Starting with equation 2.28 (Wave Equation) in Kousen's paper,

$$\frac{1}{A^2}\frac{D^2\tilde{p}}{Dt^2} - \nabla^2\tilde{p} = 2\bar{\rho}\frac{dV_x}{dx}\frac{\partial\tilde{v}_r}{\partial x} \tag{1}$$

lets look at the no flow case. In the case of sheared flow, $dV_x/dx = 0$ the right hand side will be zero

$$\frac{1}{A^2}\left(\frac{\partial^2\tilde{p}}{\partial t^2} + \vec{V}\cdot\vec{\nabla}(\tilde{p})\right) - \nabla^2\tilde{p} = 0$$

Substituting the definitions for ∇ and ∇^2 in cylindrical coordinates gives,

$$\frac{1}{A^2}\left(\frac{\partial^2\tilde{p}}{\partial t^2} + \vec{V}\cdot\left(\frac{\partial\tilde{p}}{\partial t} + \frac{1}{\tilde{r}}\frac{\partial\tilde{p}}{\partial\tilde{r}} + \frac{\partial\tilde{p}}{\partial\theta} + \frac{\partial\tilde{p}}{\partial x}\right)\right) - \left(\frac{\partial^2\tilde{p}}{\partial t^2} + \frac{1}{\tilde{r}}\frac{\partial\tilde{p}}{\partial r} + \frac{1}{\tilde{r}^2}\frac{\partial^2\tilde{p}}{\partial\theta^2} + \frac{\partial^2\tilde{p}}{\partial x^2}\right) = 0$$

Setting $\vec{V} = 0$,

$$\frac{1}{A^2}\left(\frac{\partial^2\tilde{p}}{\partial t^2}\right) - \left(\frac{\partial^2\tilde{p}}{\partial t^2} + \frac{1}{\tilde{r}}\frac{\partial\tilde{p}}{\partial r} + \frac{1}{\tilde{r}^2}\frac{\partial^2\tilde{p}}{\partial\theta^2} + \frac{\partial^2\tilde{p}}{\partial x^2}\right) = 0$$

Recall, $\tilde{p} = p/\bar{\rho}A^2$. To dimensionalize the equation, this is substituted and both sides are multiplied by $\bar{\rho}A^2$,

$$\frac{1}{A^2}\left(\frac{\partial^2 p}{\partial t^2}\right) - \left(\frac{\partial^2 p}{\partial t^2} + \frac{1}{\tilde{r}}\frac{\partial p}{\partial r} + \frac{1}{\tilde{r}^2}\frac{\partial^2 p}{\partial\theta^2} + \frac{\partial^2 p}{\partial x^2}\right) = 0$$

The process of separation of variables(seperation indeterminatarum) was first written and formalized by John Bernoulli in a letter to Leibniz. The method of separation of variables requires an assumed solution as well as initial and boundary conditions. For a partial differential equation, the assumed solution can be a linear combination of solutions to a system of ordinary differential equations that comprises the partial differential equation. Since p is a function of four variables, the solution is assumed to be a linear combination of four solutions. Each solution is assumed to be Euler's identity, a common ansant for linear partial differential equations and boundary conditions.

Defining,

$$p(x, r, \theta, t) = X(x)R(r)\Theta(\theta)T(t) \tag{2}$$

where,

$$X(x) = A_1 e^{ik_x x} + B_1 e^{-ik_x x}$$
$$\Theta(\theta) = A_2 e^{ik_\theta\theta} + B_2 e^{-ik_\theta\theta}$$
$$T(t) = A_3 e^{i\omega t} + B_3 e^{-i\omega t}$$

The next step is to rewrite the wave equation in terms of X, R, Θ, and T. To further simplify the result, each term is divided by p. Before the substitution, the derivatives of the assumed solutions need to be evaluated.

