} \times G(\mathbb{R})_{\mathrm{sr}} \to \mathbb{C}$, which is implicitly defined and unique up multiplication by the constant -1, and shows that there exists a constant $c(\Delta)$ depending on Δ such that for all $\delta \in G(\mathbb{R})_{\mathrm{sr}}$ the following character identity holds

$$c(\Delta) \sum_{\pi \in \Pi_\varphi(G)} \langle \pi, s \rangle \Theta_\pi(\delta) = \sum_\gamma \Delta(\gamma, \delta) \sum_{\sigma \in \Pi_{\varphi'}(H)} \Theta_\sigma(\gamma), \tag{1.1}$$

where γ runs over the set of (representatives for) the stable conjugacy classes of strongly regular semi-simple elements of $H(\mathbb{R})$. The proof of this identity is rather complicated due to the implicit nature of the functions $\pi \mapsto \langle \pi, - \rangle$ and Δ, and the constant $c(\Delta)$.

In [LS87], Langlands and Shelstad succeeded in making the functions Δ explicit. More precisely, they explicitly defined such functions for all local fields of characteristic zero, in such a way that they satisfy a global product formula related to the stabilization of the Arthur-Selberg trace formula. These functions still remained ambiguous up to multiplication by a non-zero complex root of unity (of order 2 over $\mathbb{R}$). In a subsequent paper [LS90, Theorem 2.6.A] they proved that these explicitly defined functions must coincide up to scalar with those implicitly defined by Shelstad earlier. This result was indirect and relied on Shelstad's identity (1.1). This prompted Arthur to pose the question [Art08] of rederiving Shelstad's identity (1.1) directly in terms of the factors of [LS87], and without using the arguments of [LS90] and the implicit work of [She82]. This was undertaken by Shelstad in her papers [She08a] and [She10]. There she defines explicitly the so called "spectral transfer factors" $\Delta(\sigma, \pi)$ and "compatibility factors" $\Delta(\sigma, \pi, \gamma, \delta)$, where σ and π are discrete series representations of $H(\mathbb{R})$ and $G(\mathbb{R})$ respectively, $\gamma \in H(\mathbb{R})_{\mathrm{sr}}$, $\delta \in G(\mathbb{R})_{\mathrm{sr}}$. The spectral transfer factor is, just like the geometric transfer factor $\Delta(\gamma, \delta)$, well-defined only up to a non-zero complex scalar multiple. The compatibility factor $\Delta(\sigma, \pi, \gamma, \delta)$ is canonical. Its purpose is to link the arbitrary choices for normalizations of geometric and spectral factors, by saying that a chosen normalization of $\Delta(\gamma, \delta)$ is compatible with a chosen normalization of $\Delta(\sigma, \pi)$ if the identity

$$\Delta(\sigma, \pi) = \Delta(\sigma, \pi, \gamma, \delta) \cdot \Delta(\gamma, \delta)$$

holds for all possible $\sigma, \pi, \gamma, \delta$, see [She10, §4]. Just like the variable γ matters only up to stable conjugacy, the variable σ matters only up to L-packets, so we may write $\Delta(\varphi', \pi)$ in place of $\Delta(\sigma, \pi)$ whenever $\sigma \in \Pi_{\varphi'}(H)$. With this notation, Shelstad proves the following modern version of her classical result

$$\sum_{\pi \in \Pi_\varphi(G)} \Delta(\varphi', \pi) \Theta_\pi(\delta) = \sum_\gamma \Delta(\gamma, \delta) \sum_{\sigma \in \Pi_{\varphi'}(H)} \Theta_\sigma(\gamma). \tag{1.2}$$

The advantage of this result is that the terms $\Delta(\gamma, \delta)$ and $\Delta(\varphi', \pi)$ are now explicitly constructed. The relationship between the spectral factor $\Delta(\varphi', \pi)$

and the character $\langle \pi, s\rangle$ in the previous formulation is

$$\frac{\Delta(\varphi', \pi_1)}{\Delta(\varphi', \pi_2)} = \frac{\langle s, \pi_1\rangle}{\langle s, \pi_2\rangle}.$$

Note that both fractions are well-defined, since the ambiguity of all objects cancels in the fractions. This relative identity is weaker than what is desired: a version of (1.1) without implicit constants and arbitrary choices.

The next step towards that goal was given by the "generic packet conjecture" formulated by Shahidi in [Sha90], which in its strong form states that any tempered L-packet containes a unique member that is generic with respect to a fixed Whittaker datum $\mathfrak{w}$. The validity of this conjecture for real groups can be extracted from the work of Kostant and Vogan. This prompted Kottwitz and Shelstad to single out [KS99, §5.3] a normalization of the geometric factor $\Delta(\gamma, \delta)$ depending on $\mathfrak{w}$ when G is a quasi-split group. Shelstad was able to prove [She08b, Theorem 11.5] that the compatibly normalized spectral transfer factor $\Delta(\varphi, \pi)$ is equal to 1 when π is that unique generic constituent. If we normalize the pairing $\langle \pi, s\rangle$ to equal 1 for that same π, then for general $\pi \in \Pi_\varphi(G)$ one has

$$\Delta(\varphi', \pi) = \langle \pi, s\rangle$$

and this finally leads to the clean formulation

$$\sum_{\pi \in \Pi_\varphi(G)} \langle \pi, s\rangle \Theta_\pi(\delta) = \sum_\gamma \Delta(\gamma, \delta) \sum_{\sigma \in \Pi_{\varphi'}(H)} \Theta_\sigma(\gamma) \tag{1.3}$$

of the character identities, where $\pi \mapsto \langle \pi, -\rangle$ is an injection of $\Pi_\varphi(G)$ into $(S_\varphi/Z(\widehat{G})^\Gamma)^*$, uniquely determined to make the above identity true, and $\Delta(\gamma, \delta)$ is the Whittaker normalization of the geometric transfer factor.

The main problem with (1.3) was that it was only available for quasi-split groups (slightly more generally, quasi-split K-groups), because other groups lack the concept of a Whittaker datum. For those, Statement (1.2) was still the only option.

The ideas that led to the resolution of this problem originated in the work of Adams-Barbasch-Vogan [ABV92]. There the authors showed that the theory becomes more balanced if one considers all groups in a given inner class together. However, one had to rigidify the concept of an inner form appropriately. An easy, but incomplete, way to do this was introduced in [Vog93], under the name "pure inner form". It was shown in [Kal11, §2.2] that pure inner forms can be used to normalize transfer factors beyond the quasi-split case. Further ideas were needed to cover all inner forms. These ideas were based on the notion of Galois gerbe from [LR87], and began taking shape in Kottwitz's study [Kot97] of isocrystals with additional structure. They led to the construction of rigid inner forms and the resulting normalizations of transfer factors for all groups in [Kal16b] for characteristic zero local fields, and in [Dil20] for positive characteristic local fields. Using those, it was shown in [Kal16b, §5.6] that a suitable formulation of (1.3) holds for all real groups. More precisely, there is a bijection $\pi \mapsto \langle \pi, -\rangle$ between the union of the packets $\Pi_\varphi(G)$ as G varies over all rigid inner forms of a given group, and $\pi_0(S_\varphi^+)^*$, where S_φ^+ is the preimage of S_φ in the universal cover of $\widehat{G}$. This bijection is normalized so that $\langle \pi, -\rangle = 1$ when π is the unique generic member in the L-packet for φ on the unique quasi-split inner form. Then (1.3) hold for all G, provided $\Delta(\gamma, \delta)$ has been normalized using the rigid inner form data as in [Kal16b, §5.3]. This claim was proved in [Kal16b, §5.6] for all tempered φ, not just the discrete ones, using the results of Shelstad from [She10] and [She08b]. Thus, while the statement was clean, the proof was again roundabout and involved the construction and study of objects, such as $\Delta(\varphi', \pi)$ and $\Delta(\sigma, \pi, \gamma, \delta)$, that were not needed for the statement. Another issue with the proof was that it relied on [She08b, Theorem 11.5], whose proof was not direct, but rather involved a series of reductions steps to the case of $\mathrm{SL}_2(\mathbb{R})$, which was then left to the reader as a "well-known computation", but, at least to the authors of this article, that computation was not known in any form other than the result for general groups that we will formulate and prove in this note, see Proposition 3.2.2 and Lemma 5.5.1.

The final issue in the classical set-up that we want to discuss is that of the assumed existence of an L-embedding ${}^LH \to {}^LG$. In general such an L-embedding does not exist. Instead, one is given an endoscopic datum $(H, s, \mathcal{H}, \eta)$, where η is an L-embedding $\mathcal{H} \to {}^LG$, but there is generally no L-isomorphism ${}^LH \to \mathcal{H}$. In the classical set-up one makes an arbitrary choice of a z-extension $H_1 \to H$ and an L-embedding $\mathcal{H} \to {}^LH_1$ and then works not with representations σ of $H(\mathbb{R})$, but rather with representations σ_1 of $H_1(\mathbb{R})$ that transform under certain non-trivial character of the kernel $H_1(\mathbb{R}) \to H(\mathbb{R})$. This brings yet another amount of arbitrariness into the theory.

Work of Adams and Vogan [AV92] introduced the idea that groups of the form $\mathcal{H}$ should be understood as L-groups of topological covers of $H(\mathbb{R})$. In [Kal22] it was shown that, in the setting of an endoscopic datum $(H, s, \mathcal{H}, \eta)$, there is a natural double cover $H(\mathbb{R})_\pm \to H(\mathbb{R})$ whose L-group is canonically identified with $\mathcal{H}$. In other words, there is a canonical L-embedding ${}^LH_\pm \to {}^LG$. Therefore, instead of assuming the existence of an (arbitrary) L-embedding ${}^LH \to {}^LG$ and working with the L-packet $\Pi_{\varphi'}(H)$, or choosing a z-extension $H_1 \to H$ and an (again arbitrary) L-embedding $\mathcal{H} \to {}^LH_1$ and working with the L-packet $\Pi_{\varphi_1}(H_1)$, one ought to consider an L-packet $\Pi_{\varphi'}(H_\pm)$ consisting of genuine representations of $H(\mathbb{R})_\pm$, where $\varphi' : W_\mathbb{R} \to {}^LH_\pm$ is the factorization of φ through the canonical L-embedding ${}^LH_\pm \to {}^LG$. Here the word "genuine" means that the non-trivial element ϵ in the kernel of $H(\mathbb{R})_\pm \to H(\mathbb{R})$ acts on the representation by -1. This allows the identity (1.3) to be stated without the assumption of an existence of an L-embedding ${}^LH \to {}^LG$, and without an arbitrary choice of a z-extension. At this point, the identity (1.3) becomes fully canonical and does not depend on any choices. It is also worth pointing out that explicit construction given in [LS87] of the transfer factor Δ is rather complicated and involves multiple factors, each of which depends on auxiliary data. In contrast, the construction of Δ in the setting of the canonical cover $H(\mathbb{R})_\pm$, which is given in [Kal22, §4.3], simplifies noticeably and becomes the product of two natural invariants, one of which has a close relationship to the simple form of the transfer factor for Lie algebras derived by Kottwitz in [Kot99].

After this overview of the classical statements and constructions and the historical development of ideas, let us briefly describe the approach taken in this note. We spend about the first half of the note to give in §2 a review of much of the background that is needed for the main argument, in order to make the note as self-contained as possible. Then, in §3, we discuss how to detect that an essentially discrete series representation is generic for a particular fixed Whittaker datum. This goes beyond the classical work of Kostant [Kos78] and Vogan [Vog78], which treats the question of when a representation is generic with respect to *some* Whittaker datum. The main result of that section, Proposition 3.2.2, describes a simple answer to this question that is entirely analogous to the answer for supercuspidal representations of p-adic groups. As far as we know this result has not yet been recorded in the literature. It allows us to give a direct and explicit proof of Shelstad's [She08b, Theorem 11.5].

In §4 we give the construction of the L-packet for a discrete series parameter. Here we use systematically the device of admissible embeddings of tori into reductive groups, and the existence of a canonical L-embedding from the L-group of the natural double cover of a maximal torus into the L-group of G. This provides a canonical factorization of the L-parameter into the L-group of that double cover, hence a genuine character of that double cover. From the genuine character we can produce explicitly a function on the set of regular elements of that torus, and basic results of Harish-Chandra provide the desired discrete series representation. The internal structure of the resulting L-packet is a simple consequence of Tate-Nakayama duality. This approach is very conceptual and avoids any explicit calculations using cocharacters and Galois cohomology. It is also parallel to the constructions employed in the p-adic case, such as in [Kal19b].

In §5 we prove (1.3). Our proof is direct and does not involve considerations of spectral transfer factors or compatibility factors, and does not appeal to prior expositions. Instead, work directly with the factor $\Delta(\gamma, \delta)$, which in the setting of the canonical double cover $H(\mathbb{R})_\pm$ is the product of two natural invariants. We show how one of these invariants simply measures the difference between the factorizations of φ and φ' through the L-groups of the natural double covers of the elliptic tori, while the other invariant detects which member of $\Pi_\varphi(G)$ is generic. In this way, we show that the factor $\Delta(\gamma, \delta)$ plays both a geometric and a spectral role. For the convenience of the reader, we give two versions of the proof of (1.3), one using the language of covers, and one using the classical language. As part of the proof, we derive a new formula for the term Δ_I of the transfer factor in terms of covers of tori, which is purely Lie-theoretic and does not appeal to Galois cohomology, see Proposition 5.4.1.

2 RECOLLECTIONS

2.1 The L-group

We review the L-group of a connected reductive group following [Vog93, §2].

Let F be a field. Assume first that F is separably closed. Let G be a connected reductive F-group. Given a Borel pair (T, B) of G one has the based root datum $\text{brd}(T, B, G) = (X^*(T), \Delta, X_*(T), \Delta^\vee)$, where $\Delta \subset X^*(T)$ is the set of B-simple roots for the adjoint action of T on $\text{Lie}(G)$, and $\Delta^\vee \subset X_*(T)$ are the corresponding coroots. For a second Borel pair (T', B'), there is a unique element of $T'(F)\backslash G(F)/T(F)$ that conjugates (T, B) to (T', B'). This element provides an isomorphism $\text{brd}(T, B, G) \to \text{brd}(T', B', G)$. This procedure leads to a system of based root data and isomorphisms, indexed by the set of Borel pairs of G. The limit of that system is the based root datum $\text{brd}(G)$ of G.

One can formalize the notion of a based root datum: we refer the reader to [Spr09, §7.4] for the formal notion of a root datum, to which one has to add a set of simple roots to obtain the formal notion of a based root datum. Based root data can be placed into a category, in which all morphisms are isomorphisms, for the evident notion of isomorphism of based root data. The classification of connected reductive F-groups [Spr09, Theorem 9.6.2, Theorem 10.1.1] can be stated as saying that $G \mapsto \text{brd}(G)$ is a full essentially surjective functor from the category of connected reductive F-groups and isomorphisms to the category of based root data and isomorphisms. Moreover, two morphisms lie in the same fiber of this functor if and only if they differ by an inner automorphism.

Consider now a general field F, let F^s a separable closure, $\Gamma = \text{Gal}(F^s/F)$ the Galois group. Given a connected reductive F-group G, there is a natural action of Γ on the set of Borel pairs of G_{F^s}, and this leads to a natural action of Γ on $\text{brd}(G_{F^s})$. We denote by $\text{brd}(G)$ the based root datum $\text{brd}(G_{F^s})$ equipped with this Γ-action. Given two connected reductive F-groups G_1, G_2, an isomorphism $\xi : G_{1,F^s} \to G_{2,F^s}$ is called an *inner twist*, if $\xi^{-1} \circ \sigma \circ \xi \circ \sigma^{-1}$ is an inner automorphism of G_{1,F^s} for all $\sigma \in \Gamma$. The two groups G_1, G_2 are then called inner forms of each other. The functor $G \mapsto \text{brd}(G)$ from the category of connected reductive F-groups to the category of based root data over F and isomorphisms is again essentially surjective. It maps inner twists to isomorphisms, and two inner twists map to the same isomorphism if they differ by an inner automorphism. The fiber over a given based root datum over F consists of all reductive groups that are inner forms of each other.

Given a based root datum $(X, \Delta, Y, \Delta^\vee)$ over F, its dual $(Y, \Delta^\vee, X, \Delta)$ is also a based rood datum over F. If G is a connected reductive F-group with based root datum $(X, \Delta, Y, \Delta^\vee)$, its dual $\widehat{G}$ is the unique split connected reductive group defined over a chosen base field (we will work with $\mathbb{C}$) with based root datum $(Y, \Delta^\vee, X, \Delta)$. Thus, given a Borel pair $(\widehat{T}, \widehat{B})$ of $\widehat{G}$ and a Borel pair (T, B) of G_{F^s}, one is given an identification $X_*(\widehat{T}) = X^*(T)$ that identifies the Weyl chambers associated to $\widehat{B}$ and B.

To form the L-group, one chooses a pinning $(\widehat{T}, \widehat{B}, \{Y_\alpha\})$ of $\widehat{G}$. The group of automorphisms of $\widehat{G}$ that preserve this pinning is in natural isomorphism with the group of automorphisms of $\text{brd}(\widehat{G})$, hence with that of $\text{brd}(G)$. The Γ-action on $\text{brd}(G)$ then lifts to an action on $\widehat{G}$ by algebraic automorphisms, and ${}^LG = \widehat{G} \rtimes \Gamma$.

When G is quasi-split, (T, B) is an F-Borel pair, and $(\widehat{T}, \widehat{B})$ is a Γ-stable Borel pair of $\widehat{G}$, then the identification $X_*(T) = X^*(\widehat{T})$ is Γ-equivariant.

2.2 Inner forms

Let G be a connected reductive F-group. An *inner twist* of G is a pair (G_1, ξ) where G_1 is a connected reductive F-group and $\xi : G_{F^s} \to G_{1,F^s}$ is an isomorphism, such that for each $\sigma \in \Gamma$ the automorphism $\xi^{-1}\sigma(\xi) = \xi^{-1} \circ \sigma \circ \xi \circ \sigma^{-1}$ of G_{F^s} is inner. An isomorphism of inner twists $(G_1, \xi_1) \to (G_2, \xi_2)$ is a homomorphism $f : G_1 \to G_2$ of F-groups such that $\xi_2^{-1} \circ f \circ \xi_1$ is an inner automorphism of G_{F^s}. From an inner twist (G_1, ξ) we obtain the function $\Gamma \to G/Z(G)$ given by $\sigma \mapsto \xi^{-1}\sigma(x)$. It is an element of $Z^1(F, G/Z(G))$, whose cohomology class depends only on the isomorphism class of (G_1, ξ). In this way the set of isomorphism classes of inner twists is in bijection with $H^1(F, G/Z(G))$.

Following Vogan [Vog93], a *pure inner twist* of G is a triple (G_1, ξ, z), where G_1 is a connected reductive F-group, $\xi : G_{F^s} \to G_{1,F^s}$ is an isomorphism and $z \in Z^1(\Gamma, G)$, subject to $\xi^{-1}\sigma(\xi) = \text{Ad}(\bar{z}_\sigma)$, where $\bar{z} \in Z^1(F, G/Z(G))$ is the image of z under the natural projection $G \to G/Z(G)$. An isomorphism of pure inner twists $(G_1, \xi_1, z_1) \to (G_2, \xi_2, z_2)$ is a pair (f, g) consisting of an isomorphism

$f: G_1 \to G_2$ of F-groups and $g \in G_0(F^s)$ such that $\xi_2^{-1} \circ f \circ \xi_1 = \mathrm{Ad}(g)$ and $z_2(\sigma) = g z_1(\sigma)\sigma(g)^{-1}$ for all $\sigma \in \Gamma$. The map $(G_1, \xi, z) \mapsto z$ induces a bijection from the set of isomorphism classes of pure inner twists to $H^1(F, G)$.

There is a lighter notation in which inner twists and pure inner twists can be recorded. It is grounded on the fact that, if (G_1, ξ) is an inner twist of G, with $\bar{z} \in Z^1(F, G/Z(G))$ given by $\bar{z}(\sigma) = \xi^{-1}\sigma(\xi)$, and we let $G_{\bar{z}}$ be the algebraic F-group obtained from G by twisting the F-structure by $\bar{z}$, then ξ becomes an isomorphism of F-groups $G_{\bar{z}} \to G_1$, in fact an isomorphism of inner twists $(G_{\bar{z}}, \mathrm{id}) \to (G_1, \xi)$. Therefore the map $\bar{z} \mapsto (G_{\bar{z}}, \mathrm{id})$ induces an injection from $Z^1(F, G/Z(G))$ into the class of all inner twists of G which meets every isomorphism class. Two elements of $Z^1(F, G/Z(G))$ map into the same isomorphism class if and only if they lie in the same orbit for the action of $G(F^s)$ on $Z^1(F, G/Z(G))$ given by $(g \cdot z)(\sigma) = g z(\sigma)\sigma(g)^{-1}$. The orbits space for this action is $H^1(F, G/Z(G))$, and this gives the same identification between that orbit space and the set of isomorphism classes of inner twists as above.

The same simplification applies to the notion of pure inner twist. There we work with the set $Z^1(F, G)$ and obtain the embedding $z \mapsto (G_z, \mathrm{id}, z)$ from that set into the class of pure inner twists of G. Again the orbit space for the action of $G(F^s)$ on $Z^1(F, G)$ by $(g \cdot z)(\sigma) = g z(\sigma)\sigma(g)^{-1}$ equals $H^1(F, G)$ and is identified with the set of isomorphism classes of pure inner twists.

We now take $F = \mathbb{R}$, hence $F^s = \mathbb{C}$. Then $\Gamma = \{1, c\}$, where c is complex conjugation. An element of $Z^1(F, G)$ can be identified with the image of c, which is an element of $G(\mathbb{C})$. We can also think of this as an element of the coset $G(\mathbb{C}) \rtimes c \subset G(\mathbb{C}) \rtimes \Gamma$. The elements of $Z^1(F, G)$ correspond precisely to the elements of $G(\mathbb{C}) \rtimes c$ of order 2.

