Continuation Value is All You Need

"Drop-In" Deep Learning for HA Models with Aggregate Shocks

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Introduction

Introduction

- Heterogeneous agent (HA) models with aggregate shocks are infeasible to solve in general using conventional methods
- Neural Value Function Iteration (nVFI): a deep learning global solution method
  - Far more general than existing methods
  - Current limitations: Discrete time, rational expectations

Can globally solve models of:

- Search and matching, space, strategic interaction, rich frictions, behavioural biases, costly adjustment, segmented markets, multiple assets, OLG, any combination of these, and more!

Idea

nVFI key idea:

1. Train a neural network to approximate the end-of-period value function ("continuation value") conditional on beginning-of-period aggregate state
2. Use conventional methods for everything else

Philosophy: Rely on neural networks **as little as possible**.

Literature

- Projection/Local Perturbation Methods:
  - Bhandari et al. (2023), Bilal (2023), Auclert et al. (2021), Winberry (2018), etc.
- Deep Policy Function Approximation:
  - Han et al. (2024), Azinovic et al. (2022), Maliar et al. (2021), etc.
  - Continuous-Time: Gu et al. (2024), Fernández-Villaverde et al. (2023), etc.
- Non-Neural Value Function Approximation:
  - Krusell and Smith (1998), Hull (2015)
  - nVFI extends these methods with deep learning and massive generality

Key Departure

Key departure from literature on ML for HA models with aggregate shocks:

- Use **conventional** methods to solve for **everything** but estimated continuation values:
- In particular, use conventional methods to solve the **policy** function
- Also: prices, all intermediate value functions, evolution of state, etc.

One perspective:

- nVFI is Krusell-Smith, replacing their "finite-moments value function" with a "neural net value function"

Another perspective:

- nVFI is Approximate Dynamic Programming (ADP) applied to heterogeneous-agent general equilibrium models

Overview

1. Simple implementation for Krusell-Smith model
2. Accuracy evaluation
3. Full generality
4. Implementation details

nVFI Application to Krusell-Smith Model

Purpose of Section

- This section describes an implementation of nVFI for the Krusell-Smith model
- The next section describes how to add massive generality to this approach

Krusell-Smith Model

Time is discrete. A measure one distribution μ_t of infinitely-lived households i vary by:

- Capital k_{it}, endogenously; and
- Productivity z_{it}, exogenously, subject to Markov chain transitions.

Each household i maximizes

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \log(c_{it})\right]$$

s.t.

$$c_{it} + k_{it+1} = (1 + r_t - \delta)k_{it} + w_t z_{it}$$

Market

There is one good. Production is Cobb-Douglas in capital K_t and labour L_t,

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

where A_t is an aggregate TFP shock following a Markov process, and

$$K_t = \int k_{it} d\mu_t(i), \qquad L_t = \int z_{it} d\mu_t(i).$$

Markets are competitive, so that,

$$w_t = (1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha}, \qquad r_t = \alpha\left(\frac{K_t}{L_t}\right)^{\alpha-1}.$$

Recursive Household Problem

The household's problem can be formulated recursively as,

$$V(k, z; \Gamma, A) = \max_{c,k'} \log(c_t) + \beta \mathbb{E}\left[V(k', z'; \Gamma, A)\right]$$

s.t.

$$c + k' = (1 + r - \delta)k + wz$$

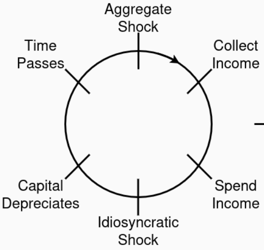
Timing Definitions

To set things up for nVFI, some timing-related definitions:

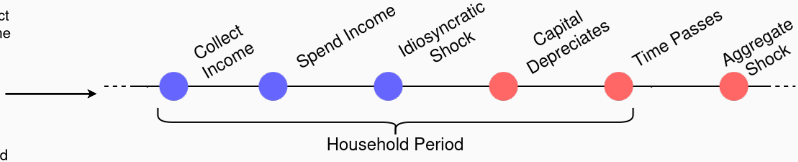
1. **Household Period**
2. **Idiosyncratic** and **Aggregate State**
3. **Start-of-Period** and **End-of-Period Value Functions**

Household Period

Denote the spell between aggregate shocks the "household period,"



Model As A Circle



One Period of Model

State Variables

The idiosyncratic (household) state at start and end of period t,

$$x_{it}^{\text{start}} \equiv (k_{it}, z_{it}) \in X^{\text{start}}$$
$$x_{it}^{\text{end}} \equiv (k_{it+1}, z_{it+1}) \in X^{\text{end}}$$

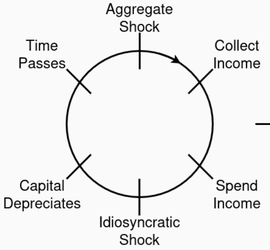
Aggregate state at start and end of household period t,

$$\Lambda_t^{\text{start}} \equiv (\mu_t, A_t)$$
$$\Lambda_t^{\text{end}} \equiv (\mu_{t+1}, A_t)$$

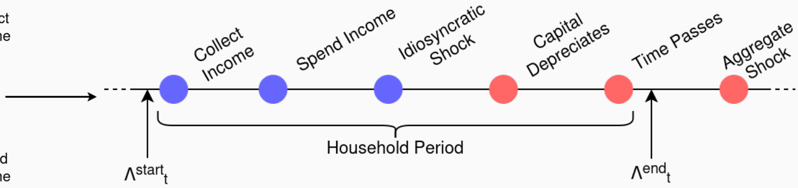
Note: Aggregate shock A_t occurs **between** Λ_t^{end} and $\Lambda_{t+1}^{\text{start}}$!

