Problem Set I

Industrial Organization II

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January 22, 2017

1

a

We know the joint distribution of the errors is given by

$$F(\epsilon_1, \epsilon_2) = \exp\{-[\exp(-\rho^{-1}\epsilon_1) + \exp(-\rho^{-1}\epsilon_2)]^{\rho}\}$$

Then

$$\begin{aligned} &\rho = 1 \Rightarrow \\ &\quad = \exp\{-[\exp(-\epsilon_1) + \exp(-\epsilon_2)]\} \\ &\quad = \exp\{-\exp(-\epsilon_1)\} + \exp\{-\exp(-\epsilon_2)\} \end{aligned} \tag{1}$$

Now note that the marginal distribution of ϵ_1 is given by

$$\begin{aligned} F(\epsilon_1) &= \lim_{\epsilon_2 \to \infty} F(\epsilon_1, \epsilon_2) \\ &= \exp\{-\exp(-\epsilon_1)\} \end{aligned}$$

And therefore

$$(1) \Rightarrow F(\epsilon_1, \epsilon_2) = F(\epsilon_1)F(\epsilon_2) \Leftrightarrow \epsilon_1 \perp\!\!\!\perp \epsilon_2 \blacksquare$$

b

The utility from each alternative is given by $u_{i,j} = \delta_j + \epsilon_{i,j}$. Now consider the probability that the maximum utility from $\{1, 2\}$ is less than some arbitrary constant U:

$$P(\max\{u_{i,1}, u_{i,2}\} \leq U) = P(\epsilon_1 \leq U - \delta_1, \epsilon_2 \leq U - \delta_2)$$

Therefore

$$\begin{aligned}
F_{max}(U) &= \exp\{-[\exp(-\rho^{-1}(U - \delta_1)) + \exp(-\rho^{-1}(U - \delta_2))]^{\rho}\} \\
&= \exp\{-[\exp(-\rho^{-1}U)(\exp(\rho^{-1}\delta_1) + \exp(\rho^{-1}\delta_2)]^{\rho}\} \\
&= \exp\{-\exp(-U)[\exp(\rho^{-1}\delta_1) + \exp(\rho^{-1}\delta_2)]^{\rho}\} \\
&= \exp\{-\exp(-U)\exp\{\log(\exp(\rho^{-1}\delta_1) + \exp(\rho^{-1}\delta_2))^{\rho})\}\} \\
&= \exp\{-\exp(\log(\exp(\rho^{-1}\delta_1) + \exp(\rho^{-1}\delta_2))^{\rho}) - U)\} \qquad (2)
\end{aligned}$$

Then (2) $\Rightarrow \max\{u_{i,1}, u_{i,2}\} \sim T1EV(\log(\exp(\rho^{-1}\delta_1) + \exp(\rho^{-1}\delta_2))^{\rho}))$.

Moreover, $(\max\{u_{i,1}, u_{i,2}\} - u_{i,0}) \sim Logistic(\log(\exp(\rho^{-1}\delta_1) + \exp(\rho^{-1}\delta_2))^{\rho}) - \delta_0)$. Thus

$$\begin{aligned}
P(i\ chooses\ 0) &= P(\max\{u_{i,1}, u_{i,2}\} - u_{i,0} \leq 0) \\
&= \frac{\exp\{\delta_0 - \log(\exp(\rho^{-1}\delta_1) + \exp(\rho^{-1}\delta_2))^{\rho})\}}{1 + \exp\{\delta_0 - \log(\exp(\rho^{-1}\delta_1) + \exp(\rho^{-1}\delta_2))^{\rho})\}} \\
&= \frac{\exp\{\delta_0\}}{\exp\{\delta_0\} + [\exp(\rho^{-1}\delta_1) + \exp(\rho^{-1}\delta_2)]^{\rho}}
\end{aligned}$$