In [ABV92] the notion of strong real form was introduced. This is an element $\delta \in G(\mathbb{C}) \rtimes c$, such that δ^2 is a finite order element of $Z(G)(\mathbb{C})$. In [Kal16b] this notion was given a cohomological interpretation and was extended to non-archimedean local fields. Following the latter reference, a *rigid inner twist* of G is a triple (G_1, ξ, z), where G_1 is a connected reductive $\mathbb{R}$-group, $\xi : G_{\mathbb{C}} \to G_{1,\mathbb{C}}$ is an isomorphism and $z \in Z^1_{\text{bas}}(\mathcal{E}_R, G)$, subject to $\xi^{-1}\sigma(\xi) = \mathrm{Ad}(\bar{z}_\sigma)$. Here $1 \to u_{\mathbb{R}}(\mathbb{C}) \to \mathcal{E}_R \to \Gamma \to 1$ is a certain extension of the Galois group, $Z^1_{\text{bas}}(\mathcal{E}_{\mathbb{R}}, G)$ denotes the group of continuous 1-cocycles $\mathcal{E}_{\mathbb{R}} \to G(\mathbb{C})$ whose restriction to $u_{\mathbb{R}}(\mathbb{C})$ takes values in $Z(G)(\mathbb{C})$, and $\bar{z}$ is again the image of z under the natural projection map $G \to G/Z(G)$; it factors through the quotient $\mathcal{E}_{\mathbb{R}} \to \Gamma$. An isomorphism of pure inner twists $(G_1, \xi_1, z_1) \to (G_2, \xi_2, z_2)$ is a pair (f, g) consisting of an isomorphism $f : G_1 \to G_2$ of $\mathbb{R}$-groups and $g \in G_0(\mathbb{C})$ such that $\xi_2^{-1} \circ f \circ \xi_1 = \mathrm{Ad}(g)$ and $z_2(e) = g z_1(e)\sigma_e(g)^{-1}$ for all $e \in \mathcal{E}_{\mathbb{R}}$, where $\sigma_e \in \Gamma$ is the image of e. The set of isomorphism classes of rigid inner twists is in bijection with $H^1_{\text{bas}}(\mathcal{E}_{\mathbb{R}}, G)$. The equivalence between the cohomology classes in $H^1_{\text{bas}}(\mathcal{E}_{\mathbb{R}}, G)$ and the elements $G(\mathbb{C}) \rtimes c$ whose square is a finite order element of $Z(G)(\mathbb{C})$ is less obvious than in the Galois case, and is discussed in [Kal16b, §5.2].

Each of these three notions partitions the set of isomorphism classes of connected reductive $\mathbb{R}$-groups into equivalence classes. The notion of inner twist is the most classical, stemming from the classification of reductive groups, and each equivalence class has exactly one quasi-split member. Unfortunately, this notion is not rigid enough for representation theory – the group of automorphisms of an inner twist (G_1, ξ) is $(G_1/Z(G_1))(\mathbb{R})$, and the conjugation by such an element can be an outer automorphism of the Lie group $G_1(\mathbb{R})$.

The notion of pure inner twist is sufficiently rigid – the group of automorphisms of a pure inner twist (G_1, ξ, z) is $G_1(\mathbb{R})$, hence acts by inner automorphisms of $G_1(\mathbb{R})$. Unfortunately, the equivalence classes induced by this notion are generally smaller, and not all of them contain a quasi-split member.

The notion of a rigid inner twist is again sufficiently rigid, and in addition the equivalence classes in induces coincide with those induced by the notion of inner twist. This will be the notion that we will use.

For further discussion of this topic we refer the reader to [Vog93] and [Kal16a].

Given an inner twist $\xi : G \to G_1$, the isomorphism ξ induces an isomorphism $\mathrm{brd}(G) \to \mathrm{brd}(G_1)$, which is Γ-equivariant, even though ξ itself is not. This leads to an identification of dual groups $\widehat{G} = \widehat{G}_1$ and L-groups ${}^L G = {}^L G_1$.

2.3 Admissible embeddings of tori

Let F be a field and let G be a connected reductive F-group. Let $\widehat{G}$ be its dual group, defined over any base field, which we take to be $\mathbb{C}$, equipped with a Γ-action.

Let S be an F-torus. We recall from [Kal19b, §5.1] that, given a Γ-stable $\widehat{G}$-conjugacy class $\widehat{J}$ of embeddings $\widehat{S} \to \widehat{G}$ whose images are maximal tori, there is an associated Γ-stable $G(F^s)$-conjugacy class J of embeddings $S \to G$. To obtain J, we first assume that G is quasi-splut. Fix Γ-stable Borel pairs $(\widehat{T}, \widehat{B})$ and (T, B) in $\widehat{G}$ and G, respectively. Thus we have the identifications $X_*(T) = X^*(\widehat{T})$ and $\Omega(T, G) = \Omega(\widehat{T}, \widehat{G})$ as Γ-modules. There is $\widehat{\jmath} \in \widehat{J}$ with image $\widehat{T}$ and we obtain the isomorphism $X_*(T) = X^*(\widehat{T}) \to X^*(\widehat{S}) = X_*(S)$, hence an isomorphism $j : S_{F^s} \to T_{F^s}$. We let J be the $G(F^s)$-conjugacy class of the composition of j with the inclusion $T \to G$.

For $\sigma \in \Gamma$ the embedding $\sigma(\widehat{\jmath}) = \sigma_{\widehat{G}} \circ \widehat{\jmath} \circ \sigma_{\widehat{S}}$ is $\widehat{G}$-conjugate to $\widehat{\jmath}$, but has the same image because $\widehat{T}$ is Γ-stable, so there exists $w \in \Omega(\widehat{T}, \widehat{G})$ such that $\sigma(\widehat{\jmath}) = w \circ \widehat{\jmath}$. By construction of j we have $\sigma(j) = w \circ j$, using $\Omega(T, G) = \Omega(\widehat{T}, \widehat{G})$, and conclude that J is Γ-stable. Since all Γ-stable Borel pairs of $\widehat{G}$ are conjugate under $\widehat{G}^\Gamma$, and all Γ-stable Borel pairs of G are conjugate under $G(F)$, the construction of J does not depend on the choices of Borel pairs.

Now drop the assumption that G is quasi-split. We consider an inner twist $\xi : G_0 \to G$ with G_0 quasi-split. It gives an identification $\widehat{G}_0 = \widehat{G}$ and we obtain from $\widehat{J}$ a Γ-stable $G_0(F^s)$-conjugacy class J_0 of embeddings $S \to G_0$. Composing with ξ we obtain the desired $G(F^s)$-conjugacy class J of embeddings $S \to G$. It is Γ-stable, because the $G(F^s)$-conjugacy class of ξ is.

This completes the construction of J. We will refer to it as the set of *admissible* embeddings $S \to G$. If we want to record the group G we will write J^G.

Write $J(F)$ for the set of Γ-fixed points in J, i.e. the set of embeddings $S \to G$ defined over F. When $G = G_0$ is quasi-split, a result of Kottwitz [Kot82, Corollary 2.2] guarantees that this set is non-empty. For a general G this set may be empty. The group $G(F)$ acts on $J(F)$ by conjugation and we will write $J(F)/G(F)$ for the set of orbits under this action.

Given any two elements $j_1, j_2 \in J(F)$ there exists $g \in G(F^s)$ such that $j_2 = \mathrm{Ad}(g) \circ j_1$. The map $\sigma \mapsto j_1^{-1}(g^{-1}\sigma(g))$ is a 1-cocycle of Γ valued in $S(F^s)$. Its class is independent of g and will be denoted by $\mathrm{inv}(j_1, j_2)$. For a fixed $j \in J(F)$ the map $j_2 \mapsto \mathrm{inv}(j_1, j_2)$ is a bijection between $J(F)/G(F)$ and $\ker(H^1(j_1) : H^1(F, S) \to H^1(F, G))$.

One can combine multiple inner forms in this discussion to obtain a more uniform picture. This requires the use of pure or rigid inner twists. Fix a quasi-split group G_0. We start with the case of pure inner twists and consider tuples (G, ξ, z, j), where (G, ξ, z) is a pure inner twist of G_0 and $j \in J^G(F)$. An isomorphism $(G_1, \xi_1, z_1, j_1) \to (G_2, \xi_2, z_2, j_2)$ between such tuples is an isomorphism $(f, g) : (G_1, \xi_1, z_1) \to (G_2, \xi_2, z_2)$ of inner twists that satisfies $j_2 = f \circ j_1$. Let $\mathcal{J}(F)$ be the category whose objects are these tuples and whose morphisms are these isomorphisms. Given two tuples as above there exists $g \in G_0(F^s)$ such that $j_2 = \xi_2 \circ \mathrm{Ad}(g) \circ \xi_1^{-1} \circ j_1$. The map $\sigma \mapsto j_1^{-1}(g^{-1}z_2(\sigma)\sigma(g)z_1(\sigma)^{-1})$ [which side should z_1 be on?] is a 1-cocycle $\Gamma \to S(F^s)$ whose class $\mathrm{inv}(j_2, j_1)$ depends only on the isomorphism classes of the tuples. Fixing the tuple (G_1, ξ_1, z_1, j_1), the map $(G_2, \xi_2, z_2, j_2) \mapsto \mathrm{inv}(j_1, j_2)$ induces a bijection from the set of isomorphism classes $[\mathcal{J}(F)]$ to $H^1(F, S)$.

The individual groups can be extracted from the combined picture as follows. For a fixed (G, ξ, z) we can view the set $J^G(F)$ as the set of objects in a category, with set of morphisms $j_1 \to j_2$ given by $\{g \in G(F) | j_2 = \mathrm{Ad}(g) \circ j_1\}$. Then we obtain an embedding (fully faithful functor) from $J^G(F)$ to $\mathcal{J}(F)$. The category $\mathcal{J}(F)$ decomposes into blocks, indexed by $H^1(F, G_0)$, and each block is equivalent to $J^G(F)$ for some (G, ξ, z). In particular, the set of isomorphism classes $[\mathcal{J}(F)]$ is the disjoint union $\bigcup_{H^1(F,G_0)}(J^G(F)/G(F))$. If we choose $j_0 \in J^{G_0}(F)$, then under the bijection $[\mathcal{J}(F)] \to H^1(F, S)$ given by $(G, \xi, z, j) \mapsto \mathrm{inv}(j, j_0)$, an individual $J^G(F)/G(F)$ coming from (G, ξ, z) is mapped bijectively onto the fiber of $H^1(j_0) : H^1(F, S) \to H^1(F, G)$ over the class of z.

The bijection $[\mathcal{J}(F)] \to H^1(F, S)$ coming from fixing (G, ξ, z, j) can be understood as the orbit map for a natural action of $H^1(F, S)$ on $[\mathcal{J}(F)]$ that does not depend on any choices. To see this action it is easier to consider the simplified notation, where a pure inner twist of G_0 is understood simply as an element $z \in Z^1(F, G_0)$, in the sense that it corresponds to (G_z, id, z), where G_z is the F-group obtained by twisting the rational structure of G_0 by z. Then $\mathcal{J}$ consists of pairs (z, j), where $z \in Z^1(F, G_0)$ and $j \in J^{G_0}$. Such a pair lies in $\mathcal{J}(F)$ if and only if $j \in J^{G_z}(F)$, which is explicitly given as $\mathrm{Ad}(z(\sigma))\sigma_{G_0} \circ j \circ \sigma_S^{-1} = j$. The group $G_0(F^s)$ acts on $\mathcal{J}$ by $g \cdot (z, j) = (gz(\sigma)\sigma(g)^{-1}, \mathrm{Ad}(g) \circ j)$. This action preserves $\mathcal{J}(F)$ and the orbit space is $[\mathcal{J}(F)]$. We introduce the action of $Z^1(F, S)$ on $\mathcal{J}(F)$ as $x \cdot (z, j) = (j(x) \cdot z, j)$ for $x \in Z^1(F, S)$ and $(z, j) \in \mathcal{J}(F)$. One checks directly that the actions of $Z^1(F, S)$ and $G(F^s)$ on $\mathcal{J}(F)$ commute and that the action of $S(F^s)$ on $Z^1(F, S)$ is compatible with the action of $Z^1(F, S)$ on $\mathcal{J}(F)$ in the sense that $(s \cdot x) \cdot (z, j) = j(s) \cdot (x \cdot (z, j))$. Thus we obtain an action of $H^1(F, S)$ on $[\mathcal{J}(F)]$.

Lemma 2.3.1. *The above action is simply transitive.*

Proof. For simplicity, let $x \in Z^1(F, S)$, $g \in G_0(F^s)$, and $(z, j) \in \mathcal{J}(F)$ be such that $g \cdot (z, j) = x \cdot (z, j)$. Then $j = \mathrm{Ad}(g) \circ j$ from which we see that $g = j(s)$ for some $s \in S(F^s)$. We also have $j(s)z(\sigma)\sigma_{G_0}(j(s))^{-1} = j(x(\sigma)) \cdot z(\sigma)$ and multiplying on the right by $z(\sigma)^{-1}$ and using that $(z, j) \in \mathcal{J}(F)$ we conclude $j(s \cdot \sigma_S(s)^{-1}) = j(x(\sigma))$, thus $[x] = 1$ in $H^1(F, S)$, as desired.

For transitivity, we need to show that any two elements of $[\mathcal{J}(F)]$ are in the same $H^1(F,S)$ orbit. Since the members of J^{G_0} are all conjugate under $G_0(F^s)$ we may represent the two elements of $[\mathcal{J}(F)]$ by (z_1,j) and (z_2,j) with the same j. The fact that both lie in $\mathcal{J}(F)$ implies that $\mathrm{Ad}(z_1(\sigma))\circ\sigma_{G_0}\circ j = \mathrm{Ad}(z_2(\sigma))\circ\sigma_{G_0}\circ j$. It follows that $\sigma_{G_0}^{-1}(z_2(\sigma)^{-1}z_1(\sigma))$ lies in the image of j, and we write it as $j(\sigma_S^{-1}(x(\sigma)))$ for some $x(\sigma)\in S(F^s)$. Using that $(z_1,j),(z_2,j)\in\mathcal{J}(F)$ we see that $j(x(\sigma)) = z_1(\sigma)z_2(\sigma)^{-1}$. From this one easily checks that $x\in Z^1(F,S)$ and concludes $(z_1,j)=x\cdot(z_2,j)$, as desired. □

The procedure that produced J from $\widehat{J}$ is invertible. Given a Γ-stable $G(F^s)$-conjugacy class J of embeddings $S\to G$, we compose with ξ^{-1} to obtain a Γ-stable $G_0(F^s)$-conjugacy class J_0 of embeddings $S\to G_0$. Pick a Γ-stable Borel pairs (T,B) of G_0 and $(\widehat{T},\widehat{B})$ of $\widehat{G}$. Choose $j\in J_0$ giving an isomorphism $S\to T$ and use it to get identifications $X^*(\widehat{S}) = X_*(S) = X_*(T) = X^*(\widehat{T})$. The resulting isomorphism $\widehat{S}\to\widehat{T}$ is not Γ-equivariant, but translates the Γ-structure on $\widehat{S}$ to a twist by $\Omega(\widehat{T},\widehat{G})$ of the Γ-structure on $\widehat{T}$. Therefore the composition of this isomorphism with the tautological inclusion $\widehat{T}\to\widehat{G}$ has the property that its $\widehat{G}$-conjugacy class $\widehat{J}$ is Γ-stable.

The same discussion applies to the setting of rigid inner forms when the base field is local. One should only replace $H^1(F,-)$ with $H^1(u_F\to\mathcal{E}_F, Z(G)\to -)$.

2.4 Representations and elements in inner forms

In §2.3 we reviewed the idea of grouping the various admissible embeddings of a torus S into the different inner forms of a fixed group G_0, thereby obtaining a set $\mathcal{J}(F)$ with an action of $G_0(F^s)$ on it. Here we follow the same idea, but group elements or representations.

We focus on $F=\mathbb{R}$, although the same procedure applies for any F of interest. Let $\tilde{\Pi}$ be the set of pairs (z,π), where $z\in Z^1(F,G_0)$ and π is an isomorphism class of irreducible admissible representations of $G_z(F)$, where G_z is again the $\mathbb{R}$-group obtained by twisting the $\mathbb{R}$-structure of G_0 by z. The group $G_0(\mathbb{C})$ acts on this set by $g\cdot(z,\pi) = (gz(\sigma)\sigma(g)^{-1}, \pi\circ\mathrm{Ad}(g)^{-1})$. Note that $\mathrm{Ad}(g): G_z\to G_{gz(\sigma)\sigma(g)^{-1}}$ is an isomorphism of algebraic $\mathbb{R}$-groups. Let $\Pi = \tilde{\Pi}/G_0(\mathbb{C})$. This is the set of isomorphism classes of irreducible admissible representations of pure inner forms of G_0. The pairs $(z,\pi)\in\tilde{\Pi}$ with a fixed z correspond to all isomorphism classes of representations of $G_z(\mathbb{R})$.

Similarly, consider the set of pairs (z,δ), where $z\in Z^1(F,G_0)$ and $\delta\in G_z(F)$ is a regular semi-simple element. The group $G_0(\mathbb{C})$ acts on this set by $g\cdot(z,\delta) = (gz(\sigma)\sigma(g)^{-1}, g\delta g^{-1})$. If two pairs (z_1,δ_1) and (z_2,δ_2) are in the same orbit for this action, we will call them *stably conjugate*. Just as with the case of admissible embeddings, there is a cohomological invariant $\mathrm{inv}((z_1,\delta_1),(z_2,\delta_2))\in H^1(F,S_1)$, where $S_1\subset G_0$ is the centralizer of δ_1, a maximal $\mathbb{R}$-torus of G_{z_1}. This invariant is the class of the 1-cocycle $z_2\cdot z_1^{-1}$ [check again that this is the correct order]. If $z_1=z_2$, then the element of $G_0(\mathbb{C})$ that conjugates them must lie in $G_z(\mathbb{R})$, and we see that δ_1 and δ_2 are $G_z(\mathbb{R})$ conjugate; we will call them *rationally* conjugate. With this in mind, we call the orbit space for this action the set of rational classes of regular semi-simple elements of pure inner forms.

2.5 Weyl denominators

Let G be a connected reductive $\mathbb{R}$-group and $T \subset G$ a maximal $\mathbb{R}$-torus. One can consider the function $T(\mathbb{R}) \to \mathbb{R}$ defined as

$$D_T(t) = D_T^G(t) = \prod_{\alpha \in R(T,G)} (1 - \alpha(t)).$$

In this paper we will normalize orbital integrals and characters by multiplying them by $|D_T(t)|^{1/2}$. Thus, for $t \in T(\mathbb{R})$ strongly regular and $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$ we set

$$O_\gamma(f) = |D_T(t)|^{1/2} \int_{G(\mathbb{R})/T(\mathbb{R})} f(g\gamma g^{-1}) dg/dt,$$

while for π admissible representation of $G(\mathbb{R})$ we have the normalized character function $\Theta_\pi : G(\mathbb{R})_{\text{sr}} \to \mathbb{C}$ determined by

$$\text{tr}(\pi(f)) = \Theta_\pi(f) = \int_{G(\mathbb{R})} |D_T(t)|^{-1/2} \Theta_\pi(g) f(g) dg.$$

This has the advantage that the functions $O_\gamma(f)$ and $\Theta_\pi(\gamma)$ remain bounded as γ approaches singular elements in $T(\mathbb{R})$.

We note here that the distribution $f \mapsto O_\gamma(f)$ depends on the choice of measures dg on $G(\mathbb{R})$ and dt on $T(\mathbb{R})$, the distribution $f \mapsto \Theta_\pi(f)$ depends on dg, while the function $\gamma \mapsto \Theta_\pi(\gamma)$ does not depend on any choices.

An important role in this paper will be played by a function D_B which has the property that $|D_B| = |D_T|^{1/2}$. To see this function, let us interpret D_T as an element of the group ring $\mathbb{Z}[Q]$, where $Q \subset X^*(T) \otimes \mathbb{Q}$ is the root lattice, we we write the group operation on Q multiplicatively. Given a Borel $\mathbb{C}$-subgroup $B \subset G$ containing T we write $\alpha > 0$ when α is a B-positive root, and define

$$D'_B = \prod_{\alpha > 0} (1 - \alpha^{-1}) \in \mathbb{Q}[Q], \qquad D_B = \prod_{\alpha > 0} (\alpha^{1/2} - \alpha^{-1/2}) \in \mathbb{Q}[Q]. \tag{2.1}$$

In $\mathbb{Q}[Q]$ we have the identity

$$D_B = \rho \cdot D'_B, \tag{2.2}$$

where $\rho = \prod_{\alpha > 0} \alpha^{1/2} \in Q$. This implies

$$D_T = D'_B \cdot D'_{\bar{B}} = D_B \cdot D_{\bar{B}},$$

where $\bar{B}$ is the Borel subgroup opposite to B. Moreover, for $w \in \Omega(T, G)$ we have

$$wD_B = D_{w^{-1}Bw} = \text{sgn}(w) D_B. \tag{2.3}$$

In particular, $|D_B|$ is independent of the choice of B and hence $|D_T|^{1/2} = |D_B|$, provided we can interpret D_B as a function on $T(\mathbb{R})$.

It is clear that D'_B is a function on $T(\mathbb{R})$. If we want to interpret D_B as a function of $T(\mathbb{R})$, the occurrence of $\alpha^{1/2}$ in the formula causes a problem. From (2.2) we see that D_B will be a function of $T(\mathbb{R})$ if and only if ρ is, which is equivalent to the element ρ lying in $X^*(T)$. This is always the case when G is semi-simple and simply connected, but can fail in general. To remedy this situation, one can introduce a double cover of $T(\mathbb{R})$, which will be discussed in the next section.

2.6 Double covers of tori and L-embeddings

Let G be a connected reductive $\mathbb{R}$-group and let $T \subset G$ be a maximal torus. An obstruction to the element D_B defining a function on $T(\mathbb{R})$ is the fact that $\rho \in \frac{1}{2}X^*(T)$ may not lie in $X^*(T)$. To remedy this, Adams–Vogan introduce in [AV92], [AV16] the ρ-double cover $T(\mathbb{R})_\rho$ as the pull-back of the diagram

$$T(\mathbb{R}) \xrightarrow{\rho^2} \mathbb{C}^\times \xleftarrow{(-)^2} \mathbb{C}^\times,$$

which comes equipped with a natural character $\rho : T(\mathbb{R})_\rho \to \mathbb{C}^\times$, namely the projection onto the right factor $\mathbb{C}^\times$. By construction we have an exact sequence

$$1 \to \{\pm 1\} \to T(\mathbb{R})_\rho \to T(\mathbb{R}) \to 1$$

and ρ is a genuine character, i.e. $\rho(-x) = -\rho(x)$ for $x \in T(\mathbb{R})_\rho$, where $-x$ denote the product of x and the element -1.

