ONLINE APPENDIX:

TRADEOFFS AND COMPARISON COMPLEXITY

F Additional Proofs

F.1 Characterization Results

Proof of Theorem 2.

The proof of necessity is routine. Theorem 1 covers sufficiency for the $n \geq 3$ case. We now show sufficiency in the case where $n = 2$; assume that M1–M6 hold. Note that Claim 1 in the proof of Theorem 1 continues to hold in this case; that is, that for any $z \in \mathbb{R}^n$ satisfying $\sum_k z_k \geq 0$, $\rho(z, 0) = \rho(d^+(z)e_1 - d^-(z)e_2, 0)$. To see this, note that if $z_1 \geq 0$, $z_2 \geq 0$, the desired equality follows from Dominance; if not then either i) $z_1 > 0$, $z_2 < 0$ or ii) $z_1 < 0$ and $z_2 > 0$. In case i), the equality is immediate since $z = d^+e_1 + d^-e_2$, which in conjunction with Exchangeability, implies the desired equality for case ii). Following the steps in Claims 2 and 3 in the proof of Theorem 1 completes the proof of sufficiency. Note that the argument for uniqueness in Theorem 1 holds for $n = 2$, and so uniqueness holds as well.

□

Proof of Theorem 3.

The proof of necessity of M1, M4–M5, and M7 are routine. To see that M3 (Moderate Transitivity) is necessary, consider x, y, z with $\rho(x, y) \geq 1/2$ and $\rho(y, z) \geq 1/2$. If $d_{L1}(x, y), d_{L1}(y, z)$, $d_{L1}(x, z) > 0$, then the restriction of ρ to $\{x, y, z\}$ belongs to the moderate utility class studied in He and Natenzon (2023a) and so by Theorem 1 of this paper we can conclude that this restriction satisfies Moderate Transitivity. There are four additional cases to consider. Case 1: suppose $d_{L1}(x, y) = 0$. We then have $\rho(x, z) = \rho(y, z)$ and $\rho(x, y) = 1/2$, so either $\rho(x, z) > \min\{\rho(x, y), \rho(y, z)\}$ or $\rho(x, z) = \rho(y, z) = \rho(x, z)$. Case 2: $d_{L1}(y, z) = 0$. We then have $\rho(x, z) = \rho(x, y)$ and $\rho(y, z) = 1/2$, and so again either $\rho(x, z) > \min\{\rho(x, y), \rho(y, z)\}$ or $\rho(x, z) = \rho(y, z) = \rho(x, z)$. Case 3: $d_{L1}(x, z) = 0$. Here we have $\rho(x, z) = 1/2$, and $\rho(x, y) = \rho(z, y) \geq 1/2$ and $\rho(y, z) \geq 1/2$, which implies $\rho(y, z) = \rho(x, y) = 1/2$; we therefore have $\rho(x, y) = \rho(y, z) = \rho(x, z)$. Finally, consider $d_{L1}(x, y) = d_{L1}(x, z) = d_{L1}(y, z) = 0$; here we have $\rho(x, y) = \rho(y, z) = \rho(x, z)$, and so Moderate Transitivity holds in all cases.

To see that M8 (Tradeoff Congruence) is necessary, take $(x,y),(y,z) \in \mathscr{D}$ congruent such that $\rho(x,y),\rho(y,z) \geq 1/2$. Note that if $d_{L1}(x,z)=0$, then $\rho(x,y)=1/2$ and since ρ satisfies Moderate Transitivity we have $\rho(x,y)=\rho(y,z)=1/2$ and we are done. Now consider the case where $\rho(x,z) \neq 0$. Note that

$$\begin{aligned}
\rho(x,z) &= G\left(\frac{\sum_k (u_k(x_k)-u_k(z_k))}{\sum_k |u_k(x_k)-u_k(z_k)|}\right) \\
&= G\left(\frac{\sum_k (u_k(x_k)-u_k(y_k)+u_k(y_k)-u_k(z_k))}{\sum_k |u_k(x_k)-u_k(y_k)+u_k(y_k)-u_k(z_k)|}\right) \\
&= G\left(\frac{U(x)-U(y)+U(y)-U(z)}{d_{L1}(x,y)+d_{L1}(y,z)}\right)
\end{aligned}$$

Where the final equality holds because congruence implies that $u_k(x_k)-u_k(y_k)$ and $u_k(y_k)-u_k(z_k)$ must either be both positive or negative. This implies that if either $d_{L1}(x,y)=0$ or $d_{L1}(y,z)=0$, we are done. Now consider the case where $d_{L1}(x,y), d_{L1}(y,z) > 0$, and suppose $\rho(y,z) \leq \rho(x,y)$; this implies $\frac{U(y)-U(z)}{d_{L1}(y,z)} \leq \frac{U(x)-U(y)}{d_{L1}(x,y)}$. The above implies

$$\begin{aligned}
\rho(x,z) &= G\left(\frac{\frac{U(x)-U(y)}{d_{L1}(y,z)} + \frac{U(y)-U(z)}{d_{L1}(y,z)}}{\frac{d_{L1}(x,y)}{d_{L1}(y,z)}+1}\right) \\
&\leq G\left(\frac{\frac{U(x)-U(y)}{d_{L1}(y,z)} + \frac{U(x)-U(y)}{d_{L1}(x,y)}}{\frac{d_{L1}(x,y)}{d_{L1}(y,z)}+1}\right) \\
&= \rho(x,y)
\end{aligned}$$

and so $\rho(x,z) \leq \max\{\rho(x,y),\rho(y,z)\}$ when $\rho(y,z) \leq \rho(x,y)$. The argument for the case where $\rho(y,z) \geq \rho(x,y)$ is analogous.

Now we show sufficency. Let $\succeq$ be the stochastic preference relation induced by ρ. $\succeq$ satisfies coordinate independence and inherits continuity from ρ, and since we have at least 3 non-null attributes, we invoke Debreu (1983) to conclude that $\succeq$ has an additively separable representation: there exists $u_i : X_i \to \mathbb{R}$, continuous, such that

$$x \succeq y \iff \sum_k u_k(x_k) \geq \sum_k u_k(y_k)$$

Since all attributes are non-null and the X_k are connected, each $u_k(X_k)$ is a non-trivial interval of $\mathbb{R}$. Since the representation is unique up to cardinal transformations, we can without loss assume that for each $k \in I$, $u_k(X_k)$ contains 0, and furthermore, since $u_k(X_k)$ is a non-trivial interval, that $u_k(X_k)$ contains a non-trivial open interval around 0. For all $k \in I$, let $\overline{u}_k = \sup u_k(X_k)$ and $\underline{u}_k = \inf u_k(X_k)$, taken with respect to the extended real line, and let $\Delta_k = \overline{u}_k - \underline{u}_k$. For all $x \in X$, define $\tilde{x} = (u_1(x_1), ..., u_k(x_k)) \in \mathbb{R}^n$. Begin by noting the following result.

Lemma 1. *For $x, y \in X$ with $\tilde{x} = \tilde{y}$: $\rho(x,z) = \rho(y,z)$ for all $z \in X$.*

Proof. Fix such an x, y, and take any $z \in X$. Note that $x \sim y$ by hypothesis. First consider the case where $x \sim y \succeq z$. Since (x, y) and (y,z) are congruent, and likewise (y, x) and (x,z) are congruent, Tradeoff Congruence implies

$$\rho(x,z) \leq \max\{\rho(y,z), \rho(x,y)\} = \rho(y,z)$$
$$\rho(y,z) \leq \max\{\rho(x,z), \rho(y,x)\} = \rho(x,z)$$

and so $\rho(y,z) = \rho(x,z)$. Analogously, consider the case where $z \succeq x \sim y$. Since (z, x) and (x, y) are congruent and likewise (z, y) and (y, x) are congruent, we have

$$\rho(z,x) \leq \max\{\rho(z,y), \rho(y,x)\} = \rho(z,y)$$
$$\rho(z,y) \leq \max\{\rho(z,x), \rho(x,y)\} = \rho(z,x)$$

and so $\rho(z,x) = \rho(z,y) \implies \rho(x,z) = \rho(y,z)$. $\square$

Let $\tilde{X} = \{\tilde{x} \in \mathbb{R}^n : x \in X\}$. Let $\tilde{\mathscr{D}} = \{(a,b) \in \tilde{X} : a \neq b\}$ and define $\phi : \tilde{\mathscr{D}} \to \mathscr{D}$ satisfying $\phi(a,b) \in \{(x,y) \in \mathscr{D} : \tilde{x} = a, \tilde{y} = b\}$, and define $\tilde{\rho} : \tilde{\mathscr{D}} \to [0,1]$ by $\tilde{\rho}(a,b) = \rho(\phi(a,b))$. Lemma 1 implies that $\tilde{\rho}$ is a binary choice rule on $\tilde{\mathscr{D}}$ and does not depend on the selection made by ϕ: in particular, we have $\tilde{\rho}(\tilde{x},\tilde{y}) = \rho(x,y)$ for all $(x,y) \in \mathscr{D}$. This in turn implies that $\tilde{\rho}$ inherits our axioms M1,M3–M5, M7–M8. Note that if there exists a strictly increasing, continuous function G such that

$$\tilde{\rho}(a,b) = G\left(\frac{\sum_k (a_k - b_k)}{\sum_k |a_k - b_k|}\right)$$

for all $(a,b) \in \mathscr{D}$, we are done, as this implies that for any $(x,y) \in \mathscr{D}$ such that $\tilde{x} \neq \tilde{y} \iff \sum_k |u_k(x_k) - u_k(y_k)| > 0$,

$$\rho(x,y) = \tilde{\rho}(\tilde{x},\tilde{y}) = G\left(\frac{\sum_k (u_k(x_k) - u_k(y_k))}{\sum_k |u_k(x_k) - u_k(y_k)|}\right)$$

and furthermore for $(x,y)\in\mathscr{D}$ such that $\tilde{x}=\tilde{y}$, we have $x\sim y \implies \rho(x,y)=1/2$, and so ρ has an additively separable L_1-complexity representation.

In what follows, we will work with $\tilde{\rho}$ defined on $\tilde{X}$ and suppress the $\sim$ in our notation. Say that ρ defined on this domain is

- *Translation invariant* if for all $x,x',y,y'\in X, z\in\mathbb{R}^n$ such that $x'=x+z$, $y'=y+z$, $\rho(x',y')=\rho(x,y)$.
- *Scale invariant* if for all $x,x',y,y'\in X$ such that $x'=cx, y'=cy$ for $c>0$, $\rho(x',y')=\rho(x,y)$.
- *Translation invariant** if for all $x,x',y,y'\in X, z\in\mathbb{R}^n$ such that $x'=x+z$, $y'=y+z$, and additionally $x_k=y_k$ for some $k\in I$, $\rho(x',y')=\rho(x,y)$.
- *Scale invariant** if for all $x,x',y,y'\in X$ such that $x'=cx, y'=cy$ for $c>0$, and additionally $x_k=y_k$ for some $k\in I$, $\rho(x',y')=\rho(x,y)$.
- *Translation invariant*† if for all $x,x',y,y'\in X, z\in\mathbb{R}^n$ such that $x'=x+z$, $y'=y+z$, and additionally $x_k=y_k$ for some $k\in I$ such that $|x_i-y_i|<\Delta_k$ for all $i\in I$, $\rho(x',y')=\rho(x,y)$.
- *Scale invariant*† if for all $x,x',y,y'\in X$ such that $x'=\lambda x, y'=\lambda y$ for $\lambda\in(0,1)$, and additionally $x_k=y_k$ for some $k\in I$ such that $|x_i-y_i|<\Delta_k$ for all $i\in I$, $\rho(x',y')=\rho(x,y)$.

First, note that Separability and Simplification imply translation invariance†.

Lemma 2. *Suppose ρ satisfies Separability and Simplification. Then ρ satisfies translation invariance*†.