Timing

Denote the spell between aggregate shocks the "household period,"



Model As A Circle



One Period of Model

Value Functions

V^{start} is the familiar value function V, viewed from the **start** of the period.

$$V^{\text{start}}(x_t^{\text{start}}, \Lambda_t^{\text{start}}) \equiv \mathbb{E}\left[\sum_{s=0}^{\infty} \beta^s u_{i,t+s} \;\middle|\; \Lambda_t^{\text{start}}, x_{it}^{\text{start}} = x_t^{\text{start}}\right].$$

V^{end} is the value function viewed from the **end** of the period,

$$\begin{aligned} V^{\text{end}}(x_t^{\text{end}}, \Lambda_t^{\text{start}}) &\equiv \mathbb{E}\left[\sum_{s=0}^{\infty} \beta^s u_{i,t+s+1} \;\middle|\; \Lambda_t^{\text{start}}, x_{it}^{\text{end}} = x_t^{\text{end}}\right]. \\ &= \mathbb{E}_{A_{t+1}}\left[V^{\text{start}}\left(x_t^{\text{end}}, (\mu_{t+1}, A_{t+1})\right) \mid A_t\right] \end{aligned}$$

Note that V^{end} can be conditioned on Λ_t^{start} rather than Λ_t^{end} because no aggregate information is revealed during the household period.

Recursive Global Solution in Two Equations

Within-Period Problem (WPP):

$$V^{\text{start}}(x^{\text{start}}, \Lambda^{\text{start}}) = \max_{c,k'} \log(c) + \beta \mathbb{E}_{z'}\left[V^{\text{end}}((k', z'), \Lambda^{\text{start}}) \mid z\right] \quad \forall x^{\text{start}} \in X^{\text{start}} \tag{1}$$
$$\text{s.t.} \quad c + k' = (1 + r(\Lambda^{\text{start}}) - \delta)k + w(\Lambda^{\text{start}})z$$

Continuation Value Problem (CVP):

$$V^{\text{end}}(x^{\text{end}}, \Lambda^{\text{start}}) = \mathbb{E}_{A_{t+1}}\left[V^{\text{start}}\left(x^{\text{end}}, (\mu_{t+1}, A_{t+1})\right) \mid A_t\right] \quad \forall x^{\text{start}} \in X^{\text{start}}. \tag{2}$$

WPP solvable with conventional methods **within** Λ^{start}. No curse of dimensionality!

CVP solvable by neural network **across** Λ^{start}, also overcomes curse of dimensionality!

(Later: Massive complexity by adding complexity to WPP.)

Neural Network Value Function Approximator

Let $\mathcal{V}^{\text{end}}$ be a neural network approximator for V^{end} with tunable parameters θ:

$$\widehat{V_\theta^{\text{end}}}(x^{\text{end}}, \Lambda^{\text{start}}) \equiv \mathcal{V}^{\text{end}}(x^{\text{end}}, \Lambda^{\text{start}}; \theta).$$

Crucially, we can approximate the RHS of the CVP by simulation and backwards induction:

1. Estimate μ' by solving, for all x^{start},
$$\text{argmax}_{c,k'} \log(c) + \beta \mathbb{E}_{z'}\left[\widehat{V_\theta^{\text{end}}}((k', z'), \Lambda^{\text{start}}) \mid z\right].$$
2. For each $x^{\text{end}\prime}$ and possible A', estimate **end of next period** $\widehat{V_\theta^{\text{end}}}(x^{\text{end}\prime}, (\widehat{\mu'}, A'))$.
3. Compute $\forall x^{\text{start}\prime} \in X^{\text{start}}$,
$$\widehat{V_\theta^{\text{start}}}\left(x^{\text{start}\prime}, (\widehat{\mu'}, A')\right) = \max_{c',k''} \log(c') + \beta \mathbb{E}_{z''}\left[\widehat{V_\theta^{\text{end}}}((k'', z''), (\widehat{\mu'}, A')) \mid z'\right].$$
4. CVP: $\widehat{V_\theta^{\text{end}}}(x^{\text{end}}, \Lambda^{\text{start}}) = \mathbb{E}_{A'}\left[\widehat{V_\theta^{\text{start}}}(x^{\text{end}}, (\widehat{\mu'}, A')) \mid A\right].$

Continuation Value is All You Need

- Note that the global solution is characterized by the WPP and the CVP problem.
- Furthermore, the WPP is solvable using conventional methods! That is, it remains only to solve the CVP.
- In machine learning terms, we want to minimize the loss function,

$$\int \left(\widehat{V_\theta^{\text{end}}}(x^{\text{end}}, \Lambda^{\text{start}}) - \mathbb{E}_{A'}\left[\widehat{V_\theta^{\text{start}}}\left(x^{\text{end}}, (\widehat{\mu}', A')\right) \mid A\right]\right)^2 dx^{\text{end}}.$$

nVFI for Krusell-Smith Overview

1. Initialize θ somehow.
2. Draw N samples Λ_i^{start} somehow.
3. For each i, compute,

$$\varepsilon_i(\Lambda_i^{\text{start}};\theta) = \int \left(\widehat{V_\theta^{\text{end}}}(x^{\text{end}},\Lambda^{\text{start}}) - \mathbb{E}_{A'}\left[\widehat{V_\theta^{\text{start}}}\left(x^{\text{end}},(\widehat{\mu}',A')\right) \mid A\right]\right)^2 dx^{\text{end}}.$$

