Spectral Solver for PDE

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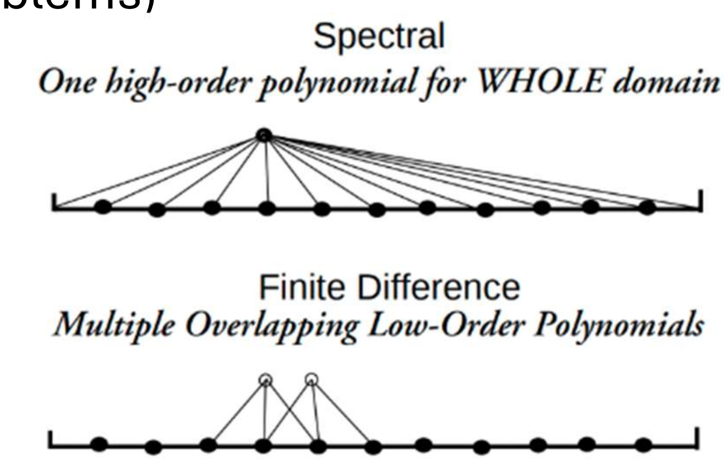
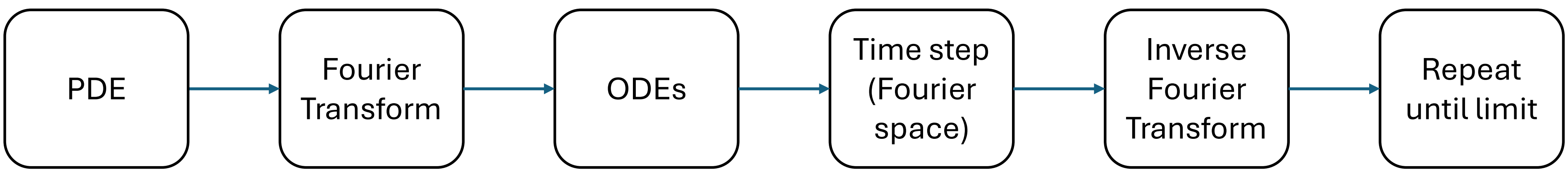
Why Spectral Solvers?

- Traditional methods (finite difference)
  - Approximates using neighbours (local)
- Spectral methods
  - Approximates entire solution across grid
  - Exponentially faster convergence (smooth problems)



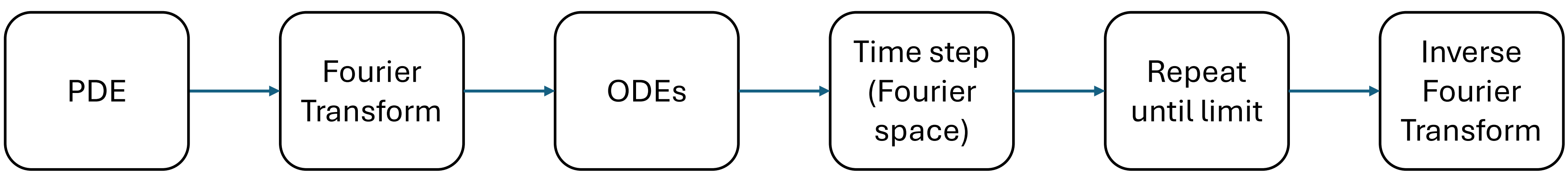
Pseudo-Spectral method workflow

- Transform Spatial domain to Fourier Space (FFT)
- Solve the simple ODEs for Fourier coefficients
- Advance in time
- Inverse transform to physical space (IFFT)
- Repeat until Limit



