

A Thesis

entitled

Verification and Validation Method for
an Acoustic Mode Prediction Code for Turbomachinery Noise

by

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Submitted to the Graduate Faculty as partial fulfillment of the requirements for the
Masters of Science Degree in Mechanical Engineering

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Over the last 20 years, there has been an increase in computational fluid dynamic codes that have made numerical analysis more and more readily available, allowing turbomachine designers to create more novel designs. However, as airport noise limitations become more restrictive over time, reducing aircraft takeoff and landing noise remains a prominent issue in the aviation community. One popular method to reduce aircraft noise is using acoustic liners placed on the walls of the engine inlet and exhaust ducts. These liners are designed to reduce the amplitude of acoustic modes emanating from the bypass fan as they propagate through the engine. The SWIRL code is a frequency-domain linearized Euler equation solver that is designed to predict the effect of acoustic liners on acoustic modes propagating in realistic sheared and swirling mean flows, guiding the design of more efficient liner configurations. The purpose of this study is to validate SWIRL using the Method Of Manufactured Solutions (MMS). This study also investigated the effect of the integration and spatial differencing methods on the convergence for a given Manufactured Solution. In addition, the effect of boundary condition implementation was tested. The improved MMS convergence rates shown for these tests suggest that the revised SWIRL code provides more accurate solutions with less computational effort than the original formulation.

For my friends and family, who have always believed in my potential when I did not believe it myself.

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List of Symbols

A	mean flow speed of sound
A_T	speed of sound at the duct radius
\tilde{A}	dimensionless speed of sound, $\frac{A}{A_T}$
D/Dt	material derivative, $\partial/\partial t + V \cdot \nabla$
D_N	derivative matrix using N points
$\mathbf{e}_x, \mathbf{e}_\theta$	s

List of Abbreviations

CFD	Computational Fluid Dynamics
GLE	Gauss' law for electricity: $\nabla \cdot E = \frac{\rho}{\varepsilon_0} = 4\pi k\rho$
HHS	Department of Health and Human Services
IaR	I am root

Preface

Chapter 1

Background

1.1 Introduction

During the 1960s, the increased demand for commercialized aircraft transport introduced jet engines to support large cargo and passengers. Consequently, this rise in innovation resulted in high volume engine noise. After 1975, efforts to reduce aircraft noise eliminated the noise pollution for 90% of the population [1]. However, since the early 2000s, the advancement in noise reduction technologies has been moderately increasing, leaving a requirement for drastic improvement in aeroacoustic modeling and treatment strategies to compete with the demand for quiet subsonic flight. A turbomachine's general flow condition includes a series of axial, tangential, and radial velocity components that vary depending on the location of concern. The swirling flow between fan stages has been an area of interest due to the potential for acoustic treatment in a location previously avoided for its flow complexity, among other reasons.

In general, jet engine designers can model flow within a turbomachine with the Navier Stokes Equations, a set of partial differential equations that describe the mass, momentum and energy of a given viscous fluid. It is common in practice to utilize the Euler equations, a closely related set of PDEs that model inviscid fluid, as they

provide an approximation for higher Reynold number flows where viscosity does not play a critical role. A popular approach to modeling sound propagation within a flow is to “linearize” the Euler equations, which decomposes the flow solution into a mean and fluctuating component (insert refs). Another method decomposes the flow into vortical and potential parts (ref Golubev & Atassi). In either case, this presents an initial value problem and for certain flows and domains, can obtain analytical solutions. Swirling flow has been a difficult problem to investigate in comparison to flows parallel to the wall domain of a duct [2]. . .

Chapter 2

Chapter 2: Literature Review

2.1 Introduction

2.1.1 Define your topic and provide an appropriate context for reviewing the literature

Following the discovery of the acoustic impact of swirling flow, there has been a concerted effort to model sound propagation under such conditions. This has lead to a number of publications that attempt to discern the acoustic signature of swirling flow through a cylindrical domain.

2.1.2 Establish reasoning - i.e. point - of - view for reviewing the literature

A large amount of aircraft noise was reduced from 1975-2000, effectively eliminating the noise pollution for 90% of the population [?]. Since the early 2000's, the advancement in noise reduction technologies has been gradual, leaving a requirement for drastic improvement in aeroacoustic treatment strategies to compete with the demand of quiet subsonic flight. A previous theoretical review by Envia has been suggested that a non linear time domain computation could capture the source gen-

eration (incident turbulence) in addition to the broadband noise. Such a process would have the capabilities of solving all components of noise generation in an individual calculation [?]. This would at minimum, require an “LES-type” fidelity code, which can tend to be computationally expensive. A promising option is NASA GRC’s Broadband Aeroacoustic Stator Simulation (BASS). BASS is a high-order, high accuracy computational aeroacoustics (CAA) code which has been used to study non linear provides mean an extensive study of non linear phenomena in turbomachinery flow, and in particular, mechanisms of noise generation that are produced from unsteady disturbances. This code allows for a wide variety of finite differencing and time marching schemes as well as artificial dissipation methods. This effective computational tool allows for an in-depth modeling of realistic velocity profiles that would be representative of flow produced from a rotor blade row. In recent work, a new method of implementing realistic, three-dimensional rotor wakes free from acoustics was validated [?]. This provides a means of studying acoustic responses non linear swirling flows within pragmatic geometric configurations, while simultaneously allow for the modeling of sources generation produced from the incident turbulence.

2.1.3 Explain the order/sequence of the review

This review will compare studies that have considered the applicability of unsteady linearized Euler equations on cylindrical and annular ducts. First the work of Kousen will briefly summarized [?, ?]. Secondly, the alternate approached shown by Golubev and Atassi used in [?, ?] were also compared. These works have utilized a standard normal mode approach to determine the modal response of inviscid, compressible, swirling flow within a cylindrical and annular ducts. Results and findings have revealed three categories of associated wave - modes, acoustic, nearly convected, and nearly sonic. Detailed examination of the literature indicate that the hierarchical system was not initially apparent. A qualitative description of the numerical methods

used to evaluate the eigensystem of solutions will be described.

One key question that remains, is how well do unsteady linearized equations capture the mechanisms of noise generation within realistic turbomachinery flow? There exists a copious amount of published work describing the use of wave equations to describe the governing behavior of ducted sound propagation(See [?] for extensive preliminary review). However, turbomachines inherently rely on high velocities, high temperatures to maximize efficiencies. Such mechanisms need a more thorough formulation so that a complete acoustic description of the flow can be provided. Large contributions were first made by Kousen and Atassi and as a result they will be used for comparison.

2.1.4 State the scope (what is included and what is not)

This review will be a qualitative report of the findings, but a quantitative comparison "meta-analysis" should be done for potentially ideal test cases.

2.1.5 Work by Kousen

The study was expanded by [?, ?, ?] who included cases of solid body swirl. Wundrow studied the swirling potential flows using Goldstein's disturbance velocity decomposition [?] using a numerical approach and found that the solutions were more accurate and found more efficiently [?]. Kousen expanded these efforts by including [?] the effect unsteady disturbances in the presence of forced solid body-swirl and free-vortex flow without the use of potential flow theory. Using normal mode analysis along with a radial spatial differencing scheme, the wave modes produced within cylindrical and annular ducts was reported. Results show (figure 4.7) two distinct families of modes, purely convected and acoustic modes. The presence of axial shear in combination solid body flows can endue coupling between these modes, which in

theory would not appear unless viscous terms were present in the eigenvalue analysis. Lack of available swirl flow results “hampered” validation attempts. However, this eigenvalue approach was utilized with a new quasi-3D formulation to find the axial wavenumbers from solid body flow was proposed in his next work .

In [?], A quasi 3D formulation was validated and used to predict the appearance of modes due to the interaction of a rotor with spatially uniform steady and unsteady flow. These modes were first classified by [?] as “spinning modes”. In addition, further investigation was done on the results shown in [?]. The axial wavenumbers that were previously found to be purely convective were shown to be in part , “nearly convective” (shear) pressure modes. These were found to propagate in the axial direction with no loss in amplitude, thus never satisfying the cut-off condition.

2.2 Golubev and Atassi’s work

Similarly, a narrow annulus is once again studied but with a different theoretical and computational approach. The governing equations, similar to Wundrow [?] were still then linearized in terms of potential and rotation. As suggested by Case [?], a Fourier series analysis was used to conduct the normal mode analysis to find the corresponding wave numbers of the eigensystem. The findings show these fall into further classification which were previously denoted as purely convective and in part, nearly-convective wave modes. These two classifications of purely convective disturbance can be split into their “nearly-convected vorticity dominated” and “nearly-sonic pressure dominated” parts. These new results show the appearance of “nearly-sonic pressure dominated modes” which can propagate at varying phase speeds throughout the duct in both directions. The imposed Doppler shift from asymmetrical modes cause the sound propagate in the opposite direction of the mean flow swirl. A weak coupling relation relates the two and allows for the presence of

vorticity - pressure mode coupling. Together, these nearly convected modes can be “identified with the purely convective gusts in a non swirling flow”. For the second group, these “nearly convected” vorticity dominated modes are further split as these disturbances approach the *critical layer*, i.e. the location at which the viscous effects of the boundary layer begin to influence the coupling between modes. When both solid body and free vortex induced rotations are in the same direction, no instabilities arise from outside the critical layer. It was shown in later works that the influence of centrifugal and Coriolis forces created by the mean swirl prevent the decomposition of modes into their potential, rotational and entropic components. The paper ultimately proposes “ a generalized definition for incident rotational waves(gusts) is proposed which accounts for both the eigenmodes and the initial value solutions”

2.3 Conclusion

- The conclusion summarizes the key findings of the review in general terms. Notable commonalities between works, whether favorable or not , may be included here.

This review discusses the development of the unsteady linearized equations, and how improvements in the modal analysis capture more families of mechanisms of noise generation within non-uniform swirling flow turbomachinery flow.

- This section is the reviewer’s opportunity to justify a research proposal. Therefore, the idea should be clearly re-stated and supported according to the findings of the review.

This literature review should be expanded to describe specific details in the methodologies and validation techniques reported by [?, ?, ?, ?].

Chapter 3

Chapter 3: Theory

3.1 No Flow

Starting with equation 2.28 (Wave Equation) in Kousen's paper,

$$\frac{1}{A^2} \frac{D^2 \tilde{p}}{Dt^2} - \nabla^2 \tilde{p} = 2\bar{\rho} \frac{dV_x}{dx} \frac{\partial \tilde{v}_r}{\partial x} \quad (3.1)$$

lets look at the no flow case. In the case of sheared flow, $dV_x/dx = 0$ the right hand side will be zero

$$\frac{1}{A^2} \left(\frac{\partial^2 \tilde{p}}{\partial t^2} + \vec{V} \cdot \vec{\nabla}(\tilde{p}) \right) - \nabla^2 \tilde{p} = 0$$

Substituting the definitions for ∇ and ∇^2 in cylindrical coordinates gives,

$$\frac{1}{A^2} \left(\frac{\partial^2 \tilde{p}}{\partial t^2} + \vec{V} \cdot \left(\frac{\partial \tilde{p}}{\partial t} + \frac{1}{\tilde{r}} \frac{\partial \tilde{p}}{\partial \tilde{r}} + \frac{\partial \tilde{p}}{\partial \theta} + \frac{\partial \tilde{p}}{\partial x} \right) \right) - \left(\frac{\partial^2 \tilde{p}}{\partial t^2} + \frac{1}{\tilde{r}} \frac{\partial \tilde{p}}{\partial \tilde{r}} + \frac{1}{\tilde{r}^2} \frac{\partial^2 \tilde{p}}{\partial \theta^2} + \frac{\partial^2 \tilde{p}}{\partial x^2} \right) = 0$$

Setting $\vec{V} = 0$,

$$\frac{1}{A^2} \left(\frac{\partial^2 \tilde{p}}{\partial t^2} \right) - \left(\frac{\partial^2 \tilde{p}}{\partial t^2} + \frac{1}{\tilde{r}} \frac{\partial \tilde{p}}{\partial r} + \frac{1}{\tilde{r}^2} \frac{\partial^2 \tilde{p}}{\partial \theta^2} + \frac{\partial^2 \tilde{p}}{\partial x^2} \right) = 0$$

Recall, $\tilde{p} = p/\bar{\rho}A^2$. To dimensionalize the equation, this is substituted and both sides are multiplied by $\bar{\rho}A^2$,

$$\frac{1}{A^2} \left(\frac{\partial^2 p}{\partial t^2} \right) - \left(\frac{\partial^2 p}{\partial t^2} + \frac{1}{\tilde{r}} \frac{\partial p}{\partial r} + \frac{1}{\tilde{r}^2} \frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial x^2} \right) = 0$$

The process of separation of variables(seperation indeterminatarum) was first written and formalized by John Bernoulli in a letter to Leibniz. The method of separation of variables requires an assumed solution as well as initial and boundary conditions. For a partial differential equation, the assumed solution can be a linear combination of solutions to a system of ordinary differential equations that comprises the partial differential equation. Since p is a function of four variables, the solution is assumed to be a linear combination of four solutions. Each solution is assumed to be Euler's identity, a common ansatz for linear partial differential equations and boundary conditions.

Defining,

$$p(x, r, \theta, t) = X(x)R(r)\Theta(\theta)T(t) \tag{3.2}$$

where,

$$X(x) = A_1 e^{ik_x x} + B_1 e^{-ik_x x}$$

$$\Theta(\theta) = A_2 e^{ik_\theta \theta} + B_2 e^{-ik_\theta \theta}$$

$$T(t) = A_3 e^{i\omega t} + B_3 e^{-i\omega t}$$

The next step is to rewrite the wave equation in terms of X , R , Θ , and T . To further simplify the result, each term is divided by p . Before the substitution, the derivatives of the assumed solutions need to be evaluated.

3.1.0.1 Temporal Derivatives

$$\begin{aligned} \frac{\partial p}{\partial t} &= \frac{\partial}{\partial t} (XR\Theta T) \\ &= XR\Theta \frac{\partial T}{\partial t} \end{aligned}$$

$$\begin{aligned} \frac{1}{p} \frac{\partial p}{\partial t} &= \frac{1}{XR\Theta T} \left(XR\Theta \frac{\partial T}{\partial t} \right) \\ &= \frac{1}{T} \frac{\partial T}{\partial t} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} &= \frac{\partial^2}{\partial t^2} (XR\Theta T) \\ &= XR\Theta \frac{\partial^2 T}{\partial t^2} \end{aligned}$$

$$\begin{aligned}\frac{1}{p} \frac{\partial^2 p}{\partial t^2} &= \frac{1}{XR\Theta T} \left(XR\Theta \frac{\partial^2 T}{\partial t^2} \right) \\ &= \frac{1}{T} \frac{\partial^2 T}{\partial t^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{\partial}{\partial t} (A_3 e^{i\omega t} + B_3 e^{-i\omega t}) \\ &= \frac{\partial}{\partial t} (A_3 e^{i\omega t}) + \frac{\partial}{\partial t} (B_3 e^{-i\omega t}) \\ &= i\omega A_3 e^{i\omega t} - i\omega B_3 e^{i\omega t}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 T}{\partial t^2} &= \frac{\partial^2}{\partial t^2} (i\omega A_3 e^{i\omega t} + i\omega B_3 e^{-i\omega t}) \\ &= \frac{\partial^2}{\partial t^2} (i\omega A_3 e^{i\omega t}) + \frac{\partial^2}{\partial t^2} (-i\omega B_3 e^{-i\omega t}) \\ &= (i\omega)^2 A_3 e^{i\omega t} - (i\omega)^2 B_3 e^{i\omega t}\end{aligned}$$

$$\begin{aligned}\frac{1}{T} \frac{\partial^2 T}{\partial t^2} &= (i\omega)^2 \\ &= -\omega^2\end{aligned}$$

3.1.0.2 Radial Derivatives

$$\begin{aligned}\frac{\partial p}{\partial r} &= \frac{\partial}{\partial r} (XR\Theta T) \\ &= X\Theta T \frac{\partial R}{\partial r}\end{aligned}$$

$$\begin{aligned}\frac{1}{p} \frac{\partial p}{\partial r} &= \frac{1}{XR\Theta T} \left(X\Theta T \frac{\partial R}{\partial r} \right) \\ &= \frac{1}{R} \frac{\partial R}{\partial r}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 p}{\partial r^2} &= \frac{\partial^2}{\partial r^2} (XR\Theta T) \\ &= X\Theta T \frac{\partial^2 R}{\partial r^2}\end{aligned}$$

$$\begin{aligned}\frac{1}{p} \frac{\partial^2 p}{\partial r^2} &= \frac{1}{XR\Theta T} \left(X\Theta T \frac{\partial^2 R}{\partial r^2} \right) \\ &= \frac{1}{R} \frac{\partial^2 R}{\partial r^2}\end{aligned}$$

The radial derivatives will be revisited once the remaining derivatives are evaluated,

