

Analytical Solution for Duct Mode Propagation in Uniform Flow

Swirl Validation

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This document shows the analytical duct mode solution as well as a numerical comparison.

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0.1 Introduction - Turbomachinery Noise

Turbomachinery noise generation occurs from pressure fluctuations from the series of fans within it's annular duct. While the jet that is produced from this stream of air freely radiates to the observer, the pressure fluctuations produced from the rotor may or may not propagate out of the inlet and exhaust and radiate to the observer. The production of this propagation can be characterized by standing waves referred to as modes, in particular, duct modes because the mode itself is dependent on the geometry of the column of air within the annular duct, as well as the speed of the flow moving through it

0.2 Duct Modes

The pressure field within a duct is governed by the convective wave equation, a second order ODE as a function of radius. Goldstein's *Aeroacoustics* shows that the wave equation can be obtained by taking the divergence of the momentum equation and subtracting the material derivative of the continuity equation, however, the same equation can be obtained by resubstitution of the continuity and momentum equations into the energy equations after the fluctuations have also been substituted for the primitive variables. (See Appendix for derivation)

The solution of the convective wave equation are eigenvalues and eigenvectors which may or may not correspond to acoustic disturbances fall into two groups. One group corresponding to the acoustics propagation and the other group corresponding to the convection speed of the flow. Both are modes that are a result from the pressure distribution from within the cylindrical domain.

Modes can be categorized based on the sign of the axial wavenumber and if it is complex in value. For example, for the uniform axial flow case, propagating modes are defined by axial wavenumbers, k_x , that have a real-part only, yielding the assumed fluctuation to resemble Euler's Formula ($e^{ik_x x}$). On the other hand, if the k_x is complex, then the mode will resemble an exponentially decaying function since the imaginary number cancels, leaving a minus sign in front of the axial wavenumber. These two distinctions are referred to as "cut-on" and "cut-off" in the field of ducted sound propagation. Furthermore, the sign of the imaginary part will change the direction of the mode's decay. If k_x is positive, the decay rate occurs in the negative direction. Conversely, if k_x is negative, the decay occurs in the positive direction. The axial wavenumber for uniform axial flow is,

$$k_x = \frac{-M_x k \pm \sqrt{k^2 - (1 - M_x^2) J_{m,n}'^2}}{(1 - M_x^2)}. \quad (1)$$

where M_x is the axial Mach number, k is the temporal (referred to as reduced) frequency, and $J_{m,n}'$ is the derivative of the Bessel function of the first kind. The \pm accounts for both upstream and downstream modes.

The condition for propagation is such that the axial wavenumber is larger than a "cut-off" value

$$k_{x,real} = \frac{\pm M_x k}{(M_x^2 - 1)}. \quad (2)$$

Every term that is being raised to the one half i.e. square rooted must be larger than zero to keep axial wavenumber from being imaginary. The mode will propagate or decay based on this condition. Recall that the mode is of the form

$$e^{ik_x x} \quad (3)$$

if k_x has a real part, $k_{x,real}$ and an imaginary part $ik_{x,imag}$ then,

$$= e^{ik_x x} \quad (4)$$

$$= e^{i(k_{x,real} + ik_{x,imag})x} \quad (5)$$

$$= \underbrace{e^{ik_{x,real}x}}_{\text{amplitude exponential decay}} \underbrace{e^{-k_{x,imag}x}} \quad (6)$$

Although the “cut-off” decay to nearly zero rapidly, the rate at which this occurred was not much of a concern earlier on in turbomachinery design. As nacelles continue to grow shorter, a mode that is “cut-off” may make it outside the duct.

For this work a desired amplitude was arbitrarily chosen for a mode, $y_{desired}$ and then the axial location at which this occurred, $x_{desired}$ which can be compared against a desired length for a nacelle. Since SWIRL assumes an infinitely long duct, there is nothing limiting the modes propagation with respect to nacelle length. For example, if the desired amplitude is one percent, then $x_{desired}$ is 0.46,

$$\begin{aligned} 0.01 &= e^{-10x_{desired}}, \\ -\frac{\ln|0.01|}{10} &= x_{desired}, \\ -\frac{\ln|0.01|}{10} &= 0.4605170185988091. \end{aligned}$$

In general,

$$\begin{aligned} y_{desired} &= e^{-k_{x,imag}x_{desired}} \\ -\frac{\ln|y_{desired}|}{k_{x,imag}} &= x_{desired} \end{aligned}$$

Decaying Mode Example $y = \exp(10x)$

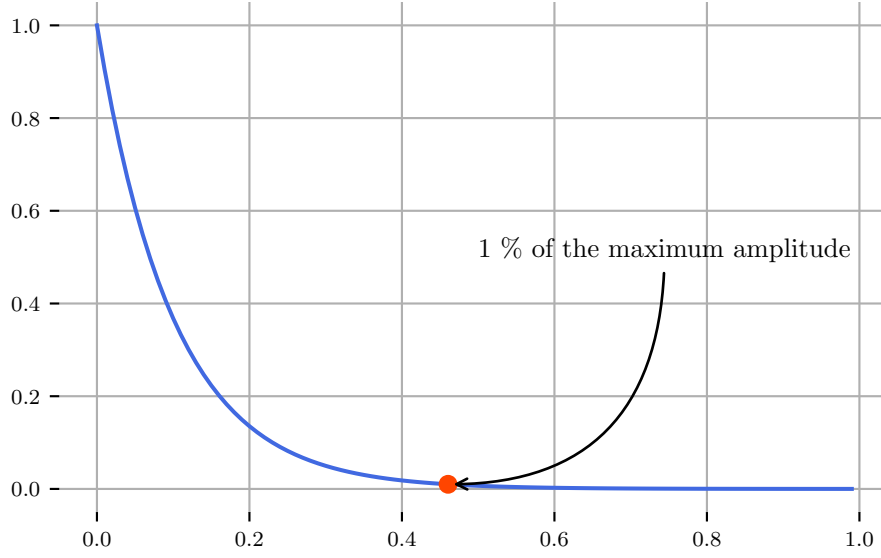


Figure 1: Decaying mode with $k_x = 0 + 10j$ and unit amplitude. One percent of the maximum amplitude is identified for nacelle length comparison