4. Compute the gradient,

$$g_i \longleftarrow \frac{\partial}{\partial\theta}\varepsilon_i(\Lambda_i^{\text{start}};\theta)$$

5. Update $\theta \longleftarrow \theta - \alpha\frac{1}{N}\sum_i g_i$
6. Repeat from Step 2 until convergence.

General Algorithm

Construction

1. Define a one-period model
2. Define an aggregate shock transition function
    - This transforms it into a recursive dynamic model
3. Apply nVFI to the recursive model

One-Period Model

- Consider a one-period model with initial aggregate state Λ^{start} and end-of-period payoffs $V^{\text{end}}(x^{\text{end}})$.
- Such models are widespread. They include search-and-matching, space, strategic interaction, rich frictions, behavioural biases, costly adjustment, segmented markets, multiple assets, OLG, and combination of these, and more

One-Period Model

For this one-period model, we specify that:

- You can already solve it
- There is a unique, aggregate-deterministic solution conditional on Λ^{start} and V^{end}.

Then:

- This solution should include initial value function $V^{\text{start}}(x^{\text{start}})$ and end-of-period state Λ^{end}.
- Massively general: **only require** that you have access to some function:

$$\left(\Lambda_t^{\text{end}}, V_t^{\text{start}}\right) = \Phi\left(\Lambda_t^{\text{start}}, V_t^{\text{end}}\right)$$

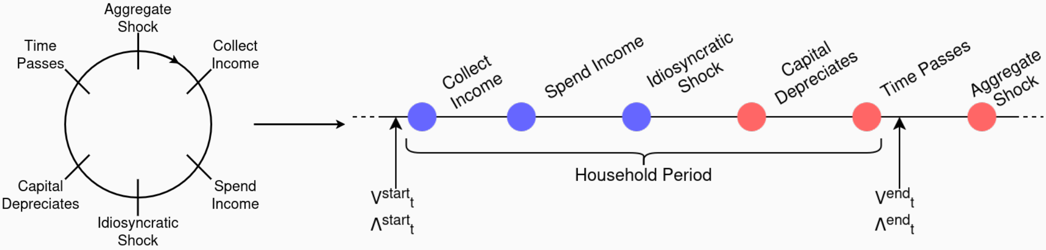
Aggregate Shock/Transition

Provide a function Ω that describes how an aggregate shock Z_t affects the aggregate state:

$$\Lambda_{t+1}^{\text{start}} = \Omega\big(\Lambda_t^{\text{end}}, Z_t\big)$$
$$Z_t \sim \mathcal{D}()$$

Timing

Denote the spell between aggregate shocks the "household period." For a given t,



Model As A Circle

One Period of Model

Denote the value function and aggregate state at start and end of household period,

$$V_t^{\text{start}}, V_t^{\text{end}} \quad \text{and} \quad \Lambda_t^{\text{start}}, \Lambda_t^{\text{end}}.$$

Definition: Λ Definition: V

Continuation Value Problem

Continuation Value Problem (CVP): Find an operator,

$$\mathcal{V} : \Lambda_t^{\text{start}} \mapsto V_t^{\text{end}}(\cdot, \Lambda_t^{\text{start}}).$$

Computationally, think of Λ_t^{start} and V_t^{end} as **vectors** whose basis is the state space X^{start} or X^{end}.

$\mathcal{V}$ Is Enough For Simulation

Given some candidate,

$$\mathcal{V}_0 : \Lambda_t^{\text{start}} \mapsto V_t^{\text{end}},$$

then you could simulate forward:

$$\begin{aligned}
\Lambda_t^{\text{end}} &= \Phi_1\big(\Lambda_t^{\text{start}}, \mathcal{V}\big(\Lambda_t^{\text{start}}\big)\big) \\
\Lambda_{t+1}^{\text{start}} &= \Omega\big(\Lambda_t^{\text{end}}, Z_t\big) \\
\Lambda_{t+1}^{\text{end}} &= \Phi_1\big(\Lambda_{t+1}^{\text{start}}, \mathcal{V}\big(\Lambda_{t+1}^{\text{start}}\big)\big) \\
&\dots
\end{aligned}$$

Simulation Operators

Given some candidate $\mathcal{V}_0$, define the simulation operators,

$$(\Lambda^{\text{end}}, V^{\text{start}}) \quad = \quad \widetilde{\Phi}\big(\Lambda^{\text{start}}; \mathcal{V}_0\big) \quad \equiv \quad \Phi\big(\Lambda^{\text{start}}, \mathcal{V}_0(\Lambda^{\text{start}})\big),$$

and

$$\Lambda^{\text{start}\prime} = \Gamma(\Lambda^{\text{start}}; \mathcal{V}_0, Z) \quad \equiv \quad \Omega(\widetilde{\Phi}_1\big(\Lambda^{\text{start}}; \mathcal{V}_0\big), Z),$$

both of which are **feasible** to compute.

