Question1. Part B)

Work out the expansion of $\log(1+x)$ **as** $\log(\frac{1+y}{1-y})$

Note: the Taylor series expansion for $\log(1+x)$ is:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n+1}x^n}{n} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$$

First, let $\frac{1+y}{1-y} = 1 + x$, then:

$$\begin{aligned} \log(1+x) &= \log(\frac{1+y}{1-y}) \\ &= \log(1+y) - \log(1-y) \end{aligned}$$

Using Talyor series to expand the expression above:

$$\begin{aligned} \log(1+y) - \log(1-y) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}y^n}{n} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-y)^n}{n} \\ &= y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots - [-y - \frac{(-y)^2}{2} + \frac{(-y)^3}{3} - \frac{(-y)^4}{4} + \ldots] \\ &= y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots + y + \frac{(-y)^2}{2} - \frac{(-y)^3}{3} + \frac{(-y)^4}{4} - \ldots \\ &= y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots + y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \ldots \\ &= 2y + 0 + \frac{2y^3}{3} + 0 + \frac{2y^5}{5} \ldots \\ &= 2(y + \frac{y^3}{3} + \frac{y^5}{5} + \ldots) \\ &= 2\sum_{n=1}^{\infty} \frac{y^{2n-1}}{2n-1} \end{aligned}$$

Now, to obtain the The following equations solves $\frac{1+y}{1-y} = 1 + x$ for y.

$$\frac{1+y}{1-y} = 1+x$$

$$\frac{2}{1-y} - 1 = 1+x$$

$$\frac{2}{1-y} = x+2$$

$$\frac{1-y}{2} = \frac{1}{x+2}$$

$$1-y = \frac{2}{x+2}$$

$$y = 1 - \frac{2}{x+2}$$

To obtain the function for `my_another_log1p(x)`, simply convert the input `x` to `y` as the expression above. Then compute the sum $2\sum_{n=1}^{\infty} \frac{y^{2n-1}}{2n-1}$.

