Representation Theorems

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1 General Results

Let X be a space of options, and let $\mathcal{D} = \{(x, y) \in X \times X : x \neq y\}$. Say $\rho : \mathcal{D} \to [0, 1]$ on is a *binary choice rule* on X if $\rho(x, y) = 1 - \rho(y, x)$.

Say that a binary choice rule ρ satisfies *moderate transitivity* if for $\rho(x, y), \rho(y, z) \geq 1/2$, $\rho(x, y) \geq \min\{\rho(x, y), \rho(y, z)\}$. Say that a binary choice rule ρ satisfies *weak transitivity* if for $\rho(x, y), \rho(y, z) \geq 1/2$, $\rho(x, y) \geq 1/2$. For some partial order $\geq$ on X, say that a binary choice rule ρ satisfies *monotonicity* with respect to $\geq$ if $\rho(x', y) \geq \rho(x, y)$ if $x' \geq x$. Say that ρ satisfies *dominance* if whenever $x \geq y$ $\rho(x, y) \geq \rho(w, z)$ for all $w, z \in X$.

Lemma A.1. If ρ defined on X satisfies moderate transitivity and dominance with respect to $\geq$, then it satisfies monotonicity with respect to $\geq$.

Proof. Take any options x, y, and suppose $x' \geq x$. Let $\succeq$ denote the stochastic order; since ρ satisfies MST, $\succeq$ is complete and transitive. By dominance, we have $x' \succeq x$. There are three cases to consider:

Case 1: $x' \succeq x \succeq y$. By moderate transitivity, $\rho(x', y) \geq \min\{\rho(x', x), \rho(x, y)\}$. But since $\rho(x', x) \geq \rho(x, y)$ by dominance, it must be the case that $\rho(x', y) \geq \rho(x, y)$.

Case 2: $x' \succeq y \succeq x$. By definition of $\succeq$, $\rho(x', y) \geq 1/2 \geq \rho(x, y)$.

Case 3: $y \succeq x' \succeq x$. Toward a contradiction, suppose that $\rho(y, x') > \rho(y, x)$. By moderate transitivity, we have $\rho(y, x) \geq \min\{\rho(y, x'), \rho(x', x)\}$, and so it must be the case that $\rho(y, x) \geq \rho(x', x)$. But this implies that $\rho(y, x') > \rho(x', x)$, which contradicts dominance. □

Say ρ on a mixture space X is *superadditive* if for any x, y, x', y' with $\rho(x, y) = \rho(x', y') \geq 1/2$, for any $\lambda \in [0, 1]$ we have $\rho(\lambda x + (1 - \lambda)x', \lambda y + (1 - \lambda)y') \geq \rho(x, y)$.

Lemma A.2. Let X be a vector space. If ρ defined on X satisfies Moderate Stochastic Transitivity, Continuity, and Linearity, then ρ is superadditive.

Proof. Note that by Linearity,

$$\rho(\lambda(x-y),0) = \rho(x,y) \geq 1/2$$
$$\rho(0,-(1-\lambda)(x'-y')) = \rho(x',y') \geq 1/2$$

Linearity and Moderate Transitivity then imply

$$\begin{aligned}\rho(\lambda x+(1-\lambda)x', \lambda y+(1-\lambda)y') &= \rho(\lambda(x-y), -(1-\lambda)(x'-y')) \\ &\geq \min\{\rho(\lambda(x-y),0), \rho(0,-(1-\lambda)(x'-y'))\} \\ &= \min\{\rho(x,y), \rho(x',y')\}\end{aligned}$$

and so $\rho(\lambda x+(1-\lambda)x', \lambda y+(1-\lambda)y') \geq \rho(x,y) = \rho(x',y')$. □

2 Multiattribute Choice Model

Consider a setting where options have k real-valued attributes; that is each option $x = (x_1, ..., x_n)$ is identified with its location in $\mathbb{R}^n$. Let $\mathcal{D} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ denote the set of all distinct pairs of distinct options in $\mathbb{R}^n$. A *binary choice rule* is a function $\rho : \mathcal{D} \to [0,1]$ such that $\rho(x,y) + \rho(y,x) = 1$ for every x, y.

A binary choice rule has an L_1-*Complexity Representation* with weights $\beta \in \mathbb{R}^n$ if

$$\rho(x,y) = G\left(\frac{U(x)-U(y)}{d_{L_1}(x,y)}\right)$$

for some continuous, increasing G symmetric around 0, where $U(x) = \sum_k \beta_k x_k$ captures the utility of option x and $d_{L_1}(x,y) = \sum_k \beta_k |x_k - y_k|$ is the L_1 distance between options x and y in value-transformed attribute space.

Let $e_k \in \mathbb{R}^n$ denote the kth standard basis vector. Also, for any two options x, y, let $y_k x$ denote the option that replaces the value of option y along attribute k with x_k. Consider the following axioms.

M1. **Continuity:** $\rho(x,y)$ is continuous on its domain.

M2. **Linearity:** $\rho(x,y) = \rho(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)z)$.

M3. **Moderate Stochastic Transitivity:** If $\rho(x,y) \geq 1/2$, $\rho(y,z) \geq 1/2$, then $\rho(x,z) \geq \min\{\rho(x,y), \rho(y,z)\}$.

M4. **Dominance**: If $\rho(y_k x, y) \geq 1/2$ for all k, then $\rho(x,y) \geq \rho(w,z)$ for any two options w, z.

M5. **Exchangeability**: If $\rho(y+\alpha e_k, y+\delta e_j) = 1/2$ and $\rho(y+\gamma e_j, y+\eta e_k) = 1/2$, then $\rho(y+\alpha e_k+\gamma e_j, y) = \rho(y+\eta e_k+\delta e_j, y)$.

M6. **Simplification**: If $\rho(x,y) \geq 1/2$: for x' with $x'_1 = y_1$, $x'_k = x_k$ for $k \geq 3$, then $\rho(x',x) \geq 1/2$ implies $\rho(x',y) \geq \rho(x,y)$.

Continuity, Linearity, and Moderate Stochastic Transitivity are standard axioms. Dominance states choice probabilities are maximally extreme when there is a dominance relation. Exchangeability states that swapping attribute labels (adjusting for attribute weights) will not affect choice, and arises from the fact in our theory similarity operates over value-transformed attribute space. Simplification says that if we "eliminate" an attribute by making x and y the same along that attribute, and re-distribute the value difference in that attribute to another attribute, the comparison becomes easier.

The following theorem states that M1-M6 characterize our model, and that the utility weights of our model are identified up to rescaling.

Theorem M1. $\rho(x, y)$ has a L_1-complexity representation if and only if it satisfies M1-M6. Also, if $\rho(x, y)$ has a L_1-complexity representation with attribute weights β, then $\rho(x, y)$ also has a L_1-complexity representation with attribute weights β' iff $\beta' = C\beta$ for $C > 0$.

2.1 Proofs

Some Preliminaries: say that (x, y), (x', y') are *congruent* if for all k, $x_k \geq y_k \iff x'_k \geq y'_k$. Say that ρ satisfies *concentration neutrality* if whenever (x, y), (x', y') are congruent with $\rho(x, y) = \rho(x', y')$, $\rho(\lambda x + (1 - \lambda)x', \lambda y + (1 - \lambda)y') = \rho(x, y)$ for $\lambda \in (0, 1)$. Say that ρ is *monotone* if $\rho(x, y)$ is monotonic in each component of x. We begin by observing that ρ satisfies a stronger form of Simplification.

Lemma M1. Suppose $\rho(x, y)$ satisfies M1-M6. If $\rho(x, y) \geq 1/2$ and $i \neq j$: if x' with $x'_i = y_i$, $x'_k = x_k$ for $k \neq i, j$, then $\rho(x', x) \geq 1/2$ implies $\rho(x', y) \geq \rho(x, y)$.

Proof. Let $\succeq$ denote the complete binary relation on $\mathbb{R}^n$ satisfying $x \succeq y$ whenever $\rho(x, y) \geq 1/2$. By weak transitivity, $\succeq$ is transitive. Since ρ satisfies Continuity and Linearity, $\succeq$ satisfies the vNM axioms and so there exists weights $\beta \in \mathbb{R}^n$ such that $U(x) = \sum_k \beta_k x_k$ represents $\succeq$. Since ρ is not constant, Dominance implies that at least two components of β is nonzero; we can without loss take all components of β to be nonzero. For the remainder of the proof, we henceforth identify each option x with its weighted attribute values, so that $U(x) = \sum_k x_k$.

Note that by Linearity, it suffices to show that for $z, z' \in \mathbb{R}^n$, $i \neq j$ such that $z'_i = z_i$, $z'_k = z_k$ for all $k \neq i, j$, with $\rho(z', z) \geq 1/2$ and $\rho(z, 0) \geq 1/2$, we have $\rho(z', 0) \geq \rho(z, 0)$. Fix such z, z', and define $\tilde{z}, \tilde{z}'$ by

$$\tilde{z}_k = \begin{cases} z_i & k = 1 \\ z_j & k = 2 \\ z_1 & k = i \\ z_2 & k = j \\ z_k & otherwise \end{cases} \qquad \tilde{z}'_k = \begin{cases} z'_i & k = 1 \\ z'_j & k = 2 \\ z'_1 & k = i \\ z'_2 & k = j \\ z'_k & otherwise \end{cases}$$

By Exchangeability, we have $\rho(\tilde{z}, 0) = \rho(z, 0)$ and $\rho(\tilde{z}', 0) = \rho(z', 0)$, and so by Simplification, we have $\rho(\tilde{z}', 0) \geq \rho(\tilde{z}, 0)$, which in turn implies $\rho(z', 0) \geq \rho(z, 0)$ as desired. □

Theorem M1. $\rho(x, y)$ has a L_1-complexity representation if and only if it satisfies M1-M6. Also, if $\rho(x, y)$ has a L_1-complexity representation with attribute weights β, then $\rho(x, y)$ also has a L_1-complexity representation with attribute weights β' iff $\beta' = C\beta$ for $C > 0$.

