FIN 4500
Chapter 2
Analytical Skills II

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Relationships between Random Variables

- In many situations (models), there are multiple random variables
- Therefore, we need to understand how random variables are related to one another
  - Covariance
  - Correlation coefficients

Covariance

- Covariance is a measure of how two random variables are related
  - It is important whenever more than one random variable is being considered

- Positive covariance between X and Y ➔
  - when X has an outcome <u>above</u> its expected value, Y will tend to have an outcome <u>above</u> its expected value,
  - when X has an outcome <u>below</u> its expected value, Y will tend to have an outcome <u>below</u> its expected value

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Covariance

- Negative covariance between X and Y ➔
  - when X has an outcome above its expected value, Y will tend to have an outcome below its expected value, and
  - when X has an outcome below its expected value, Y will tend to have an outcome above its expected value
- Zero Covariance ➔

  No relationship between two random variables.

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Covariance

- Covariance is calculated similarly to variance
    - (Variance of X = Covariance of X and X)

Covariance = weighted average of the product of the deviations of each outcome from its mean, where the weights are the probabilities of the outcomes

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Covariance

For example, we have two RVs: X and Y, as below

<u>Probability</u> <u>X</u> <u>Y</u>

$p_1\ x_1\ y_1$

$p_2\ x_2\ y_2$

$p_3\ x_3\ y_3$

$$Cov(X,Y) = [x_1-E(X)][y_1-E(Y)]p_1+[x_2-E(X)]\ [y_2-E(Y)]p_2+[x_3-E(X)][y_3-E(Y)]$$

$$p_3\ {}^{3}[xi-E(X)][yi-E(Y)]p_{��}$$

$$Cov(X,Y) = \sigma_{��=1}$$

Assume $z_i = [x_i-E(X)][y_i-E(Y)]$

$$^{3}z_i p_{��}$$

$$Cov(X,Y) = \sigma_{��=1}$$

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Example of Calculating Covariance

4 Possible Outcome Outcome

Outcomes for X for Y Probability A 9 26 0.2

B 4 30 0.3

C 10 20 0.3

D 5 24 0.2

Step 1: Calculate the expected value for X and

Y • $E(X) = 1.8 + 1.2 + 3.0 + 1.0 = 7$

• $E(Y) = 5.2 + 9.0 + 6.0 + 4.8 = 25$

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Example of Calculating Covariance

Possible Outcome Outcome

Scenario for X for Y Probability A 9 26 0.2 B 4 30 0.3 C 10 20 0.3 D 5 24 0.2

Step 2: Calculate the deviations from the expected values Step 3: Calculate the product of each deviation from the expected value and then weight by the probabilities:

Deviations from Exp Values (9-7)= 2 (26-25)= 1 (4-7)= -3 (30-25)= 5 (10-7)=3 (20-25)=-5 (5-7)= -2 (24-25)=-1

Products **2** **-15** **-15** **2**

Weight by Prob **0.4** **-4.5** **-4.5** **0.4**

Sum = Cov(X,Y) = -8.2

Step 4: Sum:

Correlation Coefficients

- The Covariance number by itself is difficult to interpret → we calculate correlation coefficients by scaling by the product of standard deviations:

- Correlation Coefficient between X and Y

$\rho_{X,Y} = Cov(X,Y) / Std(X)\ Std(Y)$

- In our example: Std(X) = 2.65 , Std(Y) = 3.92

  ➔ $\rho_{X,Y} = -8.2 / [(2.65)(3.92)] = -0.79$

Correlation Coefficients

- Correlation Coefficients are always between

|------------|------------|

-1 0 1

$\rho_{X,Y} = 1$ ➔ perfect positive correlation

$\rho_{X,Y} = -1$ ➔ perfect negative correlation

$\rho_{X,Y} = 0$ ➔ no correlation

Example: Calculate Correlation Coefficient

Economy Prob Ret on A Ret on B Good 0.2 0.20 0.10 Okay 0.6 0.10 0.05 Bad 0.2 0.00 0.06

- $E(R_A) = 10\%$ $Std(R_A) = 6.32\%$
- $E(R_B) = 6.2\%$ $Std(R_B) = 1.93\%$

- $Cov(R_A,R_B) = 0.0008$

- Correlation coefficient between R_A and $R_B = 0.652$

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Modeling Probability Distributions

- Sometimes it is useful to model Random Variables as a function of other Random Variables

- Examples:
    - Let L_i = Loss experienced in coming year by company i. • What is the cost to company i if it participates in a pooling arrangement with 5 other companies?
    - Cash Flow = (Price) X (Quantity) – Costs, where Price, Quantity and Costs are random variables
    - Return on a portfolio of securities

Analyzing a Constant times a RV

- $Y = aX$
- $E(Y) = a\,E(X)$
- Coinsurance Example:
  - L = Loss and E(L) = $1 million
  - R = retained loss = .15 L

→ E(R) = **0.15*$1,000,000 = 150,000**

Analyzing a Constant times a RV

- $Y = aX$
- $Var(Y) = a^2 Var(X)$
- $Std(Y) = a\ Std(X)$
- Coinsurance Example ($R = .15L$)

