Research Report

Jeff Severino
University of Toledo
Toledo, OH 43606
email: jeffrey.severino@rockets.utoledo.edu

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1 Current Research Direction

The current research direction is to create a code wrapper for SWIRL such that it will calculate the L2 norm of the axial wavenumber result and compare that value to the L2 norm of the analytical solutions that exist. The goal is to explore different mode filtering techniques and compare them to the DRP filter proposed by Tam. Tam and Case have shown that certain inital value problems admit themselves to instabilites which will yield spurrious modes which have no physical significance. These instabilities can be omitted by reformulating the problem with a Laplace-Fourier transform. The question is, can we identify these modes and remove them with some other technique?

2 Research Performed This Week

The first case that we would like to study is the mean flow case. The Uniform flow case can be used as a starting point and the mean flow case can easily be obtained with slight modification. Taking equation 2.28 in Kousen's paper and setting the right hand side to zero , (no velocity). We get:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 P}{\partial \theta^2} - \frac{1}{a_o}\frac{\partial^2 P}{\partial t^2} = 0 \tag{1}$$

For convenience, the partial derivatives will be expressed with subscripts. Note, I need to check the Laplacian in cylindrical coordinates.

$$P_{xx} + P_{rr} + \frac{1}{r}P_r + \frac{1}{r^2}P_{\theta\theta} - \frac{1}{a_o}P_{tt} = 0 \tag{2}$$

The method of separation of variables is a widely used approach that is efficacious is solving various partial differential equation. The method entails assuming a solution which is a product of four functions. $X(x)$ is only a function of space, $R_f(r)$ is a function of the radius, $\Theta(\theta)$ is a function of the angle, and $T(t)$ is a function of time. Each function has an assumed solution, Euler's identity. This ansant works well for linear Partial Differential Equations and boundary conditions.

$$p(x, r, \theta, t) = X(x)R_f(r)\Theta(\theta)T(t) \tag{3}$$

$$\begin{aligned} X(x) &= A_1 e^{ik_x x} + B_1 e^{-ik_x x} \\ \Theta(\theta) &= A_2 e^{ik_\theta \theta} + B_1 e^{-ik_\theta \theta} \\ T(t) &= A_3 e^{i\omega t} + B_3 e^{-i\omega t} \end{aligned} \tag{4}$$

If we divide by the assumed solution, $p(x, r, \theta, t)$, and rewrite the wave equation in terms of $X(x)$, $R_f(r)$, $\Theta(\theta)$, $T(t)$, we obtain the wave equation is a separated manner. Dots will be used for spatial derivatives where apostrophes will be used for time derivatives.

$$\frac{X''}{X} = -\frac{1}{R_f}\left(R_f'' + \frac{1}{r}R_f'\right) = -\frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{1}{a_o}\frac{\ddot{T}}{T} \tag{5}$$

The four functions within the equation have been rearranged such that each term is only a function of r, x, t, theta respectively. If we fix any term, (for instance the time term) and vary the others, then the variable that is being fixed must be a constant. This constant, λ is often referred to as an eigenvalue, which is the solution to the eigenfunction. The only non trivial solution of the wave equation (eigenfunctions) correspond to when λ is negative. This eigenvalue corresponds to the wave number in the exponent of the assumed solutions.

$$\frac{X''}{X} = \lambda \Rightarrow X'' - k_x^2 X = 0 \tag{6}$$

$$\frac{\Theta''}{\Theta} = \lambda \Rightarrow \Theta'' - k_\theta^2 \Theta = 0 \tag{7}$$

$$\frac{1}{a_o^2}\frac{\ddot{T}}{T} = -w^2 \Rightarrow \ddot{T} - k^2 T = 0 \tag{8}$$

We have now reduced the Wave Equation into a system of ordinary differential equation's. This type of system is referred to as a Sturm-Liouville Boundary Value Problem . Such a system is comprised of 2nd order linear homogeneous differential equations of the same form.