3.1.0.3 Tangential Derivatives

$$\begin{aligned}\frac{\partial p}{\partial \theta} &= \frac{\partial}{\partial t} (XR\Theta T) \\ &= XR T \frac{\partial \Theta}{\partial \theta}\end{aligned}$$

$$\begin{aligned}\frac{1}{p} \frac{\partial p}{\partial \theta} &= \frac{1}{XR\Theta T} \left(XR\Theta \frac{\partial T}{\partial \theta} \right) \\ &= \frac{1}{\Theta} \frac{\partial \Theta}{\partial \theta}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 p}{\partial \theta^2} &= \frac{\partial^2}{\partial \theta^2} (XR\Theta T) \\ &= XR T \frac{\partial^2 \Theta}{\partial \theta^2}\end{aligned}$$

$$\begin{aligned}\frac{1}{p} \frac{\partial^2 p}{\partial \theta^2} &= \frac{1}{XR\Theta T} \left(XR T \frac{\partial^2 \Theta}{\partial \theta^2} \right) \\ &= \frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial \theta^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial \Theta}{\partial \theta} &= \frac{\partial}{\partial \theta} (A_2 e^{ik_\theta \theta} + B_2 e^{-ik_\theta \theta}) \\ &= \frac{\partial}{\partial \theta} (A_2 e^{ik_\theta \theta}) + \frac{\partial}{\partial \theta} (B_2 e^{-ik_\theta \theta}) \\ &= ik_\theta A_2 e^{ik_\theta \theta} - ik_\theta B_2 e^{-ik_\theta \theta}\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \Theta}{\partial \theta^2} &= \frac{\partial^2}{\partial \theta^2} (ik_\theta A_2 e^{ik_\theta \theta} - ik_\theta B_2 e^{-ik_\theta \theta}) \\
&= \frac{\partial^2}{\partial \theta^2} (ik_\theta A_2 e^{ik_\theta \theta}) + \frac{\partial^2}{\partial \theta^2} (-ik_\theta B_2 e^{-ik_\theta \theta}) \\
&= (ik_\theta)^2 A_2 e^{ik_\theta \theta} - (ik_\theta)^2 B_2 e^{-ik_\theta \theta}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial \theta^2} &= (ik_\theta)^2 \\
&= -k_\theta^2
\end{aligned}$$

3.1.0.4 Axial Derivatives

$$\begin{aligned}
\frac{\partial p}{\partial x} &= \frac{\partial}{\partial x} (XR\Theta T) \\
&= R\Theta T \frac{\partial X}{\partial x}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{p} \frac{\partial p}{\partial x} &= \frac{1}{XR\Theta T} \left(R\Theta \frac{\partial X}{\partial x} \right) \\
&= \frac{1}{X} \frac{\partial X}{\partial x}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 p}{\partial x^2} &= \frac{\partial^2}{\partial x^2} (XR\Theta T) \\
&= R\Theta T \frac{\partial^2 X}{\partial x^2}
\end{aligned}$$

$$\begin{aligned}\frac{1}{p} \frac{\partial^2 p}{\partial x^2} &= \frac{1}{X R \Theta T} \left(R \Theta T \frac{\partial^2 X}{\partial x^2} \right) \\ &= \frac{1}{X} \frac{\partial^2 X}{\partial x^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial X}{\partial x} &= \frac{\partial}{\partial t} (A_3 e^{ik_x t} + B_3 e^{-i\omega t}) \\ &= \frac{\partial}{\partial t} (A_1 e^{ik_x x}) + \frac{\partial}{\partial t} (B_1 e^{-ik_x x}) \\ &= ik_x A_1 e^{ik_x x} - ik_x B_1 e^{ik_x x}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 X}{\partial x^2} &= \frac{\partial^2}{\partial x^2} (ik_x A_1 e^{ik_x x} + ik_x B_1 e^{-ik_x x}) \\ &= \frac{\partial^2}{\partial x^2} (ik_x A_1 e^{ik_x x}) + \frac{\partial^2}{\partial x^2} (-ik_x B_1 e^{-ik_x x}) \\ &= (ik_x)^2 A_1 e^{ik_x x} - (ik_x)^2 B_1 e^{ik_x x}\end{aligned}$$

$$\begin{aligned}\frac{1}{X} \frac{\partial^2 X}{\partial x^2} &= (ik_x)^2 \\ &= -k_x^2\end{aligned}$$

Substituting this back into the wave equation yields ,

$$\frac{1}{A^2} \left(\frac{\partial^2 p}{\partial t^2} \right) = \left(\frac{\partial^2 p}{\partial t^2} + \frac{1}{\tilde{r}} \frac{\partial p}{\partial r} + \frac{1}{\tilde{r}^2} \frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial x^2} \right)$$

$$\frac{1}{A^2} \frac{1}{T} \frac{\partial^2 T}{\partial t^2} = \frac{1}{R} \frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{1}{R} \frac{\partial R}{\partial r} + \frac{1}{r^2} \frac{1}{\Theta} \frac{\partial \Theta}{\partial \theta} + \frac{1}{X} \frac{\partial^2 X}{\partial x^2} \quad (3.3)$$

Notice that each term is only a function of its associated independent variable. So, if we vary the time, only the term on the left-hand side can vary. However, since none of the terms on the right-hand side depend on time, that means the right-hand side cannot vary, which means that the ratio of time with its second derivative is independent of time. The practical upshot is that each of these terms is constant, which has been shown. The wave numbers are the *separation constants* that allow the PDE to be split into four separate ODE's. Substituting the separation constants into Equation (3.3) gives,

$$-\frac{\omega^2}{A^2} = \frac{1}{R} \left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} \right) - \frac{k_\theta^2}{r^2} - k_x^2 \quad (3.4)$$

Note that the dispersion relation states $\omega = kA$

$$\frac{1}{R} \left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} \right) - \frac{k_\theta^2}{r^2} - k_x^2 + k^2 = 0 \quad (3.5)$$

The remaining terms are manipulated to follow the same form as *Bessel's Differential Equation* ,

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \quad (3.6)$$

The general solution to Bessel's differential equation is a linear combination of the Bessel functions of the first kind, $J_n(x)$ and of the second kind, $Y_n(x)$ [?]. The subscript n refers to the order of Bessel's equation.

$$y(x) = AJ_n(x) + BY_n(x) \quad (3.7)$$

By rearranging Equation (3.5), a comparison can be made to Equation (3.6) to

show that the two equations are of the same form.

The first step is to revisit the radial derivatives that have not been addressed. As was done for the other derivative terms, the radial derivatives will also be set equal to a separation constant, $-k_r^2$.

$$\underbrace{\frac{1}{R} \left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} \right)}_{-k_r^2} - \frac{k_\theta^2}{r^2} - k_x^2 + k^2 = 0 \quad (3.8)$$

The reader may be curious as to why the tangential separation constant k_θ is included within the definition of the radial separation constant.

Recall the ODE for the tangential direction,

$$\begin{aligned} \frac{\partial \Theta}{\partial \theta} \frac{1}{\Theta} &= -k_\theta^2 \\ \frac{\partial \Theta}{\partial \theta} \frac{1}{\Theta} + \Theta k_\theta^2 &= 0 \end{aligned}$$

where the solution is more or less,

$$\Theta(\theta) = e^{ik_\theta \theta}$$

In order to have non trivial, sensible solutions, the value of $\Theta(0)$ and $\Theta(2\pi)$ need to be the same, and this needs to be true for any multiple of 2π for a fixed r . Taking Θ to be one, a unit circle, it can be shown that the domain is only going to be an integer multiple. Therefore, there is an implied periodic azimuthal boundary condition, i.e. $0 < \theta \leq 2\pi$ and $k_\theta = m$.

Continuing with the radial derivatives...

$$-k_r^2 = \frac{1}{R} \left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} \right) - \frac{m^2}{r^2}$$

To further simplify, the chain rule is used to do a change of variables, $x = k_r r$

$$\begin{aligned} \frac{\partial R}{\partial r} &= \frac{dR}{dx} \frac{dx}{dr} \\ &= \frac{dR}{dx} \frac{d}{dr} (k_r r) \\ &= \frac{dR}{dx} k_r \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 R}{\partial r^2} &= \frac{d^2 R}{dx^2} \left(\frac{dx}{dr} \right)^2 + \frac{dR}{dr} \frac{d^2 x}{dr^2} \\ &= \frac{d^2 R}{dx^2} \frac{d}{dr} k_r^2 + k_r \frac{d^2 r}{dr^2} \\ &= \frac{d^2 R}{dx^2} \frac{d}{dr} k_r^2 \end{aligned}$$

Substituting this into Equation (3.5),

$$\left(\frac{d^2 R}{dx^2} k_r^2 + \frac{1}{r} \frac{d^2 R}{dx^2} k_r \right) + \left(k_r^2 - \frac{m^2}{r^2} \right) R \quad (3.9)$$

Dividing Equation 3.9 by k_r^2 ,

$$\left(\frac{d^2 R}{dx^2} + \frac{1}{k_r r} \frac{d^2 R}{dx^2} \right) + \left(1 - \frac{m^2}{k_r^2 r^2} \right) R \quad (3.10)$$

$$\left(\frac{d^2 R}{dx^2} + \frac{1}{x^2} \frac{d^2 R}{dx^2} \right) + \left(1 - \frac{m^2}{x^2} \right) R \quad (3.11)$$

Multiplying Equation (3.11) by x^2 gives,

$$\frac{d^2 R}{dr^2} x^2 + \frac{dR}{dr} x + (x^2 - m^2) R \quad (3.12)$$

which matches the form of Bessel's equation

Therefore, the solution goes from this,

$$y(x) = AJ_n(x) + BY_n(x) \quad (3.13)$$

to this,

$$R(r) = (AJ_n(k_r r) + BY_n(k_r r)) \quad (3.14)$$

where the coefficients A and B are found after applying radial boundary conditions.

3.1.0.5 Hard Wall boundary condition

$$\begin{aligned} \frac{\partial p}{\partial r}|_{r=r_{min}} = \frac{\partial p}{\partial r}|_{r=r_{max}} = 0 &\rightarrow \frac{\partial}{\partial r} (X\Theta T R) = 0 \\ X\Theta T \frac{\partial R}{\partial r} &= 0 \\ \frac{\partial R}{\partial r} &= 0 \end{aligned}$$

where,

$$\frac{\partial R}{\partial r}|_{r_{min}} = AJ'_n(k_r r_{min}) + BY'_n(k_r r_{min}) = 0 \rightarrow B = -A \frac{J'_n(k_r r_{min})}{Y'_n(k_r r_{min})}$$

$$\begin{aligned}
\frac{\partial R}{\partial r} &= AJ'_n(k_r r_{max}) + BY'_n(k_r r_{max}) = 0 \\
&= AJ'_n(k_r r_{max}) - A \frac{J'_n(k_r r_{min})}{Y'_n(k_r r_{min})} Y'_n(k_r r_{max}) = 0 \\
&= \frac{J'_n(k_r r_{min})}{J'_n(k_r r_{max})} - \frac{Y'_n(k_r r_{min})}{Y'_n(k_r r_{max})} = 0
\end{aligned}$$

where $k_r r$ are the zero crossings for the derivatives of the Bessel functions of the first and second kind.

In summary, the wave equation for no flow in a hollow duct with hard walls is obtained from Equation (3.8).

$$k^2 = k_r^2 + k_x^2 \quad (3.15)$$

Solving for the axial wavenumber gives,

3.2 Uniform Flow

To get the same equation but for uniform flow, the same procedure can be followed.

Starting with Equation 2.27 redimensionalized,

$$\frac{d^2 \tilde{p}}{d\tilde{r}^2} + \frac{1}{\tilde{r}} \frac{d\tilde{p}}{d\tilde{r}} + \frac{2\bar{\gamma} \left(\frac{dm_x}{d\tilde{r}} \right)}{(k - \bar{\gamma} m_x)} \frac{d\tilde{p}}{d\tilde{r}} + \left[(k - \bar{\gamma} m_x)^2 - \frac{m^2}{\tilde{r}^2} - \bar{\gamma}^2 \right] \tilde{p}$$

Let's separate the new terms from the old ones,

$$\frac{d^2 \tilde{p}}{d\tilde{r}^2} + \frac{1}{\tilde{r}} \frac{d\tilde{p}}{d\tilde{r}} + \frac{2\bar{\gamma} \left(\frac{dm_x}{d\tilde{r}} \right)}{(k - \bar{\gamma} m_x)} \frac{d\tilde{p}}{d\tilde{r}} + \left[(k - \bar{\gamma} m_x)^2 - \frac{m^2}{\tilde{r}^2} - \bar{\gamma}^2 \right] \tilde{p}$$

Recalling the non-dimensional definitions,

$$\begin{aligned}
\tilde{p} &= \frac{p}{\bar{\rho} A^2} \\
\tilde{r} &= \frac{r}{r_T} \\
\frac{\partial \tilde{p}}{\partial \tilde{r}} &= \frac{\partial \tilde{p}}{\partial r} \frac{\partial r}{\partial \tilde{r}} \\
&= \frac{\partial \tilde{p}}{\partial r} \frac{\partial}{\partial \tilde{r}} (\tilde{r} r_T) \\
&= \frac{\partial \tilde{p}}{\partial r} r_T \\
\frac{\partial^2 \tilde{p}}{\partial \tilde{r}^2} &= \frac{\partial^2 \tilde{p}}{\partial r^2} (r_T)^2 + \frac{\partial \tilde{p}}{\partial r} \frac{\partial^2 r}{\partial \tilde{r}^2} \\
&= \frac{\partial^2 \tilde{p}}{\partial r^2} (r_T)^2
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial r} \left(\frac{p}{\bar{\rho} A^2} \right) &= \frac{\left(\frac{\partial}{\partial r} (p) \bar{\rho} A^2 - \underbrace{\frac{\partial \bar{\rho} A^2}{\partial r} p}_0 \right)}{(\bar{\rho} A^2)^2} \\
&= \frac{1}{\bar{\rho} A^2} \frac{\partial p}{\partial r}
\end{aligned}$$

$$\frac{d^2 \tilde{p}}{d\tilde{r}^2} + \frac{1}{\tilde{r}} \frac{d\tilde{p}}{d\tilde{r}} - \frac{m^2}{\tilde{r}^2} \tilde{p} - \bar{\gamma}^2 \tilde{p} + \frac{2\bar{\gamma} \left(\frac{dM_x}{d\tilde{r}} \right)}{(k - \bar{\gamma} M_x)} \frac{d\tilde{p}}{d\tilde{r}} + (k - \bar{\gamma} M_x)^2 \tilde{p}$$

If there is only uniform flow, then $dM_x/dr = 0$,

$$\frac{d^2 \tilde{p}}{d\tilde{r}^2} + \frac{1}{\tilde{r}} \frac{d\tilde{p}}{d\tilde{r}} - \frac{m^2}{\tilde{r}^2} \tilde{p} - \bar{\gamma}^2 \tilde{p} + (k - \bar{\gamma} M_x)^2 \tilde{p}$$

Re-dimensionalizing,

$$\frac{1}{\bar{\rho}A^2} \left[\frac{d^2p}{dr} r_T^2 + \frac{r_T}{r} \frac{dp}{dr} r_T - \frac{m^2}{r^2} r_T^2 p - k_x^2 r_T^2 p \right] + \left(\frac{\omega}{A} r_T - k_x r_T M_x \right)^2 p$$

Expanding the last term and substituting $\omega/A = k$

$$\frac{1}{\bar{\rho}A^2} \left[\frac{d^2p}{dr} r_T^2 + \frac{r_T}{r} \frac{dp}{dr} r_T - \frac{m^2}{r^2} r_T^2 p - k_x^2 r_T^2 p \right] + \left(r_T^2 (k^2 - 2k k_x M_x + k_x^2 M_x^2) \right) p$$

Canceling out $r_T/\bar{\rho}A$ in every term

$$\frac{d^2p}{dr} + \frac{1}{r} \frac{dp}{dr} + \left[k^2 - 2k k_x M_x + k_x^2 M_x^2 - \frac{m^2}{r^2} - k_x^2 \right] p$$

Continue here,

Defining

$$-N^2 = k_x^2 M_x^2 - 2k k_x M_x - k_x^2$$

$$-N^2 = -(1 - M_x^2) k_x^2 - 2k k_x M_x$$

$$-N^2 = -\beta^2 k_x^2 - 2k k_x M_x$$

$$\frac{d^2p}{dr} + \frac{1}{r} \frac{dp}{dr} + \left[k^2 - N^2 - \frac{m^2}{r^2} \right] p$$

Let $k_r^2 = k^2 - N^2$

$$\frac{d^2 p}{dr} + \frac{1}{r} \frac{dp}{dr} + \left[k_r^2 - \frac{m^2}{r^2} \right] p$$

Looking at the radial wavenumber,

$$\begin{aligned} k_r^2 &= k^2 - N^2 \\ &= k^2 - \beta^2 k_x^2 - 2k k_x M_x \\ 0 &= -\beta^2 k_x^2 - (2M_x k) k_x + (k^2 - k_r^2) \end{aligned}$$

Where the roots to this equation are the axial wavenumber,

Applying the quadratic formula and taking

$$\begin{aligned} A &= -\beta^2 \\ B &= -2M_x k \\ C &= k^2 - k_r^2 \end{aligned}$$

Note B is negative when M_x is positive,

(I feel like N should change based on M'_x 's sign)

$$\begin{aligned} k_x &= \frac{2M_x k \pm \sqrt{4M_x^2 k^2 + 4\beta^2 (k^2 - k_r^2)}}{-2\beta^2} \\ &= \frac{-M_x k \pm \sqrt{k^2 - k_r^2}}{\beta^2} \end{aligned}$$

3.3 Annular Duct Axial Wavenumber solution

This needs to be proven,

In [?], the axial wavenumber for annular ducts is reported,

$$\frac{-(\omega - m M_\theta) M_x \pm \sqrt{(\omega - m M_\theta^2) - \beta (m^2 + \Gamma_{m,n}^2)}}{\beta a^2}$$

where

$$\Gamma_{m,n} = \frac{n^2 \pi^2}{(r_{max} - r_{min})^2}$$

3.3.1 Setting up SWIRL's Aerodynamic Model

The Euler Equations in Cylindrical Form are,

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} + \frac{v_\theta}{r} \frac{\partial \rho}{\partial \theta} + v_x \frac{\partial \rho}{\partial x} + \rho \left(\frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_x}{\partial x} \right) &= 0 \\
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_x \frac{\partial v_r}{\partial x} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} \\
\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_x \frac{\partial v_\theta}{\partial x} &= -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} \\
\frac{\partial v_x}{\partial t} + v_r \frac{\partial v_x}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_x}{\partial \theta} + v_x \frac{\partial v_x}{\partial x} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} \\
\frac{\partial p}{\partial t} + v_r \frac{\partial p}{\partial r} + \frac{v_\theta}{r} \frac{\partial p}{\partial \theta} + v_x \frac{\partial p}{\partial x} + \gamma p \left(\frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_x}{\partial x} \right) &= 0
\end{aligned}$$

SWIRL utilizes the following assumptions to simplify the aerodynamic model

- No flow in the radial direction. Consequentially, the flow is axisymmetric along the downstream direction.
- No surface or body forces
- Isentropic conditions

For steady flow, the continuity, momentum and entropy equations are

$$\nabla(\vec{V}\bar{\rho}) = 0$$

$$(\vec{V} \cdot \nabla)\vec{V}$$

$$\nabla S = 0$$

If we neglect radial velocity, the velocity vector in cylindrical coordinates are

$$\vec{V}(r, \theta, x) = V_x(r)\hat{e}_x + V_\theta(r)\hat{e}_\theta$$

3.4 Applying model to various flows

Kousen studied three specific flow configuration.

