Selling Infomation to Competitive Buyers

CS6501 Learning & Game Theory Project Report

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1 Introduction

The recent tremendous increase in the online data sources made information markets to flourish. Companies are selling a variety of information, including consumer information (e.g Oracle), credit reports (e.g FICO), and recommendations (e.g TripAdvisor). These applications have motivated research into mechanism design for information trading. A basic setting involves a monopoly information provider selling its information optimally to a single decision maker, where the goal of the information provider is to sell her private signal to the decision maker in a way that maximizes the revenue. Previous work by Chen et al. [1] designed mechanisms for information trading with a single buyer. In our project, we want to extend the setting to consider information selling to multiple buyers. The extension from a single buyer to multiple buyers is not (quite) trivial, as both seller and buyers can behave very differently under certain scenario. Jehiel et al. [3] introduced one interesting example of multiple buyers setting, known as *how to not sell the nuclear weapon*: each nuclear buyer will have to pay some amount of the money to the seller such that the buyer will not sell the nuclear weapon to others. For the example in table 1, buyer A want to pay for the seller not to recommend action b_2 to buyer B, and similarly buyer B want to pay for not recommend action a_2 to buyer A. The solution of problem abstract the interesting real world experience that the seller can take advantage of the strictly competitive (non-cooperative) nature of buyers and profit without increasing social welfare. In fact, in certain setting we will introduce in the section 4, the seller can choose to do nothing by only threatening each buyer that without realizing a proper payment, she could harm his utility by benefiting his opponents. Therefore, unionizing is important for workers in labor market, so do oil companies behave in oligopoly (OPEC).

A \ B	b_1	b_2
a_1	(0,0)	(-200, +200)
a_2	(+100,-100)	(0, 0)

Table 1: Payoff matrix for two competitive (zero-sum) nuclear weapon buyers, A and B

2 Problem Setup

We abstract the example above into the fundamental setting where N buyers are playing a Bayesian zero-sum game and both the seller's and the buyers' objectives are to maximize his or her own expected revenue. We first describe a general setting of the information-selling problem to N

competitive buyers, where a monopoly information holder (call *her* the seller) want to sell this information to N competitive decision makers, each of them (call *him* the buyer) who needs to choose an action $a_i \in A$.

The seller can observe a private signal of the state of the world $\omega \in \Omega$. Meanwhile, for each of the buyers i, he can privately observe $\theta_i \in \Theta$ (namely his buyer type). Sets A, Ω, Θ are all finite.

The utility of each buyer i against the actions of other buyers a_{-i} follows the publicly known distribution $u^i_{((\theta_1,...,\theta_N),\omega)}(a_i, a_{-i})$, where $u^i_{((\theta_1,...,\theta_N),\omega)}$ refers to the utility matrix under the signals $((\theta_1, ..., \theta_N), \omega)$ of the i'th player, and (a_i, a_{-i}) refers to a specific entry in the utility matrix, with player 1 plays a_1, player 2 plays a_2, etc. The utility function of the buyers has the competitive nature, and for simplicity, we assume the following Bayesian zero-sum game property:

$$\forall \omega \in \Omega, \theta_i \in \Theta, \quad \sum_i^N u^i_{((\theta_1,...,\theta_N),\omega)}(a_i, a_{-i}) = 0$$

Therefore, the revenue of each buyer is its expected utility minus its payment to the seller, of the seller is the total revenue from each buyer's payment.

In addition, we can also consider the following extra structures of the problem:

2.1 Buyer Budget Limit

We first consider the **unlimited public budget** setting that each buyer can make any amount of payment as long as it maximizes its revenue. We later describe the case, where each buyer i respectively has a budget limit $b_i \in B$. The budget of each buyer b can be (**public budget**) independently drawn from a publicly known prior distribution μ or (**private budget**) an unknown prior distribution private to each buyer.

2.2 Information-selling Compliance

The seller can either **(private information-selling)** choose to signal different information privately to each buyer or **(public information-selling)** be required to publicly reveal the same information signal to every buyer.