Recursive Definition of $\mathcal{V}$

The **true** $\mathcal{V} : \Lambda^{\text{start}} \mapsto V^{\text{end}}$ can be recursively defined as:

$$
\begin{aligned}
&\forall\ \Lambda^{\text{start}} \in (\text{Aggregate State Space}) \\
V^{\text{end}} \equiv \mathcal{V}(\Lambda^{\text{start}}) &= \mathbb{E}_Z\left[V^{\text{start}\prime} \;\middle|\; \Lambda^{\text{start}} \right] \\
\text{where} \quad V^{\text{start}\prime} &\equiv \widetilde{\Phi}_2\left(\Lambda^{\text{start}\prime}; \mathcal{V}\right) \\
\Lambda^{\text{start}\prime} &\equiv \Gamma(\Lambda^{\text{start}}; \mathcal{V}, Z)
\end{aligned}
$$

i.e. "Projecting 1 step forward" = "Projecting 2 steps forward and solving 1 step back".

$\mathcal{V}$ As A Fixed Point

$\mathcal{V}$ is thus a fixed point of the following “lookahead” operator:

$$\mathrm{L}\mathcal{V}_0(\Lambda^{\text{start}}) \equiv \mathbb{E}_Z\Big[\widetilde{\Phi}_2\big(\Gamma(\Lambda^{\text{start}};\mathcal{V}_0,Z);\mathcal{V}_0\big)\Big].$$

It remains only to train a neural network $\mathcal{V}_0$ to minimize:

$$\left\|\mathcal{V}_0(\Lambda^{\text{start}}) - \mathrm{L}\mathcal{V}_0(\Lambda^{\text{start}})\right\|.$$

When $\mathcal{V}$ is a neural network, this is the “Neural Bellman Equation.”

Neural Bellman Equation

Explicitly, for a neural network $\mathcal{N}_\theta$ with parameters θ,

$$\mathrm{NBE}(\theta; \Lambda^{\text{start}}) = \mathcal{N}_\theta(\Lambda^{\text{start}}) - \mathbb{E}_Z\left[\widetilde{\Phi}_2(\Gamma(\Lambda^{\text{start}}; \mathcal{N}_\theta, Z); \mathcal{N}_\theta)\right]$$

Measures the failure of $\mathcal{N}_\theta$ to produce a lookahead-consistent V^{end} projection at Λ^{start}.

Algorithm Sketch

Let $\mathcal{N}_\theta$ be a neural network with parameters θ.

1. Initialize θ somehow.
2. Draw N samples Λ_i^{start} somehow.
3. For each i:
    i. Compute $\widehat{V^{\text{end}}}_i \longleftarrow L\mathcal{N}_\theta(\Lambda_i^{\text{start}})$
    ii. Compute,
$$g_i \longleftarrow \frac{\partial}{\partial \theta} \left\| \mathcal{N}_\theta(\Lambda_i^{\text{start}}) - \widehat{V^{\text{end}}}_i \right\|^2$$
4. Update $\theta \longleftarrow \theta - \alpha \frac{1}{N} \sum_i g_i$
5. Repeat from Step 2 until convergence.

Implementation

Model Setup

- Standard Krusell-Smith Model Details
- 65 wealth gridpoints, 3 income gridpoints = 195 idiosyncratic gridpoints

Basic Neural Network

Idea: Compose matrix multiplication with elementwise nonlinear function.

This can create arbitrary nonlinear relationships between inputs and outputs.

Parameters: $\theta = (M^1, b^1, M^2, b^2)$

$$
\begin{aligned}
\text{Input: } & x^1 \\
y^1 &\longleftarrow M^1 x^1 + b^1 \\
x^2 &\longleftarrow \mathrm{elu}(y^1) \\
y^2 &\longleftarrow M^2 x^2 + b^2 \\
y^3 &\longleftarrow \mathrm{elu}(y^2) \\
\text{Output: } & y^3
\end{aligned}
$$

Where

$$
\mathrm{elu}(x)_i \equiv \begin{cases} x_i & x_i > 0 \\ e^{x_i} - 1 & x_i \leq 0 \end{cases}
$$

is an **elementwise** nonlinear function ("activation function.")

Automatic Differentiation

The real magic of neural networks: automatic differentiation.

For a “label” ℓ_x associated with each input x, and a neural network $\mathcal{N}_\theta(x)$, the gradient

$$\frac{\partial}{\partial\theta}\|\mathcal{N}_\theta(x), \ell_x\|^2$$

can be quickly and automatically computed.

- Can often thus find θ such that $\mathcal{N}_\theta(x)$ approximates/predicts ℓ_x well.
- Theorem: a sufficiently large neural net can approximate *any* θ arbitrarily well

Specific Neural Net Implementation: Step 1

Step 1: Use "Generalized Moments" mean field game-inspired method of Han, Yang, and E (2025) to summarize household state distribution μ.

$$
\begin{aligned}
&\forall x \in X : \\
\gamma_x^{GM,1} &\longleftarrow \mathrm{elu}\left(M^1_{10\times 2} \begin{pmatrix} k_x \\ z_x \end{pmatrix} + b^{GM,1} \right) \\
\gamma_x^{GM,2} &\longleftarrow \mathrm{elu}\left(M^2_{10\times 10} \gamma_x^{GM,1} + b^{GM,2} \right) \\
\gamma^{GM,3} &\longleftarrow \sum_{x \in X} \mu(x) \gamma_x^{GM,2} \\
\gamma^{GM} &\longleftarrow \mathrm{elu}\left(M^3_{10\times 10} \gamma^{GM,3} + b^{GM,3} \right)
\end{aligned}
$$

Output: γ^{GM}, containing information about aggregate state distribution.

