$e^x = \sum \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Monte Carlo

how do we pick out S_t ?

We know from earlier that the SDE

r(t)

$$\frac{dS_t}{S_t} = r\,dt + \sigma dW_t$$

Itô III on log S

with constant r and σ has the solution

drift diffusion

MILS

$$S_T = S_0 \exp\left\{\left(r - \tfrac{1}{2}\sigma^2\right) T + \sigma\phi\sqrt{T}\right\}$$

for some time horizon T; with $\phi \sim N(0,1)$; $W_t \sim N(0,t)$ and can be written $\phi\sqrt{T}$.

It is often more convenient to express in time stepping form

drift diffusion

$$S_{t+\delta t} = S_t \exp\left\{\left(r - \tfrac{1}{2}\sigma^2\right) \delta t + \sigma\phi\sqrt{\delta t}\right\}$$

Now do a Taylor series expansion of the exact solution, i.e.

$$e^{\left(r-\frac{1}{2}\sigma^2\right)\delta t + \sigma\phi\sqrt{\delta t}} \sim 1 + \left(r - \tfrac{1}{2}\sigma^2\right)\delta t + \sigma\phi\sqrt{\delta t} + \tfrac{1}{2}\sigma^2\phi^2\delta t.$$

$o(dt)$

E-M $\left[S_{t+\delta t} \sim S_t\left(1+r\delta t + \sigma\phi\sqrt{\delta t}\right)\right]$

So we have

bottom line p. 1

$$\left[S_{t+\delta t} \sim S_t\left(\underbrace{1 + r\delta t + \sigma\phi\sqrt{\delta t}}_{\text{E-M}} + \overbrace{\tfrac{1}{2}\sigma^2\left(\phi^2 - 1\right)\delta t}^{\text{extra term}} + ..\right)\right]$$

which differs from the Euler method at $O\left(\delta t\right)$ by the term $\underbrace{\frac{1}{2}\sigma^2\left(\phi^2 - 1\right)\delta t}_{\text{new 'bit'}}$.
The term

$\lim_{\delta t \to 0}$: $\delta t < \sqrt{\delta t}$

$$\frac{1}{2}\left(\phi^2 - 1\right)\delta t,$$

is called the *Milstein correction*.

The Milstein correction can be thought of as being a stochastic effect (a result of Itô's lemma in a sense).

Mod 3 exam Q1: exact sol^{n}
E-M
Milstein

Milstein Integration

Consider a process G_t

We approximate the solution of the SDE

differential form :

$$dG_t = A\left(G_t, t\right)dt + B\left(G_t, t\right)dW_t$$

EM $O(\sqrt{\delta t})$ Milstein $O(\delta t)$

This is compact form for

$$G_{t+\delta t} = G_t + \int_t^{t+\delta t} A\left(G_s, s\right) ds + \int_t^{t+\delta t} B\left(G_s, s\right) dW_s,$$

by

Milstein scheme

$$G_{t+\delta t} \approx \overbrace{G_t + A\left(G_t, t\right)\delta t + B\left(G_t, t\right)\sqrt{\delta t}\phi}^{\text{E-M}} + B\left(G_t, t\right)\frac{\partial}{\partial G_t} B\left(G_t, t\right) \cdot \underline{\tfrac{1}{2}\left(\phi^2 - 1\right)}\delta t.$$

Note: We use the same value of the random number $\phi \sim N\left(0, 1\right)$ in both of the expressions

Klaus Schmitz Abe

$$B\left(G_t, t\right)\sqrt{\delta t}\phi$$

DPhil (Oxford)

and

$$B\left(G_t, t\right)\frac{\partial}{\partial G_t} B\left(G_t, t\right) \cdot \tfrac{1}{2}\left(\phi^2 - 1\right)\delta t.$$

The error of the Milstein scheme is $O\left(\delta t\right)$ which makes it better than the Euler-Maruyama method which is $O\left(\delta t^{1/2}\right)$. The Milstein makes use of Itô's lemma to increase the accuracy of the approximation by adding the second order term.

→ $\sqrt{\delta t}$ → SDE δt

→ $\sigma/\sqrt{n}$ → CLT $\varepsilon \sim 1/\sqrt{n}$

Some texts express the scheme in difference form. So a SDE written

$$dY_t = A\left(Y_t, t\right) dt + B\left(Y_t, t\right) dW_t$$

can be discretized as

index form

$$Y_{i+1} = Y_i + A\Delta t + B\Delta W_t + \tfrac{1}{2} B \frac{\partial B}{\partial Y_i}\left(\left(\Delta W_t\right)^2 - \Delta t\right)$$

