CPSC-354 Report

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Abstract

Contents

1 Introduction

2 Week by Week

2.1 Week 1

2.1.1 Homework

What is the MU Puzzle and how do you "solve" it?:

The MU puzzle is a logic puzzle created by Douglas Hofstadter in his 1979 book "Gödel, Escher, Bach: An Eternal Golden Braid." It's designed to illustrate concepts about formal systems, computability, and the limits of rule-based reasoning. The rules are below:

Rule I: If a string ends in I, you can add U to the end (xI → xIU)
Rule II: If you have Mx, you can make Mxx (double everything after M)
Rule III: If you find III anywhere in your string, you can replace it with U (xIIIy → xUy)
Rule IV: If you find UU anywhere in your string, you can remove it (xUUy → xy)

To "solve" the puzzle, you try to apply a combination of rules step by step, creating new strings. Eventually, you'll find that MU can never be reached because the rules never allow you to remove the odd number of I's needed to get zero.

2.1.2 Exploration

Hofstadter used this puzzle to demonstrate how formal systems can have inherent limitations - some statements that seem like they should be provable within a system are actually unprovable. This connects to

Gödel's incompleteness theorems and fundamental questions about the nature of mathematical truth and computation.

Programming languages are formal systems, just like the MU puzzle. They have:

- Syntax rules (what constitutes valid code)
- Transformation rules (how expressions evaluate)
- Semantic constraints (what programs can actually compute)

The MU puzzle demonstrates that even simple rule sets can have hidden limitations - similarly, programming languages have inherent computational boundaries.

2.1.3 Questions

1. The impossibility of reaching "MU" from "MI" is provable, yet someone working within the system might not realize this. How does this relate to the halting problem and undecidable questions in programming?

2.2 Week 2

2.2.1 Homework

Consider the following list of ARSs:

1. $A = \{\}$.
2. $A = \{a\}$ and $R = \{\}$.
3. $A = \{a\}$ and $R = \{(a, a)\}$.
4. $A = \{a, b, c\}$ and $R = \{(a, b), (a, c)\}$.
5. $A = \{a, b\}$ and $R = \{(a, a), (a, b)\}$.
6. $A = \{a, b, c\}$ and $R = \{(a, b), (b, b), (a, c)\}$.
7. $A = \{a, b, c\}$ and $R = \{(a, b), (b, b), (a, c), (c, c)\}$.

Draw a picture for each of the ARSs above. Are the ARSs terminating? Are they confluent? Do they have unique normal forms?

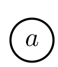
Try to find an example of an ARS for each of the possible 8 combinations. Draw pictures of these examples.

ARS 1: $A = \{\}$

Empty graph (no nodes, no edges)

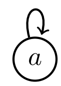
Terminating: YES **Confluent:** YES **Unique Normal Forms:** YES

ARS 2: $A = \{a\}$**,** $R = \{\}$



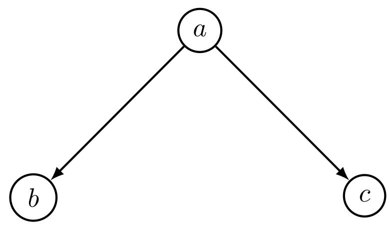
Terminating: YES **Confluent:** YES **Unique Normal Forms:** YES

ARS 3: $A = \{a\}$, $R = \{(a, a)\}$



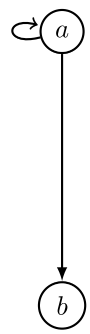
Terminating: NO **Confluent:** YES **Unique Normal Forms:** NO

ARS 4: $A = \{a, b, c\}$, $R = \{(a, b), (a, c)\}$



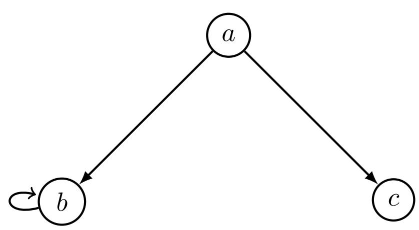
Terminating: YES **Confluent:** NO **Unique Normal Forms:** NO