Proof. Necessity of the axioms is immediate from the definition. We now show sufficiency. Note that sufficiency is immediate when ρ is constant, so we consider the case where ρ is not constant.

Let $\succeq$ denote the complete binary relation on $\mathbb{R}^n$ satisfying $x \succeq y$ whenever $\rho(x, y) \geq 1/2$. By weak transitivity, $\succeq$ is transitive. Since ρ satisfies Continuity and Linearity, $\succeq$ satisfies the vNM axioms and so there exists weights $\beta \in \mathbb{R}^n$ such that $U(x) = \sum_k \beta_k x_k$ represents $\succeq$. Since ρ is not constant, Dominance implies that at least two components of β is nonzero; we can without loss take all components of β to be nonzero. For the remainder of the proof, we henceforth identify each option x with its weighted attribute values, so that $U(x) = \sum_k x_k$. Since ρ satisfies Dominance and MST, Lemma A.1 implies that ρ satisfies monotonicity with respect to the component-wise dominance relation on $\mathbb{R}^n$, and is therefore monotone.

For $z \in \mathbb{R}^n$, Let $d^+(z) = \sum_{k:z_k \geq 0} z_k$ and $d^-(x) = \sum_{k:z_k<0} |z_k|$ denote the summed advantages and disadvantages in the comparison between z and 0. Say that z has *no dominance relationship* if $d^+(z), d^-(z) > 0$.

Claim 1. For any z with no dominance relationship satisfying $\sum_k z_k \geq 0$, $\rho(z, 0) = \rho(d^+(z)e_1 - d^-(z)e_2, 0)$.

Proof. Let $K^+ = \{k : z_k \geq 0\}$, $K^- = \{k : z_k < 0\}$, and for $i \in K^+$, $j \in K^-$, define $z^{ij} \in \mathbb{R}^n$ satisfying

$$z_k^{ij} = \begin{cases} d^+(z) & k = i \\ -d^-(z) & k = j \\ 0 & \text{otherwise} \end{cases}$$

By construction, there exists $\lambda_{ij} \in [0, 1]$, $\sum_{ij} \lambda_{ij} = 1$ such that $\sum_{ij} \lambda_{ij} z^{ij} = z$. Since ρ satisfies superadditivity by Lemma A2, we have $\rho(z, 0) \geq \rho(z^{ij}, 0)$ for some $i \in K^+, j \in K^-$. By repeated application of Lemma M1, we also have that $\rho(z, 0) \leq \rho(z^{ij}, 0)$, and so $\rho(z^{ij}, 0) = \rho(z, 0)$. Finally, by Exchangeability, $\rho(z^{12}, 0) = \rho(z^{ij}, 0) = \rho(z, 0)$. □

Define $H : \{(d^+, d^-) \in \mathbb{R}^2_+ : d^+ \geq d^-\} \to \mathbb{R}$ satisfying $H(d^+, d^-) = \rho(d^+e_1 - d^-e_2, 0)$. Claim 1 implies that for any z with no dominance relationship satisfying $\sum_k z_k \geq 0$, $\rho(z, 0) = H(d^+(z), d^-(z))$.

Claim 2 $H(d^+, d^-) = \tilde{F}\left(\frac{d^+ - d^-}{d^+ + d^-}\right)$ for some increasing, continuous $\tilde{F} : [0, 1) \to \mathbb{R}$.

Proof. Begin by showing that $H(d^+, d^-)$ satisfies

1. Homogeneity: $H(\alpha d^+, \alpha d^-) = H(d^+, d^-)$ for all $\alpha > 0$.

2. Ordering: $H(d^+, d^-)$ is increasing in d^+ and decreasing in d^-.

To see that H satisfies Homogeneity, note that due to Linearity, $H(\alpha d^+, \alpha d^-) = \rho(\alpha(d^+e_1 - d^-e_2), 0) = \rho(d^+e_1 - d^-e_2, 0) = H(d^+, d^-)$. To see that H satisfies Ordering, note that by monotonicity, $\rho(d^+e_1 - d^-e_2, 0)$ is increasing in d^+ and decreasing in d^-, and so $H(d^+, d^-)$ is also increasing in d^+ and decreasing in d^-.

Since H satisfies Homogeneity and Ordering, $H(d^+, d^-) = G(d^+/d^-)$ for some increasing function G. Let $\varphi(z) = \frac{1}{1/z+1} - \frac{1}{z+1}$; and define $\tilde{F} : [0,1) \to \mathbb{R}$ where $\tilde{F}(z) = G(\varphi^{-1}(z))$; since φ is strictly increasing and G is increasing, $\tilde{F}$ is increasing. By construction, we have $G(z) = \tilde{F}\left(\frac{1}{1/z+1} - \frac{1}{z+1}\right)$, and so $H(d^+, d^-) = \tilde{F}\left(\frac{d^+-d^-}{d^++d^-}\right)$. Finally, note that $\tilde{F}$ inherits continuity from H, which in turn inherits continuity from ρ. □

Now, let $F : (-1,1) \to \mathbb{R}$ be the symmetric extension of $\tilde{F}$ to $(-1,1)$ satisfying

$$F(z) = \begin{cases} \tilde{F}(z) & z \geq 0 \\ 1 - \tilde{F}(-z) & z < 0 \end{cases}$$

Claim 3. For any z with no dominance relationship, $\rho(z, 0) = F\left(\frac{d^+(z)-d^-(z)}{d^+(z)+d^-(z)}\right)$.

Proof. Claim 1 implies that $\rho(z, 0) = F\left(\frac{d^+(z)-d^-(z)}{d^+(z)+d^-(z)}\right)$ whenever $\sum_k z_k \geq 0$. Now consider the case where $\sum_k z_k < 0$; here we have $d^+(z) < d^-(z)$. Note that

$$\begin{aligned} \rho(z,0) &= 1 - \rho(0,z) \\ &= 1 - H(d^-(z), d^+(z)) \\ &= 1 - \tilde{F}\left(\frac{d^-(z) - d^+(z)}{d^+(z) + d^-(z)}\right) \\ &= F\left(\frac{d^+(z) - d^-(z)}{d^+(z) + d^-(z)}\right) \end{aligned}$$

as desired, where the third equality uses Claim 2. □

Due to Dominance, when z has a dominance relationship and $\sum_k z_k > 0$, $\rho(z, 0)$ takes on its maximal value, which we denote by $q > 1/2$; if instead $\sum_k z_k < 0$, $\rho(z, 0)$ takes on its minimal value of $1-q$. Claim 3 and continuity then imply that $F(1) = q$ and $F(-1) = 1-q$, and so for all z, $\rho(z, 0) = F\left(\frac{d^+(z)-d^-(z)}{d^+(z)+d^-(z)}\right)$.

Finally, take any x, y, and let $z = x - y$. Due to linearity, we have

$$\begin{aligned} \rho(x,y) &= \rho(z,0) \\ &= F\left(\frac{d^+(z) - d^-(z)}{d^+(z) + d^-(z)}\right) \\ &= F\left(\frac{\sum_k z_k}{\sum_k |z_k|}\right) \\ &= F\left(\frac{U(x) - U(y)}{d_{L_1}(x,y)}\right) \end{aligned}$$

as desired.

$\square$

We also prove a related representation theorem that does not rely on moderate transitivity (can ignore; not used in any remaining proofs).

Theorem M2. $\rho(x, y)$ has a L_1-complexity representation if and only if it satisfies M1, M2, M4, M5, and satisfies weak transitivity, concentration neutrality, and is monotone. Also, if $\rho(x, y)$ has a L_1-complexity representation with attribute weights β, then $\rho(x, y)$ also has a L_1-complexity representation with attribute weights β' iff $\beta' = C\beta$ for $C > 0$.

Proof. Necessity of the axioms is immediate from the definition. We now show sufficiency. Note that sufficiency is immediate when ρ is constant, so we consider the case where ρ is not constant.

Let $\succeq$ denote the complete binary relation on $\mathbb{R}^n$ satisfying $x \succeq y$ whenever $\rho(x, y) \geq 1/2$. By weak transitivity, $\succeq$ is transitive. Since ρ satisfies Continuity and Linearity, $\succeq$ satisfies the vNM axioms and so there exists weights $\beta \in \mathbb{R}^n$ such that $U(x) = \sum_k \beta_k x_k$ represents $\succeq$, where β is identified up to scale by $\succeq$. Since ρ is not constant, Dominance implies that at least two components of β are nonzero; we can without loss take all components of β to be nonzero. For the remainder of the proof, we henceforth identify each option x with its weighted attribute values, so that $U(x) = \sum_k x_k$.

