Mathematical Formulation of Projection Problem

Jeffrey Liang

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1 Goal

To project a set of points $\mathcal{X}$ onto the nearest ellipsoid with volume 1, such that the projection of this ellipsoid in two different views produces ellipses which minimise the error between their areas and two target areas.

2 Parametrising Ellipsoids

An ellipsoid $\mathcal{E}$ can be parametrised by a positive definite, symmetric matrix $A \in \mathbb{R}^{3\times 3}$ as follows:

$$\mathcal{E} = \left\{ \mathbf{x} \in \mathbb{R}^3 | \mathbf{x}^\top A \mathbf{x} = 1, A \succ 0 \right\}. \tag{1}$$

By performing an eigen-decomposition on A, we have that A can be represented as $\mathbf{A} = \mathbf{Q}^\top \mathbf{\Lambda} \mathbf{Q}$, where $\mathbf{Q} \in SO(3)$ is an orthonormal rotation matrix and $\mathbf{\Lambda}$ is a diagonal matrix with eigenvalues that are the inverse square of the semi-axes lengths of $\mathcal{E}$:

$$\begin{aligned}
A &= \mathbf{Q}^\top \mathbf{\Lambda} \mathbf{Q} && (2)\\
&= \mathbf{Q}^\top \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \mathbf{Q} && (3)\\
&= \mathbf{Q}^\top \begin{bmatrix} 1/a^2 & & \\ & 1/b^2 & \\ & & 1/c^2 \end{bmatrix} \mathbf{Q}, && (4)
\end{aligned}$$

where a, b, and c are the semi-axes lengths of the ellipsoid $\mathcal{E}$. As the rotation matrix can be parametrised by three Euler angles – roll ψ, pitch θ, and yaw ϕ – in addition to the three semi-axes, an ellipsoid in 3D can thus be parametrised by six degrees of freedom which we can represent as a vector $\mathbf{u} \in \mathbb{R}^6$.

3 Inner-Level Problem

Let $\mathcal{X} = \{\mathbf{x}_i\}$ be a set of sampled points $\mathbf{x}_i \in \mathbb{R}^3$ from a noisy ellipsoid. Moreover, let $Q : \mathbb{R}^3 \to SO(3)$ map three angles (roll, pitch, yaw) to the orthonormal 3D rotation matrix $\mathbf{Q}$ defined above and $\Lambda : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ map three lengths to the diagonal matrix $\mathbf{\Lambda}$ described above.

To project these points to the nearest ellipsoid with a volume of 1 unit cubed, we can minimise the least squares error subject to the volume constraint.

$$\begin{array}{ll} \underset{\text{over } \mathbf{u}\in\mathbb{R}^6}{\text{minimize}} & \sum_{i=1}^{m} \left(\mathbf{x}_i{}^\top Q(\mathbf{u}_{4:6})^\top \Lambda(\mathbf{u}_{1:3}) Q(\mathbf{u}_{4:6})\mathbf{x}_i - 1\right)^2 \\ \text{subject to} & \frac{4}{3}\pi \mathbf{u}_1\mathbf{u}_2\mathbf{u}_3 = 1 \end{array} \tag{5}$$

Or, in matrix form, letting

$$\mathbf{X} = \begin{bmatrix} | & \cdots & | \\ \mathbf{x}_1 & \cdots & \mathbf{x}_m \\ | & \cdots & | \end{bmatrix},$$

we have

$$\begin{array}{ll} \underset{\text{over } \mathbf{u}\in\mathbb{R}^6}{\text{minimize}} & \left\| \text{diag}(\mathbf{X}^\top Q(\mathbf{u}_{4:6})^\top \Lambda(\mathbf{u}_{1:3}) Q(\mathbf{u}_{4:6})\mathbf{X}) - \mathbf{1}_m \right\|^2 \\ \text{subject to} & \frac{4}{3}\pi \mathbf{u}_1\mathbf{u}_2\mathbf{u}_3 = 1 \end{array} \tag{6}$$

4 Outer-Level Problem

There are two approaches to performing the projection of the ellipsoid into two different views: parametrically and via point-sampling.

4.1 Parametric Approach

To perform the projection using parameters obtained from the inner-level problem, we employ orthographic projections. We simulate orthographic projections into two different views by rotating the projected ellipse produced from the inner-level problem differently for each view before projecting the rotated ellipsoids both onto the x-y plane.