Temporal Derivatives

$$\begin{aligned}\frac{\partial p}{\partial t} &= \frac{\partial}{\partial t}\left(XR\Theta T\right)\\ &= XR\Theta\frac{\partial T}{\partial t}\end{aligned}$$

$$\begin{aligned}\frac{1}{p}\frac{\partial p}{\partial t} &= \frac{1}{XR\Theta T}\left(XR\Theta\frac{\partial T}{\partial t}\right)\\ &= \frac{1}{T}\frac{\partial T}{\partial t}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 p}{\partial t^2} &= \frac{\partial^2}{\partial t^2}\left(XR\Theta T\right)\\ &= XR\Theta\frac{\partial^2 T}{\partial t^2}\end{aligned}$$

$$\begin{aligned}\frac{1}{p}\frac{\partial^2 p}{\partial t^2} &= \frac{1}{XR\Theta T}\left(XR\Theta\frac{\partial^2 T}{\partial t^2}\right)\\ &= \frac{1}{T}\frac{\partial^2 T}{\partial t^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{\partial}{\partial t}\left(A_3e^{i\omega t} + B_3e^{-i\omega t}\right)\\ &= \frac{\partial}{\partial t}\left(A_3e^{i\omega t}\right) + \frac{\partial}{\partial t}\left(B_3e^{-i\omega t}\right)\\ &= i\omega A_3e^{i\omega t} - i\omega B_3e^{i\omega t}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 T}{\partial t^2} &= \frac{\partial^2}{\partial t^2}\left(i\omega A_3e^{i\omega t} + i\omega B_3e^{-i\omega t}\right)\\ &= \frac{\partial^2}{\partial t^2}\left(i\omega A_3e^{i\omega t}\right) + \frac{\partial^2}{\partial t^2}\left(-i\omega B_3e^{-i\omega t}\right)\\ &= (i\omega)^2A_3e^{i\omega t} - (i\omega)^2B_3e^{i\omega t}\end{aligned}$$

$$\begin{aligned}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} &= (i\omega)^2\\ &= -\omega^2\end{aligned}$$

Radial Derivatives

$$\begin{aligned}\frac{\partial p}{\partial r} &= \frac{\partial}{\partial r}\left(XR\Theta T\right)\\ &= X\Theta T\frac{\partial R}{\partial r}\end{aligned}$$

$$
\begin{aligned}
\frac{1}{p}\frac{\partial p}{\partial r} &= \frac{1}{XR\Theta T}\left(X\Theta T\frac{\partial R}{\partial r}\right) \\
&= \frac{1}{R}\frac{\partial R}{\partial r}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 p}{\partial r^2} &= \frac{\partial^2}{\partial r^2}\left(XR\Theta T\right) \\
&= X\Theta T\frac{\partial^2 R}{\partial r^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{p}\frac{\partial^2 p}{\partial r^2} &= \frac{1}{XR\Theta T}\left(X\Theta T\frac{\partial^2 R}{\partial r^2}\right) \\
&= \frac{1}{R}\frac{\partial^2 R}{\partial r^2}
\end{aligned}
$$

The radial derivatives will be revisited once the remaining derivatives are evaluated,

Tangential Derivatives

$$
\begin{aligned}
\frac{\partial p}{\partial \theta} &= \frac{\partial}{\partial t}\left(XR\Theta T\right) \\
&= XRT\frac{\partial \Theta}{\partial \theta}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{p}\frac{\partial p}{\partial \theta} &= \frac{1}{XR\Theta T}\left(XR\Theta\frac{\partial T}{\partial \theta}\right) \\
&= \frac{1}{\Theta}\frac{\partial \Theta}{\partial \theta}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 p}{\partial \theta^2} &= \frac{\partial^2}{\partial \theta^2}\left(XR\Theta T\right) \\
&= XRT\frac{\partial^2 \Theta}{\partial \theta^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{p}\frac{\partial^2 p}{\partial \theta^2} &= \frac{1}{XR\Theta T}\left(XRT\frac{\partial^2 \Theta}{\partial \theta^2}\right) \\
&= \frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \Theta}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(A_2 e^{ik_\theta \theta} + B_2 e^{-ik_\theta \theta}\right) \\
&= \frac{\partial}{\partial \theta}\left(A_2 e^{ik_\theta \theta}\right) + \frac{\partial}{\partial \theta}\left(B_2 e^{-ik_\theta \theta}\right) \\
&= ik_\theta A_2 e^{ik_\theta \theta} - ik_\theta B_2 e^{ik_\theta \theta}
\end{aligned}
$$

