Personal Notes on

Undergraduate Physics

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Contents

1 Classical Mechanics

1.1 Damped Harmonic Oscillator

A damped harmonic oscillator is governed by

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0 \tag{1}$$

where $\omega_0 = \frac{k}{m}$ and $\gamma = \frac{b}{m}$ By solving the ode, the general solution to a damping harmonic oscillator is

$$x(t) = e^{-\frac{\gamma}{2}t}e^{-i(\pm\sqrt{\omega_0^2-\frac{\gamma^2}{4}})t} \tag{2}$$

where a few cases of damping is notable

$$\begin{aligned} \text{1st:} \quad & \omega_0^2 - \frac{\gamma^2}{4} > 0 \quad && \rightarrow \text{Underdamping} \\ \text{2nd:} \quad & \omega_0^2 - \frac{\gamma^2}{4} < 0 \quad && \rightarrow \text{Overdamping} \\ \text{3rd:} \quad & \omega_0^2 - \frac{\gamma^2}{4} = 0 \quad && \rightarrow \text{Critical damping} \end{aligned}$$

each of the cases also have a different set of solution based on the ODE properties and the fact that we only need to focus on the real part of the solution.

- Underdamped Solution:
$$x(t) = A_0 e^{-\frac{\gamma}{2}t} cos(\omega t + \phi_0)$$
- Overdamped Solution: let $\Gamma = \sqrt{\frac{\gamma^2}{2} - \omega_0^2}$
$$x(t) = A_0 e^{(-\frac{\gamma}{2}+\sqrt{\gamma^2/2-\omega_0^2})t} + Be^{(-\frac{\gamma}{2}-\sqrt{\gamma^2/2-\omega_0^2})t}$$
- Critical Damp solution:
$$x(t) = Ae^{-\frac{\gamma}{2}t} + Bte^{-\frac{\gamma}{2}t}$$

Next we can focus on the rate of energy loss in the system during the damping motion, where we bring out the key assumption that $\frac{\gamma^2}{4} << \omega_0^2$ such that the alteration to the oscillation frequency is negligible. using this assumption, we can write the velocity equation of the oscillation using the assumption above:

$$\frac{dx}{dt} = v(t) = -A_0\omega_0 e^{-\frac{\gamma}{2}t}\sin(\omega_0 t) \tag{3}$$

recalling the total energy equation, the total energy is then

$$E(t) = \frac{1}{2}kA_0^2 e^{-\gamma t} \tag{4}$$

such that $\tau = \frac{1}{\gamma}$ is defined as the lifetime of the energy.

Another important metric is the Quality Factor

Definition: **Quality Factor** is a measurement of how small energy loss of an oscillator is, it is defined by

$$Q = \frac{\omega}{\gamma} \tag{5}$$

Another definition is the number of radians through which the damped system oscillates as its energy decays to $E = \dfrac{E_0}{e}$

1.2 Driven Oscillator

Let first discuss the undamped forced oscillations govern by the ode

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos \omega t \tag{6}$$

where the general solution is

$$x(t) = A(\omega)\cos(\omega t - \delta) \tag{7}$$

For derivation simplicity we also assume instead of applying a force, we get the spring on the left to oscillate aka

$$k\xi \cos(\omega t) = F_0 \cos(\omega t)$$

after a set of transformations we can an expression for $A(\omega)$

$$A(\omega) = \frac{\xi_0}{1 - \frac{\omega^2}{\omega_0^2}} \tag{8}$$

Now we move on the the damping driven oscillator, not a lot has been changed but instead a few equations should be noted.

$$\tan\delta = \frac{\omega\gamma}{\omega_0^2 - \omega^2} \tag{9}$$

and the amplitude equation now becomes

$$A(\omega) = \frac{\xi_0\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}} \tag{10}$$

