Personal Notes on

Undergraduate Math

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Contents

1 Ordinary Differential Equation

1.1 Types of ODEs

This section includes some basic definition of ODEs and terms that are worth noting. The categories of ODEs include

- Linear/Non-linear ODE: where linear ODEs can be constructed in the format

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t)$$

- Separable vs Nonseparable: Where the ODE f(t,y) can be written in f(t,y) = p(t) q(y)
- Autonomous/ Non-autonomous: Where the ODE f(t,y) doesn't explicitely depend on the independent variable (ex. t-time)

Here are a few terms that worth remember

- Equilibrium Position: It is the position such that for an ODE f(y) = $\frac{dy}{dt}$ = 0 also called critical points, fixed points, stationary points.
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1.2 Existence & Uniqueness of ODEs

Theorem: For Linear
$y' + p(t)y = g(t), y(t_0) = y_0$
Assume p and g are continuous on an open interval
$t_0 \in (\alpha, \beta)$
Then there exists a unique solution in that interval.
Yet, if a first order linear ODE is autonomous, a unique solutions exists and is valid for all t. Where the equilibrium for the following system of first order ODE

$$\frac{dx}{dt} = Ax + b$$

is

$$x = -A^{-1}b$$

How doe the non linear ones

Theorem: For nonlinear assume that f and $\frac{df}{dy}$ are continuous in some rectangle

$$(t_0, y_0) \in (\alpha, \beta) \times (\gamma, \delta)$$

Then there exists a unique solution in some interval

$$(t_0 - h, t_0 + h) \in (\alpha, \beta)$$

1.3 Homogeneous & Non Homogeneous ODE

The following Nonhomogeneous ODE

$$\frac{dx}{dt} = Ax + b \tag{1}$$

$$x_{eq} = -A^{-1}b \tag{2}$$

can be written into a homogeneous ODE by a change of coordinate

$$x = x - x_{eq}$$

1.4 Solutions of ODEs

Assume x_1, x_2 are the solutions, the principle of superposition tells us that $c_1x_1 + c_2x_2$is also a solution for any $c_1 and c_2$

For a first order ODE, we can also obtain the solution via Eigendecomposition

$$\frac{dx}{dt} = Ax \tag{3}$$

$$x(t) = e^{\lambda t}v \tag{4}$$

where λ, v are the eigen values and corresponding eigen vectors of the solution.

However, there is one more thing to verify regarding the solutions is to see if they are linear independent, which can be done by using Wronskian.

Definition: Wronskian
is defined by

$$W[x_1, x_2](t) = |x_1 \quad x_2|$$

if $\forall t \quad W[x_1, x_2](t) \neq 0$ $x_1.x_2$ are linearly independent.
for two solutions $x_1 = e^{\lambda_1 t}v_1$ and $x_2 = e^{\lambda_2 t}v_2$ The Wronskian between them is

$$W[\vec{x}_1, \vec{x}_2](t) = \begin{vmatrix} v_{11}e^{\lambda_1 t} & v_{12}e^{\lambda_2 t} \\ v_{21}e^{\lambda_1 t} & v_{22}e^{\lambda_2 t} \end{vmatrix} = \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix} e^{(\lambda_1+\lambda_2)t}, \quad \text{non-zero because } \lambda_1 \neq \lambda_2$$

and Wronskian is calculated by

1.4.1 Complex EigenValue Solutions

If the eigenvalues are complex numbers, the general solution would become

$$\mathbf{x}_1(t) = e^{(\mu+i\nu)t}\mathbf{v}_1, \quad \mathbf{x}_2(t) = e^{(\mu-i\nu)t}\bar{\mathbf{v}}_1 \tag{5}$$

Where the eigenvalues and eigenvectors must appear in conjugate pair.

- Let $\mathbf{v}_1 = a + ib$.
- Rearranging first solution:

$$\begin{aligned} \mathbf{x}_1(t) &= e^{\mu t}\big(\cos \nu t + i \sin \nu t\big)(a + ib) \\ &= e^{\mu t}\Big(a \cos \nu t - b \sin \nu t\Big) + ie^{\mu t}\Big(a \sin \nu t + b \cos \nu t\Big) \\ &= \underbrace{e^{\mu t}\Big(a \cos \nu t - b \sin \nu t\Big)}_{\mathbf{u}(t)} + \underbrace{ie^{\mu t}\Big(a \sin \nu t + b \cos \nu t\Big)}_{\mathbf{w}(t)} \end{aligned}$$

- $\mathbf{u}(t)$ and $\mathbf{w}(t)$ solve ODE and form a fundamental set of solutions.
- General solution:

$$\mathbf{x} = c_1\mathbf{u}(t) + c_2\mathbf{w}(t)$$

1.4.2 Repeated Eigenvalues

For the case of repeated eigen values Let the ODE be

- ODE:

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

- Assume $\lambda_1 = \lambda_2$ and there's one independent eigenvector $\mathbf{v}_1$.