While the double cover $T(\mathbb{R})_\rho$ appears to depend on ρ, this is actually not so. Indeed, for any other Borel $\mathbb{C}$-subgroup B' we have $\rho'/\rho \in X^*(T)$, which allows us to define the genuine character $\rho' : T(\mathbb{R})_\rho \to \mathbb{C}^\times$ as $\rho \cdot (\rho'/\rho)$. Combining this character with the natural projection $T(\mathbb{R})_\rho \to T(\mathbb{R})$ gives a map from $T(\mathbb{R})_\rho$ to the diagram defining $T(\mathbb{R})_{\rho'}$, hence an isomorphism $T(\mathbb{R})_\rho \to T(\mathbb{R})_{\rho'}$.

To emphasize the independence of $T(\mathbb{R})_\rho$ on ρ, and emphasize the dependence on the ambient group G, we will write $T(\mathbb{R})_G$ for this cover. For each Borel $\mathbb{C}$-subgroup B we have the genuine character $\rho_B : T(\mathbb{R})_G \to \mathbb{C}^\times$.

In this paper we will be particularly interested in the case when T is elliptic. Then there is a different way to obtain the ρ-cover that generalizes to all local fields, as discussed in [Kal19a]. One first defines the "big cover" of $T(\mathbb{R})$ as follows. Each root provides a homomorphism $\alpha : T(\mathbb{R}) \to \mathbb{S}^1$. Combining these homomorphisms for a pair $A = \{\alpha, -\alpha\}$ provides a homomorphism $A : T(\mathbb{R}) \to \mathbb{S}^1_-$, where $\mathbb{S}^1_- \subset \mathbb{S}^1 \times \mathbb{S}^1$ is the kernel of the product map $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$. Define the "big cover" as the pull-back of

$$T(\mathbb{R}) \xrightarrow{(\alpha)} \prod \mathbb{S}^1_- \xleftarrow{z/\bar{z}} \prod (\mathbb{C}^\times/\mathbb{R}_{>0})_- \tag{2.4}$$

where the products run over the set of pairs $A = \{\alpha, -\alpha\}$ consisting of a root and its negative, and $(\mathbb{C}^\times/\mathbb{R}_{>0})_-$ denotes analogously the anti-diagonal in $(\mathbb{C}^\times/\mathbb{R}_{>0}) \times (\mathbb{C}^\times/\mathbb{R}_{>0})$. The result is an extension

$$1 \to \prod \{\pm 1\} \to T(\mathbb{R})_{GG} \to T(\mathbb{R}) \to 1.$$

Under the isomorphism $\mathbb{S}^1 \to \mathbb{C}^\times/\mathbb{R}_{>0}$ the map $z/\bar{z}$ becomes the squaring map, and we see that $T(\mathbb{R})_{GG}$ is equipped with a character $\alpha^{1/2} : T(\mathbb{R})_{GG} \to \mathbb{S}^1$ for each root α, and that $\beta^{1/2} = (\alpha^{1/2})^{-1}$ whenever $\beta = \alpha^{-1}$. There is an obvious surjective homomorphism $T(\mathbb{R})_{GG} \to T(\mathbb{R})_G$ whose kernel is the kernel of the multiplication map $\prod\{\pm 1\} \to \{\pm 1\}$. The function $\alpha^{1/2} - \alpha^{-1/2}$ is well-defined on the big cover, while for any choice Borel $\mathbb{C}$-subgroup V the function

$$D_B := \prod_{\alpha > 0} (\alpha^{1/2} - \alpha^{-1/2})$$

descends to the double cover $T(\mathbb{R})_G$.

The action of the Weyl group $\Omega_G(T)(\mathbb{R})$ lifts naturally to an action on $T(\mathbb{R})_G$, even on $T(\mathbb{R})_{GG}$, because $\Omega_G(T)(\mathbb{R})$ acts naturally on each term in (2.4). The identity (2.3) holds for this function.

What makes the double cover $T(\mathbb{R})_G$ very useful in the setting of the Langlands program is the fact that there is an associated L-group ${}^LT_G$, as well as a *canonical* $\widehat{G}$-conjugacy class of L-embeddings ${}^LT_G \to {}^LG$, cf. [Kal19a, §4.1]. The property of the L-group ${}^LT_G$ is that the set of $\widehat{T}$-conjugacy classes of L-homomorphisms $W_\mathbb{R} \to {}^LT_G$ is in natural bijection with the set of genuine characters of $T(\mathbb{R})_G$. Therefore, any L-parameter for G that factors through the image of the embedding of ${}^LT_G$ provides in a canonical way an $\Omega_G(T)(\mathbb{R})$-orbit of genuine characters of $T(\mathbb{R})_G$.

In contrast to ${}^LT_G$, there is generally no canonical L-embedding ${}^LT \to {}^LG$. In fact, if the Galois form of ${}^LT$ is used, there is generally no L-embedding ${}^LT \to {}^LG$ at all, let alone a canonical one. If the Weil form of ${}^LT$ is used, then there always do exist L-embeddings ${}^LT \to {}^LG$, but there is generally no canonical choice. If one chooses a genuine character of $T(\mathbb{R})_G$, then the pointwise product of its L-parameter $W_\mathbb{R} \to {}^LT_G$ with the natural inclusion $\widehat{T} \to {}^LT_G$ does lead to an L-isomorphism ${}^LT \to {}^LT_G$ between the Weil forms of the L-groups for $T(\mathbb{R})$ and $T(\mathbb{R})_G$. Composing this isomorphism with the canonical L-embedding ${}^LT_G \to {}^LG$ provides an L-embedding ${}^LT \to {}^LG$, and every L-embedding arises from this construction. A convenient choice for a genuine character on $T(\mathbb{R})_G$ is the character ρ associated to some Borel $\mathbb{C}$-subgroup of G containing T.

It is worth pointing out that all L-embeddings ${}^LT \to {}^LG$, as well as all L-embeddings ${}^LT_G \to {}^LG$, that extend a fixed embedding $\widehat{\jmath} : \widehat{T} \to \widehat{G}$, have the same image, namely

$$\{x \in N_{{}^LG}(\widehat{T}) \,|\, x \cdot \widehat{\jmath}(t) \cdot x^{-1} = \widehat{\jmath}(\sigma_x(t)) \;\forall t \in \widehat{T}\}, \tag{2.5}$$

where $\sigma_x \in \Gamma$ is the image of x under the natural projection ${}^LG \to \Gamma$.

2.7 Discrete series representations

Let G be a connected reductive $\mathbb{R}$-group. A discrete series representation of $G(\mathbb{R})$ is one which, after possibly tensoring with a continuous character $G(\mathbb{R}) \to \mathbb{C}^\times$, has a unitary central character, and such that every matrix coefficient is square-integrable on $G(\mathbb{R})/Z_G(\mathbb{R})$ (since the central character is unitary, the absolute value of a matrix coefficient is trivial on $Z_G(\mathbb{R})$).

In the literature such representations are sometimes called "essentially discrete series", or "relative discrete series", or "essentially square-integrable", and the term "discrete series" is reserved for those that have unitary central character. We have opted for the shorter formulation in order to avoid repeating the word "essentially".

Harish-Chandra has shown that the set of isomorphism classes of essentially square-integrable representations is in bijection with the set of $G(\mathbb{R})$-conjugacy classes of pairs (S, τ), where $S \subset G$ is an elliptic maximal torus, and τ is a genuine character of the double cover $S(\mathbb{R})_G$ whose differential is regular. This bijection is characterized by the fact that the character function of the representation corresponding to (S, τ), evaluated at a regular element $\delta \in S(\mathbb{R})$, is given by

$$(-1)^{q(G)} \sum_{w \in N(S,G)(\mathbb{R})/S(\mathbb{R})} \frac{\tau}{d_\tau}(w\delta) = (-1)^{q(G)} \sum_{w \in N(S,G)(\mathbb{R})/S(\mathbb{R})} \frac{\tau'}{d'_\tau}(w\dot{\delta}). \quad (2.6)$$

We explain the notation. Pull back τ to a character of $S_{\text{sc}}(\mathbb{R})_G$, where S_{sc} is the preimage of S in the universal cover G_{sc} of the derived subgroup of G. The cover $S_{\text{sc}}(\mathbb{R})_G$ splits canonically, because ρ is divisible by 2 in $X^*(S_{\text{sc}})$. Therefore τ provides a character τ_{sc} of $S_{\text{sc}}(\mathbb{R})$. This being a compact torus, τ_{sc} is an algebraic character, i.e. an element of $X^*(S_{\text{sc}})$, and coincides with its differential, which is still regular. Thus τ specifies a choice of positive roots, i.e. a Borel $\mathbb{C}$-subgroup B containing S. Write D_τ in place of D_B for the Weyl denominator (2.1). Since our convention (cf. §2.5) is to normalize orbital integrals and characters by the absolute value of this denominator, we will only need $d_\tau = \arg D_\tau$. Both τ and d_τ are genuine functions of $S(\mathbb{R})_G$, so their quotient τ/d_τ descends to $S(\mathbb{R})$.

In the second sum we have set $\tau' = \tau \cdot \rho_B^{-1}$, and $d'_\tau = d_\tau \cdot \rho_B^{-1}$, cf. (2.2). In this way, both numerator and denominator are functions of $S(\mathbb{R})$. Note that ρ_B takes values in $\mathbb{S}^1$ because S is elliptic.

It is worth pointing out that Harish-Chandra worked with a slightly different notion of a reductive group, adapted to the language of real Lie groups. His arguments often assumed the group to be connected (in the analytic topology) and *acceptable*, which means that half-sum of the positive roots is analytically integral, i.e. it becomes a character of the complex points of a maximal torus. These conditions are inconvenient when one works with groups of the form $G(\mathbb{R})$ for a connected reductive algebraic $\mathbb{R}$-group G, which are in general neither connected in the analytic topology nor acceptable. However, one can reduce to the setting of Harish-Chandra by the following lemma, based on the fact that $G(\mathbb{R})$ is both connected and acceptable when G is a simply connected semi-simple algebraic $\mathbb{R}$-group.

Lemma 2.7.1. *Let π be an irreducible discrete series representation of $G(\mathbb{R})$. Let $G(\mathbb{R})'$ denote the image of the natural map $G_{sc}(\mathbb{R}) \to G(\mathbb{R})$. The restriction of π to $Z_G(\mathbb{R}) \cdot G(\mathbb{R})'$ is a finite direct sum of irreducible representations. If π' is any one of them, then its restriction to $G_{sc}(\mathbb{R})$ is discrete series, and π equals the induction of π'. In particular, the character of π is supported on $Z_G(\mathbb{R}) \cdot G(\mathbb{R})'$.*

Proof. Since π is irreducible and $Z_G(\mathbb{R}) \cdot G(\mathbb{R})'$ is of finite index in $G(\mathbb{R})$, the restriction of π is a finite direct sum of irreducible representations. They all share the same central character, namely that of π. Therefore, each of them remains irreducible upon restriction to $G(\mathbb{R})'$. If π' is one of them, the square-integrability of its matrix coefficients is inherited from that of π, and its inflation to $G_{\text{sc}}(\mathbb{R})$ is still square-integrable. By Harish-Chandra's theory, there exists an anisotropic maximal torus S_{sc} of G_{sc} and a regular character λ of $S_{\text{sc}}(\mathbb{R})$, with the property that if ρ is half of the sum of the positive roots for the chamber containing λ, then $\lambda - \rho$ still lies in that chamber.

The group $G(\mathbb{R})$ acts by conjugation on $G(\mathbb{R})'$, hence also on its representations. We claim that the stabilizer of π' for this action is $Z_G(\mathbb{R}) \cdot G(\mathbb{R})'$. Indeed, if an element $g \in G(\mathbb{R})$ stabilizes π, it stabilizes the $G(\mathbb{R})'$-orbit of λ, hence we can modify it to assume it fixes λ. Then it lies in the centralizer of λ, which is an elliptic maximal torus $S(\mathbb{R})$. Both $S_{\text{ad}}(\mathbb{R})$ and $S_{\text{sc}}(\mathbb{R})$ are compact tori, hence connected, hence the isogeny $S_{\text{sc}}(\mathbb{R}) \to S_{\text{ad}}(\mathbb{R})$ is surjective. It follows that the element g lies in $Z_G(\mathbb{R}) \cdot S(\mathbb{R})'$, where $S(\mathbb{R})'$ is the image of $S_{\text{sc}}(\mathbb{R}) \to S(\mathbb{R})$.

We have proved that the stabilizer of π' in $G(\mathbb{R})$ is $Z_G(\mathbb{R}) \cdot G(\mathbb{R})'$. Therefore the induction of π' to $G(\mathbb{R})$ is irreducible, and since there is a non-trivial map between it and π, this map must be an isomorphism. $\square$

With this translation to the setting of connected acceptable groups, the discussion of discrete series representations and their characters comes from [HC65, Theorem 3].

2.8 Endoscopic groups and double covers

The notion of endoscopic data is introduced in [LS87, §1.2], and is a variation of the notion of endoscopic pairs or endoscopic triples discussed in [Kot84] and [Kot86].

It can be described equivalently as follows. An endoscopic datum for G is a tuple $(H, s, \mathcal{H}, \eta)$ consisting of

(1) a quasi-split connected reductive group H,

(2) an extension $1 \to \widehat{H} \to \mathcal{H} \to \Gamma \to 1$ of topological groups,

(3) a semi-simple element $s \in Z(\widehat{H})$, and

(4) an L-embedding $\mathcal{H} \to {}^L G$.

It is required that

(a) the extension $\mathcal{H}$ admits a splitting by a continuous group homomorphism $\Gamma \to \mathcal{H}$,

(b) the homomorphism $\Gamma \to \mathrm{Out}(\widehat{H})$ provided by $\mathcal{H}$ coincides with the one provided by the extension ${}^L H$,

(c) η identifies $\widehat{H}$ with the identity component of the centralizer of $\eta(s)$ in $\widehat{G}$,

(d) there exists $z \in Z(\widehat{G})$ such that $s\eta^{-1}(z) \in Z(\widehat{H})^\Gamma$.

The map η produces a Γ-equivariant embedding $Z(\widehat{G}) \to Z(\widehat{H})$, that we will use without explicit notation.

An isomorphism $(H_1, s_1, \mathcal{H}_1, \eta_1) \to (H_2, s_2, \mathcal{H}_2, \eta_2)$ is an element $g \in \widehat{G}$ that satisfies the following properties. First, $\mathrm{Ad}(g)\eta_1(\mathcal{H}_1) = \eta_2(\mathcal{H}_2)$. In particular, $\eta_2^{-1} \circ \mathrm{Ad}(g) \circ \eta_1$ is an L-isomorphism $\mathcal{H}_1 \to \mathcal{H}_2$, and restricts to a Γ-equivariant isomorphism $Z(\widehat{H}_1) \to Z(\widehat{H}_2)$. The second condition is that the resulting isomorphism $\pi_0(Z(\widehat{H}_1)/Z(\widehat{G})) \to \pi_0(Z(\widehat{H}_2)/Z(\widehat{G}))$ maps the coset of s_1 to the coset of s_2.

In this paper we are working with pure (resp. rigid) inner twists, and this necessitates a slight refinement of the notion of endoscopic datum. A *pure refined* endoscopic datum is one in which it is required $s \in Z(\widehat{H})^\Gamma$ in point (3), and this eliminates the need for condition (d). An isomorphism of such data is required to map the coset of s_1 to the coset of s_2 under $\pi_0(Z(\widehat{H}_1)^\Gamma) \to \pi_0(Z(\widehat{H}_2)^\Gamma)$, without dividing by $Z(\widehat{G})$. A *rigid refined* endoscopic datum replaces $s \in Z(\widehat{H})^\Gamma$ by $\dot{s} \in Z(\widehat{\bar{H}})^+$. An isomorphism of such data is required to map the coset of $\dot{s}_1$ to the coset of $\dot{s}_2$ under $\pi_0(Z(\widehat{\bar{H}_1})^+) \to \pi_0(Z(\widehat{\bar{H}_2})^+)$.

Given an L-parameter $\varphi : W_\mathbb{R} \to {}^LG$ and a semi-simple element $s \in S_\varphi$, where $S_\varphi = \mathrm{Cent}(\varphi, \widehat{G})$, one obtains a pure refined endoscopic datum as follows. Set $\widehat{H} = \mathrm{Cent}(s, \widehat{G})^\circ$. The homomorphism $\varphi : W_\mathbb{R} \to \mathrm{Cent}(s, \widehat{G}) \to \mathrm{Out}(\widehat{H})$ factors through the projection $W_\mathbb{R} \to \Gamma$. There is a unique (up to isomorphism) quasi-split connected reductive $\mathbb{R}$-group H with dual group $\widehat{H}$ such that the homomorphism $\Gamma \to \mathrm{Out}(H) = \mathrm{Out}(\widehat{H})$ induced by the $\mathbb{R}$-structure of H matches the one induced by φ. Set $\mathcal{H} = \widehat{H} \cdot \varphi(W_\mathbb{R})$, and let η be the tautological inclusion $\mathcal{H} \to {}^LG$. In the rigid setting, the same construction works starting with $\dot{s} \in S_\varphi^+$, where S_φ^+ is the preimage in $\widehat{\bar{G}}$ of S_φ.

By construction the parameter φ takes values in $\mathcal{H}$. However, the extensions $\mathcal{H}$ and ${}^LH$ of Γ by $\widehat{H}$ need not be isomorphic, and even if they are, there is no natural isomorphism between them. Therefore, φ is *not* a parameter for H in any natural way. There are two ways to remedy this situation.

The classical approach is to choose a z-extension $H_1 \to H$ and an L-embedding $\mathcal{H} \to {}^LH_1$ that extends the natural embedding $\widehat{H} \to \widehat{H}_1$. These choices (which always exist, cf. [KS99, §2.2]) are called a z-pair. They provide a parameter φ_1 for H_1.

An approach introduced in [Kal22] is to extend the theory of double covers of tori from [Kal19a] to the setting of quasi-split connected reductive groups. The datum $\mathcal{H}$ then leads to a *canonical* double cover $H(F)_\pm$ of $H(F)$ and a *canonical* isomorphism ${}^LH_\pm \to \mathcal{H}$. In this way, φ naturally becomes a parameter for $H(F)_\pm$, and there are no choices involved.

2.9 Transfer of orbital integrals

We continue with an endoscopic datum $(H, s, \mathcal{H}, \eta)$ for G. In the pure setting we demand $s \in Z(\widehat{H})^\Gamma$, and in the rigid setting we demand $s \in Z(\widehat{\bar{H}})^+$. We will review here the transfer of orbital integrals, which is dual to the transfer of characters that is the subject of this paper.

As explained in §2.8, in the classical setting order to formulate endoscopic transfer (both geometric and spectral), one has to make an arbitrary choice of a z-pair (H_1, η_1), where $H_1 \to H$ is a surjective homomorphism of algebraic groups whose kernel is an induced torus, and $\eta_1 : \mathcal{H} \to {}^LH_1$ is an L-embedding.

Alternatively, [Kal22] shows that there is a canonical double cover $H(F)_\pm \to H(F)$ and a canonical isomorphism $L^H_\pm \to \mathcal{H}$, whose composition with η then becomes an L-embedding ${}^LH_\pm \to {}^LG$. In this framework, no auxiliary choices are needed.

The transfer of orbital integrals and characters between G and H is governed by the transfer factor. In the classical case, it is a function

$$\Delta : H_1(\mathbb{R})^{\mathrm{sr}} \times G(\mathbb{R})^{\mathrm{sr}} \to \mathbb{C}$$

that depends on the z-pair datum, while in the setting of covers it is a function

$$\Delta : H(R)^{\text{sr}}_{\pm} \times G(\mathbb{R})^{\text{sr}} \to \mathbb{C}$$

that is genuine in the first argument. In the classical case, it is given as the product

$$\epsilon \cdot \Delta_I^{-1} \Delta_{II} \Delta_{III_1}^{-1} \Delta_{III_2}.$$

The individual factors are defined in [LS87], except for ϵ, which is defined in the more general twisted setting in [KS99, §5.3], and Δ_{III_1}, whose relative definition is given in [LS87], but whose absolute definition is given in [Kal11] in the setting of pure inner forms, and in [Kal16b] in the setting of rigid inner forms. The inverses appear due to the conventions of [KS, (1.0.4)], which we will use in this paper. The term Δ_{IV} is missing because we have normalized orbital integrals and characters by the Weyl denominator. We will not review the construction of the individual pieces here, as it has been reviewed in various other places, such as [Kal, §3.5,§4.2,§4.3]. The individual factors depend on auxiliary data, known as a-data and χ-data. The total factor depends on a choice of Whittaker datum, and z-datum.

In the case of covers, the transfer factor becomes the product

$$\epsilon \cdot \Delta_I^{-1} \cdot \Delta_{III}.$$

The terms Δ_I and Δ_{III} are slightly different from the original ones, and are defined in [Kal22, §4.3]. Neither of them depends on auxiliary data, although they are defined on certain covers of tori, and one could argue that the elements of those covers count as auxiliary data.

Recall the notation $O_\gamma(f)$ for normalized orbital integrals from §2.5. We define the stable orbital integral

$$SO_\gamma(f) = \sum_{\gamma'} O_{\gamma'}(f),$$

where γ' runs over a set of representatives for the $G(\mathbb{R})$-conjugacy classes of those elements that are stably conjugate, i.e. $G(\mathbb{C})$-conjugate, to γ.

In the setting of covers, orbital integrals and stable orbital integrals are defined analogously, using the fact that every admissible isomorphism $T \to T'$ between maximal tori of G maps $R(T,G)$ to $R(T',G)$ and hence lifts canonically to an isomorphism $T(\mathbb{R})_G \to T'(\mathbb{R})_G$. We apply this to the isomorphism induced by conjugation or stable conjugation.

We now state the theorem asserting transfer of orbital integrals. It is a fundamental result of Shelstad, [She82], [She08a]. We state two versions, one using the cover $H(\mathbb{R})_\pm$ and one using a z-pair (H_1, η_1).

