Personal Notes on

PHY365

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Abstract

This is a set of lecture notes on PHY365 Quantum information, Instructed by Professor Daniel James.

Contents

1 Lecture 2: Introduction to Quantum Computing

Classically information is determined by the number of binary digits in classical computer. For example

Example 1: "To be or not to be" takes 18 letters so 18 bytes = 144 bits

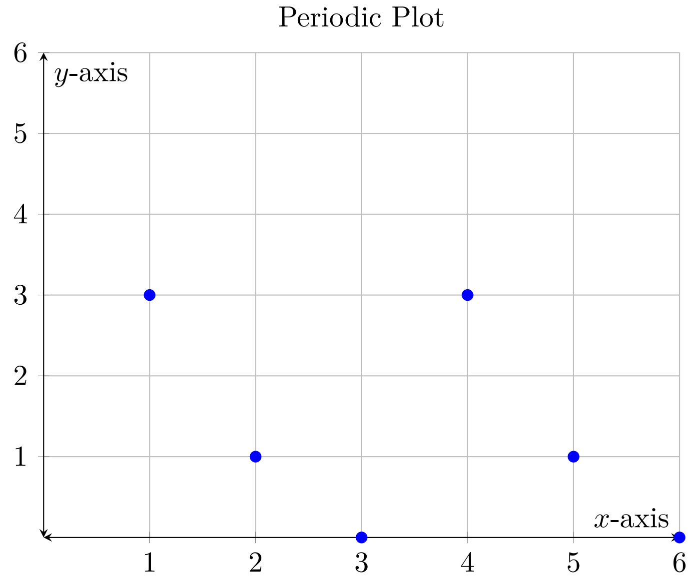
Quantumly we use qubit such that $\langle H|, \langle T| => \langle 1|, \langle 0| Whichwecanuseittointerpretastateofaqubit |\psi\rangle = \alpha |1\rangle + \beta |0\rangle, \quad \alpha^2 + \beta^2 = 1$

For two qubits, the interpretation is similar

$$|\psi\rangle = \alpha |01\rangle + \beta |00\rangle + \gamma |10\rangle + \delta |11\rangle$$

Similar for three qubits, yielding 8 possible configurations. 8 - 1 resulting in information of 7 bits. Since you can always rule out the signs(++ is same as –). For n qubits we can store $2^n - 1$ bits of classical information.

Lets define a periodic function f(x)(integer) like the function below



To find the period of this function, we classically evaluate for lots of x values, store the outcomes, and compare to deduce the period. You need to evaluate 2^{L-1} times to find out the period, and the number of x is $[0, 2^L - 1]$.

Now with quantum computer, we can do the same tasks with qubits. Lets say we have 2^L qubits to store x and more qubits to store f(x). In the form of

$$|0000...0000\rangle |00.....1....00\rangle -> |x\rangle |0\rangle$$

the first term is the argument register and the second term is the function register, where you consistently flip qubits until you reach the state of x. Then we use a transformation to convert the state into

$$\hat{U_+} |x\rangle |0\rangle => |x\rangle |f(x)\rangle$$

which classically iteratively doing for 2^{L-1} times to find the right state. However, if we store all the possibilities of $|x\rangle |0\rangle$ instead, then the total information

$$\frac{1}{2^{N/2}} \sum |x\rangle |0\rangle$$

can be converted into the form of $|x\rangle |f(x)\rangle$ with only one operation. That is just an interpretation of the advantage of quantum computer, that I have no clue why this works like this.And eventually the $|x\rangle$ is the measured period of the function.

2 Lecture 3: QM of Quantum Computing

Quantum computer is built out of qubits that are, assuming 2 dimensional input 0,1, aligned in row with each qubit representing a state of 0 or 1.

Example 2: For a system of 4 qubits, here are some possible configurations.

$$\begin{aligned}|0\rangle_4, |0\rangle_3, |0\rangle_2, |0\rangle_1 &= |0000\rangle \\ |0\rangle, |0\rangle, |0\rangle, |1\rangle &= |0001\rangle \\ \ldots.\end{aligned}$$

mathematically, we describe the state as tensor products

$$|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle = \bigotimes_{p=1}^{N} |x_p\rangle$$

this is essentially a mapping of all the possibilities in a vector space, such that we are getting 2^4 possible solutions in the above example.