0.3 Analytical Test Case

σ	0.25
k	10
m	2
M_x	0.3

Table 1: Validation test case parameters, Uniform Flow Annular Duct

0.3.1 Axial Wavenumber

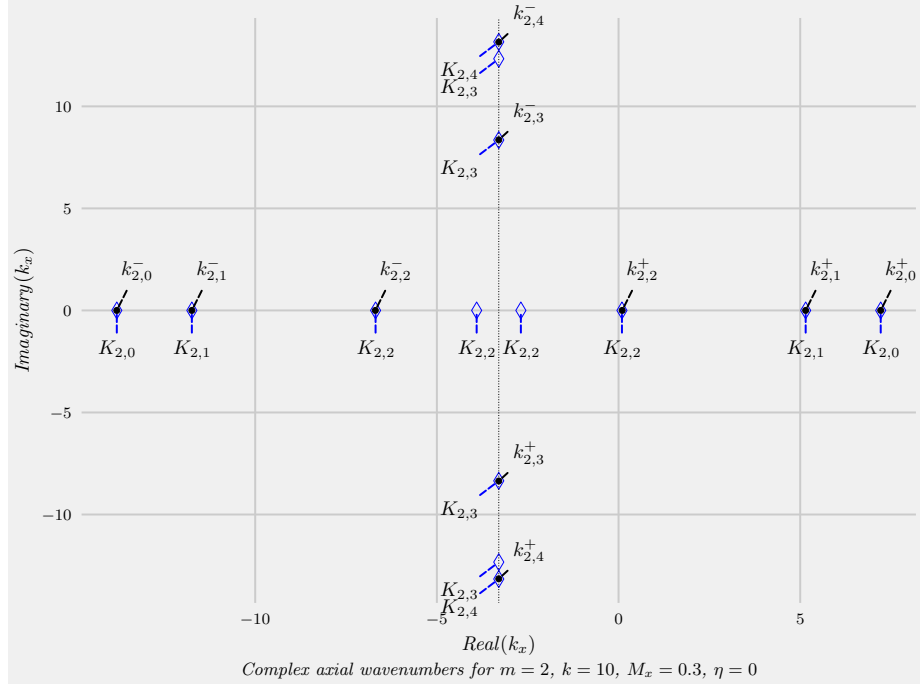


Figure 2: Analytical Solution vs Numerical Approximation using 32 points

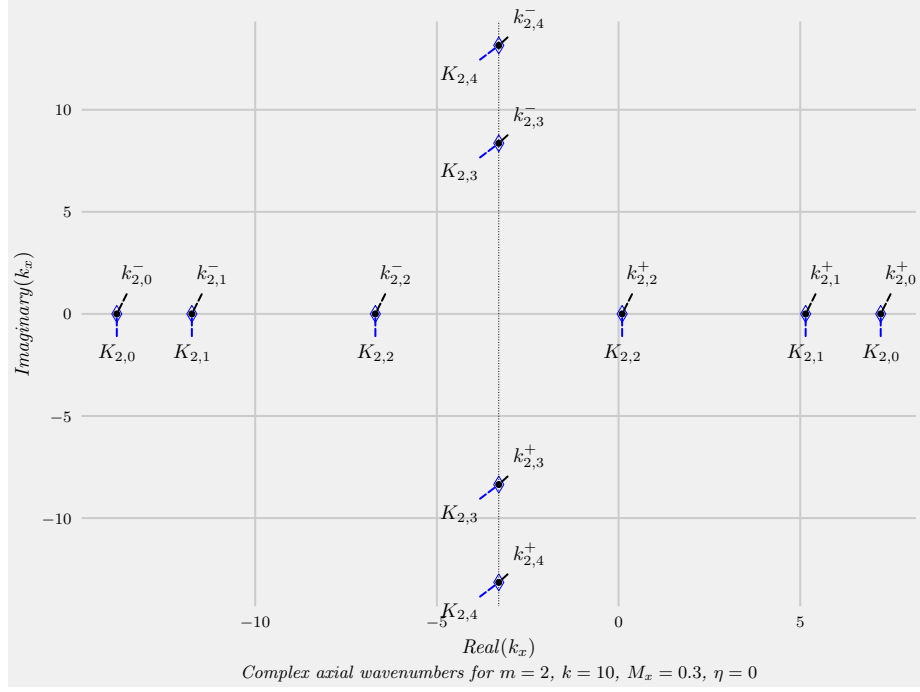


Figure 3: Analytical Solution vs Numerical Approximation using 64 points

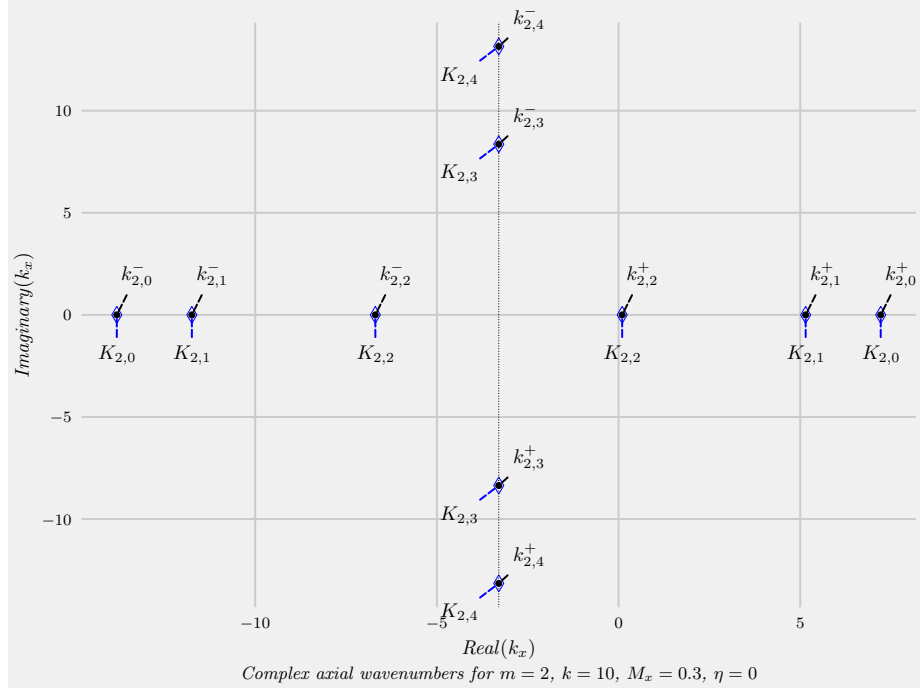


Figure 4: Analytical Solution vs Numerical Approximation using 128 points

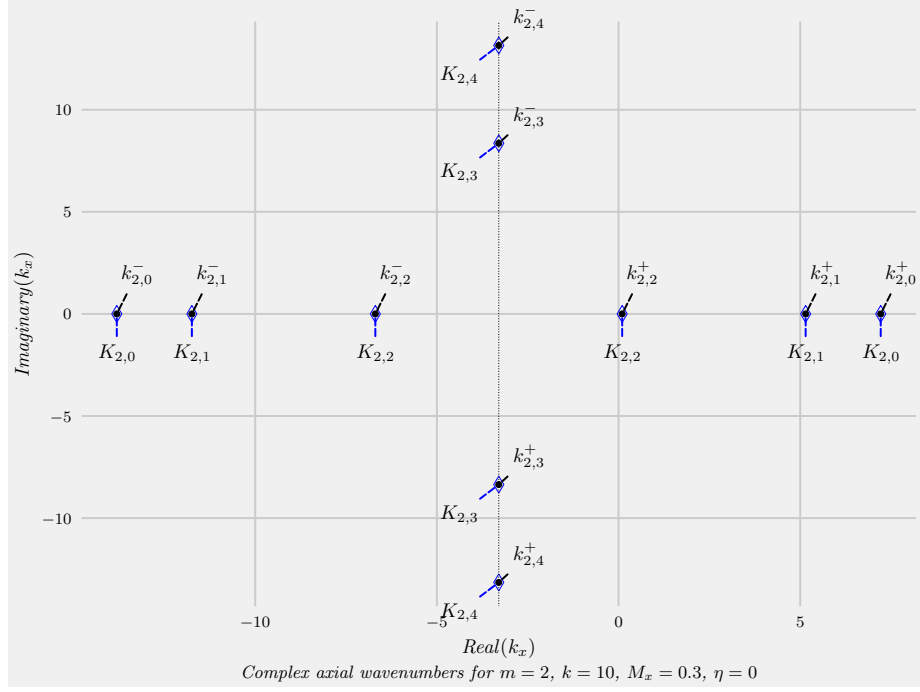


Figure 5: Analytical Solution vs Numerical Approximation using 256 points

0.3.2 Propagating Radial Modes

Fourth Order, Radial Mode 0

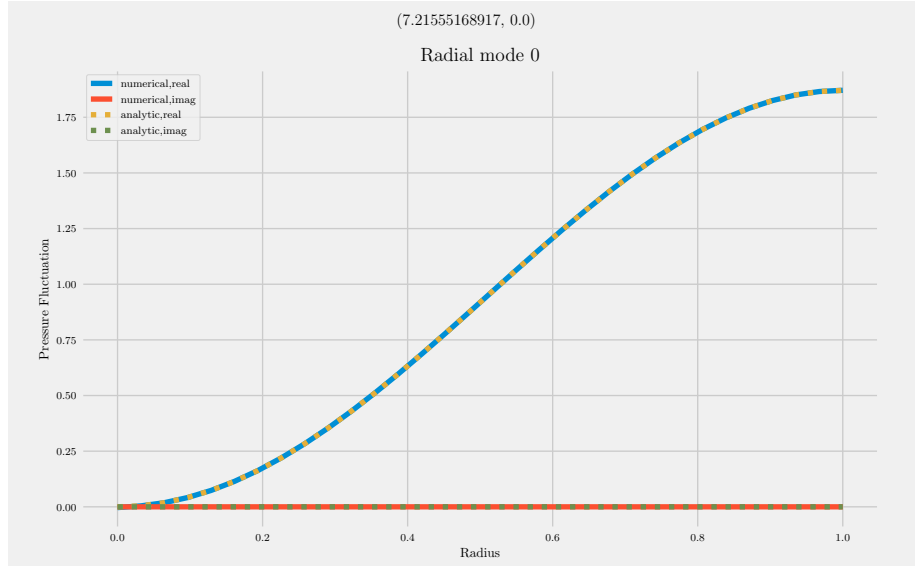