Similarly the power absorbed during forced oscillations can be modeled by

$$P(t) = bv_0(\omega)^2 \sin^2(\omega t - \delta)dt$$

where the average power is

$$\overline{P}(\omega) = \frac{b[v_0(\omega)]^2}{2}$$

where the maximum power is

$$\overline{P}_{max}(\omega) = \frac{F_0^2}{2m\gamma}$$

and

$$\omega_{fwhh} = 2\Delta\omega = \gamma = \frac{\omega_0}{Q}$$

1.3 Traveling Wave

A general equation for a sinusoidal wave is

$$y(x,t) = A\sin(kx \pm \omega t + \phi_0) \tag{11}$$

$k = \dfrac{2\pi}{\lambda}$ is the wave vector number, $\omega = \dfrac{2\pi}{T}$ is the angular frequency.

To determine whether an equation is a wave we need to verify by the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{12}$$

where v is the speed of the wave λf The energy of the wave is given by:

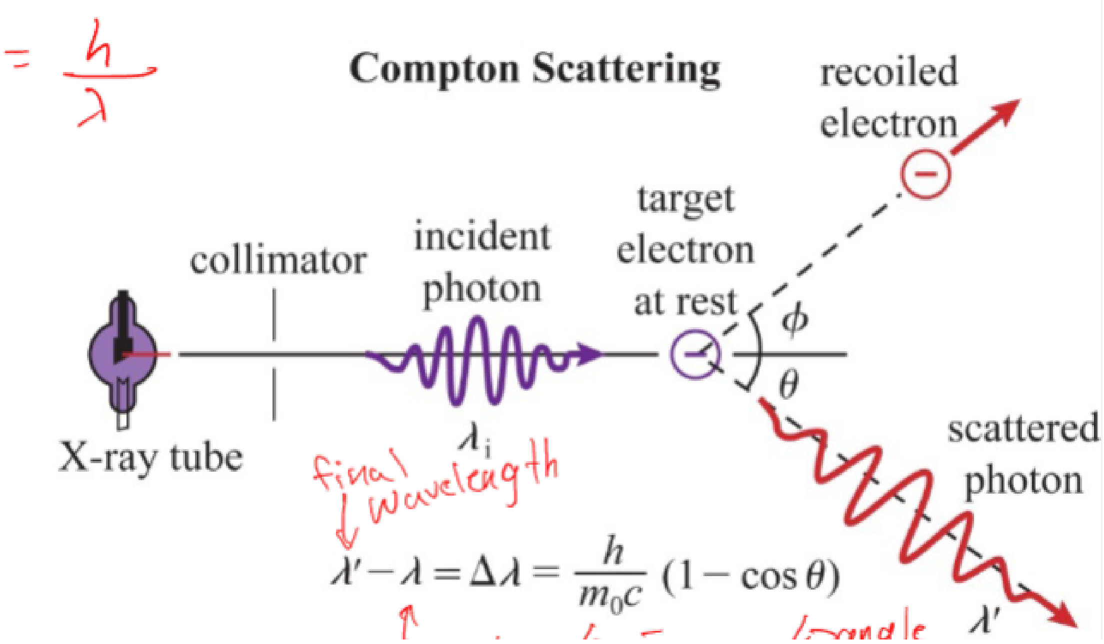
$$E_{total} = \frac{1}{2}\mu\omega^2 A^2 \lambda \tag{13}$$

and the power is

$$P(x,t) = A^2 \mu v \omega^2 \sin^2(kx - \omega t + \phi_0) \tag{14}$$

2 Modern Physics

2.1 Compton Effect



Compton effect is a part of the photoelectric phenomenon where a injected light of certain wavelength λ can result in a scattering of collision like those in classical mechanics(assuming two sphere of same mass colliding elastically). The initial energy of the system is solely $E_i = hf$ of the photon and eventually turns into $E_f = hf_{final} + K_e$ where K_e is the kinetic energy of the electron.

2.2 Quantization of Atomic Energy

Emission and Absorbance of light occurs when electron jumps from one state to another, and the energy difference should be precisely the energy of the photon.