Proof. Begin by noting that for $x',y',x,y\in X$, $z\in\mathbb{R}^n$ with $x'=x+z$, $y'=y+z$, and $x_k=y_k$ for some $k\in I$ such that $|x_i-y_i|<\Delta_k$ for all $i\in I$: for any $E\subseteq I$, $x+\sum_{j\in E}z_{\{j\}}$ and $y+\sum_{j\in E}z_{\{j\}}$ will be in our domain, with $\left(x+\sum_{j\in E}z_{\{j\}}\right)_k=\left(y+\sum_{j\in E}z_{\{j\}}\right)_k$ and with $\left|\left(x+\sum_{j\in E}z_{\{j\}}\right)_i-\left(y+\sum_{j\in E}z_{\{j\}}\right)_i\right|<\Delta_k$ for all i. Since we can translate x and y by each component $z_{\{j\}}$ attribute-by-attribute, it suffices to show that for any $x,y\in X$ with $x_k=y_k$ where $|x_i-y_i|<\Delta_k$ for all $i\in I$, $z\in\mathbb{R}^n$, $j\in I$ such that $x+z_{\{j\}}$ and $y+z_{\{j\}}$ belong to our domain, $\rho(x+z_{\{j\}},y+z_{\{j\}})=\rho(x,y)$. Fix such an $x,y\in X$, $z\in\mathbb{R}^n$, $k,j\in I$.

Note that if $j = k$, Separability gives us the desired result. Now suppose $j \neq k$. Suppose that $x_j \geq y_j$ (the argument for $x_j < y_j$ is analogous). For any $i \in I$, $a \in (\underline{u}_i, \overline{u}_i)$, $w \in X$, let $a_{\{i\}}w \in X$ denote the option equal to a for attribute $k = i$ and equal to w_k for all other attributes. Since by hypothesis $|x_i - y_i| < \Delta_k$ for all i, there exists some $b \in (\underline{u}_k, \overline{u}_k)$ such that $|x_i - y_i| < \overline{u}_k - b$ for all i. By Separability, we have $\rho(b_{\{k\}}x, b_{\{k\}}y) = \rho(x, y)$. Now consider $x' \in \mathbb{R}^n$ satisfying

$$x'_i = \begin{cases} y_j & i = j \\ b + (x_j - y_j) & i = k \\ x_i & \text{otherwise} \end{cases}$$

By construction, $b + (x_j - y_j) < \overline{u}_k$, and so $x' \in X$. Applying simplification twice, we have $\rho(x', b_{\{k\}}y) = \rho(b_{\{k\}}x, b_{\{k\}}y)$. Since $x'_j = (b_{\{k\}}y)_j$ by construction, Separability in turn implies that $\rho(x' + z_{\{j\}}, b_{\{k\}}y + z_{\{j\}}) = \rho(x', b_{\{k\}}y)$. Again applying Simplification twice, we have $\rho(b_{\{k\}}x + z_{\{j\}}, b_{\{k\}}y + z_{\{j\}}) = \rho(x' + z_{\{j\}}, b_{\{k\}}y + z_{\{j\}})$. A final application of Separability yields $\rho(x + z_{\{j\}}, y + z_{\{j\}}) = \rho(b_{\{k\}}x + z_{\{j\}}, b_{\{k\}}y + z_{\{j\}})$, and the chain of equalities yields $\rho(x + z_{\{j\}}, y + z_{\{j\}}) = \rho(x, y)$ as desired. □

The next result says that scale invariance* is implied by translation invariance† and our other axioms.

Lemma 3. *Suppose ρ satisfies translation invariance†, Continuity, Moderate Transitivity, and Tradeoff Congruence. Then ρ satisfies scale invariance*.*

Proof. First, show that invariance† holds for half-mixtures and then extend the result to arbitrary mixtures using continuity. In particular, we want to show that for $x, y \in X$ with $x_k = y_k$ for some k such that $|x_i - y_i| < \Delta_k$ for all i, $\rho(x, y) = \rho(\frac{1}{2}x, \frac{1}{2}y)$. Without loss, suppose that $x \succeq y$. By translation invariance†, we have $\rho(x, \frac{1}{2}x + \frac{1}{2}y) = \rho(\frac{1}{2}x + \frac{1}{2}y, y) = \rho(\frac{1}{2}x, \frac{1}{2}y)$. Since $(x, \frac{1}{2}x + \frac{1}{2}y)$ and $(\frac{1}{2}x + \frac{1}{2}y, y)$ are congruent and $x \succeq \frac{1}{2}x + \frac{1}{2}y \succeq y$, by Tradeoff Congruence and Moderate Transitivity, we have $\rho(x, y) = \rho(x, \frac{1}{2}x + \frac{1}{2}y) = \rho(\frac{1}{2}x, \frac{1}{2}y)$ as desired.

We now show that for any $x, y \in X$ with $x_k = y_k$ and $|x_i - y_i| < \Delta_k$ for all $i \in I$, for any $n \in \mathbb{N}$, $\rho(x, y) = \rho(\alpha x, \alpha y)$ for all $\alpha \in \{\frac{1}{2^n}, \frac{2}{2^n}, \ldots, \frac{2^n}{2^n}\}$. Note that if $x \sim y$, then the result holds by definition of $\succeq$ and we are done. Now suppose that $x \not\sim y$, and assume without loss that $x \succ y$. Proceed inductively; given what we have shown above, the statement is true for $n = 1$. Now suppose the statement is true for some n; we wish to show that for any $m \in \{1, \ldots, 2^{n+1}\}$, $\rho(\frac{m}{2^{n+1}}x, \frac{m}{2^{n+1}}y) = \rho(x, y)$. Note that for any $m \leq 2^n$ we have

$\rho(\frac{m}{2^{n+1}}x, \frac{m}{2^{n+1}}y) = \rho(\frac{m}{2^n}x, \frac{m}{2^n}y) = \rho(x, y)$ using our result on half-mixtures and by inductive hypothesis.

Now consider $m \in \{2^n+1, ..., 2^{n+1}\}$. Note that by translation invariance† and by inductive hypothesis, we have $\rho(\frac{m}{2^{n+1}}x, \frac{1}{2}y + \frac{m-2^n}{2^{n+1}}x) = \rho(\frac{1}{2}x, \frac{1}{2}y) = \rho(x, y)$. Also, by translation invariance† and inductive hypothesis, we have $\rho(\frac{1}{2}y + \frac{m-2^n}{2^{n+1}}x, \frac{m}{2^{n+1}}y) = \rho(\frac{m-2^n}{2^{n+1}}x, \frac{m-2^n}{2^{n+1}}y) = \rho(x, y)$. These two equalities and Moderate Transitivity imply that $\rho(\frac{m}{2^{n+1}}x, \frac{m}{2^{n+1}}y) \geq \rho(x, y)$.

Toward a contradiction, suppose $\rho(\frac{m}{2^{n+1}}x, \frac{m}{2^{n+1}}y) > \rho(x, y)$. Translation invariance† then implies $\rho(x, \frac{2^{n+1}-m}{2^{n+1}}x + \frac{m}{2^{n+1}}y) > \rho(x, y)$. By translation invariance† and the result shown above, we also have $\rho(\frac{2^{n+1}-m}{2^{n+1}}x + \frac{m}{2^{n+1}}y, y) = \rho(\frac{2^{n+1}-m}{2^{n+1}}x, \frac{2^{n+1}-m}{2^{n+1}}y) = \rho(x, y)$. But since Moderate Transitivity implies that $\rho(x, y) > \rho(\frac{2^{n+1}-m}{2^{n+1}}x + \frac{m}{2^{n+1}}y, y)$, we have a contradiction. This proves the statement for $n+1$, and so by induction the statement holds for any n. By taking limits and by Continuity of ρ, we can then conclude that scale invariance† holds.

Now we show that scale invariance* holds. Fix any $x, y \in X$ where $x_k = y_k$ for some k. Without loss, assume $x \succeq y$. First, show that $\rho(x, y) = \rho(\lambda x, \lambda y)$ for any $\lambda \in (0, 1)$. Note that there exists some $N \in \mathbb{N}$ such that $\frac{1}{N}|x_i - y_i| < \Delta_k$ for all i. For $n \in \{0, 1, ..., N\}$, define $w^n \in X$ by $w^n = \frac{n}{N}x + \frac{N-n}{N}y$. Now consider the sequence of comparisons (w^N, w^{N-1}), $(w^{N-1}, w^{N-2}), ..., (w^1, w^0)$. Since $w^n - w^{n-1} = \frac{1}{N}(x - y)$ for all n, we have $w^n \succeq w^{n-1}$ for all n, and additionally $|w_i^n - w_i^{n-1}| < \Delta_k$ for all i, and so translation invariance† implies that for all n, $\rho(w^n, w^{n-1}) = \rho(w^n - (\frac{N-n}{N}y + \frac{n-1}{N}x), w^{n-1} - (\frac{N-n}{N}y + \frac{n-1}{N}x)) = \rho(\frac{1}{N}x, \frac{1}{N}y)$. Sequential applications of Moderate Transitivity and Tradeoff Congruence yield, respectively

$$\rho(x, y) \geq \min\{\rho(w^N, w^{N-1}), \rho(w^{N-1}, w^{N-2}), ..., \rho(w^1, w^0)\}$$
$$\rho(x, y) \leq \max\{\rho(w^N, w^{N-1}), \rho(w^{N-1}, w^{N-2}), ..., \rho(w^1, w^0)\}$$

and so we have $\rho(x, y) = \rho(\frac{1}{N}x, \frac{1}{N}y)$. An analogous argument, taking the sequence of comparisons $(\lambda w^N, \lambda w^{N-1})$, $(\lambda w^{N-1}, \lambda w^{N-2}), ..., (\lambda w^1, \lambda w^0)$, yields $\rho(\lambda x, \lambda y) = \rho(\lambda \frac{1}{N}x, \lambda \frac{1}{N}y)$. By scale invariance†, noting again that $\frac{1}{N}|x_i - y_i| < \Delta_k$ for all i, we have $\rho(\lambda \frac{1}{N}x, \lambda \frac{1}{N}y) = \rho(\frac{1}{N}x, \frac{1}{N}y)$ and so $\rho(x, y) = \rho(\lambda x, \lambda y)$ as desired.

We have therefore shown that for any $x, y \in X$ with $x_k = y_k$ for some k, $\lambda \in (0, 1)$, $\rho(x, y) = \rho(\lambda x, \lambda y)$. Finally, fix some $c > 0$ and $x, y \in X$ with $x_k = y_k$ for some k and $cx, cy \in X$; we wish to show that $\rho(x, y) = \rho(cx, cy)$. If $c \leq 1$, we are done by the result established above. If instead $c > 1$, the above result implies that $\rho(cx, cy) = \rho(\frac{1}{c}cx, \frac{1}{c}cy) = \rho(x, y)$. $\square$

Scale invariance* allows us to strengthen translation invariance† to translation invariance*.

Lemma 4. *Suppose ρ satisfies translation invariance† and scale invariance*. Then ρ satisfies translation invariance*.*

Proof. Take $x, y \in X$ with $x_k = y_k$ for some k, and $z \in \mathbb{R}^n$ such that $x+z, y+z \in X$. There exists some $\lambda \in (0,1)$ such that $\lambda |x_i - y_i| < \Delta_k$ for all i; fix such a λ. We then have

$$\begin{aligned}\rho(x,y) &= \rho(\lambda x, \lambda y) \\ &= \rho(\lambda(x+z), \lambda(y+z)) \\ &= \rho(x+z, y+z)\end{aligned}$$

where the first and third equalities use scale invariance* and the second equality uses translation invariance†. □

We now show that scale invariance*, translation invariance*, and Tradeoff Congruence imply translation invariance.