$$\begin{aligned}\frac{\partial^2 \Theta}{\partial \theta^2} &= \frac{\partial^2}{\partial \theta^2}\left(ik_\theta A_2 e^{ik_\theta \theta} - ik_\theta B_2 e^{ik_\theta \theta}\right)\\ &= \frac{\partial^2}{\partial \theta^2}\left(ik_\theta A_2 e^{ik_\theta \theta}\right) + \frac{\partial^2}{\partial \theta^2}\left(-ik_\theta B_2 e^{-ik_\theta \theta}\right)\\ &= (ik_\theta)^2 A_2 e^{ik_\theta \theta} - (ik_\theta)^2 B_2 e^{ik_\theta \theta}\end{aligned}$$

$$\begin{aligned}\frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} &= (ik_\theta)^2\\ &= -k_\theta^2\end{aligned}$$

Axial Derivatives

$$\begin{aligned}\frac{\partial p}{\partial x} &= \frac{\partial}{\partial x}(XR\Theta T)\\ &= R\Theta T\frac{\partial X}{\partial x}\end{aligned}$$

$$\begin{aligned}\frac{1}{p}\frac{\partial p}{\partial x} &= \frac{1}{XR\Theta T}\left(R\Theta\frac{\partial X}{\partial x}\right)\\ &= \frac{1}{X}\frac{\partial X}{\partial x}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 p}{\partial x^2} &= \frac{\partial^2}{\partial x^2}(XR\Theta T)\\ &= R\Theta T\frac{\partial^2 X}{\partial x^2}\end{aligned}$$

$$\begin{aligned}\frac{1}{p}\frac{\partial^2 p}{\partial x^2} &= \frac{1}{XR\Theta T}\left(R\Theta T\frac{\partial^2 X}{\partial x^2}\right)\\ &= \frac{1}{X}\frac{\partial^2 X}{\partial x^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial X}{\partial x} &= \frac{\partial}{\partial t}\left(A_3 e^{ik_x t} + B_3 e^{-i\omega t}\right)\\ &= \frac{\partial}{\partial t}\left(A_1 e^{ik_x x}\right) + \frac{\partial}{\partial t}\left(B_1 e^{-ik_x x}\right)\\ &= ik_x A_1 e^{ik_x x} - ik_x B_1 e^{ik_x x}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 X}{\partial x^2} &= \frac{\partial^2}{\partial x^2}\left(ik_x A_1 e^{ik_x x} + ik_x B_1 e^{-ik_x x}\right)\\ &= \frac{\partial^2}{\partial x^2}\left(ik_x A_1 e^{ik_x x}\right) + \frac{\partial^2}{\partial x^2}\left(-ik_x B_1 e^{-ik_x x}\right)\\ &= (ik_x)^2 A_1 e^{ik_x x} - (ik_x)^2 B_1 e^{ik_x x}\end{aligned}$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = (ik_x)^2$$
$$= -k_x^2$$

Substituting this back into the wave equation yields ,

$$\frac{1}{A^2}\left(\frac{\partial^2 p}{\partial t^2}\right) = \left(\frac{\partial^2 p}{\partial t^2} + \frac{1}{\tilde{r}}\frac{\partial p}{\partial r} + \frac{1}{\tilde{r}^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial x^2}\right)$$

$$\frac{1}{A^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{1}{R}\frac{\partial R}{\partial r} + \frac{1}{r^2}\frac{1}{\Theta}\frac{\partial \Theta}{\partial \theta} + \frac{1}{X}\frac{\partial^2 X}{\partial x^2} \tag{3}$$

Notice that each term is only a function of its associated independent variable. So, if we vary the time, only the term on the left-hand side can vary. However, since none of the terms on the right-hand side depend on time, that means the right-hand side cannot vary, which means that the ratio of time with its second derivative is independent of time. The practical upshot is that each of these terms is constant, which has been shown. The wave numbers are the *separation constants* that allow the PDE to be split into four separate ODE's. Substituting the separation constants into Equation (3) gives,

$$-\frac{\omega^2}{A^2} = \frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) - \frac{k_\theta^2}{r^2} - k_x^2 \tag{4}$$