ARS 5: $A = \{a, b\}$, $R = \{(a, a), (a, b)\}$



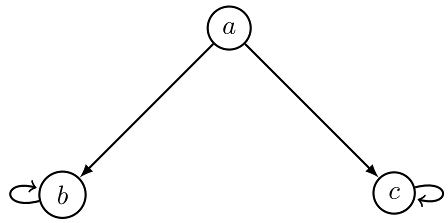
Terminating: NO **Confluent:** NO **Unique Normal Forms:** NO

ARS 6: $A = \{a, b, c\}$, $R = \{(a, b), (b, b), (a, c)\}$



Terminating: NO **Confluent:** NO **Unique Normal Forms:** NO

ARS 7: $A = \{a, b, c\}$, $R = \{(a, b), (b, b), (a, c), (c, c)\}$



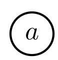
Terminating: NO **Confluent:** NO **Unique Normal Forms:** NO

8 Combinations Table

Confluent	Terminating	Unique NF	Example
True	True	True	$A = \{a\},\ R = \{\}$
True	True	False	$A = \{\},\ R = \{\}$
True	False	True	$A = \{a, b\},\ R = \{(a, b), (b, b)\}$
True	False	False	$A = \{a\},\ R = \{(a, a)\}$
False	True	True	$A = \{a, b, c, d\},\ R = \{(a, b), (a, c), (c, d)\}$
False	True	False	$A = \{a, b, c\},\ R = \{(a, b), (a, c)\}$
False	False	True	$A = \{a, b, c\},\ R = \{(a, b), (b, a), (a, c), (c, a)\}$
False	False	False	$A = \{a, b, c\},\ R = \{(a, b), (b, b), (a, c)\}$

Examples for 8 Combinations

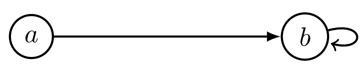
Example 1: Confluent=T, Terminating=T, Unique Normal Forms=T $A = \{a\}$, $R = \{\}$



Example 2: Confluent=T, Terminating=T, Unique Normal Forms=F $A = \{\}$, $R = \{\}$

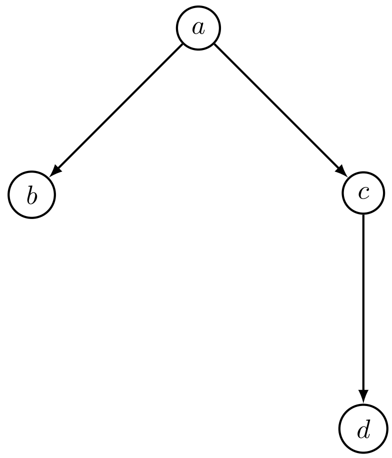
Empty graph

Example 3: Confluent=T, Terminating=F, Unique Normal Forms=T $A = \{a, b\}$, $R = \{(a, b), (b, b)\}$

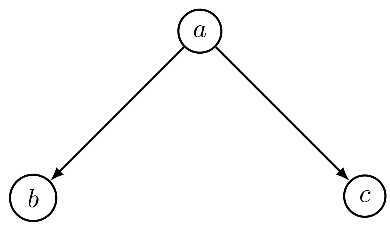


Example 4: Confluent=T, Terminating=F, Unique Normal Forms=F $A = \{a\}$, $R = \{(a, a)\}$

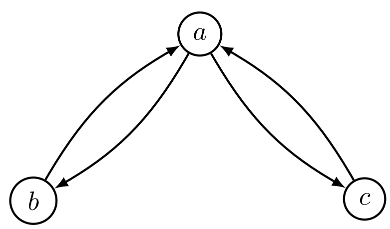
Example 5: Confluent=F, Terminating=T, Unique Normal Forms=T $A = \{a, b, c, d\}$, $R = \{(a, b), (a, c), (c, d)\}$