Lemma 5. *Suppose ρ satisfies translation invariance*, scale invariance*, Simplification, Tradeoff Congruence, and Moderate Transitivity. Then ρ satisfies translation invariance.*

Proof. Take any $x, y \in X$, $w \in \mathbb{R}^n$ such that $x+w, y+w \in X$. We want to show that $\rho(x+w, y+w) = \rho(x,y)$. Without loss, assume that $x \succeq y$. Note that if $x \geq y$, we are done by Dominance, so consider the case where $x \ngeq y$. Let $z = x - y \in \mathbb{R}^n$. If $z_k = 0$ for some k, then by translation invariance* we are done, so consider the case where $z_k \neq 0$ for all k. It must then be the case that there exist distinct indices $i, j \in I$ such that $\text{sgn}(z_i) = \text{sgn}(z_j) \neq 0$. Define $z^i, z^j \in \mathbb{R}^n$ such that

$$z^i_k = \begin{cases} z_i + z_j & k = i \\ 0 & k = j \\ z_k & \text{otherwise} \end{cases} \qquad z^j_k = \begin{cases} 0 & k = i \\ z_i + z_j & k = j \\ z_k & \text{otherwise} \end{cases}$$

Letting $\lambda = \frac{z^i}{z^i + z_j} \in (0,1)$, note that by construction $z = \lambda z^i + (1-\lambda) z^j$. Now fix any $v \in X$ such that $z + v, v \in X$; note that $z + v \in X \implies (1-\lambda) z^j + v \in X$. Since each $u_k(X_k)$ contains a non-trivial open interval around 0, there exists $\gamma \in (0,1)$ such that $\gamma z^i, \gamma z^j \in X$. We then have

$$\begin{aligned}\rho(z+v,(1-\lambda)z^j+v) &= \rho(\gamma(z+v),\gamma((1-\lambda)z^j+v)) \\ &= \rho(\gamma\lambda z^i,0) \\ &= \rho(\gamma z^i,0) \\ &= \rho(\gamma z^j,0) \\ &= \rho(\gamma(1-\lambda)z^j,0) \\ &= \rho(\gamma((1-\lambda)z^j+v),\gamma v) \\ &= \rho((1-\lambda)z^j+v,v)\end{aligned}$$

Where the first three equalities follow from scale invariance* and translation invariance*, noting that by construction, $(1-\lambda)z^j_j = z_j$, the fourth equality follows from two applications of Simplification, and the final three equalities follow from translation invariance* and scale invariance*, noting that $z^j_i = 0$.

By construction, $(z+v,(1-\lambda)z^j+v)$ and $((1-\lambda)z^j+v,v)$ are congruent, since $[z+v]-[(1-\lambda)z^j+v] = \lambda z^i$ and $[(1-\lambda)z^j+v]-v = (1-\lambda)z^j$, and since for all k, either $z^j_k, z^i_k \geq 0$ or $z^j_k, z^i_k \leq 0$. Furthermore, since $\sum_k z^i_k = \sum_k z^j_k = \sum_k z_k \geq 0$, we have $z+v \succeq (1-\lambda)z^j+v$ and $(1-\lambda)z^j+v \succeq v$. We then have

$$\begin{aligned}\rho(z+v,v) &= \rho(z+v,(1-\lambda)z^j+v) \\ &= \rho(\gamma z^i,0)\end{aligned}$$

Where the first equality follows from Tradeoff Congruence and Moderate Transitivity, and the second equality follows from the chain of equalities above. Since this equality holds for all v such that $z+v, v \in X$, substituting $v = y$ and $v = y+w$ yields $\rho(x,y) = \rho(x+w,y+w)$ as desired. □

Lemma 6. *Suppose ρ satisfies translation invariance, Continuity, Moderate Transitivity, and Tradeoff Congruence. Then ρ satisfies scale invariance.*

Proof. Fix any $x, y \in X$, and without loss suppose $x \succeq y$. Note that by translation invariance, we have $\rho(x,\frac{1}{2}x+\frac{1}{2}y) = \rho(\frac{1}{2}x+\frac{1}{2}y,y) = \rho(\frac{1}{2}x,\frac{1}{2}y)$. Since $(x,\frac{1}{2}x+\frac{1}{2}y)$ and $(\frac{1}{2}x+\frac{1}{2}y,y)$ are congruent and $x \succeq \frac{1}{2}x+\frac{1}{2}y \succeq y$, by Tradeoff Congruence and Moderate Transitivity, we have $\rho(x,y) = \rho(x,\frac{1}{2}x+\frac{1}{2}y) = \rho(\frac{1}{2}x,\frac{1}{2}y)$.

The proof for extending the result on half-mixtures to arbitrary mixtures and then to arbitrary rescaling follows an analogous argument as in the proof for Lemma 3, invoking translation invariance whenever translation invariance† is invoked in that proof. $\square$

Using Lemmas 2–6, we conclude that ρ satisfies scale and translation invariance. Linearly extend ρ to $\mathbb{R}^n$ as follows. Define $\overline{\mathscr{D}} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$, and define $\overline{\rho} : \overline{\mathscr{D}} \to [0,1]$ such that for any $(x,y) \in \mathscr{D}$, $\overline{\rho}(x,y) = \rho(x,y)$, and for any $(x,y) \in \overline{\mathscr{D}} \setminus \mathscr{D}$, $\rho(x,y) = \rho(\lambda x, \lambda y)$ for some $\lambda \in (0,1)$ such that $\lambda x, \lambda y \in X$. Since X contains an open a ball around the origin, this extension is well-defined. Furthermore, since ρ satisfies scale and translation invariance, so does $\overline{\rho}$, and so $\overline{\rho}$ satisfies M2 (Linearity). Noting that for any finite collection of options $A \subseteq \mathbb{R}^n$, there exists $\lambda \in (0,1)$ such that $\lambda x \in X$ for all $x \in A$, by scale invariance of $\overline{\rho}$ it is straightforward to show that $\overline{\rho}$ is a binary choice rule and satisfies M1, M3–M5. Theorem 1 then implies that there exists G continuous, strictly increasing, such that for all $(x,y) \in \overline{\mathscr{D}}$,

$$\overline{\rho}(x,y) = G\left(\frac{\sum_k (x_k - y_k)}{\sum_k |x_k - y_k|}\right)$$

which in turn implies that for all $(x,y) \in \mathscr{D}$,

$$\rho(x,y) = \overline{\rho}(x,y) = G\left(\frac{\sum_k (x_k - y_k)}{\sum_k |x_k - y_k|}\right)$$

which yields the desired representation.

Finally, we show uniqueness. Suppose that ρ has additively separable L_1 complexity representations $((u_i)_{i=1}^n, G)$ and $((u_i')_{i=1}^n, G')$. Let $\succeq$ denote the stochastic order on X induced by ρ. Since G and G' are strictly increasing and symmetric around 0, we have for all $x, y \in X$

$$x \succeq y \iff \sum_k u_k(x_k) \geq \sum_k u_k(y_k) \iff \sum_k u_k'(x_k) \geq \sum_k u_k'(y_k)$$

and U, U' both represent $\succeq$, where $U(x) = \sum_k u_k(x_k)$ and $U'(x) = \sum_k u_k'(x_k)$. Debreu (1983) then implies that there exists $C > 0$, $b_k \in \mathbb{R}$ such that $u_k' = C u_k + b_k$ for all k. This implies that for all $x, y \in X$,

$$G\left(\frac{\sum_k (u_k(x_k) - u_k(y_k))}{\sum_k |u_k(x_k) - u_k(y_k)|}\right) = G'\left(\frac{\sum_k (u_k(x_k) - u_k(y_k))}{\sum_k |u_k(x_k) - u_k(y_k)|}\right)$$

By assumption, there exist two non-null indices; without loss, we assume indices 1 and 2 are non-null. Since u_1, u_2 are continuous and X_1 and X_2 are connected, $u_1(X_1)$ and $u_2(X_2)$

are intervals in $\mathbb{R}^n$. Since we have shown that the u_k are unique up to affine transformations, we can without loss assume that for all $\mu \in [0,1]$, there exist $x_1^\mu \in X_1$ and $y_1^\mu \in X_2$ such that $u_1(x_1^\mu) = u_2(x_2^\mu) = \mu$.

Fix some $\overline{x} \in X$. For any $\alpha, \gamma \in [0,1]$, note that for $x, y \in X$ with

$$x_k = \begin{cases} x_1^\alpha & k=1 \\ x_2^0 & k=2 \\ \overline{x}_k & \text{otherwise} \end{cases} \qquad y_k = \begin{cases} x_1^0 & k=1 \\ x_2^\gamma & k=2 \\ \overline{x}_k & \text{otherwise} \end{cases}$$

we have

$$\rho(x,y) = G\left(\frac{\alpha-\gamma}{\alpha+\gamma}\right) = G'\left(\frac{\alpha-\gamma}{\alpha+\gamma}\right)$$

Since for any $r \in [-1,1]$ there exists $\alpha, \gamma \in [0,1]$ such that $\frac{\alpha-\gamma}{\alpha+\gamma} = r$, we must have $G' = G$.

Proof of Theorem 4.

Necessity of the axioms is immediate from the definition; we now show sufficiency.

Let $\succeq$ denote the stochastic order on X induced by ρ. By Moderate Transitivity, $\succeq$ is transitive. Since ρ satisfies Continuity and Independence, $\succeq$ satisfies the vNM axioms and so there exists a utility function $u : \mathbb{R} \to \mathbb{R}$ such that $U(x) = \sum_w u(w) f_x(w)$ represents $\succeq$; Dominance implies that u is strictly increasing.

Fix any four distinct prizes $w_a, w_b, w_c, w_d \in \mathbb{R}$ such that $u(w_a) > u(w_b) > u(w_c) > u(w_d)$. Consider any two lotteries $x, y \in X$. Enumerate $S_x \cup S_y \cup \{w_a, w_b, w_c, w_d\}$ by $w_1, w_2, ..., w_{n+1}$, where $w_1 < w_2 < ... < w_{n+1}$, and let $K = \{1, ..., n, n+1\}$. Let $X(K)$ denote the set of finite-state lotteries with support on $\{w_1, w_2, ..., w_{n+1}\}$. With some abuse of notation, we let a, b, c, d denote the indices in K corresponding to prizes w_a, w_b, w_c, w_d. We have $u(w_1) < u(w_2) < ... < u(w_{n+1})$. With some abuse of notation, for any $z \in X(K)$, let $F_z(k) = \sum_{w \le w_k} f_z(w)$ denote the value of the CDF of z at support point w_k, and let $u(k) = u(w_k)$.

Identify each lottery $z \in X(K)$ with its *utility-weighted* CDF vector $\tilde{z} \in \mathbb{R}^n$, where

$$\tilde{z}_k = -F_z(k)(u(k+1) - u(k))$$

for $k = 1, 2, ..., n$. Note that for any $x, y \in X(K)$,

$$\frac{\sum_k (\tilde{x}_k - \tilde{y}_k)}{\sum_k |\tilde{x}_k - \tilde{y}_k|} = \frac{U(x) - U(y)}{d_{CDF}(x, y)}$$

We now seek to extend the space of utility-weighted CDF vectors to $\mathbb{R}^n$ in order to apply Theorem 1. Let $\mu \in X(K)$ denote the lottery that is uniform over K; that is $F_\mu(k) = \frac{k}{n+1}$. Consider the set

$$V = \{a \in \mathbb{R}^n : a_k = \alpha(\tilde{x}_k - \tilde{\mu}_k) : x \in X(K), \alpha > 0)\}.$$

Lemma 7. $V = \mathbb{R}^n$.

Proof. By definition we have $V \subseteq \mathbb{R}^n$. To see that $\mathbb{R}^n \subseteq V$, take any $a \in \mathbb{R}^n$. We will show that $a \in V$. Define

$$\begin{aligned}
\beta &= \max_{k \in \{2,3,\ldots,n\}} (n+1)\left[a_k/(u(k+1) - u(k)) - a_{k-1}/(u(k) - u(k-1))\right] \\
\gamma &= (n+1)[a_1/(u(2) - u(1))] \\
\eta &= -(n+1)[a_n/(u(n+1) - u(n))]
\end{aligned}$$

and fix any $\alpha > \max\{\beta, \gamma, \eta, 0\}$. Define $H : K \to \mathbb{R}$ given by

$$H(k) = \begin{cases} F_\mu(k) - \dfrac{a_k/(u(k+1) - u(k))}{\alpha} & k < n+1 \\ 1 & k = n+1 \end{cases}$$

Since $\alpha > \beta$, we have $H(k+1) - H(k) \geq 0$ for all $k = 1, ..., n$, and since $\alpha > \eta$, we have $1 = H(n+1) - H(n) \geq 0$, and so H is increasing. Furthermore, since $\alpha > \gamma$, $H(1) \geq 0$, and so H is positive on its domain. Since $H(n+1) = 1$, H is the CDF of a lottery in $X(K)$, which we denote by x. Note that by construction, for all $k = 1, ..., n$ we have

$$\begin{aligned}
\alpha(\tilde{x}_k - \tilde{\mu}_k) &= \alpha\left(-F_\mu(k)(u(k+1) - u(k)) + \frac{a_k}{\alpha} + F_\mu(k)(u(k+1) - u(k))\right) \\
&= a_k
\end{aligned}$$

which implies that $a \in V$. $\square$

For any $a, b \in V$, let

$$L(a,b) = \{(x,y) \in X(K) \times X(K) : a = \alpha(\tilde{x} - \tilde{\mu}), b = \alpha(\tilde{y} - \tilde{\mu}),\ \alpha > 0\}.$$

Lemma 8. *Let $W \subseteq V$ finite. Then there exists $\alpha > 0$ such that for all $a \in W$, $a = \alpha(\tilde{x} - \tilde{\mu})$ for some $x \in X(K)$.*

Proof. Enumerate the elements of W by $\{a^1, a^2, ..., a^l\}$. For all $m = \{1, 2, ..., l\}$, there exists $\alpha^m > 0$, $z^m \in X(K)$ such that $a^m = \alpha^m(\tilde{z}^m - \tilde{\mu})$. Let $\alpha = \max_m \alpha^m$, and for all m, define $x^m \in X(K)$ satisfying $(\alpha^m/\alpha)z^m + (1 - \alpha^m/\alpha)\mu$, and notice that $a^m = \alpha(\tilde{x}^m - \tilde{\mu})$. $\square$

Define some $\phi : V \times V \to X(K) \times X(K)$ that takes an arbitrary selection from $L(a,b)$; Lemma 8 implies $L(a,b)$ is non-empty, ϕ is well-defined. For $\hat{\mathscr{D}} = \{(a,b) \in V \times V : a \neq b\}$, define $\hat{\rho} : \hat{\mathscr{D}} \to [0,1]$ by $\hat{\rho}(a,b) = \rho(\phi(a,b))$.

