$$T(n) = \begin{cases} 1 & n = 1 \\ 2T(\lfloor \frac{n}{2} \rfloor) + n & \text{otherwise} \end{cases}$$

Prove by induction that $T(n) = O(n^2)$

$\exists\, c, n_0 > 0$ tq $\forall\, n \geq n_0 \quad 0 \leq T(n) \leq cn^2$

Usando las siguiente $C = \underline{2}$, $n_0 = \underline{1}$

$P(n) : T(n) \geq cn^2$

$P(K) : T(K) \geq cK^2$

C.B. $\underline{n=1}$

$T(1) = 1 \leq 2 \cdot 1^2 = 2$

H.I. $P(1) \ldots P(K-1)$ son T $\quad K > 1$

¿P_K?

Tenemos : $T(K) = 2T(\lfloor \frac{K}{2} \rfloor) + K$

Como $K \geq 2$ $\quad \lfloor \frac{K}{2} \rfloor \geq 1$ y $\lfloor \frac{K}{2} \rfloor < K \longrightarrow P(\lfloor \frac{K}{2} \rfloor) = T$

$$T(K) = 2T(\lfloor \tfrac{K}{2} \rfloor) + K \leq 2\left[2\left(\lfloor \tfrac{K}{2} \rfloor\right)^2\right] + K$$

$$\leq 4\lfloor \tfrac{K}{2} \rfloor^2 + K \leq 4\left(\tfrac{K}{2}\right)^2 + K$$

$$= K^2 + K \leq K^2 + K^2 = 2K^2$$

Prove by induction that $T(n) = \Omega(n \lg n)$

$P(n): \; T(n) \geq c n \lg n$

$k \geq 4 \rightarrow \lfloor \frac{k}{2} \rfloor < k \longrightarrow P(\lfloor \frac{k}{2} \rfloor) = T$

$T(k) = 2T(\lfloor \frac{k}{2} \rfloor) + k$

$\geq 2\left(c \lfloor \frac{k}{2} \rfloor \lg \lfloor \frac{k}{2} \rfloor\right) + k$

$\geq 2c\left(\frac{k}{2} - 1\right) \lg\left(\frac{k}{2} - 1\right) + k$

$\geq 2c\left(\frac{k}{2} - 1\right) \lg\left(\frac{k}{4}\right) + k$

$k \geq 4$

$\lfloor \frac{k}{2} \rfloor \geq \frac{k}{2} - 1 \geq \frac{k}{4}$

$= c(k-2)(\lg k - \lg 4) + k$

$= c(k-2)(\lg k - 2) + k$

$= c(k \lg k - 2\lg k - 2k + 4) + k$

$= ck\lg k + k + 4 - 2\lg k - 2k \geq ck \lg k$

$\lg k \leq k$

Sea: $n_0 = 1$ y

$c = \min\left\{ \frac{1}{8}, \frac{T(2)}{2\lg 2}, \frac{T(3)}{3 \lg 3} \right\}$

C.B.

$$\begin{cases} T(1) \geq c \cdot 0 \\ T(2) \geq c \cdot 2 \lg 2 \\ T(3) \geq c \cdot 3 \lg 3 \end{cases}$$

Esto último cumple cuando $c \leq \frac{1}{8}$

Master theorem

$9T(\lfloor n/4 \rfloor) + n^2 - n$

$\log_4 9 = b$

$4^b = 9$

b 1

2

$$T(n)\begin{cases} n & 1 \leq n \leq 2 \\ 6T\left(\lfloor \frac{n}{3} \rfloor\right) + n^2 - n & cc. \end{cases}$$

$n = 3^k$ $\qquad 3^2$ $\qquad 3^k \cdot 2 \cdot 3 \qquad 3^{k+1} \cdot 2$

$m = k$

$$\left(T(3^k) - \cancel{6T(3^{k-1})} = 3^{2k} - 3^k\right) \times 6^{k-m}$$

$$\left(\cancel{T(3^{k-1})} - \cancel{6T(3^{k-2})} = 3^{2(k-1)} - 3^{k-1}\right) \times 6^1 \rightarrow \quad k-(k-1) \quad 3^{2(k-1)} \cdot 3^{k-1} \cdot 2^{k-1}$$

$$\left(\cancel{T(3^{k-2})} - \cancel{6T(3^{k-3})} = 3^{2(k-2)} - 3^{k-2}\right) \times 6^2 \quad k-(k-2) \quad 3^{3(k-1)} \cdot 2^{k-1}$$

$\vdots$

$$(3^3 \cdot 2)^{k-1}$$

$\vdots \; k-(k-1)$

$m = 1$

$$\left(\cancel{T(3)} - 6T(1) = 3^2 - 3\right) \times 6^{k-1}$$

$$6^0 T(3^k) - 6^k T(1) = \sum_{i=1}^{k} (3^3 \cdot 2)^{k-i} - 3^{i+1} \cdot 2$$

$n = 3^m$

$T(3^k) = 6T(3^{k-1}) + (3^k)^2 - 3^k$

$a_k = 6^{m-k}\, T(3^k)$

$6^{m-k}\left[T(3^k) - 6T(3^{k-1}) = 9^k - 3^k\right]$

$$\sum_{k=1}^{m} a_k - a_{k-1} = \sum_{k=1}^{m} (9^k - 3^k) \cdot 6^{m-k}$$

$$a_m - a_0 = \sum_{k=1}^{m} 9^k \cdot 6^{m-k} - \sum_{k=1}^{m} 3^k \cdot 6^{m-k}$$

$$= \sum_{k=1}^{m} 6^m \cdot \left(\frac{9}{6}\right)^k - \sum_{k=1}^{m} 6^m \cdot \left(\frac{3}{6}\right)^k$$

$\alpha = \frac{9}{6} \quad \beta = \frac{3}{6}$

$$= 6^m \left(\sum_{k=1}^{m} \alpha^k - \sum_{k=1}^{m} \beta^k \right)$$

$$= 6^m \left(\frac{\alpha^{m+1} - \alpha}{\alpha - 1} - \frac{\beta^{m+1} - \beta}{\beta - 1} \right)$$

$$= 6^m \left(\frac{\left(\frac{3}{2}\right)^{m+1} - \left(\frac{3}{2}\right)}{\frac{3}{2} - 1} - \frac{\left(\frac{1}{2}\right)^{m+1} - \left(\frac{1}{2}\right)}{\frac{1}{2} - 1} \right)$$

$$= 6^m \left(2\left[\left(\frac{3}{2}\right)^{m+1} - \left(\frac{3}{2}\right)\right] + 2\left[\left(\frac{1}{2}\right)^{m+1} - \left(\frac{1}{2}\right)\right] \right)$$

$$= 2\left(\frac{3}{2}\right) 6^m \left(\frac{3}{2}\right)^m - 2 \cdot 6^m \left(\frac{3}{2}\right) + 6^m \cdot 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^m - 6^m \cdot 2 \cdot \left(\frac{1}{2}\right)$$

$$= 3 \cdot 9^m - 3 \cdot 6^m + 3^m - 6^m$$

induction

$\frac{S^{n+1} - S^a}{S-1}$

$\sum_{i=0}^{n} S^i = \quad + \quad - \sum_{i=0}^{a-1} S^i = \frac{S^a - 1}{S-1}$

$\sum_{i=0}^{n} S^i = \frac{S^{n+1} - 1}{S-1} \qquad \frac{S^{n+1} - \cancel{1} - S^a + \cancel{1}}{S-1}$

$S^0 + S^1 + \cdots S^{a-1}$