Figure 6: Normalized radial mode number 0, 0th identified mode

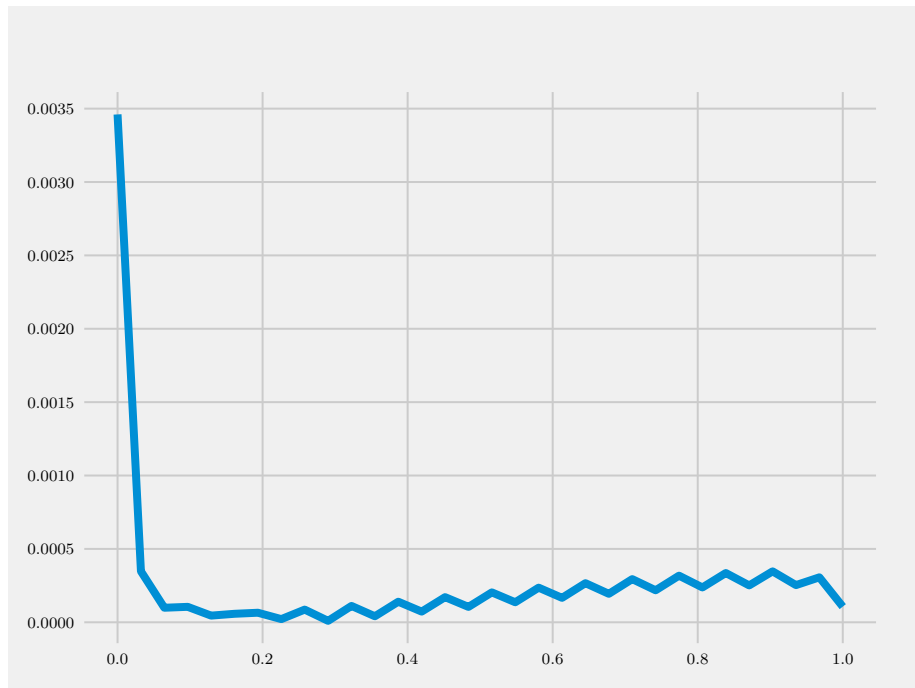


Figure 7: Normalized radial mode number 0 error, 0th identified mode

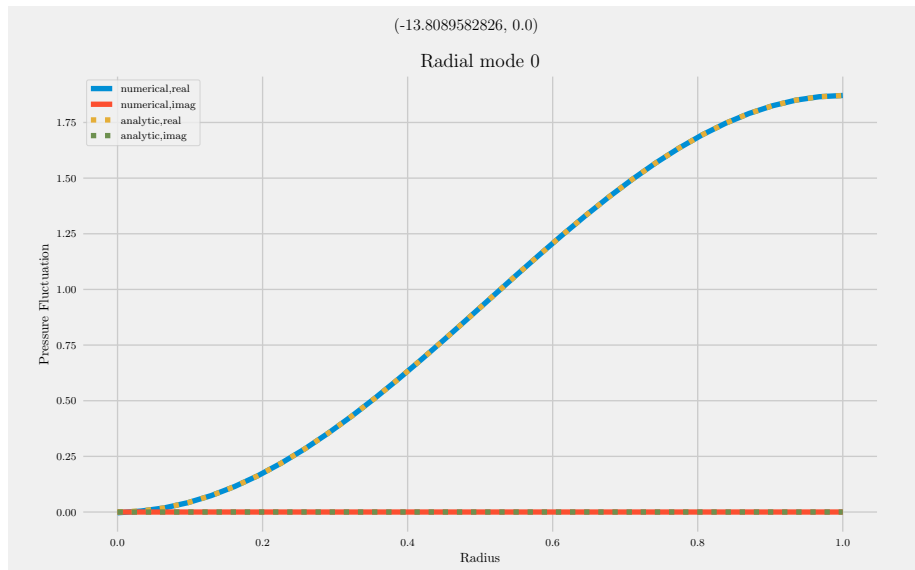


Figure 8: Normalized radial mode number 0, 1th identified mode

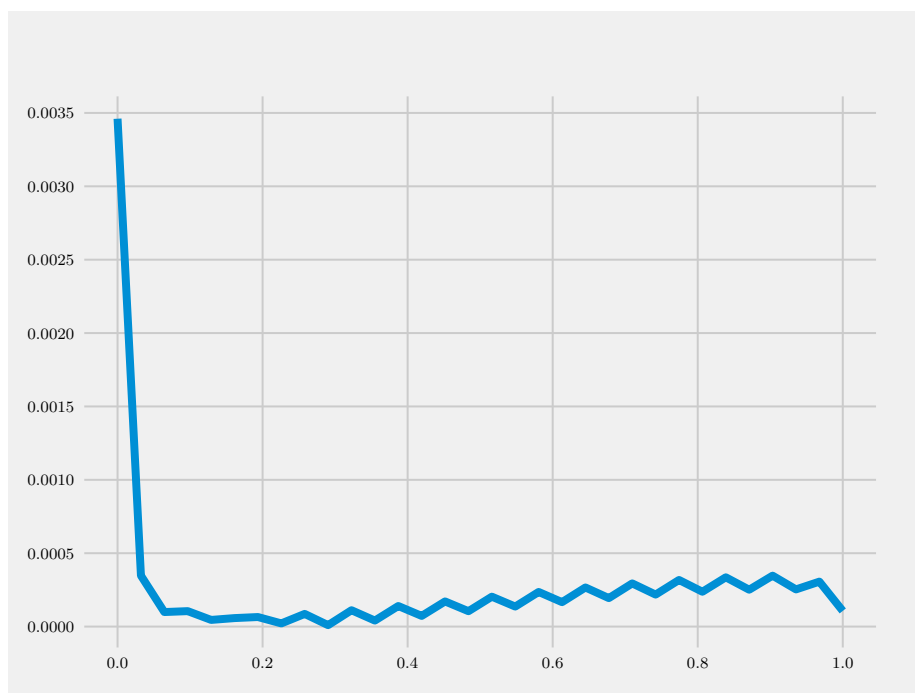


Figure 9: Normalized radial mode number 0 error, 1th identified mode

Fourth Order, Radial Mode 1

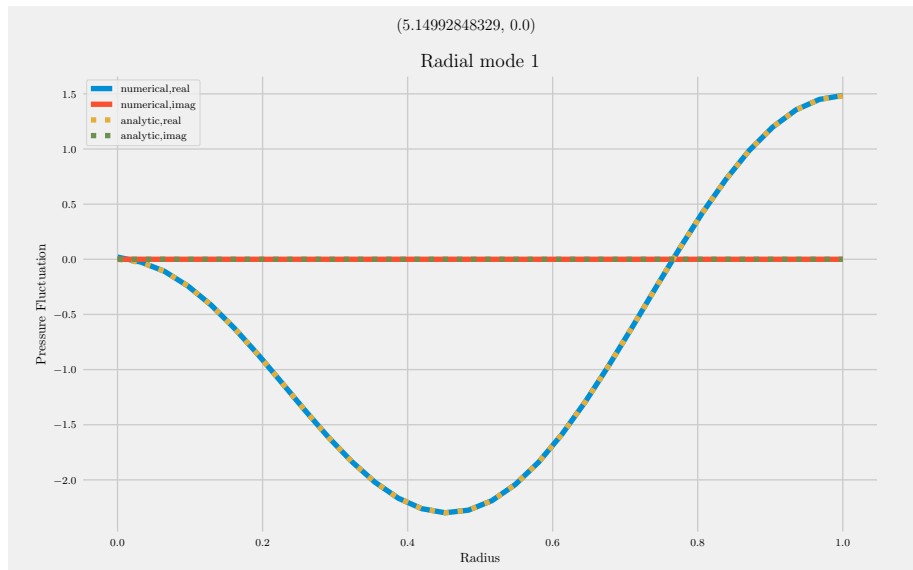


Figure 10: Fourth Order, Case number 0

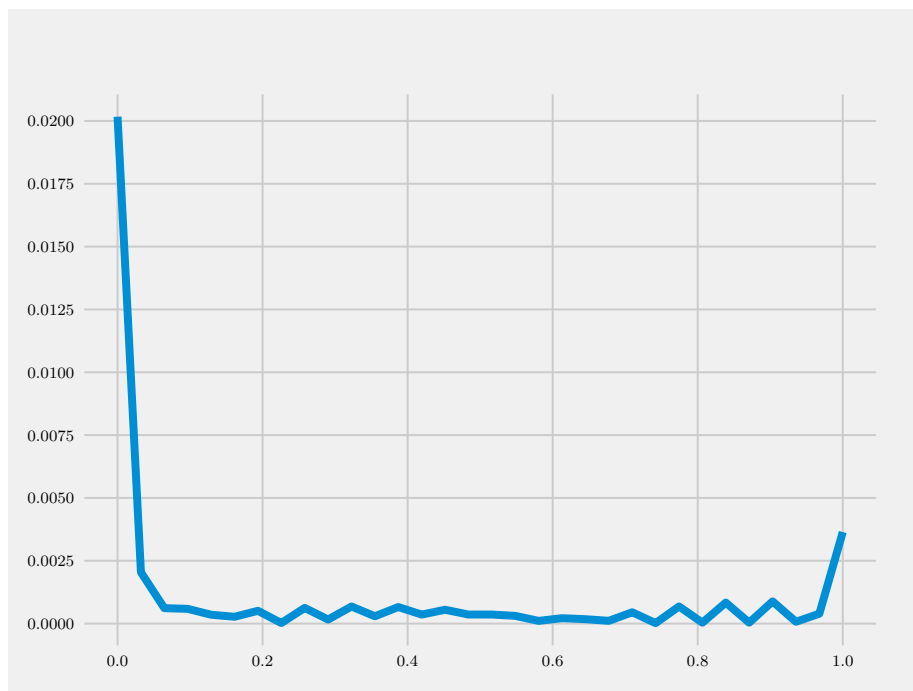


Figure 11: Normalized radial mode number 1 error, 0th identified mode

Fourth Order, Radial Mode 2

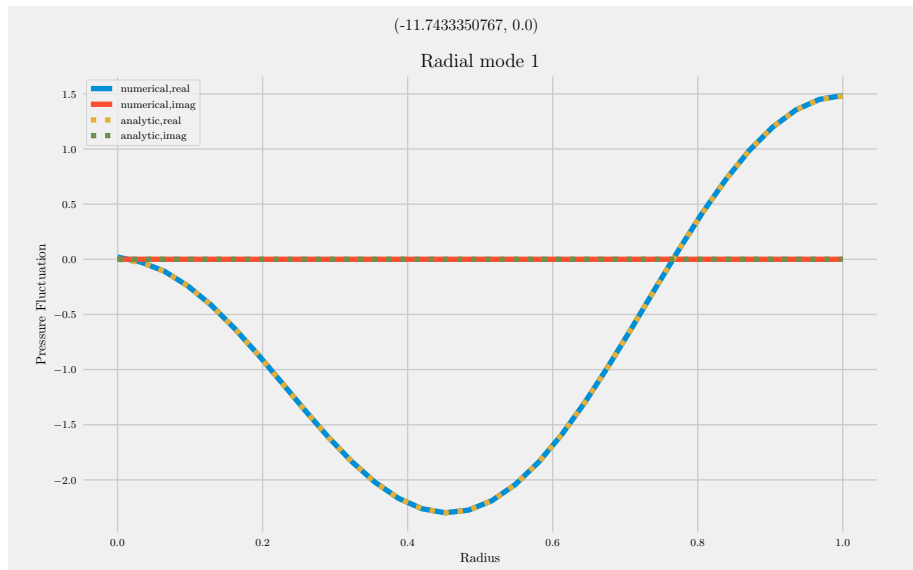


Figure 12: Fourth Order, Case number 1

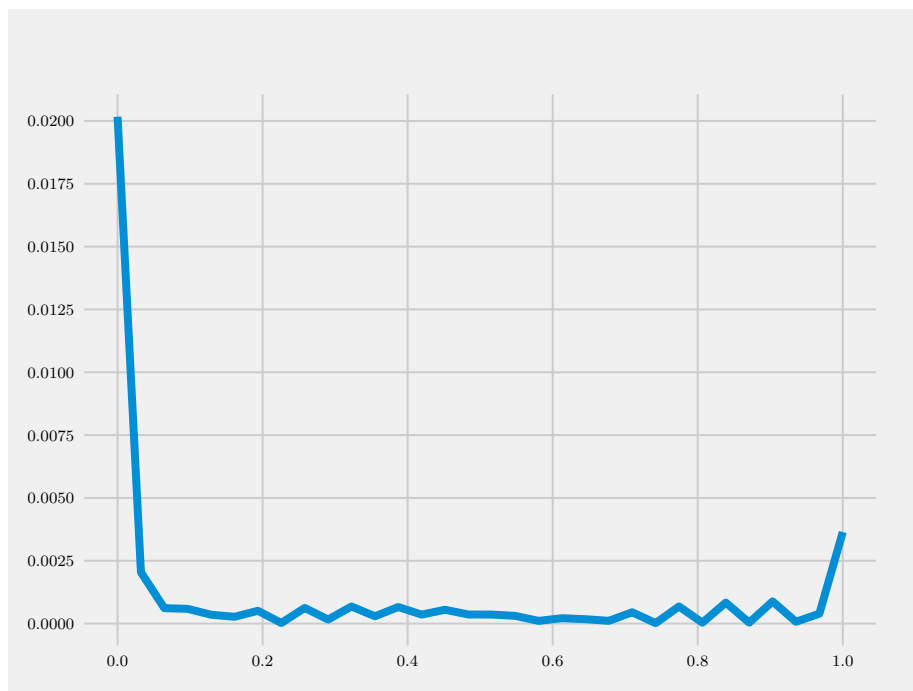