The fluid at the wall must satisfy a no slip condition . As a result, the normal velocity components of the acoustic pressure at the wall are equal to 0.

$$P_r = p(x, R, \theta, t) = 0 \tag{9}$$

Because of θ's periodic nature, k_θ is only equal to an integer. (k_θ = m). There for the radial wavenumber simplifies to the azimuthal mode order, m.

$$-k_x^2 + \frac{1}{R_f}\Big(R''_f + \frac{1}{r}R'_f\Big) - \frac{m^2}{r^2} + k^2 = 0 \tag{10}$$

The remaining radial terms R_f are manipulated to represent a form that is similar to Bessel's Equation. The general solution to R_f can be expressed using Bessel functions. This constant will be k_r

$$\begin{gathered} \tfrac{1}{R_f}\Big(\ddot{R}_f + \tfrac{1}{r}\dot{R}_f\Big) - \tfrac{m^2}{r^2} = k_r \\ \Big(\ddot{R}_f + \tfrac{1}{r}\dot{R}_f\Big) + \Big(k_r - \tfrac{m^2}{r^2}\Big)R_f(r) = 0 \end{gathered} \tag{11}$$

Let's redefine x in terms of the radial wavenumber and radius,

$$\tag{12}$$

If we substitute these variables for R_f,

$$R'{}_f = \frac{dR_f}{dr} = \frac{dR_f}{dx}\frac{dx}{dr} = \frac{dR_f}{dx}k_r \tag{13}$$

Similarly,

$$R''{}_f = \frac{d^2R_f}{dr^2} = \frac{d^2R_f}{dx^2}\frac{d^2x}{dr^2} = \frac{d^2R_f}{dx^2}k_r^2 \tag{14}$$

By re-substituting back into (11),

$$\Big(\frac{d^2R_f}{dx^2}k_r^2 + \frac{1}{r}\frac{dR_f}{dx}k_r\Big) + \Big(k_r - \frac{m^2}{r^2}\Big)R_f(x) = 0 \tag{15}$$

canceling k_r and multiplying by x^2 yields,

$$\Big(x^2\frac{d^2R_f}{dx^2} + x\frac{dR_f}{dx}\Big) + x^2\Big(1 - \frac{m^2}{r^2}\Big)R_f(x) = 0 \tag{16}$$

if we compare to:

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \tag{17}$$

where

$$y(x) = AJ_m(x) + BY_m(x) \tag{18}$$

J_m is the Bessel function of the first kind whereas Y_m is the Bessel function of the second kind. This is also referred to as the Neumann function. The subscript m refers to the order of the function. Bessel him self proved that there exists infinite solutions for non negative integers m . The coefficients A and B are determined after boundary conditions have been applied. As we switch $R_f(x)$ with $R_f(r)$, the radial wavenumber appears within the Bessel function.

$$R_f(r) = AJ_m(k_r r) + BY_m(k_r r) \tag{19}$$

Since the solution must be finite as x approaches zero, it is observed from

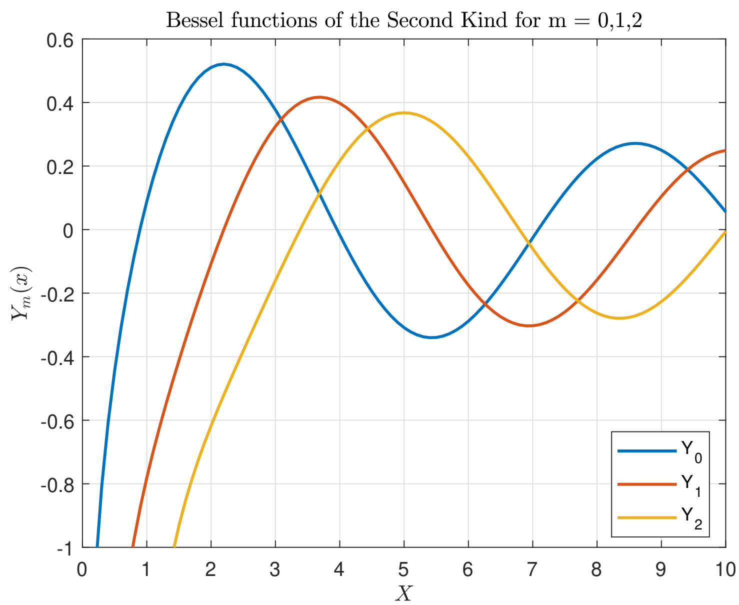
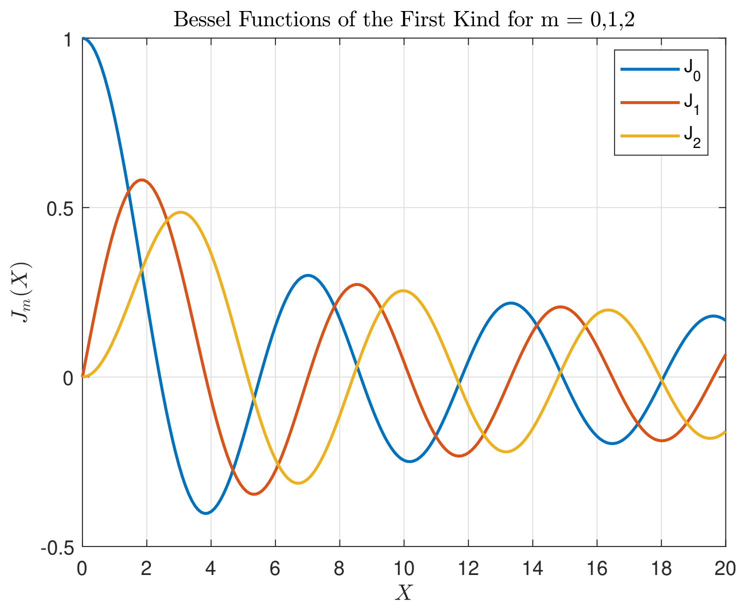


