

**Ejercicio 1.** Let  $X$  be a random variable that represents the number of heads in two tosses of a fair coin. What is the value of  $E[X^2]$ ? What is the value of  $E[X]^2$ ?

| $\Omega$ | $\times$ | $\mathbb{Z}_0^+$ | $(\cdot)^2$ | $\mathbb{Z}_0^+$ |
|----------|----------|------------------|-------------|------------------|
| cc       |          | 2                |             | 4                |
| cs       |          | 1                |             | 1                |
| sc       |          | 1                |             | 1                |
| ss       |          | 0                |             | 0                |

$$\begin{aligned}
 E[X^2] &= \sum_{c=0}^{\infty} c P(X^2=c) = \cancel{0 P(X^2=0)} + 1 P(X^2=1) + 4 P(X^2=2) \\
 &= P(X^2=1) + 4 P(X^2=4) \\
 &= P((X=-1) \cup (X=1)) \\
 &= P(X=1) + 4 P(X=2) \\
 &= \frac{2}{4} + 4 \left( \frac{1}{4} \right) = 1,5
 \end{aligned}$$

Con variable indicadora:

$X = \#$  caras al lanzar 2 monedas

$X_1 = \#$  caras de la moneda 1

$$X = X_1 + X_2$$

$X_2 = \#$  " " " " 2

$$\begin{aligned}
 E[X^2] &= E[(X_1 + X_2)^2] = E[X_1^2 + X_1 X_2 + X_2 X_1 + X_2^2] = E[X_1^2] + 2E[X_1 X_2] + E[X_2^2] \\
 &= P(X_1^2=1) + 2P(X_1 X_2=1) + P(X_2^2=1) \\
 &= P(X_1=1) + 2P((X_1=1) \cap (X_2=1)) + P(X_2=1) \\
 &= P(X_1=1) + 2P(X_1=1)P(X_2=1) + P(X_2=1) \\
 &= \frac{1}{2} + 2 \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} = \frac{3}{2}
 \end{aligned}$$

$$E[X] = E[X_1 + X_2] = E[X_1] + E[X_2] = P(X_1=1) + P(X_2=1) = \frac{1}{2} + \frac{1}{2} = 1$$

$$E[X]^2 = 1^2 = 1$$

$$\text{Var}(X) = E[(X - E[X])^2]$$

$$= E[X^2 - 2XE[X] + E[X]^2]$$

$$= E[X^2] - E[2XE[X]] + E[E[X]^2]$$

$$= E[X^2] - 2E[X] \cdot E[X] + E[X]^2$$

$$= E[X^2] - 2E[X]^2 + E[X]^2$$

$$= E[X^2] - E[X]^2 \geq 0$$

$E[E[X]^2]$  : el promedio de constantes es igual a la constante

número

**Ejercicio 2.** Let  $X$  be a random variable that represents the sum of the results in the roll of  $n$  dice. What is the value of  $E[X]$ ?

$$\Omega = \{1, \dots, 6\}^n$$

$$|\Omega| = 6^n$$

$X$  = Suma de los  $n$  caras

¿ $E[X]$ ?

$$E[X] = \sum_{c=0}^{\infty} cP(X=c)$$

→ No es indicador, pero es más sencilla

•  $X_i$  = La cara del dado  $i$

$$X = X_1 + X_2 + \dots + X_n = \sum_{i=1}^n X_i$$

$$E[X] = \sum_{i=1}^n E[X_i]$$

$$E[X_i] = \sum_{c=0}^{\infty} cP(X_i=c) \rightarrow \sum_{c=1}^6 cP(X_i=c)$$

$$= \sum_{c=1}^6 c \cdot \frac{1}{6} = \frac{6(6+1)}{2} \cdot \frac{1}{6} = \frac{7}{2}$$

$$E[X] = \sum_{i=1}^n \frac{7}{2} = 3.5n$$



**Ejercicio 5.** Consider the following algorithm that determines the largest and smallest element of a vector  $v[1 \dots n]$  with distinct positive numbers.

**LARGESTSMALLEST**( $v, n$ )

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1: largest = v[1]
2: smallest = v[1]
3: for i = 2 to n
4:   if v[i] > largest
5:     largest = v[i]
6:   else
7:     if v[i] < smallest
8:       smallest = v[i]
9: return largest, smallest

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siempre se ejecuta

la línea 7 se ejecuta siempre al else

Assume the input of the algorithm is a permutation of 1 to  $n$  chosen uniformly from all permutations of 1 to  $n$ . What is the expected number of comparisons executed in line 7 of the algorithm? What is the expected number of assignments performed in line 8 of the algorithm?

•  $\Omega = \{ \text{permutaciones } (1, \dots, n) \}$

•  $X = \#$  comparaciones ejecutadas de la línea 7 al ejecutar el algoritmo

$X_i = \#$  comparaciones en la  $i$ -ésima iteración  $\rightarrow$  variable indicadora

$$X_i = \begin{cases} 1 & \text{se ejecuta 7} \\ 0 & \text{no} \end{cases}$$

$$X = X_2 + X_3 + \dots + X_n = \sum_{i=2}^n X_i$$

$$E[X] = \sum_{i=2}^n E[X_i] = \sum_{i=2}^n P(X_i = 1) = \sum_{i=2}^n P(\text{en la } i\text{-ésima iteración se ejecutamos la línea 7})$$

$$= P(\text{en la } i\text{-ésima iteración se ejecutamos la línea 7})$$

$$P(A^c) = 1 - P(A)$$

$$= P(\text{" " " " " " " línea 6})$$

$$= P(\text{" " " " " la condicional de la línea 4 es falsa})$$

$$= P(\text{" " " " " } (v[i] > \text{mayor}) = F)$$

$$= P(\text{" " " " " } v[i] < \max(v[1:i-1]))$$

$$= P(\text{" " " " " el } \underline{\max v[1:i]} \text{ NO esté en la posición } i)$$

$$= 1 - P(\text{" " " " " el máx } v[1:i] \text{ SÍ esté en la posición } i)$$

$$= 1 - \frac{1}{i} = \frac{i-1}{i}$$

$$i=2 \quad \frac{1-1}{2} = 0$$

$$E[X] = \sum_{i=2}^n E[X_i] = \sum_{i=2}^n \left(1 - \frac{1}{i}\right) = \sum_{i=2}^n \left(1 - \frac{1}{i}\right)$$

$$= \sum_{i=2}^n 1 - \sum_{i=2}^n \frac{1}{i} = n - \ln n$$

HIRING PROBLEM

colocar el máximo en todas las posiciones menos el último

Per conteo =  $\frac{(i-1) \times (i-1)!}{i!} = \frac{i-1}{i}$

los  $i-1$  números restantes permutados

$i$  permutaciones