Figure 13: Normalized radial mode number 1 error, 1th identified mode

Fourth Order, Radial Mode 3

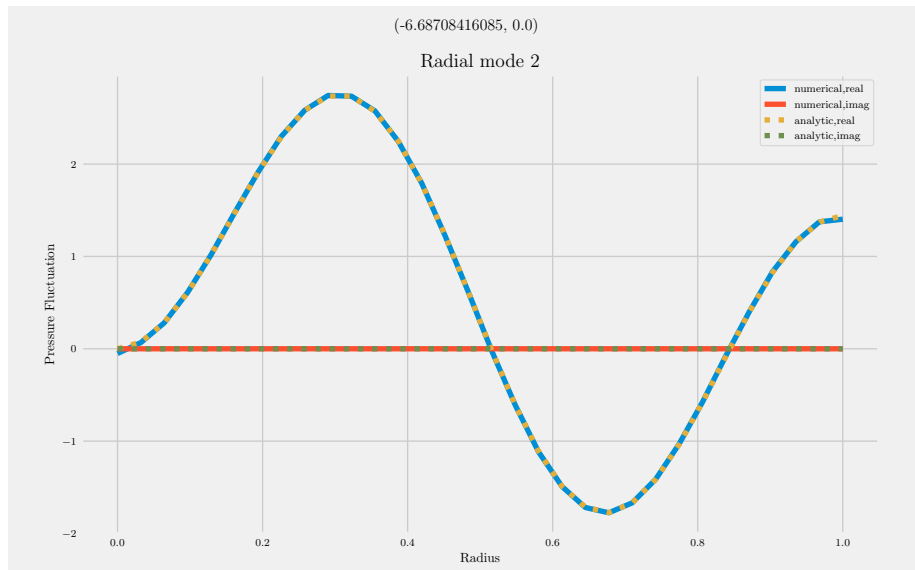


Figure 14: Fourth Order, Case number 0

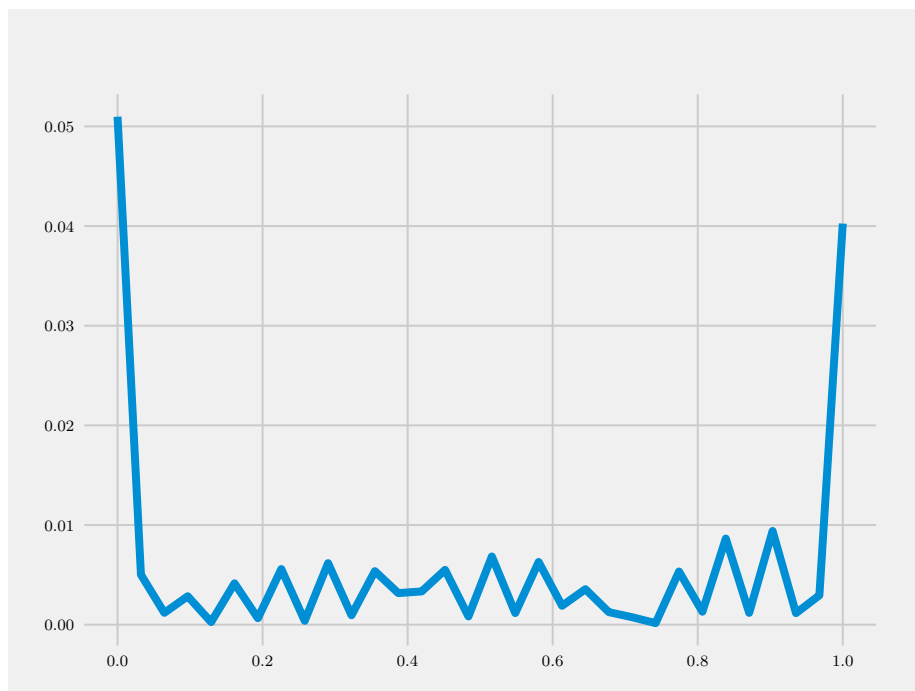


Figure 15: Normalized radial mode number 2 error, 0th identified mode

Fourth Order, Radial Mode 4

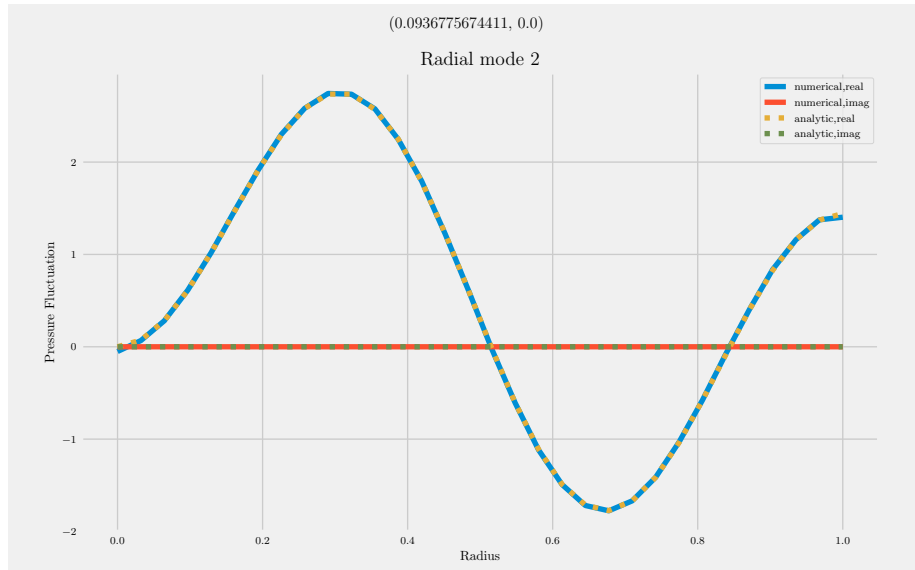


Figure 16: Fourth Order, Case number 1

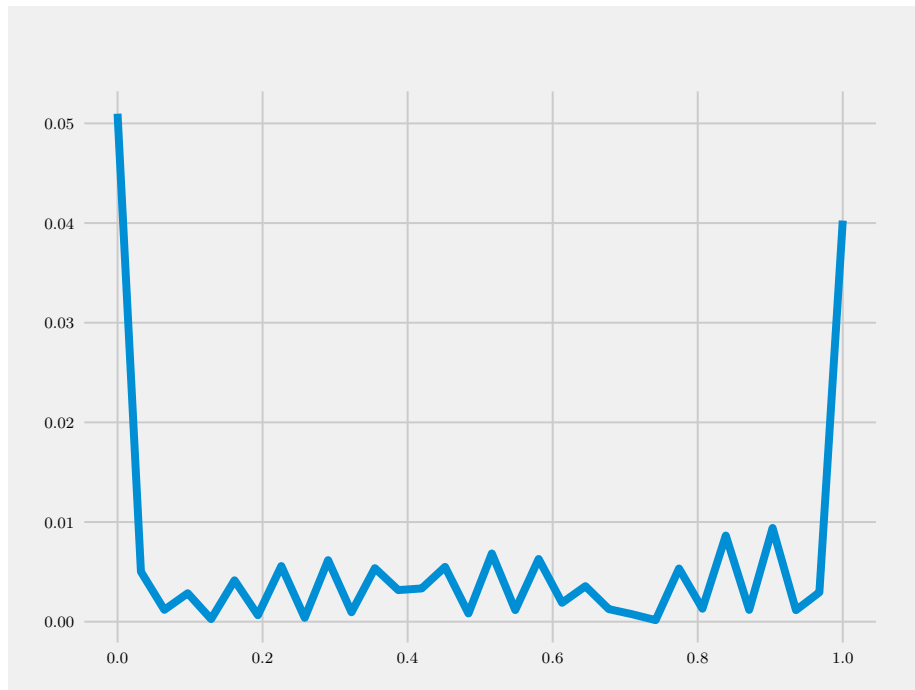


Figure 17: Normalized radial mode number 2 error, 1th identified mode

0.3.3 64 points

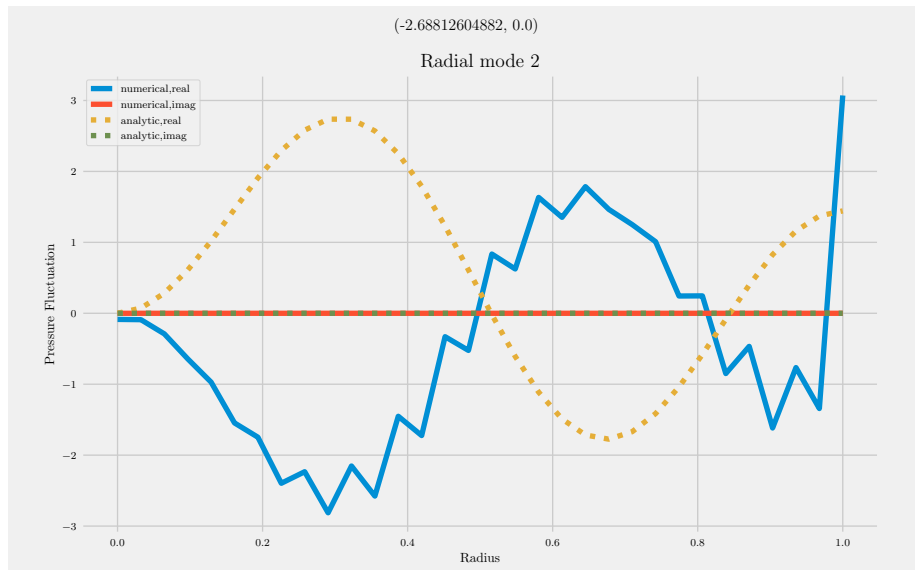


Figure 18: Fourth Order, Case number 2

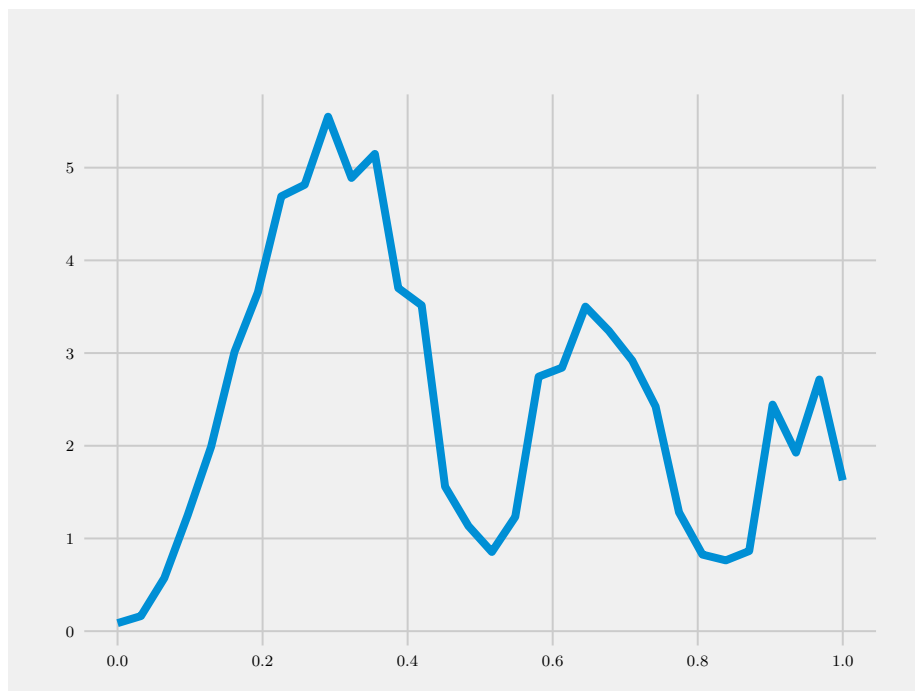


Figure 19: Normalized radial mode number 2 error, 2th identified mode