Full Spectral method workflow

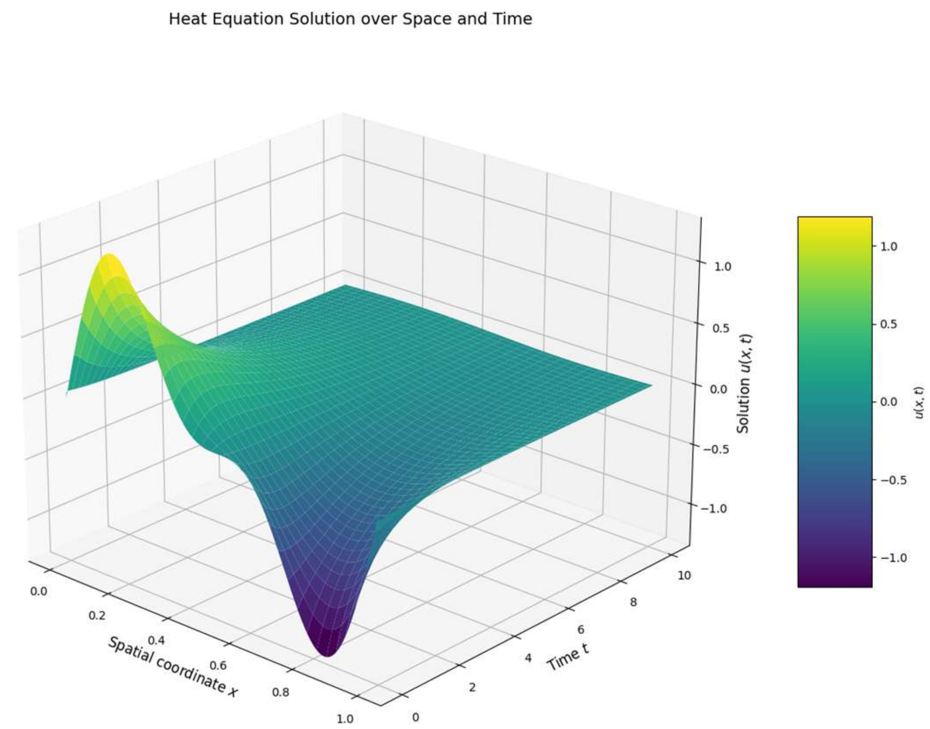
- Transform Spatial & Time domain to Fourier Space (FFT)
- Solve the resulting ODEs with spatial coordinates
- Step in time within Fourier space
- Repeat time stepping until limit
- Transform back to spatial domain (IFFT)



1D Heat

$$u_t = \alpha \nabla^2 u$$

- Fourier transform in x
- Solve ODE
- IFFT back to space
- Time step

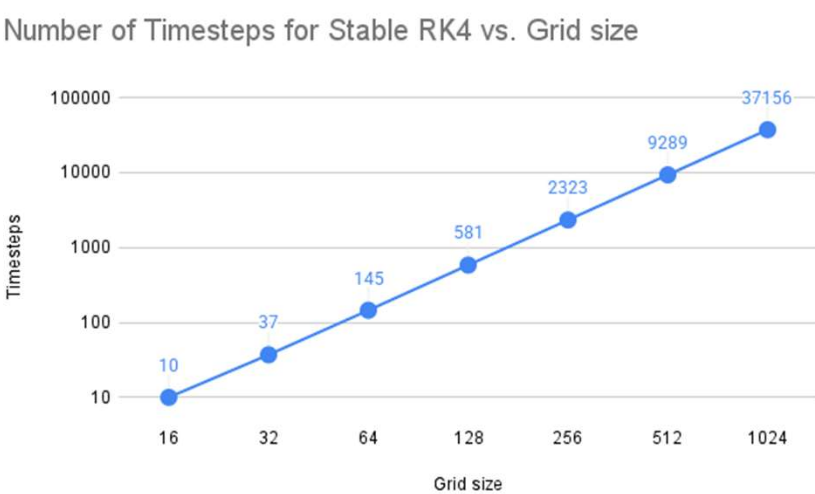


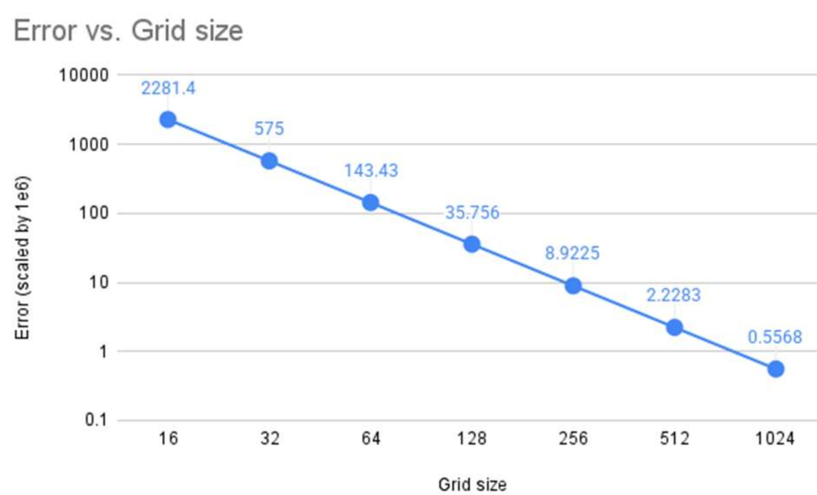
1D Heat convergence

- Start with initial condition
  - $u(x, 0) = \sin(2 \cdot \pi \cdot x)$
- Derive an analytical solution
  - $u(x, t) = e^{-4\pi^2 \alpha t} \cdot u(x, 0)$
- Used full spectral method for convergence testing
- Step forward in time until $t = 1$
- Compute the Euclidian norm of the error