2.2.1 Hydrogen emission

For a hydrogen-like atom(1 electron), the wavelength emission can be estimated by:

$$\frac{1}{\lambda} = R(\frac{1}{n'^2} - \frac{1}{n^2})$$

where R = 0.0110 1/nm. and n represents the principle quantum numbers of the corresponding states.

2.2.2 Bohr Model

An early approach done by Bohr is the Bohr model to estimate energy level in hydrogen-like atom. Giving the following equations:

- Radius of orbit

$$r_n = n^2 \frac{h^2 \epsilon_0}{\pi m e^2 Z} = 0.0529 n^2$$

- Energy Level:

$$E_n = -\frac{13.6eV Z^2}{n^2}$$

Z is the number of proton. From the equation above it is evident that the energy different equation goes as:

$$\Delta E = -13.6eV(\frac{1}{n'^2} - \frac{1}{n^2})$$

where n' is the final quantum number and n is the initial quantum number.

2.3 Matter Wave

Matter Wave is a fundamental concept to demonstrate conditions when particle start behaving wave-like properties. The wavelength of the object is defined as:

$$\lambda = \frac{h}{p}$$

where p is the momentum of the particle. This relation corresponds to the angular momentum of electron in Bohr's model $L = nh/2\pi$

2.4 Uncertainty Principle

the momentum of a particle is defined by the **de Broglie formula**

$$p = \frac{h}{\lambda} = \frac{2\pi h}{\lambda}$$

the uncertainty principle state that the more precisely determined a position is the less precisely its momentum:

$$\sigma_x \sigma_p \geq \frac{h}{2}$$

the sigma represents the standard deviation.

2.5 Special Relativity

Two famous postulates of special Relativity are:

- The laws of physics are the same in every inertial frame of reference
- The speed of light is constant in all inertial frames of reference.
- If S is an inertial frame and if a second frame S' moves with constant velocity relative to S, then S' is also an inertial frame.

One experiment to verify the second law is the Michelson-Morley Experiment. Basically use an interferometer to compare light speed under Earth motion at different directions. eventually no diffraction pattern observed so the speed of light is constant.

2.5.1 Time Dilation & Length Contraction

For the famous case of light bouncing in a training moving in velocity v relative to ground. The time observed by another frame that has a relative velocity with the train is

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \Delta t' \tag{15}$$

and $\Delta t' = \dfrac{2d}{c}$ is the time observed by the frame that has no relative velocity with the event(or the light bouncing).

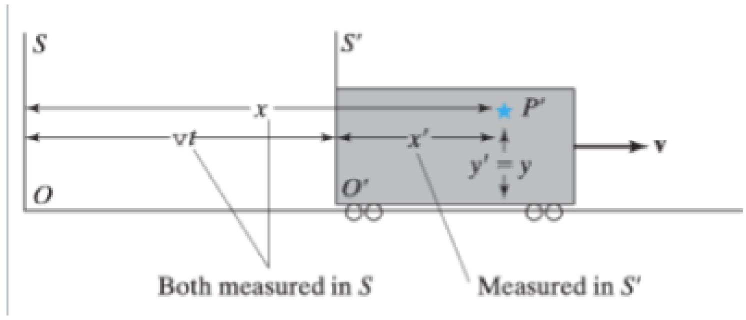
Similar phenomenon occurs on spatial scale of length.

$$l = \frac{l'}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}} l' \tag{16}$$

where l' is the length in rest frame(proper length) and l is the observed frame.

2.5.2 Lorentz Transformation

This is a more general approach to finding relative velocity while accounting for the relativistic effect of the event.



for a visualization of the event given here. The result of length contraction generates a new form of relative speed in different frame.

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \tag{17}$$

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}} \tag{18}$$

where u'_x, u_x are the speed of the object measured in frame S' and S, and v is the velocity of S' relative to S.

