CSC473 Assignment 3

Jeff Blair: 1002177057
jeffrey.blair@mail.utoronto.ca

Jaryd Hunter: 1002725893
jaryd.hunter@mail.utoronto.ca

March 2020

Question 1

a. Let P be the transition matrix for this Markov Chain, then since the transition will choose one vertex u.a.r. and then one colour u.a.r. then $P_{i,j} = \begin{cases} \frac{1}{nk} & \text{if it's possible to transition from } i \text{ to } j \\ \frac{c}{nk} & \text{if } i = j \\ 0 & \text{if it's not possible to transition from } i \text{ to } j \end{cases}$

Since it's possible to transition from one state to another only if they differ in colour on a single vertex, then the probability of transitioning from state i to j is the same as transitioning from j to i since it will be the same probability in both cases to choose the same vertex and to choose the colour of the other state. Thus, P is symmetric.

Then we can solve the time reversible condition $\forall x, y \in \{1, ..., |\Omega|\} : \pi_x P_{x,y} = \pi_y P_{y,x}$, since P is symmetric, $\pi_x = \pi_y$, thus π must be the uniform distribution.

b. Proof for $P(Z_t = z - 1 | Z_{t-1} = z) \geq \frac{z(k-6)}{nk}$:

For $Z_t = z - 1$, $U \in V$ must be a node which satisfies $C_t(U) \neq D_t(U)$. There are $\frac{z}{n}$ such nodes.

Also, I must be chosen to be a colour which is not any of the colours assigned to nodes adjacent to U in both C and D. Since the maximum degree for each node is 3 then there are at most 6 colours which I cannot be chosen as.

Thus $P(Z_t = z - 1 | Z_{t-1} = z) \geq \frac{z}{n} \cdot \frac{(k-6)}{k} = \frac{z(k-6)}{nk}$

Proof for $P(Z_t = z + 1 | Z_{t-1} = z) \leq \frac{6z}{nk}$:

For Z_t to increase the following conditions must be satisfied $C(U) = D(U)$ and $\exists v \in v : (U, v) \in E : D_t(v) \neq C_t(v)$ and $I = C_t(v) \vee I = D_t(v)$.

For any pair of vertices U and v there are 2 colours which could be chosen which would cause the differences to increase $I = C_t(v) \vee I = D_t(v)$.

Since the degree of every node is at most 3, $3z \geq |\{(U, v) \in E : C_t(U) = D_t(U) \wedge C_t(v) \neq D_t(v)\}|$.

Let the event A_i, be the event that $Z_t = z + 1$ because of a conflict on edge $i \in \{(U, v) \in E : C_t(U) = D_t(U) \wedge C_t(v) \neq D_t(v)\}$.

Then, $P(Z_t = z + 1 | Z_{t-1} = z) = P(\cup_i A_i) \leq \sum_i^{3z} \frac{1}{n} \cdot \frac{2}{k} = \sum_i^{3z} \frac{2}{nk} = \frac{6z}{nk}$

c. Let $V_{i,t}$ be and indicator variable if vertex $i \in G$ is the same in both the colouring C_t and D_t. Then

$$
\begin{aligned}
\mathbb{E}[V_{i,0}] &= (1 - \frac{1}{13}) \\
\mathbb{E}[V_{i,1}] &\leq (1 - \frac{1}{13})(1 - \frac{7}{13n}) && \text{By } \star \\
&\leq 1 - \frac{1}{13n} \\
\mathbb{E}[V_{i,t}] &\leq (1 - \frac{1}{13n})^t \\
P(Z_t) &\leq P(U_i V_{i,t}) \leq \sum_{i=1}^{n} P(V_{i,t}) \leq \sum_{i=1}^{n} (1 - \frac{1}{13n}) && \text{union bound} \\
\mathbb{E}[Z_t] &\leq n(1 - \frac{1}{13n})^t \leq ne^{\frac{-t}{13n}} && \text{linearity of expectation}
\end{aligned}
$$

$\star$: Since degree is at most 3 per vertex, then there are at most 6 colours when chosen that don't form a valid colouring for both C and D so there is a $\frac{7}{13}$ chance of choosing a colour which is valid for both.