1D Heat convergence: RK4

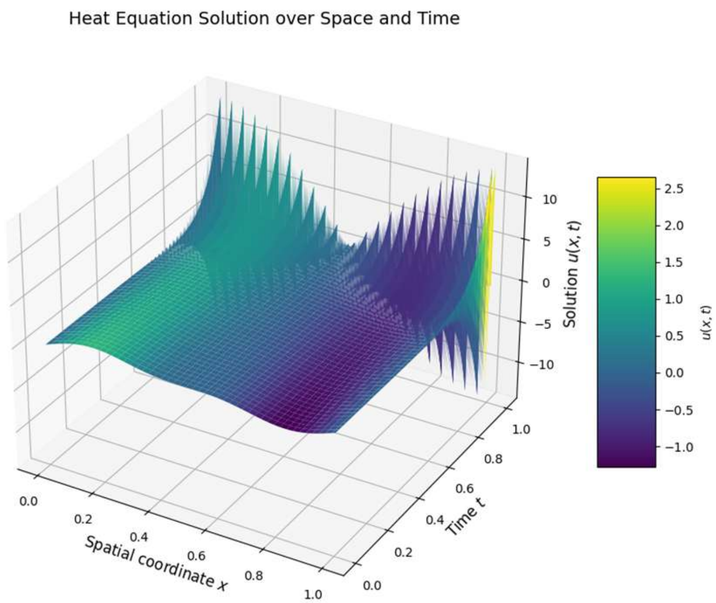
- RK4 does not converge for insufficiently small dt
  - Does not meet stability criteria
- However, once a small enough dt is reached, the error does not improve by further decreasing dt