- axial shear flow
- solid body swirl
- free vortex swirl

3.4.1 Axial Shear Flow

In Kousen's paper, axial sheared flows through a constant area duct was also investigated. The only effect on the velocity gradient occurs along the x axis. All other primitive variables (pressure and density which is \propto speed of sound) are constant. As a result, the only changes that occur are in the x direction. This implies that $\partial/\partial\theta = 0$. For the conservation of mass,

$$\nabla(\vec{V}\bar{\rho}) = \left(\underbrace{\frac{\partial(\bar{\rho}v_r)}{\partial r}}_{v_r=0} + \underbrace{\frac{1}{r}\frac{\partial\bar{\rho}v_\theta}{\partial\theta}}_{\frac{\partial}{\partial\theta}} + \frac{\partial\bar{\rho}v_x}{\partial x} \right) = \frac{\partial\bar{\rho}v_x}{\partial x}$$

Conservation of Momentum in the radial direction becomes:

$$(\vec{V} \cdot \nabla)\vec{V} = v_r \cancel{\frac{\partial v_r}{\partial r}} + \frac{v_\theta}{r} \cancel{\frac{\partial v_r}{\partial\theta}} - \frac{v_\theta^2}{r} + v_x \cancel{\frac{\partial v_r}{\partial x}} = -\frac{1}{\rho} \frac{\partial p}{\partial r}$$

$$\frac{v_\theta^2}{r} = \frac{1}{\rho} \frac{\partial p}{\partial r}$$

$$\frac{\rho v_\theta^2}{r} = \frac{\partial p}{\partial r}$$

θ direction

$$(\vec{V} \cdot \nabla)\vec{V} = v_r \cancel{\frac{\partial v_\theta}{\partial r}} + \frac{v_\theta}{r} \cancel{\frac{\partial v_\theta}{\partial\theta}} + \frac{v_r v_\theta}{r} + v_x \frac{\partial v_\theta}{\partial x} = -\frac{1}{\rho r} \frac{\partial p}{\partial\theta}$$

Dividing v_x to the other side,

$$\frac{\partial v_\theta}{\partial x} = 0$$

Similarly for the x direction,

$$\frac{\partial v_x}{\partial x} = 0$$

In regards to the entropy equation, having an isentropic flow, $\nabla S = 0$ implies $a^2 = \frac{\nabla \bar{p}}{\nabla \bar{\rho}}$

3.5 Accounting for solid body swirl

If the flow contains a swirling component, then the primitive variables are nonuniform through the flow, and mean flow assumptions are not valid. To account to this, we integrate the momentum equation in the radial direction with respect to the radius.

Equation (2.5) in [?] is

$$P = \int_{\tilde{r}}^1 \frac{\bar{\rho} V_\theta^2}{\tilde{r}} d\tilde{r}$$

where \tilde{r} is the radius dimensional radius normalized by the tip diameter $r_t = r_{max}$

To show the work, we will start with the dimensional form of the equation,

$$\frac{\bar{\rho} v_\theta^2}{r} = \frac{\partial p}{\partial r}$$

Applying separation of variables

$$\int_r^{r_{max}} \frac{\bar{\rho} v_\theta^2}{r} \partial r = - \int_{P(r)}^{P(r_{max})} \partial p$$

Since $\tilde{r} = r/r_{max}$

$$r = \tilde{r} r_{max}$$

taking total derivatives (applying chain rule)

$$dr = d(\tilde{r}r_{max}) = d(\tilde{r})r_{max}$$

Substituting these back in and evaluating the right hand side,

$$\int_{\tilde{r}}^1 \frac{\bar{\rho} v_{\theta}^2}{\tilde{r}} d\tilde{r} = P(1) - P(\tilde{r})$$

For reference the minimum value of \tilde{r} is

$$\sigma = \frac{r_{max}}{r_{min}}$$

For

$$\frac{\partial a^2}{\partial r} = \frac{\partial}{\partial r} \left(\frac{\gamma P}{\rho} \right)$$

Using the quotient rule, we can extract the definition of the speed of sound.

$$\begin{aligned} &= \frac{\partial P}{\partial r} \frac{\gamma \bar{\rho}}{\bar{\rho}^2} - \left(\frac{\gamma P}{\bar{\rho}^2} \right) \frac{\partial \bar{\rho}}{\partial r} \\ &= \frac{\partial P}{\partial r} \frac{\gamma}{\bar{\rho}} - \left(\frac{a^2}{\bar{\rho}} \right) \frac{\partial \bar{\rho}}{\partial r} \\ \text{Using } \partial P / a^2 = \partial \rho \rightarrow &= \frac{\partial P}{\partial r} \frac{\gamma}{\bar{\rho}} - \left(\frac{1}{\bar{\rho}} \right) \frac{\partial \bar{P}}{\partial r} \\ &\frac{\partial a^2}{\partial r} = \frac{\partial P}{\partial r} \frac{\gamma - 1}{\bar{\rho}} \\ \text{or.. } \frac{\bar{\rho}}{\gamma - 1} \frac{\partial a^2}{\partial r} &= \frac{\partial P}{\partial r} \end{aligned}$$

Going back to the radial momentum equation, and rearranging the

$$\begin{aligned}
\frac{\bar{\rho} v_\theta^2}{r} &= \frac{\partial P}{\partial r} \\
\frac{\bar{\rho} v_\theta^2}{r} &= \frac{\bar{\rho}}{\gamma - 1} \frac{\partial a^2}{\partial r} \\
\frac{v_\theta^2}{r} (\gamma - 1) &= \frac{\partial a^2}{\partial r} \\
\text{Dividing both sides by } a^2 \rightarrow \frac{M_\theta}{r} (\gamma - 1) &= \frac{\partial a^2}{\partial r} \frac{1}{a^2}
\end{aligned}$$

$$\begin{aligned}
\text{Integrating both sides } \int_r^{r_{max}} \frac{M_\theta}{r} (\gamma - 1) \partial r &= \int_{a^2(r)}^{a^2(r_{max})} \frac{1}{a^2} \partial a^2 \\
\int_r^{r_{max}} \frac{M_\theta^2}{r} (\gamma - 1) \partial r &= \ln(a^2(r_{max})) - \ln(a^2(r)) \\
\int_r^{r_{max}} \frac{M_\theta^2}{r} (\gamma - 1) \partial r &= \ln\left(\frac{a^2(r_{max})}{a^2(r)}\right)
\end{aligned}$$

Defining non dimensional speed of sound $\tilde{a} = \frac{a(r)}{a(r_{max})}$

$$\begin{aligned}
\int_r^{r_{max}} \frac{M_\theta}{r} (\gamma - 1) \partial r &= \ln\left(\frac{1}{\tilde{a}^2}\right) \\
&= -2\ln(\tilde{a}) \\
\tilde{a}(r) &= \exp\left[-\int_r^{r_{max}} \frac{M_\theta}{r} \frac{(\gamma - 1)}{2} \partial r\right] \\
\text{replacing r with } \tilde{r} \rightarrow \tilde{a}(r) &= \exp\left[-\int_r^{r_{max}} \frac{M_\theta}{r} \frac{(\gamma - 1)}{2} \partial r\right] \\
\tilde{a}(\tilde{r}) &= \exp\left[\left(\frac{1 - \gamma}{2}\right) \int_{\tilde{r}}^1 \frac{M_\theta}{\tilde{r}} \partial \tilde{r}\right]
\end{aligned}$$

3.5.1 Linearizing the governing equations

3.5.1.1 Linearizing Conservation of Mass

To linearize the Euler equations, we substitute each flow variable with its equivalent mean and perturbation components. Note that the mean term is only a function of space whereas the perturbation component is a dependent on both space and time (functional dependence is not explicitly written with each variable). Assuming that we can divide the variable into a known laminar flow solution to the Navier-Stokes equations and a small amplitude perturbation solution:

$$v_r = V_r(x) + v'_r \tag{3.16}$$

$$v_\theta = V_\theta + v'_\theta \tag{3.17}$$

$$v_x = V_x + v'_x \tag{3.18}$$

$$p = \bar{p} + p' \tag{3.19}$$

$$\rho = \bar{\rho} + \rho' \tag{3.20}$$

Starting with continuity,

$$\begin{aligned}
& \frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} + \frac{v_\theta}{r} \frac{\partial \rho}{\partial \theta} + v_x \frac{\partial \rho}{\partial x} + \rho \left(\frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_x}{\partial x} \right) = 0 \\
& \frac{\partial \bar{\rho} + \rho'}{\partial t} + (V_r + v'_r) \frac{\partial \bar{\rho} + \rho'}{\partial r} + \frac{V_\theta + v'_\theta}{r} \frac{\partial \bar{\rho} + \rho'}{\partial \theta} + (V_x + v'_x) \frac{\partial \bar{\rho} + \rho'}{\partial x} + \\
& (\bar{\rho} + \rho') \left(\frac{1}{r} \frac{\partial(r(V_r + v'_r))}{\partial r} + \frac{1}{r} \frac{\partial(V_\theta + v'_\theta)}{\partial \theta} + \frac{\partial(V_x + v'_x)}{\partial x} \right) = 0 \\
& \frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \rho'}{\partial t} + \\
& V_r \frac{\partial \bar{\rho}}{\partial r} + v'_r \frac{\partial \bar{\rho}}{\partial r} + V_r \frac{\partial \rho'}{\partial r} + v'_r \frac{\partial \rho'}{\partial r} + \\
& \frac{1}{r} \left(V_\theta \frac{\partial \bar{\rho}}{\partial \theta} + v'_\theta \frac{\partial \bar{\rho}}{\partial \theta} + V_\theta \frac{\partial \rho'}{\partial \theta} + v'_\theta \frac{\partial \rho'}{\partial \theta} \right) + \\
& V_x \frac{\partial \bar{\rho}}{\partial x} + v'_x \frac{\partial \bar{\rho}}{\partial x} + V_x \frac{\partial \rho'}{\partial x} + v'_x \frac{\partial \rho'}{\partial x} + \\
& \bar{\rho} \left(\frac{1}{r} \left(\frac{\partial(rV_r)}{\partial r} + \frac{\partial(rv'_r)}{\partial r} + \frac{\partial V_\theta}{\partial \theta} + \frac{\partial v'_\theta}{\partial \theta} \right) + \frac{\partial V_x}{\partial x} + \frac{\partial v'_x}{\partial x} \right) + \\
& \rho' \left(\frac{1}{r} \left(\frac{\partial(rV_r)}{\partial r} + \frac{\partial(rv'_r)}{\partial r} + \frac{\partial V_\theta}{\partial \theta} + \frac{\partial v'_\theta}{\partial \theta} \right) + \frac{\partial V_x}{\partial x} + \frac{\partial v'_x}{\partial x} \right) = 0
\end{aligned}$$

Important things to note

- The small disturbances are infinitesimal (thus linearized)
- Neglect second order terms.
- The continuity equation is comprised of mean velocity components. This is subtracted off in each of the governing equations

Blue will be used for terms that are removed after subtracting in the original continuity equation, green will be used to cancel higher(2nd) order terms. Red will be used if we take the radial velocity to be zero.

$$\begin{aligned}
&= \cancel{\frac{\partial \bar{\rho}}{\partial t}} + \frac{\partial \rho'}{\partial t} + \\
&\quad \cancel{V_r \frac{\partial \bar{\rho}}{\partial r}} + \cancel{v'_r \frac{\partial \bar{\rho}}{\partial r}} + \cancel{V_r \frac{\partial \rho'}{\partial r}} + \cancel{v'_r \frac{\partial \rho'}{\partial r}} + \\
&\quad \frac{1}{r} \left(\cancel{V_\theta \frac{\partial \bar{\rho}}{\partial \theta}} + \cancel{v'_\theta \frac{\partial \bar{\rho}}{\partial \theta}} + V_\theta \frac{\partial \rho'}{\partial \theta} + v'_\theta \frac{\partial \rho'}{\partial \theta} \right) + \\
&\quad \cancel{V_x \frac{\partial \bar{\rho}}{\partial x}} + \cancel{v'_x \frac{\partial \bar{\rho}}{\partial x}} + V_x \frac{\partial \rho'}{\partial \theta} + \cancel{v'_x \frac{\partial \rho'}{\partial x}} + \\
&\quad \bar{\rho} \left(\frac{1}{r} \left(\cancel{\frac{\partial(rV_r)}{\partial r}} + \frac{\partial(rv'_r)}{\partial r} + \cancel{\frac{\partial V_\theta}{\partial \theta}} + \frac{\partial v'_\theta}{\partial \theta} \right) + \cancel{\frac{\partial V_x}{\partial x}} + \frac{\partial v'_x}{\partial x} \right) + \\
&\quad \rho' \left(\frac{1}{r} \left(\cancel{\frac{\partial(rV_r)}{\partial r}} + \cancel{\frac{\partial(rv'_r)}{\partial r}} + \cancel{\frac{\partial V_\theta}{\partial \theta}} + \cancel{\frac{\partial v'_\theta}{\partial \theta}} \right) + \cancel{\frac{\partial V_x}{\partial x}} + \cancel{\frac{\partial v'_x}{\partial x}} = 0 \right)
\end{aligned}$$

$$\boxed{\frac{\partial \rho'}{\partial t} + \frac{V_\theta}{r} \frac{\partial \rho'}{\partial \theta} + V_x \frac{\partial \rho'}{\partial x} + \bar{\rho} \left(\frac{1}{r} \left(\frac{\partial r v'_r}{\partial r} + \frac{\partial v'_\theta}{\partial \theta} \right) + \frac{\partial v'_x}{\partial x} \right) = 0}$$

3.5.1.2 Linearizing the Conservation of Momentum in the r direction

Starting with the mean momentum equation

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_x \frac{\partial v_r}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial r}$$

Looking at the left hand side first

$$\begin{aligned} & \frac{\partial(V_r + v'_r)}{\partial t} + (V_r + v'_r) \frac{\partial(V_r + v'_r)}{\partial r} + \frac{V_\theta + v'_\theta}{r} \frac{\partial(V_r + v'_r)}{\partial \theta} - \frac{(V_\theta + v'_\theta)^2}{r} + (V_x + v'_x) \frac{\partial(V_r + v'_r)}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial r} \\ & \quad \frac{\partial V_r}{\partial t} + \frac{\partial v'_r}{\partial t} + \\ & \quad \frac{V_r}{r} \frac{\partial V_r}{\partial r} + \frac{v'_r}{r} \frac{\partial V_r}{\partial r} + \frac{V_r}{r} \frac{\partial v'_r}{\partial r} + \frac{v'_r}{r} \frac{\partial v'_r}{\partial r} + \\ & \quad \frac{1}{r} \left(V_\theta \frac{\partial V_r}{\partial \theta} + v'_\theta \frac{\partial V_r}{\partial \theta} + V_\theta \frac{\partial v'_r}{\partial \theta} + v'_\theta \frac{\partial v'_r}{\partial \theta} \right) - \\ & \quad \frac{1}{r} \left(V_\theta^2 + 2V_\theta v'_\theta + v'^2_\theta \right) + \\ & \quad \frac{V_x}{r} \frac{\partial V_r}{\partial x} + \frac{v'_x}{r} \frac{\partial V_r}{\partial x} + V_x \frac{\partial v'_r}{\partial x} + v'_x \frac{\partial v'_r}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial r} \\ & \quad \frac{\partial v'_r}{\partial t} + V_r \frac{\partial v'_r}{\partial r} + \frac{V_\theta}{r} \frac{\partial v'_r}{\partial \theta} - \frac{2V_\theta v'_\theta}{r} + V_x \frac{\partial v'_r}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial r} \end{aligned}$$

