Personal Notes on

Quantum Optics

Jeffrey Ming Han Li

jeffreyli2288@Outlook.com

Abstract

This note covers the content in Introductory Quantum Optics by Christopher C.Gerry and Peter L.Knight.

Contents

1 Quantization of Fields

1.1 Single-mode field

Consider a potential well between z = 0 and z = L, a single-mode field satisfying Maxwell's equations and the boundary conditions is given by

$$E_x(z,t) = (\frac{2\omega^2}{V\epsilon_0})^{1/2} q(t)\sin(kz)$$

where ω is the frequency of the mode, k is the wave number, V is the effective volume o the cavity and q(t) is a time-dependent factor having the dimension of length. The magnetic field is

$$B_y(z,t) = \left(\frac{\mu_0\varepsilon_0}{k}\right)\left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} \dot{q}(t)\cos(kz)$$

For the classical field, the Hamiltonian H of the field is given by

$$H = \frac{1}{2}\int dV \left[\varepsilon_0 \mathbf{E}^2(\mathbf{r},t) + \frac{1}{\mu_0}\mathbf{B}^2(\mathbf{r},t)\right] = \frac{1}{2}\int dV \left[\varepsilon_0 E_x^2(z,t) + \frac{1}{\mu_0}B_y^2(z,t)\right]$$

which is

$$H = \frac{1}{2}(p^2 + \omega^2 q^2)$$

This shows that a single-mode field is formally a harmonic oscillator of unit mass. By replacing the q and p as operators that follow the canonical commutation relation, we can express the electric field and magnetic field. Recall that the annihilation($\hat{a}$) and creation($\hat{a}^\dagger$ operators are

$$\hat{a} = (2\hbar\omega)^{-1/2}(\omega\hat{q} + i\hat{p})$$

$$\hat{a}^\dagger = (2\hbar\omega)^{-1/2}(\omega\hat{q} - i\hat{p})$$

The fields are

$$\hat{E}_x(z,t) = \mathcal{E}_0(\hat{a} + \hat{a}^\dagger)\sin(kz),$$

$$\hat{B}_y(z,t) = \mathcal{B}_0\frac{1}{i}(\hat{a} - \hat{a}^\dagger)\cos(kz)$$

where $\mathcal{B}_0 = (\mu_0/k)(\epsilon_0\hbar\omega^3/V)^{1/2}$ and $\mathcal{E}_0 = (\hbar\omega/\epsilon_0 V)^{1/2}$ represents the electric and magnetic field "per photon" (not quite). Also recall that

- $[\hat{a}, \hat{a}^\dagger] = 1$
- $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$
- **Heisenberg's Equation:** for an arbitrary operator $\hat{O}$, having no explicit time dependence:

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{O}\right]$$

For annihilation operator, this becomes

$$\begin{aligned}\frac{d\hat{a}}{dt} &= \frac{i}{\hbar}\left[\hat{H}, \hat{a}\right] \\ &= \frac{i}{\hbar}\left[\hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}), \hat{a}\right] \\ &= i\omega(\hat{a}^\dagger\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{a}) \\ &= i\omega\left[\hat{a}^\dagger, \hat{a}\right]\hat{a} = -i\omega\hat{a}\end{aligned}$$

the expression has the solution

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t} \quad and \quad \hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{-i\omega t}$$

an alternative way is to implement **Baker-Hausdorf Lemma** on the generalized solution to the $\hat{O}$ operator differential equation. Let $|n\rangle$ denote an energy eigenstate of the single mode field with the energy eigenvalue E_n such that

$$\hat{H}|n\rangle = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})|n\rangle = E_n|n\rangle$$

multiplying both sides with the creation operator and using the commutation relations, we can obtain the equation.

