

Ejercicio 4. For each of the following exercises:

- Solve the problems explicitly, ignoring the floor operator.
- Bound both the upper and lower limits using the previous item. Conclude the order Θ of the recurrence. You can use that it is given all these recurrences are increasing.
- Check, using induction, that the Θ notation found in the previous item is correct. You must do this directly from the definition, without using the results of previous items.
- Check using master Theorem if applicable.

Suppose in each case, that $T(1) = 1$ and $T(0) = 0$.

(a) $T(n) = 2T(\lfloor n/2 \rfloor) + n^2$.

(b) $T(n) = 2T(n-1) + 3n - 2$.

(c) $T(n) = 4T(\lfloor n/2 \rfloor) + n$.

(d) $T(n) = 2T(\lfloor n/2 \rfloor) + n^3$.

(e) $T(n) = 7T(\lfloor n/2 \rfloor) + n^2$.

$$\sum_{k=1}^m a_k - a_{k-1} = a_m - a_0$$

$$\sum_{k=a}^n x^k = \frac{x^{n+1} - x^a}{x - 1} = \frac{x^a - x^{n+1}}{1 - x}$$

Master theorem

$$T(n) = aT\left(\frac{n}{b}\right) + \theta(n^k)$$

$$\bullet \frac{\lg a}{\lg b} > k \rightarrow \theta(n^{\lg a / \lg b})$$

$$\bullet \frac{\lg a}{\lg b} = k \rightarrow \theta(n^k \lg n)$$

$$\bullet \frac{\lg a}{\lg b} < k \rightarrow \theta(n^k)$$

$$a) T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n^2$$

$$T(n) - 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) = n^2 \quad n = 2^m$$

$$k=m \quad (T(2^m) - 2T(2^{m-1}) = 2^{2m}) \cdot 2^0$$

$$k=m-1 \quad (T(2^{m-1}) - 2T(2^{m-2}) = 2^{2(m-1)}) \cdot 2^1$$

$$k=m-2 \quad (T(2^{m-2}) - 2T(2^{m-3}) = 2^{2(m-2)}) \cdot 2^2$$

$$\vdots$$

$$k=k \quad (T(2^k) - 2T(2^{k-1}) = 2^{2k}) \cdot 2^{m-k}$$

$$\vdots$$

$$k=1 \quad (T(2^1) - 2T(2^0) = 2^2) \cdot 2^{m-1}$$

$$a_k = 2^{m-k} T(2^k)$$

$$a_{k-1} = 2^{m-k+1} T(2^{k-1})$$

$$\begin{aligned} T(2^m) - 2^m T(2^0) &= \sum_{k=1}^m 2^{2k} \cdot 2^{m-k} \\ &= 2^m \sum_{k=1}^m 2^{2k} \cdot 2^{-k} \\ &= 2^m \sum_{k=1}^m 2^k \\ &= 2^m \left(\frac{2^{m+1} - 2}{2 - 1} \right) \\ &= 2^m \cdot 2^m \end{aligned}$$

$$T(2^m) - 2^m = 2^{2m}$$

$$T(2^m) = 2^{2m} + 2^m$$

$$T(n) = n^2 + n$$

b) (*) $2^k \leq n < 2^{k+1}$ Como la recurrencia es creciente

$$\underline{T(2^k) \leq T(n) < T(2^{k+1})}$$

$$C_1 n^2 \leq 2^{2k} + 2^k \leq T(n)$$

$$n^2 < (2^{k+1})^2 \quad (*)$$

$$n^2 < 2^{2k+2}$$

$$n^2 < 2^2 \cdot 2^{2k} \quad \forall k \geq 1$$

$$\frac{1}{4} n^2 < 2^{2k} \leq 2^{2k} + 2^k \leq T(n)$$

$$\therefore T(n) \geq \frac{1}{4} n^2 \quad \forall n \geq 2$$

$$T(n) = \Omega(n^2)$$

$$T(n) < 2^{2(k+1)} + 2^{k+1}$$

$$2^2 \cdot 2^{2k} + 2 \cdot 2^k \leq C_2 n^2$$

$$2^2 \cdot 2^{2k} + 2 \cdot 2^k \leq 2^3 \cdot 2^{2k}$$

$$\cancel{2^k} (2^2 \cdot 2^k + 2) \leq 2^3 \cdot 2^k \cdot \cancel{2^k}$$

$$2^2 \cdot 2^k + 2 \leq 2^3 \cdot 2^k \quad \forall k \geq 2$$

$$2^3 \cdot 2^k \leq 2^3 n^2$$

$$\therefore T(n) \leq 8 n^2 \quad \forall n \geq 4$$

$$T(n) = O(n^2)$$

$$T(n) = \Theta(n^2) \quad \forall n \geq 4$$

$$(2) \quad T(n) = 2T(n-1) + 3n - 2$$

$$= 2(2T(n-2) + 3(n-1) - 2) + 3n - 2$$

$$= 2^2 T(n-2) + 2 \cdot 3(n-1) - 2 \cdot 2 + 3n - 2$$

$$= 2^2 (2T(n-3) + 3(n-2) - 2) + 2 \cdot 3(n-1) + 3n - 2^2 - 2$$

$$= 2^3 T(n-3) + 2^2 \cdot 3(n-2) + 2 \cdot 3(n-1) + 3n - 2^3 - 2^2 - 2$$

$$= 2^i T(n-i) + \sum_{k=0}^{i-1} 2^k \cdot 3(n-k) - \sum_{k=1}^i 2^k$$

$$= 2^n T(0) + \sum_{k=0}^{n-1} 2^k \cdot 3(n-k) - \sum_{k=1}^n 2^k$$

$$= 2^n + 3 \sum_{k=0}^{n-1} 2^k n - k 2^k - \sum_{k=1}^n 2^k$$

$$= 2^n + 3 \left(n \sum_{k=0}^{n-1} 2^k - \sum_{k=0}^{n-1} k 2^k \right) - \sum_{k=1}^n 2^k$$

$$= 2^n + 3(n(2^n - 1) -$$

$$0 + 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + 5 \cdot 2^5$$

$$\frac{2}{(1-2)^2}$$