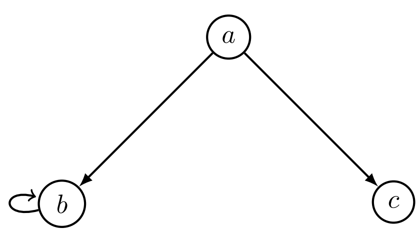
Example 6: Confluent=F, Terminating=T, Unique Normal Forms=F $A = \{a, b, c\}$, $R = \{(a, b), (a, c)\}$



Example 7: Confluent=F, Terminating=F, Unique Normal Forms=T $A = \{a, b, c\}$, $R = \{(a, b), (b, a), (a, c), (c, a)\}$



Example 8: Confluent=F, Terminating=F, Unique Normal Forms=F $A = \{a, b, c\}$, $R = \{(a, b), (b, b), (a, c)\}$



2.2.2 Exploration

Abstract Reduction Systems provide a mathematical foundation for understanding computation and rewriting. The properties of termination, confluence, and unique normal forms are fundamental to understanding how programming languages behave:

- **Termination** ensures that computations eventually halt
- **Confluence** guarantees that the order of operations doesn't affect the final result

- **Unique Normal Forms** means every expression has a single, well-defined simplified form

These concepts directly apply to programming language design, where we want predictable evaluation strategies and guaranteed termination for certain classes of programs.

2.2.3 Questions

1. How do the termination properties of ARSs relate to the halting problem in computation?
2. Why might a programming language designer prefer confluent systems over non-confluent ones?

2.3 Week 3

2.3.1 Homework

Consider the rewrite rules:

- ab → ba
- ba → ab
- aa → (empty string)
- b → (empty string)

2.3.2 Sample Reductions

Reducing abba:

abba → baba (using ab → ba)

baba → bbaa (using ab → ba)

bbaa → baa (using b → empty)

baa → aa (using b → empty)

aa → empty (using aa → empty)

Reducing bababa:

bababa → bbaaaba (using ab → ba twice)

bbaaaba → baaaba (using b → empty)

baaaba → aaaba (using b → empty)

aaaba → aba (using aa → empty)

aba → baa (using ab → ba)

baa → aa (using b → empty)

aa → empty (using aa → empty)

2.3.3 Analysis

Why is the ARS not terminating?

The first two rules ab → ba and ba → ab create cycles. You can apply these rules forever, going back and forth between ab and ba.

Find two strings that are not equivalent. How many non-equivalent strings can you find?

Two strings that are not equivalent: "a" and "empty string". The string "a" cannot be reduced further, while other strings can reduce to empty.

Equivalence classes and normal forms: There are exactly 2 equivalence classes:

1. Strings that reduce to empty string
2. Strings that reduce to "a"

The normal forms are: empty string and "a"

Modified terminating ARS: To make it terminating, always eliminate b's first, then eliminate aa's, then do swapping only if needed.

Questions about strings that can be answered using the ARS:

1. "Given a string, does it contain an even number of a's?"
2. "Given a string, does it contain an odd number of a's?"
3. "Are two strings equivalent under this rewrite system?"

2.4 Exercise 5b

2.4.1 Modified Rewrite Rules

Same as Exercise 5, but change aa → empty to aa → a:

- ab → ba
- ba → ab
- aa → a (pairs of a become single a)
- b → (empty string)

2.4.2 Sample Reductions

Reducing abba:

abba → baba (using ab → ba)

baba → bbaa (using ab → ba)

bbaa → baa (using b → empty)

baa → aa (using b → empty)

aa → a (using aa → a)

Reducing bababa:

bababa → bbaaaba (using ab → ba twice)

bbaaaba → baaaba (using b → empty)

baaaba → aaaba (using b → empty)

aaaba → aaba (using aa → a)

aaba → abaa (using ab → ba)

abaa → baaa (using ab → ba)

baaa → aaa (using b → empty)

aaa $\to$ aa (using aa $\to$ a)

aa $\to$ a (using aa $\to$ a)

2.4.3 Analysis

Why the ARS is not terminating: Same as Exercise 5 - the rules ab $\to$ ba and ba $\to$ ab create infinite cycles.