Theorem 2.9.1. *Let $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$.*

1. *There exists a genuine function $f^{H_\pm} \in \mathcal{C}_c^\infty(H(\mathbb{R})_\pm)$ such that for all $\dot\gamma \in H(F)^{sr}_\pm$*
$$SO_{\dot\gamma}(f^{H_\pm}) = \sum_\delta \Delta(\dot\gamma, \delta) O(\delta, f).$$

2. *Assume chosen a z-pair (H_1, η_1). There exists a genuine function $f^{H_1} \in \mathcal{C}_c^\infty(H_1(\mathbb{R}))$ such that for all $\gamma_1 \in H_1(F)^{sr}$*
$$SO_{\gamma_1}(f^{H_1}) = \sum_\delta \Delta(\gamma_1, \delta) O_\delta(f).$$

In both cases δ runs over the set of $G(\mathbb{R})$-conjugacy classes in $G(\mathbb{R})^{sr}$.

Definition 2.9.2. The functions f and $f^{H_\pm}$ are called *matching*. The functions f and f^{H_1} are called *matching*.

Remark 2.9.3. Since the orbital integral $O_\delta(f)$ depends on the choice of measures on $G(\mathbb{R})$ and $T(\mathbb{R})$, $T = \mathrm{Cent}(\delta, G)$, the concept of matching functions also depends on the choice of measures on $G(\mathbb{R})$, $H(\mathbb{R})$, and all tori in G and H. There is a way to synchronize the various tori, see Remark 5.1.2.

We note further that the normalization of orbital integrals we are using (§2.5) cancels the normalization of the transfer factor we are using (it is missing the Δ_{IV} piece) and therefore the notion of matching functions of Definition 2.9.2 is the same as that obtained by using un-normalized orbital integrals and a transfer factor containing the Δ_{IV} piece.

For the computations of this paper it would be useful to review some basic properties of the transfer factor. The first property concerns the behavior of Δ under stable conjugacy in the variables γ and δ. If $\dot\gamma$, resp. γ_1, is replaced by a stable conjugate, then the value of Δ doesn't change. On the other hand, if δ is replaced by a stable conjugate, the value of Δ changes in the following way

$$\Delta(\dot\gamma, (z_2, \delta_2)) = \Delta(\dot\gamma, (z_1, \delta_1)) \cdot \langle \mathrm{inv}((z_1, \delta_1), (z_2, \delta_2)), \widehat{\jmath}^{-1}(s)\rangle. \tag{2.7}$$

This is in the setting of covers and we refer to [Kal22, Lemma 4.3.1]. In the classical setting, where $\dot\gamma$ is replaced by γ_1, we refer to [Kal, Definitions 4.2.7,4.3.11], or [LS87, §4.1].

To explain this formula let us denote by $\gamma \in H(F)$ the image of $\dot\gamma$ resp. γ_1, and let $S \subset H$ be the centralizer of $\gamma \in H(F)$. Let $T \subset G_{z_1}$ be the centralizer of δ_1. Using the discussion of §2.3 we obtain from the inclusion $S \to H$ a Γ-stable $\widehat{H}$-conjugacy class of embeddings $\widehat{S} \to \widehat{H}$. Composing this with the inclusion $\eta : \widehat{H} \to \widehat{G}$ we obtain a Γ-stable $\widehat{G}$-conjugacy class of embeddings $\widehat{S} \to \widehat{G}$, hence conversely a Γ-stable $G(F^s)$-conjugacy class of embeddings $S \to G$. Among them, there is a unique one that maps γ to δ_1. It is automatically defined over F, and we call it j. Its dual-inverse $\widehat{\jmath}^{-1}$ is an isomorphism $\widehat{S} \to \widehat{T}$. Composing with the canonical inclusion $Z(\widehat{H}) \to \widehat{S}$ we can use it to transport the element $s \in Z(\widehat{H})^\Gamma$ to $\widehat{T}^\Gamma$, where it can be paired with elements of $H^1(F, T)$ by Tate-Nakayama duality. This is in the setting of pure inner twists, and the setting of rigid inner twists is analogous.

The second property is related to the definition of Δ_{III}. We will first present an idealized situation. Assume that an L-embedding ${}^LH \to {}^LG$ exists and has been fixed. Assume further that L-embeddings ${}^LS \to {}^LH$ and ${}^LT \to {}^LG$ have been fixed. These always exist if we use the Weil-forms of the L-groups, but are not unique. The isomorphism $j : S \to T$ from the previous paragraph induces an L-isomorphism ${}^LS \to {}^LT$. We now have four maps that fit in a square, but this square has no reason to commute. More precisely, there does exist an L-isomorphism ${}^LS \to {}^LT$ that does make the square commute, but it is not necessarily the one induced from j. Rather, it differs from it by multiplication by an element $a \in H^1(W_\mathbb{R}, \widehat{S})$. This element is the L-parameter of a character of $S(\mathbb{R})$, which we may denote by $\langle a, - \rangle$. By definition

$$\Delta_{III}(\gamma, \delta) = \langle a, \gamma \rangle.$$

Even though δ doesn't appear on the right, it influences the construction via the map j, which depends on δ.

The key property of this definition that we will need is the following. Assume given L-parameters $\varphi_S : W_\mathbb{R} \to {}^LS$ and $\varphi_T : W_\mathbb{R} \to {}^LT$, with the property that composing them with the L-embeddings ${}^LS \to {}^LH$, ${}^LH \to {}^LG$, and ${}^LT \to {}^LG$, provides a commutative triangle. If θ_S and θ_T are the characters of $S(\mathbb{R})$ and $T(\mathbb{R})$ corresponding to these parameters, then we have

$$\Delta_{III}(\gamma, \delta) = \theta_T(\delta)/\theta_S(\gamma). \tag{2.8}$$

To see this one observes that in the diagram

$$\begin{tikzcd} & {}^LS \arrow[r] \arrow[dd, "a"'] & {}^LH \arrow[dd] \\ W_\mathbb{R} \arrow[ur, "\varphi_S"] \arrow[dr, "\varphi_T"'] & & \\ & {}^LT \arrow[r] & {}^LG, \end{tikzcd}$$

the square commutes by definition of a, hence the trangle commutes by assumptions on φ_S and φ_T.

We now comment on the practical, rather than idealized, situation. In the classical setting an L-embedding ${}^LH \to {}^LG$ doesn't always exist. Instead, one has to choose a z-pair (H_1, η_1) and then has ${}^LG \leftarrow \mathcal{H} \to {}^LH_1$. This makes the first variable of Δ_{III} an element of $S_1(\mathbb{R})$, where S_1 is the preimage of S in H_1. Moreover, the L-embeddings ${}^LS \to {}^LH$ and ${}^LT \to {}^LG$ are specified in terms of auxiliary data, called χ-data, and the term $\Delta_I II$ depends on that choice, as well as on the choice of z-pair.

In the setting of covers, all L-embeddings are canonical, and no auxiliary data need to be chosen, but the L-groups that are involved are those of certain covers. Therefore, Δ_{III}, while independent of any choices, has first variable coming from the double cover $S(\mathbb{R})_\pm$ of $S(\mathbb{R})$ that is the pull-back of $S(\mathbb{R})$ under the canonical double cover $H(\mathbb{R})_\pm \to H(\mathbb{R})$. With these provios, the obvious analog of (2.8) still holds.

Finally, we will need to review the factor ϵ and compute it in the case of the base field $\mathbb{R}$, where the computation is rather straightforward. Consider the universal maximal torus T_0^G of G and T_0^H of H. The complexified character modules $V_G := X^*(T_0^G) \otimes_\mathbb{Z} \mathbb{C}$ and $V_H := X^*(T_0^H) \otimes_\mathbb{Z} \mathbb{C}$ are self-dual Artin representations of the same dimension. Choose an $\mathbb{R}$-pinning of the quasi-split form G_0 of G and a non-trivial additive character $\Lambda : \mathbb{R} \to \mathbb{C}$, which combine to the Whittaker datum fixed for G_0. Then

$$\epsilon = \epsilon(1/2, V_G - V_H, \Lambda),$$

where we have use Langlands' convention [Tat79, (3.6.4)] for the ϵ-factor. The pinning is used in the construction of G_0.

Lemma 2.9.4. *Let $\Lambda(x) = e^{irx}$ with $r > 0$. Then*

$$\epsilon = (-1)^{q(H)-q(G_0)} i^{r_G/2 - r_H/2},$$

where r_G is the number of roots in the absolute root system of G, and r_H is the analogous number for H.

Proof. Let $A_0^G \subset T_0^G$ be the maximal split torus ($X_*(A_0^G) = X_*(T_0^G)^\Gamma$), and $S_0^G \subset T_0^G$ the maximal anisotropic torus ($X^*(S_0^G) = X^*(T_0^G)/X^*(T_0^G)^\Gamma$). Then $X^*(T_0^G)_\mathbb{C} = X^*(A_0^G)_\mathbb{C} \oplus X^*(S_0^G)_\mathbb{C}$, where we have abbreviated $\otimes_\mathbb{Z}\mathbb{C}$ by the subscript $\mathbb{C}$. One has $\epsilon(1/2, \mathbf{1}, \Lambda) = 1$ and $\epsilon(1/2, \mathrm{sgn}, \Lambda) = i$ according to [Tat79, (3.2.4)], hence

$$\epsilon(1/2, V_G, \Lambda) = i^{d-\dim(A_0^G)},$$

where $d = \dim(T_0^G)$. We use the same computation for H and conclude

$$\epsilon = \frac{\epsilon(1/2, X^*(T_0^G)_\mathbb{C}, \Lambda)}{\epsilon(1/2, X^*(T_0^H)_\mathbb{C}, \Lambda)} = \frac{i^{d-\dim(A_0^G)}}{i^{d-\dim(A_0^H)}} = i^{\dim(A_0^H)-\dim(A_0^G)}.$$

The Iwasawa decomposition $\mathrm{Lie}(G_0) = \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{k}$ shows $2q(G_0) = \dim(\mathfrak{a}) + \dim(\mathfrak{n}) = \dim(A_0^G) + r_G/2$. We note that $q(G_0)$ is an integer, becaus e G_0 has an elliptic maximal torus, and $q(G_0)$ equals the number of positive non-compact roots with respect to any Weyl chamber. Therefore

$$\dim(A_0^H) - \dim(A_0^G) = 2(q(H) - q(G_0)) + (r_G/2 - r_H/2). \qquad \square$$

2.10 The Weyl integration formula and its stable analog

In this section we work with an arbitrary local field F of characteristic zero. While our intended application is $F = \mathbb{R}$, the arguments work for an arbitrary F and do not admit any significant simplification for $F = \mathbb{R}$, so we take this opportunity to record the statement for a general F.

Let G be a connected reductive F-group. Given a maximal torus $T \subset G$, write $\Omega(T, G) = N(T, G)/T$ for its absolute Weyl group. This is a finite F-group and we may consider the group $\Omega(T, G)(F)$ of its F-points. Write $\Omega_F(T, G) = N(T, G)(F)/T(F)$. This is an abstract finite group and we have $\Omega_F(T, G) \subset \Omega(T, G)(F)$. The inclusion is often proper.

Theorem 2.10.1 (Weyl integration formula)**.** *Let f be a smooth compactly supported functon on $G(F)$. Let $\mathcal{T}$ be a set of representatives for the $G(F)$-conjugacy classes of maximal tori of G. Then*

$$\int_{G(F)} f(g)dg = \sum_{T \in \mathcal{T}} |\Omega_F(T, G)|^{-1} \int_{T(F)_{sr}} |D_T(\gamma)|^{1/2} O_\gamma(f) d\gamma.$$

In this theorem we have chosen a Haar measure dg on $G(\mathbb{R})$, and a Haar measure $d\gamma$ on $T(\mathbb{R})$ for any $T \in \mathcal{T}$. The orbital integral $O_\gamma(f)$ is formed with respect to the quotient measure $dg/d\gamma$. The power $1/2$ occurs in the Weyl discriminant because O_γ has been normalized in §2.5.

Corollary 2.10.2 (Stable Weyl integration formula)**.** *Let f be a smooth compactly supported functon on $G(F)$. Let $\mathcal{ST}$ be a set of representatives for the stable conjugacy classes of maximal tori of G. Then*

$$\int_{G(F)} f(g)dg = \sum_{T \in \mathcal{ST}} |\Omega(T, G)(F)|^{-1} \int_{T(F)_{sr}} |D_T(\gamma)|^{1/2} SO_\gamma(f) d\gamma.$$

Proof. Let $T \in \mathcal{ST}$ and let $T_1, \ldots, T_n \in \mathcal{T}$ be those tori that are stably conjugate to T. In fact, it is known that $n = 1$ (a special feature of F) but we will not need to know that.

It is enough to show that

$$|\Omega(T,G)(F)|^{-1}\int_{T(F)_{\mathrm{sr}}}|D_T(\gamma)|^{1/2}SO_\gamma(f)d\gamma=\sum_{i=1}^n|\Omega_F(T_i,G)|^{-1}\int_{T_i(F)_{\mathrm{sr}}}|D_{T_i}(\gamma)|^{1/2}O_{\gamma_i}(f)d\gamma_i.$$

Two elements of $T(F)_{\mathrm{sr}}$ are stably conjugate if and only if they are conjugate under $\Omega(T,G)(F)$, and are rationally conjugate if and only if they are conjugate under $\Omega_F(T,G)$. Since $SO_\gamma(f)$ and $|D_T(\gamma)|$ are invariant under stable conjugacy in γ, and $O_\gamma(f)$ is invariant under rational conjugacy in γ, and since the action of $\Omega(T,G)(F)$ on $T(F)_{\mathrm{sr}}$ is free, we can write the above identity as

$$\int_{T(F)_{\mathrm{sr}}/\Omega(T,G)(F)}|D_T(\gamma)|^{1/2}SO_\gamma(f)d\gamma=\sum_{i=1}^n\int_{T_i(F)_{\mathrm{sr}}/\Omega_F(T_i,G)}|D_{T_i}(\gamma)|^{1/2}O_{\gamma_i}(f)d\gamma_i.$$

The domain of integration on the left is the set of elements of $T(F)_{\mathrm{sr}}$ up to stable conjugacy, and the domain of integration on the right is the set of elements of $T_i(F)_{\mathrm{sr}}$ up to rational conjugacy.

Choose $g_1,\dots,g_n\in G(\mathbb{C})$ so that $\mathrm{Ad}(g_i):T\to T_i$ is an isomorphism of F-tori and induces an isomorphism of F-groups $\Omega(T,G)\to\Omega(T_i,G)$. For any $\gamma\in T(F)_{\mathrm{sr}}$, the set

$$\bigcup_{i=1}^n[\Omega_F(T_i,G)\backslash\Omega(T_i,G)(F)]\cdot\mathrm{Ad}(g_i)\gamma$$

represents the $G(F)$-conjugacy classes in the stable class of γ. Therefore

$$SJ(\gamma,f)=\sum_{i=1}^n\sum_{w\in\Omega_F(T_i,G)\backslash\Omega(T_i,G)(F)}J(w\mathrm{Ad}(g_i)\gamma,f).$$

Moreover $D_T(\gamma)=D_{T_i}(\mathrm{Ad}(g_i)\gamma)$. As γ runs over all elements of $T(F)_{\mathrm{sr}}/\Omega(T,G)(F)$, $\mathrm{Ad}(g_i)\gamma$ runs over all elements of $T_i(F)_{\mathrm{sr}}/\Omega(T_i,G)(F)$. □

Remark 2.10.3. As discussed in the proof, we can rewrite the above formulas as

$$\int_{G(F)}f(g)dg=\sum_{T\in\mathcal{T}}\int_{T(F)_{\mathrm{sr}}/\Omega_F(T,G)}|D_T(\gamma)|^{1/2}O_\gamma(f)d\gamma$$

and

$$\int_{G(F)}f(g)dg=\sum_{T\in\mathcal{ST}}\int_{T(F)_{\mathrm{sr}}/\Omega(T,G)(F)}|D_T(\gamma)|^{1/2}SO_\gamma(f)d\gamma$$

respectively.

3 Genericity of discrete series representations

From the work of Kostant [Kos78] and Vogan [Vog78] it is known that every essentially discrete series representation is generic for some Whittaker datum. In this section we will prove a more precise statement (Proposition 3.2.2) that gives a link between the representation and the Whittaker datum. It provides (Proposition 3.2.2) an exact analog of a result in the p-adic case, originally due to DeBacker and [DR10, Proposition 4.10] under unramifiedness assumptions, but whose proof applies in much greater generality, see [Kal19b, Lemma 6.2.2] and [FKS21, §4.4]. This allows us to prove the strong form of Shahidi's generic packet conjecture [Sha90, §9] for discrete series packets, originally established by Shelstad [She08b] using a less direct argument. Note that the general form of the conjecture (for tempered L-packets) reduces immediately to the case of discrete L-packets.

3.1 Whittaker data and Kostant sections

We assume that G is a quasi-split connected reductive $\mathbb{R}$-group. Suppose B is a Borel $\mathbb{R}$-subgroup of G, N its unipotent radical, and $\mathfrak{n} = \mathrm{Lie}(N)$. Since $N(\mathbb{R})$ is a connected Lie group, any character $\eta : N(\mathbb{R}) \to \mathbb{C}^\times$ is determined by its differential $d\eta : \mathfrak{n}(\mathbb{R}) \to \mathbb{C}$, which is an $\mathbb{R}$-linear form. We have $\eta(\exp(Y)) = e^{\langle d\eta, Y\rangle}$ for $Y \in \mathfrak{n}(\mathbb{R})$. In fact, the real Lie group $N(\mathbb{R})$ is connected, nilpotent, and simply connected (see [Kna02, Theorem 6.46], where the assumption that G is semi-simple is unnecessary) and hence the exponential map $\exp : \mathfrak{n}(\mathbb{R}) \to N(\mathbb{R})$ is a diffeomorphism by [Kna02, Theorem 1.127].

Given a maximal torus $T \subset B$ defined over $\mathbb{R}$, the space $\mathfrak{n}(\mathbb{R})$ decomposes as the direct sum of relative root spaces, with respect to the action of the split part of T.

An $\mathbb{R}$-linear form $\mathfrak{n}(\mathbb{R}) \to \mathbb{C}$ can be obtained as the restriction of a $\mathbb{C}$-linear form $\mathfrak{n}(\mathbb{C}) \to \mathbb{C}$. Let $\bar{N}$ be the unipotent radical of B that is T-opposite to N. Since the Killing form κ induces a non-degenerate $\mathbb{C}$-linear pairing $\mathfrak{n} \times \bar{\mathfrak{n}} \to \mathbb{C}$, such forms are in 1-1 correspondence with elements of $\bar{\mathfrak{n}}(\mathbb{C})$. Given $X \in \bar{\mathfrak{n}}(\mathbb{C})$ we thus obtain the character $\eta_X(\exp(Y)) = e^{\kappa(X,Y)}$. It is unitary if and only if $X \in i\bar{\mathfrak{n}}(\mathbb{R})$.

We say a character $\eta : N(\mathbb{R}) \to \mathbb{C}^\times$ is *non-degenerate* if its differential $d\eta$ restricts non-trivially to any relative root space in $\mathfrak{n}(\mathbb{R})$. Since all choices of T are conjugate under $N(\mathbb{R})$, this condition is independent of T. For η_X with $X \in i\bar{\mathfrak{n}}(\mathbb{R})$ we can write $X = \sum_\alpha X_{-\alpha}$ where the sum runs over the set of absolute positive roots and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$. Then η_X is non-degenerate if and only if $X_{-\alpha} \neq 0$ for all simple roots α, i.e. if and only if X is a regular nilpotent element.

By a *Whittaker datum* we mean a $G(\mathbb{R})$-conjugacy class of pairs (B, η) where B is a Borel subgroup defined over $\mathbb{R}$ and η is a non-degenerate unitary character of $N(\mathbb{R})$. We write $\mathfrak{w} = [(B, \eta)]$ for the $G(\mathbb{R})$-conjugacy class of (B, η).

Suppose $X \in i\mathfrak{g}(\mathbb{R})$ is a regular nilpotent element. Let $\overline{B}$ be the unique Borel subgroup containing X. Then $\overline{B}$ is defined over $\mathbb{R}$. Let B be an $\mathbb{R}$-Borel subgroup that is opposite to $\bar{B}$, i.e. such that $B \cap \bar{B}$ is a maximal torus. The Whittaker datum defined by X is $\mathfrak{w}_X = [(B, \eta_X)]$.

Lemma 3.1.1. *The Whittaker datum $\mathfrak{w}_X$ depends only on the $G(\mathbb{R})$-conjugacy class of X.*

Proof. The choice of B is equivalent to a choice of a maximal torus $T \subset \bar{B}$ defined over $\mathbb{R}$, because T is determined by B as $T = B \cap \bar{B}$ and B is determined by T as the unique T-opposite of $\bar{B}$. But all maximal $\mathbb{R}$-tori in $\bar{B}$ are conjugate under $\bar{B}(\mathbb{R})$ by [Bor91, Theorem 19.2]. □

Here is another way to obtain Whittaker data. Let $\mathcal{P} = (T, B, \{X_\alpha\})$ be an $\mathbb{R}$-pinning of G. The element X_α specifies an isomorphism u_α from $\mathbb{G}_a$ to the root group U_α of G associated to the absolute root α, namely the unique isomorphism whose differential maps 1 to X_α. If N is the unipotent radical of B then the composition $\prod_{\alpha\in\Delta} U_\alpha \to N \to N/[N,N]$ is an isomorphism of complex algebraic groups, which we compose with (u_α). The inverse of the result, composed with the product map $\prod_\alpha \mathbb{G}_a \to \mathbb{G}_a$, becomes a homomorphism of algebraic groups $N/[N,N] \to \mathbb{G}_a$ defined over $\mathbb{R}$, which leads to a homomorphism $\eta_{\mathcal{P}} : N(\mathbb{R}) \to \mathbb{R}$.

Now fix a non-trivial unitary character Λ of $\mathbb{R}$. Then $\eta_{\mathcal{P},\Lambda} = \Lambda \circ \eta_{\mathcal{P}}$ is a non-degenerate unitary character of $N(\mathbb{R})$. The $G(\mathbb{R})$-conjugacy class of $\eta_{\mathcal{P},\Lambda}$ only depends on the $G(\mathbb{R})$-conjugacy class of $\mathcal{P}$. Write $\mathfrak{w}_{\mathcal{P},\Lambda}$ for the Whittaker data $[(B, \eta_{\mathcal{P},\Lambda}]$.