For $z \in \mathbb{R}^n$, let $d^+(z) = \sum_{k:z_k \geq 0} z_k$ and $d^-(z) = \sum_{k:z_k<0} |z_k|$ denote the summed advantages and disadvantages in the comparison between z and 0. Say that z has no *dominance relationship* if $d^+(z), d^-(z) > 0$.

Claim 1. For any z with no dominance relationship satisfying $\sum_k z_k \geq 0$, $\rho(z, 0) = \rho(d^+(z)e_1 - d^-(z)e_2, 0)$.

Proof. Let $K^+ = \{k : z_k \geq 0\}$. For all $k \in K^+$, let w^k be the choice option satisfying

$$w_l^k = \begin{cases} d^+(z) & l = k \\ z_l & l \notin K^+ \\ 0 & \text{otherwise} \end{cases}$$

That is, w^k concentrates all of z's advantages into attribute k. Fix any $k, k' \in K^+$. Since by construction we have $w^k \sim w^{k'}$, Exchangeability implies that $\rho(w^k, 0) = \rho(w^{k'}, 0)$. Since z is a convex combination of the $(w^k)_{k \in K^+}$, and since z is advantage-congruent with each w^k, Concentration Neutrality in turn implies that $\rho(z, 0) = \rho(w^i, 0)$ for some $i \in K^+$.

Now let $K^- = \{k : z_k < 0\}$. For all $k \in K^-$, let z^k be the choice option satisfying

$$z_l^k = \begin{cases} -d^-(z) & l = k \\ d^+(z) & l = i \\ 0 & \text{otherwise} \end{cases}$$

That is, z^k concentrates all of w^i's disadvantages into attribute k. Since by construction $z^k \sim z^{k'}$ for all $k, k' \in K^-$, by Exchangeability, we have $\rho(z^k, 0) = \rho(z^{k'}, 0)$ for all $k, k' \in K^-$. Since w^i is a convex combination of the $(z^k)_{k \in K^-}$, and since w^i is advantage-congruent with each z^k, Concentration Neutrality in turn implies that $\rho(w^i, 0) = \rho(z^j, 0)$ for some $j \in K^-$. Exchangeability then implies that $\rho(z^j, 0) = \rho(d^+(z)e_1 - d^-(z)e_2, 0)$, and so we have $\rho(z, 0) = \rho(d^+(z)e_1 - d^-(z)e_2, 0)$. □

Define $H : \{(d^+, d^-) \in \mathbb{R}^2_+ : d^+ \geq d^-\} \to \mathbb{R}$ satisfying $H(d^+, d^-) = \rho(d^+e_1 - d^-e_2, 0)$. Claim 1 implies that for any z with no dominance relationship satisfying $\sum_k z_k \geq 0$, $\rho(z, 0) = H(d^+(z), d^-(z))$.

Claim 2 $H(d^+, d^-) = \tilde{F}\left(\frac{d^+ - d^-}{d^+ + d^-}\right)$ for some increasing, continuous $\tilde{F} : [0, 1) \to \mathbb{R}$.

Proof. Begin by showing that $H(d^+, d^-)$ satisfies

1. Homogeneity: $H(\alpha d^+, \alpha d^-) = H(d^+, d^-)$ for all $\alpha > 0$.
2. Ordering: $H(d^+, d^-)$ is increasing in d^+ and decreasing in d^-.

To see that H satisfies Homogeneity, note that due to Linearity, $H(\alpha d^+, \alpha d^-) = \rho(\alpha(d^+e_1 - d^-e_2), 0) = \rho(d^+e_1 - d^-e_2, 0) = H(d^+, d^-)$. To see that H satisfies Ordering, note that by monotonicity, $\rho(d^+e_1 - d^-e_2, 0)$ is increasing in d^+ and decreasing in d^-, and so $H(d^+, d^-)$ is also increasing in d^+ and decreasing in d^-.

Since H satisfies Homogeneity and Ordering, $H(d^+, d^-) = G(d^+/d^-)$ for some increasing function G. Let $\varphi(z) = \frac{1}{1/z+1} - \frac{1}{z+1}$; and define $\tilde{F} : [0, 1) \to \mathbb{R}$ where $\tilde{F}(z) = G(\varphi^{-1}(z))$; since φ is strictly increasing and G is increasing, $\tilde{F}$ is increasing. By construction, we have $G(z) = \tilde{F}\left(\frac{1}{1/z+1} - \frac{1}{z+1}\right)$, and so $H(d^+, d^-) = \tilde{F}\left(\frac{d^+ - d^-}{d^+ + d^-}\right)$. Finally, note that $\tilde{F}$ inherits continuity from H, which in turn inherits continuity from ρ. □

Now, let $F : (-1, 1) \to \mathbb{R}$ be the symmetric extension of $\tilde{F}$ to $(-1, 1)$ satisfying

$$F(z) = \begin{cases} \tilde{F}(z) & z \geq 0 \\ 1 - \tilde{F}(-z) & z < 0 \end{cases}$$

Claim 3. For any z with no dominance relationship, $\rho(z, 0) = F\left(\frac{d^+(z) - d^-(z)}{d^+(z) + d^-(z)}\right)$.

Proof. Claim 1 implies that $\rho(z, 0) = F\left(\frac{d^+(z) - d^-(z)}{d^+(z) + d^-(z)}\right)$ whenever $\sum_k z_k \geq 0$. Now consider the case where $\sum_k z_k < 0$; here we have $d^+(z) < d^-(z)$. Note that

$$\begin{aligned} \rho(z, 0) &= 1 - \rho(0, z) \\ &= 1 - H(d^-(z), d^+(z)) \\ &= 1 - \tilde{F}\left(\frac{d^-(z) - d^+(z)}{d^+(z) + d^-(z)}\right) \\ &= F\left(\frac{d^+(z) - d^-(z)}{d^+(z) + d^-(z)}\right) \end{aligned}$$

as desired, where the third equality uses Claim 2. □

Due to Dominance, when z has a dominance relationship and $\sum_k z_k > 0$, $\rho(z,0)$ takes on its maximal value, which we denote by $q > 1/2$; if instead $\sum_k z_k < 0$, $\rho(z,0)$ takes on its minimal value of $1-q$. Claim 3 and continuity then imply that $F(1) = q$ and $F(-1) = 1-q$, and so for all z, $\rho(z,0) = F\left(\frac{d^+(z)-d^-(z)}{d^+(z)+d^-(z)}\right)$.

Finally, take any x, y, and let $z = x - y$. Due to linearity, we have

$$\begin{aligned}\rho(x,y) &= \rho(z,0) \\ &= F\left(\frac{d^+(z)-d^-(z)}{d^+(z)+d^-(z)}\right) \\ &= F\left(\frac{\sum_k z_k}{\sum_k |z_k|}\right) \\ &= F\left(\frac{U(x)-U(y)}{d_{L_1}(x,y)}\right)\end{aligned}$$

as desired. □

3 Risk Model

Now consider lottery choice. Let $S \subseteq \mathbb{R}$; we will consider choice over simple lotteries (lotteries with finite support) on S. In particular, let a simple lottery x be identified by the function $f_x : S \to [0,1]$ such that $f_x(s) > 0$ for finitely many s, with $\sum_s f_x(s) = 1$, and let $F_x(s) = \sum_{s'<s} f_x(s)$ denote the CDF of x, and $S_x = \{s \in S : f_x(s) > 0\}$ denote the support of s. Let $L(S)$ denote the set of simple lotteries on S. For $x, y \in L(S)$, $\lambda \in [0,1]$, define $\lambda x + (1-\lambda)y \in L(S)$ to be the lottery with pdf $\lambda f_x + (1-\lambda) f_y$. For $s \in S$, we will also with some abuse of notation let s denote the degenerate lottery that places all mass on s.

Let $\mathcal{D} = \{x, y \in L(S) \times L(S) : x \neq y\}$ denote the set of all pairs of distinct simple lotteries on S. A *binary choice rule* is a function $\rho : \mathcal{D} \to [0,1]$ such that $\rho(x,y) + \rho(y,x) = 1$ for all $(x,y) \in \mathcal{D}$.

A binary choice rule has a *CDF-Complexity Representation* with Bernoulli utility function $u : Z \to \mathbb{R}$ if u is increasing and

$$\rho(x,y) = F\left(\frac{U(x)-U(y)}{d_{CDF}(x,y)}\right)$$

for some continuous, increasing F symmetric around 0, where $U(x) = \sum_z u(s) f_x(s)$ and $d_{CDF}(x,y) = \int_{u(S)} |F_x(u^{-1}(v)) - F_y(u^{-1}(v))|\, dv = \int_0^1 |u(F_x^{-1}(q)) - u(F_y^{-1}(q))|\, dq$ is the (generalized) CDF distance.

Let $\geq$ denote the partial order on $L(Z)$ corresponding to first order stochastic dominance. Finally, for $(x_1, x_2, y_1, y_2, s) \in S$, $q_1, q_2 \in (0,1)$ such that $q_1 + q_2 \leq 1$, say that a comparison (x,y) is *simple* with parameters $(x_1, x_2, y_1, y_2, s; q_1, q_2)$ if **i)** $x = q_1 x_1 + q_2 x_2 + (1-q_1-q_2)s$,

$y = qy_1 + (1-q)y_2 + (1-q_1-q_2)s$, **ii)** $x_1 \leq y_1$, $y_2 \leq x_2$, and **iii)** $(x_1, y_1), (y_2, x_2) \subseteq \mathbb{R}$ are disjoint; that is, the non-common payoffs of one lottery sandwiches the other.