Lemma 9. *$\hat{\rho}$ is uniquely identified by ρ. That is, for any $a, b \in V$: for any $(x,y), (x',y') \in L(a,b)$,. $\rho(x,y) = \rho(x',y')$ and so $\hat{\rho}$ does not depend on the choice of ϕ. Also, $\hat{\rho}$ is a binary choice rule, that is, $\hat{\rho}(a,b) = 1 - \hat{\rho}(b,a)$.*

Proof. Fix some $a, b \in V$, and suppose $(x,y), (x',y') \in L(a,b)$. It suffices to show that $\rho(x,y) = \rho(x',y')$. Since $(x,y), (x',y') \in L(a,b)$, there exists $\alpha, \alpha' > 0$ such that

$$\begin{aligned} a &= \alpha(\tilde{x} - \tilde{\mu}) = \alpha'(\tilde{x}' - \tilde{\mu}) \\ b &= \alpha(\tilde{y} - \tilde{\mu}) = \alpha'(\tilde{y}' - \tilde{\mu}) \end{aligned}$$

Without loss, we can take $\alpha' > \alpha$. For $\lambda = \frac{\alpha}{\alpha'}$, the above inequalities directly imply that

$$\begin{aligned} x' &= \lambda x + (1 - \lambda)\mu \\ y' &= \lambda y + (1 - \lambda)\mu \end{aligned}$$

and so by Independence of ρ, $\rho(x,y) = \rho(x',y')$.

Finally to see that $\hat{\rho}$ is a binary choice rule, take any $a, b \in V$. By Lemma 8, there exists $\alpha > 0$, $x, y \in X(K)$ such that $a = \alpha(\tilde{x} - \tilde{\mu})$, $b = \alpha(\tilde{y} - \tilde{\mu})$; we have

$$\begin{aligned} \hat{\rho}(a,b) &= \rho(x,y) \\ &= 1 - \rho(y,x) \\ &= 1 - \hat{\rho}(b,a) \end{aligned}$$

as desired. $\square$

Lemma 10. $\hat{\rho}(a,b) \geq 1/2 \iff \sum_k a_k \geq \sum_k b_k$., *and* $\hat{\rho}$ *satisfies M1–M5.*

Proof. Fix any $a,b,c,a',b' \in V$. By Lemma 8, there exists $\alpha > 0$, $x,y,z,x',y' \in X(K)$ such that $a = \alpha(\tilde{x}-\tilde{\mu}), b = \alpha(\tilde{y}-\tilde{\mu}), c = \alpha(\tilde{z}-\tilde{\mu}), a' = \alpha(\tilde{x}'-\tilde{\mu}), b' = \alpha(\tilde{y}'-\tilde{\mu})$.

To show the first claim, note that $\hat{\rho}(a,b) \geq 1/2 \iff \rho(x,y) \geq 1/2 \iff U(x) \geq U(y) \iff \sum_k \tilde{x}_k \geq \sum_k \tilde{y}_k \iff \sum_k a_k \geq \sum_k b_k$.

To see that $\hat{\rho}$ satisfies Continuity, note that $\hat{\rho}$ inherits continuity from ρ. To see that $\hat{\rho}$ satisfies Linearity, take any $\lambda \in [0,1]$.Note that by construction, $\lambda a + (1-\lambda)c = \alpha(\lambda\tilde{x} + (1-\lambda)\tilde{z}-\tilde{\mu})$ and $\lambda b + (1-\lambda)c = \alpha(\lambda\tilde{y} + (1-\lambda)\tilde{z}-\tilde{\mu})$, and so

$$\begin{aligned}
\hat{\rho}(\lambda a + (1-\lambda)c, \lambda b + (1-\lambda)c) &= \rho(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z) \\
&= \rho(x,y) \\
&= \hat{\rho}(a,b)
\end{aligned}$$

where the first and final equalities follow from Lemma 9, and the second equality follows from Independence of ρ.

To show that $\hat{\rho}$ satisfies Moderate Transitivity, suppose that $\hat{\rho}(a,b) \geq 1/2$, $\hat{\rho}(b,c) \geq 1/2$. This implies that $\rho(x,y) \geq 1/2$, $\rho(y,z) \geq 1/2$, and so Moderate Transitivity of ρ implies that $\rho(x,z) \geq \min\{\rho(x,y), \rho(y,z)\}$, which in turn implies that $\hat{\rho}(a,c) \geq \min\{\rho(a,b), \rho(b,c)\}$, and so $\hat{\rho}$ satisfies Moderate Transitivity.

To show that $\hat{\rho}$ satisfies Dominance, by Lemma 9, it suffices to show that if $a_k \geq b_k$ for all k, then $x \geq y$. To see this, suppose that $a_k \geq b_k$ for all k; this implies that $\tilde{x}_k \geq \tilde{y}_k$ for all k, which in turn implies that $F_x(k) \leq F_y(k)$ for all k, and so $x \geq y$.

Finally, to see that $\hat{\rho}$ satisfies Simplification, consider $a,b \in V$ with $\rho(a,b) \geq 1/2$ and a' satisfying $a'_i = b_i$, $a'_k \neq b_k$ for all $k \neq i,j$ for $i \neq j$, with $\rho(a',a) \geq 1/2$.

By Lemma 8, there exists $\alpha > 0$, $x,x',y \in X(K)$ such that $a = \alpha(\tilde{x}-\tilde{\mu})$, $a' = \alpha(\tilde{x}'-\tilde{\mu})$, $b = \alpha(\tilde{y}-\tilde{\mu})$, and Lemma 9 implies that $\rho(x,y) \geq 1/2$ and $\rho(x',x) \geq 1/2$. Define $\hat{x},\hat{x}',\hat{y}$ by $\hat{x} = 1/2x + 1/2\mu$, $\hat{x}' = 1/2x' + 1/2\mu$, and $\hat{y} = 1/2y + 1/2\mu$. By construction that $S_{\hat{x}} = S_{\hat{x}'} = S_{\hat{y}} = \{w_1, ..., w_{n+1}\}$, and so in particular $S_{\hat{x}'} \subseteq S_{\hat{x}} \cup S_{\hat{y}}$. Independence implies that

$\rho(\hat{x},\hat{y}) \geq 1/2$, $\rho(\hat{x}',\hat{x}) \geq 1/2$. Moreover, since $a'_i = b_i$, we have $F_{\hat{x}'}(w_i) = F_{\hat{y}}(w_i)$, and since $a'_k = a_k$ for all $k \neq j,i$, we have $F_{\hat{x}'}(w) = F_{\hat{x}}(w)$ for all $w \in S_{\hat{x}} \cup S_{\hat{y}}/\{w_i,w_j\}$. Since ρ satisfies Simplification, we have $\rho(\hat{x}',\hat{y}) \geq \rho(\hat{x},\hat{y})$. Independence then implies $\rho(x',y) \geq \rho(x,y)$, and so applying Lemma 9, we have $\hat{\rho}(a',b) \geq \hat{\rho}(a,b)$, and so $\hat{\rho}$ satisfies Simplification. □

Using Lemma 10, Theorem 1 then implies that there exists a continuous, strictly increasing $G : [-1,1] \to [0,1]$, symmetric around 0, such that for all $a,b \in \mathbb{R}^n$ we have

$$\hat{\rho}(a,b) = G\left(\frac{\sum_k (a_k - b_k)}{\sum_k |a_k - b_k|}\right)$$

Lemma 9 then implies that for any $x,y \in X(K)$, we have

$$\begin{aligned}\rho(x,y) &= \hat{\rho}(\tilde{x}-\tilde{\mu},\tilde{y}-\tilde{\mu})\\ &= G\left(\frac{\sum_k(\tilde{x}_k - \tilde{y}_k)}{\sum_k |\tilde{x}_k - \tilde{y}_k|}\right)\\ &= G\left(\frac{U(x)-U(y)}{d_{CDF}(x,y)}\right)\end{aligned}$$

Let $\mathscr{K} = \{K \subseteq S : |K| < \infty, \{w_a,w_b,w_c,w_d\} \subseteq K\}$. The above implies that for any $K \in \mathscr{K}$, there exists a continuous, strictly increasing $G_K : [-1,1] \to [0,1]$ such that for all $x,y \in X(K)$,

$$\rho(x,y) = G_K\left(\frac{U(x)-U(y)}{d_{CDF}(x,y)}\right)$$

All that remains is to show that for any $K,K' \in \mathscr{K}$, $G_K = G_{K'}$. To see this, fix any $K,K' \in \mathscr{K}$, and for $\alpha \geq 0$, $\gamma \geq 0$, consider $x,y \in X$ with

$$x = \begin{cases} w_b & \text{w.p. } 1 \end{cases} \qquad y = \begin{cases} w_c & \text{w.p. } \dfrac{\alpha/(u(w_b)-u(w_c))}{\alpha/(u(w_b)-u(w_c))+\gamma/(u(w_a)-u(w_b))} \\ w_a & \text{w.p. } \dfrac{\gamma/(u(w_a)-u(w_b))}{\alpha/(u(w_b)-u(w_c))+\gamma/(u(w_a)-u(w_b))} \end{cases}$$

Note that x,y belong to both K and K', and so

$$\rho(x,y) = G_K\left(\frac{U(x)-U(y)}{d_{CDF}(x,y)}\right) = G_{K'}\left(\frac{U(x)-U(y)}{d_{CDF}(x,y)}\right)$$

and since $\frac{U(x)-U(y)}{d_{CDF}(x,y)} = \frac{\alpha-\gamma}{\alpha+\gamma}$, for any $r \in [-1,1]$ we can choose $\alpha,\gamma \geq 0$ such that $\frac{U(x)-U(y)}{d_{CDF}(x,y)} = r$, we must have $G_K = G_{K'}$.

Finally, to show uniqueness, suppose, (G,β) and (G',u') both represent ρ. Define the stochastic preference relation $\succeq$ as before. Since G and G' are both increasing and symmetric around 0, $U(x)=\sum_s f_x(w)u(w)$ and $U'(x)=\sum_s f_x(w)u'(w)$ both represent $\succeq$, which satisfies the vNM axioms, we can invoke vNM to conclude that there exists $C>0$, $b\in\mathbb{R}$ such that $u'=Cu+b$. This in turn implies that for all $x,y\in X$, we have

$$
\begin{aligned}
G\left(\frac{\sum_s(f_x(w)u(w)-f_y(w)u(w))}{\int_0^1 u(F_x^{-1}(q))-u(F_y^{-1}(q))|\,dq}\right) &= G'\left(\frac{\sum_s(f_x(w)u'(w)-f_y(w)u'(w))}{\int_0^1 u'(F_x^{-1}(q))-u'(F_y^{-1}(q))|\,dq}\right)\\
&= G'\left(\frac{\sum_s(f_x(w)u(w)-f_y(w)u(w))}{\int_0^1 u(F_x^{-1}(q))-u(F_y^{-1}(q))|\,dq}\right)
\end{aligned}
$$

Now consider $x,y\in X$ with

$$
x=\begin{cases} w_b & \text{w.p. } 1\end{cases} \qquad y=\begin{cases} w_c & \text{w.p. } \dfrac{\alpha/(u(w_b)-u(w_c))}{\alpha/(u(w_b)-u(w_c))+\gamma/(u(w_a)-u(w_b))}\\[2ex] w_a & \text{w.p. } \dfrac{\gamma/(u(w_a)-u(w_b))}{\alpha/(u(w_b)-u(w_c))+\gamma/(u(w_a)-u(w_b))}\end{cases}
$$

since $\frac{U(x)-U(y)}{d_{CDF}(x,y)}=\frac{\alpha-\gamma}{\alpha+\gamma}$, for any $r\in[-1,1]$ we can choose $\alpha,\gamma\geq 0$ such that $\frac{U(x)-U(y)}{d_{CDF}(x,y)}=r$, we must have $G'=G$.

