Ejercicio 1. Let X be a random variable that represents the number of heads in two tosses of a fair coin. What is the value of $E[X^2]$? What is the value of $E[X]^2$?

$$\Omega \xrightarrow{X} \mathbb{Z}_0^+ \xrightarrow{(\ )^2} \mathbb{Z}_0^+$$

CC	2	4
CS	1	1
SC	1	1
SS	0	0

$$E[X^2] = \sum_{c=0}^{\infty} c\,P(X^2=c) = \cancel{0\,P(X^2=0)} + 1\,P(X^2=1) + 4\,P(X^2=2)$$

$$= P(X^2=1) + 4P(X^2=4)$$

$$= P((X=-1) \cup (X=1))$$

$$= P(X=1) + 4P(X=2)$$

$$= \frac{2}{4} + 4\left(\frac{1}{4}\right) = 1{,}5$$

Con variable indicadora:

$X = \#$ caras al lanzar 2 monedas

$X_1 = \#$ caras de la moneda 1 $\qquad X = X_1 + X_2$

$X_2 = \#$ " " " " 2

$$E[X^2] = E[(X_1+X_2)^2] = E[X_1^2 + X_1X_2 + X_2X_1 + X_2^2] = E[X_1^2] + 2E[X_1X_2] + E[X_2^2]$$

$$= P(X_1^2=1) + 2P(X_1X_2=1) + P(X_2^2=1)$$

$$= P(X_1=1) + 2P((X_1=1)\cap(X_2=1)) + P(X_2=1)$$

$$= P(X_1=1) + 2P(X_1=1)\,P(X_2=1) + P(X_2=1)$$

$$= \frac{1}{2} + \cancel{2}\cdot\frac{1}{\cancel{2}}\cdot\frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

$$E[X] = E[X_1+X_2] = E[X_1] + E[X_2] = P(X_1=1) + P(X_2=1) = \frac{1}{2} + \frac{1}{2} = 1$$

$$E[X]^2 = 1^2 = 1$$

$$Var(X) = E[(X - E[X])^2]$$
$$= E[X^2 - 2X E[X] + E[X]^2]$$
$$= E[X^2] - E[2X E[X]] + E[E[X]^2]$$
$$= E[X^2] - 2E[X] \cdot E[X] + E[X]^2$$
$$= E[X^2] - 2E[X]^2 + E[X]^2$$
$$= E[X^2] - E[X]^2 \geq 0$$

número

$E[E[X]^2]$: el promedio de constantes es igual a la constante

Ejercicio 2. Let X be a random variable that represents the sum of the results in the roll of n dice. What is the value of $E[X]$?

$\Omega = \{1 - 6\}^n$ $\quad |\Omega| = 6^n$ $\quad X =$ Suma de las n caras $\quad$ ¿$E[X]$?

$$E[X] = \sum_{c=0}^{\infty} cP(X=c)$$

No es indicadora pero es más sencilla

- $X_i =$ La cara del dado i

$$X = X_1 + X_2 + \dots + X_n = \sum_{i=1}^{n} X_i$$

$$E[X] = \sum_{i=1}^{\infty} E[X_i]$$

$$E[X_i] = \sum_{c=0}^{\infty} cP(X_i=c) \longrightarrow \sum_{c=1}^{6} cP(X_i=c)$$

$$= \sum_{c=1}^{6} c \cdot \frac{1}{6} = \frac{6(6+1)}{2} \cdot \frac{1}{6} = \frac{7}{2}$$

$$E[X] = \sum_{i=1}^{n} \frac{7}{2} = 3.5n$$

Ejercicio 5. Consider the following algorithm that determines the largest and smallest element of a vector $v[1 \dots n]$ with distinct positive numbers.

LARGESTSMALLEST(v, n)

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1: largest = v[1]
2: smallest = v[1]
3: for i = 2 to n
4:   if v[i] > largest
5:     largest = v[i]
6:   else
7:     if v[i] < smallest
8:       smallest = v[i]
9: return largest, smallest
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siempre se ejecuta

la línea 7 se ejecuta siempre al else

Assume the input of the algorithm is a permutation of 1 to n chosen uniformly from all permutations of 1 to n. What is the expected number of comparisons executed in line 7 of the algorithm? What is the expected number of assignments performed in line 8 of the algorithm?

- $\Omega = \{$ permutaciones $(1, \dots, n) \}$
- $X = \#$ comparaciones ejecutadas de la línea 7 al ejecutar el algoritmo

$X_i = \#$ comparaciones en la i-ésima iteración $\longrightarrow$ variable indicadora

$$X_i = \begin{cases} 1 & : \text{se ejecuta 7} \\ 0 & : \text{cc} \end{cases}$$

$$X = X_2 + X_3 + \dots + X_n = \sum_{i=2}^{n} X_i$$

$$E[X] = \sum_{i=2}^{n} E[X_i] = \sum_{i=2}^{n} P(X_i = 1) = \sum_{i=2}^{n} P(\text{En la i-ésima iteración ejecutamos la línea 7})$$

$= P($En la i-ésima iteración ejecutamos la línea 7$)$ $\qquad P(A^c) = 1 - P(A)$

$= P($ " " " " " línea 6$)$

$= P($ " " " " la condicional de la línea 4 es falsa$)$

$= P($ " " " " $(V[i] > \text{mayor}) = F)$ $\qquad$ HIRING PROBLEM

$= P($ " " " " $V[i] < \max(V[1:i-1]))$

$= P($ " " " " el $\max V[1:i]$ NO esté en la posición $i)$

$= 1 - P($ " " " " el $\max V[1:i]$ Si está en la posición $i)$

$$= 1 - \frac{1}{i} = \frac{i-1}{i}$$

$$i=1 \quad 1 - \frac{1}{1} = 0$$

$$E[X] = \sum_{i=2}^{n} E[X_i] = \sum_{i=2}^{n} \left(1 - \frac{1}{i}\right) = \sum_{i=1}^{n} \left(1 - \frac{1}{i}\right)$$

$$= \sum_{i=1}^{n} 1 - \sum_{i=1}^{n} \frac{1}{i} = n - \lg n$$

Colocar el máximo en todo los espacios menos el último

Los $i-1$ números restantes permutados

$$\text{Por conteo} = \frac{\overbrace{(i-1)} \times \overbrace{(i-1)!}}{i!} = \frac{i-1}{i}$$

i permutaciones