Finally, call a subset of prizes $B \subseteq S$ *consequential* if for all distinct $s, s' \in B$, either $\rho(s, s') > 1/2$ or $\rho(s', s) > 1/2$. Consider the following axioms:

L1. **Continuity:** $\rho(x, y)$ is continuous on its domain.

L2. **Independence:** $\rho(x, y) = \rho(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z)$ for $\lambda \in (0, 1)$.

L3. **Moderate Stochastic Transitivity:** If $\rho(x, y) \geq 1/2$, $\rho(y, z) \geq 1/2$, then $\rho(x, z) \geq \min\{\rho(x, y), \rho(y, z)\}$.

L4. **Dominance:** $x \geq y$, then $\rho(x, y) \geq \rho(w, z)$ for any $w, z \in L(S)$.

L5. **Simple Exchangeability:** Let (x, y) be a simple comparison with parameters $(x_1, x_2, y_1, y_2, s; q_1, q_2)$. Then for any simple comparison (x', y') with parameters $(x_1', x_2, y_1', y_2, s; q_1', q_2')$, where

$$\rho\left(py_1 + (1-p)x_1', px_1 + (1-p)y_1'\right) = 1/2$$

for $p = \frac{q_1}{q_1 + q_1'}$, $\rho(x, y) = \rho(x', y')$.

L6. **Simplification**. If $\rho(x, y) \geq 1/2$: for x' with $f_{x'}(s_n) = f_y(s_n)$ and $f_{x'}(s) = f_x(s)$ for all $s < s_{n-1}$, where s_n, s_{n-1} are the largest and second largest payoffs in $S_x \cup S_y$, then $\rho(x', x) \geq 1/2$ implies $\rho(x', y) \geq \rho(x, y)$.

Theorem L1. If S contains a subset of consequential prizes of size 4, ρ satisfies L1-L7 iff it has a CDF-Complexity Representation.

3.1 Proofs

Theorem L1. If S contains a subset of consequential prizes of size 4, ρ satisfies L1-L7 iff it has a CDF-Complexity Representation.

Proof. Necessity of the axioms is immediate from the definition; we now show sufficiency. Sufficiency is immediate when ρ is constant, so we consider the case where ρ is not constant.

Let $\succeq$ denote the complete binary relation on $L(S)$ satisfying $x \succeq y$ whenever $\rho(x, y) \geq 1/2$. By Weak Stochastic Transitivity, $\succeq$ is transitive. Since ρ satisfies Continuity and Independence, $\succeq$ satisfies the vNM axioms and so there exists a utility function $u : S \to \mathbb{R}$ such that $U(x) = \sum_s u(s) f_x(s)$ represents $\succeq$; Dominance implies that u is increasing.

Since S contains a consequential subset of prizes of size 4, there exists four distinct prizes $s_a, s_b, s_c, s_d \in S$ such that $u(s_a) > u(s_b) > u(s_c) > u(s_d)$. Consider any two lotteries $x, y \in L(S)$ for which there exists no dominance relationship. Enumerate $S_x \cup S_y \cup \{s_a, s_b, s_c, s_d\}$ by $s_1, s_2, ..., s_{n+1}$, where $s_1 < s_2 < ... < s_{n+1}$, and let $K = \{1, ..., n, n+1\}$. With some abuse of notation, we let a, b, c, d denote the indices in K corresponding to prizes s_a, s_b, s_c, s_d. We'll for now make our lives easier and assume that $u(s_0) < u(s_1) < ... < u(s_n)$ (should be straightforward to show that this is without loss by relabeling states). With some abuse of notation, for any $w \in L(K)$, let $F_w(k) = \sum_{s \leq s_k} f_w(s)$ denote the value of the CDF of x and y at support point s_k, and let $u(k) = u(s_k)$.

Note that the following form of Exchangeability holds.

Lemma L.1. Suppose x, y are simple lotteries satisfying

$$x = \begin{cases} s_{i+1} & \text{w.p. } q \\ s_j & \text{w.p. } 1-q \end{cases} \qquad y = \begin{cases} s_i & \text{w.p. } q \\ s_{j+1} & \text{w.p. } 1-q \end{cases}$$

Then for $k \neq j$ and x', y' satisfying

$$x' = \begin{cases} s_{k+1} & \text{w.p. } q' \\ s_j & \text{w.p. } 1-q' \end{cases} \qquad y' = \begin{cases} s_k & \text{w.p. } q' \\ s_{j+1} & \text{w.p. } 1-q' \end{cases}$$

for $\frac{q'}{1-q'} = \frac{q}{1-q} \frac{u(i+1)-u(i)}{u(k+1)-u(k)}$, we have $\rho(x', y') = \rho(x, y)$.

Proof. Take such x, y, x', y'. We consider two cases:

Case 1. $u(k+1) - u(k) \leq u(i+1) - u(i)$. Let $\alpha = \frac{u(i+1)-u(i)}{u(k+1)-u(k)} \geq 1$, and consider $\tilde{x}$, $\tilde{y}$ satisfying

$$\tilde{x} = \begin{cases} s_{i+1} & \text{w.p. } \frac{q}{1-q+\alpha q} \\ s_j & \text{w.p. } \frac{1-q}{1-q+\alpha q} \\ s_l & \text{otherwise} \end{cases} \qquad \tilde{y} = \begin{cases} s_i & \text{w.p. } \frac{q}{1-q+\alpha q} \\ s_{j+1} & \text{w.p. } \frac{1-q}{1-q+\alpha q} \\ s_l & \text{otherwse} \end{cases}$$

Note that by Independence, $\rho(\tilde{x}, \tilde{y}) = \rho(x, y)$. Furthermore, note that by construction, $q' = \frac{1-q}{1-q+\alpha q}$, and

$$\begin{aligned} (u(i+1) - u(i)) \cdot \frac{q}{1-q+\alpha q} &= (u(k+1) - u(k)) \cdot \frac{\alpha q}{1-q+\alpha q} \\ &= (u(k+1) - u(k)) \cdot q' \end{aligned}$$

and so by Simple Exchangeability, we have $\rho(\tilde{x}, \tilde{y}) = \rho(x', y')$, which implies $\rho(x, y) = \rho(x', y')$.

Case 2. $u(k+1) - u(k) > u(i+1) - u(i)$. Let $\beta = \frac{u(k+1)-u(k)}{u(i+1)-u(i)} > 1$, and consider $\tilde{x}'$, $\tilde{y}'$ satisfying

$$\tilde{x}' = \begin{cases} s_{k+1} & \text{w.p. } \frac{q'}{1-q'+\beta q'} \\ s_j & \text{w.p. } \frac{1-q'}{1-q'+\beta q'} \\ s_l & \text{otherwise} \end{cases} \qquad \tilde{y}' = \begin{cases} s_k & \text{w.p. } \frac{q'}{1-q'+\beta q'} \\ s_{j+1} & \text{w.p. } \frac{1-q'}{1-q'+\beta q'} \\ s_l & \text{otherwse} \end{cases}$$

Note that by Independence, $\rho(\tilde{x}', \tilde{y}') = \rho(x', y')$. Furthermore, note that by construction, $1 - q = \frac{1-q'}{1-q'+\beta q'}$, and

$$\begin{aligned}(u(k+1) - u(k)) \cdot \frac{q'}{1 - q' + \beta q'} &= (u(k+1) - u(k)) \cdot 1/\beta q \\ &= (u(i+1) - u(i)) \cdot q\end{aligned}$$

and so by Simple Exchangeability, we have $\rho(\tilde{x}', \tilde{y}') = \rho(x, y)$, which implies $\rho(x, y) = \rho(x', y')$. □

Now identify each lottery $w \in L(K)$ with its *utility-weighted* CDF vector $\tilde{w} \in \mathbb{R}^n$, where

$$\tilde{w}_k = -F_w(k)(u(k+1) - u(k))$$

Note that for any $x, y \in L(K)$,

$$\frac{\sum_k (\tilde{x}_k - \tilde{y}_k)}{\sum_k |\tilde{x}_k - \tilde{y}_k|} = \frac{U(x) - U(y)}{d_{CDF}(x, y)}$$

We now seek to extend the space of utility-weighted CDF vectors to $\mathbb{R}^n$ in order to apply Theorem 1. Let $\mu \in L(K)$ denote the lottery that is uniform over K; that is $F_\mu(k) = \frac{k}{n+1}$. Consider the set

$$V = \{a \in \mathbb{R}^n : a_k = \alpha(\tilde{x}_k - \tilde{\mu}_k) : x \in L(K), \alpha > 0)\}$$

Lemma L.2. $V = \mathbb{R}^n$, and in particular, V is a linear space.