Non-equivalent strings: Two strings that are not equivalent: "a" and "empty string". We can find exactly 2 non-equivalent strings.

Equivalence classes and normal forms: There are exactly 2 equivalence classes:

1. Strings with even number of a's $\to$ reduce to empty
2. Strings with odd number of a's $\to$ reduce to "a"

The normal forms are: empty string and "a"

Modified terminating ARS: To make it terminating, use the same priority as Exercise 5:

1. b $\to$ empty (eliminate all b's first)
2. aa $\to$ a (reduce pairs of a's)
3. ab $\to$ ba (only if needed)

Questions about strings that can be answered using the ARS:

1. "Given a string, does it contain an even number of a's?"
2. "Given a string, does it contain an odd number of a's?"
3. "Are two strings equivalent under this rewrite system?"

2.4.4 Exploration

2.4.5 Questions

1. If two completely different sets of rewrite rules (Exercise 5 vs 5b) produce the same equivalence classes, what does this tell us about the relationship between implementation and specification in computer science?
2. If "abab" and "bbaa" are equivalent under this system, but clearly different as strings, what does "equivalence" really mean? Is mathematical equivalence different from everyday sameness?

2.5 Week 4

2.5.1 Homework

2.5.2 Exploration

2.5.3 Questions

2.6 Week 5

2.6.1 Homework

2.6.2 Exploration

2.6.3 Questions

2.7 Week 6

2.7.1 Homework

2.7.2 Exploration

2.7.3 Questions

2.8 Week 7

2.8.1 Homework

2.8.2 Exploration

2.8.3 Questions

2.9 Week 8

2.9.1 Homework

Natural Number Game - Tutorial World: Levels 5-8

Level 5: Simplifying with `add_zero` Prove that $a + (b + 0) + (c + 0) = a + b + c$.

```
rw [add_zero]
rw [add_zero]
rfl
```

Level 6: Targeted rewriting Prove that $a + (b + 0) + (c + 0) = a + b + c$ using explicit arguments.

```
rw [add_zero c]
rw [add_zero b]
rfl
```

Level 7: `succ_eq_add_one` Prove that for all natural numbers n, $\mathrm{succ}(n) = n + 1$.

```
rw [one_eq_succ_zero]
rw [add_succ]
rw [add_zero]
rfl
```

Level 8: Proving $2 + 2 = 4$ Prove that $2 + 2 = 4$.

```
nth_rewrite 2 [two_eq_succ_one]
rw [add_succ]
rw [one_eq_succ_zero]
rw [add_succ]
rw [add_zero]
rw [<- three_eq_succ_two]
rw [<- four_eq_succ_three]
rfl
```

2.9.2 Natural Language Proof

Level 8: Proving $2 + 2 = 4$

Proof. We want to prove that $2 + 2 = 4$ using only the Peano axioms and previously established theorems about natural numbers.

We begin by expanding the second 2 in the left-hand side using its definition. By the theorem `two_eq_succ_one`, we know that $2 = \text{succ}(1)$. Rewriting the second occurrence of 2, we obtain:

$$2 + \text{succ}(1) = 4$$

Next, we apply the fundamental recursion axiom for addition, `add_succ`, which states that for any natural numbers a and b, we have $a + \text{succ}(b) = \text{succ}(a + b)$. Applying this axiom gives us:

$$\text{succ}(2 + 1) = 4$$

Now we expand 1 using its definition. By `one_eq_succ_zero`, we know that $1 = \text{succ}(0)$. Substituting this yields:

$$\text{succ}(2 + \text{succ}(0)) = 4$$

We apply `add_succ` again to the inner addition:

$$\text{succ}(\text{succ}(2 + 0)) = 4$$

By the base case axiom for addition, `add_zero`, which states that $n + 0 = n$ for any natural number n, we can simplify $2 + 0$ to 2:

$$\text{succ}(\text{succ}(2)) = 4$$

Now we recognize this expression in terms of known number definitions. By the definition `three_eq_succ_two`, we know that $3 = \text{succ}(2)$. Using this in reverse (indicated by the backwards arrow in the formal proof), we can rewrite $\text{succ}(2)$ as 3:

$$\text{succ}(3) = 4$$

Finally, by the definition `four_eq_succ_three`, we know that $4 = \text{succ}(3)$. Using this in reverse, we obtain:

$$4 = 4$$

This is true by the reflexivity of equality: any object is equal to itself.

Therefore, $2 + 2 = 4$. $\square$

2.9.3 Exploration

The proof of $2 + 2 = 4$ is a remarkable example of how even the simplest arithmetic facts require rigorous justification when building mathematics from first principles. Several profound insights emerge from this exercise:

- **Numbers as constructions**: In the Peano axioms, numbers are not primitive objects but are constructed iteratively from zero using the successor function. The number 2 is defined as succ(succ(0)), 3 as succ(2), and 4 as succ(3). This constructive approach ensures that all natural numbers can be built systematically.
- **Addition as recursion**: Addition is not defined by a lookup table but by two recursive rules: the base case $n+0 = n$ and the recursive case $n+\text{succ}(m) = \text{succ}(n+m)$. The proof of $2+2 = 4$ essentially "executes" this recursive definition step by step.
- **The role of definitions**: Much of the proof consists of unfolding and refolding definitions. We expand 2 into succ(1), then 1 into succ(0), perform the addition, and finally recognize the result as 3 and then 4. This shows that definitions are not just abbreviations but active components of reasoning.
- **Computational content of proofs**: This proof has a computational interpretation. Each rewrite step corresponds to a computation step, and the entire proof traces the execution of the addition algorithm. This connection between proofs and programs is central to the Curry-Howard correspondence.
- **Nothing is obvious in formal systems**: What seems trivial in everyday mathematics ($2 + 2 = 4$) requires multiple logical steps when formalized. This explicitness is both a strength (eliminates ambiguity and hidden assumptions) and a weakness (can obscure high-level mathematical intuition).

This exercise bridges the gap between our intuitive understanding of arithmetic and the formal foundations required for computer-verified mathematics and programming language semantics.

2.9.4 Questions

1. Why does proving $2 + 2 = 4$ require eight steps when it seems inherently true?
2. When should a programming language prioritize precision over simplicity?

2.10 Week 9

2.10.1 Homework

Natural Number Game - Addition World: Level 5 (`add_right_comm`)

Theorem Statement Prove that for all natural numbers a, b, and c, we have $(a + b) + c = (a + c) + b$.

This theorem is called *right commutativity* because it states that the second and third terms can be swapped when grouped with the first term.

Solution 1: Using Induction **Lean Proof:**

```
theorem add_right_comm (a b c : N) : (a + b) + c = (a + c) + b := by
  induction c with d hd
  case zero =>
    rw [add_zero]
    rw [add_zero]
    rfl
  case succ =>
    rw [add_succ]
    rw [add_succ]
    rw [hd]
```

```
  rw [succ_add]
  rfl
```

Mathematical Proof:

Proof. We prove $(a+b)+c=(a+c)+b$ by induction on c.

Base Case ($c=0$): We need to show $(a+b)+0=(a+0)+b$.