2.3 Signaling Scheme

A signaling scheme is a randomized mapping from the seller's signal set Ω to a set of signals Σ, which can be fully described by the likelihood function $p_{((\theta_1,...,\theta_n),\omega)}(\sigma)$, $\omega \in \Omega, \sigma \in \Sigma$, where $p_{((\theta_1,...,\theta_n),\omega)}(\sigma)$ is the probability of sending signal σ given seller signal ω and buyers' signal $\theta_1, ..., \theta_n$. We will prove in section 3.2 that when we only have two players, $\forall \sigma \in \Sigma$, σ corresponds two behaviors a_1, a_2 that corresponds to two behaviors recommended to each player respectively.

2.4 Obedient Constraint

When the seller recommends signals σ to both players, they do not have to obey the recommendation. They could choose the action that maximize their own expected utility. However, *obedience constraint* will be an intrinsic constraint of the optimal selling strategy shown in section 3.1. That is, the recommended action must indeed be an optimal action for the buyer's reported type or, more formally, for each player i,

$$a_i = \arg\max_{a'_i \in A_i} \sum_\omega u_{i((\theta_1,...,\theta_n),\omega)}(a_i, a_{-i}) \frac{\mu(\omega) p_{((\theta_1,...,\theta_N),\omega)}(a_i, a_{-i})}{\sum_{\omega' \in \Omega} \mu(\omega') p_{((\theta_1,...,\theta_N),\omega')}(a_i, a_{-i})}.$$

3 Private Information-selling

We first describe the optimal strategy for the seller and buyers, then prove its optimality and show computation to solve this strategy.

3.1 Optimal Selling Strategy

Definition 1. A consulting mechanism with direct payment for N buyers **(CM-dirP^N)** *as follows:*

1. *The seller commits to a payment amount $t^i_{[\theta]}$ for each buyer i and an action recommendation policy described as mapping $p_{[\theta]} : \omega^N \to \Delta A^N$ for each buyer type θ, where $p_{[\theta]}(w, [a])$ denoting the probability of recommending actions $[a]$ respectively to buyers each of type in $[\theta]$, given seller signal ω. $p_{[\theta]}$ is required to be obedient i.e. conditioned on any recommended action, it must indeed be an optimal action for the buyer given his information.*
2. *The seller asks each of the buyer i to report his type $\hat{\theta}_i$, and let $[\hat{\theta}] = [\hat{\theta}_1, \ldots, \hat{\theta}_N]$.*
3. *The seller charges each buyer i an amount $t^i_{[\hat{\theta}_i]}$*
4. *Based on her signal ω, the seller samples a vector of action $[a] \sim p_{[\hat{\theta}]}$, where and recommend respectively to each buyer.*

Theorem 1. *When each buyer's signal θ is independent from ω, there always exists an optimal mechanism that is **(CM-dirP^N)***

Proof. The proof directly generalizes from the Revelation Principle [1]. According to Peski [4], in Bayesian zero-sum games, revealing more information to any player privately always (weakly) increases that player's expected utility. Intuitively, for each buyer, they are guaranteed to lose more if they do not pay for the full information that minimizes its loss (maximize his utility). In another word, for the seller, they can at most charge each buyer the exact amount of the surplus that they would suffer but no more. And the buyer would always want to make the payment. Meanwhile, none of the participants have the incentive to abort in the middle of the mechanism as only reaching the end of the mechanism maximizes their revenue.

□

3.2 Computing Optimal Mechanisms

For simplicity of notation, we describe the mechanism in two players setting, which can be easily generalize to N player settings.

Lemma 1. *The optimal private information signal scheme can without loss of generality use $m \times n$ signals; each signal (i, j) recommends action i to player 1 and action j to player 2.*

Proof. Let M be any IC pricing mapping mechanism with direct payment. Let $S(\theta_1, \theta_2)$ be the signal scheme for two players with type θ_1 and θ_2 respectively. Let $s_1, ..., s_m$ be the leaves of $S(\theta_1, \theta_2)$, each contains a pair of signals. Let $i_1, ..., i_n$ be pairs of optimal actions of player 1 and 2 with type θ_1 and θ_2 when the signal sent by $S(\theta_1, \theta_2)$ is realized to $s_1, ..., s_m$ respectively. In each action pair i_k, let i^1_k refers to the action of player 1 and i^2_k refers to the action of player 2.