Now looking at the right side, Expanding the $1/\rho$ using a Taylor series approximation

$$\begin{aligned}
\frac{1}{\bar{\rho} + \rho'} &= \frac{1}{\bar{\rho}} + \left(\frac{1}{\bar{\rho} + \rho'} - \frac{1}{\bar{\rho}} \right) \\
&= \frac{1}{\bar{\rho}} + \left(\frac{\bar{\rho}}{\bar{\rho}(\bar{\rho} + \rho')} - \frac{1}{\bar{\rho}} \frac{\bar{\rho} + \rho'}{\bar{\rho} + \rho'} \right) \\
&= \frac{1}{\bar{\rho}} - \left(\frac{\bar{\rho} - \bar{\rho} + \rho'}{\bar{\rho}(\bar{\rho} + \rho')} \right) \\
&= \frac{1}{\bar{\rho}} - \frac{\rho'}{\bar{\rho}} \underbrace{\left(\frac{1}{\bar{\rho} + \rho'} \right)}_{\text{This is what we're solving for!}} \\
&= \frac{1}{\bar{\rho}} - \frac{\rho'}{\bar{\rho}} \underbrace{\left[\frac{1}{\bar{\rho}} + \left(\frac{1}{\bar{\rho} + \rho'} - \frac{1}{\bar{\rho}} \right) \right]}_{\text{This is from step 1}} \\
&= \frac{1}{\bar{\rho}} - \frac{\rho'}{\bar{\rho}^2} + \underbrace{\left[\left(\frac{\rho'}{\bar{\rho}} \right)^2 \frac{1}{\bar{\rho} + \rho'} \right]}_{\text{These are higher order terms that will go to } \infty}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\rho} \frac{\partial p}{\partial r} &= \left(-\frac{1}{\bar{\rho}} + \frac{\rho'}{\bar{\rho}^2} \right) \left(\frac{\partial \bar{p} + p'}{\partial r} \right) \\
\frac{1}{\rho} \frac{\partial p}{\partial r} &= -\cancel{\frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial r}} - \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial r} + \frac{\rho'}{\bar{\rho}^2} \frac{\partial \bar{p}}{\partial r} + \cancel{\frac{\rho'}{\bar{\rho}^2} \frac{\partial p'}{\partial r}} \\
\frac{1}{\rho} \frac{\partial p}{\partial r} &= -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial r} + \frac{\rho'}{\bar{\rho}^2} \frac{\partial \bar{p}}{\partial r}
\end{aligned}$$

$$\boxed{\frac{\partial v'_r}{\partial t} + \frac{V_\theta}{r} \frac{\partial v'_r}{\partial \theta} - \frac{2V_\theta v'_\theta}{r} + V_x \frac{\partial v'_r}{\partial x} = \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial r} + \frac{\rho'}{\bar{\rho}^2} \frac{\partial \bar{p}}{\partial r}}$$

3.5.1.3 Linearizing the Conservation of Momentum in the θ direction

Starting with the mean momentum equation

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_x \frac{\partial v_\theta}{\partial x} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta}$$

Looking at the left hand side first

$$\begin{aligned} & \frac{\partial(V_\theta + v'_\theta)}{\partial t} + (V_r + v'_r) \frac{\partial(V_\theta + v'_\theta)}{\partial r} + \\ & \frac{V_\theta + v'_\theta}{r} \frac{\partial(V_\theta + v'_\theta)}{\partial \theta} + \frac{(V_r + v'_r)(V_\theta + v'_\theta)}{r} + (V_x + v'_x) \frac{\partial(V_\theta + v'_\theta)}{\partial x} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} \\ & \quad \frac{\cancel{\partial V_\theta}}{\cancel{\partial t}} + \frac{\partial v'_\theta}{\partial t} + \\ & \quad \cancel{V_r \frac{\partial V_\theta}{\partial r}} + \underbrace{v'_r \frac{\partial V_\theta}{\partial r}}_{v'_r=0} + \cancel{V_r \frac{\partial v'_\theta}{\partial r}} + \cancel{v'_r \frac{\partial v'_\theta}{\partial r}} + \\ & \quad \frac{1}{r} \left(\cancel{V_\theta \frac{\partial V_\theta}{\partial \theta}} + \cancel{v'_\theta \frac{\partial V_\theta}{\partial \theta}} + V_\theta \frac{\partial v'_\theta}{\partial \theta} + \cancel{v'_\theta \frac{\partial v'_\theta}{\partial \theta}} \right) + \\ & \quad \frac{1}{r} (\cancel{V_r V_\theta} + v'_r V_\theta + \cancel{V_r v'_\theta} + \cancel{v'_r v'_\theta}) + \\ & \quad \cancel{V_x \frac{\partial V_\theta}{\partial x}} + \cancel{v'_x \frac{\partial V_\theta}{\partial x}} + V_x \frac{\partial v'_\theta}{\partial x} + \cancel{v'_x \frac{\partial v'_\theta}{\partial x}} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} \\ & \quad \frac{\partial v'_\theta}{\partial t} + v'_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial v'_\theta}{\partial \theta} + \frac{v'_r V_\theta}{r} + V_x \frac{\partial v'_\theta}{\partial x} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} \end{aligned}$$

Now looking at the right side, Expanding the $1/\rho$ using a Taylor series approximation

$$\begin{aligned} -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} &= \left(-\frac{1}{\bar{\rho}} + \frac{\rho'}{\bar{\rho}^2} \right) \left(\frac{\partial \bar{p} + p'}{\partial \theta} \right) \\ \frac{1}{\rho r} \frac{\partial p}{\partial \theta} &= -\frac{1}{\cancel{\bar{\rho}}} \frac{\partial \bar{p}}{\partial \theta} - \frac{1}{\bar{\rho} r} \frac{\partial p'}{\partial \theta} + \frac{\cancel{\rho'}}{\bar{\rho}^2 r} \frac{\partial \bar{p}}{\partial \theta} + \frac{\cancel{\rho'}}{\bar{\rho}^2 r} \frac{\partial p'}{\partial \theta} \\ \frac{1}{\rho r} \frac{\partial p}{\partial \theta} &= -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial \theta} \end{aligned}$$

$$\boxed{\frac{\partial v'_\theta}{\partial t} + v'_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial v'_\theta}{\partial \theta} + \frac{v'_r V_\theta}{r} + V_x \frac{\partial v'_r}{\partial x} = -\frac{1}{\bar{\rho} r} \frac{\partial p'}{\partial \theta}}$$

3.5.1.4 Linearizing the Conservation of Momentum in the x direction

Starting with the mean momentum equation

$$\frac{\partial v_x}{\partial t} + v_r \frac{\partial v_x}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_x}{\partial \theta} + v_x \frac{\partial v_x}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$

$$\begin{aligned} \frac{\partial(V_x + v'_x)}{\partial t} + (V_r + v'_r) \frac{\partial(V_x + v'_x)}{\partial r} + \frac{V_\theta + v'_\theta}{r} \frac{\partial(V_x + v'_x)}{\partial \theta} + (V_x + v'_x) \frac{\partial(V_x + v'_x)}{\partial x} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} \\ &\quad \frac{\cancel{V_x}}{\cancel{\partial t}} + \frac{\partial v'_x}{\partial t} + \\ &\quad \cancel{V_r} \frac{\cancel{\partial V_x}}{\partial r} + v'_r \frac{\partial V_x}{\partial r} + \cancel{V_r} \frac{\cancel{\partial v'_x}}{\partial r} + v'_r \frac{\partial v'_x}{\partial r} + \\ &\quad \frac{1}{r} \left(\cancel{V_\theta} \frac{\cancel{\partial V_x}}{\partial \theta} + \cancel{v'_\theta} \frac{\cancel{\partial V_x}}{\partial \theta} + V_\theta \frac{\partial v'_x}{\partial \theta} + \cancel{v'_\theta} \frac{\cancel{\partial v'_x}}{\partial \theta} \right) + \\ &\quad \cancel{V_x} \frac{\cancel{\partial V_x}}{\partial x} + \cancel{v'_x} \frac{\cancel{\partial V_x}}{\partial x} + V_x \frac{\partial v'_x}{\partial x} + \cancel{v'_x} \frac{\cancel{\partial v'_x}}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \\ &\quad \boxed{\frac{\partial v'_x}{\partial t} + v'_r \frac{\partial V_x}{\partial r} + \frac{V_\theta}{r} \frac{\partial v'_x}{\partial \theta} + V_x \frac{\partial v'_x}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x}} \end{aligned}$$

$$\begin{aligned}
&\rightarrow -\frac{1}{\rho} \frac{\partial p}{\partial x} = \left(-\frac{1}{\bar{\rho}} + \frac{\rho'}{\bar{\rho}^2} \right) \left(\frac{\partial \bar{p} + p'}{\partial x} \right) \\
&\frac{1}{\rho} \frac{\partial p}{\partial x} = -\frac{1}{\cancel{\bar{\rho}}} \frac{\partial \cancel{\bar{p}}}{\partial x} - \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x} + \frac{\rho'}{\cancel{\bar{\rho}^2} r} \frac{\partial \cancel{\bar{p}}}{\partial x} + \frac{\rho'}{\cancel{\bar{\rho}^2} r} \frac{\partial \cancel{p'}}{\partial x} \\
&\frac{1}{\rho} \frac{\partial p}{\partial x} = -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x}
\end{aligned}$$

$$\boxed{\frac{\partial v'_x}{\partial t} + v'_r \frac{\partial V_x}{\partial r} + \frac{V_\theta}{r} \frac{\partial v'_x}{\partial \theta} + V_x \frac{\partial v'_x}{\partial x} = -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x}}$$

3.5.1.5 Linearizing the Energy Equation

$$\frac{\partial p}{\partial t} + v_r \frac{\partial p}{\partial r} + \frac{v_\theta}{r} \frac{\partial p}{\partial \theta} + v_x \frac{\partial p}{\partial x} + \gamma p \left(\frac{1}{r} \frac{\partial r v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_x}{\partial x} \right) = 0$$

$$\begin{aligned} \frac{\partial(\bar{P} + P')}{\partial t} + (V_r + v'_r) \frac{\partial(\bar{P} + P')}{\partial r} + \frac{(V_\theta + v'_\theta)}{r} \frac{\partial(\bar{P} + p')}{\partial x} + (V_x + v'_x) \frac{\partial(\bar{P} + P')}{\partial x} + \dots \\ \gamma(\bar{P} + P') \left(\frac{1}{r} \frac{\partial r(V_r + v'_r)}{\partial r} + \frac{1}{r} \frac{\partial(V_\theta + v'_\theta)}{\partial \theta} + \frac{\partial(V_x + v'_x)}{\partial x} \right) = 0 \end{aligned}$$

$$\begin{aligned} & \frac{\partial \bar{P}}{\partial t} + \frac{\partial P'}{\partial t} + \\ & V_r \frac{\partial \bar{P}}{\partial r} + V_r \frac{\partial P'}{\partial r} + V'_r \frac{\partial \bar{P}}{\partial r} + V'_r \frac{\partial P'}{\partial r} + \\ & \frac{V_\theta}{r} \frac{\partial \bar{P}}{\partial x} + \frac{V_\theta}{r} \frac{\partial P'}{\partial x} + \frac{v'_\theta}{r} \frac{\partial \bar{P}}{\partial x} + \frac{v'_\theta}{r} \frac{\partial P'}{\partial x} + \\ & V_x \frac{\partial \bar{P}}{\partial x} + V_x \frac{\partial P'}{\partial x} + v'_x \frac{\partial \bar{P}}{\partial x} + v'_x \frac{\partial P'}{\partial x} + \\ & \gamma \bar{P} \left(\frac{1}{r} \frac{\partial r V_r}{\partial r} + \frac{1}{r} \frac{\partial r v'_r}{\partial r} + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial v'_\theta}{\partial \theta} + \frac{\partial V_x}{\partial x} + \frac{\partial v'_x}{\partial x} \right) \\ & \gamma P' \left(\frac{1}{r} \frac{\partial r V_r}{\partial r} + \frac{1}{r} \frac{\partial r v'_r}{\partial r} + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial v'_\theta}{\partial \theta} + \frac{\partial V_x}{\partial x} + \frac{\partial v'_x}{\partial x} \right) \end{aligned}$$

$$\boxed{\frac{\partial p'}{\partial t} + v'_r \frac{\partial P}{\partial r} + \frac{V_\theta}{r} \frac{\partial p'}{\partial \theta} + V_x \frac{\partial p'}{\partial x} + \gamma P \left(\frac{1}{r} \frac{\partial(r v'_r)}{\partial r} + \frac{1}{r} \frac{\partial v'_\theta}{\partial \theta} + \frac{\partial v'_x}{\partial x} \right) = 0}$$

The linearized Euler equations are,

$$\begin{aligned}
\frac{\partial \rho'}{\partial t} + \frac{V_\theta}{r} \frac{\partial \rho'}{\partial \theta} + V_x \frac{\partial \rho'}{\partial x} + \bar{\rho} \left(\frac{1}{r} \left(\frac{\partial r v'_r}{\partial r} + \frac{\partial v'_\theta}{\partial \theta} \right) + \frac{\partial v'_x}{\partial x} \right) &= 0 \\
\frac{\partial v'_r}{\partial t} + \frac{V_\theta}{r} \frac{\partial v'_\theta}{\partial \theta} - \frac{2V_\theta v'_\theta}{r} + V_x \frac{\partial v'_r}{\partial x} &= -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial r} + \frac{\rho'}{\bar{\rho}^2} \frac{\partial \bar{p}}{\partial r} \\
\frac{\partial v'_\theta}{\partial t} + v'_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial v'_\theta}{\partial \theta} + \frac{v'_r V_\theta}{r} + V_x \frac{\partial v'_r}{\partial x} &= -\frac{1}{\bar{\rho} r} \frac{\partial p'}{\partial \theta} \\
\frac{\partial v'_x}{\partial t} + v'_r \frac{\partial V_x}{\partial r} + \frac{V_\theta}{r} \frac{\partial v'_x}{\partial \theta} + V_x \frac{\partial v'_x}{\partial x} &= -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x} \\
\frac{\partial p'}{\partial t} + v'_r \frac{\partial P}{\partial r} + \frac{V_\theta}{r} \frac{\partial p'}{\partial \theta} + V_x \frac{\partial p'}{\partial x} + \gamma P \left(\frac{1}{r} \frac{\partial (r v'_r)}{\partial r} + \frac{1}{r} \frac{\partial v'_\theta}{\partial \theta} + \frac{\partial v'_x}{\partial x} \right) &= 0
\end{aligned}$$

Recalling:

$$\frac{\partial P}{\partial r} = \frac{\bar{\rho} V_\theta^2}{r}$$

$$\gamma P = \bar{\rho} A^2$$

$$\rho' = \frac{1}{A^2} p'$$

We can rearrange the equations to reflect Equations 2.33-2.36. Note that the momentum equation in the θ and x directions remain unchanged. The term $\frac{\partial(rv'_r)}{\partial r} = \frac{\partial(r)}{\partial r} v'_r + \frac{\partial v'_r}{\partial r} r$ in the Energy equation

$$\begin{aligned}
\frac{1}{\bar{\rho} A^2} \left(\frac{\partial p'}{\partial t} + \frac{V_\theta}{r} \frac{\partial p'}{\partial \theta} + V_x \frac{\partial p'}{\partial x} \right) + \frac{V_\theta^2}{A^2 r} v'_r + \frac{\partial v'_r}{\partial r} + \frac{v'_r}{r} + \frac{1}{r} \frac{\partial v'_\theta}{\partial \theta} + \frac{\partial v'_x}{\partial x} &= 0 \\
\frac{\partial v'_r}{\partial t} + \frac{V_\theta}{r} \frac{\partial v'_\theta}{\partial \theta} - \frac{2V_\theta v'_\theta}{r} + V_x \frac{\partial v'_r}{\partial x} &= \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial r} + \frac{V_\theta}{\bar{\rho} r A^2} p' \\
\frac{\partial v'_\theta}{\partial t} + v'_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial v'_\theta}{\partial \theta} + \frac{v'_r V_\theta}{r} + V_x \frac{\partial v'_\theta}{\partial x} &= -\frac{1}{\bar{\rho} r} \frac{\partial p'}{\partial \theta} \\
\frac{\partial v'_x}{\partial t} + v'_r \frac{\partial V_x}{\partial r} + \frac{V_\theta}{r} \frac{\partial v'_x}{\partial \theta} + V_x \frac{\partial v'_x}{\partial x} &= -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x}
\end{aligned}$$

3.6 Substituting Pertubation Variables

One key assumption is that the perturbation quantites, \tilde{p} , $\tilde{v}_r, \tilde{v}_\theta$, and \tilde{v}_x , are all exponential and that they are solely a function of radius,

$$v'_r = v_r(r)e^{i(k_x x + m\theta - \omega t)}$$

$$v'_\theta = v_\theta(r)e^{i(k_x x + m\theta - \omega t)}$$

$$v'_x = v_x(r)e^{i(k_x x + m\theta - \omega t)}$$

$$p' = p(r)e^{i(k_x x + m\theta - \omega t)}$$

Substituting the quantities into the linearized equations will give us the final governing equations. Starting with the Conservation of Momentum in the r direction,

$$\frac{\partial v'_r}{\partial t} + \frac{V_\theta}{r} \frac{\partial v'_r}{\partial \theta} - \frac{2V_\theta}{r} v'_r + V_x \frac{\partial v'_r}{\partial x} = \frac{\partial p'}{\partial r} + \frac{\rho'}{\bar{\rho}^2} \frac{\partial P}{\partial r}$$