Lemma 3.1.2. *1. Fix a non-trivial unitary character Λ of $\mathbb{R}$. The map $\mathcal{P} \mapsto \mathfrak{w}_{\mathcal{P},\Lambda}$ is a bijection from $G(\mathbb{R})$-conjugacy classes of real pinnings to Whittaker data.*

2. Set $\Lambda(x) = e^{ix}$. Then $\mathfrak{w}_{\mathcal{P},\Lambda} = \mathfrak{w}_{\overline{X}}$, where $\mathcal{P} = (T, B, \{X_\alpha\})$ and $\overline{X} = i\sum_{\alpha\in\Delta} X_{-\alpha}$, and $X_{-\alpha}$ satisfies $[X_\alpha, X_{-\alpha}] = H_\alpha$.

Proof. Since all Borel subgroups of $G(\mathbb{R})$ are $G(\mathbb{R})$-conjugate, fix a Borel subgroup B defined over $\mathbb{R}$. Then the first statement is equivalent to: there is a bijection between the sets of positive root vectors $\{X_\alpha\}$, modulo conjugation by $T(\mathbb{R})$, and non-degenerate characters of $N(\mathbb{R})$, modulo the action of $T(\mathbb{R})$. We leav the straightforward verification of this bijection to the reader.

Set $X_- = \sum_{\alpha\in\Delta} X_{-\alpha}$. Then X_- is a regular nilpotent element of the Lie algebra of G and is fixed by the Galois action, hence an $\mathbb{R}$-point. It lies in $\bar{\mathfrak{n}}$, the Lie algebra of the unipotent radical of the Borel subgroup T-opposite to B. Thus $\bar{X} \in i\bar{\mathfrak{n}}(\mathbb{R})$ is also regular nilpotent.

Given $Y \in \mathfrak{n}(\mathbb{R})$ write it as $\sum_{\alpha\in\Delta} y_\alpha \cdot X_\alpha + Y'$ with $Y' \in [\mathfrak{n}, \mathfrak{n}](\mathbb{R})$. Then we have

$$\kappa(Y, X_-) = \sum_{\alpha\in\Delta} y_\alpha \kappa(X_\alpha, X_{-\alpha}) = \sum_{\alpha\in\Delta} y_\alpha,$$

therefore

$$\eta_{\bar{X}}(\exp(Y)) = e^{\kappa(Y,\bar{X})} = e^{i\kappa(Y,X_-)} = \Lambda(\sum_{\alpha\in\Delta} y_\alpha) = \eta_{\mathcal{P},\Lambda}(\exp(Y)).$$

□

It is helpful to understand the set of all Whittaker data. Let

$$Q(G(\mathbb{R})) = G_{\mathrm{ad}}(\mathbb{R})/\mathrm{ad}(G(\mathbb{R})).$$

This is a finite abelian 2-group. Now $G_{\mathrm{ad}}(\mathbb{R})$ acts by automorphisms on $G(\mathbb{R})$, which induces an action of $Q(G(\mathbb{R}))$ on the set of Whittaker data.

Lemma 3.1.3. *$Q(G(\mathbb{R}))$ acts simply transitively on the set of Whittaker data.*

Proof. We use Lemma 3.1.2(1) and the fact that $\mathcal{P} \mapsto \mathfrak{w}_{\mathcal{P},\Lambda}$ is $G_{\mathrm{ad}}(\mathbb{R})$-equivariant, to reduce to the transitivity of the action of $G_{\mathrm{ad}}(\mathbb{R})$ on the set of $\mathbb{R}$-pinnings. Since all Borel pairs over $\mathbb{R}$ are conjugate under $G(\mathbb{R})$, this reduces the statement to the simple transitivity of the action of $T_{\mathrm{ad}}(\mathbb{R})$ on the possible choices of non-zero simple root vectors. If the simple root α is real, then so is the corresponding fundamental coweight ϖ_α, which then gives a 1-parameter subgroup $\mathbb{R}^\times \to T_{\mathrm{ad}}(\mathbb{R})$, which clearly acts simply transitively on $\mathfrak{g}_\alpha(\mathbb{R}) \smallsetminus \{0\}$. If the simple root is complex, then a pair $(X_\alpha, X_{\sigma\alpha})$ contributing to the pinning is an element of $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma\alpha})(\mathbb{R}) = \mathbb{C}$. The fundamental coweight ϖ_α induces a 1-parameter subgroup $\mathbb{C}^\times \to T_{\mathrm{ad}}(\mathbb{R})$ which again acts simply transitively on $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma\alpha})(\mathbb{R})$. □

Suppose $X \in \mathfrak{g}$ is a regular nilpotent element. Choose an $SL(2)$-triple $[X, H, Y]$ [reference]. The *Kostant Section* $\mathcal{K}(X)$ of X is the affine space $X + \mathrm{Cent}_{\mathfrak{g}}(Y)$. Kostant showed [Kos63] that the Kostant section meets every regular orbit in a unique point. If the triple lies in $\mathfrak{g}(\mathbb{R})$, then $\mathcal{K}(X)$ is Galois stable. The Kostant section $\mathcal{K}(X)$ depends on a choice of triple, but any two such choices that lie in $\mathfrak{g}(\mathbb{R})$ are $G(\mathbb{R})$-conjugate, and the $G(\mathbb{R})$-conjugacy class of $\mathcal{K}(X)$ only depends on the $G(\mathbb{R})$-conjugacy class of X.

Suppose $X \in \mathfrak{g}(\mathbb{R})$ and $\mathcal{O}$ is a regular orbit which is defined over $\mathbb{R}$. Then, since X is also defined over $\mathbb{R}$, the unique point in $\mathcal{K}(X) \cap \mathcal{O}$ is contained in $\mathfrak{g}(\mathbb{R})$. The $G(\mathbb{R})$-orbit of $\mathcal{K}(X) \cap \mathcal{O}$ depends only on the $G(\mathbb{R})$-orbit of X.

3.2 Generic discrete series representations

Definition 3.2.1. Suppose $\mathfrak{w} = [(B, \eta)]$ is a Whittaker datum. We say that a representation π of G is $\mathfrak{w}$-generic if there is a non-zero smooth vector v in the space of π such that $\pi(x)(v) = \eta(x)v$ for all $x \in N(\mathbb{R})$. We say π is generic if it is $\mathfrak{w}$-generic for some $\mathfrak{w}$.

Suppose π is an irreducible essentially discrete series representation and write $\pi = \pi(S, \tau)$ as in Section 2.7. Let H_π be the element of $i\mathfrak{s}(\mathbb{R})$ corresponding to $d\tau$ via κ. The $G(\mathbb{R})$-conjugacy class of H_π is well defined. The purpose of this subsection is to prove the following result.

Proposition 3.2.2. *Let $\mathfrak{w} = \mathfrak{w}_X$ be a Whittaker datum for some regular nilpotent element $X \in i\mathfrak{g}(\mathbb{R})$. Let π be an essentially discrete series representation. If π is $\mathfrak{w}$-generic then $H_\pi \in i\mathfrak{s}(\mathbb{R})$ is $G(\mathbb{R})$-conjugate to an element of the Kostant section of X.*

We begin with some preparations. If W is a subset of a real or complex vector space V, define $\mathrm{AC}(W)$, the *asymptotic cone* of W as in [BV80, Proposition 3.7], [AV21, Definition 2.9]:

$$\mathrm{AC}(W) = \{v \in V \mid \exists t_i \in \mathbb{R}_{>0}, t_i \to 0, w_i \in W, \lim_{i \to \infty} t_i w_i = v\}$$

This is a closed cone.

Lemma 3.2.3. *Let $\mathcal{O}_{\mathbb{R}} \subset \mathfrak{g}(\mathbb{R})$ be a regular $G(\mathbb{R})$-orbit. Assume there exists a regular nilpotent element $X \in \mathrm{AC}(\mathcal{O}_{\mathbb{R}})$. Then $\mathcal{K}(X)$ meets $\mathcal{O}_{\mathbb{R}}$.*

Proof. By [Kos63], $\mathcal{K}(X)$ is transversal to any $G(\mathbb{C})$-orbit in $\mathfrak{g}(\mathbb{C})$. Therefore the set $\mathrm{Ad}(G(\mathbb{C}))\mathcal{K}(X)$ contains an open ball around X in $\mathfrak{g}(\mathbb{C})$. This remains valid over $\mathbb{R}$ as well: $\mathrm{Ad}(G(\mathbb{R}))\mathcal{K}(X)(\mathbb{R})$ contains an open ball around X in $\mathfrak{g}(\mathbb{R})$.

Choose $t_i \in \mathbb{R}_{>0}$ with $t_i \to 0$ and $w_i \in \mathcal{O}_{\mathbb{R}}$ such that $t_i w_i \to X$. We conclude that there exists $g \in G(\mathbb{R})$ such that $t_i w_i \in \mathrm{Ad}(g)\mathcal{K}(X)(\mathbb{R})$ for some i. This is equivalent to $\mathrm{Ad}(g)^{-1} w_i \in t_i^{-1}(\mathcal{K}(X))(\mathbb{R})$. Since $\mathcal{O}_{\mathbb{R}}$ is a $G(\mathbb{R})$-orbit, we see $\mathcal{O}_{\mathbb{R}} \cap t_i^{-1}(\mathcal{K}(X))(\mathbb{R}) \neq \varnothing$.

We now use the elementary facts that $t\mathcal{K}(X) = \mathcal{K}(tX)$ and $\mathrm{Ad}(h)\mathcal{K}(X) = \mathcal{K}(\mathrm{Ad}(h)X)$ for $t \in \mathbb{R}$ and $h \in G(\mathbb{C})$. Also it is well known that the $G(\mathbb{R})$-orbit of X is a cone. (To see this, apply the real version of the Jacobson-Morozov theorem [CM93, Theorem 9.2.1] to find a homomorphism $\phi : SL_2 \to G$ such that $d\phi\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = X$. Then $\mathrm{Ad}(\phi(\mathrm{diag}(t,\frac{1}{t}))(X) = t^2X$.) The latter of these facts implies that there exists $h \in G(\mathbb{R})$ with $t_i^{-1}X = \mathrm{Ad}(h)X$, and we see $t_i^{-1}\mathcal{K}(X) = \mathrm{Ad}(h)\mathcal{K}(X)$. Thus $\mathcal{O}_{\mathbb{R}}$ intersects $\mathcal{K}(X)(\mathbb{R})$ as claimed. □

Proof of Proposition 3.2.2. Suppose π is $\mathfrak{w}_X$-generic. We define the wave-front set $\mathrm{WF}(\pi)$ as in [Mat92, Section 3]. By definition this is a subset of $i\mathfrak{g}(\mathbb{R})^*$ consisting of nilpotent elements. We have the direct sum decomposition $\mathfrak{g} = \mathfrak{z}\oplus\mathfrak{g}'$, where $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$. Then $\mathrm{WF}(\pi)$ is a union of nilpotent orbits in $i\mathfrak{g}'(\mathbb{R})^*$.

We have $X \in \mathrm{WF}(\pi)$ according to [Mat92, Theorem A] and moreover $\mathrm{WF}(\pi) = \mathrm{AC}(G(\mathbb{R})\cdot H_\pi)$ according to [HHO16, Theorem 1.2]. The proposition then follows from Lemma 3.2.3 applied with $i\mathfrak{g}'(\mathbb{R})^*$ in place of $\mathfrak{g}(\mathbb{R})$. □

Remark 3.2.4. The map $\pi \to H_\pi$ depends on the choice of κ. However, different choices of κ differ only on the center of $\mathfrak{g}$. Since any Kostant section is stable under the action of the center by translations, this choice is immaterial for Proposition 3.2.2.

We now mention a supplementary result, which is well known to the experts, but not easy to find in the literature.

Lemma 3.2.5. *Suppose $\pi = \pi(S,\tau)$ and $\pi' = \pi(S,\tau')$ are generic discrete series representations, which are $\mathfrak{w},\mathfrak{w}'$-generic, respectively. Then $\mathfrak{w} = \mathfrak{w}'$ if and only if $d\tau$ and $d\tau'$ are in the same Weyl chamber. In particular a generic discrete series representation is $\mathfrak{w}$-generic for a unique Whittaker datum $\mathfrak{w}$.*

Since we don't use this result we skip the proof, and refer the reader to a forthcoming paper on Whittaker models for real groups by the first author and Alexandre Afgoustidis.

3.3 An example with SL_2

Set $G = \mathrm{SL}(2,\mathbb{R})$. Let $\mathfrak{s}(\mathbb{C}) = \{t_z \mid z \in \mathbb{C}\}$ where

$$t_z = \begin{pmatrix}0 & z\\ -z & 0\end{pmatrix}$$

Then $\mathfrak{s}(\mathbb{R}) = \{t_x \mid x \in \mathbb{R}\}$.

For $z \in \mathbb{C}$ define $\lambda_z \in \mathfrak{s}(\mathbb{C})^*$ by $\lambda_z(t_x) = xz$. The positive root is $\alpha(t_z) = 2iz$, i.e.

$$\alpha = \lambda_{2i}, \quad \rho = \lambda_i.$$

and

$$^\vee\alpha = t_{-i}, \quad {}^\vee\rho = t_{-i/2}$$

In particular

$$^\vee\rho(-i) = t_{-\frac{1}{2}} = \begin{pmatrix}0 & -\frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}$$

Define $H_\lambda \in \mathfrak{s}(\mathbb{C})$ so that $\kappa(H_\lambda, t_x) = \lambda(t_x)$. Then $H_{\lambda_z} = t_{-\frac{z}{8}}$:

$$H_{\lambda_z} = t_{\frac{-z}{8}} = \begin{pmatrix}0 & -\frac{z}{8}\\ \frac{z}{8} & 0\end{pmatrix}$$

Take $z = ik$, so
$$H_{\lambda_{ik}} = \begin{pmatrix} 0 & -\frac{ik}{8} \\ \frac{-ik}{8} & 0 \end{pmatrix}$$

Let $\pi(\lambda_{ik})$ be the discrete series representation with Harish-Chandra parameter λ_{ik} ($k \in \mathbb{Z}_{\neq 0}$). If $\pi = \pi(\lambda)$ let $H_\pi = H_\lambda \in i\mathfrak{s}(\mathbb{R})^*$.

For an $SL(2)$-triple we can take $\{X_\alpha, X_{-\alpha}, t_{-i}\}$ where
$$X_\alpha = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad X_{-\alpha} = \overline{X_\alpha}.$$

The Killing form satisfies
$$\kappa(t_x, t_y) = -8xy.$$

Conjugating this by $\mathrm{diag}(x, \frac{1}{x})$ takes it to $\begin{pmatrix} 0 & -x^2\frac{ik}{8} \\ \frac{ik}{x^2} & 0 \end{pmatrix}$, and taking the limit we see
$$\mathrm{AC}(G(\mathbb{R}) \cdot H_{\lambda_{ik}}) = \begin{cases} \mathbb{R}^+ * \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} & k > 0 \\ \mathbb{R}^+ * \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} & k < 0 \end{cases}$$

Therefore
$$\mathrm{WF}(\pi(\lambda_{ik})) = \begin{cases} \mathbb{R}^+ * \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} & k > 0 \\ \mathbb{R}^+ * \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} & k < 0 \end{cases}$$

Note that
$$\mathcal{K}\begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} = \{\begin{pmatrix} 0 & -i \\ z & 0 \end{pmatrix} \mid z \in \mathbb{C}\}$$

and
$$\mathcal{K} \cap i\mathfrak{g}(\mathbb{R}) = \{\begin{pmatrix} 0 & -i \\ iy & 0 \end{pmatrix} \mid y \in \mathbb{R}\}$$

In particular
$$\mathcal{K}\begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} \cap i\mathfrak{g}(\mathbb{R}) \ni H_\pi = \begin{pmatrix} 0 & -\frac{ik}{8} \\ \frac{ik}{8} & 0 \end{pmatrix} \quad (k > 0)$$

as required by Proposition 3.2.2. Similarly
$$\mathcal{K}\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \cap i\mathfrak{g}(\mathbb{R}) \ni H_\pi = \begin{pmatrix} 0 & -\frac{ik}{8} \\ \frac{ik}{8} & 0 \end{pmatrix} \quad (k < 0).$$

Using Lemma 3.2.5 amounts to
$$\mathcal{K}(-iX) = \mathcal{K}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \ni {}^\vee\rho(-i) = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad (k > 0)$$

and
$$\mathcal{K}(-iX) = \mathcal{K}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ni {}^\vee\rho(-i) = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \quad (k < 0)$$

Note that the coroot ${}^\vee\alpha : \mathbb{C}^\times \to S$ is given by

$$ {}^\vee\alpha(e^z) = \begin{pmatrix} \cos(z) & \sin(z) \\ -\sin(z) & \cos(z) \end{pmatrix} $$

or more algebraically

$$ {}^\vee\alpha(z) = \begin{pmatrix} \frac{z+\frac{1}{z}}{2} & \frac{z-\frac{1}{z}}{2i} \\ -\frac{z-\frac{1}{z}}{2i} & \frac{z+\frac{1}{z}}{2} \end{pmatrix}. $$

4 Construction of L-packet and internal structure

Let G_0 be a quasi-split connected reductive $\mathbb{R}$-group with dual group $\widehat{G}$ and L-group ${}^LG$. Let $[\varphi] : W_\mathbb{R} \to {}^LG$ be a $\widehat{G}$-conjugacy class of discrete Langlands parameters.

4.1 Factorization of a parameter

Before we state the next lemma, we introduce the following notation. Given $a, b \in \mathbb{C}$ with $a - b \in \mathbb{Z}$ and $z \in \mathbb{C}^\times$ we define

$$ z^a \cdot \bar{z}^b := |z|^{a+b} \cdot (z/|z|)^{a-b}. $$

More generally, if $\widehat{T}$ is a complex torus and $\lambda, \mu \in X_*(\widehat{T})_\mathbb{C}$ with $\lambda - \mu \in X_*(\widehat{T})$, and $z \in \mathbb{C}^\times$, we define

$$ \lambda(z) \cdot \mu(\bar{z}) \in \widehat{T} $$

to be the unique element characterized by

$$ \chi(\lambda(z) \cdot \mu(\bar{z})) = z^{\langle \chi, \lambda \rangle} \cdot \bar{z}^{\langle \chi, \mu \rangle}, \qquad \forall \chi \in X^*(\widehat{T}). $$

It is well known that every continuous group homomorphism $\mathbb{C}^\times \to \mathbb{C}^\times$ is of the form $z \mapsto z^a \cdot \bar{z}^b$ for unique $a, b \in \mathbb{C}$ with $a - b \in \mathbb{Z}$. Therefore, every continuous group homomorphism $\mathbb{C}^\times \to \widehat{T}$ is of the form $z \mapsto \lambda(z) \cdot \mu(\bar{z})$ for unique $\lambda, \mu \in X_*(\widehat{T})_\mathbb{C}$ with $\lambda - \mu \in X_*(\widehat{T})$.

Lemma 4.1.1. *Choose any representative φ within the conjugacy class $[\varphi]$.*

1. *There exists a maximal torus $\widehat{T} \subset \widehat{G}$ that is normalized by the image of φ.*
2. *There exist $\lambda, \mu \in X_*(\widehat{T})_\mathbb{C}$ with $\lambda - \mu \in X_*(\widehat{T})$ such that, for all $z \in \mathbb{C}^\times$,*
$$ \varphi(z) = \lambda(z) \cdot \mu(\bar{z}) \in \widehat{T}. $$
3. *The action of $\Gamma = W_\mathbb{R}/\mathbb{C}^\times$ on $\widehat{T}$ by conjugation via φ induces multiplication by -1 on $X_*(\widehat{T}/Z(\widehat{G}))$.*
4. *For all $\alpha \in R(\widehat{T}, \widehat{G})$, the a-priori complex number $\langle \lambda, \alpha \rangle$ is a non-zero half-integer, and equals $-\langle \mu, \alpha \rangle$.*
5. *$\widehat{T} = Cent(\varphi(\mathbb{C}^\times), \widehat{G})$, thus $\widehat{T}$ is uniquely determined by φ.*

Remark 4.1.2. 1. Note that 4. implies that the images $\lambda', \mu' \in X_*(\widehat{T}/Z(\widehat{G}))_\mathbb{C}$ lie in $\frac{1}{2}X_*(\widehat{T}/Z(\widehat{G}))$ and satisfy $\mu' = -\lambda'$. Although we will not need it, it can be shown that in fact λ', μ' are integral, i.e. they lie in $X_*(\widehat{T}/Z(\widehat{G}))$.

2. Furthermore, 4. implies that λ' is a regular element of $X_*(\widehat{T}/Z(\widehat{G}))_{\mathbb{R}}$, and hence determines a Weyl chamber. Let Δ be the set of simple roots for that chamber. Once the integrality of λ' is established, its regularity implies $\langle\alpha,\lambda'\rangle \geq 1$, and hence $\lambda' - \rho \in X_*(\widehat{T}/Z(\widehat{G}))$ is still a dominant integral element of the chamber.

Proof of Lemma 4.1.1. (1) We consider ${}^LG = \widehat{G} \rtimes \Gamma$ as a disconnected algebraic group. Let $A \subset {}^LG$ denote the Zariski closure of the image of φ and let $B \subset A$ denote the Zariski closure of $\varphi(\mathbb{C}^\times)$; the latter lies in $\widehat{G}$.

We claim that A consists of semi-simple elements. Since $A^2 \subset B$ it is enough to show that B consists of semi-simple elements. This is equivalent to showing that the adjoint action of B on the Lie algebra of $\widehat{G}$ is semi-simple. Thus consider $\mathrm{Ad} : \widehat{G} \to \mathrm{GL}(\widehat{\mathfrak{g}})$. The subgroup $\varphi(\mathbb{C}^\times)$ of $\widehat{G}$ and consists of elements which commute with each other, and are semi-simple elements by definition of φ, cf. [Bor79, §8]. Therefore their actions on $\widehat{\mathfrak{g}}$ can be simultaneously diagonalized, i.e. their images in $\mathrm{GL}(\widehat{\mathfrak{g}})$ lie in a common torus. The preimage of this torus in $\widehat{G}$ is a closed subgroup consisting of commuting semi-simple elements and contains B, proving the claim.

Now B is normal in A of index 2, B/B° is a finite abelian group, and B° is a connected abelian algebraic groups consisting of semi-simple elements, hence a torus. We conclude that A is supersolvable according to [SS70, Definition 5.14], and [SS70, Theorem 5.16] implies that A normalizes a maximal torus $\widehat{T} \subset \widehat{G}$.