Fourth Order, Radial Mode 0

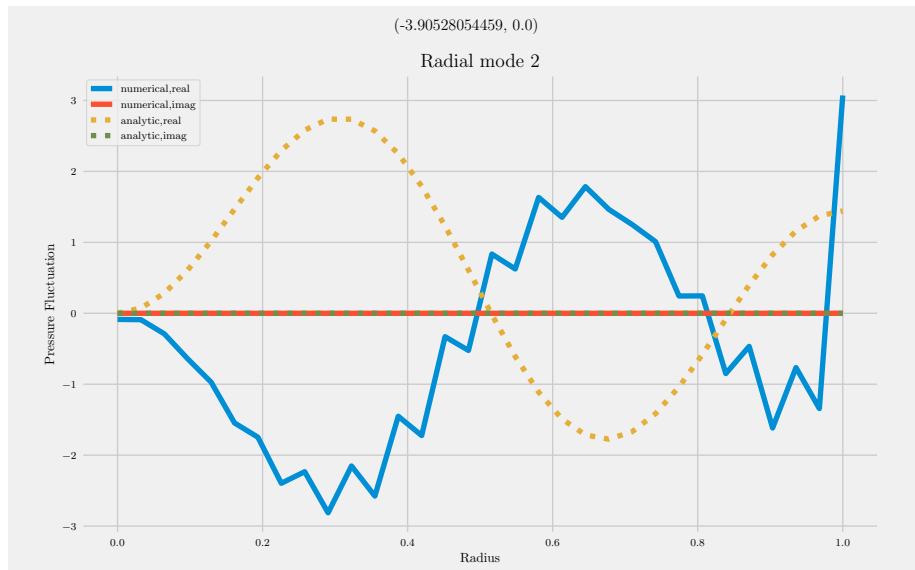


Figure 20: Fourth Order, Case number 3

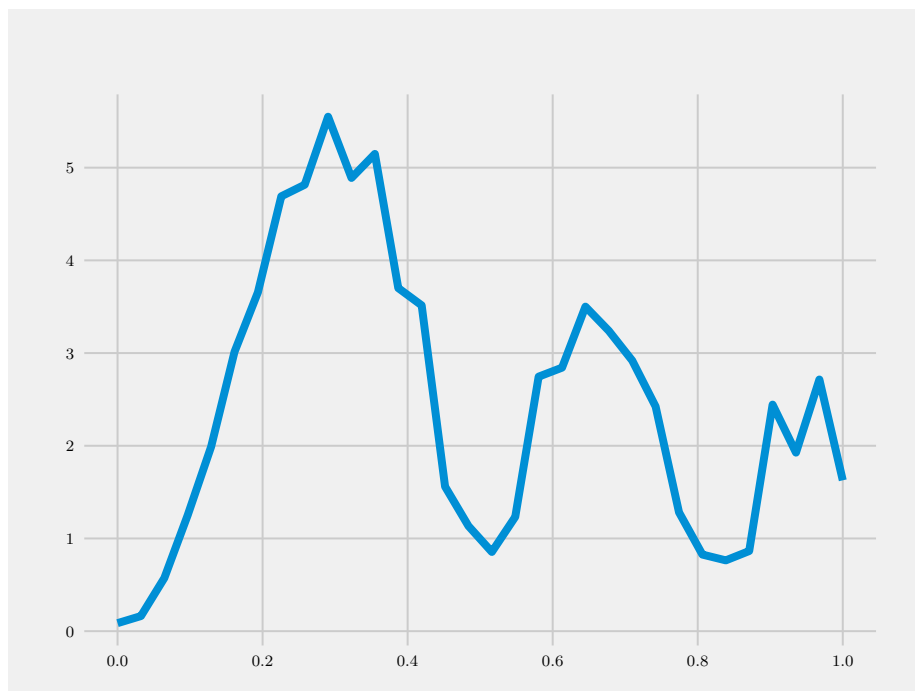


Figure 21: Normalized radial mode number 2 error, 3th identified mode

Fourth Order, Radial Mode 1

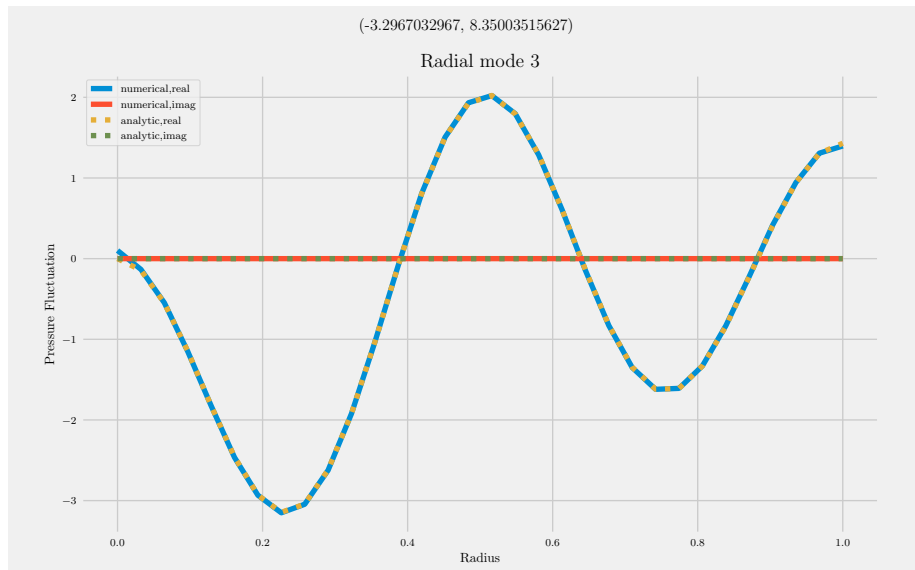


Figure 22: Fourth Order, Case number 0

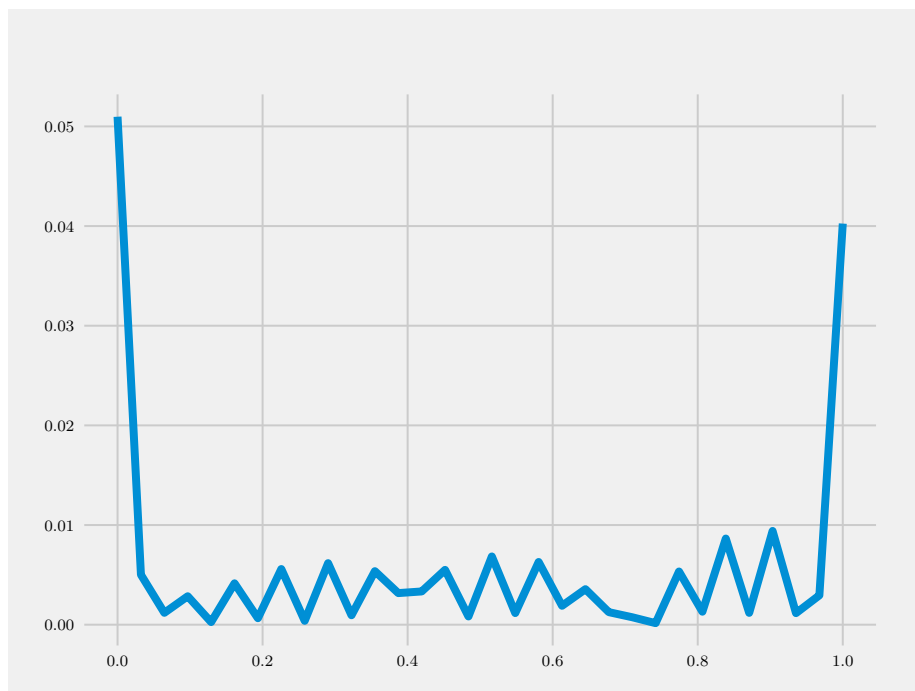


Figure 23: Normalized radial mode number 2 error, 0th identified mode

Fourth Order, Radial Mode 2

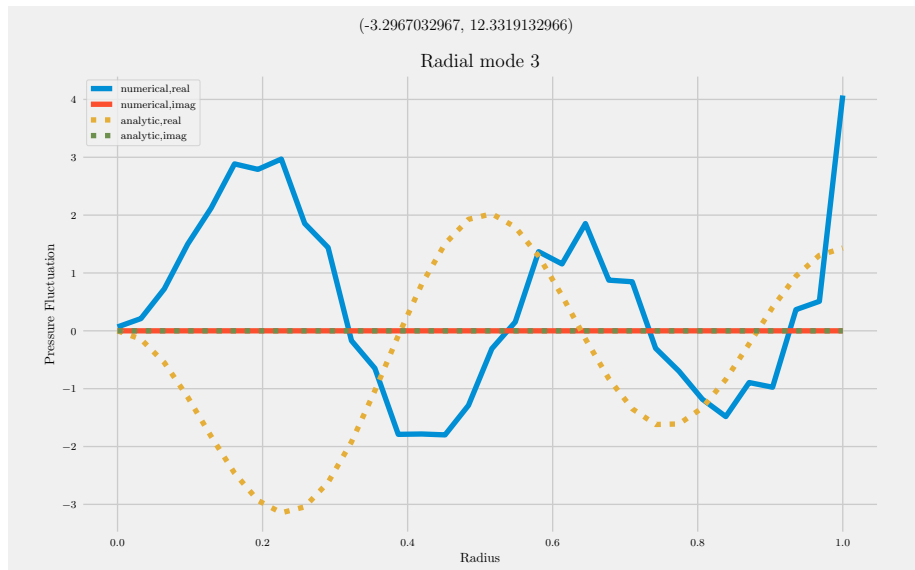


Figure 24: Fourth Order, Case number 1

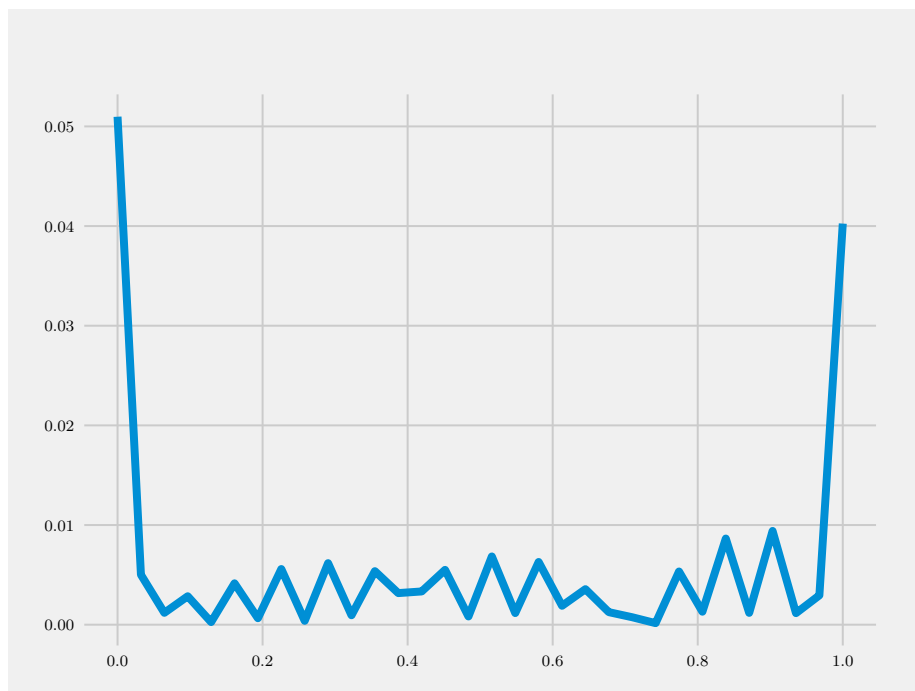


Figure 25: Normalized radial mode number 2 error, 1th identified mode

Fourth Order, Radial Mode 3

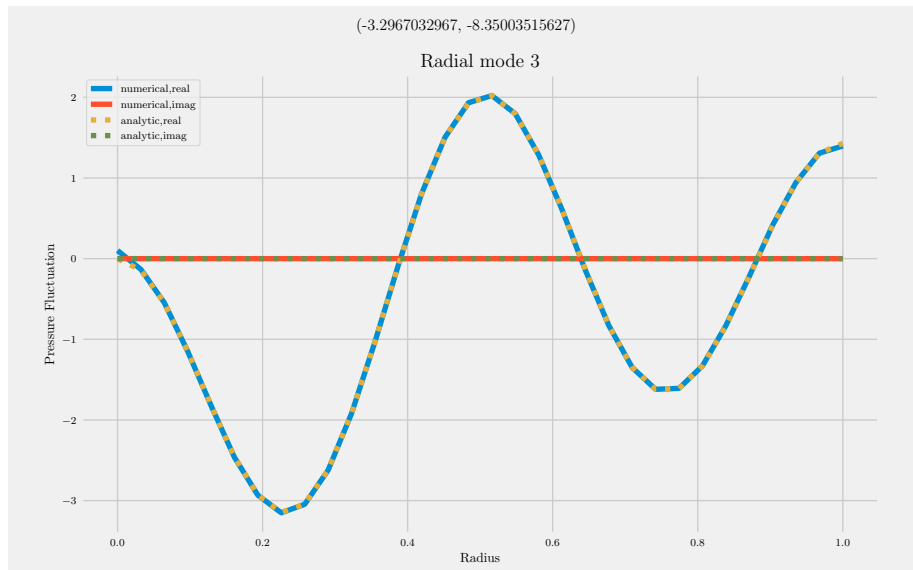