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the quantum state is a vector space defined by the basis for example, a 2 dimensional vector space for one qubit is

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

where α, β are compelx valued components. If you have N qubits, you have 2^N dimensions. Also

$$|\alpha|^2 + |\beta|^2 = 1$$

and

$$|\psi\rangle = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}, \quad \langle\psi| = (\alpha^*, \beta^*)$$

Mathamtically, if we combine a bra and a ket symbols, it would be

$$\langle\psi|\psi\rangle = (\alpha^*, \beta^*)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = 1$$

Operators and Gate

Now we can change the state of qubits from

$$|\psi\rangle \to |\psi'\rangle \quad \alpha \to \alpha' \quad \beta \to \beta'$$

One big assumption is that **Quantum Mechanics is linear**(experimentally verified) such that

$$\begin{aligned}\alpha' &= U_{00}\alpha + U_{01}\beta \\ \beta' &= U_{10}\alpha + U_{11}\beta\end{aligned}$$

similarly for β'

$$\hat{U} = \begin{pmatrix}U_{00} & U_{01} \\ U_{10} & U_{11}\end{pmatrix}$$

which is the operator defined in quantum computing such that

$$\boxed{|\psi\rangle = \hat{U} |\psi'\rangle}$$

but now how does it work for bra instead of ket

$$\begin{aligned}\alpha'^* &= U_{00}^*\alpha^* + U_{01}^*\beta^* \\ \beta'^* &= U_{10}^*\alpha^* + U_{11}^*\beta^*\end{aligned}$$

which is now in matrix form

$$(\alpha'^*, \beta'^*) = (\alpha^*, \beta^*) \begin{pmatrix} U_{00}^* & U_{01}^* \\ U_{10}^* & U_{11}^* \end{pmatrix}$$

and the new matrix operator is straight up $\hat{U}^\dagger$ its hermitian Adjoint. Eventually, the symbols simplified to

$$\boxed{\langle\psi'| = \langle\psi| \hat{U}^\dagger} \tag{1}$$

Lets say we have multiple operators now such that

$$\hat{A}\hat{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

and the hermitian adjoint of this is

$$(\hat{A}\hat{B})^\dagger = \begin{bmatrix} e^*a^* + g^*b^* & e^*c^* + g^*d^* \\ f^*a^* + h^*b^* & f^*b^* + h^*d^* \end{bmatrix}$$

also

$$\boxed{(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger}$$

- This is true for any dimension of operator
- Operators must be unitary

  $$|\alpha'|^2 + |\beta'|^2 = 1$$

  Therefore, the operators must be unitary such that

  $$\hat{U}^\dagger \hat{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

  By using the ideaa for

  - $|\psi\rangle = |0\rangle$ $|\psi\rangle = |1\rangle$, $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$

  we can find that the matrix is just a identity matrix

3 Lecture 4: More into Operators

Bit Flip

A hermitian operator is the one where you take the hermitian conjugate of the matrix but get the same matrix again

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hat{X}^\dagger$$

Some other identities include

- Unitary:
$$\hat{X}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
- Bit Flip Gate: It can make $|0\rangle \rightarrow |1\rangle$ and vise versa

Phase Flip

A state can be written as

$$|\alpha|e^{i\phi_a} |0\rangle + |\beta|e^{i\phi_b} |1\rangle = e^{i\phi_a}(|\alpha| |0\rangle + |\beta|e^{i(\phi_b - \phi_a)} |1\rangle)$$