Looking at the left hand side (LHS) of the equation, the derivatives are:

$$\frac{\partial v'_r}{\partial t} = \underbrace{\frac{\partial v_r(r)}{\partial t}}_0 e^{i(k_x x + m\theta - \omega t)} + v_r(r) (-i\omega e^{i(k_x x + m\theta - \omega t)})$$

$$\frac{\partial v'_r}{\partial \theta} = \underbrace{\frac{\partial v_r(r)}{\partial \theta}}_0 e^{i(k_x x + m\theta - \omega t)} + v_r(r) (im e^{i(k_x x + m\theta - \omega t)})$$

$$\frac{\partial v'_r}{\partial x} = \underbrace{\frac{\partial v_r(r)}{\partial x}}_0 e^{i(k_x x + m\theta - \omega t)} + v_r(r) (ik_x e^{i(k_x x + m\theta - \omega t)})$$

Similarly for the right hand side (RHS),

$$\frac{\partial p'}{\partial r} = \frac{\partial P(r)}{\partial r} e^{i(k_x x + m\theta - \omega t)} + P(r) \underbrace{\frac{\partial}{\partial r}}_0 e^{i(k_x x + m\theta - \omega t)}$$

Recalling $p'/\rho' = A^2 \rightarrow \rho' = \frac{1}{A^2} p' \frac{\partial \bar{P}}{\partial r} = \frac{\bar{\rho} v_\theta^2}{r}$

$$\frac{\rho'}{\bar{\rho}^2} \frac{\partial \bar{P}}{\partial r} = \frac{V_\theta^2}{A^2} \frac{1}{\bar{\rho} r} p'$$

After substituting and canceling common terms,

$$\begin{aligned} v_r (-i\omega \cancel{e^{i(k_x x + m\theta - \omega t)}}) + \frac{V_\theta}{r} v_r (im \cancel{e^{i(k_x x + m\theta - \omega t)}}) - \frac{2V_\theta}{r} v_r \cancel{e^{i(k_x x + m\theta - \omega t)}} + V_x (v_r (ik_x \cancel{e^{i(k_x x + m\theta - \omega t)}})) \\ = \left(-\frac{1}{\bar{\rho}} \frac{\partial P(r)}{\partial r} \cancel{e^{i(k_x x + m\theta - \omega t)}} + \frac{V_\theta^2}{A^2} \frac{1}{\bar{\rho} r} P(r) \cancel{e^{i(k_x x + m\theta - \omega t)}} \right) \end{aligned}$$

$$\left(-i\omega + \frac{imV_\theta}{r} + ik_x V_x\right) v_r - \frac{2V_\theta}{r} v_\theta = -\frac{1}{\bar{\rho}} \frac{\partial P}{\partial r} + \frac{V_\theta^2}{A^2} \frac{1}{\bar{\rho} r} p$$

Continuing with conservation of momentum in the θ direction,

$$\frac{\partial v'_\theta}{\partial t} + v'_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial v'_\theta}{\partial \theta} + \frac{v'_r V_\theta}{r} + V_x \frac{\partial v'_\theta}{\partial x} = -\frac{1}{\bar{\rho} r} \frac{\partial p'}{\partial \theta}$$

$$\frac{\partial v'_\theta}{\partial t} = \underbrace{\frac{\partial v_\theta(r)}{\partial t}}_0 e^{i(k_x x + m\theta - \omega t)} + v_\theta(r) (-i\omega e^{i(k_x x + m\theta - \omega t)})$$

$$\frac{\partial v'_\theta}{\partial \theta} = \underbrace{\frac{\partial v'_\theta(r)}{\partial \theta}}_0 e^{i(k_x x + m\theta - \omega t)} + v_\theta(r) (im e^{i(k_x x + m\theta - \omega t)})$$

$$\frac{\partial v'_\theta}{\partial x} = \underbrace{\frac{\partial v_\theta(r)}{\partial x}}_0 e^{i(k_x x + m\theta - \omega t)} + v_\theta(r) (ik_x e^{i(k_x x + m\theta - \omega t)})$$

$$\frac{\partial p'}{\partial \theta} = \underbrace{\frac{\partial P(r)}{\partial \theta}}_0 e^{i(k_x x + m\theta - \omega t)} + P(r) \underbrace{\frac{\partial}{\partial \theta} e^{i(k_x x + m\theta - \omega t)}}_{me^{i(k_x x + m\theta - \omega t)}}$$

After substituting and canceling common terms

$$\left(-i\omega + \frac{imV_\theta}{r} + ik_x V_x\right) v_\theta + \left(\frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r}\right) v_\theta = -\frac{m}{\bar{\rho} r} p$$

Next, the conservation of momentum in the x direction,

$$\frac{\partial v'_x}{\partial t} + v'_r \frac{\partial V_x}{\partial r} + \frac{V_\theta}{r} \frac{\partial v'_x}{\partial \theta} + V_x \frac{\partial v'_x}{\partial x} = -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x}$$

$$\frac{\partial v'_x}{\partial t} = \underbrace{\frac{\partial v_x(r)}{\partial t}}_0 e^{i(k_x x + m\theta - \omega t)} + v_x(r) (-i\omega e^{i(k_x x + m\theta - \omega t)})$$

$$\frac{\partial v'_x}{\partial \theta} = \underbrace{\frac{\partial v_x(r)}{\partial \theta}}_0 e^{i(k_x x + m\theta - \omega t)} + v_x(r) (im e^{i(k_x x + m\theta - \omega t)})$$

$$\frac{\partial v'_x}{\partial x} = \frac{\partial v_x(r)}{\partial x} e^{i(k_x x + m\theta - \omega t)} + v_x(r) (ik_x e^{i(k_x x + m\theta - \omega t)})$$

$$\frac{\partial p'}{\partial x} = 0 + ik_x p e^{i(k_x x + m\theta - \omega t)}$$

$$\boxed{\left(-i\omega + \frac{imV_\theta}{r} + ik_x V_x \right) v_x + \frac{\partial V_x}{\partial r} v_r = -\frac{ik_x}{\bar{\rho}} p}$$

Continuing with the Conservation of Energy,

$$\frac{1}{\bar{\rho} A^2} \left(\frac{\partial p'}{\partial t} + \frac{V_\theta}{r} \frac{\partial p'}{\partial \theta} + V_x \frac{\partial p'}{\partial x} \right) + \frac{V_\theta^2}{A^2 r} v'_r + \frac{\partial v'_r}{\partial r} + \frac{1}{r} \frac{\partial v'_\theta}{\partial \theta} + \frac{\partial v'_x}{\partial x} = 0$$

Left hand side (LHS) derivatives:

$$\frac{\partial p'}{\partial t} = \underbrace{\frac{\partial p(r)}{\partial t}}_0 e^{i(k_x x + m\theta - \omega t)} + p(r) (-i\omega e^{i(k_x x + m\theta - \omega t)})$$

$$\frac{\partial p'}{\partial \theta} = \underbrace{\frac{\partial p(r)}{\partial \theta}}_0 e^{i(k_x x + m\theta - \omega t)} + p(r) (im e^{i(k_x x + m\theta - \omega t)})$$

$$\frac{\partial p'}{\partial x} = \underbrace{\frac{\partial p(r)}{\partial x}}_0 e^{i(k_x x + m\theta - \omega t)} + p(r) (ik_x e^{i(k_x x + m\theta - \omega t)})$$

$$\frac{\partial v'_r}{\partial r} = \frac{\partial v_r(r)}{\partial r} e^{i(k_x x + m\theta - \omega t)} + v_r(r) \underbrace{\frac{\partial}{\partial r} (e^{i(k_x x + m\theta - \omega t)})}_0$$

$$\frac{\partial v'_\theta}{\partial \theta} = \frac{\partial v_\theta(r)}{\partial \theta} e^{i(k_x x + m\theta - \omega t)} + v_\theta(r) (im e^{i(k_x x + m\theta - \omega t)})$$

$$\frac{\partial v'_x}{\partial x} = \underbrace{\frac{\partial v_x(r)}{\partial x}}_0 e^{i(k_x x + m\theta - \omega t)} + v_x(r) (ik_x e^{i(k_x x + m\theta - \omega t)})$$

After substituting and canceling common terms,

$$\begin{aligned} & \frac{1}{\bar{\rho}A^2} \left(p(r) (-i\omega e^{i(k_x x + m\theta - \omega t)}) + \frac{V_\theta}{r} p(r) (ime^{i(k_x x + m\theta - \omega t)}) + V_x p(r) (ik_x e^{i(k_x x + m\theta - \omega t)}) \right) + \\ & \frac{V_\theta^2}{A^2 r} v'_r + \frac{\partial v_r(r)}{\partial r} e^{i(k_x x + m\theta - \omega t)} + \frac{1}{r} (v_\theta(r) (ime^{i(k_x x + m\theta - \omega t)})) + v_x(r) (ik_x e^{i(k_x x + m\theta - \omega t)}) = 0 \end{aligned}$$

$$\boxed{\frac{1}{\bar{\rho}A^2} \left(-i\omega + \frac{imV_\theta}{r} + ik_x V_x \right) p(r) + \frac{V_\theta^2}{A^2 r} v_r + \frac{v_r}{r} + \frac{\partial v_r(r)}{\partial r} + \frac{im}{r} v_\theta(r) + ik_x v_x(r) = 0}$$

The Linearized Euler equations now become

$$r\text{-direction: } i \left(-\omega + \frac{m}{r} + k_x V_x \right) v_r - \frac{2\bar{v}_\theta}{r} v_\theta = -\frac{1}{\bar{\rho}} \frac{\partial P}{\partial r} + \frac{V_{\theta^2}}{A^2} \frac{1}{\bar{\rho}r} p$$

$$\theta\text{-direction: } i \left(-\omega + \frac{m}{r} + k_x V_x \right) v_\theta + \left(\frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r} \right) v_\theta = -\frac{m}{\bar{\rho}r} p$$

$$x\text{-direction: } i \left(-\omega + \frac{mV_\theta}{r} + k_x V_x \right) v_x + \frac{\partial V_x}{\partial r} v_r = -\frac{ik_x}{\bar{\rho}} p$$

$$\text{Energy: } \frac{1}{\bar{\rho}A^2} \left(-i\omega + \frac{imV_\theta}{r} + ik_x V_x \right) p(r) + \frac{V_\theta^2}{A^2 r} v_r + \frac{v_r}{r} + \frac{\partial v_r(r)}{\partial r} + \frac{im}{r} v_\theta(r) + ik_x v_x(r) = 0$$

3.7 Non-Dimensionalization

Defining

$$r_T = r_{max}$$

$$A_T = A(r_{max})$$

$$k = \frac{\omega r_T}{A_T}$$

$$\bar{\gamma} = k_x r_T$$

$$\tilde{r} = \frac{r}{r_T}$$

$$\frac{\partial}{\partial r} = \frac{\partial \tilde{r}}{\partial r} \frac{\partial}{\partial \tilde{r}} = \frac{1}{r_T} \frac{\partial}{\partial \tilde{r}}$$

$$V_\theta = M_\theta A$$

$$V_x = M_x A$$

$$\tilde{A} = \frac{A}{A_T}$$

$$v_x = \tilde{v}_x A$$

$$v_r = \tilde{v}_r A$$

$$v_\theta = \tilde{v}_\theta A$$

$$p = \tilde{p} \bar{\rho} A^2$$

$$r\text{-direction: } i \left(-\omega + \frac{mV_\theta}{r} + k_x V_x \right) v_r - \frac{2\bar{v}_\theta}{r} v_\theta = -\frac{1}{\bar{\rho}} \frac{\partial P}{\partial r} + \frac{V_\theta^2}{A^2} \frac{1}{\bar{\rho} r} p$$

$$\theta\text{-direction: } i \left(-\omega + \frac{m}{r} + k_x V_x \right) v_\theta + \left(\frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r} \right) v_\theta = -\frac{m}{\bar{\rho} r} p$$

$$x\text{-direction: } i \left(-\omega + \frac{mV_\theta}{r} + k_x V_x \right) v_x + \frac{\partial V_x}{\partial r} v_r = -\frac{ik_x}{\bar{\rho}} p$$

$$\text{Energy: } \frac{1}{\bar{\rho} A^2} i \left(-\omega + \frac{mV_\theta}{r} + k_x V_x \right) p(r) + \frac{V_\theta^2}{A^2 r} v_r + \frac{v_r}{r} + \frac{\partial v_r(r)}{\partial r} + \frac{im}{r} v_\theta(r) + ik_x v_x(r) = 0$$

Substituting the non dimensional quantities, and noting r_T and A^2 in each term, the radial momentum equation becomes,

$$i \left[-\frac{k}{\bar{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma} M_x \right] \tilde{v}_r - \frac{2M_\theta \tilde{v}_\theta}{\tilde{r}} = -\frac{1}{\bar{\rho} A^2} \frac{\partial \tilde{p} \bar{\rho} A^2}{\partial \tilde{r}} + M_\theta \frac{\tilde{p}}{\tilde{r}}$$

Similarly for the θ and x directions:

$$i \left[-\frac{k}{\bar{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma} M_x \right] \tilde{v}_\theta + \left(\frac{M_\theta}{\tilde{r}} + \frac{1}{A} \frac{\partial M_\theta A}{\partial \tilde{r}} \right) \tilde{v}_r = \frac{im}{\tilde{r}} \tilde{P}$$

$$i \left[-\frac{k}{\bar{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma} M_x \right] \tilde{v}_x + \frac{1}{A} \frac{\partial M_x A}{\partial \tilde{r}} \tilde{v}_r = -i\bar{\gamma} \tilde{P}$$

and for the Energy equation:

$$i \left[-\frac{k}{\bar{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma} M_x \right] \tilde{p} + \frac{M_\theta^2}{\tilde{r}} \tilde{v}_r + \frac{1}{A} \frac{\partial (\tilde{v}_r A)}{\partial \tilde{r}} + \frac{\tilde{v}_r}{\tilde{r}} + \frac{im}{\tilde{r}} \tilde{v}_\theta + i\bar{\gamma} \tilde{v}_x = 0$$

Expanding mean derivatives (using product rule) $\frac{\partial \tilde{p} \bar{\rho} A^2}{\partial \tilde{r}}$

$$i \left[-\frac{k}{\bar{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma} M_x \right] \tilde{v}_r - \frac{2M_\theta \tilde{v}_\theta}{\tilde{r}} = - \left(\frac{\partial \tilde{p}}{\partial \tilde{r}} + \frac{\tilde{p}}{\bar{\rho} A^2} \frac{\partial \bar{\rho} A^2}{\partial \tilde{r}} \right) + M_\theta \frac{\tilde{p}}{\tilde{r}}$$

$$i \left[-\frac{k}{\tilde{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma}M_x \right] \tilde{v}_\theta + \left(\frac{M_\theta}{\tilde{r}} + \frac{1}{A} \frac{\partial M_\theta A}{\partial \tilde{r}} \right) \tilde{v}_r = \frac{im}{\tilde{r}} \tilde{P}$$

$$i \left[-\frac{k}{\tilde{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma}M_x \right] \tilde{v}_x + \frac{1}{A} \frac{\partial M_x A}{\partial \tilde{r}} \tilde{v}_r = -i\bar{\gamma}\tilde{P}$$

$$i \left[-\frac{k}{\tilde{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma}M_x \right] \tilde{p} + \frac{M_\theta^2}{\tilde{r}} \tilde{v}_r + \frac{\partial \tilde{v}_r}{\partial \tilde{r}} + \frac{1}{A} \frac{\partial A}{\partial \tilde{r}} \tilde{v}_r + \frac{\tilde{v}_r}{\tilde{r}} + \frac{im}{\tilde{r}} \tilde{v}_\theta + i\bar{\gamma}\tilde{v}_x = 0$$

$$\frac{1}{A} \frac{\partial A}{\partial \tilde{r}}$$

Recall, $\partial/\partial r = (1/r_T)(\partial/\partial \tilde{r})$,

$$\begin{aligned} \frac{1}{A} \frac{\partial A}{\partial \tilde{r}} &= r_T \left(\frac{1}{A} \frac{\partial A}{\partial r} \right) \\ &= \frac{r_T}{A^2} \left(A \frac{\partial A}{\partial r} \right) \end{aligned}$$

Using the trick, $\frac{\partial}{\partial r} \left(\frac{A^2}{2} \right) = A \frac{\partial A}{\partial r}$

$$\begin{aligned} &= \frac{r_T}{A^2} \left(\frac{\partial}{\partial r} \left(\frac{A^2}{2} \right) \right) \\ &= \frac{r_T}{2A^2} \frac{\partial A^2}{\partial r} \end{aligned}$$

Using the definition derived earlier (Needs eqn reference) $\frac{\partial A^2}{\partial r} = \frac{\gamma-1}{2} \frac{v_\theta^2}{r}$

$$\begin{aligned} &= \frac{r_T}{2A^2} \frac{\gamma-1}{2} \frac{v_\theta^2}{r} \\ &= \frac{\gamma-1}{2} \frac{M_\theta^2}{\tilde{r}} \end{aligned}$$

$$\begin{aligned} \frac{\partial \rho A^2}{\partial \tilde{r}} &= \gamma \frac{\partial p}{\partial \tilde{r}} \\ &= \gamma \frac{\rho v_\theta^2}{\tilde{r}} \\ &= r_T \gamma \frac{\rho v_\theta^2}{r} \end{aligned}$$

$$\frac{1}{\rho A^2} \frac{\partial \rho A^2}{\partial r} = \frac{\gamma M_\theta^2}{\tilde{r}}$$