- First solution:

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1, \quad (\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_1 = 0$$

- Second solution:

$$\mathbf{x}_2(t) = te^{\lambda_1 t}\mathbf{v}_1 + e^{\lambda_1 t}\mathbf{w}, \quad (\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{w} = \mathbf{v}_1$$

- General solution:

$$\mathbf{x} = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$$

2 First Order ODE

2.0.1 Integrating Factors

This is the most basic method to solve only **First Order Linear ODE**. For an ODE like

$$\frac{dy}{dt} + p(t)y = g(t)$$

The integrating factor I is determined by

$$I(t) = e^{\int p(t)dt}$$

and using this integrating factor we can obtain the general equation to be

$$y(t) = \frac{1}{I(t)}\left[\int_{t_0}^{t} I(t)g(t)dt + C\right] \tag{6}$$

2.0.2 Particular Solution

Integrating factor is the simplist method to solve a single Linear ODE, however learning how to solve a nonhomogeneous system is also worth noting.

Theorem: Let X_p be a particular solution of the non homogeneous system on an interval I and let

$$X_H = \alpha_1 X1 + \alpha_2 X_2 + ...\alpha_n X_n$$

The general solution on the same interval $X' = AX$. Then the general solution reads $X = X_p + X_H$ on I

2.0.3 Exponential Function

The key in this method is to draw parallel relationship with X $= e^{Ct}$ while we replaced C with an array A

Theorem: Let A be an nxn matrix with constant entries. We define the exponential function matrix as

$$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots + \frac{A^kt^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{A^kt^k}{k!}.$$

This series is absolutely convergent for any value of $t \in \mathbb{R}$.
Properties of e^{At} are listed:

- The exponential function of the zero matrix is the identity matrix,

$$e^{0\times x} = I_{n\times n}.$$

- The matrix A commutes with its exponential function,

$$e^{tA}A = Ae^{tA} \quad \text{for any } t.$$

- The exponential function of the sum of two matrices is the product of the corresponding exponential functions if the matrices commute,

$$e^{(A+B)t} = e^{At}e^{Bt}.$$

- The derivative of the matrix exponential is given by:

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

Given these features, it is now easy to verify that $X(t) = e^{At}C$ is a general solution to the system $X' = AX$. Indeed,

$$X'(t) = [e^{At}C]' = Ae^{At}C = AX.$$

2.0.4 Finding the Exponential Matrix Function

For various cases of matrix A there are methods to target them specifically

- Diagonal Case:

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}, \quad e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix}.$$

- Non-Diagonal Case: There are two methods to solve this case we can explain case by case with an example:
  - Method 1: **Example 9.8.** Find e^{At} where $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$. The general solution to the system $X' = AX$ was derived in Lecture 11 and is given by

$$X(t) = \alpha_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \alpha_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}, \quad \alpha_1, \alpha_2 \text{ arbitrary constants.}$$

We know the expression of $X(t)$ and X_0, as the latter can be fixed. How to find $e^{At} = (C_1, C_2)$ where C_1 and C_2 are column vectors of e^{At}? We use **(58)** and choose the vector X_0 so as to get C_1.

$$X_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies \begin{cases} \alpha_1 + 3\alpha_2 = 1 \\ -\alpha_1 + 2\alpha_2 = 0 \end{cases}$$

and $\alpha_1 = \frac{2}{5}$, $\alpha_2 = \frac{1}{5}$.

Combining this and **(58)**, we arrive at

$$C_1 = \begin{pmatrix} \frac{2}{5}e^{-t} + \frac{3}{5}e^{4t} \\ -\frac{2}{5}e^{-t} + \frac{2}{5}e^{4t} \end{pmatrix}.$$

Now we wish to get C_2 via a similar procedure and we choose $X_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

This leads to

$$X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies \begin{cases} \alpha_1 + 3\alpha_2 = 0 \\ -\alpha_1 + 2\alpha_2 = 1 \end{cases}$$

and $\alpha_1 = -\frac{3}{5}$, $\alpha_2 = \frac{2}{5}$.