Suppose s_j, s_k both equal $\bar{i^1} \in [m]$ and $\bar{i^2} \in [m]$ for some j and k, we show that we can without loss merge the two leaves s_j, s_k. That is, whenever the seller moves to s_j, she moves to s_k instead. The transition function of $S(\theta_1, \theta_2)$ is updated as follows: $\pi'_{((\theta_1,\theta_2),\omega)}(s_k) \leftarrow \pi_{((\theta_1,\theta_2),\omega)}(s_j)+$

$\pi_{(\theta_1,\theta_2),\omega}(s_k)$ and $\pi_{((\theta_1,\theta_2),\omega}(s_j) \leftarrow 0$, for all ω. The payments keep unchanged. Therefore, the seller's revenue will not change if the buyer still truthfully reports.

We claim that this new mechanism will not change the utility of both players. First of all, the optimal action for the both players at leaf $s+k$ is still $\bar{i}$. This is because, by definition, we have

$$\bar{i^1} = \arg\max_{i^1 \in [n]} \sum_{\omega, i^2} \mu(\omega)\pi_{((\theta_1,\theta_2),\omega)}(s_j)u_{((\theta_1,\theta_2),\omega)}(i^1, i^2)$$

$$\bar{i^2} = \arg\min_{i^2 \in [m]} \sum_{\omega, i^1} \mu(\omega)\pi_{((\theta_1,\theta_2),\omega)}(s_j)u_{((\theta_1,\theta_2),\omega)}(i^1, i^2)$$

$$\bar{i^1} = \arg\max_{i^1 \in [n]} \sum_{\omega, i^2} \mu(\omega)\pi_{((\theta_1,\theta_2),\omega)}(s_k)u_{((\theta_1,\theta_2),\omega)}(i^1, i^2)$$

$$\bar{i^2} = \arg\min_{i^2 \in [m]} \sum_{\omega, i^1} \mu(\omega)\pi_{((\theta_1,\theta_2),\omega)}(s_k)u_{((\theta_1,\theta_2),\omega)}(i^1, i^2)$$

Therefore $\arg\max_{i^1\in[n]} \sum_{\omega,i^2} \mu(\omega)\pi'_{((\theta_1,\theta_2),\omega)}u_{((\theta_1,\theta_2),\omega)}(i^1,i^2)$ must equal to $\bar{i^1}$, and $\arg\min_{i^2\in[m]} \sum_{\omega,i^1} \mu(\omega)\pi_{((\theta_1,\theta_2),\omega)}u_{((\theta_1,\theta_2),\omega)}(i^1,i^2)$ must also equal $\bar{i^2}$.

As a result, the expected utility of both buyers will not change as for player 1,

$$\begin{aligned}
&\sum_{\omega,i^2} \mu(\omega)\pi_{((\theta_1,\theta_2),\omega)}(s_j)u_{(\theta_1,\theta_2),\omega}(\bar{i^1}, i^2) + \sum_{\omega,i^2} \mu(\omega)\pi_{(\theta_1,\theta_2),\omega}(s_k)u_{((\theta_1,\theta_2),\omega}(\bar{i^1}, i^2) \\
=&\sum_{\omega,i^2} \mu(\omega)\pi'_{(\theta_1,\theta_2),\omega}(s_j)u_{(\theta_1,\theta_2),\omega}(\bar{i^1}, i^2)
\end{aligned}$$

and for player 2,

$$\begin{aligned}
&\sum_{\omega,i^1} \mu(\omega)\pi_{((\theta_1,\theta_2),\omega)}(s_j)u_{(\theta_1,\theta_2),\omega}(i^1, \bar{i^2}) + \sum_{\omega,i^1} \mu(\omega)\pi_{(\theta_1,\theta_2),\omega}(s_k)u_{((\theta_1,\theta_2),\omega}(i^1, \bar{i^2}) \\
=&\sum_{\omega,i^2} \mu(\omega)\pi'_{(\theta_1,\theta_2),\omega}(s_j)u_{(\theta_1,\theta_2),\omega}(i^1, \bar{i^2})
\end{aligned}$$