2.5.3 Doppler Effect

A moving source changes the frequency of the wave emitted and a moving receiver detects a different frequencies. Considering an electromagnetic waves, the frequency observed is governed by the equations

$$f_{\text{obs}} = \sqrt{\frac{1+\beta}{1-\beta}} f_{\text{source}} \qquad \text{Source approaching observer} \tag{19}$$

$$f_{\text{obs}} = \sqrt{\frac{1-\beta}{1+\beta}} f_{\text{source}} \qquad \text{Source moving away from observer} \tag{20}$$

where $\beta = \dfrac{v}{c}$ and v is the relative velocity between the source and the observer.

2.5.4 Relative Mass and Momentum

Knowing the velocity under relativistic effect, the momentum of the object can be determined by

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{21}$$

where m is the rest mass of the object and v is the velocity of the rest frame of reference relative to the ground. Similarly, the kinetic energy of an object is then

$$K = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - mc^2 \tag{22}$$

and the total energy is then

$$E = K + mc^2 => E^2 = (pc)^2 + (mc^2)^2 \tag{23}$$

3 Quantum Mechanics

3.1 Schrodinger equation

Schrodinger Equation:

$$ih\frac{\partial\psi}{\partial t} = -\frac{h^2}{2m}\frac{\partial\psi^2}{\partial t^2} + V\psi$$

Definition: The average value of j or the expectation value:

$$< j >= \frac{\sum jN(j)}{N} = \sum jjP(j)$$

for a continuous distribution we use probability density $\rho(x)$ so the probability of a particle between a and b is:

$$\int_b^a \rho(x)dx$$

here are a few equations to take note of:

- Expected Position:

$$< x >= \int_{+\infty}^{-\infty} x|\psi(x,t)|^2 dx$$

- Expected momentum:

$$< p >= \int_{+\infty}^{-\infty} -ih(\psi^* \frac{\partial \psi}{\partial x})dx$$

a few important things i learned:

- if LHS is dependent on t and RHS dependent on x, the only possibility is that both sides are constant.
- if we want an operator just sub in the form of the p - momentum
- c_n is the probability of measurement of the energy would return the value E_n

The Schrodinger equation can be solved by **seperation of variables**

$$\Psi(x,t) = \psi(x)\phi(t)$$

and by separating variables we got :

$$E = ih\frac{\partial \phi}{\partial t}$$

in other words:

$$-\frac{h^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi = E\psi$$

There are 3 reasons to use separable solutions:

- Stationary States:

  $$\Psi(x,t) = \psi(x)e^{-iEt/h}$$

  where the wavefunction does not depend on time:

  $$|\Psi(x,t)|^2 = |\psi(x)|^2$$

- States of Defined energy: Where the total Hamiltonian of the system is a constant:

  $$H = \frac{p^2}{2m} + V(x)$$

  where the hamiltonian operator is:

  $$\hat{H} = -\frac{h^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

  The time independent Schrodinger equation is:

  $$\hat{H}\psi(x) = E\psi(x)$$

  where the variance of H is 0 aka all the measurements should return the same energy E.

- General Solution is linear combination of separable solutions: A collection of solutions($\psi_1(x), \psi_2(x), \psi_3(x)$) can be combined to form a general solution: each with a defined separation constant(E_1, E_2, E_3) thus there is a different wave function for each allowed energy.

$$\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_n t/h}$$

  and every time dependent solution can be written in this form.

To solve schrodinger equation, with a given potential V and $\Psi(x,0)$ we can always fit the t = 0 state by:

$$\Psi(x,0) = \sum_n c_n \psi_n(x)$$

with an appropriate choise of c_n we can always fit the initial state.

and $|c_n|^2$ is the probability of measurement of the energy would return the value E_n and the sum should be equivalent to 1. With the expectation hamiltonian being:

$$< H >= \sum_n |c_n|^2 E_n$$

3.1.1 Infinite Square Well

suppose we need a boundary condition for the wave function to be 0 at the boundaries of the well.