Figure 26: Fourth Order, Case number 2

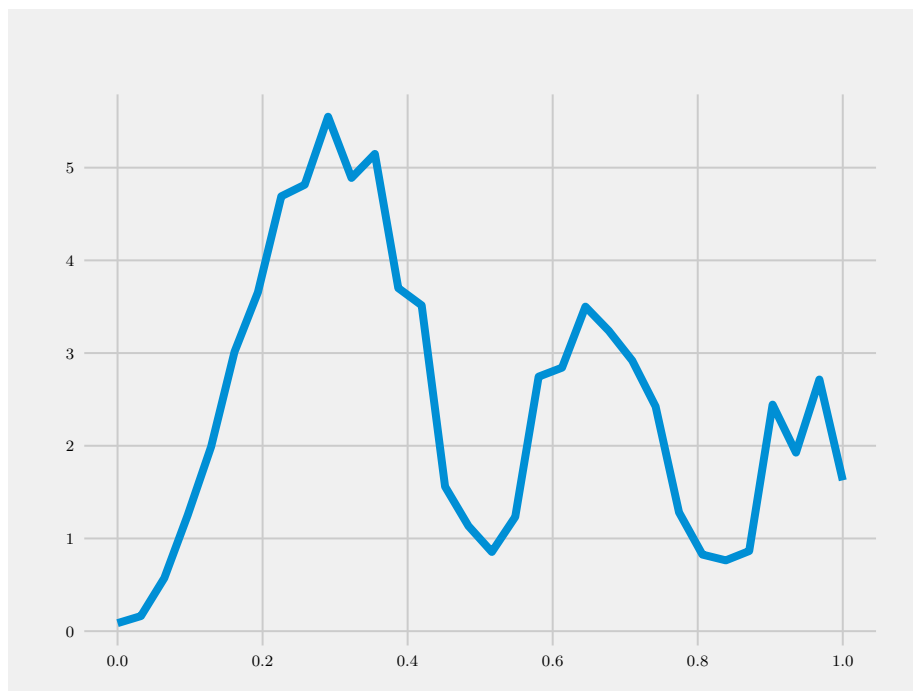


Figure 27: Normalized radial mode number 2 error, 2th identified mode

Fourth Order, Radial Mode 4

0.3.4 128 points

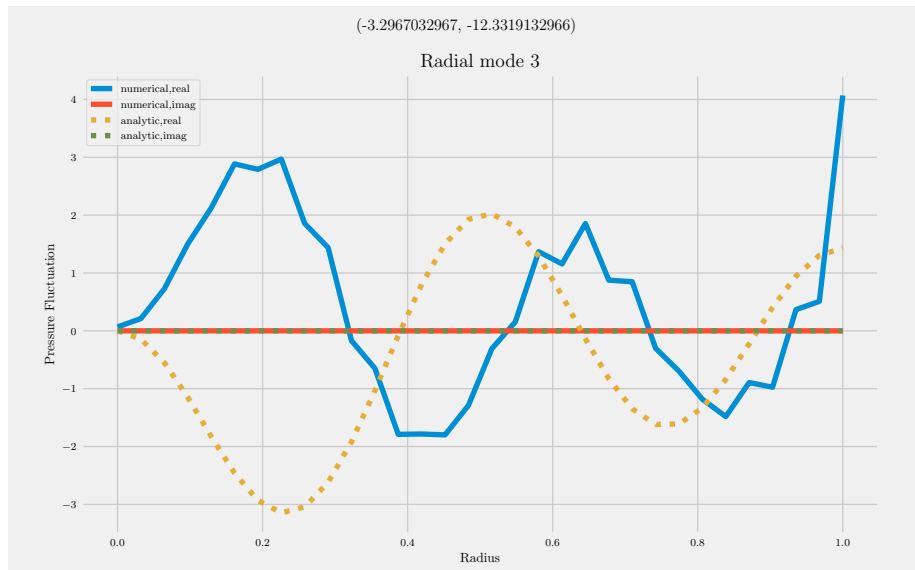


Figure 28: Fourth Order, Case number 3

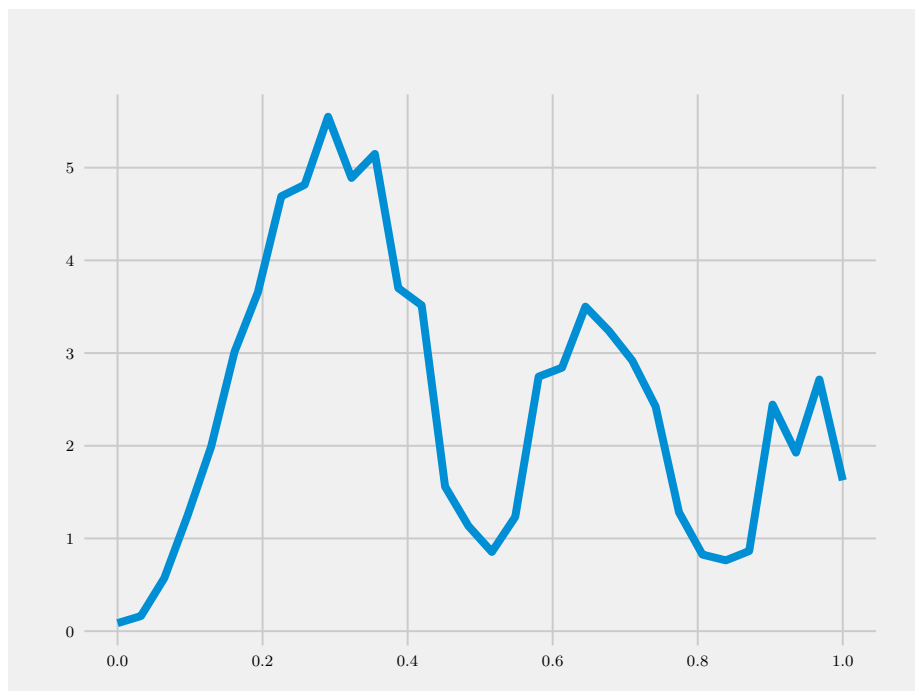


Figure 29: Normalized radial mode number 2 error, 3th identified mode

Fourth Order, Radial Mode 0

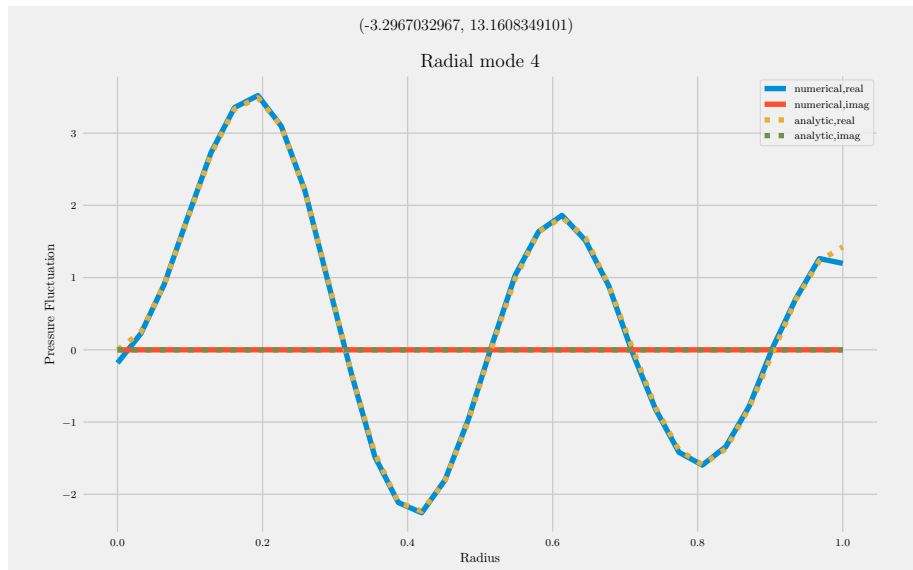


Figure 30: Fourth Order, Case number 0

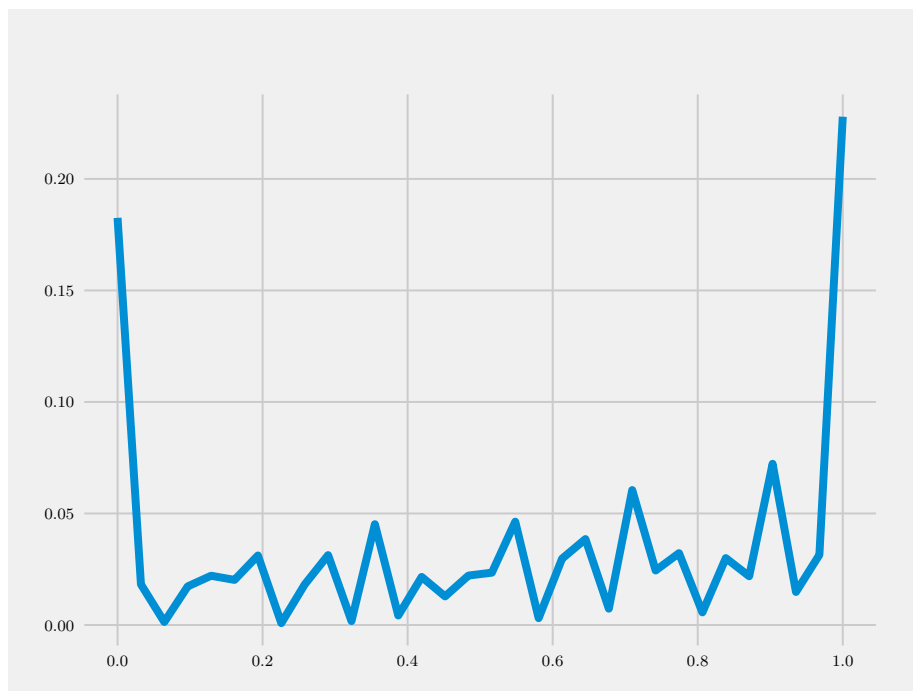


Figure 31: Normalized radial mode number 4 error, 0th identified mode

Fourth Order, Radial Mode 1

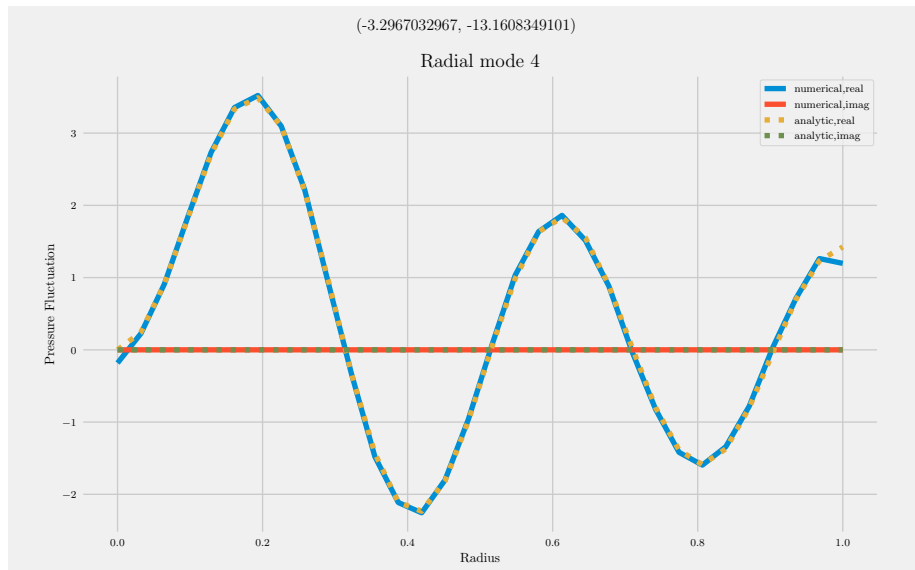


Figure 32: Fourth Order, Case number 1