Yet this ϕ_a is the Global Phase of the system that is not measurable. A phase flip gate is doing where $\phi_b - \phi_a = \phi$

$$\phi \rightarrow \phi + \pi$$

and what rally happens to the state is like

$$|\psi'\rangle = (|\alpha| |0\rangle + |\beta|e^{i(\phi+\pi)} |1\rangle)$$

which is equivalent to

$$|\psi'\rangle = (|\alpha| |0\rangle - |\beta|e^{i(\phi)} |1\rangle)$$

and this operator is

$$\boxed{\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

A combination of these two yields another gate

$$\hat{Z}\hat{X} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \hat{Y} = \text{spin half operator}$$

they are so called the Pauli Operators

Hadamard Gate

It is defined as

$$\hat{H} = \frac{1}{\sqrt{2}}(\hat{X} + \hat{Z}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The effect is that when it is applied to a single qubit, the qubit will take all the possible values at once

$$\hat{H} |0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\hat{H} |1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Some interesting properties is that

$$\hat{H}\hat{X}\hat{H} = \hat{Z}$$

Arbitrary 2x2 Unitary Matrix

Just going through all the 2x2 unitary matrixes we have so far

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\dagger} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

if the matrix is unitary we must have

$$\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

assume the entries are all real. And solving the determinant of the matrix we find out that

$$|ad - bc|^2 = 1$$

such that

$$ad - bc = e^{i\phi} = 1 = \text{Global Phase}$$

another name is like **SU(2)** States

Therefore for any arbitrary Unitary matrix, it must have the form

$$\hat{U} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

where $|a|^2 + |b|^2 = 1$

Lets say $|0\rangle, |1\rangle$ are the computational basis then having a unitary operation on the states is like

$$\hat{U}|\psi\rangle = \alpha(\hat{U}|0\rangle) + \beta(\hat{U}|1\rangle) = \alpha|u\rangle + \beta|u_2\rangle$$

therefore we can either understand unitary operators as changing the phase or changing the Basis directly.

In a formal writting manner

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha[a|0\rangle - b^*|1\rangle] + \beta[b|0\rangle + a^*|1\rangle]$$

where the first term is essentially $|u\rangle$ and the second term is $|u_2\rangle$

4 Lecture 5: Time evolution of Qubits

It is obvious from pervious lecture that

$$|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$$

essentially a ladder operator in time domain. Furthermore, taking a derivative to the equation

$$\frac{d}{dt}\,|\psi(t)\rangle = \frac{\partial \hat{U}}{\partial t}\,|\psi(0)\rangle = \frac{\partial \hat{U}}{\partial t}\hat{U}^\dagger\,|\psi(t)\rangle$$

familier expression

$$\hat{U}\hat{U}^\dagger = 1$$

the chain rule of this expression yields a conclusion that

$$\frac{\partial \hat{U}}{\partial t}\hat{U}^\dagger = -\hat{U}\frac{\partial \hat{U}^\dagger}{\partial t} = \left(\frac{\partial \hat{U}}{\partial t}\hat{U}^\dagger\right)^\dagger$$

which we realize that it is an **anti hermitian operator**

$$\left(\frac{\partial \hat{U}}{\partial t}\hat{U}^\dagger\right) = \frac{\hat{H}}{i\hbar}$$

and now the first equation can be written in

$$\boxed{\frac{\partial}{\partial t}\,|\psi(t)\rangle = \frac{1}{i\hbar}\hat{H}\,|\psi(t)\rangle} \tag{2}$$

which is the Schrodinger Equation. lets say

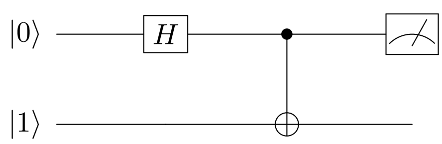
$$|\psi(t)\rangle = e^{-i\omega t}\,|\psi(0)\rangle$$

the equation becomes

$$\hbar\omega\,|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle$$

and $\hat{H}$ is straight up the Hamiltonian.

Quantum Circuit Diagram



Quantum Measurement

Here's a famous quote
"If the particle is in a state $|\psi\rangle$, measurement of the variable Ω will yield one of the eigenvalues ω with the probability $P(w) = |\langle\omega|\psi\rangle|^2$. The state of the system will change from $|\psi\rangle \to |\omega\rangle$ as a result.
For a quantum computer, we are measureing either 0 or 1. Therefore, the variable might be

$$\hat{\Omega} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

if we measure $\omega = 0$ we are measuring $|0\rangle$ and vice versa. This operator essentially tells us the eigenvalues correspond to one of the states.

