A Result of Harish Chandra

Let G be a connected real reductive group. We let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. Let G_r be the open set of regular semisimple elements of G. Suppose H is a Cartan subgroup of G, and set $\mathfrak{h} = \mathrm{Lie}(H) \otimes \mathbb{C}$ and $H_r = H \cap G_r$. Define

$$\Delta(h) = (\det(\mathrm{Ad}(h)^{-1} - 1)|_{\mathfrak{g}/\mathfrak{h}} \quad (a, h \in H_r).$$

For later use we also define

$$\Delta_a(h) = \Delta(ah) = (\det(\mathrm{Ad}(ah)^{-1} - 1)|_{\mathfrak{g}/\mathfrak{h}} \quad (h \in H_r).$$

Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra. An element $X \in \mathfrak{U}(\mathfrak{g})$ defines a left-invariant differential operator $\tau(X)$. For $x \in \mathfrak{g}$ the definition is

$$\tau(X)(f)(g) = \frac{d}{dt} g \exp(tX)|_{t=0}.$$

Let $L_g(f)(h) = f(gh)$ $(g, h \in G, f \in C^\infty(G))$. A differential operator D being left-invariant means

$$D(L_g f)(h) = D(f)(gh). \tag{0.1}$$

It is customary to drop the τ notation, and write Xf instead of $\tau(X)f$. I prefer to keep it for reasons that should become clear. Let $\mathfrak{Z}$ be the center of $\mathfrak{U}(\mathfrak{g})$. Let W be the Weyl group of H in G, and let $\gamma : \mathfrak{Z} \to \mathfrak{U}(\mathfrak{h})^W$ be the Harish Chandra isomorphism. Let G_r be the open set of regular semisimple elements of G, and let C^∞ be the smooth function on G_r.

Proposition 0.0.1. *Suppose $f \in C^\infty(G_r)$ is a class function. Then*

$$(\tau(z)f)(h) = |\Delta(h)|^{-\frac{1}{2}} \tau(\gamma(z))(|\Delta|^{\frac{1}{2}} f)(h) \quad (h \in H_r).$$

A less pedantic way of writing this is to drop τ and identify $\mathfrak{U}(\mathfrak{g})$ with the left-invariant differential operators on G. Then the conclusion is

$$(zf) = |\Delta(h)|^{-\frac{1}{2}} \gamma(z)(|\Delta|^{\frac{1}{2}} f)$$

as functions on H_r. Before giving the proof we establish some notation and preliminary results.

Suppose D is a differential operator on G. Then D has a local expression $D_g \in \mathfrak{U}(\mathfrak{g})$ for any $g \in G$. See [HC56, Section 4]. By definition this means

$$D(f)(g) = \tau(D_g)(f)(g) \quad (g \in G) \tag{0.2}$$

Note that D is left-invariant if and only if

$$D_g = D_h \quad \text{(for all } g, h \in G) \tag{0.3}$$

To see this, note that for $h, g \in G$, left-invariance of D says:

$$D(f)(hg) = D(L_h f)(g).$$

The left hand side is $\tau(D_{hg})(f)(hg)$. The right hand side is

$$\tau(D_g)(L_h(f))(g) = \tau(D_g)(f)(hg)$$

since $\tau(D_g)$ is (obviously) left-invariant. Therefore $\tau(D_{hg}) = \tau(D_g)$, and therefore, $D_{hg} = D_g$ for all g, h.

Suppose $a \in G$ is a regular semisimple element, and set $H = \mathrm{Cent}_G(a)$. Suppose U is an open set in G containing a, and set

$$U_H = a^{-1}U \cap H_r. \tag{0.4}$$

This is an open neighborhood of 1 in H_r. Suppose D is a differential operator on U. Define $\delta_a(D)$, a differential operator on U_H by [HC65, Section 4]. Note that if $y \in U_H$ then $\delta_a(D)_y \in \mathfrak{U}(\mathfrak{h})$.

See [HC64, Section 8] for the definition of locally invariant functions. The restriction of a class function on G_r to any open set is locally invariant.