Next we argue that the utility of any other buyers with type θ'_1, θ'_2 will not increase when he misreport θ_1, θ_2. This is because for player 1, his utility from the original two leaves s_j, s_k is

$$\max_{i^1\in[n]} \sum_{\omega,i^2} \mu(\omega)\pi_{((\theta_1,\theta_2),\omega)}(s_j)u_{((\theta'_1,\theta_2),\omega)}(i^1,i^2) \text{ and } \max_{i^1\in[n]} \sum_{\omega,i^2} \mu(\omega)\pi_{((\theta_1,\theta_2),\omega)}(s_k)u_{((\theta'_1,\theta_2),\omega)}(i^1,i^2)$$

and for player 2, his utility from the original two leaves s_j, s_k is

$$\min_{i^2\in[m]} \sum_{\omega,i^1} \mu(\omega)\pi_{((\theta_1,\theta_2),\omega)}(s_j)u_{((\theta_1,\theta'_2),\omega)}(i^1,i^2) \text{ and } \min_{i^2\in[m]} \sum_{\omega,i^1} \mu(\omega)\pi_{((\theta_1,\theta_2),\omega)}(s_k)u_{((\theta_1,\theta'_2),\omega)}(i^1,i^2)$$

For buyer 1, his sum is at least $\max_{i^1\in[n]} \sum_{\omega,i^2} \mu(\omega)[\pi_{((\theta_1,\theta_2),\omega)}(s_j)+\pi_{((\theta_1,\theta_2),\omega)}(s_k)]u_{((\theta'_1,\theta_2),\omega)}(i^1,i^2)$, and for buyer 2, his sum is at most $\min_{i^2\in[m]} \sum_{\omega,i^1} \mu(\omega)[\pi_{((\theta_1,\theta_2),\omega)}(s_j)+\pi_{((\theta_1,\theta_2),\omega)}(s_k)]u_{((\theta_1,\theta'_2),\omega)}(i^1,i^2)$.

To sum up, for each buyer, the new mechanism will not change his utility and will not increase the other buyers' utility when they misreport θ, and thus will remain IC and IR. Moreover, the revenue does not change. We can perform such merging operation until each leaf corresponds to a different action $i^1\ in[n]$ and $i^2 \in [m]$. To that end, each signal can be viewed as an "honest" action recommendation which will indeed maximizes the buyer's expected utility. This is precisely a consulting mechanism, as desired. $\square$

As a result, the optimal consulting mechanism can be computed via the following linear program (LP) with $p_{((\theta_1,\theta_2),\omega)}(a_1,a_2), t^1_{(\theta_1,\theta_2)}, t^2_{(\theta_1,\theta_2)}$ as variables.

