A RESULT FROM HARISH CHANDRA

We assume G is a real, connected, semisimple group. Surely these assumptions can be weakened, but this is the setting of some of the results we want to quote from Harish Chandra. We let $\mathfrak{g} = \mathrm{Lie}(G)$ and $G_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$.

Let G_r be the open set of regular semisimple elements if G (I don't think we need strongly regular here). Let H be a Cartan subgroup of G, and set $H_r = H \cap G_r$. Define

$$\Delta(h) = (\det(\mathrm{Ad}(h)^{-1} - 1)|_{\mathfrak{g}/\mathfrak{h}} \quad (h \in H_r).$$

(I'm pretty sure this is $\pm$ the function denoted D_T in section 2.5).

Let $\mathfrak{Z}$ be the center of the universal enveloping algebra, and let $\gamma : \mathfrak{Z} \to S(\mathfrak{h})^W$ be the Harish Chandra isomorphism.

For $Z \in \mathfrak{Z}$ write $\partial(Z)$ for the corresponding differential operator on $\mathfrak{g}$. Similarly for $Y \in S(\mathfrak{h})^W$ we have the differential operators $\delta(Y)$ on $\mathfrak{h}$.

If f is a class function on G_r write $\overline{f}$ for its restriction to H_r.

Lemma 0.0.1. *Suppose $f : G_r \to \mathbb{C}$ is a smooth class function. Then*

$$\partial(z)(f)(h) = |\Delta(h)|^{-\frac{1}{2}}\partial(\gamma(z))(|\Delta|^{\frac{1}{2}}\overline{f})(h) \quad (h \in H_r).$$

Here is a super short sketch of the proof. In [HC65, Section 14] Harish Chandra defines a bijection $\phi \to f_\phi$ between functions ϕ on an open set in $\mathfrak{g}$, and functions on $U_G = \exp(U) \in G$, where U is a sufficiently small open set. We can choose this set so that $\exp : U \to U_G$ is an analytic diffeomorphism, and U_G contains a given element $h \in H_r$.

The third-to-last formula on page 476 of [HC65] states, for any invariant function ϕ on U:

$$\partial(z)(f_\phi)(h) = |\Delta(h)|^{-\frac{1}{2}}\partial(\gamma(z))(|\Delta|^{\frac{1}{2}}\overline{f_\phi})(h) \quad (h \in H_r).$$

The result follows since $\exp|_U$ is an analytic diffeomorphism and $\phi \to f_\phi$ is a bijection.

There are a few, but not very many, details to check, and I'll write this out.

It *does not* appear to me that you get rid of the $|\Delta|^{\frac{1}{2}}$ terms, so perhaps we need to look at the application and see if they are actually there somewhere. If not... then maybe I need to see if this does indeed simplify.

REFERENCES

[HC65] Harish-Chandra, *Invariant eigendistributions on a semisimple Lie algebra*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 27, 5–54. MR 180630