Two views were chosen as we believed this was sufficient to constrain the problem such that the ellipsoid matched the desired properties.

The output of the inner-level problem gives us a vector

$$\begin{aligned}\mathbf{u}^* = \quad & \operatorname{argmin}\left\|\operatorname{diag}(\mathbf{X}^\top Q(\mathbf{u}_{4:6})^\top \Lambda(\mathbf{u}_{1:3}) Q(\mathbf{u}_{4:6})\mathbf{X}) - \mathbf{1}_m\right\|^2 \\ & \text{subject to} \tfrac{4}{3}\pi \mathbf{u}_1\mathbf{u}_2\mathbf{u}_3 = 1.\end{aligned}$$

Using this $\mathbf{u}^*$, we have the projected ellipsoid with volume 1, and can apply the rotation matrices to the resulting $A(\mathbf{u}^*) = Q(\mathbf{u}_{4:6})^\top \Lambda(\mathbf{u}_{1:3}) Q(\mathbf{u}_{4:6})$. The rotations are performed by two rotation matrices, $\mathbf{R}_1, \mathbf{R}_2 \in SO(3)$ respectively. Each are applied as a right conjugation to the matrix $A(\mathbf{u}^*)$ in a change-of-basis operation.

Thus, we get two rotated ellipsoids, $\mathcal{E}_1$ and $\mathcal{E}_2$, parametrised by the following corresponding matrices:

$$\mathcal{E}_1 : \mathbf{R}_1^\top A(\mathbf{u}^*)\mathbf{R}_1 \qquad \mathcal{E}_2 : \mathbf{R}_2^\top A(\mathbf{u}^*)\mathbf{R}_2. \tag{7}$$

To apply the projection onto the x-y plane, it is *not* as simple as multiplying the projection matrix $\mathbf{P}$,

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{8}$$

as this will give us the cross-section with the x-y plane instead. To obtain the projection, we need to take the Schur complement of the bottom-right entry of the 3D matrix. Let $S : \mathbb{R}^{3\times 3} \to \mathbb{R}^{2\times 2}$ return the Schur complement described above. Then, the two 2D ellipses ε_1 and ε_2 whose areas we are interested in are parametrised by the matrices $\hat{\mathbf{A}}_1$ and $\hat{\mathbf{A}}_2$:

$$\varepsilon_1 : \hat{\mathbf{A}}_1 = S\big(\mathbf{R}_1^\top A(\mathbf{u}^*)\mathbf{R}_1\big) \qquad \varepsilon_2 : \hat{\mathbf{A}}_2 = S\big(\mathbf{R}_2^\top A(\mathbf{u}^*)\mathbf{R}_2\big). \tag{9}$$

The determinants of these 2×2 matrices $\hat{\mathbf{A}}_1$ and $\hat{\mathbf{A}}_2$ would give the product of the eigenvalues of each matrix. As the eigenvalues would be the squared inverses of the semi-axes of ε_1 and ε_2, the inverse square root of the product of eigenvalues gives the product of the semi-axes lengths. Therefore, the area of an ellipse can be expressed as:

$$\pi ab = \frac{\pi}{\frac{1}{\sqrt{a^2b^2}}} = \frac{\pi}{\sqrt{\lambda_1\lambda_2}} = \frac{\pi}{\sqrt{\det \hat{\mathbf{A}}_i}}. \tag{10}$$

Given two target areas t_1 and t_2 for the projected ellipses ε_1 and ε_2, our final minimisation problem is thus:

$$\begin{aligned} \underset{\text{over}\mathcal{X}=\{\mathbf{x}_i\}}{\text{minimize}} \quad & \left\|\frac{\pi}{\sqrt{\det \hat{A}_1(\mathbf{u}^*)}} - t_1\right\|^2 + \left\|\frac{\pi}{\sqrt{\det \hat{A}_2(\mathbf{u}^*)}} - t_2\right\|^2 \\ \text{subject to} \quad & \mathbf{u}^* = \operatorname{argmin} \left\|\operatorname{diag}(\mathbf{X}^\top Q(\mathbf{u}_{4:6})^\top \Lambda(\mathbf{u}_{1:3}) Q(\mathbf{u}_{4:6})\mathbf{X}) - \mathbf{1}_m\right\|^2 \ . \\ & \quad \text{subject to } \tfrac{4}{3}\pi\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3 = 1 \end{aligned}$$

4.2 Sampling Approach