$$V(x) = \begin{cases} 0, & 0 \leq x \leq a, \\ \infty, & \text{otherwise} \end{cases}$$

and the equation of time-independent schrodinger equation is:

$$-\frac{h^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = E\psi$$

which is a simple harmonic oscillator equation. and the solution is:

$$\psi(x) = A\sin(kx)$$

where k is the wave number: $k = \dfrac{\sqrt{2mE}}{h}$ since the boundary condition requires the wave function to be 0 at the boundaries. So ka must be a multiple of π: By subbing the new equation into the original time-independent schrodinger equatiom, the possible values of E in each mode is:

$$E_n = \frac{n^2\pi^2h^2}{2ma^2} = \frac{h^2k_n^2}{2m}$$

and the solutions is:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

ψ_1 is called ground state and states with energies increase in proportion to n^2 is called excited States.

and here are a few interesting properties:

- They are alternately even and odd.
- each successive state has one more node.
- They are mutually orthogonal such that:

$$\int \psi_m(x)^* \psi_n(x) dx = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

- They are complete, such that any other fucntion f(x) can be expressed as a linear combination of them.

To find c_n we can use fourier trick: let

$$\psi(x) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} c_n sin(\frac{n\pi}{a}x)$$

to find the probability factor for one state -m - is:

$$c_m = \int \psi_m^*(x)\psi(x)dx$$

in addition, the stationary states are:

$$\Psi_n(x,t) = \sqrt{\frac{2}{a}} sin(\frac{n\pi}{a}x)e^{-i(n^2\pi^2 h/2ma^2)t}$$

Therefore, the coefficient is:

$$c_n = \sqrt{\frac{2}{a}} \int_0^a sin(\frac{n\pi}{a}x)\Psi(x,0)dx$$

3.1.2 Harmonic Oscillator

Any simple harmonic oscillator can be approximated as:

$$V(x) = \frac{1}{2}V''(x_0)(x-x_0)^2$$

and for quantum problem we can further simplify it into:

$$V(x) = \frac{1}{2}mw^2x^2$$

and sub it into the time independent schrodinger equation:

$$-\frac{h}{2m}\frac{\partial\psi^2}{\partial x^2} + \frac{1}{2}mw^2x^2\psi = E\psi$$

and there are two solutions

- power series method
- Ladder operators method

Algebraic Method: Ladder Method we can rewrite time independent schrodinger equation into this form:

$$\frac{1}{2m}(\hat{p}^2 + m^2w^2\hat{x}^2)\psi = E\psi$$

and the basic idea is to factor the hamiltonian. here we have the famous **ladder operator**

$$\hat{a}_+ = \frac{1}{\sqrt{2m\hbar w}}(-i\hat{p} + mw\hat{x})$$

$$\hat{a}_- = \frac{1}{\sqrt{2m\hbar w}}(i\hat{p} + mw\hat{x})$$

and the relationship bewteen hamiltonian is:

$$\hat{a}_-\hat{a}_+ = \frac{1}{h\omega}\hat{H} + \frac{1}{2}$$

if we reverse the order of multiplication in the left hand side, the sign of that one have reverses. aka the **commutator** of the two operators is:

$$[\hat{a}_-, \hat{a}_+] = 1$$

by subbing the new hamiltonian equation into the solution we obtain:

$$h\omega(\hat{a}_+\hat{a}_- + \frac{1}{2})\psi = E\psi$$

Definition: If ψ satisfies the schrodinger equation with energy E, then $\hat{a}_+\psi$ also satisfies the schrodinger equation with energy E + hω

$$\hat{H}(a_+\psi) = (E + h\omega)(a_+\psi)$$

Idea: HOWEVER, it is not guaranteed that the new solution is normalize.