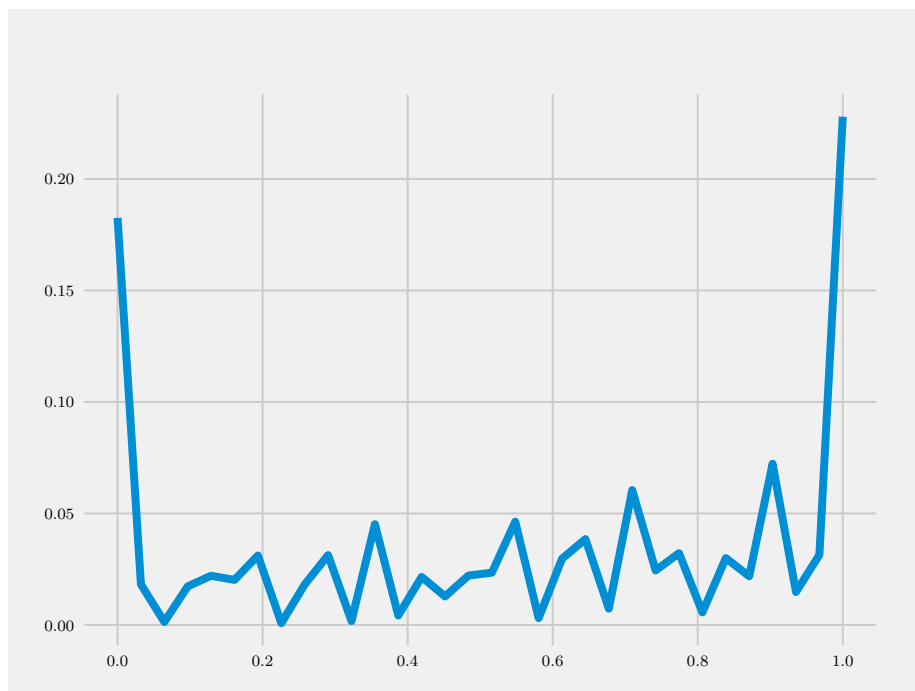


Figure 33: Normalized radial mode number 4 error, 1th identified mode

Fourth Order, Radial Mode 2

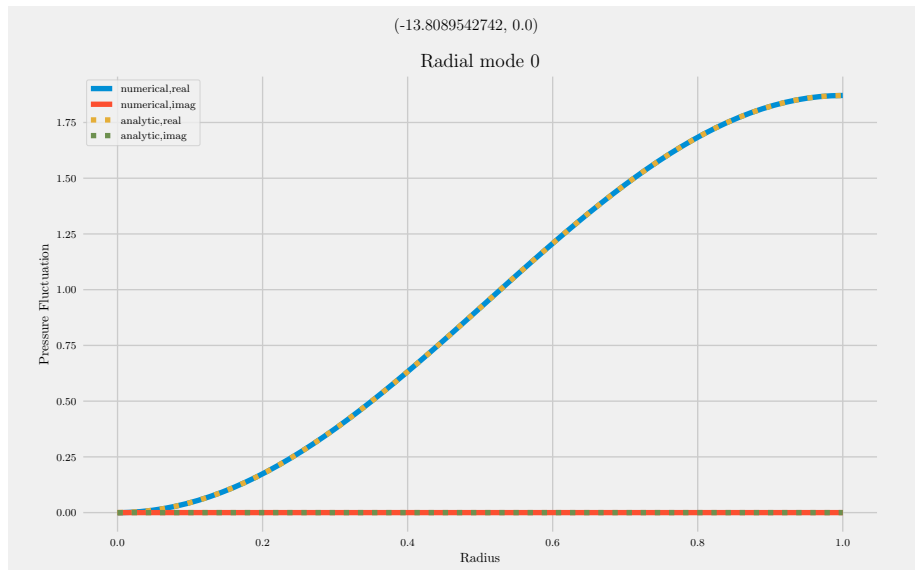


Figure 34: Fourth Order, Case number 0

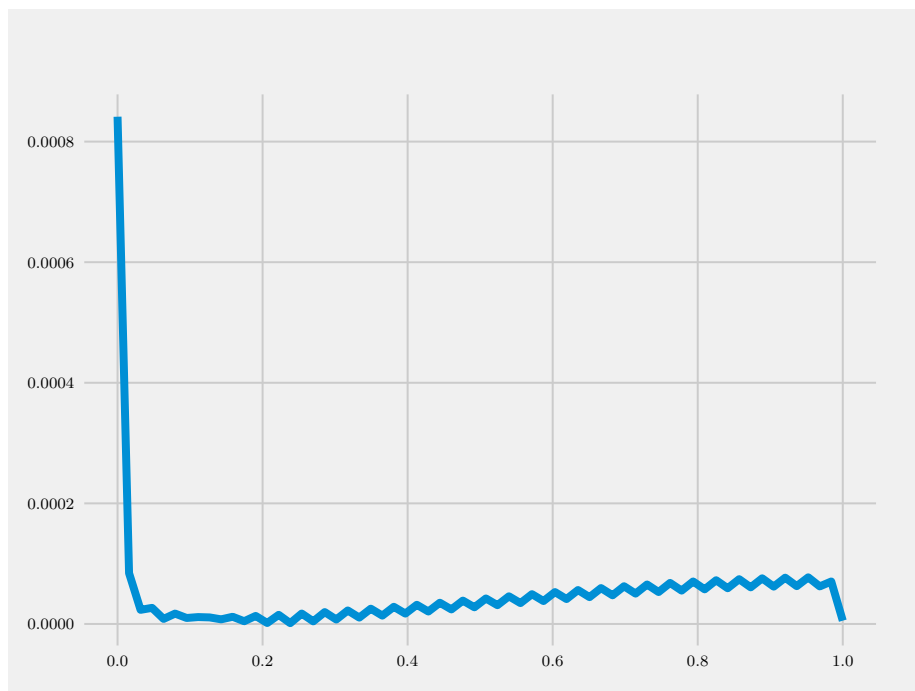


Figure 35: Normalized radial mode number 0 error, 0th identified mode

Fourth Order, Radial Mode 3

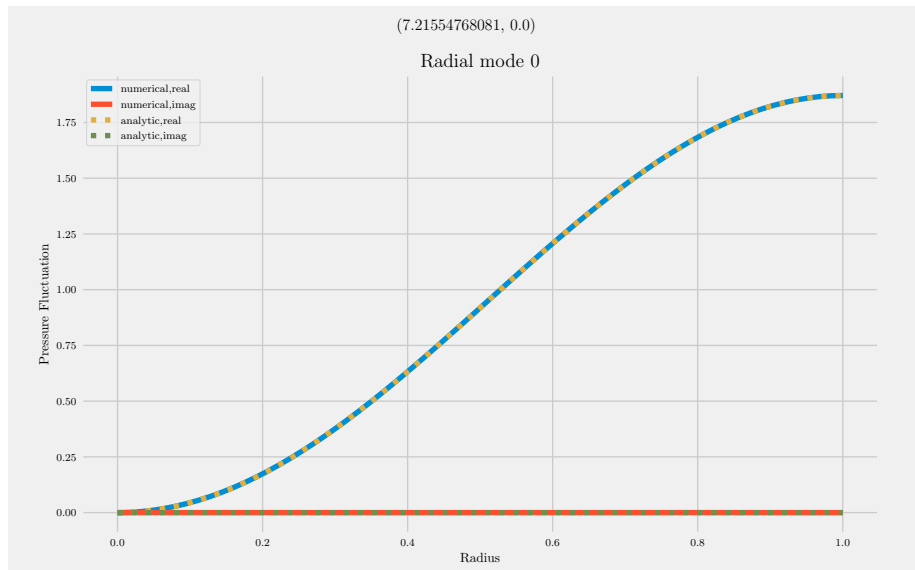


Figure 36: Fourth Order, Case number 1

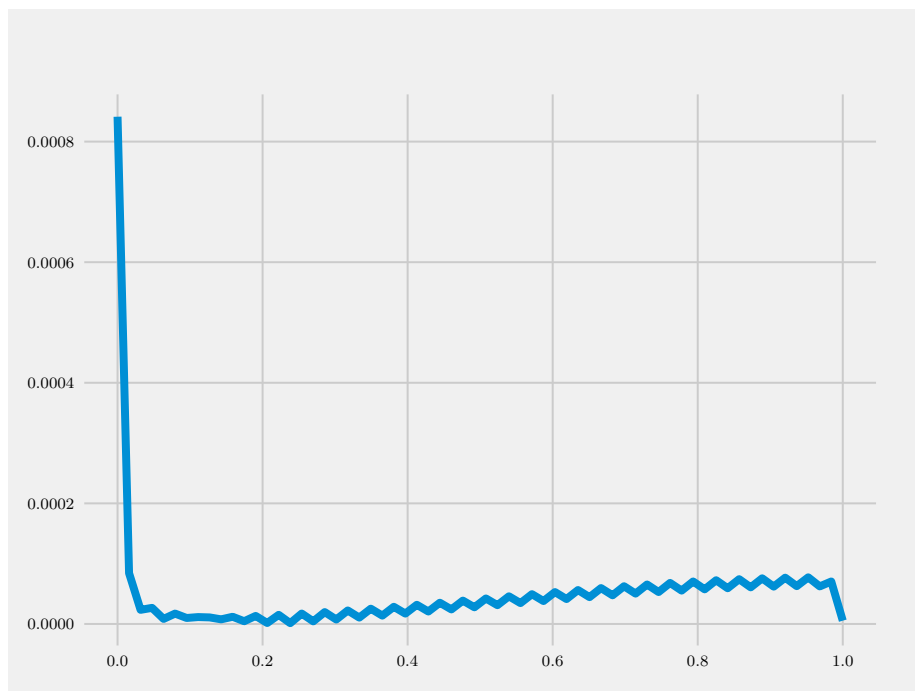


Figure 37: Normalized radial mode number 0 error, 1th identified mode

Fourth Order, Radial Mode 4

0.3.5 256 points

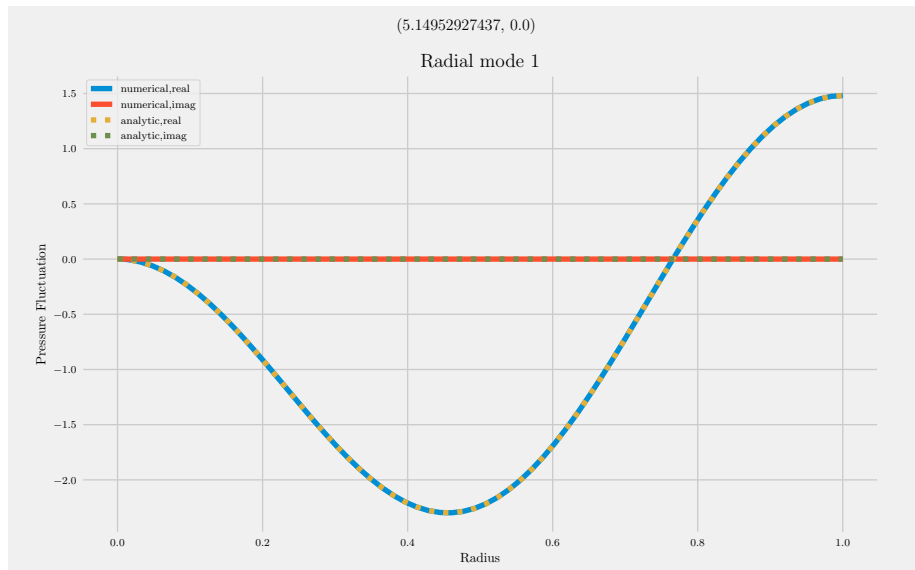


Figure 38: Fourth Order, Case number 0

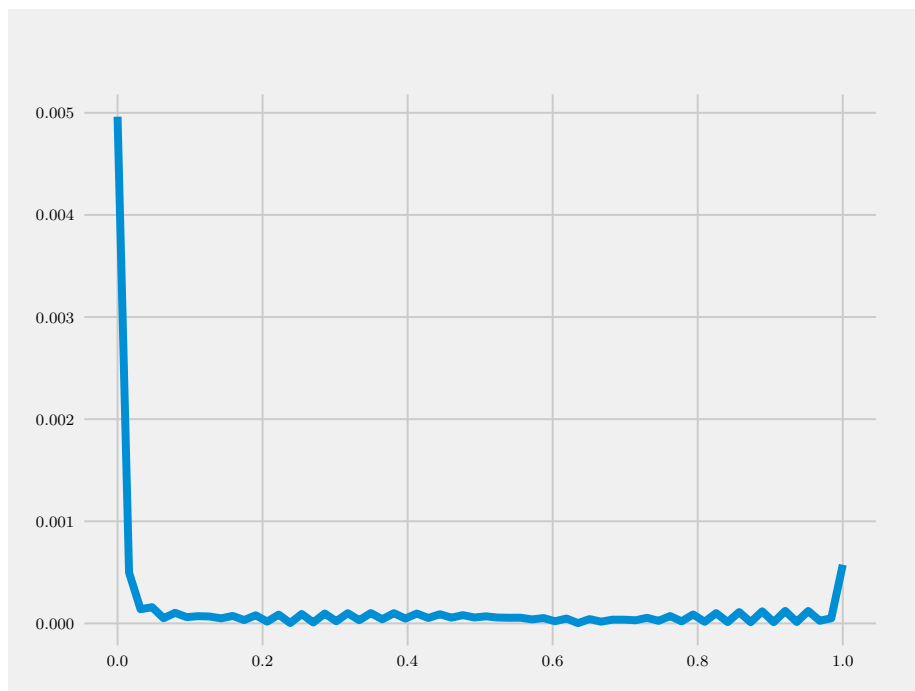