Hence,

$$C_2 = \begin{pmatrix} -\frac{3}{5}e^{-t} + \frac{3}{5}e^{4t} \\ \frac{3}{5}e^{-t} + \frac{2}{5}e^{4t} \end{pmatrix}, \quad e^{At} = \begin{pmatrix} \frac{2}{5}e^{-t} + \frac{3}{5}e^{4t} & -\frac{3}{5}e^{-t} + \frac{3}{5}e^{4t} \\ -\frac{2}{5}e^{-t} + \frac{2}{5}e^{4t} & \frac{3}{5}e^{-t} + \frac{2}{5}e^{4t} \end{pmatrix}. \tag{59}$$

– Method 2: We assume A is diagonalizable and can be written in the form

$$A = PDP^{-1}$$

or if A has n independent eigenvalues.

$$P = (V_1, V_2, \dots, V_n), \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

where V_i, $i = 1, 2, \dots, n$ is the eigenvector of A corresponding to the eigenvalue λ_i. Observe that

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1} \quad \text{and} \quad A^3 = A^2A = (PDP^{-1})(PDP^{-1}) = PD^3P^{-1}.$$

Iterating this formula, we arrive at $A^k = PD^kP^{-1}$ for any positive integer k. Consequently, we have that

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = P \sum_{k=0}^{\infty} \frac{t^k D^k}{k!} P^{-1} = Pe^{Dt}P^{-1}. \tag{60}$$

Let us reconsider the above example where $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$. The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 4$ with the eigenvalues $V_1 = A \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $V_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, respectively. Therefore, for this particular example we have

$$P = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 2/5 & -3/5 \\ 1/5 & 1/5 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}.$$

Applying formula (60), one obtains

$$e^{At} = Pe^{Dt}P^{-1} = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{4t} \end{pmatrix} \begin{pmatrix} 2/5 & -3/5 \\ 1/5 & 1/5 \end{pmatrix}.$$

Simplifying, we get:

$$e^{At} = \begin{pmatrix} 3/5e^{-t} + 2/5e^{4t} & -3/5e^{-t} + 3/5e^{4t} \\ -2/5e^{-t} + 2/5e^{4t} & 3/5e^{-t} + 2/5e^{4t} \end{pmatrix}.$$

3 Second Order ODE

For a second order ode to have a unique solution.

Theorem: For a second order ODE

$$a(t)y'' + b(t)y' + c(t)y = f(t)$$

$\frac{b}{a}, \frac{c}{a}, \frac{f}{a}$ are all continuous on a common interval I containing the initial conditions. Then the 2nd Order ODE admits a unique solution which exists throughout the interval I.

We can now try to turn a system of 2nd order ode into a system of first-order ODE

$$x_1' = x_2, \quad x_2' = y'' = -\frac{b}{a}x_2 - \frac{c}{a}x_1.$$

In matrix form, we write

$$X' = AX, \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 \\ -\dfrac{c}{a} & -\dfrac{b}{a} \end{pmatrix}.$$

The wronskian in this case is defined as

$$W(X_1, X_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t).$$

3.1 Homogeneous Solution

For a second order ODE of

$$ay'' + by' + cy = 0$$

The $(y, y')^T$ is a solution vector of the system

$$X' = AX, A = \begin{pmatrix} 0 & 1 \\ -\dfrac{c}{a} & -\dfrac{b}{a} \end{pmatrix}$$

The eigenvalues are the solutions to the polynomial

$$P(\lambda) = a\lambda^2 + b\lambda + c$$

and the corresponding eigenvectors are

$$V_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$$

if there is repeated eigen values, the general solution become

$$\mathbf{X}(t) = \alpha_1 \begin{pmatrix} 1 \\ \lambda \end{pmatrix} e^{\lambda t} + \alpha_2 \begin{pmatrix} t \\ 1 + \lambda t \end{pmatrix} e^{\lambda t}$$

If the eigenvalues are pair of complex conjugate

$$\lambda = \alpha - i\beta$$

The solution becomes

$$y(t) = \alpha_1 e^{\alpha t} \cos \beta t + \alpha_2 e^{\alpha t} \sin \beta t$$

3.2 Non-Homogeneous Solution

The first method is by guessing the particular solution and combine it with the general solution. However there is a different way which is by the Variation of parameters

3.2.1 Variation of Parameters

Theorem: *A particular solution to system (72) is given by the formula*

$$X_p(t) = \Phi(t) \int \Phi(t)^{-1} F(t)\, dt \tag{75}$$

where $\Phi(t)$ is the fundamental matrix of $X' = AX$. In addition, if the column vectors of this matrix are X_1 and X_2, then the general solution to (72) is

$$X(t) = \alpha_1 X_1(t) + \alpha_2 X_2(t) + X_p(t). \tag{76}$$

where Φ is essentially the Wronskian of the solutions of the general solution.

Based on this *If* $\{y_1, y_2\}$ *form a fundamental set of solution to the homogeneous ODE* $y'' + a(t)y' + b(t)y = 0$ *then a particular solution to the nonhomogeneous ODE* $y'' + a(t)y' + b(t)y = f(t)$ *is given by*

$$y_p(t) = -y_1(t) \int \frac{y_2(t)f(t)}{W(y_1, y_2)(t)}\, dt + y_2(t) \int \frac{y_1(t)f(t)}{W(y_1, y_2)(t)}\, dt \tag{77}$$

and the general solution is

$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t) + y_p(t). \tag{78}$$