We now provide an alternative proof of (1). The image of $\mathbb{C}^\times$ is connected in the analytic topology, abelian, and consists of semisimple elements. It follows [reference?] that $\widehat{L} = \mathrm{Cent}_{\widehat{G}}(\varphi(\mathbb{C}^\times))$ is a Levi subgroup. In particular $\widehat{L}$ is connected, reductive, and contains a maximal torus $\widehat{S}$ of $\widehat{G}$. Since $\varphi(\mathbb{C}^\times)$ is contained in the identity component of the center of $\widehat{L}$, it is contained in $\widehat{S}$. Choose an element $g \in \varphi(W_{\mathbb{R}})\backslash\varphi(\mathbb{C}^\times)$. Then g normalizes $\widehat{L}$, and since $g^2 \in \varphi(\mathbb{C}^\times)$ conjugation by g acts as an involution on $\widehat{L}$ and is thus induces a semi-simple automorphism of $\widehat{L}$. According to [Ste68, Theorem 7.5], it stablizes a Borel pair in $\widehat{L}$. The maximal torus $\widehat{T}$ in that Borel pair is a maximal torus of $\widehat{G}$, containing $\varphi(\mathbb{C}^\times)$ (since the latter is contained in the center of of $\widehat{L}$), and fixed by the action of g, hence and therefore normalized by the image of φ.

(2) We continue with a maximal torus $\widehat{T} \subset \widehat{G}$ normalized by the image of φ. Thus $\varphi(\mathbb{C}^\times)$, which is a subgroup of $\widehat{G}$, lies in $N(\widehat{T},\widehat{G})$. By continuity of φ for the analytic topology, the subset $\varphi(\mathbb{C}^\times)$ of $\widehat{G}$ is connected in the analytic topology. The projection map $N(\widehat{T},\widehat{G}) \to \Omega(\widehat{T},\widehat{G})$ is continuous in the analytic topology, from which we conclude that the image of $\varphi(\mathbb{C}^\times)$ in $\Omega(\widehat{T},\widehat{G})$ is trivial, and hence $B \subset \widehat{T}$. In particular, $\varphi(z) \in \widehat{T}$ for all $z \in \mathbb{C}^\times$. The discussion before the statement of the lemma provides unique $\lambda,\mu \in X_*(\widehat{T}) \otimes_{\mathbb{Z}} \mathbb{C}$ with $\lambda - \mu \in X_*(\widehat{T})$ such that

$$\varphi(z) = \lambda(z)\cdot\mu(\bar{z}), \qquad \forall z \in \mathbb{C}^\times.$$

For the remainder of the proof, we assume without loss of generality that $\widehat{G}$ is adjoint.

(3) Since $\varphi(\mathbb{C}^\times) \subset \widehat{T}$ by (2), the action of $W_\mathbb{R}$ on $\widehat{T}$ by conjugation via φ factors through $W_\mathbb{R}/\mathbb{C}^\times = \Gamma$. If the induced action on $X_*(\widehat{T})$ stabilizes some $0 \neq \nu \in X_*(\widehat{T})$, then the image of φ would be contained in the proper parabolic subgroup P_ν determined by nu, contradicting the assumed discreteness of φ. Therefore, the involution of $X_*(\widehat{T})$ induced by the action of the non-trivial element of Γ is given by multiplication by -1.

(4) Using (3) we see that, for all $z \in \mathbb{C}^\times$,

$$\mu(z)\lambda(\bar{z}) = \varphi(\bar{z}) = \varphi(\sigma \cdot z \cdot \sigma^{-1}) = \mathrm{Ad}(\varphi(\sigma))(\lambda(z)\mu(\bar{z})) = (-\lambda)(z) \cdot (-\mu)(\bar{z}),$$

which shows $\lambda = -\mu$. Thus $\langle \lambda, \alpha \rangle = -\langle \mu, \alpha \rangle$ for all $\alpha \in R(\widehat{T}, \widehat{G})$. Moreover, since $\lambda - \mu \in X_*(\widehat{T})$, we conclude $2\lambda \in X_*(\widehat{T})$.

If there is some $\alpha \in R(\widehat{T}, \widehat{G})$ with $\langle \lambda, \alpha \rangle = 0$, then all elements of the root subgroups U_α and $U_{-\alpha}$ are fixed by $\varphi(\mathbb{C}^\times)$, while these two root subgroups are interchanged by $\varphi(\sigma)$ for any $\sigma \in W_\mathbb{R}$ projecting to the non-trivial element of Γ. The semi-simple group of rank 1 generated by U_α and $U_{-\alpha}$ is thus stable under the action of $W_\mathbb{R}$ via conjugation by φ, and this action descends to Γ and is thus given by an involution. This involution is necessarily inner, i.e. it coincides with the conjutation action of an element of this group. The subgroup of fixed points is thus at least of dimension 1 (it contains a maximal torus in this 3-dimensional subgroup). This subgroup lies in $\mathrm{Cent}(\varphi, \widehat{G})$ and contradicts the discreteness of φ.

(5) From (2) and (4) we have $\varphi(z) = (2\lambda)(z/|z|) \in \widehat{T}$ and 2λ is a regular element of $X_*(\widehat{T})$. Since the image of $z \mapsto z/|z|$ is the unit circle $\mathbb{S}^1$ of $\mathbb{C}^\times$, which is Zariski dense in the 1-dimensional torus $\mathbb{C}^\times$, the centralizer of the subgroup $(2\lambda)(\mathbb{S}^1)$ of $\widehat{G}$ is the same as that of $(2\lambda)(\mathbb{C}^\times)$, which in turn equals $\widehat{T}$. $\square$

We continue with a chosen representative φ of the conjugacy class $[\varphi]$. Lemma 4.1.1 provides the maximal torus $\widehat{T} \subset \widehat{G}$ normalized by φ and containing $\varphi(\mathbb{C}^\times)$. The composition of φ with the projection $N(\widehat{T}, {}^LG) \to \Omega(\widehat{T}, {}^LG) = N(\widehat{T}, {}^LG)/\widehat{T}$ factors through a homomorphism $\xi : \Gamma \to \Omega(\widehat{T}, {}^LG)$. Let $\widehat{S}$ denote the Γ-module with underlying abelian group $\widehat{T}$ and Γ-structure given by $\mathrm{Ad} \circ \xi$. Let S be the $\mathbb{R}$-torus whose dual is $\widehat{S}$, i.e. the $\mathbb{R}$-torus determined by $X^*(S) = X_*(\widehat{S})$ as Γ-modules.

By construction we have $R(\widehat{T}, \widehat{G}) \subset X^*(\widehat{T}) = X^*(\widehat{S}) = X_*(S)$, and we write $R^\vee(S, G)$ for this set. Analogously we have a subset $R(S, G) \subset X^*(S)$. Both of these subsets are Γ-stable and according to Lemma 4.1.1 the action of σ on $R(S, G)$ is by negation. Thus $S/Z(G)$ is anisotropic, where $Z(G) \subset S$ is the joint kernel of all elements if $R(S, G)$.

Let $S(\mathbb{R})_G$ be the double cover of $S(\mathbb{R})$ reviewed in §2.6, associated to the subset $R(S, G) \subset X^*(S)$. As discussed there, there is a canonical $\widehat{G}$-conjugacy class of L-embeddings ${}^LS_G \to {}^LG$. Inside of this class, there is a unique $\widehat{S}$-conjugacy class, call it ${}^Lj$, whose restriction to $\widehat{S}$ is the tautological embedding $\widehat{S} \to \widehat{G}$. The image of this L-embedding is described in (2.5), and contains the image of φ by construction. Thus $\varphi = {}^Lj \circ \varphi_S$ for a unique $\widehat{S}$-conjugacy class of L-homomorphisms $\varphi_S : W_\mathbb{R} \to {}^LS_G$. According to [Kal19a, Theorem 3.15], φ_S corresponds to a genuine character $\tau : S(\mathbb{R})_G \to \mathbb{C}^\times$.

Finally, the tautological inclusion $\widehat{T} \subset \widehat{G}$ provides an embedding $\widehat{\jmath} : \widehat{S} \to \widehat{G}$. While this embedding is not Γ-equivariant, its $\widehat{G}$-conjugacy class is, because the embedding $\widehat{T} \to \widehat{G}$ is Γ-equivariant and the Γ-structures of $\widehat{G}$ and $\widehat{S}$ differ by twisting by $\widehat{G}$.

The construction of $(S, \tau, \widehat{\jmath})$ depended on the choice of φ within its conjugacy class. The next lemma shows that this dependence is irrelevant.

Lemma 4.1.3. *If $(S_1, \tau_1, \widehat{\jmath}_1)$ and $(S_2, \tau_2, \widehat{\jmath}_2)$ are two pairs obtained from two different choices φ_1 and φ_2 of elements of the $\widehat{G}$-conjugacy class $[\varphi]$, there exists a unique isomorphism $S_1 \to S_2$ which identifies τ_1 with τ_2 and whose dual intertwines $\widehat{\jmath}_1$ and $\widehat{\jmath}_2$. It is given by conjugation by an element of $\widehat{G}$.*

Proof. The uniqueness claim is clear from the compatibility with $\widehat{\jmath}_i$. We show existence. Let $\widehat{T}_i$ be the centralizer of $\varphi_i(\mathbb{C}^\times)$, a maximal torus according to Lemma 4.1.1. Choose any $g \in \widehat{G}$ such that $\mathrm{Ad}(g) \circ \varphi_1 = \varphi_2$. Then $\mathrm{Ad}(g)\widehat{T}_1 = \widehat{T}_2$ and the isomorphism $\mathrm{Ad}(g) : \widehat{T}_1 \to \widehat{T}_2$ translates the Γ-action induced by $\mathrm{Ad}\circ\varphi_1$ to that induced by $\mathrm{Ad}\circ\varphi_2$. Therefore, $\mathrm{Ad}(g) : \widehat{S}_1 \to \widehat{S}_2$ is Γ-equivariant. Tautologically, it intertwines $\widehat{\jmath}_1$ and $\widehat{\jmath}_2$.

The dual isomorphism $S_2 \to S_1$ identifies the subsets $R(S_1, G) \subset X^*(S_1)$ and $R(S_2, G) \subset X^*(S_2)$, hence lifts canonically to an isomorphism of double covers $S_2(\mathbb{R})_G \to S_1(\mathbb{R})_G$. Dually the isomorphism $\widehat{S}_1 \to \widehat{S}_2$ extends canonically to an isomorphism ${}^LS_{1,G} \to {}^LS_{2,G}$, which commutes with the canonical L-embeddings into ${}^LG$. Therefore, it translates the factorization of φ_1 through ${}^LS_{1,G}$ to the factorization of φ_2 through ${}^LS_{2,G}$, which implies that the isomorphism $S_2(\mathbb{R})_G \to S_1(\mathbb{R})_G$ identifies τ_2 with τ_1. □

4.2 Construction of the L-packet

The natural embedding $\widehat{S} \to \widehat{G}$ is not Γ-equivariant, but its $\widehat{G}$-conjugacy class is. From §2.3 we obtain the category $\mathcal{J}$ of embeddings of S into all pure (or rigid) inner forms of G_0.

Consider $(G, \xi, z, j) \in \mathcal{J}(\mathbb{R})$. As discussed in §2.7, there exists a unique essentially discrete series representation π_j of $G(\mathbb{R})$ associated to the pair (S, τ), transported to G via j. According to §2.7 the representation π_j depends on the $G(\mathbb{R})$-conjugacy class of j, and two distinct such conjugacy classes produce two non-isomorphic representations.

Definition 4.2.1. We define the pure (resp rigid) compound L-packet

$$\tilde{\Pi}_\varphi = \{(G, \xi, z, \pi_j) | (G, \xi, z, j) \in \mathcal{J}(F)\} \subset \tilde{\Pi},$$

$$\Pi_\varphi = \tilde{\Pi}_\varphi / G_0(\mathbb{C}) \subset \Pi.$$

For each pure (or rigid) inner twist (G, ξ, z) of G_0, we define

$$\Pi_\varphi(G, \xi, z) = \{\pi | (G, \xi, z, \pi) \in \tilde{\Pi}_\varphi\}.$$

Lemma 4.2.2. *The set of representations $\Pi_\varphi((G, \xi, z))$ equals $\{\pi_j | j \in J^G(\mathbb{R})/G(\mathbb{R}))\}$. In particular, it is independent of z. It coincides with the set $\Pi_\varphi(G)$ constructed by Langlands in [Lan89, §3].*

Proof. As discussed in §2.3, the set of $(G, \xi, z, j) \in \mathcal{J}(\mathbb{R})$ with fixed triple (G, ξ, z) corresponds to $J^G(\mathbb{R})$, in particular is independent of z. Since the images of the members of $J^G(\mathbb{R})$ are elliptic maximal tori, and all such are conjugate under $G(\mathbb{R})$, we can choose representatives of $J^G(\mathbb{R})/G(\mathbb{R})$ that all have the same image, call it $S' \subset G$. Then these representatives are a single orbit under $\Omega(S', G)$. In other words, if τ' is the transport of τ under one admissible embedding $j : S \to G$ with image S', then all others make out the $\Omega(S', G)$-orbit of τ'. Since we have specified the representations π_j by their character values on regular elements of $S'(\mathbb{R})$ in the same way as in [Lan89, §3] or [AV16, §4]. $\square$

We have thus recovered the L-packets constructed by Langlands. At the moment they do not depend on the datum z in the triple (G, ξ, z). This datum will play a role in the internal parameterization of these packets, to which we turn next.

4.3 Internal structure of the compound packet

Recall from Lemma 2.3.1 that the abelian group $H^1(\Gamma, S)$ in the pure case (resp. $H^1(u \to W, Z(G_0) \to S)$ in the rigid case) acts simply transitively on the set $\mathcal{J}(\mathbb{R})/G_0(\mathbb{C})$. By construction we have a bijection $\mathcal{J}(\mathbb{R})/G_0(\mathbb{C}) \to \Pi_\varphi$. At the same time, Lemma 4.1.1 provides an identification $S_\varphi = \widehat{S}^\Gamma$, hence by Tate-Nakayama duality $\pi_0(S_\varphi)^* = \pi_0(\widehat{S}^\Gamma)^* = H^1(\Gamma, S)$. Analogously, in the rigid setting we obtain $\pi_0(S_\varphi^+)^* = H^1(u \to W, Z(G_0) \to S)$. This provides a simply transitive action of the abelian group $\pi_0(S_\varphi)^*$ in the pure setting, and $\pi_0(S_\varphi^+)^*$ in the rigid setting, on the set Π_φ.

Lemma 4.3.1. *The set $\Pi_\varphi((G_0, 1, 1))$ contains a unique $\mathfrak{w}$-generic member.*

Proof. Choose a G_0-equivariant extension κ to $\mathfrak{g}$ of the Killing form on $[\mathfrak{g}, \mathfrak{g}]$. Its pull-back to $\mathfrak{s}$ along any embedding $j \in J^{G_0}$ is the same, since all these embeddings are conjugate under G_0.

Let $H \in i\mathfrak{s}(\mathbb{R})$ be the element such that $\kappa(H, Y) = d\tau(Y)$ for all $Y \in \mathfrak{s}(\mathbb{R})$. For $j \in J^{G_0}(\mathbb{R})$, the element $dj(H) \in i\mathfrak{g}(\mathbb{R})$ is associated to the representation π_j as in Proposition 3.2.2, which implies that π_j is $\mathfrak{w}$-generic if and only if $dj(H)$ meets the associated Kostant section. But as j varies over $J^{G_0}(\mathbb{R})/G_0(\mathbb{R})$, the element $dj(H)$ varies over the $G_0(\mathbb{R})$-classes in a fixed stable class. Therefore, $dj(H)$ meets any Kostant section for precisely one $j \in J^{G_0}(\mathbb{R})/G_0(\mathbb{R})$. At the same time, $j \mapsto \pi_j$ is a bijection from $j \in J^{G_0}(\mathbb{R})/G_0(\mathbb{R})$ to $\Pi_\varphi((G_0, 1, 1))$ by construction of the latter. $\square$

Taking the unique $\mathfrak{w}$-generic member of $\Pi((G_0, 1, 1)) \subset \Pi_\varphi$, provided by Lemma 4.3.1, as a base-point, the simply-transitive action turns into the bijection from

$$\iota_{\mathfrak{w}} : \pi_0(S_\varphi)^* \to \Pi_\varphi, \quad \text{resp.} \quad \iota_{\mathfrak{w}} : \pi_0(S_\varphi^+)^* \to \Pi_\varphi.$$

We can summarize the construction and internal structure of Π_φ as follows. We use the language of pure inner forms; that of rigid inner forms is entirely analogous. We use the simplified notation to write $\tilde{\Pi}$ for the set of pairs (z, π), where $z \in Z^1(\mathbb{R}, G_0)$ and π an isomorphism class of representations of the twist G_z, and $\Pi = \tilde{\Pi}/G_0(\mathbb{C})$. Then $\tilde{\Pi}_\varphi \subset \tilde{\Pi}$ is the image of the map $\mathcal{J}(F) \to \tilde{\Pi}$

sending (z, j) to (z, π_j). The group $Z^1(\mathbb{R}, S)$ acts on $\mathcal{J}(\mathbb{R})$ by $x \cdot (z, j) = (x \cdot z, j)$. If $j_{\mathfrak{w}} : S \to G_0$ is an embedding for which $\pi_{j_{\mathfrak{w}}}$ is $\mathfrak{w}$-generic, then we obtain the map $Z^1(\mathbb{R}, S) \to \mathcal{J}(\mathbb{R})$ sending x to $(x, j_{\mathfrak{w}})$. Composing this with the map $\mathcal{J}(\mathbb{R}) \to \tilde{\Pi}_\varphi$ and taking quotient under the action of $G_0(\mathbb{C})$ produces the bijection $H^1(\mathbb{R}, S) \to \Pi_\varphi$, which, together with the isomorphism $\pi_0(S_\varphi)^* = H^1(\mathbb{R}, S)$ produces the desired bijection $\iota_{\mathfrak{w}} : \pi_0(S_\varphi)^* \to \Pi_\varphi$.

4.4 Dependence on the choice of Whittaker datum

In order to obtain the bijection from $\pi_0(S_\varphi)^*$ (resp,. $\pi_0(S_\varphi^+)^*$) to Π_φ, we had to choose a Whittaker datum $\mathfrak{w}$ and apply Lemma 4.3.1. Another Whittaker datum is of the form $\mathfrak{w}' = \mathrm{Ad}(\bar{g})\mathfrak{w}$ with $\bar{g} \in G_{0,\mathrm{ad}}(\mathbb{R})$. According to Proposition 3.2.2, if π is $\mathfrak{w}$-generic, then $\mathrm{Ad}(\bar{g})\pi$ is $\mathfrak{w}'$-generic. If $j : S \to G_0$ is the embedding with $\pi = \pi_j$, then $\mathrm{Ad}(\bar{g})\pi_j$ is associated to the embedding $\mathrm{Ad}(\bar{g}) \circ j$.

As described in §4.3, the bijection $\pi_0(S_\varphi)^* \to \Pi_\varphi$ is the composition of the orbit map for the action of $\pi_0(S_\varphi)^* = H^1(\mathbb{R}, S)$ on $\mathcal{J}(\mathbb{R})/G_0(\mathbb{C})$ through the embedding j with the $G_0(\mathbb{C})$-equivariant bijection $\mathcal{J}(F) \to \tilde{\Pi}_\varphi$. Neither the latter bijection nor the action of $H^1(\mathbb{R}, S)$ on $\mathcal{J}(\mathbb{R})/G_0(\mathbb{C})$ depend on $\mathfrak{w}$, only the particular point $j \in \mathcal{J}(F)$ does. The orbit map through j is given by $x \mapsto (x, j)$, while the orbit map through $\mathrm{Ad}(\bar{g}) \circ j$ is given by $x \mapsto (x, \mathrm{Ad}(\bar{g}) \circ j)$. But $(x, \mathrm{Ad}(\bar{g}) \circ j) = g(g^{-1}x\sigma(g), j)g^{-1}$, for any $g \in G_0(\mathbb{C})$ lifting $\bar{g} \in G_{0,\mathrm{ad}}(\mathbb{R})$.

Now $z(\sigma) = g^{-1}\sigma(g)$ belongs to $Z^1(\mathbb{R}, Z(G_0))$. We conclude that $(x, \mathrm{Ad}(\bar{g}), j)$ is equivalent modulo the action of $G_0(\mathbb{C})$ to $(z \cdot x, j)$. This shows that the bijection $H^1(\mathbb{R}, S) \to \Pi_\varphi$ normalized via $\mathfrak{w}' = \mathrm{Ad}(g)\mathfrak{w}$ is obtained form the bijection normalized via $\mathfrak{w}$ by shifting the latter via multiplication by $[z] \in H^1(\mathbb{R}, Z(G_0))$.

Now consider the identification $\pi_0(S_\varphi)^* = H^1(\mathbb{R}, S)$. It can be extended to a commutative diagram

$$\begin{tikzcd} \pi_0(S_\varphi)^* \arrow[r, "\cong"] & H^1(\mathbb{R}, S) \\ H^1(W_\mathbb{R}, \widehat{G}_{\mathrm{sc}} \to \widehat{G})^* \arrow[u] \arrow[r, "\cong"] & H^1(\mathbb{R}, Z(G_0)). \arrow[u] \end{tikzcd}$$

We have used here the $W_\mathbb{R}$-cohomology of the crossed module $\widehat{G}_{\mathrm{sc}} \to \widehat{G}$ and the duality between it and the Γ-cohomology of the crossed module $G \to G_{\mathrm{ad}}$. The latter is however quasi-isomorphic to $Z(G_0)$. The left vertical map is obtained from the long exact cohomology sequence for the crossed module $\widehat{G}_{\mathrm{sc}} \to \widehat{G}$ endowed with $W_\mathbb{R}$-action given by $\mathrm{Ad} \circ \varphi$. The edge map $S_\varphi = H^0(W_\mathbb{R}, \varphi, \widehat{G}) \to H^1(W_\mathbb{R}, \widehat{G}_{\mathrm{sc}} \to \widehat{G})$ factors through $\pi_0(S_\varphi)$. Note that, the cohomology of $\widehat{G}_{\mathrm{sc}} \to \widehat{G}$ is canonically the same whether we take the $W_\mathbb{R}$-action coming from the dual groups, or the one coming from $\mathrm{Ad} \circ \varphi$, because the two structures differ by a homotopically trivial twist in the sense of [Kal11, §2.4]. For an alternative construction without resorting to crossed modules we refer to [Kal13, §4].