$\square$

Proof of Theorem 5.

For $x,y\in X$, $a,b\in\mathbb{R}$, define $ax+by\in X$ to be the payoff stream with the payoff function am_x+bm_y. Let $\phi^\tau\in X$ be the payoff stream that pays off 1 at time τ and 0 otherwise. We start by observing a Lemma.

Lemma 11. *Suppose $U:X\to\mathbb{R}$ is linear. Then there exists $d:[0,\infty)\to\mathbb{R}$ such that $U(x)=\sum_t d(t)m_x(t)$.*

Proof. Let $d:[0,\infty)\to\mathbb{R}$ satisfying $d(t)=U(\phi^t)$. Take any $x\in X$. Note that $x=\sum_{t\in T_x} m_x(t)\phi^t$, and so inductive application of linearity implies $U(x)=\sum_t d(t)m_x(t)$ as desired. $\square$

Necessity of the axioms is immediate from the definitions; we now show sufficiency. Let $\succeq$ denote the complete binary relation on X induced by ρ. By Moderate Transitivity, $\succeq$ is transitive. Since ρ satisfies Continuity and Independence, by Theorem 8 in Herstein and Milnor (1953), $\succeq$ is represented by a linear $U : X \to \mathbb{R}$, and Lemma 11 in turn implies the existence of a $d : [0, \infty) \to \mathbb{R}$ such that $U(x) = \sum_t d(t)m_x(t)$. Dominance implies that $d(t)$ is positive and strictly decreasing. Extend d to $[0, \infty) \cup \{+\infty\}$ by taking $d(\infty) = 0$.

Fix any $t^a, t^b, t^c, t^d \in [0, \infty)$, $t^a < t^b < t^c < t^d$; we have $d(t^a) < d(t^b) < d(t^c) < d(t^d)$. Now consider any $x, y \in X$. Let $T = \{0, t^a, t^b, t^c, t^d\} \cup T_x \cup T_y$, and enumerate $T \cup \{\infty\}$ in increasing order by $\{t_1, t_2, ..., t_n, t_{n+1}\}$; we have $d(t_1) < d(t_2) < ... < d(t_{n+1})$. Let $X(T) = \{x \in X : T_x \subseteq T\}$ denote the set of payoff flows with support in T. Note that all $w \in X(T)$ corresponds to a unique $\tilde{w} \in \mathbb{R}^n$ satisfying $\tilde{w}_k = M_x(t_k)(d(t_k) - d(t_{k+1}))$. Denote by $\tilde{\rho}$ the induced preference on $\mathbb{R}^n$ satisfying $\tilde{\rho}(\tilde{x}, \tilde{y}) = \rho(x, y)$.

Claim 1. $\tilde{\rho}(\tilde{x}, \tilde{y}) \geq 1/2$ iff $\sum_k \tilde{x}_k \geq \sum_k \tilde{y}_k$. $\tilde{\rho}$ satisfies M1-M5.

Proof. Note that since $\sum_k \tilde{w}_k = \sum_t d(t)m_w(t)$ for all $w \in X(T)$, we have $\sum_k \tilde{x}_k \geq \sum_k \tilde{y}_k \iff \sum_t d(t)m_x(t) \geq \sum_t d(t)m_y(t) \iff \rho(x, y) \geq 1/2 \iff \tilde{\rho}(\tilde{x}, \tilde{y}) \geq 1/2$.

It is immediate that $\tilde{\rho}$ inherits Continuity, Linearity, and Moderate Stochastic Transitivity from ρ. Dominance follows from the fact that for all $x, y \in X(T)$, $M_x(t) \geq M_y(t)$ for all t if and only if $\tilde{x}_k \geq \tilde{y}_k$ for all k.

Finally, to see that $\tilde{\rho}$ satisfies Simplification, take any $\tilde{x}, \tilde{y} \in \mathbb{R}^n$ with $\tilde{\rho}(\tilde{x}, \tilde{y}) \geq 1/2$ and $i \neq j$, and consider $\tilde{x}'$ satisfying $\tilde{x}'_i = \tilde{y}_i$, $\tilde{x}'_k = \tilde{x}_k$ for $k \neq i, j$, and with $\tilde{\rho}(\tilde{x}', \tilde{x}) = 1/2$. By construction, we have $\rho(x, y) \geq 1/2$, $\rho(x', x) \geq 1/2$. Since $m_x(t), m_y(t) \neq 0$ for finitely many t, there exists $\eta \in \mathbb{R}$ such that $m_x(t) + \eta \neq 0$ and $m_y(t) + \eta \neq 0$ for all t. Let $z \in X(T)$ denote the payoff flow with $m_z(t) = \eta$ for all $t \in T$, and $m_z(t) = 0$ otherwise. Define $\hat{x}, \hat{x}', \hat{y} \in X$ by $\hat{x} = x + z$, $\hat{x}' = x' + z$, $\hat{y} = y + z$. By Linearity of ρ, we have $\rho(\hat{x}, \hat{y}) \geq 1/2$, $\rho(\hat{x}', \hat{x}) \geq 1/2$. Note that by construction, $T_{\hat{x}} = T_{\hat{x}'} = T_{\hat{y}} = \{t_1, ..., t_n\}$, and so the support of $\hat{x}'$ is contained in $T_{\hat{x}} \cup T_{\hat{y}}$. Furthermore, $\tilde{x}'_i = \tilde{y}_i$ implies $M_{\hat{x}'}(t_i) = M_{\hat{y}}(t_i)$, and $\tilde{x}'_k = \tilde{y}_k$ for all $k \neq i, j$ implies $M_{\hat{x}'}(t) = M_{\hat{x}}(t)$ for all $t \in T_{\hat{x}} \cup T_{\hat{y}} / \{t_i, t_j\}$, and so since ρ satisfies Simplification, we have $\rho(\hat{x}', \hat{y}) \geq \rho(\hat{x}, \hat{y})$. Linearity of ρ then implies that $\rho(x', y) \geq \rho(x, y)$, and so by definition of $\tilde{\rho}$ we have $\tilde{\rho}(\tilde{x}', \tilde{y}) \geq \tilde{\rho}(\tilde{x}, \tilde{y})$ as desired. $\square$

Using Claim 1, Theorem 1 then implies that there exists a continuous, strictly increasing

$G : [-1,1] \to [0,1]$, symmetric around 0, such that for all $x, y \in X(T)$ $\tilde{x}, \tilde{y} \in \mathbb{R}^n$, we have

$$\begin{aligned}\rho(x,y) &= \tilde{\rho}(\tilde{x},\tilde{y}) \\ &= G\left(\frac{\sum_k(\tilde{x}_k - \tilde{y}_k)}{\sum_k|\tilde{x}_k - \tilde{y}_k|}\right) \\ &= G\left(\frac{U(x)-U(y)}{d_{CPF}(x,y)}\right)\end{aligned}$$

Let $\mathscr{T} = \{T \subseteq [0,\infty) : |T| < \infty, \{0, t^a, t^b, t^c, t^d\} \subseteq T\}$. The above implies that for all $T \in \mathscr{T}$, there exists a continuous, strictly increasing $G_T : [-1,1] \to [0,1]$, symmetric around 0 such that for any $x, y \in X(T)$,

$$\rho(x,y) = G_T\left(\frac{U(x)-U(y)}{d_{CPF}(x,y)}\right)$$

Since for any $x, y \in X$, there exists some $T \in \mathscr{T}$ such that $x, y \in X(T)$, all that remains to show that All that remains is to show that $G_T = G_{T'}$ for any $T, T' \in \mathscr{T}$. To see this, fix any $T, T' \in \mathscr{T}$, and consider $x, y \in X$ with

$$m_x(t) = \begin{cases}\alpha/(d(t_a)-d(t_b)) & t = t_a \\ \gamma/(d(t_b)-d(t_c)) & t = t_c \\ 0 & \text{otherwise}\end{cases} \qquad m_y(t) = \begin{cases}\alpha/(d(t_a)-d(t_b)) + \gamma/(d(t_b)-d(t_c)) & t = t_b \\ 0 & \text{otherwise}\end{cases}$$

for some $\alpha \geq 0$, $\gamma \geq 0$. Note that x, y belong to both T and T', and so we have

$$\rho(x,y) = G_T\left(\frac{U(x)-U(y)}{d_{CPF}(x,y)}\right) = G_{T'}\left(\frac{U(x)-U(y)}{d_{CPF}(x,y)}\right)$$

and since $\frac{U(x)-U(y)}{d_{CPF}(x,y)} = \frac{\alpha-\gamma}{\alpha+\gamma}$, for any $r \in [-1,1]$ we can choose $\alpha, \gamma \geq 0$ such that $\frac{U(x)-U(y)}{d_{CPF}(x,y)} = r$, we must have $G_T = G_{T'}$.

Finally, to show uniqueness, suppose (G,d) and (G',d') both represent ρ. Define the stochastic preference relation $\succeq$ as before. Since G, G' are both strictly increasing, symmetric around 0, both $U(x) = \sum_t d(t)m_x(t)$ and $U'(x) = \sum_t d'(t)m_x(t)$ both represent $\succeq$. Since $d \geq 0$ and and d, d' are both strictly decreasing, we have $d(0), d'(0) > 0$. Fix any $t \in (0,\infty)$, and let $\lambda_t = d(t)/d(0)$. By construction, $U(\phi^t) = U(\lambda_t\phi^0)$, and so $\phi^t \sim \lambda_t\phi^0$. Since U' also represents $\succeq$, we have $U'(\phi^t) = U'(\lambda_t\phi^0) \implies d'(t) = \lambda_t d'(0)$, and so $d'(t) = Cd(t)$

for all $t \in [0, \infty)$, where $C = d'(0)/d(0) > 0$. This in turn implies that for all $x, y \in X$, $\{t_0, t_1, ..., t_n\}$ containing $\{0, \infty\} \cup T_x \cup T_y$,

$$\begin{aligned} G\left(\frac{U(x) - U(y)}{d_{CPF}(x, y)}\right) &= G'\left(\frac{\sum_k (d'(t_k) m_x(t_k) - d'(t_k) m_y(t_k))}{\sum_k |M_x(t_k) - M_y(t_k)|(d'(t_k) - d'(t_{k+1}))}\right) \\ &= G'\left(\frac{\sum_k (d(t_k) m_x(t_k) - d(t_k) m_y(t_k))}{\sum_k |M_x(t_k) - M_y(t_k)|(d(t_k) - d(t_{k+1}))}\right) \\ &= G'\left(\frac{U(x) - U(y)}{d_{CPF}(x, y)}\right) \end{aligned}$$

Consider $x, y \in X$ with

$$m_x(t) = \begin{cases} \alpha/(d(t_a) - d(t_b)) & t = t_a \\ \gamma/(d(t_b) - d(t_c)) & t = t_c \\ 0 & \text{otherwise} \end{cases} \qquad m_y(t) = \begin{cases} \alpha/(d(t_a) - d(t_b)) + \gamma/(d(t_b) - d(t_c)) & t = t_b \\ 0 & \text{otherwise} \end{cases}$$

for some $\alpha \geq 0$, $\gamma \geq 0$. Since $\frac{U(x)-U(y)}{d_{CPF}(x,y)} = \frac{\alpha-\gamma}{\alpha+\gamma}$, for any $r \in [-1, 1]$ we can choose $\alpha, \gamma \geq 0$ such that $\frac{U(x)-U(y)}{d_{CPF}(x,y)} = r$, we must have $G' = G$.

