Group Velocity Notes - Jeffrey Epstein - October 10, 2017

Let $U(t)$ and $T(x)$ be unitary representations of the time- and space-translation groups, respectively, on some Hilbert space. Assume moreover that

$$[U(t), T(x)] = 0 \tag{1}$$

for all t, x, as is the case if the Hamiltonian that generates $U(t)$ is translation-invariant. Then the two families of operators may be simultaneously diagonalized, so that we can write down an eigenbasis $|\omega, k, a\rangle$ with the properties

$$\begin{aligned} U(t)\,|\omega, k, a\rangle &= e^{-i\omega t}\,|\omega, k, a\rangle \\ T(x)\,|\omega, k, a\rangle &= e^{-ikx}\,|\omega, k, a\rangle\,. \end{aligned} \tag{2}$$

In other words, ω and k are "good quantum numbers". The a label accounts for degeneracies. Now:

$$U(t)\,|\omega, k, a\rangle = e^{-i\omega t}\,|\omega, k, a\rangle = e^{-ik(\frac{\omega}{k}t)}\,|\omega, k, a\rangle = T\left(\frac{\omega}{k}t\right)|\omega, k, a\rangle\,. \tag{3}$$

In other words, applying time-translation by t to a state $|\omega, k, a\rangle$ is equivalent to applying space-translation by vt for "phase velocity" $v = \omega/k$. If we now time-evolve a superposition $|\psi\rangle$ of states $|\omega_i, k_i, a_i\rangle$ with $\omega_i/k_i = v$ for all i, we find

$$\begin{aligned} U(t)\,|\psi\rangle = U(t)\sum_i \alpha_i\,|\omega_i, k_i, a_i\rangle &= \sum_i \alpha_i U(t)\,|\omega_i, k_i, a_i\rangle = \sum_i \alpha_i T(vt)\,|\omega_i, k_i, a_i\rangle \\ &= T(vt)\sum_i \alpha_i\,|\omega_i, k_i, a_i\rangle = T(vt)\,|\psi\rangle\,. \end{aligned} \tag{4}$$

This means that on a subspace spanned by states with the same phase velocity, we have $U(t) = T(vt)$. We can express this in shorthand as

$$U(t)|_{\mathcal{H}(v)} = T(vt)|_{\mathcal{H}(v)}. \tag{5}$$

Physically, what this means is that if we set up a superposition of states with the same group velocity, say some kind of localized particle excitation, that excitation will travel with velocity v without experiencing any deformation.

In the particular case of the Heisenberg Hamiltonian, the Hamiltonian and the translation operator both also commute with the total Z operator, which we can think of as a "particle number" operator. This means that we can also diagonalize this operator in the $|\omega, k, a\rangle$ basis, which we might now parametrize as $|\omega, k, n, a\rangle$, with n the particle number.

A point to make note of is that the k label in the eigenbasis is *not* the same as the *set* of momenta needed to specify a multi-particle state in the Fourier bosons approach. The k here is such that

$$T(x)\,|\omega, k, n, a\rangle = e^{-ikx}\,|\omega, k, n, a\rangle\,, \tag{6}$$

whereas in the Fourier boson approach, states are generated from the vacuum $|\Omega\rangle$ as

$$|q_1, q_2, \ldots, q_n\rangle = b^\dagger_{q_1} b^\dagger_{q_2} \ldots b^\dagger_{q_n}\,|\Omega\rangle\,. \tag{7}$$

The momentum k of the eigenstate can be expressed in terms of the single-particle momenta q_i by examining how these states transform under translation:

$$\begin{aligned} T(x)\,|q_1, q_2, \ldots, q_n\rangle &= T(x) b^\dagger_{q_1} b^\dagger_{q_2} \ldots b^\dagger_{q_n}\,|\Omega\rangle \\ &= T(x) b^\dagger_{q_1} T(x)^\dagger T(x) b^\dagger_{q_2} T(x)^\dagger \ldots T(x) b^\dagger_{q_n} T(x)^\dagger T(x)\,|\Omega\rangle \\ &\vdots \\ &= e^{-ik(q_1, q_2, \ldots, q_n)}\,|q_1, q_2, \ldots, q_n\rangle \end{aligned} \tag{8}$$

and reading off the relationship. You should find that the total momentum k of the state is just the sum of the single-particle momenta q_i. In fact, you can also see that there is a conservation of momentum law if you add interactions to the Hamiltonian, as long as they are translation-invariant.