Specific Neural Net Implementation: Step 2

Step 2: Combine state distribution summary γ^{GM} with idiosyncratic state x and aggregate productivity A.

$$\begin{aligned}
\gamma_x^{HH,1} &\longleftarrow \mathrm{elu}\left(M_{10\times 2}^{HH,1}\begin{pmatrix} k_x \\ z_x \end{pmatrix} + b^{HH,1}\right) \\
\gamma^{A,1} &\longleftarrow \mathrm{elu}\left(M_{10\times 1}^{A,1}A + b^{A,1}\right) \\
\gamma^{A} &\longleftarrow \mathrm{elu}\left(M_{10\times 10}^{A,2}\gamma^{A,1} + b^{A,2}\right) \\
\gamma_x^{HH,2} &\longleftarrow \gamma_x^{HH,1} + \gamma^{GM} + \gamma^{A}
\end{aligned}$$

Output: $\gamma_x^{HH,2}$, containing information about idiosyncratic x and aggregate $\Lambda^{\text{start}} = (\mu, A)$.

Specific Neural Net Implementation: Step 3

Step 3: Put $\gamma_x^{HH,2}$ through three more layers:

$$\gamma^{HH,3} \longleftarrow \text{elu}\Big(M_{8\times 10}^{HH,3}\gamma_x^{HH,2} + b^{HH,3}\Big)$$
$$\gamma^{HH,4} \longleftarrow \text{elu}\Big(M_{8\times 8}^{HH,4}\gamma^{HH,3} + b^{HH,4}\Big)$$
$$\gamma^{HH,5} \longleftarrow \text{elu}\Big(M_{5\times 8}^{HH,5}\gamma^{HH,4} + b^{HH,5}\Big)$$
$$\widehat{V}(x;\Lambda^{\text{start}},\theta) \equiv M_{1\times 5}^{HH,6}\gamma^{HH,5} + b^{HH,6}$$

Output: $\widehat{V}(x;\Lambda^{\text{start}},\theta)$, an estimate of the state-contingent value $V(x;\Lambda^{\text{start}})$.

Krusell-Smith Comparison

The Krusell-Smith *method* is essentially the same, but with $\widehat{V}$ given by,

$$\overline{k} \longleftarrow \sum_{x \in X} \mu(x) k_x$$

$$\gamma^{HH} \longleftarrow \begin{pmatrix} v(k,1;\overline{k},z_g) & v(k,1;\overline{k},z_b) & v(k,0;\overline{k},z_g) & v(k,0;\overline{k},z_b) \end{pmatrix}'$$

$$\widehat{V}^{KS}(x;A,\mu) \equiv \begin{pmatrix} \mathbb{1}_{[A=1,z_x=z_g]} & \mathbb{1}_{[A=1,z_x=z_b]} & \mathbb{1}_{[A=0,z_x=z_b]} & \mathbb{1}_{[A=0,z_x=z_b]} \end{pmatrix} \gamma^{HH}$$

No need to interpret as "predicting the law of motion of aggregate capital." It's just value function approximation!

Data Representation

- Represent V_t^{start}, ${\mu^{\text{start}}}_t$ by data arrays $A_t^{V^{\text{start}}}$, $A_t^{\mu^{\text{start}}}$. Similarly, $A_t^{V^{\text{end}}}$, $A_t^{\mu^{\text{end}}}$.
- Φ is then implemented as a function taking arrays to arrays:

$$\Phi : \left(A^{V^{\text{end}}}, A^{\Lambda^{\text{start}}}, S_t^{\text{start}}\right) \mapsto \left(A^{V^{\text{start}}}, A^{\Lambda^{\text{end}}}, S_t^{\text{end}}\right)$$

  where S_t^{start} and S_t^{end} are other state variables.
- Implemented **conventionally** (no neural net)

Algorithm Sketch

Let $\mathcal{N}_\theta$ be a neural network with parameters θ.

1. Initialize θ somehow. $\longleftarrow$ We Are Here
2. Draw N samples Λ_i^{start} somehow.
3. Compute $\widehat{V^{\text{end}}}_i \longleftarrow L\mathcal{N}_\theta(\Lambda_i^{\text{start}})$.
4. For each i, compute,
$$g_i \longleftarrow \frac{\partial}{\partial\theta} \left\| \mathcal{N}_\theta(\Lambda_i^{\text{start}}) - \widehat{V^{\text{end}}}_i \right\|^2$$
5. Update $\theta \longleftarrow \theta - \alpha \frac{1}{N} \sum_i g_i$
6. Repeat from Step 2 until convergence.

Initializing θ

Starting with a completely random θ may lead to terrible initial simulations.

Several options here:

1. Pre-train θ to minimize

$$\left\| \mathcal{N}_\theta(\Lambda^{\text{start}})(x) - \left(k_x + \frac{z_x}{1-\beta} \right) \right\|$$

2. Pre-train θ on $(\Lambda^{\text{start}}, V^{\text{end}})$ data-label pairs from:
   2.1 Deterministic version of model
   2.2 Krusell-Smith *method* solution of full model

Algorithm Sketch

Let $\mathcal{N}_\theta$ be a neural network with parameters θ.

