Transformation from Cartesian to Cylindrical Coordinates

Ray Hixon
University of Toledo
email: Duane.Hixon@utoledo.edu

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1 Coordinate System Definitions

The definition of the cylindrical coordinate system is:

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

Similarly,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

Then,

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial f}{\partial \theta} \end{aligned}$$

Finding the definitions:

$$\begin{aligned} \frac{d}{dx}\left(\tan^{-1} u\right) &= \frac{1}{1+u^2}\frac{du}{dx} \\ \frac{d}{dx}\left(\sqrt{u}\right) &= \frac{1}{2\sqrt{u}}\frac{du}{dx} \end{aligned}$$

gives:

$$\begin{aligned} \frac{d}{dx}\left(\tan^{-1}\frac{y}{x}\right) &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) \\ &= \frac{-x^2}{x^2+y^2}\left(\frac{y}{x^2}\right) \\ &= \frac{-y}{x^2+y^2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{-y}{r^2} \\
&= \frac{-\sin\theta}{r} \\
\frac{d}{dy}\left(\tan^{-1}\frac{y}{x}\right) &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) \\
&= \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right) \\
&= \frac{x}{x^2+y^2} \\
&= \frac{x}{r^2} \\
&= \frac{\cos\theta}{r} \\
\frac{d}{dx}\left(\sqrt{x^2+y^2}\right) &= \frac{1}{2\sqrt{x^2+y^2}}(2x) \\
&= \frac{x}{r} \\
&= \cos\theta \\
\frac{d}{dy}\left(\sqrt{x^2+y^2}\right) &= \frac{1}{2\sqrt{x^2+y^2}}(2y) \\
&= \frac{y}{r} \\
&= \sin\theta
\end{aligned}
$$

and:

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta} \\
\frac{\partial f}{\partial y} &= \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}
\end{aligned}
$$

2 Nabla operator

In Cartesian coordinates, the $\vec{\nabla}$ operator is defined:

$$
\vec{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z
$$

Noting that:

$$
\begin{aligned}
\vec{e}_x &= \cos\theta \vec{e}_r - \sin\theta \vec{e}_\theta \\
\vec{e}_y &= \sin\theta \vec{e}_r + \cos\theta \vec{e}_\theta
\end{aligned}
$$

Substituting in gives:

$$
\begin{aligned}
\vec{\nabla} &= \begin{pmatrix} \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)(\cos\theta \vec{e}_r - \sin\theta \vec{e}_\theta) \\ + \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)(\sin\theta \vec{e}_r + \cos\theta \vec{e}_\theta) \\ + \frac{\partial}{\partial z}\vec{e}_z \end{pmatrix} \\
&= \begin{pmatrix} \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)(\cos\theta \vec{e}_r - \sin\theta \vec{e}_\theta) \\ + \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)(\sin\theta \vec{e}_r + \cos\theta \vec{e}_\theta) \\ + \frac{\partial}{\partial z}\vec{e}_z \end{pmatrix} \\
&= \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{e}_\theta + \frac{\partial}{\partial z}\vec{e}_z
\end{aligned}
$$

2.1 Useful definitions

Given:

$$
\begin{aligned}
u &= v_r \cos\theta - v_\theta \sin\theta \\
v &= v_r \sin\theta + v_\theta \cos\theta \\
w &= w
\end{aligned}
$$

Then,

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \\
&= \cos\theta \frac{\partial}{\partial r}(v_r \cos\theta - v_\theta \sin\theta) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}(v_r \cos\theta - v_\theta \sin\theta) \\
&= \sin^2\theta \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \sin\theta\cos\theta\left(-\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\right) + \cos^2\theta \frac{\partial v_r}{\partial r} \\
\frac{\partial u}{\partial y} &= \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}
\end{aligned}
$$

$$
\begin{aligned}
&= \sin\theta \frac{\partial}{\partial r}\left(v_r\cos\theta - v_\theta\sin\theta\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(v_r\cos\theta - v_\theta\sin\theta\right) \\
&= \sin^2\theta\left(-\frac{\partial v_\theta}{\partial r}\right) + \sin\theta\cos\theta\left(\frac{\partial v_r}{\partial r} - \frac{v_r}{r} - \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}\right) + \cos^2\theta\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r}\right) \\
\frac{\partial u}{\partial z} &= \frac{\partial}{\partial z}\left(v_r\cos\theta - v_\theta\sin\theta\right) \\
&= \cos\theta\frac{\partial v_r}{\partial z} - \sin\theta\frac{\partial v_\theta}{\partial z} \\
\frac{\partial v}{\partial x} &= \cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta} \\
&= \cos\theta\frac{\partial}{\partial r}\left(v_r\sin\theta + v_\theta\cos\theta\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(v_r\sin\theta + v_\theta\cos\theta\right) \\
&= \sin^2\theta\left(-\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{v_\theta}{r}\right) + \sin\theta\cos\theta\left(\frac{\partial v_r}{\partial r} - \frac{v_r}{r} - \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}\right) + \cos^2\theta\left(\frac{\partial v_\theta}{\partial r}\right) \\
\frac{\partial v}{\partial y} &= \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial\theta} \\
&= \sin\theta\frac{\partial}{\partial r}\left(v_r\sin\theta + v_\theta\cos\theta\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(v_r\sin\theta + v_\theta\cos\theta\right) \\
&= \sin^2\theta\left(\frac{\partial v_r}{\partial r}\right) + \sin\theta\cos\theta\left(\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r}\right) + \cos^2\theta\left(\frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}\right) \\
\frac{\partial v}{\partial z} &= \frac{\partial}{\partial z}\left(v_r\sin\theta + v_\theta\cos\theta\right) \\
&= \sin\theta\frac{\partial v_r}{\partial z} + \cos\theta\frac{\partial v_\theta}{\partial z} \\
\frac{\partial w}{\partial x} &= \cos\theta\frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial\theta} \\
\frac{\partial w}{\partial y} &= \sin\theta\frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial\theta} \\
\frac{\partial w}{\partial z} &= \frac{\partial w}{\partial z}
\end{aligned}
$$