$$
\begin{aligned}
& \max \sum_{(\theta_1,\theta_2)} \mu(\theta_1,\theta_2)\cdot(t^1_{(\theta_1,\theta_2)}+t^2_{(\theta_1,\theta_2)}) \\
& \text{s.t.} \\
\textbf{OB-1:}\quad \text{for } a_1,a_1',\theta_1, \quad & \sum_{a_2,\theta_2,\omega} \mu(\omega)\mu(\theta_2)p_{((\theta_1,\theta_2),\omega)}(a_1,a_2)u_{((\theta_1,\theta_2),\omega)}(a_1,a_2) \\
& \geq \sum_{a_2,\theta_2,\omega} \mu(\omega)\mu(\theta_2)p_{((\theta_1,\theta_2),\omega)}(a_1,a_2)u_{((\theta_1,\theta_2),\omega)}(a_1',a_2) \\
\textbf{OB-2:}\quad \text{for } a_2,a_2',\theta_2, \quad & \sum_{a_1,\theta_1,\omega} \mu(\omega)\mu(\theta_1)p_{((\theta_1,\theta_2),\omega)}(a_1,a_2)u_{((\theta_1,\theta_2),\omega)}(a_1,a_2) \\
& \leq \sum_{a_1,\theta_1,\omega} \mu(\omega)\mu(\theta_1)p_{((\theta_1,\theta_2),\omega)}(a_1,a_2)u_{((\theta_1,\theta_2),\omega)}(a_1,a_2') \\
\textbf{IC-1:}\quad \text{for } \theta_1,\theta_1', \quad & \sum_{\omega,\theta_2,a_1,a_2} \mu(\omega)\mu(\theta_2)p_{((\theta_1,\theta_2),\omega)}(a_1,a_2)u_{((\theta_1,\theta_2),\omega)}(a_1,a_2) - t^1_{(\theta_1,\theta_2)} \\
& \geq \sum_{a_1} \max_{a_1'} \sum_{\omega,\theta_2,a_2} \mu(\omega)\mu(\theta_2)p_{((\theta_1',\theta_2),\omega)}(a_1,a_2)u_{((\theta_1,\theta_2),\omega)}(a_1',a_2) - t^1_{(\theta_1',\theta_2)} \\
\textbf{IC-2:}\quad \text{for } \theta_2,\theta_2', \quad & \sum_{\omega,\theta_1,a_1,a_2} \mu(\omega)\mu(\theta_1)p_{((\theta_1,\theta_2),\omega)}(a_1,a_2)u_{((\theta_1,\theta_2),\omega)}(a_1,a_2) - t^2_{(\theta_1,\theta_2)} \\
& \leq \sum_{a_2} \min_{a_2'} \sum_{\omega)\mu(\theta_1)\mu(a_2} \mu(\omega,\theta_1)p_{((\theta_1,\theta_2'),\omega)}(a_1,a_2)u_{((\theta_1,\theta_2),\omega)}(a_1,a_2') - t^2_{(\theta_1,\theta_2')} \\
\textbf{IR-1:},\quad \text{for } \theta_1,a_1', \quad & \sum_{\omega,\theta_2,a_1,a_2} \mu(\omega)\mu(\theta_2)p_{((\theta_1,\theta_2),\omega)}(a_1,a_2)u_{(\theta_1,\theta_2),\omega}(a_1,a_2) - t^1_{\theta_1,\theta_2} \\
& \geq \sum_{\omega,\theta_2,a_2} \mu(\omega)\mu(\theta_2)u_{(\theta_1,\theta_2),\omega}(a_1',a_2) \\
\textbf{IR-2:},\quad \text{for } \theta_2,a_2', \quad & \sum_{\omega,\theta_1,a_1,a_2} \mu(\omega)\mu(\theta_1)p_{((\theta_1,\theta_2),\omega)}(a_1,a_2)u_{(\theta_1,\theta_2),\omega}(a_1,a_2) - t^2_{\theta_1,\theta_2} \\
& \leq \sum_{\omega,\theta_1,a_1} \mu(\omega)\mu(\theta_1)u_{(\theta_1,\theta_2),\omega}(a_1,a_2') \\
\text{for } \theta_1,\theta_2,\omega, \quad & \sum_{a_1,a_2} p_{((\theta_1,\theta_2),\omega)}(a_1,a_2) = 1 \\
\text{for } \theta_1,\theta_2,\omega,(a_1,a_2), \quad & 0 \leq p_{((\theta_1,\theta_2),\omega)}(a_1,a_2) \leq 1
\end{aligned}
$$

where θ_1, θ_2 are the reported type from player 1 and 2 respectively, $t^1_{(\theta_1,\theta_2)}, t^2_{(\theta_1,\theta_2)}$ are the payment of player 1 and 2 respectively under the condition that player 1 reports θ_1 and player 2 reports θ_2, and a_1, a_2 arebe the signal from the seller to player 1 and player 2 respectively.

The prior distribution of ω is a public knowledge, represented as $\mu(\omega)$. Also, the distribution of θ_1 and θ_2 are public knowledge and are represented as $\mu(\theta_1), \mu(\theta_2)$. π denotes a signaling scheme, which is a mapping function $\omega \times (\theta_1,\theta_2) \to (a_1,a_2)$ under obedience constraint. The mapping is also a distribution and let $p_{((\theta_1,\theta_2),\omega)}(a_1,a_2)$ denotes the probability that π gives out signal (a_1,a_2), given ω and the reported types θ_1, θ_2.

The utility matrix is determined by three parameters, $\omega, \theta_1, \theta_2$. Since two buyers are playing zero-sum game, $u_{((\theta_1,\theta_2),\omega)}$ represents the utility matrix for buyer 1 and the utility matrix for buyer 2 would be $-u_{((\theta_1,\theta_2),\omega)}$ automatically. An entry $u_{((\theta_1,\theta_2),\omega)}(a_1, a_2)$ is the utility of buyer 1 when he takes action a_1 and buyer 2 takes action a_2.