□

F.2 Other Results

Proof of Proposition 6

Note that since H is strictly increasing,

$$\begin{aligned} \max_{g \in \Gamma(x,y)} \tau^{L1}_{xy}(g) &= \max_{g \in \Gamma(x,y)} H\left(\frac{|EU(x) - EU(y)|}{\sum_{w_x, w_y} |g(w_x, w_y)(u(w_x) - u(w_y))|}\right) \\ &= H\left(\frac{|EU(x) - EU(y)|}{\min_{g \in \Gamma(x,y)} \sum_{w_x, w_y} |g(w_x, w_y)(u(w_x) - u(w_y))|}\right) \end{aligned}$$

Let $\tilde{x}$ and $\tilde{y}$ denote the utility-valued lotteries induced by x and y, defined by the quantile functions $F_{\tilde{x}}^{-1}(q) = u(F_x^{-1}(q))$ and $F_{\tilde{y}}^{-1}(q) = u(F_y^{-1}(q))$ for all $q \in [0,1]$. Note that

$$\begin{aligned}
\min_{g\in\Gamma(x,y)} \sum_{w_x,w_y} |g(w_x,w_y)(u(w_x)-u(w_y))| &= \min_{g\in\Gamma(\tilde{x},\tilde{y})} \sum_{w_x,w_y} g(w_x,w_y)|(w_x-w_y)| \\
&= \int_{-\infty}^{\infty} |F_{\tilde{x}}(w)-F_{\tilde{y}}(w)|\,dw \\
&= \int_0^1 |F_{\tilde{x}}^{-1}(q)-F_{\tilde{y}}^{-1}(q)|\,dq \\
&= d_{CDF}(x,y)
\end{aligned}$$

Where the second equality follows from Vallender (1974), since $\min_{g\in\Gamma(\tilde{x},\tilde{y})} \sum_{w_x,w_y} |g(w_x,w_y)(w_x-w_y)|$ is the 1-Wassertein metric between the distributions $F_{\tilde{x}}$ and $F_{\tilde{y}}$, the third equality follows from a change of variables, and the final equality follows from the definition of $\tilde{x}$, $\tilde{y}$.

□

Proof of Proposition 7

Note that since H is strictly increasing,

$$\begin{aligned}
\max_{b\in B(x,y)} \tau_{xy}^{L1}(b) &= \max_{b\in B(x,y)} H\left(\frac{|DU(x)-DU(y)|}{\sum_{t_x,t_y}|b(t_x,t_y)(d(t_x)-d(t_y))|}\right) \\
&= H\left(\frac{|DU(x)-DU(y)|}{\min_{b\in B(x,y)}\sum_{t_x,t_y}|b(t_x,t_y)(d(t_x)-d(t_y))|}\right)
\end{aligned}$$

All that remains is to show that for $d_{L1}^b(x,y) \equiv \sum_{t_x,t_y}|b(t_x,t_y)d(t_x)-d(t_y)|$, we have $\min_{b\in B(x,y)} d_{L1}^b(x,y) = d_{CPF}(x,y)$.

Without loss, normalize $d(0) = 1$, and fix any x, y. Let $\overline{w} = \sum_t m_x(t) + \sum_t m_y(t)$ denote the total payoff delivered by both x and y. Let $\overline{B}(x,y)$ contain all $b \in B(x,y)$ satisfying $b(t_x,t_y) > 0$ for all t_x, t_y. Note that this implies that for all $b \in \overline{B}(x,y)$, we have $\sum_{t_x,t_y} b(t_x,t_y) \leq \overline{w}$. Since x and y have positive payouts, we have

$$\max_{b\in B_{x,y}} d_{L1}^b(x,y) = \max_{b\in\overline{B}_{x,y}} d_{L1}^b(x,y)$$

We will now show that $\max_{b\in\overline{B}_{x,y}} d_{L1}^b(X,Y) = d_{CPP}(x,y)$. For all $b \in \overline{B}(X,Y)$, consider a joint density $\tilde{b}$ over $[0,1]^2$ with mass function satisfying

$$\tilde{b}(w_x, w_y) = \begin{cases} b(d^{-1}(w_x), d^{-1}(w_y)/\overline{w} & w_x \neq 0 \text{ or } w_y \neq 0 \\ 1 - \sum_{\{(t_x,t_y):\neg(t_x=\infty,t_y=\infty)\}} b(t_x, t_y)/\overline{w} & w_x = w_y = 0 \end{cases}$$

Note that $\tilde{b}$ is well-defined since $b(t_x, t_y) > 0$ for all t_x, t_y and $\sum_{t_x,t_y} b(t_x, t_y)/\overline{w} \leq 1$ by construction.

Let $\tilde{b}_x$ and $\tilde{b}_y$ denote the marginal distributions of $\tilde{b}$. Note that for all $t \in [0, \infty)$, we have

$$\begin{aligned} \tilde{b}_x(d(t)) &= \sum_{w_y} \tilde{b}(d(t), w_y) \\ &= \sum_{t_y} \tilde{b}(d(t), d(t_y))/\overline{w} \\ &= \sum_{t_y} b(t, t_y)/\overline{w} \\ &= m_x(t)/\overline{w} \end{aligned}$$

where the third equality follows from the fact that $\sum_{t_y} b(t, t_y) = m_x(t)$ for all $t \in [0, \infty)$, and so

$$\tilde{b}_x(w) = h_x(w) \equiv \begin{cases} m_x(d^{-1}(w))/\overline{w} & w \in (0,1] \\ 1 - \sum_t m_x(t)/\overline{w} & w \in 0 \end{cases}$$

A similar argument implies that

$$\tilde{b}_y(w) = h_y(w) \equiv \begin{cases} m_y(d^{-1}(w))/\overline{w} & w \in (0,1] \\ 1 - \sum_t m_y(t)/\overline{w} & w \in 0 \end{cases}$$

Let $\tilde{B}(x, y)$ denote the set of joint densities $g(w_x, w_y)$ over $[0,1]^2$ with marginals given by $g_x = h_x$ and $g_y = h_y$. The above implies that for all $b \in \overline{B}(x, y)$, $\tilde{b} \in \tilde{B}(x, y)$. We will now show that for all $g \in \tilde{B}(x, y)$, there exists $b \in \overline{B}(x, y)$ such that $\tilde{b} = g$.

Fix any $g \in \tilde{B}(x,y)$, and define $b : \mathbb{R}_+ \cup \{+\infty\} \times \mathbb{R}_+ \cup \{+\infty\} \to \mathbb{R}$ by

$$b(d^{-1}(w_x), d^{-1}(w_y)) = \begin{cases} g(w_x, w_y) \cdot \overline{w} & w_x \neq 0 \text{ or } w_y \neq 0 \\ 0 & w_x = w_y = 0 \end{cases}$$

for all $w_x, w_y \in [0,1]^2$. By construction, $\sum_{t_x,t_y} b(t_x,t_y) \leq \overline{w}$ and $b(t_x,t_y) > 0$. Furthermore, for all $t \in [0,\infty)$ we have

$$\begin{aligned}\sum_{t_y} b(t,t_y) &= \sum_{w_y} b(t, d^{-1}(w_y)) \\ &= \sum_{w_y} g(d(t), w_y) \cdot \overline{w} \\ &= h_x(d(t)) \cdot \overline{w} \\ &= m_x(t)\end{aligned}$$

where the third equality follows from the fact that $g_x = h_x$ and the last equality follows from the definition of h_x. We similarly have $\sum_{t_x} b(t_x,t) = my(t)$ for all $t \in [0,\infty)$, and so $b \in \overline{B}(x,y)$. Note that by construction, $\tilde{b} = g$ as desired. Now since

$$\begin{aligned}d^b_{L1}(x,y) &= \sum_{t_x,t_y} b(t_x,t_y)|d(t_x) - d(t_y)| \\ &= \overline{w} \sum_{w_x,w_y} \tilde{b}(w_x,w_y)|w_x - w_y|\end{aligned}$$

the fact that for any $b \in \overline{B}(x,y)$, $\tilde{b} \in \tilde{B}(x,y)$ and that for any $g \in \tilde{B}(x,y)$, there exists $b \in \overline{B}(x,y)$ s.t. $\tilde{b} = g$ implies that

$$\begin{aligned}\min_{b \in \overline{B}(x,y)} d^b_{L1}(x,y) &= \min_{g \in \tilde{B}(x,y)} \overline{w} \sum_{w_x,w_y} g(w_x,w_y)|w_x - w_y| \\ &= \overline{w} \int_0^1 |H_x(w) - H_y(w)|\, dw\end{aligned}$$

where the second line follows from Vallender (1974), for H_x and H_y the CDFs of h_x, h_y. Enumerate the elements of T_{xy} by $0 = t_0, t_1, ..., t_n = \infty$ and let $w_k = d(t_k)$ for all $k =$

$0, 1, ..., n$. Note that for all $k = 1, ..., n$,

$$\begin{aligned} H_x(w_k) &= \sum_{j=k}^{n-1} m_x(d^{-1}(w_j))/\overline{w} + 1 - \sum_{j=1}^{n-1} m_x(t_j)/\overline{w} \\ &= 1 - \sum_{j=1}^{k-1} m_x(t_j)/\overline{w} \\ &= 1 - M_x(t_{k-1})/\overline{w} \end{aligned}$$

By a similar argument for H_y, we have

$$H_x(w_k) = \begin{cases} 1 - M_x(t_{k-1})/\overline{w} & k \geq 1 \\ 1 & k = 0 \end{cases} \quad H_y(w_k) = \begin{cases} 1 - M_y(t_{k-1})/\overline{w} & k \geq 1 \\ 1 & k = 0 \end{cases}$$

We therefore have

$$\begin{aligned} \min_{b \in \overline{B}(x,y)} d_{L1}^b(x, y) &= \overline{w} \sum_{k=1}^{n} |H_x(w_k) - H_y(w_k)|(w_{k-1} - w_k) \\ &= \sum_{k=1}^{n} |M_x(t_{k-1}) - M_y(t_{k-1})|(d(t_{k-1}) - d(t_k)) \\ &= d_{CPF}(x, y) \end{aligned}$$

as desired.

$\square$

Proof of Proposition 8

Suppose a multinomial choice rule ρ is represented by (Q, v, τ) and (Q', v', τ'). With some abuse of notation, let ρ also denote the binary choice rule induced by the restriction of ρ to binary menus.

Let $\succeq$ denote the stochastic order induced by ρ. Since ρ is represented by (Q, v, τ), we have $\rho(x, y) = \Phi(\text{sgn}(v(x) - v(y))\tau(x, y))$, and so $x \succeq y$ iff $v(x) \geq v(y)$. Similarly, since ρ is represented by (Q', v', τ'), $x \succeq y$ iff $v'(x) \geq v'(y)$. This implies that for any x, y, we have $v(x) = v(y) \iff x \sim y \iff v'(x) = v'(y)$, and so the transformation $\phi : v(X) \to \mathbb{R}$ satisfying $\phi(v(x)) = v'(x)$ for all $x \in X$ is well defined. To see that ϕ is strictly increasing, suppose not; there exists $x, y \in X$ such that $v(x) > v(y)$ but $\phi(v(x)) \leq \phi(v(y))$; the former implies that $x \succ y$ but the latter implies that $y \succeq x$, a contradiction.

To see that $\tau = \tau'$, fix any $(x,y) \in \mathscr{D}$. First consider the case where $v(x) = v(y)$; by definition of τ, $\tau(x,y) = 0$. But since $v(x) = v(y) \implies v'(x) = v'(y)$, we also have $\tau'(x,y) = 0$. Now consider the case where $v(x) \neq v(y)$; without loss, assume $v(x) > v(y)$. By the above result, we have $\text{sgn}(v(x) - v(y)) = \text{sgn}(v'(x) - v'(y)) = 1$, which in turn implies that $\rho(x,y) = \Phi(\tau(x,y)) = \Phi(\tau'(x,y))$. Since Φ is strictly increasing, we have $\tau(x,y) = \tau'(x,y)$, and so $\tau = \tau'$ as desired.