2.2 Divergence

In Cartesian,

$$
\vec{\nabla}\cdot\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}
$$

$$
\begin{aligned}
&= \left(\begin{array}{c} \sin^2\theta \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \sin\theta\cos\theta \left(-\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\right) + \cos^2\theta \frac{\partial v_r}{\partial r} \\ + \sin^2\theta \left(\frac{\partial v_r}{\partial r}\right) + \sin\theta\cos\theta \left(\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}\right) + \cos^2\theta \left(\frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}\right) \\ + \frac{\partial w}{\partial z} \end{array} \right) \\
&= \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial w}{\partial z} \\
&= \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial w}{\partial z}
\end{aligned}
$$

2.3 Curl

In Cartesian,

$$
\begin{aligned}
\vec{\nabla} \times \vec{V} &= \left(\begin{array}{c} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \vec{e}_x \\ + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \vec{e}_y \\ + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \vec{e}_z \end{array} \right) \\
&= \left(\begin{array}{c} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) (\cos\theta \vec{e}_r - \sin\theta \vec{e}_\theta) \\ + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) (\sin\theta \vec{e}_r + \cos\theta \vec{e}_\theta) \\ + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \vec{e}_z \end{array} \right) \\
&= \left(\begin{array}{c} \left(\frac{\partial w}{\partial y}\cos\theta - \frac{\partial v}{\partial z}\cos\theta + \frac{\partial u}{\partial z}\sin\theta - \frac{\partial w}{\partial x}\sin\theta\right) \vec{e}_r \\ + \left(-\frac{\partial w}{\partial y}\sin\theta + \frac{\partial v}{\partial z}\sin\theta + \frac{\partial u}{\partial z}\cos\theta - \frac{\partial w}{\partial x}\cos\theta\right) \vec{e}_\theta \\ + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \vec{e}_z \end{array} \right) \\
&= \left(\begin{array}{c} \left(\begin{array}{c} \left(\sin\theta \frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta}\right)\cos\theta \\ - \left(\sin\theta \frac{\partial v_r}{\partial z} + \cos\theta \frac{\partial v_\theta}{\partial z}\right)\cos\theta \\ + \left(\cos\theta \frac{\partial v_r}{\partial z} - \sin\theta \frac{\partial v_\theta}{\partial z}\right)\sin\theta \\ - \left(\cos\theta \frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}\right)\sin\theta \end{array} \right) \vec{e}_r \\ + \left(\begin{array}{c} - \left(\sin\theta \frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta}\right)\sin\theta \\ + \left(\sin\theta \frac{\partial v_r}{\partial z} + \cos\theta \frac{\partial v_\theta}{\partial z}\right)\sin\theta \\ + \left(\cos\theta \frac{\partial v_r}{\partial z} - \sin\theta \frac{\partial v_\theta}{\partial z}\right)\cos\theta \\ - \left(\cos\theta \frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}\right)\cos\theta \end{array} \right) \vec{e}_\theta \\ + \left(\begin{array}{c} \left(\sin^2\theta \left(-\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\theta}{r}\right) + \sin\theta\cos\theta \left(\frac{\partial v_r}{\partial r} - \frac{v_r}{r} - \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}\right) + \cos^2\theta \left(\frac{\partial v_\theta}{\partial r}\right)\right) \\ - \left(\sin^2\theta \left(-\frac{\partial v_\theta}{\partial r}\right) + \sin\theta\cos\theta \left(\frac{\partial v_r}{\partial r} - \frac{v_r}{r} - \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}\right) + \cos^2\theta \left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}\right)\right) \end{array} \right) \vec{e}_z \end{array} \right)
\end{aligned}
$$

$$= \left(\begin{array}{c} \left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\vec{e}_r \\ +\left(\frac{\partial v_r}{\partial z} - \frac{\partial w}{\partial r}\right)\vec{e}_\theta \\ +\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r v_\theta\right) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\vec{e}_z \end{array} \right)$$