$(\phi\sqrt{\delta t})^2 - \delta t$

Applying Milstein to the earlier example of GBM

difference form

$$dS_t = \underbrace{rS_t}_{A}dt + \underbrace{\sigma S_t}_{B}dW_t$$

$(\phi^2 - 1)\,\delta t$

where

$$\begin{aligned} A\left(S_t, t\right) &= rS_t \\ B\left(S_t, t\right) &= \sigma S_t \end{aligned}$$

gives

$$\begin{aligned} S_{t+\delta t} &\sim S_t + rS_t\delta t + \sigma S_t\sqrt{\delta t}\phi + \tfrac{1}{2}\sigma S_t \frac{\partial}{\partial S_t}\sigma S_t\left(\phi^2 - 1\right)\delta t \\ &= S_t\left(\underbrace{1 + r\delta t + \sigma\phi\sqrt{\delta t}}_{\text{E-M}} + \tfrac{1}{2}\sigma^2\underbrace{\left(\phi^2 - 1\right)\delta t}_{\text{correction term}}\right) \end{aligned}$$

As another example, the CIR model for the spot rate is

$$dr_t = \underbrace{(\eta - \gamma r_t)}_{A}\, dt + \underbrace{\sqrt{\alpha r_t}}_{B} dW_t.$$

So identifying

r

$$\begin{aligned} A(r_t, t) &= \eta - \gamma r_t \\ B(r_t, t) &= \sqrt{\alpha r_t} \end{aligned}$$

and substituting into the Milstein scheme gives

$$\begin{aligned} r_{t+\delta t} &\sim r_t + (\eta - \gamma r_t)\,\delta t + \sqrt{\alpha r_t \delta t}\phi + \sqrt{\alpha r_t}\frac{\partial}{\partial r_t}\sqrt{\alpha r_t} \cdot \frac{1}{2}\left(\phi^2 - 1\right)\delta t \\ &= \underbrace{r_t + (\eta - \gamma r_t)\,\delta t + \sqrt{\alpha r_t \delta t}\phi}_{E-M} + \underbrace{\frac{1}{4}\alpha\left(\phi^2 - 1\right)\delta t.} \end{aligned}$$

Part I of the mod J exam

$r^{1/2}$ $r^{1/2}$

$r^{-1/2}$ $\frac{1}{2}$ $r^{-1/2}$ $\frac{1}{2}$

To conclude, a SDE for the process Y_t

$$dY_t = A\left(Y_t, t\right) dt + B\left(Y_t, t\right) dW_t$$

can be discretized using Milstein as

$$Y_{i+1} = Y_i + A\delta t + B\phi\sqrt{\delta t} + \tfrac{1}{2} B \frac{\partial B}{\partial Y_i}\left(\phi^2 - 1\right)\delta t,$$

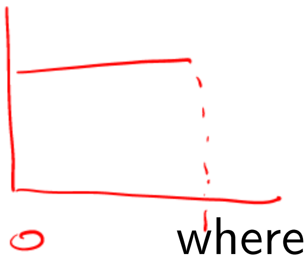
where $\frac{1}{2}\left(\phi^2 - 1\right)\delta t$ is the **Milstein correction term.** The same random number $\phi \sim N\left(0, 1\right)$ is used per time-step.

We all know Exp. $\longrightarrow \int$

Monte-Carlo methods are centred on evaluating definite integrals as expectations (or averages). Before studying this in greater detail, we consider the simple problem of estimating expectations of functions of uniformly distributed random numbers.

Motivating Example: Estimate $\theta = \mathbb{E}\left[e^{U^2}\right]$, where $\boxed{U \sim U(0,1)}$.

We note that $\mathbb{E}\left[e^{U^2}\right]$ can be expressed in integral form, i.e.



$$\mathbb{E}\left[e^{U^2}\right] = \int_0^1 e^{x^2} p(x)\, dx = \int_{-\infty}^{0} + \int_0^1 + \int_1^{\infty}$$

where $p(x)$ is the density function of a $U(0,1)$

$$\boxed{p(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}}$$

RV : X $\quad p(x)$

$$\mathbb{E}\left[f(x)\right] = \int_{-\infty}^{\infty} f(x)\, p(x) dx$$

hence

$$\boxed{\mathbb{E}\left[e^{U^2}\right] = \int_0^1 e^{x^2} dx}$$

$\hat{\theta}$ estimator for θ

$\times 1$

$\boxed{\text{Invariant}}$

Expec $\longrightarrow \int$

$\int \longrightarrow \mathbb{E}[\cdot]$?

This integral does not have an analytical solution. The theme of this section is to consider solving numerically, using simulations. We use the Monte Carlo simulation procedures:

RAND() $\sim U(0,1)$

1. Generate a sequence $\underline{U_1}, \underline{U_2}, ..., \underline{U_n} \sim U(0,1)$ where U_i are i.i.d (independent and identically distributed)

U_i	Y_i
U_1	$e^{U_1^2}$
U_2	$e^{U_2^2}$
$\vdots$	$\vdots$
U_n	$e^{U_n^2}$

for each U_i,

2. Compute $Y_i = e^{U_i^2}$ $(i = 1, ..., n)$

3. Estimate θ by

$$
\begin{aligned}
\boxed{\widehat{\theta}_n} &\equiv \frac{1}{n}\sum_{i=1}^{n} Y_i \\
&= \frac{1}{n}\sum_{i=1}^{n} e^{U_i^2}
\end{aligned}
$$

Average $= \frac{1}{n} \sum_{i=1}^{n} e^{U_i^2}$

i.e. use the sample mean of the $e^{U_i^2}$ terms.