Substituting in yields ,

$$i \left[-\frac{k}{\tilde{A}} + \frac{m M_\theta}{\tilde{r}} + \bar{\gamma} M_x \right] \tilde{v}_r - \frac{2 M_\theta \tilde{v}_\theta}{\tilde{r}} = -\frac{\partial \tilde{p}}{\partial \tilde{r}} - (\gamma - 1) \frac{\gamma M_\theta}{\tilde{r}} \tilde{p}$$

$$i \left[-\frac{k}{\tilde{A}} + \frac{m M_\theta}{\tilde{r}} + \bar{\gamma} M_x \right] \tilde{v}_\theta + \left(\frac{M_\theta}{\tilde{r}} + \frac{1}{A} \frac{\partial M_\theta A}{\partial \tilde{r}} \right) \tilde{v}_r = \frac{i m}{\tilde{r}} \tilde{p}$$

$$i \left[-\frac{k}{\tilde{A}} + \frac{m M_\theta}{\tilde{r}} + \bar{\gamma} M_x \right] \tilde{v}_x + \frac{1}{A} \frac{\partial M_x A}{\partial \tilde{r}} \tilde{v}_r = -i \bar{\gamma} \tilde{p}$$

$$i \left[-\frac{k}{\tilde{A}} + \frac{m M_\theta}{\tilde{r}} + \bar{\gamma} M_x \right] \tilde{p} + \frac{M_\theta^2}{\tilde{r}} \tilde{v}_r + \frac{\partial \tilde{v}_r}{\partial \tilde{r}} + \frac{1}{A} \frac{\partial A}{\partial \tilde{r}} v_r + \frac{\tilde{v}_r}{\tilde{r}} + \frac{i m}{\tilde{r}} \tilde{v}_\theta + i \bar{\gamma} \tilde{v}_x = 0$$

Defining, $\lambda = -i \bar{\gamma}$

and defining

$$\{\bar{x}\} = \begin{pmatrix} \tilde{v}_r \\ \tilde{v}_\theta \\ \tilde{v}_x \\ \tilde{p} \end{pmatrix}$$

The governing equations can be written in the form of $[A]x - \lambda[B]x$

$$\bar{x} = \begin{bmatrix} -i \left(\frac{\kappa}{A} - \frac{mM_\theta}{\tilde{r}} \right) - \lambda M_x & -\frac{2M_\theta}{\tilde{r}} & 0 & \frac{\partial}{\partial \tilde{r}} + \frac{\gamma-1}{\tilde{r}} \\ \frac{M_\theta}{\tilde{r}} + \frac{\partial M_\theta}{\partial \tilde{r}} + \left(\frac{\gamma-1}{2} \right) \frac{M_\theta^3}{\tilde{r}} & -i \left(\frac{\kappa}{A} - \frac{mM_\theta}{\tilde{r}} \right) - \lambda M_x & 0 & \frac{im}{\tilde{r}} \\ \frac{\partial M_x}{\partial \tilde{r}} + \left(\frac{\gamma-1}{2} \frac{M_x M_\theta^2}{\tilde{r}} \right) & 0 & -i \left(\frac{\kappa}{A} - \frac{mM_\theta}{\tilde{r}} \right) - \lambda M_x & -\lambda \\ \frac{\partial}{\partial \tilde{r}} + \frac{\gamma+1}{2} \frac{M_\theta^2}{\tilde{r}} + \frac{1}{\tilde{r}} & \frac{im}{\tilde{r}} & -\lambda & -i \left(\frac{\kappa}{A} - \frac{mM_\theta}{\tilde{r}} \right) - \lambda M_x \end{bmatrix}$$

Chapter 4

Chapter 4: Numerical Models

4.1 Numerical Integration

4.2 Introduction

The Method of Manufactured Solutions (MMS) is a process for generating an analytical solution for a code that provides the numerical solution for a given domain. The goal of MMS is to establish a manufactured solution that can be used to establish the accuracy of the code within question. For this study, SWIRL, a code used to calculate the radial modes within an infinitely long duct is being validated through code verification. SWIRL accepts a given mean flow and uses numerical integration to obtain the speed of sound. The integration technique is found to be the composite trapezoidal rule through asymptotic error analysis.

For SWIRL, the absolute bare minimum requirement is to define the corresponding flow components for the domain of interest. SWIRL assumes no flow in the radial direction, leaving only two other components, axial and tangential for a 3D cylindrical domain. Since SWIRL is also non dimensionalized, the mean flow components are defined using the Mach number. SWIRL uses the tangential mach number to obtain the speed of sound using numerical integration. The speed of sound is then used to

find the rest of the primitive variables for the given flow.

4.3 Methods

SWIRL is a linearized Euler equations of motion code that calculates the axial wavenumber and radial mode shapes from small unsteady disturbances in a mean flow. The mean flow varies along the axial and tangential directions as a function of radius. The flow domain can either be a circular or annular duct, with or without acoustic liner. SWIRL was originally written by Kousen [insert ref].

The SWIRL code requires two mean flow parameters as a function of radius, M_x , and M_θ . Afterwards, the speed of sound, \tilde{A} is calculated by integrating M_θ with respect to r . To verify that SWIRL is handling and returning the accompanying mean flow parameters, the error between the mean flow input and output variables are computed. Since the trapezoidal rule is used to numerically integrate M_θ , the discretization error and order of accuracy is computed. Since finite differencing schemes are to be used on the result of this integration, it is crucial to accompany the integration with methods of equal or less order of accuracy. This will be determined by applying another MMS on the eigenproblem which will also have an order of accuracy.

4.3.1 Theory

To relate the speed of sound to a given flow, the radial momentum equation is used. If the flow contains a swirling component, then the primitive variables are nonuniform through the flow, and mean flow assumptions are not valid.

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} - \frac{v_\theta^2}{r} v_x \frac{\partial v_r}{\partial x} = \frac{1}{\rho} \frac{\partial P}{\partial r}$$

To account to for this, the radial momentum is simplified by assuming the flow is steady, the flow has no radial component. In addition, the viscous and body forces are neglected. Then the radial pressure derivative term is set equal to the dynamic pressure term. Separation of variables is applied.

$$\frac{v_\theta^2}{r} = \frac{1}{\rho} \frac{\partial P}{\partial r}$$

$$P = \int_r^{r_{max}} \frac{\rho V_\theta^2}{r}$$

To show the work, we will start with the dimensional form of the equation and differentiate both sides. Applying separation of variables,

$$\int_r^{r_{max}} \frac{\bar{\rho} v_\theta^2}{r} \partial r = - \int_{P(r)}^{P(r_{max})} \partial p.$$

Since $\tilde{r} = r/r_{max}$,

$$r = \tilde{r} r_{max}.$$

Taking total derivatives (i.e. applying chain rule),

$$dr = d(\tilde{r} r_{max}) = d(\tilde{r}) r_{max},$$

Substituting these back in and evaluating the right hand side,

$$\int_{\tilde{r}}^1 \frac{\bar{\rho} v_{\theta}^2}{\tilde{r}} \partial \tilde{r} = P(1) - P(\tilde{r})$$

For reference the minimum value of \tilde{r} is,

$$\sigma = \frac{r_{max}}{r_{min}}$$

For the radial derivative, the definition of the speed of sound is utilized,

$$\frac{\partial A^2}{\partial r} = \frac{\partial}{\partial r} \left(\frac{\gamma P}{\rho} \right).$$

Using the quotient rule, the definition of the speed of sound is extracted,

$$\begin{aligned} &= \frac{\partial P}{\partial r} \frac{\gamma \bar{\rho}}{\bar{\rho}^2} - \left(\frac{\gamma P}{\bar{\rho}^2} \right) \frac{\partial \bar{\rho}}{\partial r} \\ &= \frac{\partial P}{\partial r} \frac{\gamma}{\bar{\rho}} - \left(\frac{A^2}{\bar{\rho}} \right) \frac{\partial \bar{\rho}}{\partial r} \end{aligned}$$

Using isentropic condition $\partial P/A^2 = \partial \rho$,

$$\begin{aligned} &= \frac{\partial P}{\partial r} \frac{\gamma}{\bar{\rho}} - \left(\frac{1}{\bar{\rho}} \right) \frac{\partial P}{\partial r} \\ \frac{\partial A^2}{\partial r} &= \frac{\partial P}{\partial r} \frac{\gamma - 1}{\bar{\rho}} \end{aligned}$$

$$\frac{\bar{\rho}}{\gamma - 1} \frac{\partial A^2}{\partial r} = \frac{\partial P}{\partial r}$$

Going back to the radial momentum equation, and rearranging the terms will simplify the expression. The following terms are defined to start the nondimensionalization.

$$\begin{aligned}
M_\theta &= \frac{V_\theta}{A} \\
\tilde{r} &= \frac{r}{r_{max}} \\
\tilde{A} &= \frac{A}{A_{r,max}} \\
A &= \tilde{A}A_{r,max} \\
r &= \tilde{r}r_{max} \\
\frac{\partial}{\partial r} &= \frac{\partial \tilde{r}}{\partial r} \frac{\partial}{\partial \tilde{r}} \\
&= \frac{1}{r_{max}} \frac{\partial}{\partial \tilde{r}}
\end{aligned}$$

Dividing by A ,

$$\frac{M_\theta^2}{r} (\gamma - 1) = \frac{\partial A^2}{\partial r} \frac{1}{A^2}$$

Now there is two options, either find the derivative of \tilde{A} or the integral of M_θ with respect to r .

1. Defining non dimensional speed of sound $\tilde{A} = \frac{A(r)}{A(r_{max})}$

$$\begin{aligned}
\int_r^{r_{max}} \frac{M_\theta}{r} (\gamma - 1) \partial r &= \ln \left(\frac{1}{\tilde{A}^2} \right) \\
&= -2 \ln(\tilde{A}) \\
\tilde{A}(r) &= \exp \left[- \int_r^{r_{max}} \frac{M_\theta (\gamma - 1)}{r} \partial r \right] \\
\text{replacing } r \text{ with } \tilde{r} \rightarrow \tilde{A}(r) &= \exp \left[- \int_r^{r_{max}} \frac{M_\theta (\gamma - 1)}{r} \partial r \right] \\
\tilde{A}(\tilde{r}) &= \exp \left[\left(\frac{1 - \gamma}{2} \right) \int_{\tilde{r}}^1 \frac{M_\theta}{\tilde{r}} \partial \tilde{r} \right]
\end{aligned}$$

2. Or we can differentiate

Solving for M_θ ,

$$M_\theta^2 = \frac{\partial A^2}{\partial r} \frac{r}{A^2 (\gamma - 1)}$$

Nondimensionalizing and substituting,

$$\begin{aligned}
M_\theta^2 \frac{(\gamma - 1)}{\tilde{r} r_{max}} &= \frac{1}{(\tilde{A} A_{r,max})^2} \frac{A_{r,max}^2}{r_{max}} \frac{\partial \tilde{A}^2}{\partial \tilde{r}} \\
M_\theta^2 \frac{(\gamma - 1)}{\tilde{r}} &= \frac{1}{\tilde{A}^2} \frac{\partial \tilde{A}^2}{\partial \tilde{r}} \\
M_\theta &= \sqrt{\frac{\tilde{r}}{(\gamma - 1) \tilde{A}^2} \frac{\partial \tilde{A}^2}{\partial \tilde{r}}}
\end{aligned} \tag{4.1}$$

4.3.2 Calculation of Observed Order-of-Accuracy

The numerical scheme used to perform the integration of the tangential velocity will have a theoretical order-of-accuracy. To find the theoretical order-of-accuracy, the discretization error must first be defined. The error, ϵ , is a function of id spacing, Δr

$$\epsilon = \epsilon(\Delta r)$$

The discretization error in the solution should be proportional to $(\Delta r)^\alpha$ where $\alpha > 0$ is the theoretical order for the computational method. The error for each grid is expressed as

$$\epsilon_{M_\theta}(\Delta r) = |M_{\theta,analytic} - M_{\theta,calc}|$$

where $M_{\theta,analytic}$ is the tangential mach number that is defined from the speed of sound we also defined and the $M_{\theta,calc}$ is the result from SWIRL. The Δr is to indicate that this is a discretization error for a specific grid spacing. Applying the same concept to the speed of sound,

If we define this error on various grid sizes and compute ϵ for each grid, the observed order of accuracy can be estimated and compared to the theoretical order of accuracy. For instance, if the numerical soution is second-order accurate and the error is converging to a value, the L2 norm of the error will decrease by a factor of 4 for every halving of the grid cell size.

Since the input variables should remain unchanged (except from minor changes from the Akima interpolation), the error for the axial and tangential mach number should be zero. As for the speed of sound, since we are using an analytic expression for the tangential mach number, we know what the theoretical result would be from the numerical integration technique as shown above. Similarly we define the discretization error for the speed of sound.

$$\epsilon_A(\Delta r) = |A_{analytic} - A_{calc}|$$

For a perfect answer, we expect ϵ to be zero. Since a Taylor series can be used to derive the numerical schemes, we know that the truncation of higher order terms is what indicates the error we expect from using a scheme that is constructed with such truncated Taylor series.

The error at each grid point j is expected to satisfy the following,

$$\begin{aligned} 0 &= |A_{analytic}(r_j) - A_{calc}(r_j)| \\ \tilde{A}_{analytic}(r_j) &= \tilde{A}_{calc}(r_j) + (\Delta r)^\alpha \beta(r_j) + H.O.T \end{aligned}$$

where the value of $\beta(r_j)$ does not change with grid spacing, and α is the asymptotic order of accuracy of the method. It is important to note that the numerical method recovers the original equations as the grid spacing approached zero. It is important to note that β represents the first derivative of the Taylor Series. Subtracting $A_{analytic}$ from both sides gives,

$$\begin{aligned} A_{calc}(r_j) - A_{analytic}(r_j) &= A_{analytic}(r_j) - A_{analytic}(r_j) + \beta(r_j)(\Delta r)^\alpha \\ \epsilon_A(r_j)(\Delta r) &= \beta(r_j)(\Delta r)^\alpha \end{aligned}$$

To estimate the order of accuracy of the accuracy, we define the global errors by calculating the L2 Norm of the error which is denoted as $\hat{\epsilon}_A$

$$\hat{\epsilon}_A = \sqrt{\frac{1}{N} \sum_{j=1}^N \epsilon(r_j)^2}$$

$$\hat{\beta}_A(r_j) = \sqrt{\frac{1}{N} \sum_{j=1}^N \beta(r_j)^2}$$

As the grid density increases, $\hat{\beta}$ should asymptote to a constant value. Given two grid densities, Δr and $\sigma \Delta r$, and assuming that the leading error term is much larger than any other error term,

$$\begin{aligned}\hat{\epsilon}_{grid1} &= \hat{\epsilon}(\Delta r) = \hat{\beta}(\Delta r)^\alpha \\ \hat{\epsilon}_{grid2} &= \hat{\epsilon}(\sigma \Delta r) = \hat{\beta}(\sigma \Delta r)^\alpha \\ &= \hat{\beta}(\Delta r)^\alpha \sigma^\alpha\end{aligned}$$

The ratio of two errors is given by,

$$\begin{aligned}\frac{\hat{\epsilon}_{grid2}}{\hat{\epsilon}_{grid1}} &= \frac{\hat{\beta}(\Delta r)^\alpha}{\hat{\beta}(\Delta r)^\alpha} \sigma^\alpha \\ &= \sigma^\alpha\end{aligned}$$

Thus, α , the asymptotic rate of convergence is computed as follows

$$\alpha = \frac{\ln \frac{\hat{\epsilon}_{grid2}}{\hat{\epsilon}_{grid1}}}{\ln(\sigma)}$$

Defining for a doubling of grid points ,

$$\alpha = \frac{\ln(\hat{\epsilon}(\frac{1}{2}\Delta r)) - \ln(\hat{\epsilon}(\Delta r))}{\ln(\frac{1}{2})}$$

Similarly for the eigenvalue problem, Initially the source terms were defined without mention of the indices of the matrices they make up. In other words, there was no fore sight on the fact that these source terms are sums of the elements within A,B, and X. To investigate the source terms in greater detail, the FORTRAN code that calls the source terms will output the terms within the source term and then sum them, instead of just their sum.

$$[A]x - \lambda[B]x = 0$$

4.4 Fairing Functions

Goal: How can we modify a manufactured solution such that the endpoints are suitable for comparison against a codes boundary condition implementation

4.5 Setting Boundary Condition Values Using a Fairing Function

4.5.1 Using β as a scaling parameter

Defining the nondimensional radius in the same way that SWIRL does:

$$\tilde{r} = \frac{r}{r_T}$$

where r_T is the outer radius of the annulus.

The hub-to-tip ratio is defined as:

$$\sigma = \frac{r_H}{r_T} = \tilde{r}_H$$

where \tilde{r}_H is the inner radius of the annular duct. The hub-to-tip ratio can also be zero indicating the duct is hollow.