Proof. Take any $a \in \mathbb{R}^n$. We will show that $a \in V$. Define

$$\begin{aligned}\beta &= \max_{k \in \{2,3,\ldots,n\}} (n+1) \left[a_k/(u(k+1) - u(k)) - a_{k-1}/(u(k) - u(k-1))\right] \\ \gamma &= (n+1)[a_1/(u(2) - u(1))] \\ \eta &= -(n+1)[a_n/(u(n+1) - u(n))]\end{aligned}$$

and fix any $\alpha > \max\{\beta, \gamma, \eta, 0\}$. Define $G : K \to \mathbb{R}$ given by

$$H(k) = \begin{cases} F_\mu(k) - \dfrac{a_k/(u(k+1) - u(k))}{\alpha} & k < n+1 \\ 1 & k = n+1 \end{cases}$$

Since $\alpha > \beta$, we have $H(k+1) - H(k) \geq 0$ for all $k = 1, ..., n$, and since $\alpha > \eta$, we have $1 = H(n+1) - H(n) \geq 0$, and so H is increasing. Furthermore, since $\alpha > \gamma$, $H(1) \geq 0$, and so H is positive on its domain. Since $H(n+1) = 1$, H is the CDF of a lottery in $L(K)$, which we denote by x. Note that by construction, for all $k = 1, ..., n$ we have

$$\begin{aligned}\alpha(\tilde{x}_k - \tilde{\mu}_k) &= \alpha \left(-F_\mu(k)(u(k+1) - u(k)) + \frac{a_k}{\alpha} + F_\mu(k)(u(k+1) - u(k))\right) \\ &= a_k\end{aligned}$$

which implies that $a \in V$. □

For any $a, b \in V$, let $L(a,b) = \{(x,y) \in L(K) \times L(K) : a = \alpha(\tilde{x} - \tilde{\mu}), b = \alpha(\tilde{y} - \tilde{\mu})\}$.

Lemma L.3. For any $W \subseteq V$ finite, there exists some $\alpha > 0$ such that for all $a \in W$, there exists $x \in L(K)$ such that $a = \alpha(\tilde{x} - \tilde{\mu})$.

Proof. Enumerate the elements of W by $\{a^1, a^2, ..., a^l\}$. For all $m = \{1, 2, ..., l\}$, there exists $\alpha^m > 0$,$w^m \in L(K)$ such that $a^m = \alpha^m(\tilde{w}^m - \tilde{\mu})$. Let $\alpha = \max_m \alpha^m$, and for all m, define $x^m = (\alpha^m/\alpha)w^m + (1 - \alpha^m/\alpha)\mu \in L(K)$, and notice that $a^m = \alpha(\tilde{x}^m - \tilde{\mu})$. □

Define some $\phi : V \times V \to L(K) \times L(K)$ that takes an arbitrary selection from $L(a,b)$; Lemma L.3 implies $L(a,b)$ is non-empty, ϕ is well-defined. For $\hat{\mathcal{D}} = \{(a,b) \in V \times V : a \neq b\}$, define $\hat{\rho} : \hat{\mathcal{D}} \to [0,1]$ by $\hat{\rho}(a,b) = \rho(\phi(a,b))$.

Lemma L.4. $\hat{\rho}$ is uniquely identified by ρ. That is, for any $a, b \in V$: for any $(x,y), (x',y') \in L(a,b)$,. $\rho(x,y) = \rho(x',y')$ and so $\hat{\rho}$ does not depend on the choice of ϕ. Also, $\hat{\rho}$ is a binary choice rule, that is, $\hat{\rho}(a,b) = 1 - \hat{\rho}(b,a)$.

Proof. Fix some $a, b \in V$, and suppose $(x,y), (x',y') \in L(a,b)$. It suffices to show that $\rho(x,y) = \rho(x',y')$. Since $(x,y), (x',y') \in L(a,b)$, there exists $\alpha, \alpha' > 0$ such that

$$\begin{aligned} a &= \alpha(\tilde{x} - \tilde{\mu}) = \alpha'(\tilde{x}' - \tilde{\mu}) \\ b &= \alpha(\tilde{y} - \tilde{\mu}) = \alpha'(\tilde{y}' - \tilde{\mu}) \end{aligned}$$

Without loss, we can take $\alpha' > \alpha$. For $\lambda = \frac{\alpha}{\alpha'}$, the above inequalities directly imply that

$$\begin{aligned} x' &= \lambda x + (1 - \lambda)\mu \\ y' &= \lambda y + (1 - \lambda)\mu \end{aligned}$$

and so by Independence of ρ, $\rho(x,y) = \rho(x',y')$.

Finally to see that $\hat{\rho}$ is a binary choice rule, take any $a, b \in V$. By Lemma L.3, there exists $\alpha > 0$, $x, y \in L(K)$ such that $a = \alpha(\tilde{x} - \tilde{\mu})$, $b = \alpha(\tilde{y} - \tilde{\mu})$; we have

$$\begin{aligned} \hat{\rho}(a,b) &= \rho(x,y) \\ &= 1 - \rho(y,x) \\ &= 1 - \hat{\rho}(b,a) \end{aligned}$$

as desired. □

Lemma L.5. $\hat{\rho}(a,b) \geq 1/2 \iff \sum_k a_k \geq \sum_k b_k$., and $\hat{\rho}$ satisfies M1-M6.

Proof. Fix any $a, b, c, a', b' \in V$. By Lemma L.3, there exists $\alpha > 0$, $x, y, z, x', y' \in L(K)$ such that $a = \alpha(\tilde{x} - \tilde{\mu}), b = \alpha(\tilde{y} - \tilde{\mu}), c = \alpha(\tilde{z} - \tilde{\mu}), a' = \alpha(\tilde{x}' - \tilde{\mu}), b' = \alpha(\tilde{y}' - \tilde{\mu})$.

To show the first claim, note that $\hat{\rho}(a,b) \geq 1/2 \iff \rho(x,y) \geq 1/2 \iff U(x) \geq U(y) \iff \sum_k \tilde{x}_k \geq \sum_k \tilde{y}_k \iff \sum_k a_k \geq \sum_k b_k$.

To see that $\hat{\rho}$ satisfies Continuity, note that $\hat{\rho}$ inherits continuity from ρ. To see that $\hat{\rho}$ satisfies Linearity, take any $\lambda \in [0,1]$.Note that by construction, $\lambda a + (1-\lambda)c = \alpha(\lambda\tilde{x} + (1-\lambda)\tilde{z} - \tilde{\mu})$ and $\lambda b + (1-\lambda)c = \alpha(\lambda\tilde{y} + (1-\lambda)\tilde{z} - \tilde{\mu})$, and so

$$\begin{aligned}\hat{\rho}(\lambda a + (1-\lambda)c, \lambda b + (1-\lambda)c) &= \rho(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z)\\ &= \rho(x,y)\\ &= \hat{\rho}(a,b)\end{aligned}$$

where the first and final equalities follow from Lemma L.4, and the second equality follows from Independence of ρ.

To show that $\hat{\rho}$ satisfies Moderate Stochastic Transitivity, suppose that $\hat{\rho}(a,b) \geq 1/2$, $\hat{\rho}(b,c) \geq 1/2$. This implies that $\rho(x,y) \geq 1/2$, $\rho(y,z) \geq 1/2$, and so Moderate Stochastic Transitivity of ρ implies that $\rho(x,z) \geq \min\{\rho(x,y), \rho(y,z)\}$, which in turn implies that $\hat{\rho}(a,c) \geq \min\{\rho(a,b), \rho(b,c)\}$, and so $\hat{\rho}$ satisfies Moderate Stochastic Transitivity.

To show that $\hat{\rho}$ satisfies Dominance, by Lemma L.4, it suffices to show that if $a_k \geq b_k$ for all k, then $x \geq y$. To see this, suppose that $a_k \geq b_k$ for all k; this implies that $\tilde{x}_k \geq \tilde{y}_k$ for all k, which in turn implies that $F_x(k) \leq F_y(k)$ for all k, and so $x \geq y$.

To show that $\hat{\rho}$ satisfies Exchangeability, take any $a \in \mathbb{R}^n$ $\alpha, \gamma, \delta, \eta \in \mathbb{R}$, $i, j \in \{1, ..., n\}$ satisfying

$$\begin{aligned}\hat{\rho}(a + \alpha e_i, a + \delta e_j) = 1/2\\ \hat{\rho}(a + \gamma e_j, a + \eta e_i) = 1/2\end{aligned}$$

By Linearity of $\hat{\rho}$, it suffices to show that $\hat{\rho}(\alpha e_i + \gamma e_j, 0) = \hat{\rho}(\delta e_j + \eta e_i, 0)$. We can without loss take $i < j$. By the first claim, of the lemma, note that the indifference conditions imply that $\delta = \alpha$, $\eta = \gamma$. Note that if $\neg(\alpha > 0, \gamma < 0)$ or $\neg(\alpha < 0, \gamma > 0)$, Dominance of $\hat{\rho}$ implies $\hat{\rho}(\alpha e_i + \gamma e_j, 0) = \hat{\rho}(\delta e_j + \eta e_i, 0)$. Now consider the case where $(\alpha > 0, \gamma < 0)$ or $(\alpha < 0, \gamma > 0)$; by Linearity of $\hat{\rho}$, we can without loss take $\alpha > 0, \gamma < 0$.