Figure **??** that Y_m approaches infinity. Since this would yield in a trivial solution, the coefficient $B = 0$. Which reduces the Bessel function to:

$$R_f(r) = AJ_m(k_r r) \tag{20}$$

Using the hard wall boundary condition at the wall, $P_r = 0$, we have the condition that $AJ_0(k_r r) = 0$. This requires that the Bessel function be zero in order for the cylindrical wave equation to be true [**?**]. The terms inside of the Bessel function would then correspond to values along the x axis which satisfy our equation. The zeros of the Bessel function are then established as $\alpha_{m,n}$ of m^{th} order and n^{th} zero index. [6 7].

$$\begin{gathered} P_r = p(x, R, \theta, t) = X\Theta T A J'_m(k_r R) = 0 \\ J'_m(k_r R) = 0 \\ k_r R = \alpha_{m,n} \\ k_r = \frac{\alpha_{m,n}}{R} \end{gathered} \tag{21}$$



By solving for the axial wave number we can begin to establish the cut-on condition. In order for acoustic propagation to occur in the duct, $k_{x,mn}$ must be real. Consequently a m, n mode will propagate if the wavenumber is larger than the radial wave number. At low frequencies, the axial wave number will be equal to the temporal wave number. Values of $\alpha_{m,n}$ are often tabulated. Various softwares such as Mathmatica have bessel zero calculators or tables already ready to pull from.

$$k_{x,mn}^2 = \pm\sqrt{k^2 - k_{r,mn}^2} \tag{22}$$

$$\omega_{m,n}^{cutoff} = \frac{a_o \alpha_{m,n}}{R} \tag{23}$$

The final eigenfunction for the duct modes:

$$p_{mn} = J_m(\frac{\alpha_{m,n} r}{R}) e^{i(k_{m,n}x + m\theta - \omega t)} \tag{24}$$

$$p_\omega = \sum_{m=-\infty}^{m=\infty} \sum_{n=0}^{n=\infty} p_{mn} \tag{25}$$

At low frequencies below the cutoff, only the fundamental mode will propagate

$$p_{0,0} = e^{kx - \omega t} \tag{26}$$

3 Issues and Concerns

The right hand side has to be included in this procedure for the axial wavenumbers to be compared. In other words, we have to account for mean flow, which is Equation 2.30 in Kousen's paper.

An issue that I have not resolved yet is to increase the precision of the tabulated values of $\alpha_{m,n}$ so that they are double precision. Mathmatica and Maple are rational languages that allows the user to dictate the number of digits desired. I recall that double precision is 12 decimal places but I will have to confirm that SWIRL is using double precision and that these are the correct number of decimal places for this precision.

4 Planned Research

The analytical solution for Mean flow will be obtained. The study shown here provides the basis as to why the Bessel functions are needed to obtain the analytical mode shape solution. This also serves as a good starting basis for obtaining the L2 Norm of an analytical solution. Ali Amr has shown this Bessel function dependency for mean flow, however the axial wavenumbers will then depend on the Mach number. I have done the work, but I need to reverify this by comparing against Kousen's analytical work.

Case also outlines a Laplace-Fourier approach that I'd like to try for couette flow but I did not get too far on that this week