Note that the dispersion relation states $\omega = kA$

$$\frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) - \frac{k_\theta^2}{r^2} - k_x^2 + k^2 = 0 \tag{5}$$

The remaining terms are manipulated to follow the same form as *Bessel's Differntial Equation* ,

$$x^2\frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \tag{6}$$

The general solution to Bessel's differential equation is a linear combination of the Bessel functions of the first kind, $J_n(x)$ and of the second kind, $Y_n(x)$ [1]. The subscript n refers to the order of Bessel's equation.

$$y(x) = AJ_n(x) + BY_n(x) \tag{7}$$

By rearranging Equation (5), a comparison can be made to Equation (6) to show that the two equations are of the same form.

The first step is to revisit the radial derivatives that have not been addressed. As was done for the other derivative terms, the radial derivatives will also be set equal to a separation constant, $-k_r^2$.

$$\underbrace{\frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) - \frac{k_\theta^2}{r^2}}_{-k_r^2} - k_x^2 + k^2 = 0 \tag{8}$$

The reader may be curious as to why the tangential separation constant k_θ is included within the definition of the radial separation constant.

Recall the ODE for the tangential direction,

$$\frac{\partial\Theta}{\partial\theta}\frac{1}{\Theta} = -k_\theta^2$$
$$\frac{\partial\Theta}{\partial\theta}\frac{1}{\Theta} + \Theta k_\theta^2 = 0$$

where the solution is more or less,

$$\Theta(\theta) = e^{ik_\theta\theta}$$

In order to have non trivial, sensible solutions, the value of $\Theta(0)$ and $\Theta(2\pi)$ need to be the same, and this needs to be true for any multiple of 2π for a fixed r. Taking Θ to be one, a unit circle, it can be shown that the domain is only going to be an integer multiple. Therefore, there is an implied periodic azimuthal boundary condition, i.e. $0 < \theta \leq 2\pi$ and $k_\theta = m$.

Continuing with the radial derivatives...

$$-k_r^2 = \frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) - \frac{m^2}{r^2}$$

To further simplify, the chain rule is used to do a change of variables, $x = k_r r$

$$\begin{aligned}\frac{\partial R}{\partial r} &= \frac{dR}{dx}\frac{dx}{dr}\\ &= \frac{dR}{dx}\frac{d}{dr}(k_r r)\\ &= \frac{dR}{dx}k_r\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 R}{\partial r^2} &= \frac{d^2R}{dx^2}\left(\frac{dx}{dr}\right)^2 + \frac{dR}{dr}\frac{d^2x}{dr^2}\\ &= \frac{d^2R}{dx^2}\frac{d}{dr}k_r^2 + k_r\frac{d^2r}{dr^2}\\ &= \frac{d^2R}{dx^2}\frac{d}{dr}k_r^2\end{aligned}$$

Substituting this into Equation (5),

$$\left(\frac{d^2R}{dx^2}k_r^2 + \frac{1}{r}\frac{d^2R}{dx^2}k_r\right) + \left(k_r^2 - \frac{m^2}{r^2}\right)R \tag{9}$$

Dividing Equation 8 by k_r^2,

$$\left(\frac{d^2R}{dx^2} + \frac{1}{k_r r}\frac{d^2R}{dx^2}\right) + \left(1 - \frac{m^2}{k_r^2 r^2}\right)R \tag{10}$$

$$\left(\frac{d^2R}{dx^2} + \frac{1}{x^2}\frac{d^2R}{dx^2}\right) + \left(1 - \frac{m^2}{x^2}\right)R \tag{11}$$

Multiplying Equation (10) by x^2 gives,

$$\frac{d^2R}{dr^2}x^2 + \frac{dR}{dr}x + \left(x^2 - m^2\right)R \tag{12}$$

which matches the form of Bessel's equation

In summary, the wave equation for no flow in a hollow duct with hard walls is obtained from Equation (**??**).

$$k^2 = k_r^2 + k_x^2 \tag{13}$$

Hard Wall boundary condition

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left(X\Theta TR\right) \tag{14}$$

Bibliography

[1] Eric W. Weisstein. "bessel differential equation." from mathworld–a wolfram web resource.