□

Proof of Proposition 9

Consider the extension of a binary choice rule ρ to a multinomial choice rule $\rho(x,A)$ satisfying $\sum_{x \in A} \rho(x,A) = 1$ in every finite menu A. Note that ρ describes a mapping from finite subsets of $\mathbb{R}^n$ to a probability measure on the Borel sigma-algebra in $\mathbb{R}^n$. Endow the set of finite subsets of $\mathbb{R}^n$ with the topology induced by the Hausdorff metric, and endow the set of probability measures on the Borel sigma-algebra with the topology of weak convergence. We now state the Gul and Pesendorfer (2006) postulates (henceforth, GP postulates). Say that ρ is *continuous* when this mapping is continuous. Say that ρ is *monotone* if $\rho(x,A) \geq \rho(x,B)$ whenever $A \subseteq B$, and that ρ is *linear* if for all $x \in$, $y \in \mathbb{R}^n$, $\alpha \in (0,1)$, $\rho(x,A) = \rho(\alpha x + (1-\alpha y), \{\alpha z + (1-\alpha)y : z \in A\})$. Say that ρ is *extreme* if $\rho(x,A) > 0$ implies that x is an extreme point of A.

Now consider a binary choice rule ρ with an L_1-complexity representation (β, G), where $G(1) = G(-1) = 1$. Note that on its domain, ρ satisfies the GP postulates of continuity, linearity, and extremeness, as in any binary menu both options are extreme points. Since the GP postulate of monotonicity makes no restrictions in binary choice behavior, ρ also satisfies this property on its domain. Therefore, there exists an extension of ρ to a multinomial choice rule that is conitnuous, monotone, linear, and extreme. By Theorem 3 of Gul and Pesendorfer (2006), there exists a random vector $\tilde{\beta}$ such that

$$\rho(x,A) = \mathbb{P}\left\{\sum_k \tilde{\beta}_k x_k \geq \sum_k \tilde{\beta}_k y_k \, \forall \, y \in A\right\}$$

which in turn implies that for all $(x,y) \in \mathscr{D}$,

$$\rho(x,y) = \mathbb{P}\left\{\sum_k \tilde{\beta}_k x_k \geq \sum_k \tilde{\beta}_k y_k\right\}$$

To see that $\mathbb{P}\left\{\text{sgn}(\tilde{\beta}_k) = \text{sgn}(\beta_k)\right\} = 1$ for all k, suppose not; let $y = \vec{0}$ and $x \in \mathbb{R}^n$ such that

$x_k = \text{sgn}(\beta_k)$ and $x_j = 0$ for all $j \neq k$. Since $G(1) = 1$ and $\beta_k x_k \geq \beta_k y_k$ for all k, we have $\rho(x, y) = 1$. However, $\mathbb{P}\left\{\text{sgn}(\tilde{\beta}_k) = \text{sgn}(\beta_k)\right\} \neq 1$ implies that $\rho(x, y) < 1$, a contradiction. $\square$

Proof of Proposition 10

Suppose that ρ has an linear differentiation representation (β, Σ, G) and that at least 3 attributes are non-null; without loss, we assume that attributes $k = 1, 2, 3$ are non-null.

Let $\tilde{\rho}$ denote the binary choice rule on $\mathbb{R}^2$ defined by the restriction of ρ to the first two dimensions, i.e. $\tilde{\rho}(\tilde{x}, \tilde{y}) = \rho((\tilde{x}, 0, ..., 0), (\tilde{y}, 0, ..., 0))$ for all $\tilde{x}, \tilde{y} \in \mathbb{R}^2$, $\tilde{x} \neq \tilde{y}$; it is immediate from the definition that $\tilde{\rho}$ has an linear differentiation representation with parameters $(\tilde{\beta}, \tilde{\Sigma}, G)$, where $\tilde{\beta} = (\beta_1, \beta_2)$ and $\tilde{\Sigma}$ is the submatrix formed from the first 2 rows and columns of Σ. Furthermore, since attributes 1 and 2 are non-null, $\tilde{\beta}_1, \tilde{\beta}_2 \neq 0$.

Fix any $\tilde{y} \in \mathbb{R}^n$, and define $B = \{\tilde{x} \in \mathbb{R}^2 : \tilde{\beta}'(\tilde{x} - \tilde{y}) = 1\}$. Note that $\arg\max_{\tilde{x}' \in B} \tilde{\rho}(\tilde{x}', \tilde{y})$ has a unique maximizer, which we denote by $\tilde{x}$: to see this, note that Proposition 1 of He and Natenzon (2023b) implies that if $\tilde{x} \in \arg\max_{\tilde{x}' \in B} \tilde{\rho}(\tilde{x}', \tilde{y})$, then $\tilde{x} - \tilde{y} = \alpha \tilde{\Sigma}^{-1} \tilde{\beta}$ for some $\alpha \neq 0$; since $\tilde{\beta}'(\tilde{x} - \tilde{y}) = 1$, it must be the case that $\alpha = 1/\tilde{\beta}' \tilde{\Sigma}' \tilde{\beta}$ and so $\tilde{x} = \tilde{y} + \frac{1}{\tilde{\beta}' \tilde{\Sigma}' \tilde{\beta}} \tilde{\Sigma}^{-1} \tilde{\beta}$.

Take any $\tilde{w} \neq \tilde{x}$ such that $\tilde{w} \in B$ and $\text{sgn}(\tilde{w}_k) = \text{sgn}(\tilde{\beta}_k)$ for $k = 1, 2$, and define $x, w, y \in \mathbb{R}^n$ where $x = (\tilde{x}, 0, ..., 0)$, $w = (\tilde{w}, 0, ..., 0)$, $y = (\tilde{y}, 0, ..., 0)$. Since $\tilde{x}$ is the unique maximizer of $\arg\max_{\tilde{x}' \in B} \tilde{\rho}(\tilde{x}', \tilde{y})$, we have $\tilde{\rho}(\tilde{w}, \tilde{y}) < \tilde{\rho}(\tilde{x}, \tilde{y})$, which in turn implies $\rho(w, y) < \rho(x, y)$. Furthermore, since by construction we have $\text{sgn}(x_k) = \text{sgn}(\beta_k)$ for $k = 1, 2$ and $x_k = 0$ for all $k > 2$, and since $\beta_1, \beta_2 \neq 0$, we have $w >_D y$.

Note that if $x \not>_D y$, we are done. If instead $x >_D y$, define $x'(\epsilon) \in \mathbb{R}^n$ by $x'(\epsilon) = (\tilde{x}_1, \tilde{x}_2, -\text{sgn}(\beta_3)\epsilon, 0, ..., 0)$; by continuity of ρ, there exists some $\epsilon > 0$ such that $\rho(w, y) < \rho(x'(\epsilon), y)$. Furthermore, since $\beta_3 \neq 0$ as the third attribute is non-null, by construction we have $x'(\epsilon) \not>_D y$ and so we are done.

$\square$

Proof of Proposition 11

Suppose ρ has an L_1-complexity representation. Theorem 2 implies that ρ satisfies moderate transitivity and dominance with respect to $>_D$, and so Lemma 1 implies that ρ satisfies monotonicity with respect to $>_D$, which in turn implies weak monotonicity.

Now suppose that ρ has a linear differentiation representation (β, Σ, G) and suppose that at least two attributes are non-null; without loss, we take these attributes to be $k = 1, 2$.

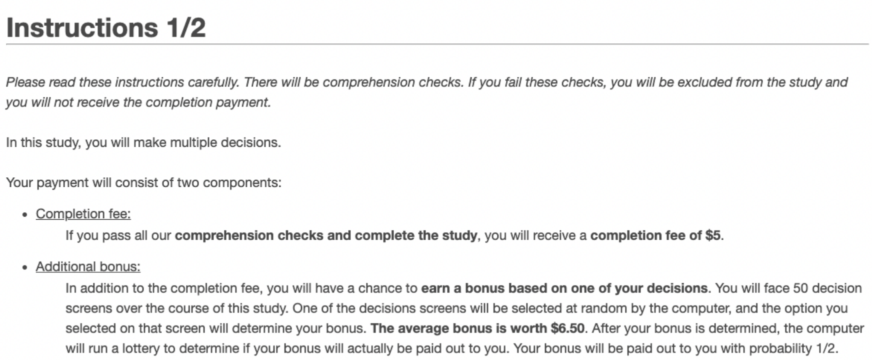
Let $\tilde{\rho}$ denote the binary choice rule on $\mathbb{R}^2$ defined by the restriction of ρ to the first two dimensions, i.e. $\tilde{\rho}(\tilde{x},\tilde{y}) = \rho((\tilde{x},0,...,0),(\tilde{y},0,...,0))$ for all $\tilde{x},\tilde{y} \in \mathbb{R}^2$, $\tilde{x} \neq \tilde{y}$; it is immediate from the definition that $\tilde{\rho}$ has an linear differentiation representation with parameters $(\tilde{\beta},\tilde{\Sigma},G)$, where $\tilde{\beta} = (\beta_1,\beta_2)$ and $\tilde{\Sigma}$ is the submatrix formed from the first 2 rows and columns of Σ. Furthermore, since attributes 1 and 2 are non-null, $\tilde{\beta}_1,\tilde{\beta}_2 \neq 0$.

Fix any $\tilde{y} \in \mathbb{R}^2$. Proposition 1 of He and Natenzon (2023b) implies that any $\tilde{x} \in \arg\max_{\tilde{x}'} \tilde{\rho}(\tilde{x}',\tilde{y})$ satisfies $\tilde{x}-\tilde{y} = \alpha\tilde{\Sigma}^{-1}\tilde{\beta}$ for some $\alpha \neq 0$; fix such a $\tilde{x}$. Since $\left\{\alpha\tilde{\Sigma}^{-1}\tilde{\beta}\right\}_{\alpha\in\mathbb{R}}$ traces a unique direction in $\mathbb{R}^2$, there exists $b_1, b_2 > 0$ such that for $b \equiv (\text{sgn}(\beta_1)\cdot b_1, \text{sgn}(\beta_2)\cdot b_2)$, we have $b \neq \alpha\tilde{\Sigma}^{-1}\tilde{\beta}$ for any $\alpha \neq 0$, which in turn implies that for $\tilde{x}' \equiv \tilde{x} + b$, $\tilde{x}' - \tilde{y} \neq \alpha\tilde{\Sigma}^{-1}\tilde{\beta}$ for any $\alpha \neq 0$, and so $\tilde{\rho}(\tilde{x}',\tilde{y}) < \tilde{\rho}(\tilde{x},\tilde{y})$.

Now define $x, x', y \in \mathbb{R}^n$ by $x = (\tilde{x},0,...,0)), x' = (\tilde{x}',0,...,0)), y = (y,0,...,0))$; by construction and the above, we have $\rho(x',y) < \rho(x,y)$. Also by construction, we have $x' >_D x$, and so ρ violates weak monotonicity.

G Experimental Interfaces

G.1 Multiattribute Binary Choice



Instructions 1/2

Please read these instructions carefully. There will be comprehension checks. If you fail these checks, you will be excluded from the study and you will not receive the completion payment.

In this study, you will make multiple decisions.

Your payment will consist of two components:

- Completion fee:
  If you pass all our **comprehension checks and complete the study**, you will receive a **completion fee of $5**.
- Additional bonus:
  In addition to the completion fee, you will have a chance to **earn a bonus based on one of your decisions**. You will face 50 decision screens over the course of this study. One of the decisions screens will be selected at random by the computer, and the option you selected on that screen will determine your bonus. **The average bonus is worth $6.50**. After your bonus is determined, the computer will run a lottery to determine if your bonus will actually be paid out to you. Your bonus will be paid out to you with probability 1/2.

Instructions 2/2

Choice task: which phone plan is a better deal?

On each decision screen, you will be presented with two cell-phone plans. Your task is to help **Amy**, a fictional customer, choose the lowest-cost plan. For the main part of the study, phone plans will consist of three components:

- Upfront cost of the device (charged annually)
- Recurring fee (paid in monthly installments)
- Data usage fee (charged per GB used)

The data usage fee is priced "per GB," and **Amy always uses 6 GB of data per month (72 GB annually)**. So, for a plan with a data usage fee of $1.00/GB, Amy would have to spend $6.00 per month on data, which amounts to $72.00 annually. Note: each plan offers the exact same services and devices; these plans *only* differ in their costs.