Note that currently, the IC constraints is not linear yet. However, they can be easily transformed into linear constraints by introducing new variables

$$
\begin{aligned}
z^1_{\theta_1,\theta_1',a_1} &= \max_{a_1'} \sum_{\omega,\theta_2,a_2} \mu(\omega)\mu(\theta_2)p_{((\theta_1',\theta_2),\omega)}(a_1,a_2)u_{((\theta_1,\theta_2),\omega)}(a_1',a_2) \\
z^2_{\theta_2,\theta_2',a_1} &= \min_{a_2'} \sum_{\omega)\mu(\theta_1)\mu(a_2} \mu(\omega,\theta_1)p_{((\theta_1,\theta_2'),\omega)}(a_1,a_2)u_{((\theta_1,\theta_2),\omega)}(a_1,a_2')
\end{aligned}
$$

Then the IC constraints can be re-formatted as follows

$$
\begin{aligned}
\textbf{IC-1:}\quad \text{for } \theta_1,\theta_1', \quad & \sum_{\omega,\theta_2,a_1,a_2} \mu(\omega)\mu(\theta_2)p_{((\theta_1,\theta_2),\omega)}(a_1,a_2)u_{((\theta_1,\theta_2),\omega)}(a_1,a_2) - t^1_{(\theta_1,\theta_2)} \\
& \geq \sum_{a_1} z^1_{\theta_1,\theta_1',a_1} - t^1_{(\theta_1',\theta_2)} \\
\textbf{IC-2:}\quad \text{for } \theta_2,\theta_2', \quad & \sum_{\omega,\theta_1,a_1,a_2} \mu(\omega)\mu(\theta_1)p_{((\theta_1,\theta_2),\omega)}(a_1,a_2)u_{((\theta_1,\theta_2),\omega)}(a_1,a_2) - t^2_{(\theta_1,\theta_2)} \\
& \leq \sum_{a_2} z^2_{\theta_2,\theta_2',a_1} - t^2_{(\theta_1,\theta_2')}
\end{aligned}
$$

Obedience Constraints **(OB-i)**: (LHS) The expected utility that player i players the recommended action a_i given that his type is θ_i is greater or equal to (RHS) the expected utility that player i players any action a_i' given that his signal is θ_i.

Incentive Compatible Constraints **(IC-i)**: (LHS) The expected utility that player i truthfully report θ_i is greater or equal to (RHS) the expected utility that player i report another signal θ_i' and receive the optimal recommendation a_i'.

Individual Rationality Constraints **(IR-i)**: (LHS) The expected utility that player i participate in this mechanism is greater or equal to (RHS) the expected utility that player i do not participate in the mechanism.

3.3 Private information-selling with limited budget

We consider two cases when we have budget for buyers: public budget and private budget.

Public budget: In this case, we only need an additional constraint on payment. Let b_1, b_2 be the budget for player 1 and player 2 respectively.

$$
\begin{aligned}
\text{For } \theta_1,\theta_2, \quad & t^1_{(\theta_1,\theta_2)} \leq b_1 \\
& t^2_{(\theta_1,\theta_2)} \leq b_2
\end{aligned}
$$

Private budget: In this case, the budget is only available for the buyer himself and not anyone else. Thus, by a similar idea from $\textbf{CM-dirP}^N$, we need to require each buyer report and deposit their budget $\hat{b_1}$ and $\hat{b_2}$ first, then recommend actions to both buyers and refund $\hat{b_1} - t^1_{\theta_1,\theta_2}, \hat{b_2} - t^2_{\theta_1,\theta_2}$. (accordingly we name as $\textbf{CM-depR}^N$). In this case, the signaling scheme π will consider the elicited budget as a parameter as well, since we want to ensure obedience constraints, incentive compatible and individual rational on π. As a result, π should be a function of $\omega \times (\theta_1, \theta_2) \times \hat{b_1} \times \hat{b_2} \to (a_1, a_2)$ under obedience constraints, incentive compatible constraints, and individual rational constraints, where $\hat{b_1}, \hat{b_2}$ will be the reported budget for player 1 and 2 respectively. $p_{((\theta_1,\theta_2),\omega,\hat{b_1},\hat{b_2})}(a_1, a_2)$ will be the probability that π gives out signal (a_1, a_2), given ω, the reported types θ_1, θ_2, and the reported budgets $\hat{b_1}, \hat{b_2}$.