  If $Std(L) = \$0.2$ million

- Std(R) = **0.15 * 0.2mil = 30,000**

Analyzing a Constant times a RV

- One additional relationship:
- Cov(aX , bY) = ab Cov(X,Y)
- Note:
  - Var(X) = Cov(X,X)
  - Var(aX) = Cov(aX,aX) = a^2 Var(X)

Expected Value of a Sum of Random Variables

- $Z = X + Y$

- $\rightarrow E(Z) = E(X) + E(Y)$

  - L = Auto Liability losses for coming year
  - P = Physical damage losses for coming year
  - $T = L + P$

  - If $E(L) = \$100$ and $E(P) = \$150$

$\rightarrow E(T) = 250$

Expected Value of a Sum of Random Variables

- R_1 = return on stock 1
- R_2 = return on stock 2
- R_P = return on portfolio
- Invest 40% in stock 1 & 60% in stock 2

$\rightarrow R_P = 0.4\ R_1 + 0.6\ R_2$

- What is the expected return on the portfolio:
  - $\rightarrow E(R_P) = 0.4*10+0.6*20 = 16\%$

Stock	Expected Ret	Standard Dev of Ret
1	10%	15%
2	20%	30%

Variance & Standard Deviation of a Sum of RV

- $Z = X + Y$
- $Var(Z) = Var(X) + Var(Y) + 2\,Cov(X,Y)$
- Example:
  - L = Auto Liability losses for coming year
  - P = Physical damage losses for coming year
  - $T = L + P$

  - If Std(L) = \$800 and Std(P) = \$200, $\rho_{L,P} = ?$

  $\rightarrow$ $Var(T) = 800^2+200^2+2(800)(200)\,\rho_{L,P}$

Variance & Standard Deviation of a Sum of RV

$$Var(T) = 800^2 + 200^2 + 2(800)(200)\,\rho_{L,P}$$

- If $\rho = 1$,
  - $Var(T) = (800 + 200)^2$

  → $Std(T) = 800 + 200$ (sum of individual std)

of the parts

- If $\rho < 1$,
  - $Std(T) <$ sum of individual std

Risk is diversified away; the total risk is less than the sum

Standard Deviation of a Sum of RVs

- Stock Portfolio example

Stock	Expected	Standard Dev of Ret	% of Wealth Invested
1	Ret 10%	15%	40%
2	20%	30%	60%

Correlation between stock 1 and 2 = ρ

- We are interested in the rv:

  - $R_P = 0.4\ R_1 + 0.6\ R_2$

Standard Deviation of a Sum of RVs

Stock 1	Expected Ret 10%	Standard Dev of Ret 15%	% of Wealth Invested 40%
2	20%	30%	60%

Correlation between stock 1 and 2 = ρ

- $\text{Std}(R_P) = \text{Var}(R_p)^{1/2} = [\text{Var}(0.4R_1 + 0.6R_2)]^{1/2}$

$= [.4^2\text{Var}(R_1) + .6^2\text{Var}(R_2) + 2(.4)(.6)\text{cov}(R_1, R_2)]^{1/2}$

$\text{Cov}(R_1, R_2) = \rho_{1,2}\,\text{Std}(R_1)\,\text{Std}(R_2)$

Portfolio Example

- $Std(R_p) = [.4^2 (.15)^2 + .6^2(.3)^2 + 2(.4)(.6)\ (.15)(.30)\ \rho_{1,2}\ ]^{1/2}$

Stock	Expected	Standard Dev of Ret	% of Wealth Invested
1	Ret 10%	15%	40%
2	20%	30%	60%

Corr coeff	Expected Ret on Portf	Standard Dev of Portf Ret
1.0	16%	24.0%
0.7	16%	22.6%
0.4	16%	21.1%

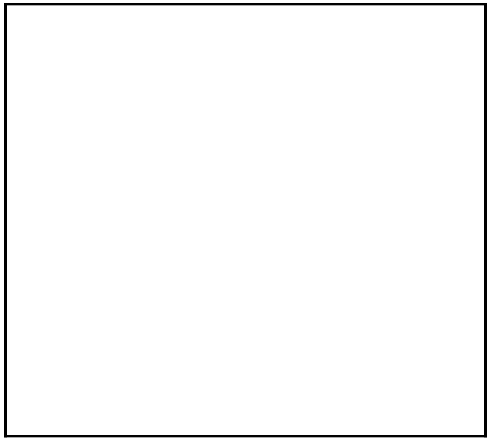
0.0	**16%**	**19.0%**
-0.4	**16%**	**16.5%**
-0.7	**16%**	**14.4%**
-1.0	**16%**	**12.0%**

Portfolio Example

- Graphically:

Expected
Return

0.16



Sum of 3 Random Variables:

- $T = L_1 + L_2 + L_3$

- $Var(T) = Var(L_1 + L_2 + L_3)$

$= Var(L_1) + Var(L_2) + Var(L_3)$

$$+ 2\mathrm{Cov}(L_1, L_2) + 2\mathrm{Cov}(L_1, L_3) + 2\mathrm{Cov}(L_2, L_3)$$