For the main part of the study, Amy's **annual phone budget** is **$700**. Your goal is to **guess which plan will leave Amy with the most money left over at the end of the year**. On each decision screen, you will be asked to make **two decisions:**

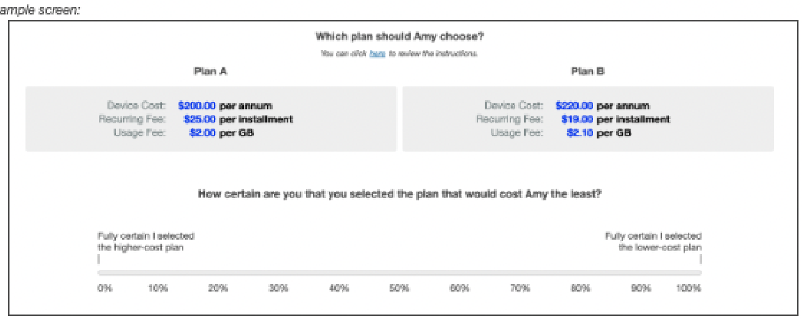
Step 1: Guess which phone plan has lower annual cost

- We will ask you to guess which plan will cost Amy less. You need to select exactly one plan.
- If this decision is randomly chosen for payment, your bonus will be 1 month's worth of Amy's total savings. This is equal to Amy's annual budget minus the cost of your chosen plan, divided by 12.
- This means that **to maximize your bonus, you should select the plan that you think will cost Amy the least over the year.**

Step 2: Indicate your certainty about your guess

- You may be uncertain over which plan actually has lower cost. Therefore, we will ask you to indicate how certain you are (in percent) that you've actually selected the lower-cost plan.
  - **For example, if you think it is 70% likely that you chose the lower-cost plan, you should set the slider to 70%.**
  - If you are certain that you chose the lower-cost plan, you should set the slider to 100%.

Example screen:



Factoring Amy's data usage, Plan A will cost her $644 over the year, whereas Plan B will cost $599, so Plan B is the lower cost plan.

Here is how your bonus would be determined in this example:

- If you selected Plan A, Amy would save $700 - $644 = $56, so your bonus would be $56 / 12 = $4.67.
- If you selected Plan B, Amy would save $700 - $599 = $101, so your bonus would be $101 / 12 = $8.42.

After your bonus is determined, the computer will randomly determine whether or not it will be paid out to you. You will actually receive your bonus 1/2 of the time.

Once you click the next button, the comprehension check questions will start!

Comprehension check

To verify your understanding of the instructions, please answer the comprehension questions below. If you get one or more of them wrong twice in a row, you will not be allowed to participate in the study and earn a completion payment. In each question, exactly one response option is correct.

You can review the instructions here.

1. To maximize your bonus, what should you select on each decision screen?

- I should try to select the phone plan bundle that I think would be the best deal for me personally.
- I should try to select the phone plan bundle that I think would be the best deal for most cell-phone users.
- I should try to select the phone plan bundle that will be best for Amy specifically.

2. How should you determine which bundle is best for Amy?

- The devices and services offered in each pair of bundles are identical, so I should try to select the bundle that will cost Amy the least given her data usage.
- I should usually select the bundle with a more expensive device, because that means Amy will get a nicer phone.
- I should always select a plan with a low per-GB data fee in case Amy's data usage increases.

3. Under the plan below, how much in **usage fees** would Amy pay **over the year**?

Device Cost:	$200.00 **per annum**
Recurring Fee:	$25.00 **per installment**
Usage Fee:	$1.00 **per GB**

- $1.00
- $6.00
- $12.00
- $72.00

4. Suppose you chose plan A on a decision screen. Which of the following statements is correct?

- If I think it is 70% likely that plan A has the lowest cost, then I should set the certainty slider to 70%.
- If I think it is 70% likely that plan A has the lowest cost, then I should set the certainty slider to 100%.
- I should always set the certainty slider somewhere in the middle, even if I'm certain which plan has the lowest cost.
- I should always set the certainty slider to 100% even if I'm not certain which plan has the lowest cost.

Disclaimer

It is up to you how you wish to work on the tasks, but **we would prefer if you did not use a calculator** to help make your decisions.

You will need to complete **50 decision tasks** in total. You may take as much time for each task as you'd like, but please remember that the study was advertised for **30 minutes** and you will only be paid on that basis.

If you find that you don't have much time, you may look at the plans and make an informed guess about which one is lower cost. Again, it is up to you to how you wish to work on the tasks.

Which plan should Amy choose?

You can click here to review the instructions.

	Plan A	Plan B
Device Cost:	$169.92 **per annum**	$169.92 **per annum**
Recurring Fee:	$10.26 **per installment**	$18.00 **per installment**
Usage Fee:	$3.49 **per GB**	$3.16 **per GB**

How certain are you that you selected the plan that would cost Amy the least?

Fully certain I selected the higher-cost plan | Fully certain I selected the lower-cost plan

0% 10% 20% 30% 40% 50% 60% 70% 80% 90% 100%

G.2 Intertemporal Binary Choice

Instructions 1/2

Please read these instructions carefully. There will be comprehension checks. If you fail these checks, you will be excluded from the study and you will not receive the completion payment.

In this study, you will make multiple decisions.

Your payment will consist of two components:

- Completion fee:
  If you pass all our **comprehension checks and complete the study**, you will receive a **completion fee of $3.50**.
- Additional bonus:
  In addition to the completion fee, you will have a chance to **earn a bonus based on one of your decisions**. You will face 50 decision screens over the course of this study. With 1/5 chance, you will be selected to win a bonus payment. If this happens, one of the decision screens will be selected at random by the computer, and the option you selected on that screen will determine your **bonus**. The **maximal bonus** you can earn in this study is **$40**.

Instructions 2/2

Choice task: which payment option would you like to receive?

On each decision screen, you will be presented with two payment options. Each option will consist of payment amounts (in dollars), along with dates at which the payments are to be received.

On **each decision screen**, you will be **asked to indicate which payment option you prefer to receive.**

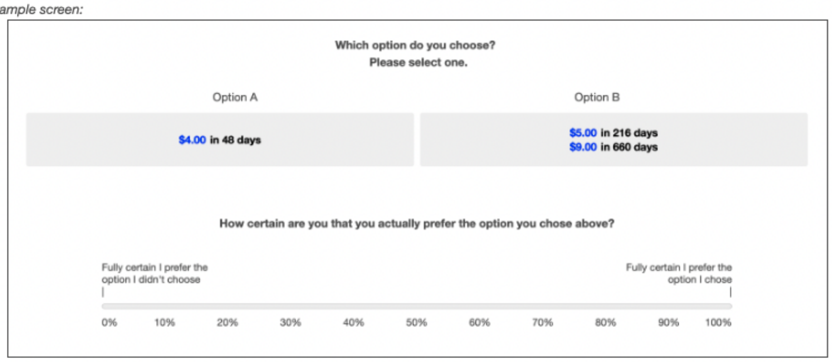
Step 1: Choose the payment option you prefer

- We will ask you to indicate which option you prefer to receive.
- If this decision is selected to determine your bonus, you will receive the payments in the option you chose, at the specified dates.

Step 2: Indicate your certainty about your choice

- You might feel uncertain about which payment option you actually prefer. Therefore, we will ask you to indicate how certain you are (in percent) that you actually prefer the option that you chose.
  - **For example, if you think it is 70% likely that you actually prefer the payment option that you chose, you should set the slider to 70%.**
  - If you are certain that you prefer the payment option you chose, you should set the slider to 100%.

Example screen:

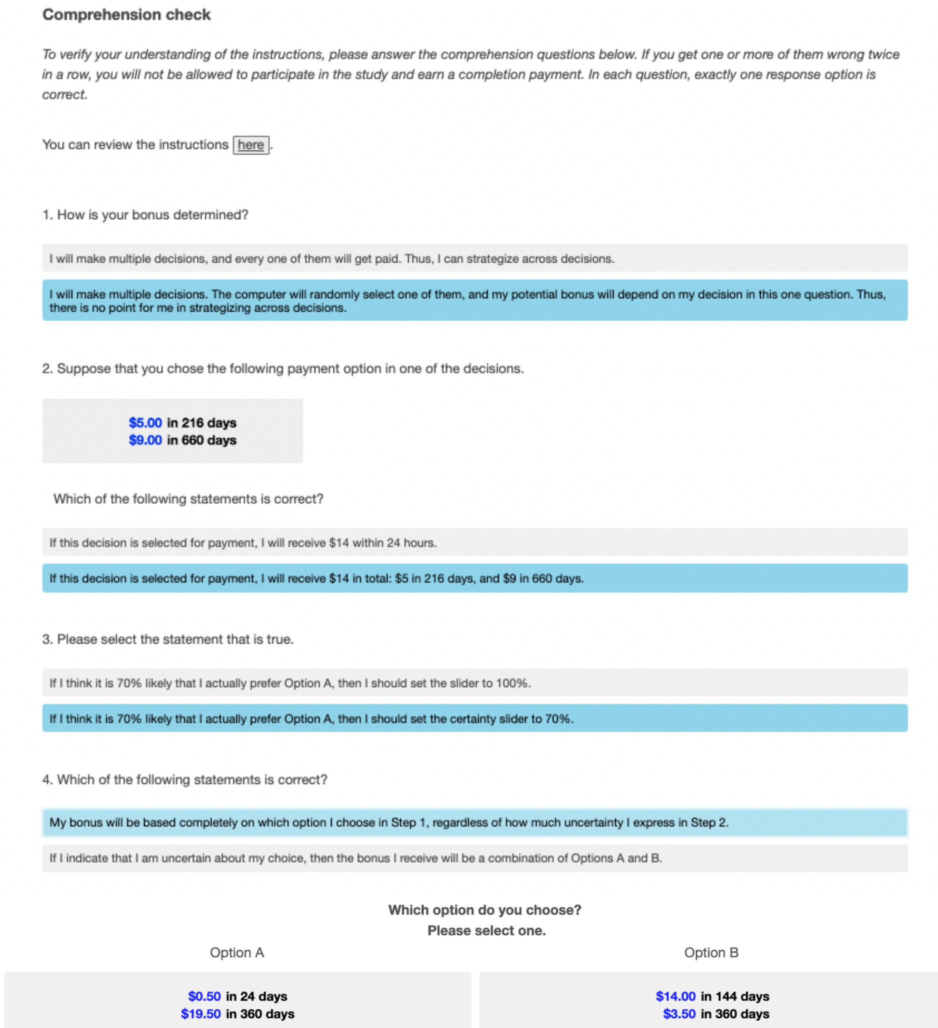


If this decision is selected to determine your bonus:

- If you selected option A, you would receive a payment of $4.00 delivered to your account in 48 days.
- If you selected option B, you would receive a payment of $5.00 delivered to your account in 216 days and an additional payment of $9.00 delivered to your account in 660 days.

If a decision is selected to determine your bonus, the payments in the option you chose will be delivered to your account within 24 hours of the specified dates. When a payment is delivered, we will also send you a reminder through Prolific to cash out the payment.

Once you click the next button, the comprehension check questions will start!



Comprehension check

To verify your understanding of the instructions, please answer the comprehension questions below. If you get one or more of them wrong twice in a row, you will not be allowed to participate in the study and earn a completion payment. In each question, exactly one response option is correct.

You can review the instructions here.

1. How is your bonus determined?

I will make multiple decisions, and every one of them will get paid. Thus, I can strategize across decisions.

I will make multiple decisions. The computer will randomly select one of them, and my potential bonus will depend on my decision in this one question. Thus, there is no point for me in strategizing across decisions.

2. Suppose that you chose the following payment option in one of the decisions.

$5.00 **in 216 days**
$9.00 **in 660 days**

Which of the following statements is correct?

If this decision is selected for payment, I will receive $14 within 24 hours.

If this decision is selected for payment, I will receive $14 in total: $5 in 216 days, and $9 in 660 days.

3. Please select the statement that is true.

If I think it is 70% likely that I actually prefer Option A, then I should set the slider to 100%.

If I think it is 70% likely that I actually prefer Option A, then I should set the certainty slider to 70%.

4. Which of the following statements is correct?

My bonus will be based completely on which option I choose in Step 1, regardless of how much uncertainty I express in Step 2.

If I indicate that I am uncertain about my choice, then the bonus I receive will be a combination of Options A and B.

Which option do you choose?
Please select one.

Option A	Option B
$0.50 **in 24 days** $19.50 **in 360 days**	$14.00 **in 144 days** $3.50 **in 360 days**

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