Now define $x, y \in L(K)$ where

$$x = \begin{cases} s_{i+1} & \text{w.p. } q \\ s_j & \text{w.p. } 1-q \end{cases} \qquad y = \begin{cases} s_i & \text{w.p. } q \\ s_{j+1} & \text{w.p. } 1-q \end{cases}$$

where $\frac{q}{1-q} = \frac{\alpha/(u(i+1)-u(i))}{-\gamma/u(j+1)-u(j)}$. By Lemma L.4, $\rho(x,y) = \hat{\rho}(\tilde{x} - \tilde{\mu}, \tilde{y} - \tilde{\mu}) = \hat{\rho}(\alpha e_i + \gamma e_j, 0)$, where the last equality obtains since $\hat{\rho}$ satisfies Linearity and since

$$\tilde{x}_k - \tilde{y}_k = \begin{cases} \dfrac{\alpha}{\alpha/(u(i+1)-u(i)) - \gamma/(u(j+1)-u(j))} & k = i \\ \dfrac{\gamma}{\alpha/(u(i+1)-u(i)) - \gamma/(u(j+1)-u(j))} & k = j \\ 0 & \text{otherwise} \end{cases}$$

Now define $x^1, y^1 \in L(K)$ satisfying

$$x^1 = \begin{cases} s_2 & \text{w.p. } q^1 \\ s_j & \text{w.p. } 1-q^1 \end{cases} \qquad y^1 = \begin{cases} s_1 & \text{w.p. } q^1 \\ s_{j+1} & \text{w.p. } 1-q^1 \end{cases}$$

where $\frac{q^1}{1-q^1} = \frac{q}{1-q}\frac{u(i+1)-u(i)}{u(2)-u(1)}$. By construction, we have

$$\frac{q^1}{1-q^1}(u(2)-u(1)) = \frac{q}{1-q}(u(i+1)-u(i))$$

and so by Lemma L.1 we have $\rho(x^1, y^2) = \rho(x, y)$. Similarly, for (x^2, y^2), (x^3, y^3), (x^4, y^4), (x^5, y^5) defined by

$$x^2 = \begin{cases} s_2 & \text{w.p. } 1 \end{cases} \qquad y^2 = \begin{cases} s_1 & \text{w.p. } q^2 \\ s_3 & \text{w.p. } 1-q^2 \end{cases}$$

$$x^3 = \begin{cases} s_n & \text{w.p. } q^3 \\ s_2 & \text{w.p. } 1-q^3 \end{cases} \qquad y^3 = \begin{cases} s_{n-1} & \text{w.p. } q^3 \\ s_3 & \text{w.p. } 1-q^3 \end{cases}$$

$$x^4 = \begin{cases} s_n & \text{w.p. } q^4 \\ s_i & \text{w.p. } 1-q^4 \end{cases} \qquad y^4 = \begin{cases} s_{n-1} & \text{w.p. } q^4 \\ s_{i+1} & \text{w.p. } 1-q^4 \end{cases}$$

$$x^5 = \begin{cases} s_{j+1} & \text{w.p. } q^5 \\ s_i & \text{w.p. } 1-q^5 \end{cases} \qquad y^5 = \begin{cases} s_j & \text{w.p. } q^5 \\ s_{i+1} & \text{w.p. } 1-q^5 \end{cases}$$

where $\frac{q^2}{1-q^2} = \frac{q^1}{1-q^1}\frac{u(3)-u(2)}{u(j+1)-u(j)}$, $\frac{q^3}{1-q^3} = \frac{q^2}{1-q^2}\frac{u(2)-u(1)}{u(n)-u(n-1)}$, $\frac{q^4}{1-q^4} = \frac{q^3}{1-q^3}\frac{u(i+1)-u(i)}{u(3)-u(2)}$, $\frac{q^5}{1-q^5} = \frac{q^4}{1-q^4}\frac{u(n)-u(n-1)}{u(j+1)-u(j)}$, Lemma L.1 implies that $\rho(x^2, y^2) = \rho(x^1, y^1)$, $\rho(x^3, y^3) = \rho(x^2, y^2)$, $\rho(x^4, y^4) = \rho(x^3, y^3)$, $\rho(x^5, y^5) = \rho(x^4, y^4)$, and so in particular, $\rho(x, y) = \rho(x^5, y^5)$. Note also that

$$\begin{aligned} \frac{q^5}{1-q^5} &= \frac{(u(i+1)-u(i))^2}{(u(j+1)-u(j))^2} \cdot \frac{q}{1-q} \\ &= \frac{\alpha/(u(j+1)-u(j))}{-\gamma/(u(i+1)-u(i)} \end{aligned}$$

which in turn implies that

$$\tilde{x}^5_k - \tilde{y}^5_k = \begin{cases} \dfrac{\alpha}{\alpha/(u(j+1)-u(j)) - \gamma/(u(i+1)-u(i))} & k = j \\ \dfrac{\gamma}{\alpha/(u(j+1)-u(j)) - \gamma/(u(i+1)-u(i))} & k = i \\ 0 & \text{otherwise} \end{cases}$$

By Lemma L.4, we have $\rho(x^5, y^5) = \hat{\rho}(\tilde{x}^5 - \tilde{\mu}, \tilde{y}^5 - \tilde{\mu}) = \hat{\rho}(\gamma e_i + \alpha e_j, 0)$, where the last equality follows from linearity of $\hat{\rho}$, and so $\hat{\rho}(\alpha e_i + \gamma e_j, 0) = \hat{\rho}(\gamma e_i + \alpha e_j, 0)$ as desired.

Finally, to see that $\hat{\rho}$ satisfies Simplification, fix any $a, b \in \mathbb{R}^n$ with $\hat{\rho}(a, b) \geq 1/2$, and let $a' \in \mathbb{R}^n$ satisfy $a'_n = b_n$, $a'_k = a_k$ for all $k \leq n-2$, with $\hat{\rho}(a', a) \geq 1/2$. By Lemma L.3, there exists $\alpha > 0$. $x, x', y \in L(K)$ such that $a = \alpha(\tilde{x} - \tilde{\mu})$, $a' = \alpha(\tilde{x}' - \tilde{\mu})$, $b = \alpha(\tilde{y} - \tilde{\mu})$, and Lemma L.4 implies that $\rho(x, y) \geq 1/2$ and $\rho(x', x) \geq 1/2$. Since $a'_n = b_n \implies F_{x'}(s_n) = F_y(s_n) \implies f_{x'}(s_n) = f_y(s_n)$, and $a'_k = a_k$ for all $k \leq n-2$ implies that $F_{x'}(s_k) = F_x(s_k)$ for all $k \leq n-2$ which in turn implies that $f_{x'}(s_k) = f_x(s_k)$ for all $k \leq n-2$, the fact that ρ satisfies Simplification implies that $\rho(x', y) \geq \rho(x, y)$. Lemma L.4 then implies that $\hat{\rho}(a', b) \geq \hat{\rho}(a, b)$, and so $\hat{\rho}$ satisfies simplification with respect to attributes $n-1, n$. Without loss, we can re-index attributes $n-1, n$ to $1, 2$, and so $\hat{\rho}$ satisfies Simplification. $\square$

Using Lemma L.5, Theorem 1 then implies that there exists a continuous, increasing $G : [-1, 1] \to [0, 1]$, symmetric around 0, such that for all $a, b \in \mathbb{R}^n$ we have

$$\hat{\rho}(a, b) = G\left(\frac{\sum_k (a_k - b_k)}{\sum_k |a_k - b_k|}\right)$$

Lemma L.4 then implies that for any $x, y \in L(K)$, we have

$$\begin{aligned}
\rho(x, y) &= \hat{\rho}(\tilde{x} - \tilde{\mu}, \tilde{y} - \tilde{\mu}) \\
&= G\left(\frac{\sum_k (\tilde{x}_k - \tilde{y}_k)}{\sum_k |\tilde{x}_k - \tilde{y}_k|}\right) \\
&= G\left(\frac{U(x) - U(y)}{d_{CDF}(x, y)}\right)
\end{aligned}$$

Let $\mathcal{K} = \{K \subseteq S : |K| < \infty, \{s_a, s_b, s_c, s_d\} \subseteq K\}$. The above implies that for any $K \in \mathcal{K}$, there exists a continuous, increasing $G_K : [-1, 1] \to [0, 1]$ such that for all $x, y \in L(K)$,

$$\rho(x, y) = G_K\left(\frac{U(x) - U(y)}{d_{CDF}(x, y)}\right)$$

All that remains is to show that for any $K, K' \in \mathcal{K}$, $G_K = G_{K'}$. To see this, fix any $K, K' \in \mathcal{K}$, and for $\alpha \geq 0$, $\gamma \geq 0$, consider $x, y \in L(S)$ with

$$x = \begin{cases} s_b & \text{w.p. } 1 \end{cases} \qquad y = \begin{cases} s_c & \text{w.p. } \dfrac{\alpha/(u(s_b) - u(s_c))}{\alpha/(u(s_b) - u(s_c)) + \gamma/(u(s_a) - u(s_b))} \\ s_a & \text{w.p. } \dfrac{\gamma/(u(s_a) - u(s_b))}{\alpha/(u(s_b) - u(s_c)) + \gamma/(u(s_a) - u(s_b))} \end{cases}$$

Note that x, y belong to both K and K', and so

$$\rho(x, y) = G_K\left(\frac{U(x) - U(y)}{d_{CDF}(x, y)}\right) = G_{K'}\left(\frac{U(x) - U(y)}{d_{CDF}(x, y)}\right)$$

and since $\frac{U(x)-U(y)}{d_{CDF}(x,y)} = \frac{\alpha-\gamma}{\alpha+\gamma}$, for any $r \in [-1, 1]$ we can choose $\alpha, \gamma \geq 0$ such that $\frac{U(x)-U(y)}{d_{CDF}(x,y)} = r$, we must have $G_K = G_{K'}$.