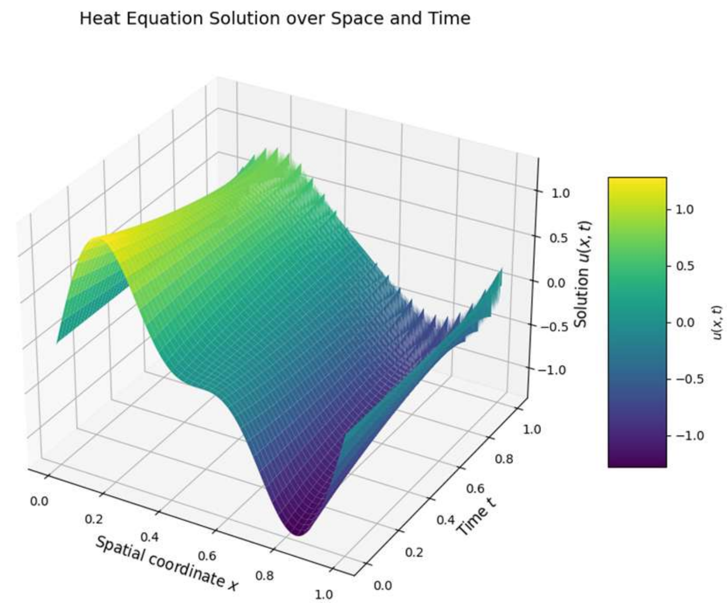


Grid spacing = 1/512

1D Heat convergence: RK4 Stability



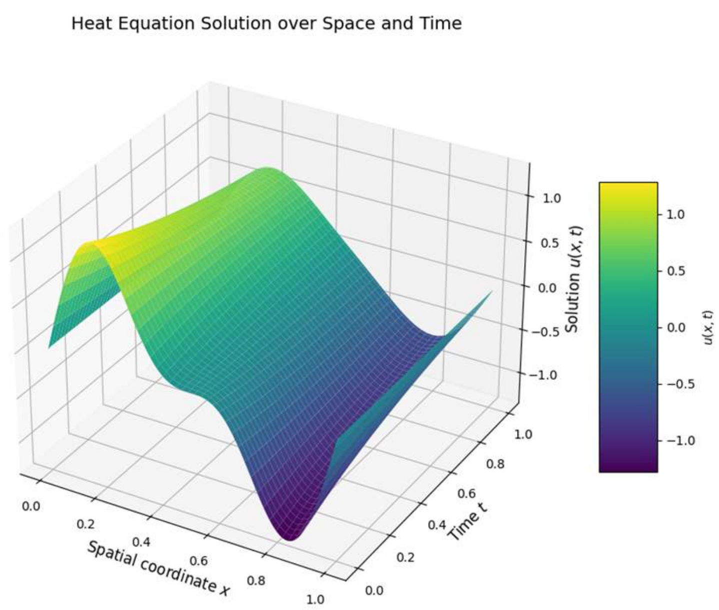
Timesteps: 9286
Error = 23.842

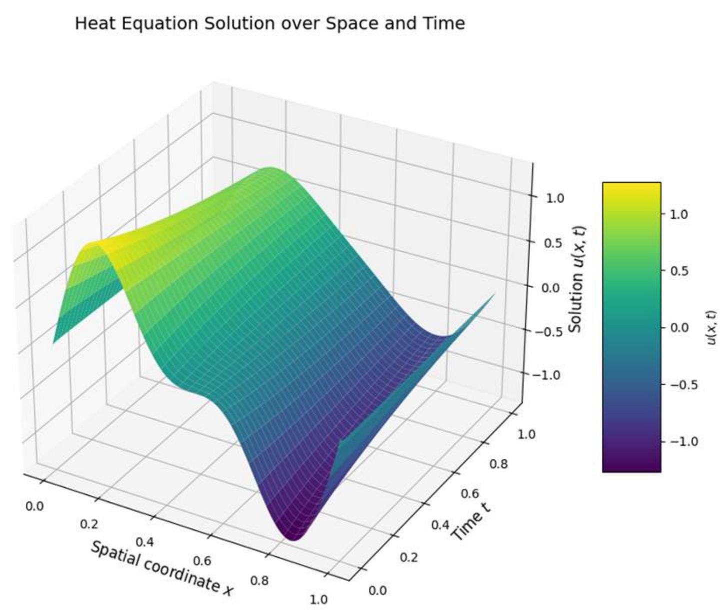


Timesteps: 9287
Error = 5.3729e-3

Grid spacing = 1/512

1D Heat convergence: RK4 Stability



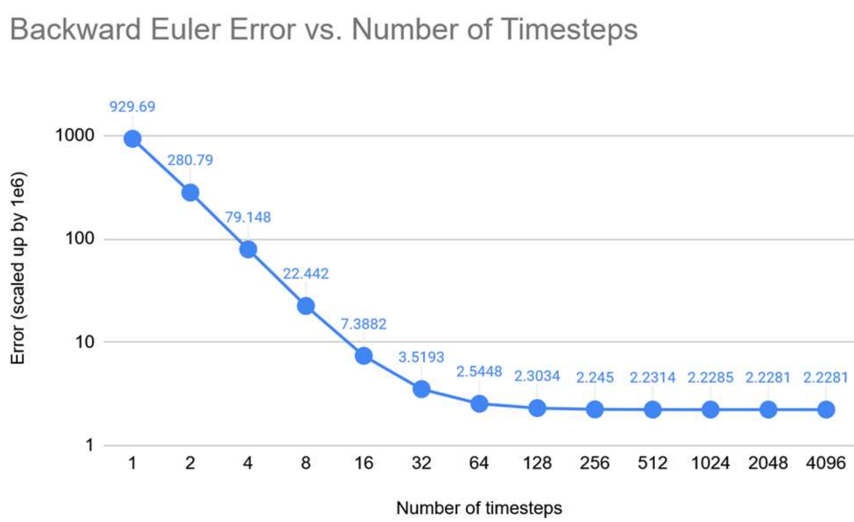


Timesteps: 9289
Error = 2.283e-6

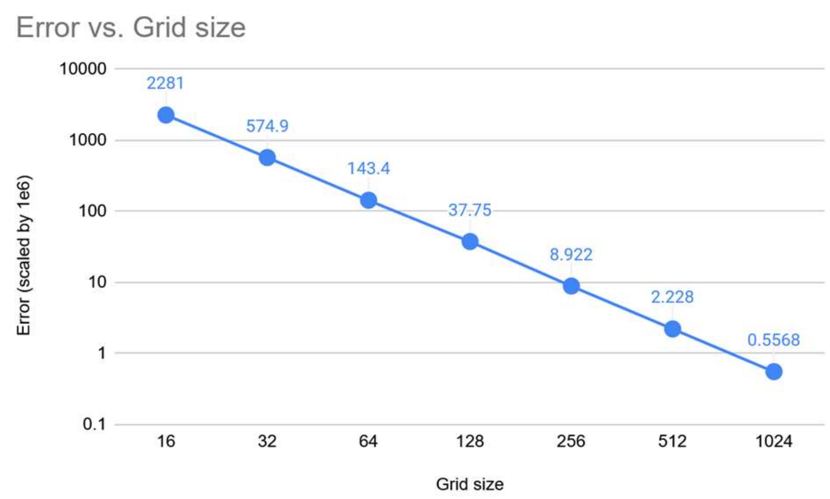
Timesteps: 20000
Error = 2.283e-6

1D Heat convergence: Backward Euler

- Converged faster than RK4
- No stability issue (implicit integrator)