If we denote by $(\mathfrak{w}, \mathfrak{w}')$ the character of $H^1(W_\mathbb{R}, \widehat{G}_{\mathrm{sc}} \to \widehat{G})$ corresponding to $[z]$, as well as its pull-back to $\pi_0(S_\varphi)$, then we obtain the formula

$$\iota_{\mathfrak{w}'}(x) = \iota_{\mathfrak{w}}(x \cdot (\mathfrak{w}, \mathfrak{w}')). \tag{4.1}$$

4.5 The case of a cover of G

We now consider G quasi-split and a cover of $G(\mathbb{R})$ coming from a character $x : \tilde{\pi}_1(G) \to \mu_n(\mathbb{R})$ as in [Kal22]. This will be relevant for endoscopic groups H, where we will be working with L-packets on the double cover $H(\mathbb{R})_\pm$. It was shown in [Kal22, §2.6] that the local Langlands correspondence for such covers follows from the case of non-covers. We will give here an alternative approach that does not pass via reduction to non-covers, and instead works directly with the covers. This is possible due to the flexibility of Harish-Chandra's results on representation theory and harmonic analysis.

The procedure is essentially the same as for the group $G(\mathbb{R})$. We start with a discrete L-parameter $\varphi : W_\mathbb{R} \to {}^L G_x$. By the same arguments as in §4.1, $\widehat{T} = \mathrm{Cent}(\varphi(\mathbb{C}^\times), \widehat{G})$ is a maximal torus of $\widehat{G}$, and the image of φ lies in $N(\widehat{T}, {}^L G_x)$. The composition of φ with the adjoint action Ad factors through Γ and induces a Γ-structure on $\widehat{T}$, which we call $\widehat{S}$. The $\widehat{G}$-conjugacy class of the inclusion $\widehat{\jmath} : \widehat{S} \to \widehat{G}$ is Γ-stable, hence leads again to the set $\mathcal{J}(F)$ of admissible embeddings of S into pure (or rigid) inner forms of G.

Let $S(\mathbb{R})_G$ be the double cover of $S(F)$ associated to the subset $R(S,G) \subset X^*(S)$. It is the same double cover that was used in §4.1. Now we have another double cover, called $S(F)_x$, coming from the pull-back of $x : \tilde{\pi}_1(G) \to \mu_n(\mathbb{C})$ under the natural map $\tilde{\pi}_1(S) \to \tilde{\pi}_1(G)$. Each admissible embedding $j : S \to G$ lifts naturally to an embedding $S(\mathbb{R})_x \to G(\mathbb{R})_x$.

Consider the Baer sum $S(\mathbb{R})_{G,x}$ of the two double covers. We have the canonical L-embedding ${}^L S_G \to {}^L G$. This embedding induces a canonical L-embedding ${}^L S_{G,x} \to {}^L G_x$. The L-parameter φ factors through it and provides an L-parameter $\varphi_S : W_\mathbb{R} \to {}^L S_{G,x}$, hence a genuine character $\tau : S(\mathbb{R})_{G,x} \to \mathbb{C}^\times$. Thus again we have a triple $(S, \tau, \widehat{\jmath})$. The analog of Lemma 4.1.3 holds, with the same proof.

For each $(G, \xi, z, j) \in \mathcal{J}(\mathbb{R})$ we use j to identify S with a maximal torus of G. We can again write the formula (2.6). Note that now τ/d_τ is not a function of $S(\mathbb{R})$ and more, but rather a function on $S(\mathbb{R})_x$, which is a subgroup of $G(\mathbb{R})_x$. Nonetheless, Harish-Chandra's results imply that there is a unique discrete series representation π_j of the real Lie group $G(\mathbb{R})_x$ whose character restricted to $S(\mathbb{R})_x$ is given by this formula. The definition of the L-packet and its internal structure are now done in the same way as in §4.1 and §4.3.

[Jeff, am I right here that Harish-Chandra's results can treat the generality of finite-order covers of reductive groups?]

[Answer: A little care is required here. If you take a non-linear cover then Cartan subgroups aren't abelian. Trouble can arise, but probably not here. I'm not sure precisely what covers we're talking about here (see the warning above). But I assume they are linear in the following sense: $G(\mathbb{R})_x$ is a subgroup of a complex reductive group (something like $G(C) \times C^\times$). Then it is in "Harish-Chandra's class" which makes everything OK. But there is something to check.]

Let $\varphi : W_\mathbb{R} \to {}^LG$ be a discrete parameter. Let $s \in S_\varphi$ (resp. $s \in S_\varphi^+$) be a semi-simple element. Let $(H, s, \mathcal{H}, \eta)$ be the (pure or rigid) refined endoscopic datum associated to the pair (φ, s), whose construction was reviewed in §2.8.

5.1 Statement of the main theorem

As discussed in §4.5 there is an associated compound L-packet Π_{φ_H}. We will be only interested in the contribution of the trivial twist $(H, 1, 1)$, and we write $\Pi_\varphi(H) \subset \Pi_{\varphi_H}$ for it. Consider the virtual character

$$S\Theta_{\varphi_H} := \sum_{\sigma \in \Pi_\varphi(H)} \langle \sigma, s\rangle \Theta_\sigma = \sum_{\sigma \in \Pi_\varphi(H)} \langle \sigma, 1\rangle \Theta_\sigma = \sum_{\sigma \in \Pi_\varphi(H)} \Theta_\sigma$$

on $H(\mathbb{R})_\pm$, where $\langle \sigma, -\rangle$ is the character of the irreducible representation of $\pi_0(S_\varphi)$ (resp. $\pi_0(S_\varphi^+)$) associated to σ by the bijection of §4.3. Let us argue the two equalities. Since $Z(\widehat{H})^\Gamma$ (resp. $Z(\widehat{\widehat{H}})^+$) acts trivially on this irreducible representation, and s belongs by construction to this group, we see $\langle \sigma, s\rangle = \langle \sigma, 1\rangle$, hence the first equality. The second comes from the fact that S_φ is abelian, because it lies in $\widehat{S}$ (and S_φ^+ lies in $\widehat{\widehat{S}}$), where $\widehat{S}$ is the torus involved in the construction of the L-packet on H. Note that, while the bijection of §4.3 depends on the choice of a Whittaker datum, the argument of §3.1 shows that the value $\langle \sigma, 1\rangle$ does not depend on this choice.

Let (G, ξ, z) be a pure (resp. rigid) inner twist of G_0. We have the virtual character on $G(\mathbb{R})$ given by

$$\Theta_\varphi^{\mathfrak{w},s} := e(G) \sum_{\pi \in \Pi_\varphi((G,\xi,z))} \langle \pi, s\rangle \Theta_\pi.$$

This virtual character does depend on $\mathfrak{w}$.

The following is the main theorem of this article. It is a fundamental result of Shelstad [She82], [She10], [She08b].

Theorem 5.1.1. *Let $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$ be a test function.*

1. *If $f^{H_\pm} \in \mathcal{C}_c^\infty(H(\mathbb{R})_\pm)$ matches f as in Definition 2.9.2, then*

$$\Theta_\varphi^{\mathfrak{w},s}(f) = S\Theta_{\varphi_H}(f^{H_\pm}).$$

2. *If $f^{H_1} \in \mathcal{C}_c^\infty(H_1(\mathbb{R}))$ matches f as in Definition 2.9.2, then*

$$\Theta_\varphi^{\mathfrak{w},s}(f) = S\Theta_{\varphi_{H_1}}(f^{H_1}).$$

Remark 5.1.2. Recall from Remark 2.9.3 that the concept of matching functions depends on choices of measures for $G(\mathbb{R})$, $H(\mathbb{R})$, and all tori in those groups. In the above theorem the distribution $\Theta_\varphi^{\mathfrak{w},s}$ depends on the choice of measure on $G(\mathbb{R})$, and $S\Theta_{\varphi_H}$ depends on the choice of measure on $H(\mathbb{R})$. But the measures on the tori are do not influence these distributions. Therefore, the validity of the claimed identity assumes that the measures on the tori of G and H are synchronized.

More precisely, if $T_H \subset H$ and $T \subset G$ are tori and $T_H \to T$ is an admissible isomorphism, we demand that it identifies the measures on $T_H(\mathbb{R})$ and $T(\mathbb{R})$. Any two admissible isomorphisms differ by conjugation by $\Omega(T,G)(\mathbb{R})$. Since this action preserves any Haar measure on $T(\mathbb{R})$, the choice of admissible isomorphism is irrelevant.

5.2 Reduction to the elliptic set

We now formulate an equivalent version of Theorem 5.1.1 that involves character functions, rather than character distributions. Note that the character functions are canonical, in particular independent of choices of mesures.

Theorem 5.2.1. *1. For every strongly regular semi-simple element $\delta \in G(\mathbb{R})$ the following identity holds*

$$\Theta_\varphi^{\mathfrak{w},s}(\delta) = \sum_{\gamma \in H(\mathbb{R})/st} \Delta[\mathfrak{w},\mathfrak{e},z](\dot\gamma,\delta) S\Theta_{\varphi_H}(\dot\gamma).$$

2. For every strongly regular semi-simple element $\delta \in G(\mathbb{R})$ the following identity holds

$$\Theta_\varphi^{\mathfrak{w},s}(\delta) = \sum_{\gamma \in H(\mathbb{R})/st} \Delta[\mathfrak{w},\mathfrak{e},\mathfrak{z},z](\gamma_1,\delta) S\Theta_{\varphi_{H_1}}(\gamma_1).$$

Lemma 5.2.2. *Theorem 5.1.1 is equivalent to Theorem 5.2.1. In fact, this equivalence holds when $\mathbb{R}$ is replaced by any local field F.*

Proof. The proofs of 1. and 2. are almost identical. We give the proof of 1.

We have $S\Theta_{\varphi_H}(f^{H_\pm}) = \int_{H(F)} |D^H(\gamma)|^{-1/2} S\Theta_{\varphi_H}(\dot\gamma) f^{H_1}(\dot\gamma) d\gamma$, where $\dot\gamma \in H(F)_\pm$ is any preimage of γ. Note that, while the factors in the integrand depends on $\dot\gamma$, the dependence cancels out in the product, because both $S\Theta_\varphi$ and $f^{H_\pm}$ are genuine.

We apply the stable Weyl integration formula (Corollary 2.10.2, but in the form of Remark 2.10.3) and use the stable invariance of $S\Theta_{\varphi_H}$ to rewrite the above as

$$\sum_{T_H \in \mathcal{ST}_H} \int_{T_H(F)_{\mathrm{sr}}/\Omega(T_H,H)(F)} S\Theta_{\varphi_H}(\dot\gamma) SO_{\dot\gamma}(f^{H_\pm}) d\gamma.$$

According to Theorem 2.9.1 this equals

$$\sum_{T_H \in \mathcal{ST}_H} \int_{T_H(F)_{\mathrm{sr}}/\Omega(T_H,H)(F)} S\Theta_{\varphi_H}(\dot\gamma) \sum_\delta \Delta(\dot\gamma,\delta) O_\delta(f) d\gamma. \tag{5.1}$$

We will now rework the indexing sets. Let $\mathfrak{X}$ be the set of pairs $([[\gamma]],[\delta])$, where $[[\gamma]]$ is a stable class of strongly regular semi-simple elements of $H(F)$, $[\delta]$ is a rational class of strongly regular semi-simple elements of $G(F)$, and γ and δ are related. Projecting onto the first coordinate provides a surjective map

$$\mathfrak{X} \to \bigcup_{T_H \in \mathcal{ST}_H} T_H(F)_{\mathrm{sr}}/\Omega(T_H,H)(F)$$

with finite fibers. Pulling back the measure on the target that is comprised of the various Haar measures on the tori $T_H(F)$, we obtain a measure on $\mathfrak{X}$, and (5.1) becomes

$$\int_{\mathfrak{X}} S\Theta_{\varphi_H}(\dot\gamma) \Delta(\dot\gamma,\delta) O_\delta(f) d([[\gamma]],[\delta]). \tag{5.2}$$

On the other hand, projecting onto the second coordinate provides a surjective map

$$\mathfrak{X} \to \bigcup_{T \in \mathcal{T}_G} T(F)_{\mathrm{sr}}/\Omega_F(T,G)$$

and pulling back the measure on the target that is comprised of the various Haar measures on the tori $T(F)$, we obtain a measure on $\mathfrak{X}$. Since the measures on the tori in H and those on the tori in G are synchronized as in Remark 5.1.2, the two measures on $\mathfrak{X}$ agree. Therefore, we can rewrite (5.2) as

$$\sum_{T \in \mathcal{T}} \int_{T(F)_{\mathrm{sr}}/\Omega_F(T,G)} O_\delta(f) \sum_{\gamma} \Delta(\dot\gamma, \delta) S\Theta_{\varphi_H}(\dot\gamma) d\delta,$$

where now γ runs over the set of stable classes of strongly regular semi-simple elements of $H(F)$. Applying the Weyl integration formula (Theorem 2.10.1, in the form of Remark 2.10.3) we can rewrite this as

$$\int_{G(F)_{\mathrm{sr}}} f(\delta)|D^G(\delta)|^{-1/2} \sum_{\gamma} \Delta(\dot\gamma, \delta) S\Theta_{\varphi_H}(\dot\gamma) d\delta. \tag{5.3}$$

Now, if we assume the validity of Theorem 5.1.1, so that $S\Theta_{\varphi_H}(f^{H_\pm}) = \Theta_\varphi(f)$ we see that

$$\int_{G(F)_{\mathrm{sr}}} f(\delta) \sum_{\gamma} \Delta(\dot\gamma, \delta) S\Theta_{\varphi_H}(\dot\gamma)|D^G(\delta)|^{-1/2} d\delta = \int_{G(F)_{\mathrm{sr}}} f(\delta)\Theta_\varphi(\delta)|D^G(\delta)|^{-1/2} d\delta$$

and conclude that

$$\Theta_\varphi(\delta) = \sum_{\gamma} \Delta(\dot\gamma, \delta) S\Theta_{\varphi_H}(\dot\gamma), \tag{5.4}$$

which is the statement of Theorem 5.2.1. Conversely, if we assume the validity of Theorem 5.2.1 then plugging (5.4) into (5.3) we obtain the validity of Theorem 5.1.1. □

Lemma 5.2.3. *If Theorem 5.2.1 holds for all elliptic δ, then it holds for all δ.*

Proof. [this proof is not finished]

Again the proofs of 1. and 2. are essentially the same.

Step 1: Let $G(\mathbb{R})'$ be the image of $G_{\mathrm{sc}}(\mathbb{R}) \to G(\mathbb{R})$. Both sides are supported on $Z_G(\mathbb{R}) \cdot G(\mathbb{R})'$, and transform by the same character of $Z_G(\mathbb{R})$. It is thus enough to compare them after pulling back to $G_{\mathrm{sc}}(\mathbb{R})$. This allows us to assume that G is semi-simple and simply connected.

Proof. Lemma 2.7.1 shows that the character of a discrete series representation of $G(\mathbb{R})$ is supported on $Z_G(\mathbb{R}) \cdot G(\mathbb{R})'$. It follows that the function $\Theta_\varphi^{\mathfrak{w},s}$ is supported on $Z_G(\mathbb{R}) \cdot G(\mathbb{R})'$ and transforms under $Z_G(\mathbb{R})$ by the common central character $\omega_\varphi : Z_G(\mathbb{R}) \to \mathbb{C}^\times$ of all members of $\Pi_\varphi(G)$. Applying the same reasoning to H we see that the function $S\Theta_{\varphi_\pm}$ is supported on $Z_{H(\mathbb{R})_\pm} \cdot H(\mathbb{R})'_\pm$, where $H(\mathbb{R})'_\pm$ is the image of the natural splitting $H_{\mathrm{sc}}(\mathbb{R}) \to H(\mathbb{R})_\pm$.

Letting $Z_G(\mathbb{R})$ act on the elliptic maximal torus of G by multiplication, the assumption that the identity of Theorem 5.2.1 holds for elliptic elements implies that both sides of that identity transform under the same character of $Z_G(\mathbb{R})$.

To show that the right hand side of that identity is supported on $Z_G(\mathbb{R}) \cdot G_{\mathrm{der}}(\mathbb{R})^0$, consider a strongly regular semi-simple element $\delta \in G(\mathbb{R})$ for which the right-hand side is non-zero. Thus there exists $\dot\gamma \in H(\mathbb{R})_\pm$ such that $S\Theta_{\varphi'}(\dot\gamma) \neq 0$ and such that $\Delta(\dot\gamma, \delta)$.

Let $T \subset G$ be the centralizer of δ and let $S \subset H$ be the centralizer of the image $\gamma \in H(\mathbb{R})$ of $\dot\gamma$. Then $\gamma \in Z_H(\mathbb{R}) \cdot S(\mathbb{R})'$, where $S(\mathbb{R})'$ is the image of $S_{\mathrm{sc}}(\mathbb{R}) \to S(\mathbb{R})$. There is a unique admissible isomorphism $j : S \to T$ mapping γ to δ. The induced isomorphism $X_*(S) \to X_*(T)$ identifies the coroot lattice $Q^\vee(S, H)$ with a sublattice of the coroot lattice $Q^\vee(T, G)$, and hence restricts to a homomorphism $X_*(S_{\mathrm{sc}}) \to X_*(T_{\mathrm{sc}})$. This means that $j : S(\mathbb{R}) \to T(\mathbb{R})$ maps $S(\mathbb{R})'$ into $T(\mathbb{R})'$. This reduces to showing that j maps $Z_H(\mathbb{R})$ to $Z_G(\mathbb{R}) \cdot T(\mathbb{R})'$. For this we recall that, since the discrete parameter φ factors through H, H is an elliptic endoscopic group, i.e. $Z_H(\mathbb{R})/Z_G(\mathbb{R})$ is compact. Thus the image of $Z_H(\mathbb{R})$ in $T(\mathbb{R})$ lies in $Z_G(\mathbb{R}) \cdot T_c(\mathbb{R})$, where $T_c(\mathbb{R})$ is the maximal anisotropic sub torus in T.

[finish] $\square$

Step 2: Both sides are invariant eigendistributions, with the same infinitesimal character, whose values on the regular set are bounded.

Proof. It is known [HC65, Theorem 3] that the un-normalized character of a discrete series representation π of a $G(\mathbb{R})$ is represented by a conjugation-invariant function Θ which is an eigenfunction for the center $\mathfrak{z}$ of the universal enveloping algebra with respect to the character $\mu : \mathfrak{z} \to \mathbb{C}$ corresponding to $d\tau$, where (S, τ) is the Harish-Chandra parameter of π, and such that the normalized function $|D_T|^{1/2}\Theta$ is bounded. These statements carry over to the left-hand side of the identity of Theorem 5.2.1, which is just a complex linear combination of such functions.

We now claim that the right-hand side of the identity of Theorem 5.2.1 is also has those properties. Invariance (under conjugation by $G(\mathbb{R})$) follows from the corresponding property of the transfer factor $\Delta(\dot\gamma, \delta)$ in the variable δ.

To see boundedness, we note first that we have already normalized the characters by $|D_T|^{1/2}$, so we need to show that the right-hand side is bounded. For any fixed δ, there are only finitely many $\dot\gamma$ with $\Delta(\dot\gamma, \delta) \neq 0$, and in fact their number is bounded by $|\Omega(T, G)|$. The values of the transfer factor are roots of unity ([Kal22, Lemma 4.3.3]). The boundedness of the right-hand side now follows from the boundedness of $S\Theta_{\varphi'}$.

[remains to check eigen]

$\square$

Step 3: We have shown that both sides of the identity of Theorem 5.2.1 are invariant functions on $G(\mathbb{R})_{\mathrm{sr}}$, transform under the same character of $\mathfrak{z}$, and are bounded (having already been normalized). We are assuming that they take equal values on all elliptic elements. Therefore [HC65, Lemma 44] implies that they are equal. $\square$

5.3 The left hand side

In this subsection we will provide a formula for the left hand side of the identity in Theorem 5.2.1, i.e. $\Theta^{\mathfrak{w}}_{\varphi,s}(\delta)$, for strongly regular semi-simple elliptic $\delta \in G(\mathbb{R})$. The end result is (5.5).

The members of $\Pi_\varphi((G,\xi,z))$ are parameterized by the set of $G(\mathbb{R})$-conjugacy classes of admissible embedding $j : S \to G$. Given such an embedding let π_j be the corresponding representation. Let $j_{\mathfrak{w}} : S \to G_0$ be the unique embedding for which $\pi_{j_{\mathfrak{w}}}$ is the unique $\mathfrak{w}$-generic member of $\Pi_\varphi((G_0,1,1))$. Then $\mathrm{inv}(j_{\mathfrak{w}}, j) \in H^1(\Gamma, S) = \pi_0(\widehat{S}^\Gamma)^* = \pi_0(S_\varphi)^*$ equals ρ_{π_j}. Therefore the left hand side becomes

$$\Theta^{\mathfrak{w}}_{\varphi,s}(\delta) = e(G)\sum_j \langle s, \mathrm{inv}(j_{\mathfrak{w}}, j)\rangle \Theta_{\pi_j}(\delta),$$

where j runs over (a set of representatives for) the set of $G(\mathbb{R})$-conjugacy classes in J_ξ. Harish-Chandra's character formula (2.6) states

$$\Theta_{\pi_j}(\delta) = (-1)^{q(G)} \sum_{w\in W_{\mathbb{R}}(G,jS)} \frac{\tau'}{d'_\tau}(j^{-1}w^{-1}\delta),$$

where we have conjugated δ within $G(\mathbb{R})$ to land in $jS(\mathbb{R})$. Combining the two formulas and using $e(G) = (-1)^{q(G_0)-q(G)}$, we obtain

$$\Theta^{\mathfrak{w}}_{\varphi,s}(\delta) = (-1)^{q(G_0)} \sum_j \langle s, \mathrm{inv}(j_{\mathfrak{w}}, j)\rangle \sum_{w\in W_{\mathbb{R}}(G,jS)} \frac{\tau'}{d'_\tau}(j^{-1}w^{-1}\delta).$$

Instead of conjugating δ to land in jS, we can conjugate j by $G(\mathbb{R})$ to achieve this, without changing π_j. With this shift in point of view we can combine the two sums and arrive at

$$\Theta^{\mathfrak{w}}_{\varphi,s}(\delta) = (-1)^{q(G_0)} \sum_j \langle s, \mathrm{inv}(j_{\mathfrak{w}}, j)\rangle \frac{\tau'}{d'_\tau}(j^{-1}w^{-1}\delta),$$

where now the sum runs over the set of those $j \in J_\xi$ whose image contains δ.