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3. Compute $\widehat{V^{\text{end}}}_i \longleftarrow L\mathcal{N}_\theta(\Lambda_i^{\text{start}})$.
4. For each i, compute,
$$g_i \longleftarrow \frac{\partial}{\partial\theta} \left\| \mathcal{N}\theta(\Lambda_i^{\text{start}}) - \widehat{V^{\text{end}}}_i \right\|^2$$
5. Update $\theta \longleftarrow \theta - \alpha \frac{1}{N} \sum_i g_i$
6. Repeat from Step 2 until convergence.

Sampling Λ^{start}

Options:

1. Draw from simple (e.g. uniform) distribution
2. Draw from ergodic distribution induced by θ
3. Draw from ergodic distribution induced by some other $\mathcal{V}_0$, in MCMC spirit
    - E.g. Krusell-Smith
4. Take current data state as starting point, sample from possible states within T periods of present

Sampling Λ^{start}

I use MCMC-style sampling:

1. Simulate forward T "burn-in" periods using
$$\Lambda^{\text{start}}_{t+1} \longleftarrow \Gamma(\Lambda^{\text{start}}_t; \mathcal{N}_\theta, Z)$$
2. For $t > T$, add Λ^{start}_t to sample if $t \equiv 0 (\text{mod } n)$ for some n

Basic challenge: Data generation depends on $\mathcal{N}_\theta$.

Algorithm Sketch

Let $\mathcal{N}_\theta$ be a neural network with parameters θ.

1. Initialize θ somehow.
2. Draw N samples Λ_i^{start} somehow.
3. Compute $\widehat{V^{\text{end}}}_i \longleftarrow L\mathcal{N}_\theta(\Lambda_i^{\text{start}})$. $\longleftarrow$ We Are Here
4. For each i, compute,
$$g_i \longleftarrow \frac{\partial}{\partial\theta}\left\|\mathcal{N}\theta(\Lambda_i^{\text{start}}) - \widehat{V^{\text{end}}}_i\right\|^2$$
5. Update $\theta \longleftarrow \theta - \alpha\frac{1}{N}\sum_i g_i$
6. Repeat from Step 2 until convergence.

Within-Period Problem

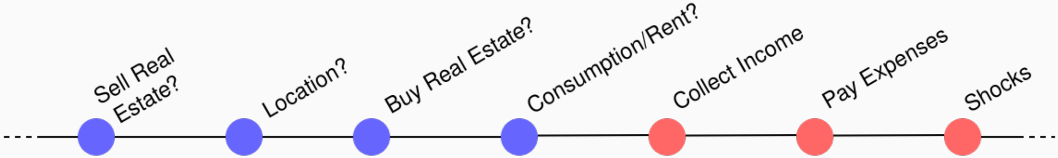
Within-period problem: find,

$$\Phi : \left(\Lambda_t^{\text{start}}, V_t^{\text{end}}\right) \mapsto \left(\Lambda_t^{\text{end}}, V_t^{\text{start}}\right).$$

Performance strategy: break into “stages”: effectively “sub-periods.”

Krusell-Smith model is easy, but we can get much richer.

Sub-Periods: "Conventionally" Solved but *Modular*



Similar decomposition to Auclert et al. (2023)

Multiplicative performance benefits:

1. Simple, optimized, prepackaged for CPU/GPU/cluster
2. Search over one dimension at a time
3. Identify and overcome bottlenecks (interpolation, grid search) in non-vectorized Julia

Stage Example

When individuals experience an idiosyncratic shock with transition matrix M,

$$V^{\text{start}} = M' V^{\text{end}}$$

and

$$\mu^{\text{end}} = M \mu^{\text{start}}.$$

Where each V is a matrix representing a value function over the **idiosyncratic** state space, and each μ is a matrix representing a distribution over the **idiosyncratic** state space.

Fixing Bottlenecks: Dynamic Information Sharing

Example: Consumption-saving by grid search

- If $MPC \geq 0$, then my optimal saving is between my wealth-neighbors'
- Don't need to search over entire axis!
- By "sharing" information between wealth-neighbors: $O(N^2) \to O(N \log N)$
- Incompatible with vectorization (Python, Matlab) but fast in Julia

Pre-Packaged Stages

Another benefit of stages: can be pre-prepared

- Possibly separate project
- Aim to provide highly efficient state implementations for CPU, GPU, and cluster

Household Problem Code Example

```
function solve_period!(prealloc, V_next, params)
    V_preshock = get_V_preshock(prealloc, V_next)

    V_consume = get_V_preconsume(V_preshock, prealloc)

    V_income = get_V_preincome(V_consume, prealloc, params)
    enforce_borrowing_constraint!(V_preincome, prealloc)

    V_premove = get_V_premove(V_preincome, prealloc, params)

    V_end = YOUR_CODE_HERE(V_premove, prealloc, params)
end
```

Solving for Prices

Options:

1. Solve analytically (e.g. Krusell-Smith)
2. Solve via rootfinding (accurate but slow)
3. Use additional neural network (e.g. Azinovic et al. (2022), Deep Equilibrium Nets)

Within-Period Problem

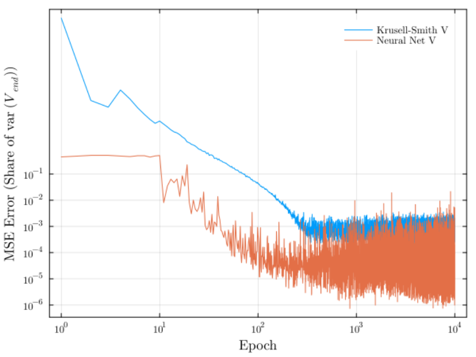
Challenge: Performance of Within-Period Solver is crucial.