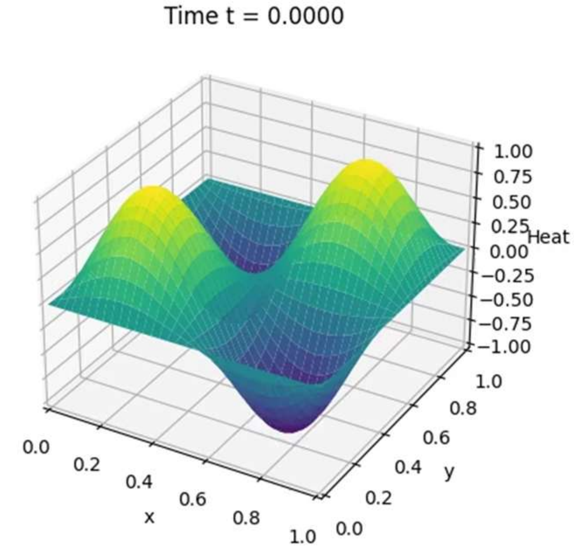
Grid spacing = 1/512



Timesteps = 8192

2D Heat

$$u_t = \alpha \nabla^2 u$$



- Spectral diffusion
- Alpha: 0.01
- Grid size: 128x128
- Time step: 0.0001
- Steps: 50000

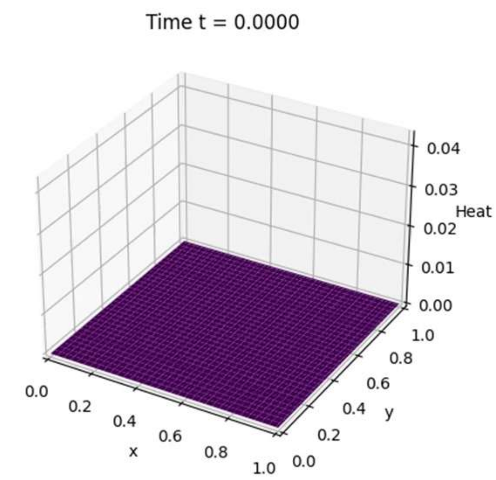
- 2D Fourier transform
- Solve ODE for each Fourier mode
- Solves in time-stepping (backward Euler and RK4)
  - Pseudo Spectral solver helps incorporate source term effects in time independently

2D Heat with source

$$u_t = \alpha \nabla^2 u + f(x,t)$$

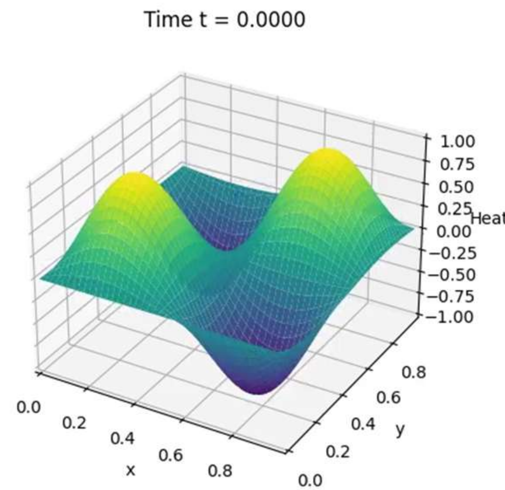
- Spectral diffusion with source
- Alpha: 0.01
- Grid size: 128x128
- Time step: 0.0001
- Steps: 50000

- 2D Fourier transform
- Solve ODE for each Fourier mode
- Solves in time-stepping (backward Euler and RK4)
  - Pseudo Spectral solver helps incorporate source term effects in time independently



2D convection

$$u_t + v \cdot \nabla u = D\nabla^2 u$$



- Transport term in physical space
- Velocity: x=1.0 y=0.5
- Grid size: 128x128
- Diffusion coefficient: 0.01
- Time step: 0.0001
- Steps: 50000

- Pseudo-spectral and full spectral implemented in RK4 and backward Euler
- Expected full spectral to be more accurate but saw no difference

2D reaction

$$u_t = D\nabla^2 u + f(u)$$

- Non linear reaction in physical space
- Logistic model $f(u) = ru(1-u)$
- Grid size: 128x128
- Diffusion coefficient: 0.01
- Time step: 0.0001
- Steps: 50000