Ejercicio 4. For each of the following exercises:

- Solve the problems explicitly, ignoring the floor operator.
- Bound both the upper and lower limits using the previous item. Conclude the order Θ of the recurrence. You can use that it is given all these recurrences are increasing.
- Check, using induction, that the Θ notation found in the previous item is correct. You must do this directly from the definition, without using the results of previous items.
- Check using master Theorem if applicable.

Suppose in each case, that $T(1) = 1$ and $T(0) = 0$.

(a) $T(n) = 2T(\lfloor n/2 \rfloor) + n^2$.

(b) $T(n) = 2T(n-1) + 3n - 2$.

(c) $T(n) = 4T(\lfloor n/2 \rfloor) + n$.

(d) $T(n) = 2T(\lfloor n/2 \rfloor) + n^3$.

(e) $T(n) = 7T(\lfloor n/2 \rfloor) + n^2$.

$$\boxed{\sum_{k=1}^{m} a_k - a_{k-1} = a_m - a_0}$$

$$\boxed{\sum_{k=a}^{n} x^k = \frac{x^{n+1} - x^a}{x - 1} = \frac{x^a - x^{n+1}}{1 - x}}$$

Master theorem

$T(n) = aT\left(\frac{n}{b}\right) + \theta(n^k)$

- $\frac{\lg a}{\lg b} > k \rightarrow \theta(n^{\lg a / \lg b})$
- $\frac{\lg a}{\lg b} = k \rightarrow \theta(n^k \lg n)$
- $\frac{\lg a}{\lg b} < k \rightarrow \theta(n^k)$

a) $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n^2$

$T(n) - 2T(\lfloor \frac{n}{2} \rfloor) = n^2 \qquad n = 2^m$

$k = m \qquad (T(2^m) - \cancel{2T(2^{m-1})} = 2^{2m}) \cdot 2^0$

$k = m-1 \qquad (\cancel{T(2^{m-1})} - \cancel{2T(2^{m-2})} = 2^{2(m-1)}) \cdot 2^1$

$k = m-2 \qquad (\cancel{T(2^{m-2})} - \cancel{2T(2^{m-3})} = 2^{2(m-2)}) \cdot 2^2$

$\vdots$

$k = k \qquad (\cancel{T(2^k)} - \cancel{2T(2^{k-1})} = 2^{2k}) \cdot 2^{m-k}$

$\vdots$

$k = 1 \qquad (\cancel{T(2^1)} - 2T(2^0) = 2^2) \cdot 2^{m-1}$

$a_k = 2^{m-k} T(2^k)$

$a_{k-1} = 2^{m-k+1} T(2^{k-1})$

$$T(2^m) - 2^m T(2^0) = \sum_{k=1}^{m} 2^{2k} \cdot 2^{m-k}$$

$$= 2^m \sum_{k=1}^{m} 2^{2k} \cdot 2^{-k}$$

$$= 2^m \sum_{k=1}^{m} 2^k$$

$$= 2^m \left(\frac{2^{m+1} - 2}{2 - 1} \right)$$

$$= 2^m \cdot 2^m$$

$$T(2^m) - 2^m = 2^{2m}$$

$$T(2^m) = 2^{2m} + 2^m$$

$$T(n) = n^2 + n$$

b) (*) $2^k \leq n < 2^{k+1}$ Como la recurrencia es creciente

$T(2^k) \leq T(n) < T(2^{k+1})$

$C_1 n^2 \leq 2^{2k} + 2^k \leq T(n)$

$n^2 < (2^{k+1})^2$ (*)

$n^2 < 2^{2k+2}$

$n^2 < 2^2 . 2^{2k}$ $\forall k \geq 1$

$\frac{1}{4} n^2 < 2^{2k} \leq 2^{2k} + 2^k \leq T(n)$

$\therefore$ $T(n) \geq \frac{1}{4} n^2$ $\forall n \geq 2$

$T(n) = \Omega(n^2)$

$T(n) < 2^{2(k+1)} + 2^{k+1}$

$2^2 2^{2k} + 2 . 2^k \leq C_2 n^2$

$2^2 2^{2k} + 2 . 2^k \leq 2^3 . 2^{2k}$

$\cancel{2^k}(2^2 . 2^k + 2) \leq 2^3 . 2^k . \cancel{2^k}$

$2^2 . 2^k + 2 \leq 2^3 . 2^k$ $\forall k \geq 2$

$2^3 . 2^k \leq 2^3 n^2$

$\therefore$ $T(n) \leq 8 n^2$ $\forall n \geq 4$

$T(n) = O(n^2)$

$T(n) = \Theta(n^2)$ $\forall n \geq 4$

(2) $T(n) = 2T(n-1) + 3n - 2$

$= 2(2T(n-2) + 3(n-1) - 2) + 3n - 2$

$= 2^2 T(n-2) + 2.3(n-1) - 2.2 + 3n - 2$

$= 2^2(2T(n-3) + 3(n-2) - 2) + 2.3(n-1) + 3n - 2^2 - 2$

$= 2^3 T(n-3) + 2^2.3(n-2) + 2.3(n-1) + 3n - 2^3 - 2^2 - 2$

$\vdots$

$= 2^i T(n-i) + \sum_{k=0}^{i-1} 2^k.3(n-k) - \sum_{k=1}^{i} 2^k$

$\vdots$

$= 2^n T(0) + \sum_{k=0}^{n-1} 2^k.3(n-k) - \sum_{k=1}^{n} 2^k$

$0 + 2^1 + 2.2^2 + 3.2^3 + 4.2^4 + 5.2^5$

$= 2^n + 3\sum_{k=0}^{n-1} 2^k n - k2^k - \sum_{k=1}^{n} 2^k$

$\frac{2}{(1-2)^2}$

$= 2^n + 3\left(n\sum_{k=0}^{n-1} 2^k - \sum_{k=0}^{n-1} k2^k\right) - \sum_{k=1}^{n} 2^k$

$= 2^n + 3(n(2^n - 1) -$