■

c

We want $P(i\ chooses\ 1|i\ does\ not\ choose\ 0)$. i does not choose 0 if his utility from either choice 1 or choice 2 is greater than his utility from choice 0. Hence $P(1|not\ 0) = P(u_{i,1} > u_{i,2}|\{u_{i,1} > u_{i.0}\} \cup \{u_{i,2} > u_{i.0}\})$. But since $u_{i,0} \perp\!\!\!\perp (u_{i,1}, u_{i,2})$, this is equivalent to $P(u_{i,1} > u_{i,2})$. Then:

$$
\begin{aligned}
P(u_{i,1} > u_{i,2}) = P(\delta_1 + \epsilon_{i,1} > \delta_2 + \epsilon_{i,2}) &= P(\epsilon_{i,2} < \delta_1 - \delta_2 + \epsilon_{i,1}) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\delta_1 - \delta_2 + \epsilon_1} f(e_1, e_2) de_1 de_2 \\
&= \int_{-\infty}^{\infty} F(e_1 + (\delta_1 - \delta_2) | \epsilon_1 = e_1) de_1
\end{aligned}
$$

where $F(e_1 + (\delta_1 - \delta_2)|\epsilon_1 = e_1) = Pr(\epsilon_2 < e_1 + (\delta_1 - \delta_2)|\epsilon_1 = e_1)$. Now notice that

$$
F(e_1, e_2) = \int_{-\infty}^{e_1} \int_{-\infty}^{e_2} f(\epsilon_1, \epsilon_2) d\epsilon_1 d\epsilon_2 \implies F(e_2 | \epsilon_1 = e_1) = \frac{\frac{\partial F(e_1, e_2)}{\partial e_1}}{f(e_1)}.
$$

The denominator is just $f(e_1) = \exp\{-\exp(-e_1)\}\exp(-e_1)$. On the other hand,

$$
\frac{\partial F(e_1, e_2)}{\partial e_1} = F(e_1, e_2) \exp(-\rho^{-1} e_1) \left[\exp(-\rho^{-1} e_1) + \exp(-\rho^{-1} e_2)\right]^{\rho - 1}
$$

Let $K \equiv 1 + \exp(-\frac{\delta_1 - \delta_2}{\rho})$. We then have that

$$
F(e_1 + (\delta_1 - \delta_2) | \epsilon_1 = e_1) = \exp\left\{-K^\rho \exp(-e_1)\right\} \exp(-e_1) K^{\rho - 1}.
$$

Thus,

$$
\begin{aligned}
Pr(i \to 1 | i \nrightarrow 0) &= \int_{-\infty}^{\infty} F(e_1 + (\delta_1 - \delta_2) | \epsilon_1 = e_1) de_1 \\
&= \int_{-\infty}^{\infty} \exp\left\{-K^\rho \exp(-e_1)\right\} \exp(-e_1) K^{\rho - 1} de_1 \\
&= \frac{1}{K}.
\end{aligned}
$$

The last equality is reached with integration by substitution.[1] Using the definition of K, we conclude that

$$
Pr(i \to 1 | i \nrightarrow 0) = \frac{\exp(\rho^{-1}\delta_1)}{\exp(\rho^{-1}\delta_1) + \exp(\rho^{-1}\delta_2)}.
$$

[1]Letting $u = -K^\rho \exp(-e_1)$, so that $du = K^\rho \exp(-e_1) de_1$.

2

First note that the elasticities given are equivalent to $\partial\pi_j/\partial p_j$ (using own- or cross-price elasticities does not matter in this case; the results are identical).

Define $q_j = \exp(-\alpha p_j + x_j'\beta + \xi_j)$. Then we have:

$$\begin{aligned}\frac{\partial \ln \pi_j}{\partial \ln p_j} &= \frac{\partial(\ln q_j - \ln(\sum_k q_k))}{\partial \ln p_j} \\ &= \frac{\partial \ln q_j}{\partial \ln p_j} - \frac{\partial \ln(\sum_k q_k)}{\partial \ln p_j}\end{aligned} \tag{3}$$

Now the first term in (3) is the Marshallian own-price elasticity. Hence (3) becomes:

$$\epsilon_{jj}^M - \frac{\partial \ln(\sum_k q_k)}{\partial \ln p_j} \tag{4}$$

Then (4) implies

$$\frac{\partial \ln \pi_j}{\partial \ln p_j} = \epsilon_{jj}^M \Leftrightarrow \sum_k q_k = c$$

With c any constant. The discrete-choice demand model underlying our choice probabilities implies that $\sum_k q_k = c$, and so the expression **is** a Marshallian price elasticity.