$$\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)(\hat{a}^\dagger|n\rangle) = (E_n + \hbar\omega)(\hat{a}^\dagger|n\rangle) = \hat{H}|n\rangle$$

note that applying an annihilation operator on eigenstate $|0\rangle$ would result in the lowest energy state which is

$$\hat{H}(\hat{a}|0\rangle) = (E_0 - \hbar\omega)(\hat{a}|0\rangle) = 0$$

therefore, the lowest energy is $E_0 = \frac{1}{2}\hbar\omega$, and the generalized form for energy in harmonic energy configuration is $E_n = \hbar\omega(n + \frac{1}{2})$. Also let the number operator be $\hat{n}$ such that $\hat{n}|n\rangle = n|n\rangle = \hat{a}^\dagger\hat{a}|n\rangle$ Here are few things to know

- **Inner Product**:

$$\begin{aligned}\left(\langle n|\hat{a}^\dagger\right)(\hat{a}|n\rangle) &= \langle n|\hat{a}^\dagger\hat{a}|n\rangle = n \\ &= \langle n-1|c_n^* c_n|n-1\rangle = \left|c_n^2\right|\end{aligned} \tag{1}$$

  taking $c_n = \sqrt{n}$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

  Therefore, the final expression for any in the harmonic energy configuration eigenstate can be expressed as

$$\boxed{|n\rangle = \frac{\hat{a}^{\dagger n}}{\sqrt{n!}}|0\rangle}$$

  must be noted that the c_n in this case is not the same as the weight of the linear combination of quantum states, it is simply a constant for the annihilation operator to function as intended.

- **States form complete set:** such that

$$\sum_{n=0}^{\infty}|n\rangle\langle n| = 1$$

  meaning the sum of all projection matrix yields an identity matrix.

1.2 Quantum Fluctuation

Note that number state $|n\rangle$ is a well-defined energy but not a state of electric field since the mean field is zero (prove by $\langle n| E_x |n\rangle = 0$. However, the square of the electric field, which contributed to the energy density is not zero

$$\begin{aligned}\langle n|\hat{E}_x^2(z,t)|n\rangle &= \mathcal{E}_0^2 \sin^2(kz)\langle n|\hat{a}^{\dagger^2} + \hat{a}^2 + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger|n\rangle \\ &= \mathcal{E}_0^2 \sin^2(kz)\langle n|\hat{a}^{\dagger^2} + \hat{a}^2 + 2\hat{a}^\dagger\hat{a} + 1|n\rangle \\ &= 2\mathcal{E}_0^2 \sin^2(kz)\left(n + \frac{1}{2}\right)\end{aligned} \tag{2}$$

Therefore, the **fluctuation** or standard deviation of the energy field is then square root of the varience

$$\left\langle\left(\Delta\hat{E}_x(z,t)\right)^2\right\rangle = \left\langle\hat{E}_x^2(z,t)\right\rangle - \left\langle\hat{E}_x(z,t)\right\rangle^2 \tag{3}$$

$$\Delta E_x = \sqrt{2\mathcal{E}_0}\sin(kz)\left(n + \frac{1}{2}\right)^{1/2} \tag{4}$$

which is the uncertainty of the field for the number state $|n\rangle$. Note that even when n =0 there is still fluctuation called vacuum fluctuations. Recall that $\mathcal{E}_0$ is the electric energy per photon, so n here might as well represent the state of the field containing n photons.

1.3 Quadrature Operators

Recall that the time dependence expression of the electric field operator is

$$\hat{E}_x = \mathcal{E}_0 \left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right)\sin(kz) \tag{5}$$

we can define the **Quadrature Operators**

$$\begin{aligned}\hat{X}_1 &= \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right)\\ \hat{X}_2 &= \frac{1}{2i}\left(\hat{a} - \hat{a}^\dagger\right)\end{aligned} \tag{6}$$

which makes the field operator to be

$$\hat{E}_x(t) = 2\mathcal{E}_0 \sin(kz)\left[\hat{X}_1\cos(\omega t) + \hat{X}_2\sin(\omega t)\right] \tag{7}$$

note that these two operators are the position and momentum operators appeared previously but scaled to be dimensionless, it satisfies:

- **Communtation**

$$\left[\hat{X}_1, \hat{X}_2\right] = \frac{i}{2}$$

- **Uncertainties**

$$\left\langle \left(\Delta\hat{X}_1\right)^2\right\rangle \left\langle \left(\Delta\hat{X}_2\right)^2\right\rangle \geq \frac{1}{16} \tag{8}$$

- **Varience:**

$$\langle n|\,\hat{X}_1\,|n\rangle = 0 = \langle n|\,\hat{X}_2\,|n\rangle$$

and

$$\begin{aligned}\langle n|\hat{X}_1^2|n\rangle &= \frac{1}{4}\langle n|\hat{a}^2 + \hat{a}^{\dagger^2} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger|n\rangle\\ &= \frac{1}{4}\langle n|\hat{a}^2 + \hat{a}^\dagger 2 + 2\hat{a}^\dagger\hat{a} + 1|n\rangle\\ &= \frac{1}{4}(2n+1) = \langle n|\hat{X}_2^2|n\rangle\end{aligned} \tag{9}$$

1.4 Multimode Fields

Here are some things to recall

- Electric and magnetic radiation fields may be given in terms of the vector potential $\mathbf{A(r,t)}$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = 0 \tag{10}$$

with the condition

$$\begin{aligned}\nabla\cdot\mathbf{A}(r,t) &= 0\\ \mathbf{E}(r,t) &= -\frac{\partial\mathbf{A}(r,t)}{\partial t}\\ \mathbf{B}(r,t) &= \nabla\times\mathbf{A}(r,t)\end{aligned}$$

- **All** physical results obtained from this model of a cubic cavity of side length L with perfectly reflecting wall comparing to the dimensions of anything inside the cavity, should be independent of the size of the cavity as we can take $L\to\infty$. The purpose of the cavity is to impose periodic boundary conditions on the faces of the cube such that $e^{ik_x x} = e^{ik_x(x+L)}$. This property follows that

$$k_x = (\frac{2\pi}{L})m_x \quad m_x = 0, \pm 1....$$

and the wave vector is then

$$\mathbf{k} = \frac{2\pi}{L}(m_x, m_y, m_z)$$

- Assuming a quasi-continuous limit(since the wavelength are small compared to L, we can find the equation by approximating all the small difference of m's by differential

$$dm = 2(\frac{V}{8\pi^3})dk_x dk_y dk_z = 2(\frac{V}{8\pi^3})k^2 dk d\Omega \quad \text{In k-space Spherical Polar Coordinates}$$

where $d\Omega = \sin(\theta)d\theta d\phi$, and by using the relation $k = \omega_k/c$ we can transform the equation above into

$$dm = 2(\frac{V}{8\pi^3})\frac{\omega^2 k}{c^3}d\omega_k d\Omega$$

where the 2 is to account for the two independent polarizations. Integrating both equations give

$$\left.\begin{array}{l}\text{the numbers of modes}\\ \text{in all directions}\\ \text{in the range } k \text{ to}\\ k+dk\end{array}\right\} = V\frac{k^2}{\pi^2}\,dk = V\rho_k\,dk$$

where $\rho_k dk$ is the mode density(number of modes per unit volume) and $\rho_k = k^2/\pi^2$ and

$$\left.\begin{array}{l}\text{the numbers of modes}\\ \text{in all directions}\\ \text{in the range } \omega_k \text{ to}\\ \omega_k+d\omega_k\end{array}\right\} = V\frac{\omega_k^2}{\pi^2 c^3}\,d\omega_k \equiv V\,\rho(\omega_k)\,d\omega_k$$

where $\rho(\omega_k) = \omega_k^2/(\pi^2 c^3)$

- The vector potential can be expressed as a superposition of all plane waves

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k},s}\mathbf{e}_{\mathbf{k}s}\left[A_{\mathbf{k}s}(t)e^{i\mathbf{k}\cdot\mathbf{r}} + A^*_{\mathbf{k}s}(t)e^{-i\mathbf{k}\cdot\mathbf{r}}\right]$$