4 Public Information-Selling

4.1 Without individual private information

In order to get a sense of this problem, let us first consider one special case of public information-selling to two competitive zero-sum buyer problem where there is no private information to each buyer. We can capture it by a menu with four possible outcomes depending on whether each buyer is willing to pay or not: $\{(\pi^1, x, y), (\pi^2, x, 0), (\pi^3, 0, y), (\pi^4, 0, 0)\}$, where each π^i is a signaling scheme and the two scalars after are the payment of buyer 1 and 2 respectively. The mechanism starts with the default signaling scheme π^1 and asks buyer 1 for a payment of x and buyer 2 for 2 a payment of y. If only buyer 1 refuses to pay, the seller will execute the signaling scheme π^2 . Similarly if buyer 2 alone or both refuse to pay, π^3 or π^4 will be executed.

We can define **Individual Rationality (IR)** as if the following condition are satisfied,

1. $u_1\left(\pi^1\right) - x \geq u_1\left(\pi^3\right) - 0$
2. $u_2\left(\pi^1\right) - y \geq u_2\left(\pi^2\right) - 0$

In another word, (π^1, x, y) is the Nash equilibrium.

Dominant-Strategy Individual Rationality (DSIR) as if the following condition are satisfied,

1. $u_1\left(\pi^1\right) - x \geq u_1\left(\pi^3\right) - 0$ and $u_1\left(\pi^2\right) - x \geq u_1\left(\pi^4\right) - 0$
2. $u_2\left(\pi^1\right) - y \geq u_2\left(\pi^2\right) - 0$ and $u_2\left(\pi^3\right) - y \geq u_2\left(\pi^4\right) - 0$

In another word, (π^1, x, y) is the dominant-strategy Nash equilibrium.

Now suppose we can calculate $\pi_1^\star$ [resp. $\pi_2^\star$] as the signaling scheme that maximizes buyer 1's [resp buyer 2's] utility, i.e., $\max u_1 = -\min u_2 = u_1\left(\pi_1^*\right)$, $\max u_2 = -\min u_1 = u_2\left(\pi_2^*\right)$

Theorem 2. *An IR information selling mechanism is optimal iff it contains the four outcomes:*

$$\{(\pi^1, x, y), (\pi_1^\star, x, 0), (\pi_2^\star, 0, y), (\pi^4, 0, 0)\},$$

where the payment $x = u_1\left(\pi^1\right) - \min_1$, $y = u_2\left(\pi^2\right) - \min_2$ *and* π^1, π^4 *are arbitrary signaling scheme. The optimal revenue is* $R_{\max} = \max u_1 + \max u_2 = \max u_1 - \min u_1$

Moreover, an DSIR information selling mechanism is optimal iff it additionally satisfies the constraint that $u_1\left(\pi^1\right) + u_1\left(\pi^4\right) = R_{\max}$

This theorem can be easily derive from the above definition of **IR** and **DSIR**. Intuitively, this theorem suggests that how much the seller can harm (by benefiting the rest) each buyer is exactly how much the seller can profit from each buyer.

4.2 Hardness Reduction

Following theorem 2, the mechanism needs to solve for $\pi_1^\star, \pi_2^\star$ for the maximum revenue. In the simplified example in table 1, we can easily identify $\pi_1^\star = (a_1, b_2), \pi_2^\star = (a_2, b_1)$ as the state of nature is determined. However, finding such optimal public signaling scheme in finite 2-player Bayesian zero-sum game is intractable (strictly harder than FPTAS) [2]. Specially, consider the following problem: Let r and c, denote the number of pure strategies of the row player and column player respectively, and a family of payoff matrices $A^\theta \in \mathbb{R}^{r\times c}$, indexed by states of nature $\theta \in \Theta = \{1, \ldots, M\}$. Computing the signal that maximizes (or minimizes) the utility of the row player can be reduced to the convex decomposition problem in regard to (α, x) the signal probability and posterior distribution of θ on the prior $\lambda \in \Delta M$ on the states of nature:

$$\arg\max_{\alpha,x} \quad u(\alpha, x) = \sum_{\sigma\in\Sigma} \left(\alpha_\sigma \max_{y\in\Delta_r} \min_{j=[c]} \left(y^\top \mathbf{E}_{\theta\sim x_\sigma} \left[A^\theta \right] \right)_j \right)$$

Now if we go back the general setting that considers the private type of each buyers, the mechanism needs additional IC constraint to ensure truthful reporting of their private type. And we here conjecture that computing the optimal public information selling mechanism in general cases can be still reduced to a polynomial times (by the size of types) of computing the optimal public signaling scheme. We leave the proof of this conjecture to (near) future work.

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