Now note that $\sum_k q_k = c \Rightarrow q_j = \exp(-\alpha p_j + x_j'\beta + \xi_j)$: that is, the term in the denominator of the logit choice probability is simply a constant. This confirms the initial guess of q_j and the result.

Furthermore, in this demand model the income elasticity η is simply equal to the share of the outside good s_0. By Slutsky's equality, $\epsilon_{jj}^H = \epsilon_{jj}^M - \eta_j = \epsilon_{jj}^M - \frac{\partial \ln(\sum_k q_k)}{\partial \ln p_j} - s_0$. Hence $\sum_k q_k = c$ is also a sufficient condition for this to be a legitimate Hicksian price elasticity (since the same conditions apply to s_0 as s_j).

3

The Lagrangian is

$$\mathcal{L} = x^{0.7}y^{0.5} + \lambda[180x + 230y - 57290] + \mu x + \gamma y$$

For the rest of the problem, we supplied IPOPT in Matlab the analytical form of the gradient of the objective function, the Jacobian of the constraints, the Hessian of the Lagrangian, and the sparsity structure of the Jacobian and Hessian. The results converge to $(x = 185, y = 103)$, which is equivalent to the analytical solution to the problem.

Attached is the code as well as three sets of initial conditions [(0,0), (5,10) and (500,500)].

4

a

The log-likelihood function is:

$$\ln \mathcal{L} = \ell = \sum_{i=1}^{N} \sum_{j=1}^{J} y_{ij} \left(x'_{ij}\beta - \ln \sum_{k=0}^{J} \exp(x'_{ik}\beta) \right)$$

b

The score function is:

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^{N} \sum_{j=1}^{J} y_{ij} \left(x'_{ij} - \frac{\sum_{k=0}^{J} \exp(x'_{ik}\beta) x'_{ik}}{\sum_{k=0}^{J} \exp(x'_{ik}\beta)} \right)$$

c

The information matrix is

$$\frac{\partial^2 \ell}{\partial \beta' \partial \beta} = - \sum_{i=1}^{N} \sum_{j=1}^{J} y_{ij} \Bigg(\sum_{k=0}^{J} \frac{1}{\left[\sum_{l=1}^{J} \exp(x'_{il}\beta)\right]^2} \Bigg[\exp(x'_{ik}\beta) \sum_{l=1}^{J} \exp(x'_{il}\beta) x_{ik} x'_{ik} \\ - \exp(x'_{ik}\beta) \sum_{l=1}^{J} \exp(x'_{il}\beta) x_{il} x'_{ik} \Bigg] \Bigg)$$

d

e

See *mlogit.py* in the attached files.

f

The maximization is called in *main.py*. Our 10 initial starting points all converge to:

$$\beta_1 = -2.492564, \quad \beta_2 = 3.068752, \quad \beta_3 = 1.992837.$$

See *results_f.csv* for more detailed optimization results.

g

Estimating equation (4.2) with OLS yields the following results:

$$\beta_1 = -2.283377, \quad \beta_2 = 2.852713, \quad \beta_3 = 1.840058.$$

They are quite similar to our MLE findings.

h

Notice that we are in the just identified case. Therefore, the GMM and OLS point estimates are identical.

The estimate of the asymptotic variance-covariance matrix will depend on the structure of the data. In our case we have i.i.d. market-product observations (this is the unit of observation), so that the second-stage GMM variance-covariance matrix is just the standard white (robust) variance covariance matrix $\hat{V}$. This is:

$$\hat{V} = \begin{pmatrix} 0.01012613 & -0.00960659 & -0.0079941 \\ -0.00960659 & 0.02132305 & 0.00698023 \\ -0.0079941 & 0.00698023 & 0.00638794 \end{pmatrix}.$$