A_{ks} is the complex amplitude of the field and e_{ks} is the real polarization vector. The sum over k simply means the sum over the set of integers m_x, m_y, m_z(all the modes) and the sum over s is the sum over the two independent polarization direction(i.e. x and y directions). In free space, the summation can be replaced by the integral

$$\sum_k \rightarrow \frac{V}{\pi^2}\int k^2 dk$$

Recall the wave equation for A field, we ahve

$$A_{ks}(t) = A_{ks}e^{-i\omega_k t}$$

so we can obtain the expression for electric and magnetic fields

$$\mathbf{E}(\mathbf{r},t) = i\sum_{\mathbf{k},s}\omega_k\mathbf{e}_{\mathbf{k}s}\left[A_{\mathbf{k}s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - A^*_{\mathbf{k}s}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right] \tag{11}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{i}{c}\sum_{\mathbf{k},s}\omega_k\left(\boldsymbol{\kappa}\times\hat{\mathbf{e}}_{\mathbf{k}s}\right)\left[A_{\mathbf{k}s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - A^*_{\mathbf{k}s}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right] \tag{12}$$

- Recall that the energy of the field is given by

$$H = \frac{1}{2}\int_V(\epsilon_0 E\cdot E + \frac{1}{\mu_0}B\cdot B)dV$$

and the periodic property gives that

$$\int_V e^{\pm i(\mathbf{k}\text{ - }\mathbf{k'})\cdot r}dV = \delta_{kk'}V$$

From which we can find the individual contribution of the electric and the magnetic field, detailed calculation can be found in Page 21 of the Introductory Quantum Optics textbook, the result is that

$$H = 2\epsilon_0 V\sum_{ks}\omega_k^2 A_{ks}(t)A^*_{ks}(t) = 2\epsilon_0 V\sum_{ks}\omega_k^2 A_{ks}A^*_{ks}$$

to Quantize the field, we must introduce the canonical variables

$$A_{\mathbf{k}s} = \frac{1}{\sqrt{2\omega_k(\varepsilon_0 V)}}\left[\omega_k q_{\mathbf{k}s} + ip_{\mathbf{k}s}\right],$$
$$A^*_{\mathbf{k}s} = \frac{1}{\sqrt{2\omega_k(\varepsilon_0 V)}}\left[\omega_k q_{\mathbf{k}s} - ip_{\mathbf{k}s}\right],$$

subbing it back to the equation we obtain the classical energy equations

$$H = \frac{1}{2}\sum_{ks}(p_{ks}^2 + \omega_k^2 q_{ks}^2) \tag{13}$$

and the canonical variables follow that

$$[\hat{q}_{\mathbf{k}s}, \hat{q}_{\mathbf{k}'s'}] = 0 = [\hat{p}_{\mathbf{k}s}, \hat{p}_{\mathbf{k}'s'}]$$
$$[\hat{q}_{\mathbf{k}s}, \hat{p}_{\mathbf{k}'s'}] = i\hbar\delta_{\mathbf{k}\mathbf{k}'}\delta_{ss'}$$

as for single mode we have that

$$\hat{a}_{\mathbf{k}s} = \frac{1}{\sqrt{2\hbar\omega_k}}\left(\omega_k\hat{q}_{\mathbf{k}s} + i\hat{p}_{\mathbf{k}s}\right),$$
$$\hat{a}^\dagger_{\mathbf{k}s} = \frac{1}{\sqrt{2\hbar\omega_k}}\left(\omega_k\hat{q}_{\mathbf{k}s} - i\hat{p}_{\mathbf{k}s}\right),$$

which satisfy

$$[\hat{a}_{\mathbf{k}s}, \hat{a}_{\mathbf{k}'s'}] = 0 = [\hat{a}^\dagger_{\mathbf{k}s}, \hat{a}^\dagger_{\mathbf{k}'s'}],$$
$$[\hat{a}_{\mathbf{k}s}, \hat{a}^\dagger_{\mathbf{k}'s'}] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{ss'}.$$