Accuracy

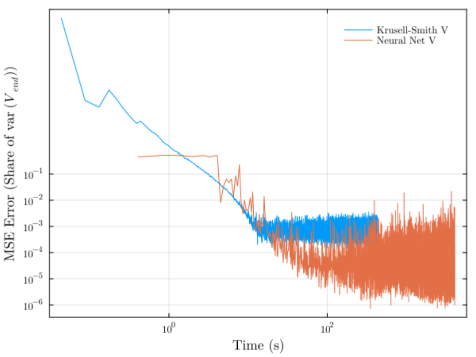
Accuracy

- Strength: $\Phi : (\Lambda_t^{\text{start}}, V_t^{\text{end}}) \mapsto (\Lambda_t^{\text{end}}, V_t^{\text{start}})$ is immediately correct from the beginning
- Only need to check Neural Bellman Equation error
- Weakness: Computing Euler Equation errors nontrivial
- Hardware: One laptop CPU thread (i9-13900H)

Learning Curve Comparison



V error as share of V variance by training epoch



Relative V error by training time (CPU)

- Both stop improving within about 300 epochs (122s for NN, 13s for KS)
- After Epoch 300, mean $\log_{10}$ error is -4.431 for NN, -3.152 for KS

Conclusion

Conclusion

- Describe a solution method for HA models with agg. shocks that overcomes curse of dimensionality (of aggregate state)
- Method uses neural nets only where needed – solution otherwise conventional
- Much complexity is offloaded to **Within-Period Problem** solver
- In other work, provide tools to implement WPP flexibly, easily, performantly

Discussion

- Key advantages:
  - Complex household problems supported
  - No need to train policy network
- Disadvantages:
  - Individual state must be low-dimensional (≤ 6 or so)
  - In more complex models, prices require inner loop around IPP or price neural net à la Azinovic et al. (2022)
- Future work:
  - Assess other error metrics, e.g. Euler equation error, risk premia
  - Compare economics of solutions

Appendix

Definition: Aggregate State

Λ_t^{start} (Λ_t^{end}) is the aggregate state at the start (end) of household period t.

$$\Lambda_t^{\text{start}} \equiv \left(\mu_t^{\text{start}}, S_t^{\text{start}}\right) \qquad\qquad \left(\Lambda_t^{\text{end}} \equiv \left(\mu_t^{\text{end}}, S_t^{\text{end}}\right)\right)$$

where at the start (end) of household period t,

- μ_t^{start} (μ_t^{end}) is the distribution of idiosyncratic states $x^{\text{start}} \in X^{\text{start}}$ ($x^{\text{end}} \in X^{\text{end}}$).
- S_t^{start} (S_t^{end}) is all other aggregate state.

Without loss of generality, parametrize the aggregate shock so that,

$$\Lambda_{t+1}^{\text{start}} = \Omega(\Lambda_t^{\text{end}}, Z_t) \quad \text{where} \quad Z_t \sim \mathcal{D}().$$

Definition: Value Function

For household i, $V_t^{\text{start}}(x_i)$ is NPV of future utility given state x_i^{start} at the beginning of t,

$$
\begin{aligned}
V_t^{\text{start}}(x_i) \quad &\equiv \quad V^{\text{start}}(x_{it}, \Lambda^{\text{start}}) \quad \equiv \quad \mathbb{E}\left[\sum_{s=0}^{\infty} \beta^s u_{i,t+s} \;\middle|\; x, \Lambda^{\text{start}} \right]. \\
V_t^{\text{end}}(x_i) \quad &\equiv \quad V^{\text{end}}(x_{it}, \Lambda^{\text{start}}) \quad \equiv \quad \mathbb{E}\left[\sum_{s=1}^{\infty} \beta^s u_{i,t+s} \;\middle|\; x, \Lambda^{\text{start}} \right].
\end{aligned}
$$

- V_t^{start} refers to the whole *function*, e.g. an array on a grid over X^{start}.
- Conditioning on Λ_t^{start} and Λ_t^{end} are equivalent by aggregate-determinism within t.

Algorithm Details

- I use a gridded CDF representation of Λ, but using a finite number of agents is also possible. However, they have to interpolate over continuation V
- Training data for each simulated period is $A_{it}^{V\text{end}}$ together with

$$\mathbb{E}\left[A_{i,t+1}^{V,\text{start}} \mid \Gamma_{it}\right] = \frac{1}{K}\sum_{k=1}^{K} \text{IPP}_V\left(\mathcal{N}(\Omega(\Gamma_{it}, k)\ ;\ \theta_i), A_{it}^{\Lambda\text{end}}, \Omega(\Gamma_{it}, k)\right)$$

- For large models, if memory is constrained, you can update θ_i as you go, accumulating gradients but not storing the entire simulation

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Neural Network Implementation

Neural net $\mathcal{N}\left(A_{\Gamma}^{\Lambda\text{start}} ; \theta\right)$ has following components:

1. Generalized-moment of Han et al. (2023):

$$\mathrm{GM}_{\Gamma} = \sum_{j \in J} \left(A_{\Gamma}^{\Lambda\text{start}}\right)_j \mathcal{N}_{\mathrm{GM}}(X_j)$$

2. One layer ($1 \Rightarrow 10$) neural net on aggregate productivity A
3. Dense feedforward neural net on input: $(X_j, \mathrm{GM}_{\Gamma}, \mathcal{N}_A(A))$
    - Three hidden layers with 8, 8, and 5 neurons
    - Elu activation