A useful and similar parameter is introduced, β , where $0 \leq \beta \leq 1$

$$\beta = \frac{r - r_H}{r_T - r_H}$$

Dividing By r_T

$$\begin{aligned}\beta &= \frac{\frac{r}{r_T} - \frac{r_H}{r_T}}{\frac{r_T}{r_T} - \frac{r_H}{r_T}} \\ &= \frac{\tilde{r} - \tilde{r}_H}{1 - \sigma}\end{aligned}$$

Suppose a manufactured solution f_{MS} with boundaries $f_{MS}(r = \sigma)$ and $f_{MS}(\tilde{r} = 1)$ is the specified analytical solution. The goal is to change the boundary conditions of the manufactured solution in such way that allows us to adequately check the boundary conditions imposed on SWIRL. Defining the manufactured solution, $f_{MS}(\tilde{r})$, where $\sigma \leq \tilde{r} \leq 1$ and there are desired values of f at the boundaries desired values are going to be denoted as f_{minBC} and f_{maxBC} . The desired changes in f are defined as:

$$\begin{aligned}\Delta f_{minBC} &= f_{minBC} - f_{MS}(\tilde{r} = \sigma) \\ \Delta f_{maxBC} &= f_{maxBC} - f_{MS}(\tilde{r} = 1)\end{aligned}$$

We'd like to impose these changes smoothly on the manufactured solution function. To do this, the fairing functions, $A_{min}(\tilde{r})$ and $A_{max}(\tilde{r})$ where:

$$f_{BCsImposed}(\tilde{r}) = f_{MS}(\tilde{r}) + A_{min}(\tilde{r})\Delta f_{minBC} + A_{max}(\tilde{r})\Delta f_{maxBC}$$

Then, in order to set the condition at the appropriate boundary, the following conditions are set,

$$A_{min}(\tilde{r} = \sigma) = 1$$

$$A_{min}(\tilde{r} = 1) = 0$$

$$A_{max}(\tilde{r} = 1) = 1$$

$$A_{max}(\tilde{r} = \sigma) = 0$$

If $A_{min}(\tilde{r})$ is defined as a function of $A_{max}(\tilde{r})$ then only $A_{max}(\tilde{r})$ needs to be defined, therefore

$$A_{min}(\tilde{r}) = 1 - A_{max}(\tilde{r})$$

It is also desirable to set the derivatives for the fairing function at the boundaries incase there are boundary conditions imposed on the derivatives of the fairing function.

$$\begin{aligned}\frac{\partial A_{max}}{\partial \tilde{r}}|_{\tilde{r}=\sigma} &= 0 \\ \frac{\partial A_{max}}{\partial \tilde{r}}|_{\tilde{r}=1} &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial A_{min}}{\partial \tilde{r}}|_{\tilde{r}=\sigma} &= 0 \\ \frac{\partial A_{min}}{\partial \tilde{r}}|_{\tilde{r}=1} &= 0\end{aligned}$$

4.5.2 Minimum Boundary Fairing Function

Looking at A_{min} first, the polynomial is:

$$A_{min}(\beta) = a + b\beta + c\beta^2 + d\beta^3$$

$$A_{min}(\tilde{r}) = a + b\left(\frac{\tilde{r} - \sigma}{1 - \sigma}\right) + c\left(\frac{\tilde{r} - \sigma}{1 - \sigma}\right)^2 + d\left(\frac{\tilde{r} - \sigma}{1 - \sigma}\right)^3$$

Taking the derivative,

$$A'_{min}(\tilde{r}) = b\left(\frac{1}{1 - \sigma}\right) + 2c\left(\frac{1}{1 - \sigma}\right)\left(\frac{\tilde{r} - \sigma}{1 - \sigma}\right) + 3d\left(\frac{1}{1 - \sigma}\right)\left(\frac{\tilde{r} - \sigma}{1 - \sigma}\right)^2$$

$$A'_{min}(\beta) = \left(\frac{1}{1 - \sigma}\right)[b + 2c\beta + 3d\beta^2]$$

Now we will use the conditons mentioned earlier as constraints to this system of equations Using the possible values of \tilde{r} ,

$$A_{min}(\sigma) = a \qquad \qquad \qquad = 1$$

$$A_{min}(1) = a + b + c + d \qquad \qquad \qquad = 0$$

$$A'_{min}(\sigma) = b \qquad \qquad \qquad = 0$$

$$A'_{min}(1) = b + 2c + 3d \qquad \qquad \qquad = 0$$

which has the solution,

$$a = 1$$

$$b = 0$$

$$c = -3$$

$$d = 2$$

giving the polynomial as:

$$A_{min}(\tilde{r}) = 1 - 3 \left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right)^2 + 2 \left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right)^3$$

4.5.3 Max boundary polynomial

Following the same procedure for A_{max} gives

$$A_{min}(\tilde{r}) = 3 \left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right)^2 - 2 \left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right)^3$$

4.5.4 Corrected function

The corrected function is then,

$$\begin{aligned} f_{BCsImposed}(\tilde{r}) &= f_{MS}(\tilde{r}) + A_{min}\Delta f_{minBC} + A_{max}\Delta f_{maxBC} \\ &= f_{MS}(\tilde{r}) + \left(1 - 3 \left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right)^2 + 2 \left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right)^3 \right) [\Delta f_{minBC}] \\ &\quad + \left(3 \left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right)^2 - 2 \left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right)^3 \right) [\Delta f_{maxBC}] \\ f_{BCsImposed}(\beta) &= f_{MS}(\beta) + \Delta f_{minBC} + (3\beta^2 - 2\beta^3) [\Delta f_{maxBC} - \Delta f_{minBC}] \end{aligned}$$

Note that we're carrying the correction throughout the domain, as opposed to limiting the correction at a certain distance away from the boundary. The application of this correction ensures that there is no discontinuous derivatives inside the domain; as suggested in Roach's MMS guidelines (insert ref)

What is meant by "just because A_{min} and its first derivative go to zero doesn't mean that the second derivatives"

4.5.5 Symbolic Sanity Checks

We want to ensure that $f_{BCsImposed}$ has the desired boundary conditions, $f_{minBC/maxBC}$ instead of the original boundary values that come along for the ride in the manufactured solutions, $f_{MS}(\tilde{r} = \sigma/1)$. In another iteration of this method, we will be changing the derivative values, so let's check the values of $\frac{\partial f_{BCsImposed}}{\partial \tilde{r}}$ to make sure those aren't effected unintentionally.

Symbolic Sanity Check 1

The modified manufactured solution, $f_{BCsImposed}$ with the fairing functions A_{min} and A_{max} substituted in is,

$$f_{BCsImposed}(\tilde{r}) = \left(3 \left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right)^2 - 2 \left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right)^3 \right) [\Delta f_{maxBC}].$$

Further simplification yields,

$$\begin{aligned} f_{BCsImposed}(\tilde{r} = \sigma) &= \left(f_{MS}(\tilde{r} = \sigma) + \Delta f_{minBC} + \left(3 \left(\frac{\sigma - \sigma}{1 - \sigma} \right)^2 - 2 \left(\frac{\sigma - \sigma}{1 - \sigma} \right)^3 \right) [\Delta f_{maxBC} - \Delta f_{minBC}] \right) \\ &= f_{MS}(\tilde{r} = \sigma) + \Delta f_{minBC} \\ &= f_{MS}(\tilde{r} = \sigma) + (f_{minBC} - f_{MS}(\tilde{r} = \sigma)) \\ &= f_{minBC} \end{aligned}$$

$$\begin{aligned}
f_{BCsImposed}(\tilde{r} = 1) &= \left(f_{MS}(\tilde{r} = 1) + \Delta f_{minBC} + \left(3 \left(\frac{1-\sigma}{1-\sigma} \right)^2 - 2 \left(\frac{1-\sigma}{1-\sigma} \right)^3 - \right) [\Delta f_{maxBC} - \Delta f_{minBC}] \right) \\
&= f_{MS}(\tilde{r} = 1) + \Delta f_{maxBC} \\
&= f_{MS}(\tilde{r} = 1) + (f_{maxBC} - f_{MS}(\tilde{r} = 1)) \\
&= f_{maxBC}
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial \tilde{r}} \left(f_{BCsImposed}(\tilde{r}) = \left(3 \left(\frac{\tilde{r}-\sigma}{1-\sigma} \right)^2 - 2 \left(\frac{\tilde{r}-\sigma}{1-\sigma} \right)^3 \right) [\Delta f_{maxBC}] \right) \\
&\frac{\partial f_{MS}}{\partial \tilde{r}} + \left(\frac{6}{1-\sigma} \right) \left(\left(\frac{\tilde{r}-\sigma}{1-\sigma} \right) - \left(\frac{\tilde{r}-\sigma}{1-\sigma} \right)^2 \right) (\Delta f_{maxBC} - \Delta f_{minBC})
\end{aligned}$$

At $\tilde{r} = \sigma$, the derivative is:

$$\begin{aligned}
&\frac{\partial f_{MS}}{\partial \tilde{r}}|_{\sigma} \\
&\frac{\partial f_{MS}}{\partial \tilde{r}}|_1
\end{aligned}$$

4.5.6 Min boundary derivative polynomial

The polynomial is of the form:

$$\begin{aligned}
B_{min}(\beta) &= a + b\beta + c\beta^2 + d\beta^3 \\
B_{min}(\tilde{r}) &= a + b \left(\frac{\tilde{r}-\sigma}{1-\sigma} \right) + c \left(\frac{\tilde{r}-\sigma}{1-\sigma} \right)^2 + d \left(\frac{\tilde{r}-\sigma}{1-\sigma} \right)^3
\end{aligned}$$

Taking the derivative,

$$B'_{min}(\tilde{r}) = b \left(\frac{1}{1-\sigma} \right) + 2c \left(\frac{1}{1-\sigma} \right) \left(\frac{\tilde{r}-\sigma}{1-\sigma} \right) + 3d \left(\frac{1}{1-\sigma} \right) \left(\frac{\tilde{r}-\sigma}{1-\sigma} \right)^2$$

$$B'_{min}(\beta) = \left(\frac{1}{1-\sigma} \right) [b + 2c\beta + 3d\beta^2]$$

Applying the four constraints gives:

$$a = 0$$

$$b = (1 - \sigma)$$

$$a + b + c + d = 0$$

$$2 + 2c + 3d = 0$$

$$c + d = -b$$

$$2c + 3d = -b$$

$$c = -2b$$

$$d = b$$

and the min boundary derivative polynomial is:

$$B_{min}(\tilde{r}) = b \left(\frac{\tilde{r}-\sigma}{1-\sigma} \right) - 2b \left(\frac{\tilde{r}-\sigma}{1-\sigma} \right)^2 + b \left(\frac{\tilde{r}-\sigma}{1-\sigma} \right)^3$$

$$= (1 - \sigma) \left(\left(\frac{\tilde{r}-\sigma}{1-\sigma} \right) - 2 \left(\frac{\tilde{r}-\sigma}{1-\sigma} \right)^2 + \left(\frac{\tilde{r}-\sigma}{1-\sigma} \right)^3 \right)$$

4.5.7 Polynomial function, max boundary derivative

The polynomial is of the form:

The polynomial is of the form:

$$B_{max}(\beta) = a + b\beta + c\beta^2 + d\beta^3$$

$$B_{max}(\tilde{r}) = a + b\left(\frac{\tilde{r} - \sigma}{1 - \sigma}\right) + c\left(\frac{\tilde{r} - \sigma}{1 - \sigma}\right)^2 + d\left(\frac{\tilde{r} - \sigma}{1 - \sigma}\right)^3$$

which has the derivative,

$$B'_{max}(\tilde{r}) = b\left(\frac{1}{1 - \sigma}\right) + 2c\left(\frac{1}{1 - \sigma}\right)\left(\frac{\tilde{r} - \sigma}{1 - \sigma}\right) + 3d\left(\frac{1}{1 - \sigma}\right)\left(\frac{\tilde{r} - \sigma}{1 - \sigma}\right)^2$$

$$B'_{max}(\beta) = \left(\frac{1}{1 - \sigma}\right)[b + 2c\beta + 3d\beta^2]$$

Applying the four constraints gives:

$$a = 0$$

$$b = 0$$

$$a + b + c + d = 0$$

$$b + 2c + 3d = (1 - \sigma)$$

working this out:

$$c + d = 0$$

$$2c + 3d = (1 - \sigma)$$

gives

$$c = -(1 - \sigma) d = (1 - \sigma)$$

and the max boundary derivative polynomial is:

$$B_{max}(\tilde{r}) = (1 - \sigma) \left(- \left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right)^2 + \left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right)^3 \right)$$

4.5.8 Putting it together

The corrected function is then:

$$\begin{aligned} f_{BCsImposed}(\tilde{r}) &= f_{MS} + B_{min}(\tilde{r}) \Delta f'_{minBC} + B_{max}(\tilde{r}) \Delta f'_{maxBC} \\ &= f_{MS} + \\ &\quad (1 - \sigma) \left(\left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right) - \left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right)^2 \right) \Delta f'_{minBC} + \\ &\quad (1 - \sigma) \left(- \left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right)^2 + \left(\frac{\tilde{r} - \sigma}{1 - \sigma} \right)^3 \right) (\Delta f'_{minBC} + \Delta f'_{maxBC}) \end{aligned}$$

Chapter 5

Results and Discussion

5.1 Verification

5.1.1 Method of Manufactured Solution Results

The method of manufactured solutions was used to determine the rate of convergence for SWIRL’s numerical integration and finite difference methods while making sure all of the variables are set to a value to ensure all terms in the aerodynamic and aeroacoustic model were tested. This established a metric for determining whether SWIRL is working as intended. Figure [1] shows the “manufactured” velocity profile used to conduct the MMS. A series of hyperbolic tangents were summed together using the formulation described in Chapter 4. This method was used to create slope changes in the velocity profile at various points along the radius in order to create a substantial amount of error when a small number of grid points are used. Since SWIRL is a subsonic code, the velocity profile was restricted such that the total mach number did not exceed one. The results from applying the trapezoidal rule to numerically integrate the tangential mach number to obtain the speed of sound is reported in Figure [3]. The expected speed of sound was obtained by using the methods outlined in Chapter 4 and the actual speed of sound is the result obtained from SWIRL’s output.