Starting with the left-hand side:

$$(a+b)+0=a+b \quad (\text{by } \texttt{add\_zero})$$

For the right-hand side:

$$(a+0)+b=a+b \quad (\text{by } \texttt{add\_zero})$$

Therefore, $(a+b)+0=(a+0)+b$. ✓

Inductive Step: Assume the inductive hypothesis: $(a+b)+d=(a+d)+b$.

We must prove: $(a+b)+\mathrm{succ}(d)=(a+\mathrm{succ}(d))+b$.

Starting with the left-hand side:

$$\begin{aligned}(a+b)+\mathrm{succ}(d)&=\mathrm{succ}((a+b)+d) \quad (\text{by } \texttt{add\_succ})\\ &=\mathrm{succ}((a+d)+b) \quad (\text{by inductive hypothesis})\end{aligned}$$

For the right-hand side:

$$\begin{aligned}(a+\mathrm{succ}(d))+b&=\mathrm{succ}(a+d)+b \quad (\text{by } \texttt{add\_succ})\\ &=\mathrm{succ}((a+d)+b) \quad (\text{by } \texttt{succ\_add})\end{aligned}$$

Both sides equal $\mathrm{succ}((a+d)+b)$, completing the inductive step.

Therefore, by mathematical induction, $(a+b)+c=(a+c)+b$ for all natural numbers a, b, and c. □

Solution 2: Using Previously Proven Theorems (No Induction) Lean Proof:

```
theorem add_right_comm (a b c : N) : (a + b) + c = (a + c) + b := by
  rw [add_assoc]
  rw [add_comm b c]
  rw [<- add_assoc]
```

Mathematical Proof:

Proof. We prove $(a+b)+c=(a+c)+b$ using the associativity and commutativity of addition.

$$\begin{aligned}(a+b)+c&=a+(b+c) \quad (\text{by associativity: } \texttt{add\_assoc})\\ &=a+(c+b) \quad (\text{by commutativity: } \texttt{add\_comm})\\ &=(a+c)+b \quad (\text{by associativity in reverse})\end{aligned}$$

Therefore, $(a+b)+c=(a+c)+b$. □

Comparison of the Two Approaches

- **Induction approach:** This solution builds the theorem from scratch using only the fundamental definition of addition (the recursive rules `add_zero` and `add_succ`). It directly proves the property by examining the structure of natural numbers. This approach is more fundamental but requires more steps.
- **Algebraic approach:** This solution leverages previously proven theorems (`add_assoc` and `add_comm`). It's more elegant and intuitive, treating addition as an abstract operation with known properties. However, it depends on having already proven those properties (likely using induction themselves).
- **Trade-off:** The inductive proof is self-contained but longer, while the algebraic proof is shorter but requires a richer theory. This mirrors a general principle in mathematics and computer science: we can either work at a low level with explicit detail, or at a high level using abstractions—each approach has its place.

2.10.2 Exploration

This proof reveals two fundamental approaches to formal reasoning:

- **Induction (constructive):** Builds the proof from scratch using the recursive definition of addition. Works at the level of number structure. Longer but self-contained.
- **Algebraic (abstract):** Reuses previously proven theorems (`add_assoc`, `add_comm`). Treats addition as an abstract operation with known properties. Shorter but dependent on prior results.

The key insight: there are multiple valid paths to the same mathematical truth. The choice between approaches mirrors software engineering trade-offs between "building from scratch" and "reusing libraries"—each has its place depending on context and goals.

2.10.3 Questions

1. The inductive proof required 10 lines while the algebraic proof required only 3. Does this mean the algebraic proof is "better"? What if we count the lines needed to prove `add_assoc` and `add_comm`?
2. In software development, we often choose between "reinventing the wheel" and "using libraries." How does this trade-off relate to our two proof strategies?

2.11 Week 10

2.11.1 Homework

2.11.2 Exploration

2.11.3 Questions

2.12 Week 11

2.12.1 Homework

2.12.2 Exploration

2.12.3 Questions

3 Synthesis

4 Evidence of Participation

5 Conclusion

References