Lemma 0.0.2. *Suppose a, H, U and U_H are as above. Suppose D is a differential operator on U, and f is a locally invariant function on U. Then for $y \in U_H$:*

$$D(f)(ay) = \tau(\delta_a(D)_y)(f)(ay)$$

This is [HC65, Lemma 18]. Some care is required here.

The statement of [HC65, Lemma 18] says

$$D(f)(ay) = \delta_a(D)(f)(ay)$$

However the right hand side isn't defined: $\delta_a(D)$ is a differential operator on U_H, but ay is in U, but not U_H. What the proof actually shows (see the last line of the proof) is that

$$\tau(D_{ay})(f)(ay) = \tau(\delta_a(D)_y)(f)(ay).$$

Note that $\tau(D_{ay})$ is a well defined left-invariant differential operator on U_H. By (0.2) the LHS equals $D(f)(ay)$, so this gives the statement of the Lemma.

Corollary 0.0.3. *Suppose f is a class function on G_r and D is a left-invariant differential operator on G_r. Suppose $a \in G_r$ is semisimple and set $H = \mathrm{Cent}_G(a)$. Then $U_H = H_r$ (see* (0.4)*), so $\delta_a(D)$ is a differential operator on H_r.*

Assume $\delta_a(D)$ is left-invariant on H_r. Then

$$D(f)(h) = \delta_a(D)(f)(h) \quad (h \in H_r)$$

Remark 0.0.4. I think D is automatically left-invariant on H_r, but I'm not sure. In any event we don't use the Corollary.

Proof. From the Lemma we have

$$D(f)(ay) = \tau(\delta_a(D)_y)(f)(ay)$$

for all $y \in U_H = H_r$. Since $\delta_a(D)$ is left-invariant so by (0.3)

$$\tau(\delta_a(D)_y)(f)(ay) = \tau(\delta_a(D)_{ay})(f)(ay) = \delta_a(D)(ay)$$

□

Lemma 0.0.5.

$$\delta_a(\tau(z))(f)(h) = |\Delta_a(h)|^{-\frac{1}{2}}\tau(\gamma(z))(|\Delta_a|^{\frac{1}{2}}f)(h)$$

This is [HC65, Lemma 13].

Proof of the Proposition. Let U be an open set containing h and let $U_H = h^{-1}U \cap H_r$ as before. Write $h = ay$ where $a \in H_r$ and $y \in U_H$. Suppose $z \in \mathfrak{Z}(\mathfrak{g})$. Then:

$$\begin{aligned}
\tau(z)(f)(ay) &= \tau(\delta_a(\tau(z))_y)(f)(ay) \quad \text{(by Lemma 0.0.5)} \\
&= \tau(\delta_a(\tau(z))_y)(L_a(f))(y) \\
&= \delta_a(\tau(z))(L_a(f))(y) \quad \text{(definition of } L_a\text{)} \\
&= |\Delta_a(y)|^{-\frac{1}{2}}\tau(\gamma(z))(|\Delta_a|^{\frac{1}{2}}L_a(f))(y) \quad \text{(by Lemma 0.0.5)} \\
&= |\Delta_a(y)|^{-\frac{1}{2}}\tau(\gamma(z))(L_a(|\Delta|^{\frac{1}{2}}f))(y) \quad \text{(elementary)} \\
&= |\Delta_a(y)|^{-\frac{1}{2}}\tau(\gamma(z))(|\Delta|^{\frac{1}{2}}f))(ay) \quad \text{(by left-invariance of } \tau(\gamma(z))\text{)}
\end{aligned}$$

The elementary step is the equality of the functions $|\Delta_a|^{\frac{1}{2}}L_a(f)$ and $L_a(|\Delta|^{\frac{1}{2}}f)$.

This proves the Proposition. □

References

[HC56] Harish-Chandra, *The characters of semisimple Lie groups*, Trans. Amer. Math. Soc. **83** (1956), 98–163. MR 80875

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[HC65] ———, *Invariant eigendistributions on a semisimple Lie group*, Trans. Amer. Math. Soc. **119** (1965), 457–508. MR 180631