As j runs over this set, $j_{\mathfrak{w}}j^{-1}(\delta)$ runs over the set of elements $\delta_0 \in S_{\mathfrak{w}}(\mathbb{R})$ that are stably conjugate to δ, where $S_{\mathfrak{w}} \subset G_0$ is the image of $j_{\mathfrak{w}}$, an elliptic maximal torus of G_0. Moreover, $j_{\mathfrak{w}}$ transports $\mathrm{inv}(j_{\mathfrak{w}}, j) \in H^1(\mathbb{R}, S)$ to $\mathrm{inv}(\delta_0, \delta) \in H^1(\mathbb{R}, S_{\mathfrak{w}})$. So we arrive at

$$\Theta^{\mathfrak{w}}_{\varphi,s}(\delta) = (-1)^{q(G_0)} \sum_{\delta_0} \langle s_{\mathfrak{w}}, \mathrm{inv}(\delta_0, \delta)\rangle \frac{\tau'_{\mathfrak{w}}}{d'_{\mathfrak{w}}}(\delta_0), \tag{5.5}$$

where the sum runs over the set of elements $\delta_0 \in S_{\mathfrak{w}}(\mathbb{R})$ that are stably conjugate to δ, and we have used the subscript $\mathfrak{w}$ to indicate various transports under $j_{\mathfrak{w}} : S \to S_{\mathfrak{w}}$.

5.4 The right hand side: covers

In this subsection we will show that the right hand side of the identity of Theorem 5.2.1(1) is also equal to (5.5).

We begin by applying (5.5) to the group H, the parameter φ_H, and the trivial endoscopic element, and obtain

$$S\Theta_{\varphi_H}(\dot\gamma) = \sum_{\dot\gamma_0} \frac{\tau_H}{d_H}(\dot\gamma_0),$$

where we have fixed an embedding $j_H : S \to H$ for which the corresponding discrete series representation of $H(\mathbb{R})_\pm$ is generic with respect to some Whittaker datum and denote by subscript H the various transports under j_H, $\dot\gamma_0$ runs over the elements of $S_H(\mathbb{R})_\pm$ that are H-stably conjugate to $\dot\gamma$, and we have used $\tau_H/d_H = \tau'_H/d'_H$.

The right hand side of Theorem 5.2.1(1) then becomes

$$\sum_{\gamma \in H(\mathbb{R})/\mathrm{st}} \Delta[\mathfrak{w}, \mathfrak{e}, z](\dot\gamma, \delta) \sum_{\dot\gamma_0} \frac{\tau_H}{d_H}(\dot\gamma_0).$$

The first sum runs over elements of $H(\mathbb{R})$ that are related to δ, up to stable conjugacy under H. Each such stable conjugacy class consists of regular semi-simple elliptic elements (because δ is such), and hence intersects $S_H(\mathbb{R})$. The second sum runs over elements $\dot\gamma_0 \in S_H(\mathbb{R})_\pm$ that lie in the H-stable class of the lift $\dot\gamma \in S_H(\mathbb{R})_\pm$ of γ. Since Δ is H-stably invariant in the first factor, its values at $\dot\gamma$ and $\dot\gamma_0$ are the same. We can combine the two sums together and obtain

$$\sum_{\gamma_0 \in S_H(\mathbb{R})} \Delta[\mathfrak{w}, \mathfrak{e}, z](\dot\gamma_0, \delta) \frac{\tau_H}{d_H}(\dot\gamma_0),$$

where now γ_0 runs over all elements of $S_H(\mathbb{R})$, equivalently all those that are related to δ, since the transfer factor vanishes for the others.

Having fixed the embeddings $j_\mathfrak{w}$ and j_H, they provide an isomorphism $S_\mathfrak{w} \to S_H$, and this isomorphism induces a bijection

$$\delta_0 \leftrightarrow \gamma_0$$

between the set of elements of $S_\mathfrak{w}(\mathbb{R})$ that are stably conjugate to δ and the set of elements of $S_H(\mathbb{R})$ related to δ. Using the basic property (2.7) of transfer factors we obtain

$$\sum_{\delta_0 \in S_\mathfrak{w}(\mathbb{R})} \Delta[\mathfrak{w}, \mathfrak{e}, z](\dot\gamma_0, \delta_0) \langle s_\mathfrak{w}, \mathrm{inv}(\delta_0, \delta)\rangle \frac{\tau_H}{d_H}(\dot\gamma_0), \tag{5.6}$$

where δ_0 runs over the elements of $S_\mathfrak{w}(\mathbb{R})$ that are stably conjugate to δ, $\gamma_0 \in S_H(\mathbb{R})$ denotes the element corresponding to δ_0 under above bijection, and $\dot\gamma_0 \in S_H(\mathbb{R})_\pm$ is an arbitrary lift of γ_0.

We now unpack the transfer factor. It is given as

$$\Delta(\dot\gamma_0, \delta_0) = \epsilon \Delta_I^{-1}(\dot\gamma_0, \dot\delta_0) \Delta_{III}(\dot\gamma_0, \dot\delta_0),$$

where we recall that the term Δ_{IV} is missing because we are working with normalized characters and orbital integrals, and $\dot\delta_0 \in S_\mathfrak{w}(\mathbb{R})_{G/H}$ is an arbitrary lift of δ_0.

We claim that

$$\Delta_{III}(\dot\gamma_0, \dot\delta_0) = \frac{\tau_\mathfrak{w}}{\tau_H}(\dot\delta_0).$$

This is the property (2.8) that was reviewed in an idelized situation. We say here a few more words about the actual situation in the setting of double covers. We have the commutative diagram

$$\begin{tikzcd}
 & {}^L S_{H,\pm} \arrow[r] \arrow[dd, "a"] & {}^L H_{\pm} \arrow[dd] \\
W_{\mathbb{R}} \arrow[ur, "\varphi_S^H"] \arrow[dr, "\varphi_S"'] \arrow[urr, bend left=50, "\varphi^H"] \arrow[drr, bend right=50, "\varphi"'] & & \\
 & {}^L S_G \arrow[r] & {}^L G,
\end{tikzcd}$$

Here ${}^L S_G$ is the L-group of the double cover $S(\mathbb{R})_G$ and the bottom horizontal map is the canonical L-embedding. We have written ${}^L S_{H,\pm}$ for the L-group of the double cover of $S_H(\mathbb{R})$ obtained as the Baer sum of the canonical double cover $S_H(\mathbb{R})_H$ coming from $R(S_H, H) \subset X^*(S_H)$ and the double cover $S_H(\mathbb{R})_\pm$ that is the preimage of $S_H(\mathbb{R})$ in the double cover $H(\mathbb{R})_\pm \to H(\mathbb{R})$. The top horizontal map is the canonical L-embedding of the L-group of $S_H(\mathbb{R})_H$ into ${}^L H$, which remains an L-embedding after passing to the $\pm$-covers. The right vertical map is the canonical L-embedding ${}^L H_\pm \to {}^L G$. The map a is the unique map making the square, hence also the triangle, commute. It is the paremter of a genuine character of the double cover of $S(\mathbb{R})$ obtained as the Baer sum of $S_H(\mathbb{R})_H$, $S_H(\mathbb{R})_\pm$, and $S(\mathbb{R})_G$, where we have identified S_H with S. This Baer sum is the same as for $S_H(\mathbb{R})_\pm$ and $S(\mathbb{R})_{G/H}$. The claimed identity now follows.

The following result completes the proof of Theorem 5.2.1(1).

Proposition 5.4.1. *For any $\dot\delta_0 \in S_{\mathfrak{w}}(\mathbb{R})_{G/H}$, the following identity holds*

$$\Delta_I(\dot\gamma_0, \dot\delta_0) = \epsilon \cdot (-1)^{q(G_0)-q(H)} \cdot \prod_{\substack{\alpha \in R(S,G/H) \\ \langle \alpha^\vee, d\tau_{\mathfrak{w}} \rangle > 0}} \arg(\alpha^{1/2}(\dot\delta_0) - \alpha^{-1/2}(\dot\delta_0)),$$

where $\dot\gamma_0 \in S_H(\mathbb{R})_{G/H}$ is the image of $\dot\delta_0$ under the isomorphism $j_H \circ j_{\mathfrak{w}}^{-1}$.

Proof. Since $d\tau_{\mathfrak{w}}$ is a regular element of $X_*(S)_{\mathbb{R}}$ the condition $\langle \alpha^\vee, d\tau_{\mathfrak{w}} \rangle > 0$ defines a positive system $R(S,G)^+$ of roots in $R(S,G)$. We will write $\alpha > 0$ when $\alpha \in R(S,G)$ belongs to that positive system. We will write $R(S,G/H)^+ = R(S,G/H) \cap R(S,G)^+$.

We first investigate how both sides vary as functions of $\dot\delta_0$. Consider another strongly regular $\dot\delta_1$. Replacing $\dot\delta_0$ with $\dot\delta_1$ in the right hand side results in multiplication by $\prod_\alpha \arg(b_\alpha)$, where the product runs again over $R(S,G/H)^+$ and $b_\alpha = (\dot\delta_{1,\alpha} - \dot\delta_{1,-\alpha})/(\dot\delta_{0,\alpha} - \dot\delta_{0,-\alpha})$. By construction $\dot\delta_{0,-\alpha} = \sigma(\dot\delta_{0,\alpha})$, and the same holds for $\dot\delta_1$, from which follows $b_\alpha \in \mathbb{R}^\times$, and hence $\arg(b_\alpha) = \mathrm{sgn}(b_\alpha)$.

We now look at the left hand side. Replacing $\dot\delta_0$ by $\dot\delta_1$ multiplies $\mathrm{inv}(\dot\delta_0, \mathrm{pin})$ by $\prod_{\alpha>0,\sigma\alpha<0} \alpha^\vee(b_\alpha)$, and hence Δ_I by the Tate-Nakayama pairing of this 1-cocycle with the endoscopic element $s_{\mathfrak{w}}$. Since the torus S is elliptic, the conditions $\alpha > 0$ and $\sigma\alpha < 0$ are equivalent, and the value of the 1-cocycle at σ

equals $\prod_{\alpha>0}\alpha^\vee(b_\alpha)$. This is the product over $\alpha>0$ of the images of the 1-cocycles $b_\alpha\in Z^1(\Gamma,R^1_{\mathbb{C}/R}\mathbb{G}_m)$ under the homomorphisms $\alpha^\vee:R^1_{\mathbb{C}/R}\mathbb{G}_m\to S$, so the change in $\Delta_{I,\pm}$ is given by $\prod_{\alpha>0}\langle b_\alpha,s_\alpha\rangle$, where s_α is the image of $s\in\widehat{S}$ under $\widehat{\alpha}:\widehat{S}\to\mathbb{C}^\times$. This is a Galois-equivariant homomorphism, with σ acting as inversion on $\mathbb{C}^\times$. Since s is σ-fixed, so is s_α, i.e. $s_\alpha\in\{\pm1\}\subset\mathbb{C}^\times$. By construction of the endoscopic group H, we have $s_\alpha=1$ precisely for $\alpha\in R(S,H)$. On the other hand, when $s_\alpha=-1$, then $\langle b_\alpha,s_\alpha\rangle=\mathrm{sgn}(b_\alpha)$.

We have thus shown that both sides of the identity multiply by the same factor upon replacing $\dot\delta_0$ by a different element $\dot\delta_1$. To establish the identity we may thus evaluate at an arbitrary element $\dot\delta_0$. For this we note that both sides descend to functions on $S_{\mathrm{ad}}(\mathbb{R})_{G/H}$, so we may assume that G is adjoint. Let $H_\mathfrak{w}$ be the element corresponding to $d\tau_\mathfrak{w}$ via some extension κ of the Killing form on $\mathfrak{g}'$ to $\mathfrak{g}$. Then $H_\mathfrak{w}\in i\cdot\mathrm{Lie}(S_\mathfrak{w})(\mathbb{R})$, so $-irH_\mathfrak{w}\in\mathrm{Lie}(S_\mathfrak{w})(\mathbb{R})$ for any $r\in\mathbb{R}$. We will choose $r>0$ small enough and set $\ddot\delta=\exp(-irH_\mathfrak{w})\in S_\mathfrak{w}(\mathbb{R})_{\pm\pm}$, where we are using the exponential map $\mathrm{Lie}(S_\mathfrak{w})(\mathbb{R})\to S_\mathfrak{w}(\mathbb{R})_{\pm\pm}$ discussed in [Kal19a, §3.7].

Considering the right hand side, we have for each $\alpha\in R(S,G)$

$$\alpha^{1/2}(\dot\delta_0)=\dot\delta_{0,\alpha}=e^{d\alpha(-irH_\mathfrak{w})/2}=e^{-ir\langle\alpha^\vee,d\tau_\mathfrak{w}\rangle/2}.$$

Then

$$\alpha^{1/2}(\dot\delta_0)-\alpha^{-1/2}(\dot\delta_0)=-2i\sin(r\langle\alpha^\vee,d\tau_\mathfrak{w}\rangle/2).$$

Choosing $r>0$ so that $0<r\langle d\alpha,d\tau_\mathfrak{w}\rangle/2<\pi$ for all $\alpha>0$ we obtain

$$\arg(\alpha^{1/2}(\dot\delta_0)-\alpha^{-1/2}(\dot\delta_0))=(-i)^{\#R(S,G/H)^+}.$$

Choose a pinning $(T_0,B_0,\{X_\alpha\})$, which, together with $\Lambda(x)=e^{ix}$, produces the chosen Whittaker datum $\mathfrak{w}$. Let $X_-=\sum_{\alpha\in\Delta}X_{-\alpha}$ and let $\bar X=iX_-$. According to Lemma 3.1.2, being generic with respect to the pinning and Λ is equivalent to being generic with respect to $\mathfrak{w}_{\bar X}$, and Proposition 3.2.2 implies that $H_\mathfrak{w}$ is $G(\mathbb{R})$-conjugate to the Kostant section $K(\bar X)$. Therefore, $-irH_\mathfrak{w}$ is $G(\mathbb{R})$-conjugate to the Kostant section $K(rX_-)$, which is the Kostant section of the rescaled pinning $(T_0,B_0,\{rX_\alpha\})$. If we construct $\Delta_I(\dot\gamma_0,\dot\delta_0)$ with respect to this rescaled pinning, then [Kal22, Lemma 4.1.4] show that $\Delta_I(\dot\gamma_0,\dot\delta_0)=1$. On the other hand, the rescaled pinning together with the character $\Lambda_r(x)=e^{r^{-1}ix}$ also produces the Whittaker datum $\mathfrak{w}$. According to Lemma 2.9.4 the right hand side equals 1. □

5.5 The right hand side: classical set-up

In this subsection we will show that the right hand side of the identity of Theorem 5.2.1(2) is also equal to (5.5). The initial arguments of §5.4, which applied to the right hand side of the identity in Theorem 5.2.1(1), have direct analogs in the setting of Therem 5.2.1(2) and show that the right hand side of that identity equals the analog of the expression (5.6), which is given by

$$\sum_{\delta_0\in S_\mathfrak{w}(\mathbb{R})}\Delta[\mathfrak{w},\mathfrak{e},z](\gamma_1,\delta_0)\langle s_\mathfrak{w},\mathrm{inv}(\delta_0,\delta)\rangle\frac{\tau'_{H_1}}{d'_{H_1}}(\gamma_1),\tag{5.7}$$

where we have used the parameter φ_{H_1} of the z-extension H_1 to obtain the character τ'_{H_1} of the torus $S_{H_1}(\mathbb{R})$.

It is the handling of the transfer factor that is slightly different. Indeed, in this setting without covers the transfer factor is given by

$$\Delta = \epsilon\Delta_I^{-1}\Delta_{II}\Delta_{III_2}.$$

The construction of the pieces involves a choice of an admissible isomorphism $S^H \to S_{\mathfrak{w}}$, which we take to be $j_{\mathfrak{w}} \circ j_H^{-1}$, as we did in §5.4. It further involves choices of χ-data and a-data for S. We take ρ-based χ-data, and $(-\rho)$-based a-data, so that $\chi_\alpha(x) = \arg(x)$ when $\alpha > 0$ and $a_\alpha = i$ when $\alpha < 0$, and where $\alpha > 0$ means $\langle \alpha^\vee, d\tau_{\mathfrak{w}}\rangle > 0$.

We claim that

$$\Delta_{III_2}(\gamma_1, \delta_0) = \frac{\tau'_{\mathfrak{w}}(\delta_0)}{\tau'_{H_1}(\gamma_1)}.$$

This was the property (2.8) that was reviewed in an idealized situation. We say here a few more words about the actual situation, still assuming that the z-pair is trivial, i.e. there exists an L-isomorphism ${}^L\eta : {}^LH \to \mathcal{H}$, the general case being entirely analogous by requiring more cumbersome notation. We then have the commutative diagram

$$\begin{tikzcd}
 & {}^LS_H \arrow[r, "\rho^H"] \arrow[dd, "a"'] & {}^LH \arrow[dd, "{}^L\eta"] \\
W_{\mathbb{R}} \arrow[ur, "\varphi_S^H"] \arrow[dr, "\varphi_S"'] \arrow[urr, bend left=60, "\varphi^H"] \arrow[drr, bend right=60, "\varphi"'] & & \\
 & {}^LS \arrow[r, "\rho"] & {}^LG,
\end{tikzcd}$$

where the horizontal arrows are the L-embeddings obtained via ρ-based and ρ^H-based χ-data, respectively. We have $\Delta_{III_2}(\gamma_1, \delta_0) = \langle a, \delta_0\rangle$, where $a \in Z^1(W_{\mathbb{R}}, \widehat{S})$ is the 1-cocycle that makes the above diagram commute, the pairing is the Langlands pairing, and $\delta_0 \in S(\mathbb{R})$ is the image of $\gamma_0 \in S^H(\mathbb{R})$ to $S(\mathbb{R})$ under the chosen fixed admissible isomorphism. The claim now follows from the above commutative diagram and the fact that τ and τ^H are the characters with parameters φ_S and φ_S^H.

Next we consider

$$\frac{\Delta_{II}(\gamma_0, \delta)}{d'_H(\gamma_0)}.$$

By definition,

$$\Delta_{II}(\gamma_0, \delta) = \frac{\Delta_{II}^G(\gamma_0, \delta)}{\Delta_{II}^H(\gamma_0, \delta)}.$$

With the chosen a-data and χ-data we have

$$\Delta_{II}^H(\gamma_0) = \prod_{\substack{\alpha \in R(S^H, H) \\ \langle \alpha, \rho^H \rangle > 0}} \arg\Big(\frac{\alpha(\gamma_0) - 1}{-i}\Big) = i^{\#R(S^H,H)/2} \cdot \prod_{\substack{\alpha \in R(S^H, H) \\ \langle \alpha, \rho^H \rangle > 0}} \arg(\alpha(\gamma_0) - 1).$$

On the other hand,

$$(\alpha(\gamma_0) - 1)(1 - \alpha(\gamma_0)^{-1}) = \alpha(\gamma_0) + \alpha(\gamma_0)^{-1} - 2 = 2(\mathrm{Re}(\alpha(\gamma_0) - 1) < 0.$$

Hence

$$\Delta_{II}^H(\gamma_0,\delta)d'_H(\gamma_0) = (-i)^{\#R(S^H,H)/2}.$$

In the same way one shows

$$\Delta_{II}^G(\gamma_0,\delta)d'_{G_0}(\delta_0)) = (-i)^{\#R(S,G_0)/2}.$$

and we conclude

$$\frac{\Delta_{II}(\gamma_0,\delta)}{d'_H(\gamma_0)} = \frac{i^{\#R(S^H,H)/2-\#R(S,G_0)/2}}{d'_{G_0}(\delta_0)}.$$

With this (5.7) becomes

$$(-1)^{q(H)}i^{\#R(S^H,H)/2-\#R(S,G_0)/2}\epsilon \sum_{\delta_0\in S_{\mathfrak{w}}(\mathbb{R})} \Delta_I(\gamma_0,\delta)^{-1}\langle s_{\mathfrak{w}}, \mathrm{inv}(\delta_0,\delta)\rangle \cdot \frac{\tau_{\mathfrak{w}}(\delta_0)}{d'_{G_0}(\delta_0)}.$$

We have

$$\Delta_I(\gamma_0,\delta)^{-1} = \langle s,\lambda\rangle^{-1}$$

where $\lambda \in H^1(\Gamma,S)$ is the splitting invariant of S relative to the chosen a-data. Using Lemma 2.9.4 we obtain

$$(-1)^{q(G_0)}\langle s,\lambda\rangle^{-1}\sum_{\delta_0}\langle s, \mathrm{inv}(\delta_0,\delta)\rangle \cdot \frac{\tau_{\mathfrak{w}}(\delta_0)}{d'_{G_0}(\delta_0)}.$$

The following lemma completes the proof.

Lemma 5.5.1. *The splitting invariant λ of S computed in terms of $(-\rho)$-based a-data is trivial.*

Proof. Let $(S_{\mathfrak{w}},\tau_{\mathfrak{w}})$ be the Harish-Chandra parameter of the generic member of $\Pi_\varphi((G_0,1,1))$. Let $H_{\mathfrak{w}} \in \mathrm{Lie}(S_{\mathfrak{w}})$ be the element determined by $\kappa(H_{\mathfrak{w}},-) = \langle d\tau_{\mathfrak{w}},-\rangle$ with respect to any extension κ to $\mathfrak{g}$ of the Killing form on $\mathfrak{g}'$. We transport $H_{\mathfrak{w}}$ to $\mathrm{Lie}(S)$ via the admissible isomorphism $j_{\mathfrak{w}}$. The element $-iH_{\mathfrak{w}} \in \mathrm{Lie}(S)$ is then Galois-fixed and thus lies in $\mathrm{Lie}(S)(\mathbb{R})$. For every $\alpha > 0$ the complex number $d\alpha(-iH_{\mathfrak{w}})$ is a positive real multiple of $-i$. Therefore we can replace the $(-\rho)$-based a-data with the a-data $d\alpha(-iH_{\mathfrak{w}})$ without changing the splitting invariant. According to [Kot99, Theorem 5.1] and Proposition 3.2.2, λ is trivial. $\square$

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