# CSC473 Assignment 4

Jeff Blair: 1002177057
jeffrey.blair@mail.utoronto.ca

Jaryd Hunter: 1002725893
jaryd.hunter@mail.utoronto.ca

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## Question 1

a. To show that the two programs are a primal and dual pair it is easiest to transform them to the standard form and then derive one from the other. Let, $b^T = c^T = \begin{bmatrix}1 & ... & 1\end{bmatrix}$ with length $|E|$, $M$ be a matrix of dimension $|V| \times |E|$, where each entry $m_{i,j} = \begin{cases} 1 \text{ if edge } j \text{ connects to vertex } i \\ 0 \text{ o.w.} \end{cases}$. Then, we can rewrite the primal in the following way:

$$\begin{aligned} &\max\ c^T x \\ Mx &\leq b \\ x_i &\geq 0 \end{aligned}$$

As given in the notes the dual will then be:

$$\begin{aligned} &\max\ b^T y \\ M^T y &\geq c \\ y_i &\geq 0 \end{aligned}$$

Since $b$ is a vector of 1's, max $b^T y$ is equivalent to $\sum_{u \in V} y_u$. Also since each edge must have one endpoint in the set $A$ and the other in the set $B$ we know there will only be two entries with 1 and the rest will be 0 for each row of $M$. And these two entries will correspond to the two endpoints of each edge, which is the same as the line $y_a + y_b \geq 1$ in the handout.

Thus, the two LP's are a Primal, Dual pair.

The complementary slackness of the two programs is as follows:

$$\begin{aligned} &\forall v \in \{1, ..., |V|\} : \begin{cases} (1 - \sum_{b:(v,b) \in E} x_{v,b}) y_v = 0, \text{ if } v \in |A| \\ (1 - \sum_{a:(a,v) \in E} x_{a,v}) y_v = 0, \text{ o.w.} \end{cases} \\ &\forall (a,b) \in E : (y_a + y_b - 1) x_{a,b} = 0 \end{aligned}$$

b. Let $M_I$ be the optimal value for the maximal matching, $V_I$ be the optimal value for the Minimal Vertex Cover, $M_L$ be the optimal value for the left LP, and $V_L$ be the optimal value for the right LP.

All possible matchings are in the domain of the left LP, so the left LP is a relaxation of the maximal matching problem. Similarly, all possible vertex covers are in the domain of the right LP, so the right LP is a relaxation of the minimal vertex cover problem.

Since, the left LP is a relaxation of maximal matching, then $M_I \leq M_L$. Similarly, the right LP is a relaxation of Minimal Vertex Cover, thus $V_I \geq V_L$.

Also note, that since the two LPs are a primal dual pair, the optimal value for each must be equal. So we have, $M_I \leq M_L = V_L \leq V_I$

Using Konig's theorem, we know that $M_I = V_I$ for any arbitrary bipartite graph, therefore $M_L = V_L$, and there exists an integer solution for both of these LPs namely the solutions to the integer problems.

c. Any vertex cover must choose a vertex adjacent to each edge, if the largest degree of any node is $\Delta$ then the addition of a single vertex can cover at most $\Delta$ edges, then a vertex cover must choose at least $\frac{|E|}{\Delta}$ vertices to cover every edge. As shown in the previous subquestions the vertex cover is the dual of a matching, so the optimal matching is equal to the smallest feasible vertex cover. Then the optimal matching must be greater than or equal to $\frac{|E|}{\Delta}$ since that is the smallest possible vertex cover given that the largest degree is $\Delta$.

# Question 2

a. Given a solution $(S, \bar{S})$ to the maximum directed cut problem, $\forall (u, v) \in (S, \bar{S})$ set $x_u = 1$ and $x_v = 0$. Additionally, $\forall (u, v) \in E \setminus (S, \bar{S})$ set $x_u = 0$ and $x_v = 1$. Fixing x, the linear program

$$\begin{aligned}
&\max \sum_{e \in E} w_e z_e \\
&s.t \\
&z_{uv} \leq x_u \\
&z_{uv} \leq 1 - x_v \\
&x_u \in \{0, 1\} \\
&0 \leq z_e \leq 1
\end{aligned}$$

is maximized when $z_{uv} = 1$ for all $(u, v) \in (S, \bar{S})$, $z_{uv} = 0$ otherwise, and $w(S, \bar{S}) = \sum_{e \in E} w_e z_e$. This is true since all $w_e$ are positive and each $z_e$ is forced to be zero for all edges not in the max cut.

Therefore the solution $(x, z)$ is feasible.

Conversely, given a feasible solution $(x, z)$ to the LP we can construct the following cut of G:

For each $(u, v) \in E$: If $x_u = 1, x_v = 0$ and $z_{uv} > 0$: add $u$ to $S$ and $v$ to $\bar{S}$.
For all other $v \in V \setminus (S, \bar{S})$ assign $v$ to $S$.

Note if any $v$ is added to $S$ then $x_v = 1$, and $v$ cannot be added to $\bar{S}$ by the constraint $z_{uv} \leq 1 - x_v$. Therefore $S$ and $\bar{S}$ are disjoint and the cut is valid.

Since all $w_e \geq 0$ and $0 \leq z_e \leq 1$:
$\implies w(S, \bar{S}) \geq \sum_{e \in E} w_e z_e$

b.

$$z_{uv} \leq x_u \tag{1}$$
$$z_{uv} \leq 1 - x_v \tag{2}$$

$$\begin{aligned}
P(u \in S, v \in \bar{S}) &\geq P(u \in S)P(v \in \bar{S}) \\
&\geq \left(\frac{1}{4} + \frac{x_u}{2}\right)\left(\frac{3}{4} - \frac{x_v}{2}\right) \\
&\geq \left(\frac{1}{4} + \frac{z_{uv}}{2}\right)\left(\frac{3}{4} - \frac{1}{2} + \frac{1}{2} - \frac{x_v}{2}\right) \text{ by (1)} \\
&\geq \left(\frac{1}{4} + \frac{z_{uv}}{2}\right)\left(\frac{1}{4} - \frac{1 - x_v}{2}\right) \\
&\geq \left(\frac{1}{4} + \frac{z_{uv}}{2}\right)\left(\frac{1}{4} - \frac{z_{uv}}{2}\right) \text{ by (2)} \\
&\geq \left(\frac{\frac{1}{2} + z_{uv}}{2}\right)^2 \\
&\geq \frac{1}{2} z_{uv}
\end{aligned}$$

$$\begin{aligned}
E[w(S, \bar{S})] &= \sum_{e \in E} w_e \cdot P(e \in (S, \bar{S})) \\
&= \sum_{(u,v) \in E} w_{uv} \cdot P(u \in S, v \in \bar{S}) \\
&\geq \sum_{(u,v) \in E} w_{uv} \cdot \frac{1}{2} z_{uv} \\
&\geq \frac{1}{2} \sum_{e \in E} w_e z_e
\end{aligned}$$