3.1.3 The Free Particle

The free particle should have been the simplest case (V(x) = 0 everywhere). It is just a motion at constant velocity. However it is kinda subtle. For the simplified Schrodinger Equation:

$$-\frac{h^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

The solution is simply a general solution for a second order ODE:

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

Taking on the standard time dependent exponential term, the general solution for the time dependent free partical equation becomes:

$$\Psi(x,t) = Ae^{ik(x-\frac{hk}{2m}t)} + Be^{-ik(x+\frac{hk}{2m}t)}$$

Which you can understand as a standing wave traveling to left or right at a constant velocity $v = \frac{hk}{2m}$ where the sign before v indicates the opposite fo the velocity("-" to the right, "+" to the left)

3.1.4 Finite Square Well

A simpler method to understand the solution to a finite square well $U(x) = U_0$ is:

$$\psi(x) = \begin{cases} Ae^{\alpha x} + Be^{-\alpha x}, & E < U_0 \\ Ce^{ikx} + De^{-ikx}, & E > U_0 \end{cases}$$

$$\alpha = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}}, \quad k = \sqrt{\frac{2m(E - U_0)}{\hbar^2}}$$

Lets say a particle comes from the left to hit the potential barrier.

The reflection coefficient is defined as

$$R = \frac{|B|^2}{|A|^2}$$

and the transmission coefficient is defined as:

$$T = 1 - R$$

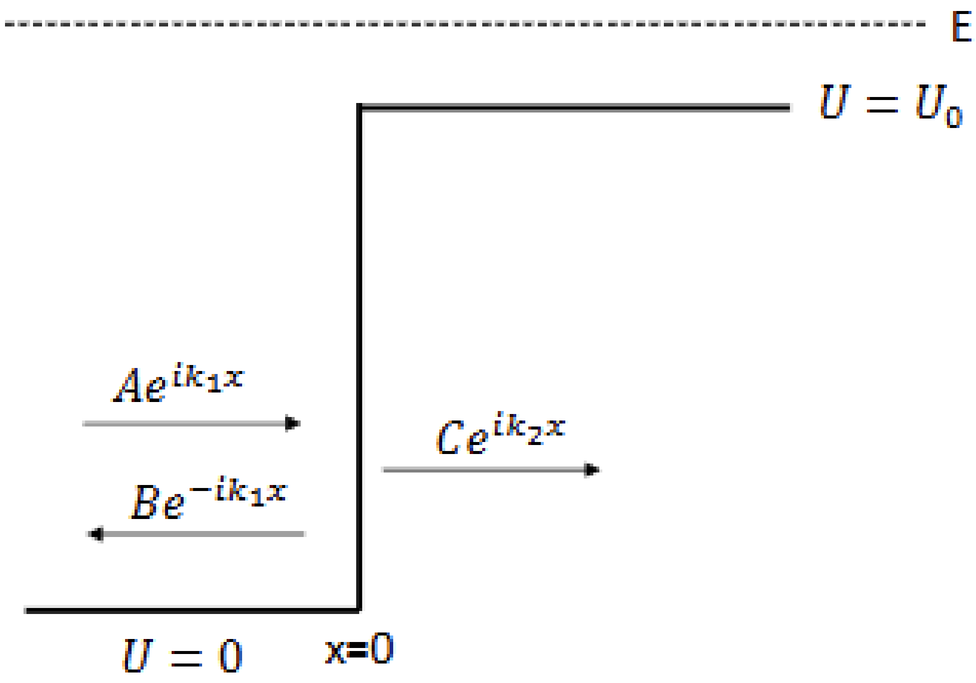
4 Quantum Information

4.1 Hilbert Space

Definition: Hilbert Space
a set of all square integrable functions on a specified interval f(x) such that

$$\int_a^b |f(x)|^2 dx < \infty \tag{24}$$

constitues a vector space of $L^2(a,b)$ namely **Hilbert Space**.



And for two functions that live in Hilbert Space, the inner product is defined as:

$$\langle f|g\rangle = \int_a^b f^*(x)g(x)dx \tag{25}$$

which follows the integral **Schwarz Inequality**:

$$|\langle f|g\rangle| \leq \sqrt{\langle f|f\rangle}\sqrt{\langle g|g\rangle} \tag{26}$$

a bit more definition for a terms:

- Normalized: if the inner project with itself is 1
- Orthogonal: if the inner product with another function is 0
- Orthonormal: if the inner product with itself is 1 and with another function is 0
- Complete: if any other function can be expressed as a linear combination of the existing functions