$f = t'^{a} f\left(\frac{y'}{t'^{b}}\right) \qquad \xi = \frac{y'}{t'^{s}}$

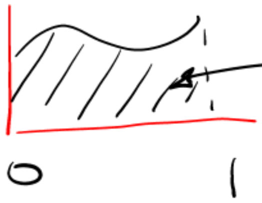
Monte Carlo Integration

When a closed form solution for evaluating an integral is not available, numerical techniques are used. The purpose of Monte Carlo schemes is to use simulation methods to approximate integrals in the form of expectations.

Suppose $f(\cdot)$ is some function such that $f : [0,1] \longrightarrow \mathbb{R}$. The basic problem is to evaluate the integral

$$I = \int_0^1 f(x)\, dx$$

i.e. diagram



Consider e.g. the earlier problem $f(x) = e^{x^2}$, for which an analytical solution cannotbe obtained.

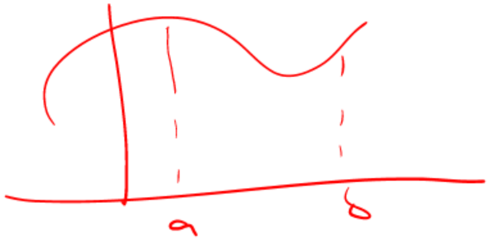
Note that if $U \sim U(0,1)$ then

$$\mathbb{E}[f(U)] = \int_0^1 f(u)\, p(u)\, du \qquad = 1$$

Python Matlab C++

Java

where the density $p(u)$ of a uniformly distributed random variable $U(0,1)$ is given earlier. Hence



$$\begin{aligned}\mathbb{E}[f(U)] &= \int_0^1 f(u)\,p(u)\,du \\ &= I.\end{aligned}$$

So the problem of estimating I becomes equivalent to the exercise of estimating $\mathbb{E}[f(U)]$ where $U \sim U(0,1)$.

$\int_1^3 x^2\,dx$

Very often we will be concerned with an arbitrary domain, other than $[0,1]$. This simply means that the initial part of the problem will involve seeking a transformation that converts $[a,b]$ to the domain $[0,1]$. We consider two fundamental cases.

$\int_0^1 x^5\,dx = 0.1667$

1. Let $f(\cdot)$ be a function s.t. $f : [a,b] \longrightarrow \mathbb{R}$ where $-\infty < a < b < \infty$. The problem is to evaluate the integral

Transf^n | Subst^n

Model problem : $$I = \int_a^b f(x)\,dx.$$

$\int_0^1$

$a + (b-a)y = x$

$(b-a)\,dy = dx$

In this case consider the following substitution

$x = a \quad y = 0$

$x = b \quad y = 1$

$$y = \frac{x-a}{b-a}$$

$(b-a)y = x - a$

which gives $dy = dx/(b-a)$. This gives

$y \sim U(0,1)$

$$\begin{aligned} I &= (b-a)\int_0^1 f\,(\overbrace{y \times (b-a) + a}^{x})\,\boxed{dy} \\ &= (b-a)\,\mathbb{E}\left[f\,(U \times (b-a) + a)\right] \end{aligned}$$

where $U \sim U(0,1)$. Hence I has been expressed as the product of a constant and expected value of a function of a $U(0,1)$ random number; the latter can be estimated by simulation.

2. Let $g(\cdot)$ be some function s.t. $g : [0, \infty) \to \mathbb{R}$ ~~where $-\infty < a < b < \infty$.~~
The problem is to evaluate the integral

$$\longrightarrow \quad I = \int_0^\infty g(x)\,dx,$$

(ge)

$\int_0^\infty$

provided $I < \infty$. So this is the area under the curve $g(x)$ between 0 and ∞. In this case use the following substitution

$x=0 \quad y=1$

$x=\infty \quad y=0$

$\int^0$

$$y = \frac{1}{1+x}$$

$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

which is equivalent to $x = -1 + \frac{1}{y}$. This gives

$\int_0^1$

$\int \boxed{\frac{1}{1+x^2}} dx$

$$\begin{aligned} dy &= -dx/\underbrace{(1+x)^2} \\ &= -y^2 dx. \end{aligned}$$

$= -\left(\frac{1}{1+x}\right)^2$

$\frac{dy}{dx} = -y^2$

The resulting problem is

$x \in [0, \infty)$ $\quad \tan^{-1}(x)$

$$\begin{aligned} I &= \int_0^1 \frac{g\left(\frac{1}{y}-1\right)}{y^2} dy \\ &= \mathbb{E}\left[\frac{g\left(-1+\frac{1}{U}\right)}{U^2}\right] \end{aligned}$$

$dx \quad \frac{1}{-y^2} dy = dx$

$\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_{0}^{\infty}$

where $U \sim U(0,1)$. Hence I has again been expressed as the expected value of a function of a $U(0,1)$ random number; to be estimated by simulation.