Let $t \geq 13n \ln(\frac{n}{\epsilon})$

$$
\begin{aligned}
P(Z_t \geq 1) &\leq ne^{\frac{-13n \ln(\frac{n}{\epsilon})}{13n}} && \text{Using Markov's inequality} \\
&= \epsilon
\end{aligned}
$$

Since, D_t is initialized uniformly at random then we know that the probability that C_t differs from D_t bounds the $d_{tv}(C_t, D_t)$. Thus, $d_{tv}(C_t, D_t) \leq P(Z_t \leq 1)$, as shown above if we let $t \geq 13n \ln(\frac{n}{\epsilon})$ we have our bound on $d_{tv}(C_t, D_t)$.

Question 2

a. **Solution:**

$$\text{Define: } \hat{x} \in \mathcal{R}^{2n} \mid \forall \hat{x}_i : \hat{x}_i \geq 0 \text{ and } x_i = \hat{x}_{2i} - \hat{x}_{2i+1} \tag{1}$$
$$\text{Define: } \hat{A} \in \mathcal{R}^{m,2n} \mid \hat{a}_{m,2_i} = -\hat{a}_{m,2_i+1} = a_{m,i} \tag{2}$$
$$\text{Define: } \hat{c} \in \mathcal{R}^{2n} \mid \hat{c}_{2i} = -\hat{c}_{2i+1} = c_i \tag{3}$$

$$\begin{aligned}
\hat{c}^T\hat{x} &= \hat{c}_0\hat{x}_0 + \hat{c}_1\hat{x}_1 + ... + \hat{c}_{2n-1}\hat{x}_{2n-1} \\
&= \hat{c}_0(\hat{x}_0 - \hat{x}_1) + \hat{c}_2(\hat{x}_2 - \hat{x}_3) + ... + \hat{c}_{2n-2}(\hat{x}_{2n-2} - \hat{x}_{2n-1}) \qquad \text{by (3)} \\
&= c_0(x_0) + c_1(x_1) + ... + c_{n-1}(x_{n-1}) \qquad \text{by (1)} \\
&= c^Tx
\end{aligned}$$

$$\hat{A}\hat{x} = \begin{bmatrix} \hat{a}_{0,0}\hat{x}_0 & + & ... & + & \hat{a}_{0,2n-1}\hat{x}_{2n-1} \\ ... & & ... & & ... \\ \hat{a}_{m-1,0}\hat{x}_0 & + & ... & + & \hat{a}_{m-1,2n-1}\hat{x} \end{bmatrix} = \begin{bmatrix} \hat{a}_{0,0}(\hat{x}_0 - \hat{x}_1) & + & ... & + & \hat{a}_{0,2n-2}(\hat{x}_{2n-2} - \hat{x}_{2n-1}) \\ ... & & ... & & ... \\ \hat{a}_{m-1,0}(\hat{x}_0 - \hat{x}_1) & + & ... & + & \hat{a}_{m-1,2n-2}(\hat{x}_{2n-2} - \hat{x}_{2n-1}) \end{bmatrix} \text{ by (2)}$$

$$= \begin{bmatrix} a_{0,0}(x_0) & + & ... & + & a_{0,n-1}(x_{n-1}) \\ ... & & ... & & ... \\ a_{m-1,0}(x_0) & + & ... & + & a_{m-1,n-1}(x_{n-1}) \end{bmatrix} = Ax \qquad \text{by (1)}$$

Then solving the linear program

$$\begin{gathered} \max_x c^Tx \\ \text{s.t} \\ Ax \leq b \\ x \in \mathcal{R}^n \end{gathered}$$

is equivalent to solving the linear program

$$\begin{gathered} \max_{\hat{x}} \hat{c}^T\hat{x} \\ \text{s.t} \\ \hat{A}\hat{x} \leq b \\ \hat{x} \in \mathcal{R}^{2n} \\ \hat{x} \geq 0 \end{gathered}$$

And its dual is the linear program

$$\begin{gathered} \min_y b^Ty \\ \text{s.t} \\ \hat{A}^Ty \geq \hat{c} \\ y \in \mathcal{R}^m \\ y \geq 0 \end{gathered}$$

b. **Solution:**

$$\begin{aligned} c = \sum_{i \in S} a_i^T &\implies \max_x c^T x = \max_x \sum_{i \in S} a_i^T x \mid Ax \leq b \\ &\implies \forall x : c^T x \leq \sum_{i \in S} b_i \end{aligned}$$

Then $c^T x$ has a global maximum at $\forall i \in S : a_i^T x = b_i$, which is solved by the vector x^*. Since x^* is a vertex in the polytope $P : \{x : Ax \leq b\}$, any change to the vector x^* either violates the condition $Ax \leq b$, or loosens one of the constraints in S. Since all constraints in S need to be tight to maximize $c^T x$, x^* is a unique solution.