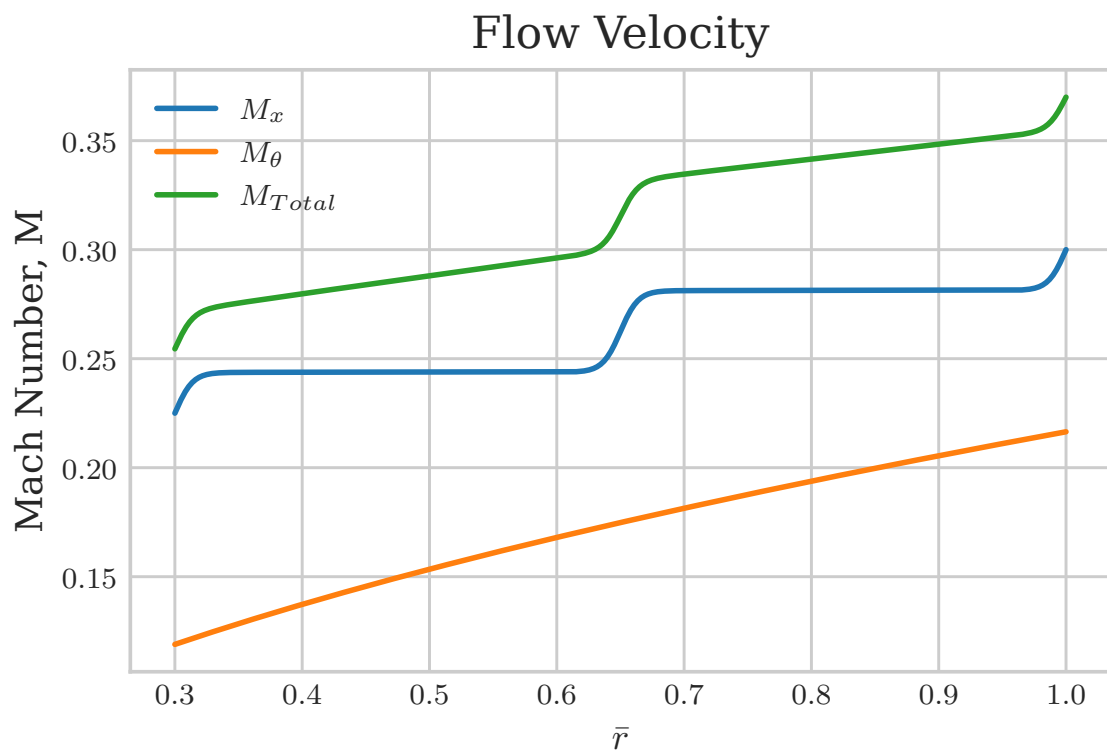


Figure 5-1: Mach distribution for method of manufactured solution case

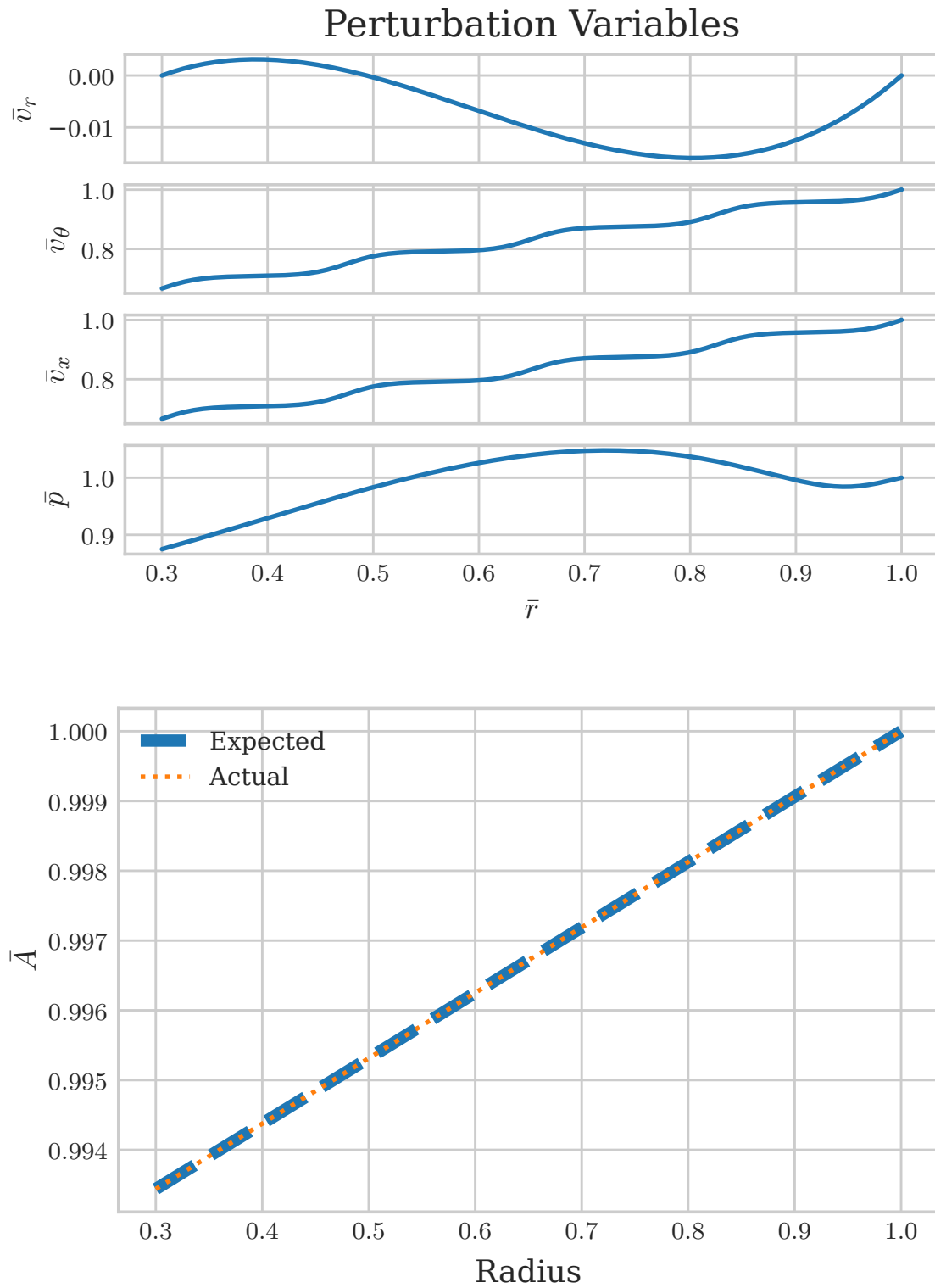


Figure 5-2: Speed of Sound from Integrating the Tangential Mach Number

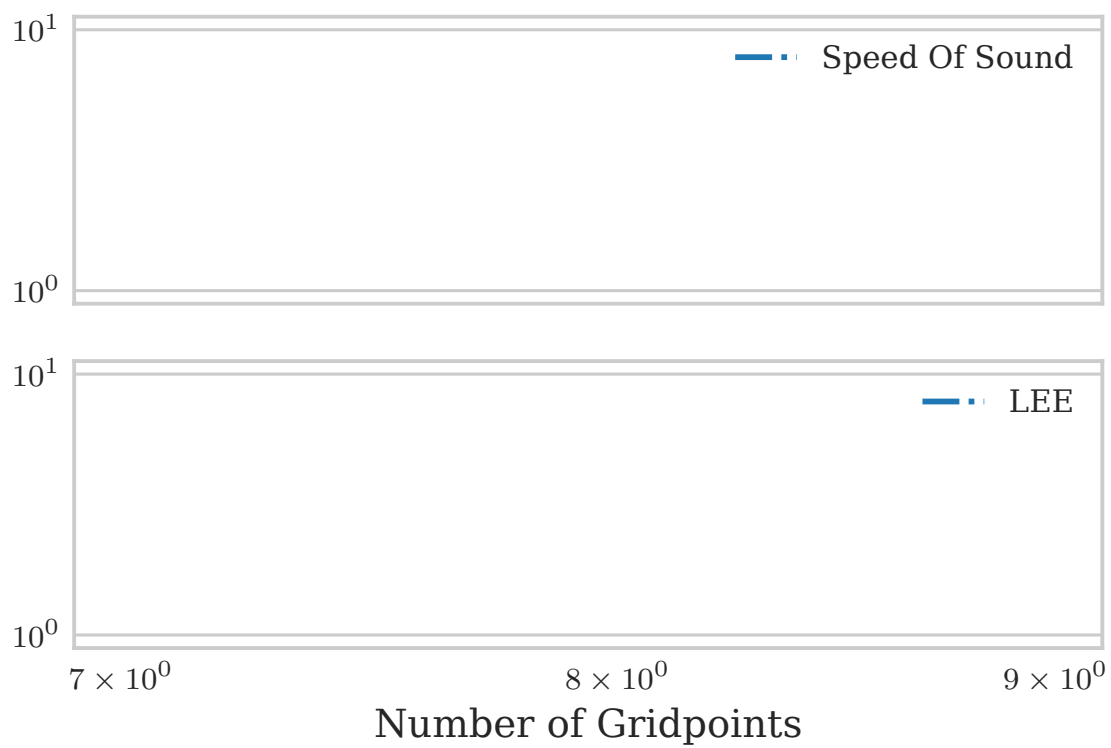


Figure 5-3: L2 Norm of the Error for the MMS as a function of Grid Points

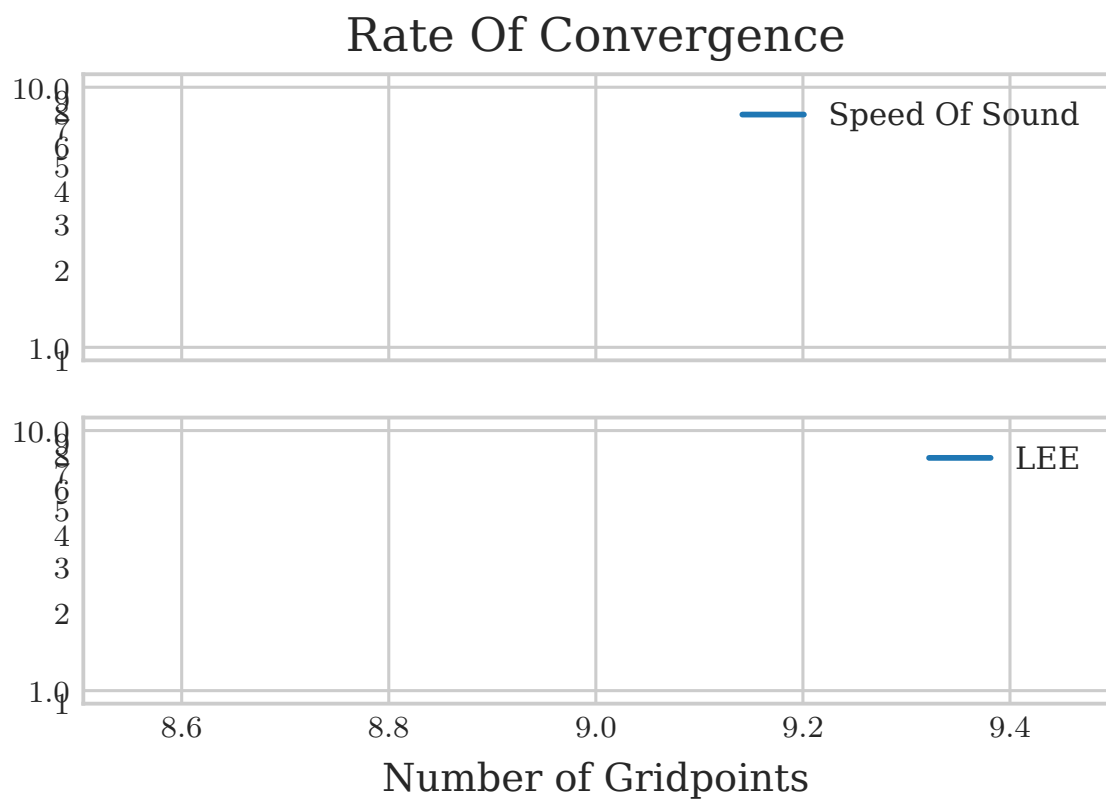


Figure 5-4: Rate of Convergence for the Speed of Sound Integration

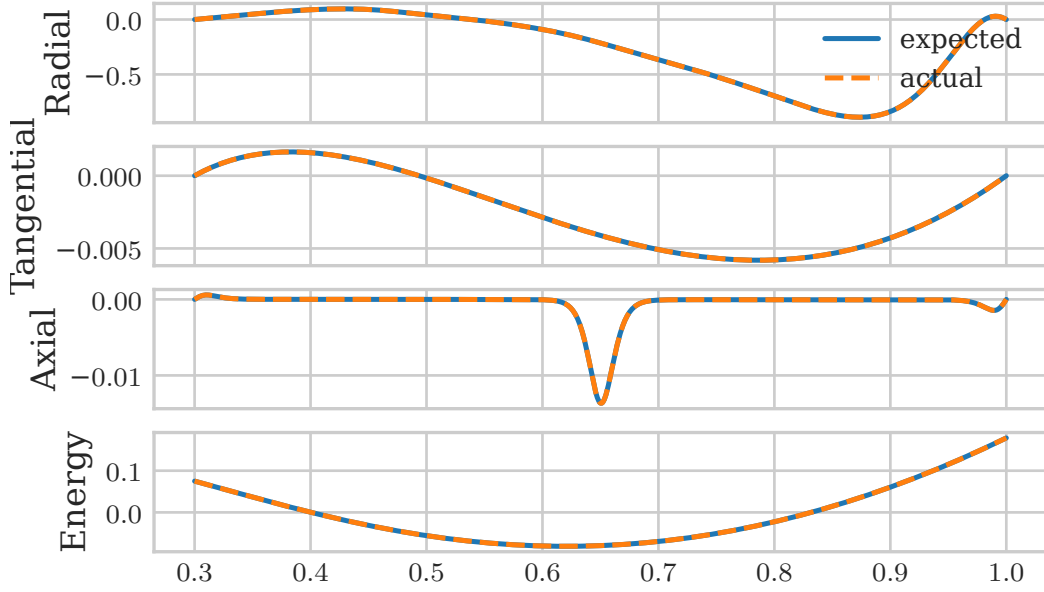


Figure 5-5: Source Term Error

Figure [2] reports the manufactured functions used for the perturbation variables. These functions were used to calculate the source terms for the linearized euler equations which is reported in Figure [4]. The boundary conditions implemented in SWIRL have also been applied to the manufactured radial velocity and pressure perturbations by implementing the fairing function method outlined in Chapter 4. After the source terms have been computed, The L2Norm of the difference between the actual and expected speed of sound and source terms are shown in Figure [5]. An asymptotic rate of convergence was calculated for 10 iterations by recomputing the L2 norm of the error at progressively denser grids. For this study, the grid spacing was halved each iteration. The results for a trapezoidal rule and a second-order central finite difference scheme reported in Figure [6]. For the speed of sound, the L2norm reached $10e-11$ (machine precision) in 8 iterations while the LEE is nearly converged but not yet at machine precision after 10 iterations.

Figure [8] show the L2 norm for the speed of sound integration used to

5.2 Validation

5.2.1 Comparison to Kousen

The following functions were used for the two MMS test cases,

$$i \tag{5.1}$$

$i++i$

5.2.2 Comparison to Maldanado

References

- [1] Department Of Transportation Federal Aviation Administration. Aviation environmental and energy policy statement - july 2012, 2012.
- [2] A. J. COOPER and N. PEAKE. Propagation of unsteady disturbances in a slowly varying duct with mean swirling flow. *Journal of Fluid Mechanics*, 445:207–234, 10 2001.

Appendix A

Theory Appendix

Appendix B

Chapter 3 Appendix

B.1 Appendix A: Speed of Sound

Sound wave is a pressure disturbance that moves with at a speed a By applying a rectangular control volume around this pressure wave, we can apply our conservation equations. We are assuming that these properties are increasing by a small increment. This is why each variable is added by a infinitesimally small term.

Recalling the conservation of mass (continuity equation), $\dot{m} = \text{constant}$

$$\dot{m}_{\text{left}} = \dot{m}_{\text{right}}$$

Recalling the definition of density, $\rho = m/\bar{V}$ and rewriting $\bar{V} = uA$ (check the units)

$$\rho a \cancel{A} = (\rho + d\rho)(a + da)\cancel{A}$$

Futher expanding gives,

$$\rho a = (\rho a + \rho da + a d\rho + da d\rho)$$

We say that $da d\rho$ is so small, we can assume it is zero. This is often referred to as "neglecting higher order terms (H.O.T)". The expression then becomes

$$\frac{da}{a} = -\frac{d\rho}{\rho}$$

For the momentum equation $P + \rho u^2 = \text{constant}$

$$P + \rho a^2 = P + dP + (\rho + d\rho)(a + da)(a + da)$$

But we just said that $\rho a = (\rho + d\rho)(a + da)$

$$P + \rho a^2 = P + dP + \rho a(a + da)$$

$$dP + \rho a da = a$$

Multiplying the second term by a and divide by a , this is essentially multiplying the second term by one.

$$dP + \rho a^2 \frac{da}{a} = a$$

recalling the relation $\frac{da}{a} = -\frac{d\rho}{\rho}$

$$dp - a^2 d\rho = 0$$

$$a^2 = \frac{dp}{d\rho}$$

Since a sound wave is a very weak wave, when it travels through a medium, it only increases the pressure and density, etc. slightly. The effect of this is that friction and heat transfer can be neglected. Since friction cannot be undone, we call this an irreversible process. Whenever there is no transfer of heat, it is called this adiabatic. Thus, the propagation of sound is an adiabatic, reversible process, otherwise called isentropic. Isentropic implies no increase in entropy, which is *not* true in the presence of shock waves.

In the case of a thermally perfect gas, we can say $P = \rho RT$

For a calorically perfect gas we can say $pv^\gamma = \text{constant}$, where v is volume per unit mass, or specific volume

Differentiating and recalling that $v = 1/\rho$

$$a = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\gamma RT}$$

$$dm = (\rho u)_2 - (\rho u)_1$$

$$D(mV) = (\rho u^2 + P)_2$$

For Steady flow,

$$d\dot{m} = (\rho u)_2 - (\rho u)_1$$

$$(\rho u)_1 = (\rho u)_2$$

Similarly, for the Momentum equation,

$$(\rho u^2 + P)_1 = (\rho u^2 + P)_2$$

Let us change the coordinate system motion for the traveling wave be independent of time, and thus corresponds to *steady state* wave propagation.

$$u_1 = \bar{u} + a - \frac{1}{2}\partial u$$

$$u_2 = \bar{u} + a + \frac{1}{2}\partial u$$

where, \bar{u} is the average flow velocity and a is the wave speed.

Substituting this back into the conservation of mass

$$(\rho u)_1 = (\rho u)_2$$

$$\left(\rho - \frac{1}{2}\partial\rho\right)\left(\bar{u} - a - \frac{1}{2}\partial u\right) = \left(\rho + \frac{1}{2}\partial\rho\right)\left(\bar{u} - a + \frac{1}{2}\partial u\right)$$

Further expanding

$$\cancel{\rho \bar{u}} - \cancel{\rho a} + \frac{1}{2} (-\rho \partial u - \bar{u} - \bar{u} \partial \rho + a \partial \rho) + \cancel{\frac{1}{4} \partial \rho \partial u} = \cancel{\rho \bar{u}} - \cancel{\rho a} + \frac{1}{2} (\rho \partial u + \bar{u} - \bar{u} \partial \rho - a \partial \rho) + \cancel{\frac{1}{4} \partial \rho \partial u}$$

$$\frac{1}{2} (-\rho \partial u - \bar{u} \partial \rho + a \partial \rho) = \frac{1}{2} (\rho \partial u + \bar{u} \partial \rho - a \partial \rho)$$

$$\rho \partial u + u \partial \rho - a \partial \rho = 0$$

$$\rho \partial u + (u - a) \partial \rho = 0$$

Momentum Equation

$$(\rho u^2 + P)_1 = (\rho u^2 + P)_2$$

B.2 Appendix B: Isentropic Waves

$$dU = dW + dQ$$

For adiabatic, reversible processes, the work done by a system with constant pressure and a change in volume is $-pdV$ and the change in heat energy is zero. Hence,

$$dU = -pdV$$

.

The change in enthalpy of such a system can be found by taking the derivative of its expression for a thermodynamic process

$$H = U + pV$$

$$dH = dU + pdV + vdP$$

$$dH = -pdV + pdV + vdP$$

$$dH = vdP$$

The specific heats at constant pressure and constant volume

$$\left(\frac{\partial U}{\partial T}\right)_v = C_v$$

$$\left(\frac{\partial H}{\partial T}\right)_p = C_p$$

$$\gamma = \frac{C_p}{C_v} = \frac{dH}{dU} = -\frac{VdP}{pdV} = -\frac{V}{dV} \frac{dP}{p}$$

Integrating both sides

$$\gamma \frac{dV}{V} = \frac{-dP}{P} \rightarrow \gamma \int \frac{1}{V} dV = - \int \frac{1}{P} dP$$

$$\gamma \ln(V) + \ln(P) = C$$

Using log rules

$$\ln(V^\gamma) + \ln(P) = C$$

$$\ln(pV^\gamma) = C$$

$$pV^\gamma = e^C = C$$

$$\frac{p}{\rho^\gamma}$$