State after the measurement is

$$|\psi\rangle \to \frac{\hat{\Pi}_0\,|\psi\rangle}{\sqrt{P(0)}}$$

the operator is called the projector operator, which is equivalent to the outer product of the states

$$\hat{\prod}_0 = |0 \otimes 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\hat{\prod}_1 = |1 \otimes 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}$$

This projection operator is also Non-Unitary

4.1 2.3 Two Qubits system

recall that a two qubit system can be written as

$$|\psi\rangle = \alpha |00\rangle + \beta |01\rangle + \gamma |01\rangle + \delta |11\rangle$$

A **Indepdent/Separable** qubits can be written in thet form

$$|\psi_{sep}\rangle = (a |0\rangle + b |1\rangle) \otimes (c |0\rangle + d |1\rangle) = ac |00\rangle + ad |01\rangle + bc |10\rangle + bd |11\rangle$$

Note that if we

$$\alpha\delta - \beta\gamma = acbd - adbc = 0$$

This is a criteria for separable states, if we obtain 0. Therefore, we define a variable so called **Concurrence**

$$C = 2|\alpha\delta - \beta\gamma|$$

- If C = 0, the qubits are separable state
- If C $\neq$ 0 the qubits are in **Entangled State.**

5 Lecture 6: More on Two Qubit system

Lets say now we apply a unitary operator on the two qubit states

$$|\psi'\rangle = (\hat{U} \otimes \hat{I})\, |\psi\rangle$$

which can be expressed in terms of the operators

$$|\psi'\rangle = \alpha(a\,|0\rangle - b^*\,|1\rangle)\,|0\rangle + \beta(a\,|0\rangle - b^*\,|1\rangle)\,|1\rangle + \gamma(b\,|0\rangle + a^*\,|0\rangle) + \delta(b\,|0\rangle + a^*\,|1\rangle)\,|1\rangle$$

and we can express the new coefficients in a matrix

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ -b^* & 0 & a^* & 0 \\ 0 & -b^* & 0 & a^* \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} a\hat{I} & b\hat{I} \\ -b^*\hat{I} & a^*\hat{I} \end{pmatrix} \tag{3}$$

The concurrence coefficient can then be written as

$$C' = 2|\alpha'\delta' - \beta'\gamma'|$$

by plugging in the expression we obtained in the previous state, we can come out with the following format

$$C' = 2|(|a|^2 + |b|^2)(\alpha\delta - \beta\gamma)| = C$$

one bit conclusion is that

- A "Local" (i.e. Single qubit) operator cannot change the entanglement!

Schmidt Decomposition

It is a form of decomposition of state such that

$$|\psi\rangle = \sum_{i,j=0} \chi_{ij}\, |ij\rangle \tag{4}$$

where

$$\chi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

This involves an important idea in physics called **Singular Value Decomposition**

$$\chi_{ij} = \sum_p u_{ip}\lambda_p V_{jp} \tag{5}$$

where U and V is the uniform matrices, λ is positive constant. This essentially tell that any matrix can be expressed as

1. Rotation in axis
2. Rescaling the vector
3. Rotate again.

using the expression, the equation we have earlier becomes

$$|\psi\rangle = \sum_p \lambda_p (\sum_i U_{ip}\,|i\rangle) \otimes (\sum_j V_{jp}\,|j\rangle) = (\hat{U} \otimes \hat{V}) \sum_p \lambda_p\,|pp\rangle = (\hat{U} \otimes \hat{V})(\lambda_0\,|00\rangle + \lambda_1\,|11\rangle) \tag{6}$$

which says that any state can be expressed as a combination of basic states with some basis change operation. where

$$\lambda_0^2 + \lambda_1^2 = 1$$

Looking at the Concurrence of this new expression

$$C = 2\lambda_0\lambda_1$$

so we can solve for the normalization coefficient lambda

$$\lambda_1 = \frac{C}{2\lambda_0}$$

$$\lambda_0^2 + \frac{(C/2)^2}{\lambda_0^2} = 1$$

so

$$\lambda_0 = \frac{1 \pm \sqrt{1 - C^2}}{2}$$

for a maximally entangled state C = 1, the state of the other qubit when you do a measurement in 1 qubit would be orthogonal to the current qubit having equal probability.