$\square$

4 Intertemporal Model

Now consider intertemporal choice. We will consider finite payoff streams over money. In particular, let a payoff stream x be identified by the payoff function $m_x : [0, \infty) \to \mathbb{R}$ such that $m_x(t) \neq 0$ for finitely many t, and let $M_x(t) = \sum_{t' \leq t} m_x(t')$ denote the *cumulative payoff function* of x. Let X denote the set of payoff streams. For $x \in X$, let $T_x = \{t : m_x(t) \neq 0\}$ denote the *support* of x. For $x, y \in X$, $a, b \in \mathbb{R}$, define $ax + by \in X$ to be the payoff stream with the payoff function $am_x + bm_y$. Let $\phi^\tau \in X$ be the payoff stream that pays off 1 at time τ and 0 otherwise.

Let $\mathcal{D} = \{x, y \in X \times X : x \neq y\}$ denote the set of all pairs of distinct simple lotteries on S. A *binary choice rule* is a function $\rho : \mathcal{D} \to [0, 1]$ such that $\rho(x, y) + \rho(y, x) = 1$ for all $(x, y) \in \mathcal{D}$. For $x, y \in X$.

A binary choice rule has a *CPF-Complexity Representation* with a discount function $d : \mathbb{R}^+ \cup \{+\infty\} \to \mathbb{R}^+$ if d is positive, decreasing, with $d(\infty) = 0$ and

$$\rho(x, y) = F\left(\frac{U(x) - U(y)}{d_{CPF}(x, y)}\right)$$

for some continuous, increasing F symmetric around 0, where $U(x) = \sum_t d(t)m_x(t)$ and $d_{CPF}(x, y) = \sum_k |M_x(t_k) - M_y(t_k)| \cdot (d(t_k) - d(t_{k+1}))$ for any $\{t_0, t_1, ..., t_n\}$ containing $\{0, \infty\} \cup T_x \cup T_y$ and for which $t_k < t_{k+1}$ for all k, is the (generalized) CPF distance. Note that if d is differentiable, we could rewrite d_{CPF} more conveniently as $d_{CPF}(x, y) = \int_0^\infty |M_x(t) - M_y(t)| \cdot (-d'(t))\, dt$.

Let $\geq$ denote the partial order on X corresponding to temporal dominance; that is $x \geq y$ if $M_x(t) \geq M_y(t)$ for all t. Say that a comparison (x, y) is *simple* with parameters $(t_1^x, t_2^x, t_1^y, t_2^y, a, b)$ if $m_x = a\phi^{t_1^x} + b\phi^{t_2^x}$, $m_x = a\phi^{t_1^y} + b\phi^{t_2^y}$, $a, b > 0$, $t_1^x \leq t_1^y$, $t_2^x \geq t_2^y$, (t_1^x, t_1^y),(t_2^x, t_2^y) disjoint.

Finally, call a subset of times $T \subseteq [0, \infty)$ *non-null* if for all distinct $t, t' \in T$, either $\rho(\phi^t, \phi^{t'}) > 1/2$ or $\rho(\phi^{t'}, \phi^t) > 1/2$. Consider the following axioms:

T1. **Continuity:** $\rho(x, y)$ is continuous on its domain.

T2. **Linearity:** $\rho(x, y) = \rho(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)z)$ for $\lambda \in (0, 1)$.

T3. **Moderate Stochastic Transitivity:** If $\rho(x, y) \geq 1/2$, $\rho(y, z) \geq 1/2$, then $\rho(x, z) \geq \min\{\rho(x, y), \rho(y, z)\}$.

T4. **Dominance:** $x \geq y$, then $\rho(x, y) \geq \rho(w, z)$ for any $w, z \in X$.

T5. **Simple Exchangeability:** Let (x, y) be a simple comparison with parameters $(t_1^x, t_2^x, t_1^y, t_2^y, a, b)$. Then for any simple comparison (x', y') with parameters $(t_1^x, t_2^{x'}, t_1^y, t_2^{y'}, a, b')$, where

$$\rho\left(b\phi^{t_2^x} + b'\phi^{t_2^{y'}}, b\phi^{t_2^y} + b'\phi^{t_2^{x'}}\right) = 1/2$$

$\rho(x, y) = \rho(x', y')$.

T6. **Simplification**. If $\rho(x,y) \geq 1/2$: for x' with $M_{x'}(t_n) = M_y(t_n)$ and $m_{x'}(t) = m_x(t)$ for all $t < t_{n-1}$, where t_n, t_{n-1} are the most and second-most delayed dates in $T_x \cup T_y$, $\rho(x', x) \geq 1/2$ implies $\rho(x', y) \geq \rho(x,y)$.

Theorem T1. Suppose there exists a non-null set of dates of size 4. ρ satisfies T1 – T7 iff it has a CPF-Complexity Representation.

4.1 Proofs

Theorem T1. Suppose there exists a non-null set of dates of size 4. ρ satisfies T1 – T7 iff it has a CPF-Complexity Representation.

Proof. We start by observing a lemma.

Lemma T1. Suppose $U : X \to \mathbb{R}$ is linear. Then there exists $d : [0,\infty) \to \mathbb{R}$ such that $U(x) = \sum_t d(t) m_x(t)$.

Proof. Let $d : [0,\infty) \to \mathbb{R}$ satisfying $d(t) = U(\phi^t)$. Take any $x \in X$. Note that $x = \sum_{t \in T_x} m_x(t)\phi^t$, and so inductive application of linearity implies $U(x) = \sum_t d(t) m_x(t)$ as desired. □

Now to prove Theorem 4: necessity of the axioms is immediate from the definitions; we now shoe sufficiency. Sufficiency is immediate when ρ is constant, so we consider the case where ρ is not constant.

Let $\succeq$ denote the complete binary relation on X satisfying $x \succeq y$ whenever $\rho(x,y) \geq 1/2$. By Weak Stochastic Transitivity, $\succeq$ is transitive. Since ρ satisfies Continuity and Independence, by Theorem 8 in Herstein and Milnor (1953), $\succeq$ is represented by a linear $U : X \to \mathbb{R}$, and Lemma 4.1 in turn implies the existence of a $d : [0,\infty) \to \mathbb{R}$ such that $U(x) = \sum_t d(t) m_x(t)$. Dominance implies that $d(t)$ is positive and decreasing. Extend d to $[0,\infty) \cup \{+\infty\}$ by taking $d(\infty) = 0$.

Since there exists a set of non-null times of size 4, there exists $t^a, t^b, t^c, t^d \in [0,\infty)$, $t^a < t^b < t^c < t^d$, for which $d(t^a) < d(t^b) < d(t^c) < d(t^d)$. Now consider any $x, y \in X$. Let $T = \{0, t^a, t^b, t^c, t^d\} \cup T_x \cup T_y$, and enumerate $T \cup \{\infty\}$ in increasing order by $\{t_1, t_2, ..., t_n, t_{n+1}\}$. We'll make our lives easier and assume that d is strictly decreasing on T, but it should be straightforward to generalize. Let $X(T) = \{x \in X : T_x \subseteq T\}$ denote the set of payoff flows with support in T. Note that all $w \in X(T)$ corresponds to a unique $\tilde{w} \in \mathbb{R}^n$ satisfying $\tilde{w}_k = M_x(t_k)(d(t_k) - d(t_{k+1}))$. Denote by $\tilde{\rho}$ the induced preference on $\mathbb{R}^n$ satisfying $\tilde{\rho}(\tilde{x}, \tilde{y}) = \rho(x,y)$.

Claim 1. $\tilde{\rho}(\tilde{x}, \tilde{y}) \geq 1/2$ iff $\sum_k \tilde{x}_k \geq \sum_k \tilde{y}_k$. $\tilde{\rho}$ satisfies M1-M6.

Proof. Note that since $\sum_k \tilde{w}_k = \sum_t d(t) m_w(t)$ for all $w \in X(T)$, we have $\sum_k \tilde{x}_k \geq \sum_k \tilde{y}_k \iff \sum_t d(t) m_x(t) \geq \sum_t d(t) m_y(t) \iff \rho(x,y) \geq 1/2 \iff \tilde{\rho}(\tilde{x}, \tilde{y}) \geq 1/2$.

It is immediate that $\tilde{\rho}$ inherits Continuity, Linearity, and Moderate Stochastic Transitivity from ρ. Dominance follows from the fact that for all $x, y \in X(T)$, $M_x(t) \geq M_y(t)$ for all t if and only if $\tilde{x}_k \geq \tilde{y}_k$ for all k.