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Krusell-Smith Model Details

- 65 wealth gridpoints
- 3 income gridpoints
- $\beta = 0.98$
- Income process by Tauchen discretization with persistence 0.95 and std 0.1
- Log-linear wealth grid from 1k to 10m
- Income states: 15.4k, 40.3k, 105.4k
- Risk aversion: 0.9
- Capital share: 0.36
- Depreciation rate: 0.025

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Krusell-Smith Method Details

- 5 aggregate capital gridpoints: 100k, 150k, 200k, 250k, 300k
- Linear extrapolation outside aggregate capital grid
- 2 aggregate productivity states: 0.5 and 1.0
- Unlike Krusell and Smith (1998), use gridded CDF population distribution representation for cleaner comparison

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Optimizations

Benchmark

- 1000 locations, 129 wealth states, 5 income types, 6 age groups = 3.87m gridpoints
- Single-thread CPU
- Language: Julia
- Strawman: Jeffrey, May 2023
- One evaluation of household problem
- Initial time: 218s (3m38s)

Low-Hanging Fruit

- **Initial**: 218s
- **Memory Preallocation**: 218s $\rightarrow$ 152s
- **(Almost) Automatic Multithreading**: 152s $\rightarrow$ 49s
- **32 Bit Precision**: 49s $\rightarrow$ 31.9s

Individual Stage Optimizations

Choose Location	25.2s $\to$ 9.78s $\to$ 0.053s
Receive Income	0.38s $\to$ 0.019s
Choose Consumption	0.498s $\to$ 0.025s
Income Shock	4.52s $\to$ 0.028s

Overall: 31.9s $\to$ 0.353s (= 0.126s listed stages + 0.228s other)

Choose Location

$$\text{Let} \qquad V^{\text{start}}_{ts\iota}(\ell) = V^{\text{start}}_{ts}(k_{\iota t-1}, z_{\iota t}, \ell, a_{\iota t}), \quad V^{\text{end}}_{ts\iota}(\ell) \text{ similar}$$

where ι indexes all household types up to location.

The i.i.d. Gumbel location preference shocks imply:

$$\exp\left(\psi V^{\text{start}}_{ts\iota}(\ell)\right) = \sum_{\ell'} \exp\left(\psi\left(V^{\text{end}}_{ts\iota}(\ell') - D_{\ell\ell'}\right)\right)$$

$$P(\ell' = \ell_0 \mid \ell) = \frac{\exp\left(\psi\left(V^{\text{end}}_{ts\iota}(\ell') - D_{\ell\ell'}\right)\right)}{\exp\left(\psi V^{\text{start}}_{ts\iota}(\ell)\right)}$$

$$\lambda^{\text{end}}_{ts\iota}(\ell) = \sum_{\ell'} P(\ell' = \ell \mid \ell)\lambda^{\text{start}}_{ts\iota}(\ell')$$

First optimization: Precompute $\exp\left(\psi V^{\text{start}}_{ts\iota}(\ell)\right)$, then $P(\ell' = \ell_0 \mid \ell)$, then $\lambda^{\text{end}}_{ts\iota}(\ell)$

Time: 25.2s $\rightarrow$ 9.78s

Choose Location

Second optimization: observe that (with $\otimes$ and $\oslash$ elementwise mult. and div.)

$$
\begin{aligned}
\widetilde{V}_{ts}^{\text{start}} &= D\widetilde{V}_{ts}^{\text{end}} \\
\Lambda_{ts}^{\text{end}} &= \widetilde{V}_{ts}^{\text{end}} \otimes \left(D'\Lambda_{ts}^{\text{start}} \oslash \widetilde{V}_{ts}^{\text{start}}\right) \\
\text{where matrices} \quad D_{\ell\ell'} &= \exp\left(-\psi D_{\ell\ell'}\right) \\
\left(\widetilde{V}_{ts}^{\text{start}}\right)_{\ell\iota} &= \exp\left(\psi V_{ts\iota}^{\text{start}}(\ell)\right) \\
\left(\widetilde{V}_{ts}^{\text{end}}\right)_{\ell\iota} &= \exp\left(\psi V_{ts\iota}^{\text{end}}(\ell)\right) \\
\left(\Lambda_{ts}^{\text{end}}\right)_{\ell\iota} &= \lambda_{ts\iota}^{\text{end}}(\ell)
\end{aligned}
$$

No matter the size of the state space, just two matrix multiplications!

Time: 9.78s $\rightarrow$ 0.053s

The Power of Matrix Multiplication

Why is matrix multiplication 200 faster than an explicit loop?

- Surprising algorithms exist to multiply two matrices in as little as $O(n^{2.371552})$ time
- Most CPUs have specialized hardware for matrix multiplication
- Pretty much exactly the same thing works for CES production functions, etc.
- Similar approach to optimizing income shocks, or any finite-state Markov process

Choose Consumption

- Strategy: Gridsearch
- If $MPC \geq 0$, then my optimal saving is between my wealth-neighbors'
- Don't need to search over entire axis!
- By "sharing" information between wealth-neighbors: $O(N^2) \rightarrow O(N \log N)$
- Incompatible with vectorization (Python, Matlab) but fast in Julia
- Similar approach for linear interpolation of many gridpoints

Global Solution

- A neural network is trained to predict V^{end}. Everything else is conventional
- In particular, no neural network used to approximate policy function