A RESULT OF HARISH CHANDRA

Let G be a connected real reductive group. We let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. Let θ be a Cartan involution of G.

Let G_r be the open set of regular semisimple elements if G. Suppose H is a θ-stable Cartan subgroup of G, and set $\mathfrak{h} = \mathrm{Lie}(H) \otimes \mathbb{C}$ and $H_r = H \cap G_r$. Define

$$\Delta(h) = (\det(\mathrm{Ad}(h)^{-1} - 1)|_{\mathfrak{g}/\mathfrak{h}} \quad (a, h \in H_r).$$

For later use we also define

$$\Delta_a(h) = \Delta(ah) = (\det(\mathrm{Ad}(ah)^{-1} - 1)|_{\mathfrak{g}/\mathfrak{h}} \quad (h \in H_r).$$

Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra. An element $X \in \mathfrak{U}(\mathfrak{g})$ defines a left-invariant differential operator $\partial(X)$ on G. Let $L_g(f)(h) = f(gh)$ $(g, h \in G, f \in C^\infty(G))$. A differential operator D being left-invariant means

$$D(L_g f)(h) = D(f)(gh). \tag{0.1}$$

Let $\mathfrak{Z}$ be the center of $\mathfrak{U}(\mathfrak{g})$. We identify $\mathfrak{U}(\mathfrak{h})$ with $S(\mathfrak{h})$. Let W be the Weyl group of H in G, and let $\gamma : \mathfrak{Z} \to S(\mathfrak{h})^W$ be the Harish Chandra isomorphism. Let G_r be the open set of regular semisimple elements of G, and let C^∞ be the smooth function on G_r.

Proposition 0.0.1. *Suppose $f \in C^\infty(G_r)$ is a class function. Then*

$$(\partial(z)f)(h) = |\Delta(h)|^{-\frac{1}{2}} \gamma(z)(|\Delta|^{\frac{1}{2}} f)(h) \quad (h \in H_r).$$

Before giving the proof we establish some notation and preliminary results.

Suppose D is a differential operator on G. Then D has a local expression $D_g \in \mathfrak{U}(\mathfrak{g})$ for any $g \in G$. See [HC56, Section 4]. By definition this means

$$\partial(D_g)(f)(g) = D(f)(g) \quad (g \in G). \tag{0.2}$$

Suppose $a \in G$ is a regular semisimple element, and set $H = \mathrm{Cent}_G(a)$. Suppose U is an open set in G containing a, and set

$$U_H = a^{-1} U \cap H_r$$

This is an open neighborhood of 1 in H_r. Suppose D is a differential operator on U. Define $\delta_a(D)$, a differential operator on U_H by [HC65, Section 4]. Note that if $y \in U_H$ then $\delta_a(D)_y \in S(\mathfrak{h})$.

See [HC64, Section 8] for the definition of locally invariant functions. The restriction of a class function on G_r to any open set is locally invariant.

Lemma 0.0.2. *Suppose a, H, U and U_H are as above. Suppose D is a differential operator on U, and f is a locally invariant function on U. Then for $y \in U_H$:*

$$D(f)(ay) = \partial(\delta_a(D)_y)(f)(ay)$$

This is [HC65, Lemma 18]. Some care is required here. Note that $\delta_a(D)$ is a differential operator on U_H (*not* on U), so $\delta_a(D)(ay)$ is not defined. However $\delta_a(D)_y \in S(\mathfrak{h})$, so $\partial(\delta_a(D)_y)$ is a well defined differential operator at ay.

The statement of [HC65, Lemma 18] says

$$D(f)(ay) = \delta_a(D)(f)(ay)$$

However the right hand side isn't defined: $\delta_a(D)$ is a differential operator on U_H, but ay is in U, but not U_H. What the proof actually shows (see the last line of the proof) is that

$$\partial(D_{ay})(f)(ay) = \partial(\delta_a(D)_y)(f)(ay).$$

By (0.2) the LHS equals $D(f)(ay)$, so this gives the statement of the Lemma.

Lemma 0.0.3.

$$\delta_a(z)(f)(h) = |\Delta_a(h)|^{-\frac{1}{2}}\gamma(z)(|\Delta_a|^{\frac{1}{2}}f)(h)$$

This is [HC65, Lemma 13].

Proof of the Proposition. Let U be an open set containing h and let $U_H = h^{-1}U \cap H_r$ as before. Write $h = ay$ where $a \in H_r$ and $y \in U_H$. Suppose $z \in \mathfrak{Z}(\mathfrak{g})$. By Lemma 0.0.2 we have

$$\partial(z)(f)(ay) = \partial(\delta_a(z)_y)(f)(ay)$$

Here we've written $\delta_a(z)$ in place of $\delta_a(\partial(z))$. We have to show the right hand side is equal to

$$|\Delta(ay)|^{-\frac{1}{2}}\gamma(z)(|\Delta|^{\frac{1}{2}}f))(ay).$$

By (0.2) the right hand side is equal to

$$\begin{aligned}\partial(\delta_a(z)_y)(L_a(f))(y) &= \delta_a(z)(L_a(f))(y)\\ &= |\Delta_a(y)|^{-\frac{1}{2}}\gamma(z)(|\Delta_a|^{\frac{1}{2}}L_a(f))(y) \quad \text{(by Lemma 0.0.3)}\end{aligned}$$

Consider the function

$$y \mapsto (|\Delta_a|^{\frac{1}{2}}L_a(f))(y) = |\Delta(ay)|^{\frac{1}{2}}L_a(f)(y) = L_a(|\Delta|^{\frac{1}{2}}f)(y)$$

Using this we see

$$|\Delta_a(y)|^{-\frac{1}{2}}\gamma(z)(|\Delta_a|^{\frac{1}{2}}L_a(f))(y) = |\Delta(ay)|^{-\frac{1}{2}}\gamma(z)(L_a(|\Delta|^{\frac{1}{2}}f))(y)$$

Now $\gamma(z)$ is invariant by left-translation, so by (0.1) this equals

$$|\Delta(ay)|^{-\frac{1}{2}}\gamma(z)(|\Delta|^{\frac{1}{2}}f))(ay)$$

This proves the Proposition. □

References

[HC56] Harish-Chandra, *The characters of semisimple Lie groups*, Trans. Amer. Math. Soc. **83** (1956), 98–163. MR 80875

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[HC65] ______, *Invariant eigendistributions on a semisimple Lie group*, Trans. Amer. Math. Soc. **119** (1965), 457–508. MR 180631