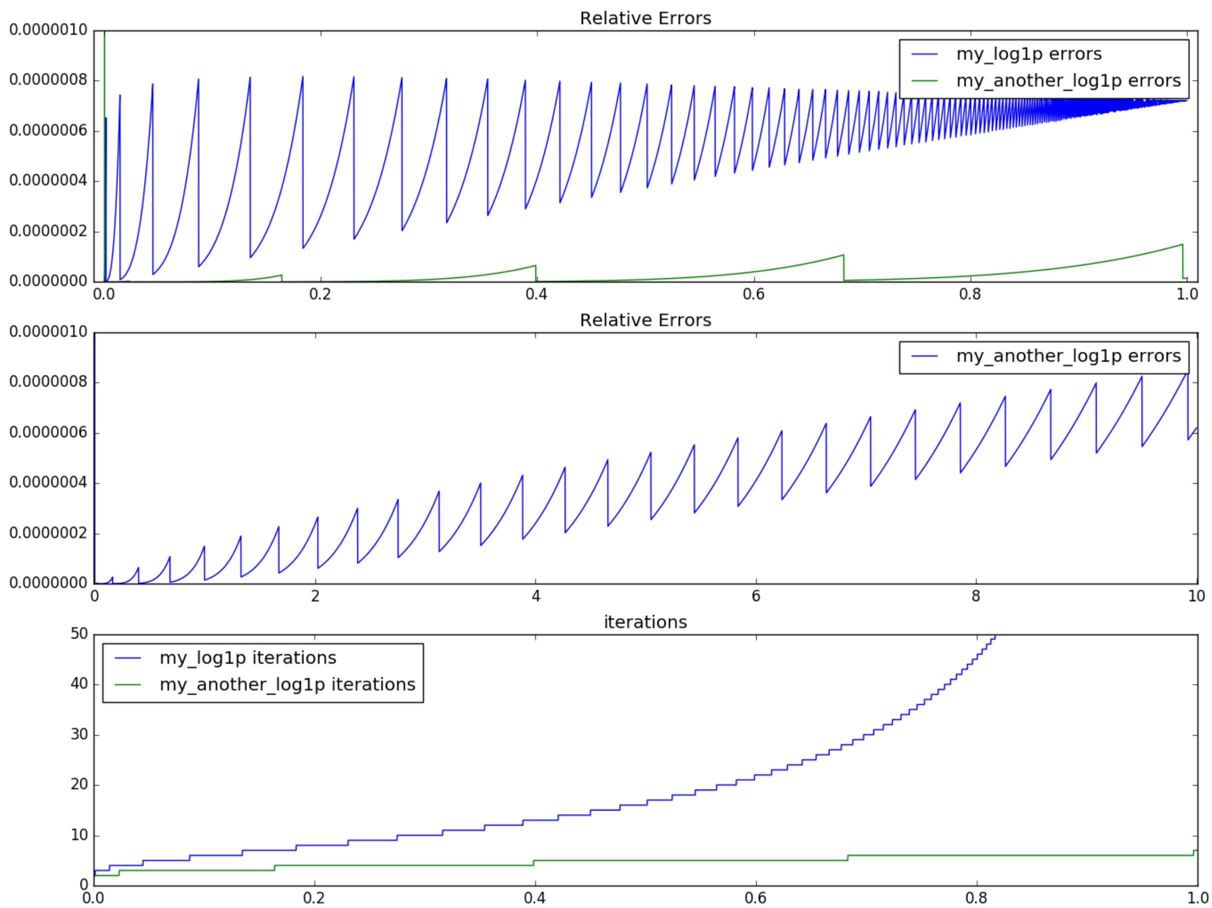

Figure 1: (top): relative error between two log1p implementations between 0 and 1, (middle): relative error with larger inputs, (bottom): number of iterations until return value.

For small values of `x` (smaller than machine eps), $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$ is able to converge to a much better result compare to $2\sum_{n=1}^{\infty}\frac{y^{2n-1}}{2n-1}$, as seen in figure 1, this is because of the fact that in the second method, we convert `x` to `y` via an addition of a constant. When `x` is smaller than machine eps, the addition $x+2=2$, resulting in an incorrect value, resulting in 100% relative error. However, the second implementation is able to produce results with much better accuracy when input of `x` greater than machine eps. Furthermore, the second implementation allows the computation of input `x` greater than 1. This is something that the first implementation is incapable of. The second implementation is able to converge to a return value much quicker than the first implementation. See Figure 1, bottom graph.

Taking a deeper look into the two formula, we can see that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}$ has an alternating term $(-1)^{n+1}$, this results in the sum alternate around the true value before converging. In contrast, $2\sum_{n=1}^{\infty}\frac{y^{2n-1}}{2n-1}$ does not have any alternating terms, hence converges to a return value much more quickly. The convergence of the two can be graphed. The experiment can be found in `q1_experiment.ipynb`.

Observe figure 2 below, the first method alternates about the actual value and finally converges many iterations later, whereas the second method converges to a return value with desired accuracy very quickly.

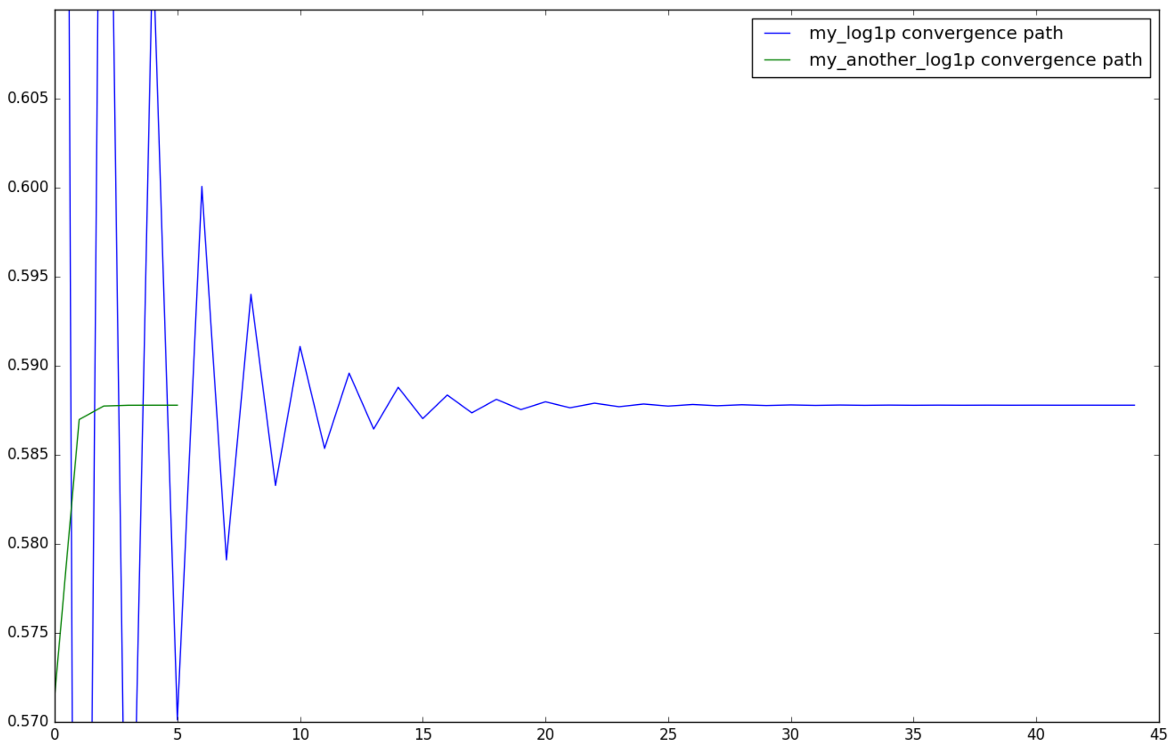

Figure 2: fix x = 0.8, the path to convergence of the two implementations.