To see that $\tilde{\rho}$ satisfies Exchangeability, take any $\tilde{y} \in \mathbb{R}^n$ $\alpha, \gamma, \delta, \eta \in \mathbb{R}$, $i, j \in \{1, ..., n\}$ satisfying

$$\tilde{\rho}(\tilde{y} + \alpha e_i, \tilde{y} + \delta e_j) = 1/2$$
$$\tilde{\rho}(\tilde{y} + \gamma e_j, \tilde{y} + \eta e_i) = 1/2$$

By Linearity of $\tilde{\rho}$, it suffices to show that $\tilde{\rho}(\alpha e_i + \gamma e_j, 0) = \tilde{\rho}(\delta e_j + \eta e_i, 0)$. We can without loss take $i < j$. Note that by the first part of the claim, the indifference conditions imply that $\delta = \alpha$, $\eta = \gamma$. Note that if $\neg(\alpha > 0, \gamma < 0)$ or $\neg(\alpha < 0, \gamma > 0)$, Dominance implies $\tilde{\rho}(\alpha e_i + \gamma e_j, 0) = \tilde{\rho}(\delta e_j + \eta e_i, 0)$. Now consider the case where $(\alpha > 0, \gamma < 0)$ or $(\alpha < 0, \gamma > 0)$; by Linearity of $\tilde{\rho}$, we can without loss take $\alpha > 0, \gamma < 0$.

Now consider the simple comparison (x, y) where

$$m_x(t) = \begin{cases} \alpha/(d(t_i) - d(t_{i+1})) & t = t_i \\ -\gamma/(d(t_j) - d(t_{j+1})) & t = t_{j+1} \\ 0 & \text{otherwise} \end{cases} \qquad m_y(t) = \begin{cases} \alpha/(d(t_i) - d(t_{i+1})) & t = t_{i+1} \\ -\gamma/(d(t_j) - d(t_{j+1})) & t = t_j \\ 0 & \text{otherwise} \end{cases}$$

Note that $\rho(x, y) = \tilde{\rho}(\tilde{x}, \tilde{y}) = \tilde{\rho}(\alpha e_i + \gamma e_j, 0)$, where the final equality follows from linearity of $\tilde{\rho}$. Now consider the simple comparison (x^1, y^1) where

$$m_{x^1}(t) = \begin{cases} \alpha/(d(t_1) - d(t_2)) & t = t_1 \\ -\gamma/(d(t_j) - d(t_{j+1})) & t = t_{j+1} \\ 0 & \text{otherwise} \end{cases} \qquad m_{y^1}(t) = \begin{cases} \alpha/(d(t_1) - d(t_2)) & t = t_2 \\ -\gamma/(d(t_j) - d(t_{j+1})) & t = t_j \\ 0 & \text{otherwise} \end{cases}$$

Since $-\gamma/(d(t_1) - d(t_2)) \cdot (d(t_1) - d(t_2)) = -\gamma/(d(t_i) - d(t_{i+1})) \cdot (d(t_i) - d(t_{i+1}))$, Simple Exchangeability implies that $\rho(x^1, y^1) = \rho(x, y)$. Similarly, for $(x^2, y^2), (x^3, y^3), (x^4, y^4), (x^5, y^5)$ defined by

$$m_{x^2}(t) = \begin{cases} \alpha/(d(t_1) - d(t_2)) & t = t_1 \\ -\gamma/(d(t_2) - d(t_3)) & t = t_3 \\ 0 & \text{otherwise} \end{cases} \qquad m_{y^2}(t) = \begin{cases} \alpha/(d(t_1) - d(t_2)) - \gamma/(d(t_2) - d(t_3)) & t = t_2 \\ 0 & \text{otherwise} \end{cases}$$

$$m_{x^3}(t) = \begin{cases} \alpha/(d(t_{n-1}) - d(t_n)) & t = t_{n-1} \\ -\gamma/(d(t_2) - d(t_3)) & t = t_3 \\ 0 & \text{otherwise} \end{cases} \qquad m_{y^3}(t) = \begin{cases} \alpha/(d(t_{n-1}) - d(t_n)) & t = t_n \\ -\gamma/(d(t_2) - d(t_3)) & t = t_2 \\ 0 & \text{otherwise} \end{cases}$$

$$m_{x^4}(t) = \begin{cases} \alpha/(d(t_{n-1}) - d(t_n)) & t = t_{n-1} \\ -\gamma/(d(t_i) - d(t_{i+1})) & t = t_{i+1} \\ 0 & \text{otherwise} \end{cases} \qquad m_{y^4}(t) = \begin{cases} \alpha/(d(t_{n-1}) - d(t_n)) & t = t_n \\ -\gamma/(d(t_i) - d(t_{i+1})) & t = t_i \\ 0 & \text{otherwise} \end{cases}$$

$$m_{x^5}(t) = \begin{cases} \alpha/(d(t_j) - d(t_{j+1})) & t = t_j \\ -\gamma/(d(t_i) - d(t_{i+1})) & t = t_{i+1} \\ 0 & \text{otherwise} \end{cases} \qquad m_{y^5}(t) = \begin{cases} \alpha/(d(t_j) - d(t_{j+1})) & t = t_{j+1} \\ -\gamma/(d(t_i) - d(t_{i+1})) & t = t_i \\ 0 & \text{otherwise} \end{cases}$$

Simple Exchangeability implies that $\rho(x^1, x^1) = \rho(x^2, y^2)$, $\rho(x^2, x^2) = \rho(x^3, y^3)$, $\rho(x^3, x^3) = \rho(x^4, y^4)$, $\rho(x^4, x^4) = \rho(x^5, y^5)$. Note that $\rho(x^5, y^5) = \tilde{\rho}(\tilde{x}^5, \tilde{y}^5) = \tilde{\rho}(\alpha e_j + \gamma e_i, 0)$, where the last equality follows from Linearity, and so we have $\tilde{\rho}(\alpha e_i + \gamma e_j, 0) = \rho(x, y) = \rho(x^5, y^5) = \tilde{\rho}(\alpha e_j + \gamma e_i, 0)$ as desired.

Finally, to see that $\tilde{\rho}$ satisfies Simplification, take any $\tilde{x}, \tilde{y} \in \mathbb{R}^n$, with $\tilde{\rho}(\tilde{x}, \tilde{y}) \geq 1/2$, and consider $\tilde{x}'$ satisfying $\tilde{x}'_n = \tilde{y}_n$, $\tilde{x}'_k = \tilde{x}_k$ for $k \leq n-2$, and with $\tilde{\rho}(\tilde{x}', \tilde{x}) = 1/2$. By construction, this implies that $\rho(x, y) \geq 1/2$, $M_{x'}(t_n) = M_y(t_n)$, $m_{x'}(t) = m_x(t)$ for all $t < t_{n-2}$, and $\rho(x', x) = 1/2$, and so since ρ satisfies Simplification, we have $\rho(x', y) \geq \rho(x, y) \implies \tilde{\rho}(\tilde{x}', \tilde{y}) \geq \tilde{\rho}(\tilde{x}, \tilde{y})$, and $\tilde{\rho}$ satisfies simplification with respect to attributes $n-1, n$. Without loss, we can re-index attributes $n-1, n$ to $1, 2$, and so $\tilde{\rho}$ satisfies Simplification. □

Using Claim 1, Theorem 1 then implies that there exists a continuous, increasing $G : [-1, 1] \to [0, 1]$, symmetric around 0, such that for all $x, y \in X(T)$ $\tilde{x}, \tilde{y} \in \mathbb{R}^n$, we have

$$\begin{aligned} \rho(x, y) &= \tilde{\rho}(\tilde{x}, \tilde{y}) \\ &= G\left(\frac{\sum_k (\tilde{x}_k - \tilde{y}_k)}{\sum_k |\tilde{x}_k - \tilde{y}_k|}\right) \\ &= G\left(\frac{U(x) - U(y)}{d_{CPF}(x, y)}\right) \end{aligned}$$

Let $\mathcal{T} = \{T \subseteq [0, \infty) : |T| < \infty, \{0, t^a, t^b, t^c, t^d\} \subseteq T\}$. The above implies that for all $T \in \mathcal{T}$, there exists a continuous, increasing $G_T : [-1, 1] \to [0, 1]$, symmetric around 0 such that for any $x, y \in X(T)$,

$$\rho(x, y) = G_T\left(\frac{U(x) - U(y)}{d_{CPF}(x, y)}\right)$$

Since for any $x, y \in X$, there exists some $T \in \mathcal{T}$ such that $x, y \in X(T)$, all that remains to show that All that remains is to show that $G_T = G_{T'}$ for any $T, T' \in \mathcal{T}$. To see this, fix any $T, T' \in \mathcal{T}$, and consider $x, y \in X$ with

$$m_x(t) = \begin{cases} \alpha/(d(t_a) - d(t_b)) & t = t_a \\ \gamma/(d(t_b) - d(t_c)) & t = t_c \\ 0 & \text{otherwise} \end{cases} \qquad m_y(t) = \begin{cases} \alpha/(d(t_a) - d(t_b)) + \gamma/(d(t_b) - d(t_c)) & t = t_b \\ 0 & \text{otherwise} \end{cases}$$

for some $\alpha \geq 0$, $\gamma \geq 0$. Note that x, y belong to both T and T', and so we have

$$\rho(x, y) = G_T\left(\frac{U(x) - U(y)}{d_{CPF}(x, y)}\right) = G_{T'}\left(\frac{U(x) - U(y)}{d_{CPF}(x, y)}\right)$$

and since $\frac{U(x)-U(y)}{d_{CPF}(x,y)} = \frac{\alpha-\gamma}{\alpha+\gamma}$, for any $r \in [-1, 1]$ we can choose $\alpha, \gamma \geq 0$ such that $\frac{U(x)-U(y)}{d_{CPF}(x,y)} = r$, we must have $G_T = G_{T'}$. □