Based on the previous definition of the energy configuration, we have the Hamiltonian operator

$$\hat{H} = \sum_{ks}\hbar\omega_k(\hat{a}^\dagger_{ks}\hat{a}_{ks} + \frac{1}{2}) = \sum_{ks}\hbar\omega_k(\hat{n}_{ks} + \frac{1}{2}) = \sum_j \hbar\omega_j(\hat{n}_j + \frac{1}{2})$$

and a multimode photon number state is just a product of all the number states of each mode

$$|n_1\rangle|n_2\rangle ... = |n_1, n_2...\rangle = |\{n_j\}\rangle$$

- The action of annihilation and creation are

$$\hat{a}_j|n_1, n_2, \ldots, n_j, \ldots\rangle = \sqrt{n_j}\,|n_1, n_2, \ldots, n_j - 1, \ldots\rangle.$$

Similarly, for the creation operator

$$\hat{a}^\dagger_j|n_1, n_2, \ldots, n_j, \ldots\rangle = \sqrt{n_j + 1}\,|n_1, n_2, \ldots, n_j + 1, \ldots\rangle.$$

In general, all the number states can be generated from the vacuum state according to

$$|\{n_j\}\rangle = \prod_j \frac{(\hat{a}^\dagger_j)^{n_j}}{\sqrt{n_j!}}|\{0\}\rangle$$

also the amplitudes A_{ks} become operators which has the form

$$\hat{A}_{ks} = (\frac{\hbar}{2\omega_k\epsilon_0 V})^{1/2}\hat{a}_{ks}$$

and thus the quantized vector potential has the form

$$\hat{\mathbf{A}}(\mathbf{r}, t) = \sum_{\mathbf{k}s}\left(\frac{\hbar}{2\omega_k\varepsilon_0 V}\right)^{\frac{1}{2}}\mathbf{e}_{\mathbf{k}s}\left[\hat{a}_{\mathbf{k}s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + \hat{a}^\dagger_{\mathbf{k}s}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right].$$

The electric field operator is then

$$\hat{\mathbf{E}}(\mathbf{r},t) = i\sum_{\mathbf{k}s}\left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{\frac{1}{2}}\mathbf{e}_{\mathbf{k}s}\left[\hat{a}_{\mathbf{k}s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - \hat{a}^\dagger_{\mathbf{k}s}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right]$$

while the magnetic field operator is

$$\hat{\mathbf{B}}(\mathbf{r},t) = \frac{i}{c}\sum_{\mathbf{k}s}(\mathbf{k}\times\mathbf{e}_{\mathbf{k}s})\left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{\frac{1}{2}}\left[\hat{a}_{\mathbf{k}s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - \hat{a}^\dagger_{\mathbf{k}s}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right],$$

noted that the time-dependent annihilation operator for a free field is given by

$$\hat{a}_{ks}(t) = \hat{a}_{ks}e^{-i\omega_k t}$$

Therefore, the electric field can be written as

$$\hat{\mathbf{E}}(\mathbf{r},t) = i\sum_{\mathbf{k}s}\left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{\frac{1}{2}}\mathbf{e}_{\mathbf{k}s}\left[\hat{a}_{\mathbf{k}s}(t)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - \hat{a}^\dagger_{\mathbf{k}s}(t)e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right] = \hat{E}^{(+)}(r,t) + \hat{E}^{(-)}(r,t)$$

note that in atomic system, the wavelength is much longer than the radius of the atom

$$\frac{\lambda}{2\pi} = \frac{1}{k} \gg |r_{atom|}$$

such that we can expand the exponent

$$e^{\pm ik\cdot r} \approx 1 \pm ik\cdot r$$

Replacing the exponential by unity, we obtain

$$\hat{E}(r,t) \approx i(\frac{\hbar\omega}{2\epsilon_0 V})^{1/2}\mathbf{e}_x[\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}]$$

this is called the dipole approximation