Figure 39: Normalized radial mode number 1 error, 0th identified mode

Fourth Order, Radial Mode 0

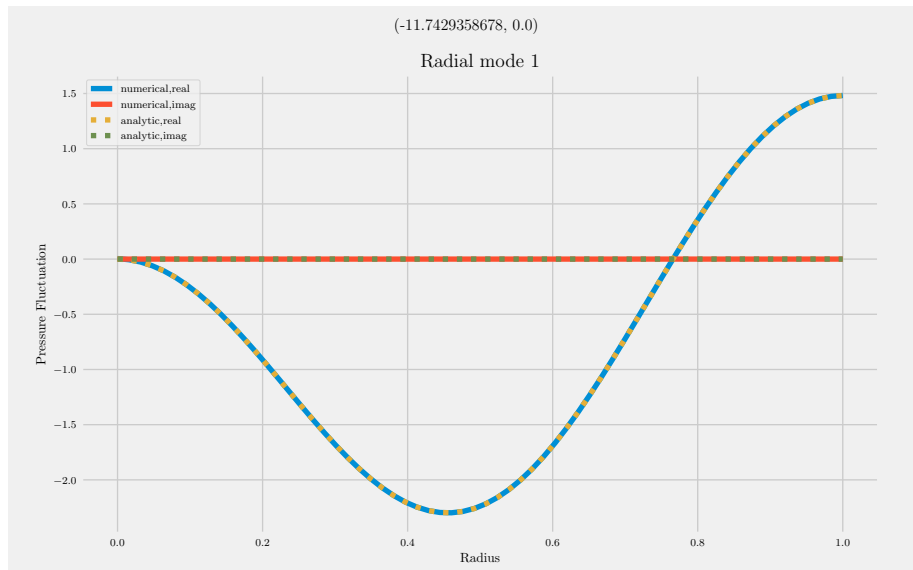


Figure 40: Fourth Order, Case number 1

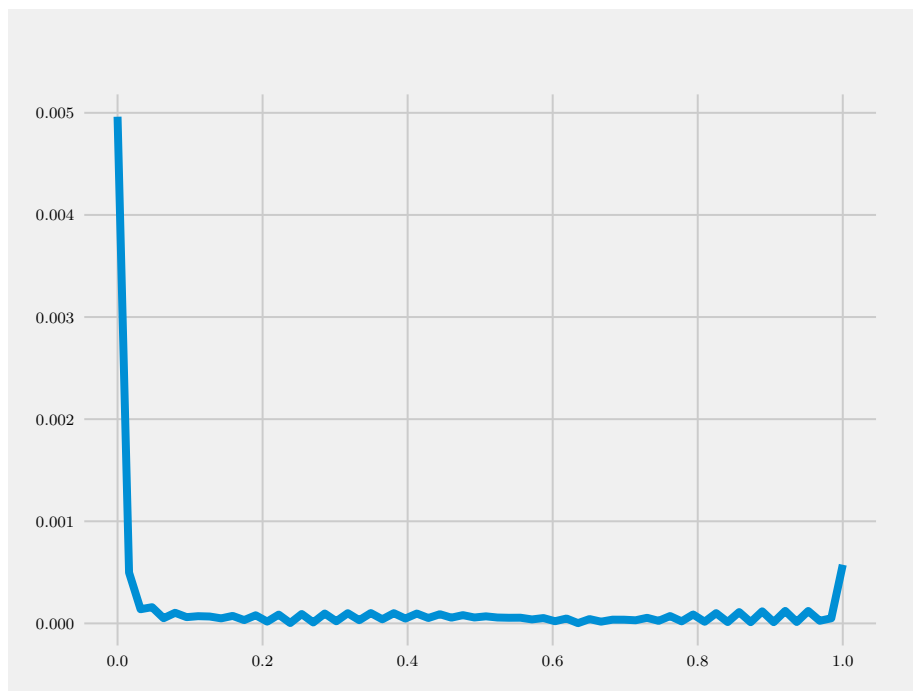


Figure 41: Normalized radial mode number 1 error, 1th identified mode

Fourth Order, Radial Mode 1

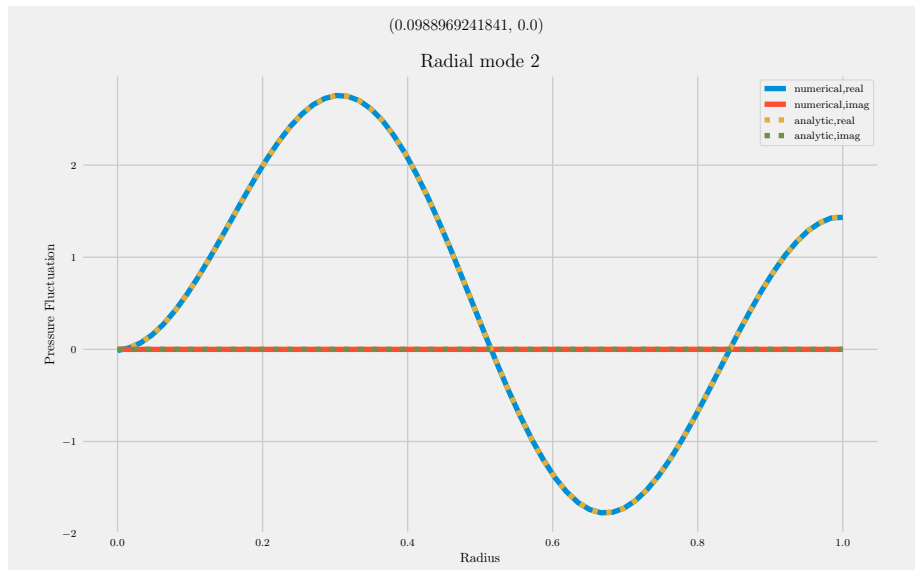


Figure 42: Fourth Order, Case number 0

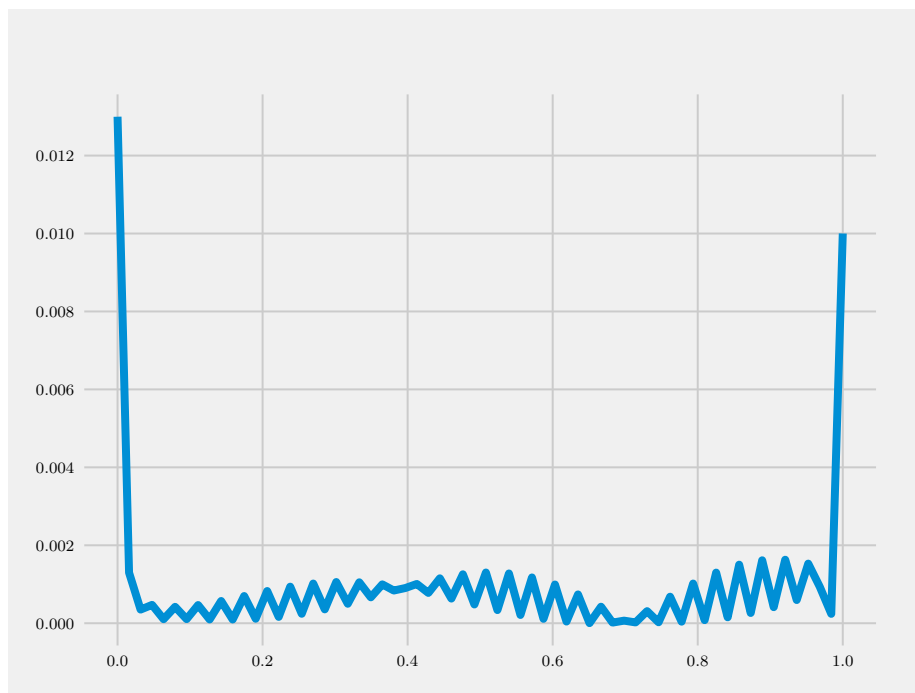


Figure 43: Normalized radial mode number 2 error, 0th identified mode

Fourth Order, Radial Mode 2

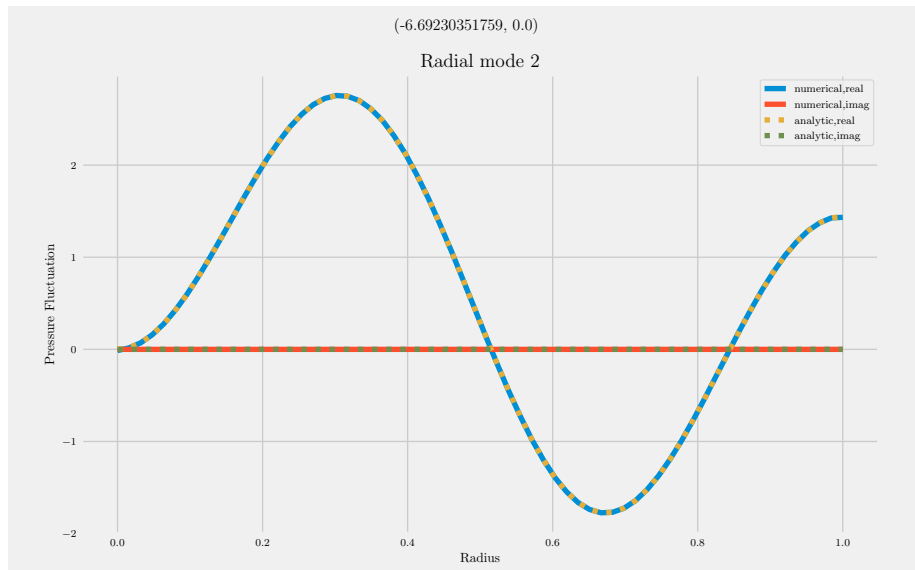


Figure 44: Fourth Order, Case number 1

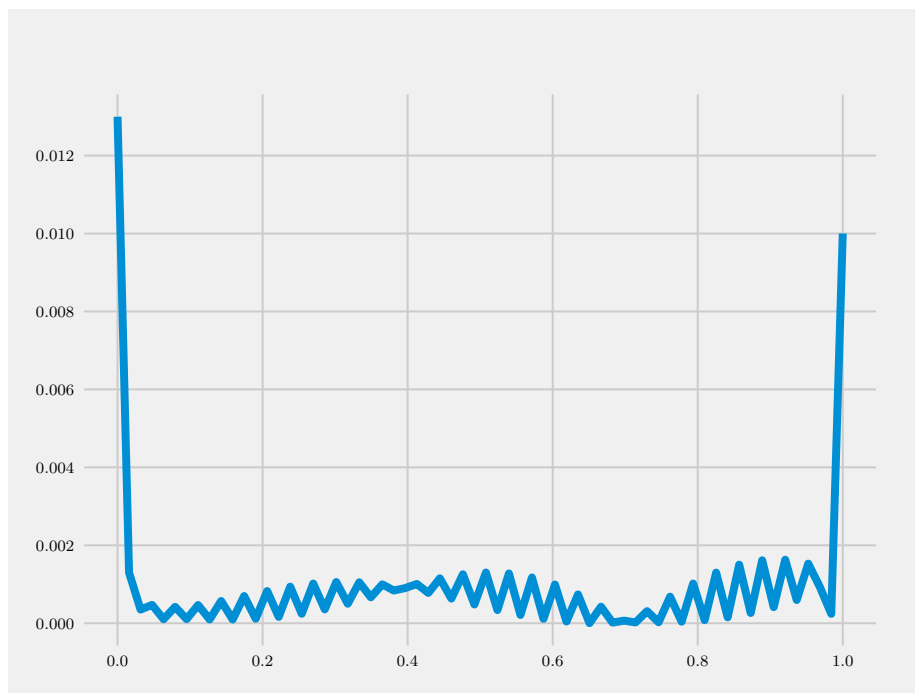


Figure 45: Normalized radial mode number 2 error, 1th identified mode

Fourth Order, Radial Mode 3

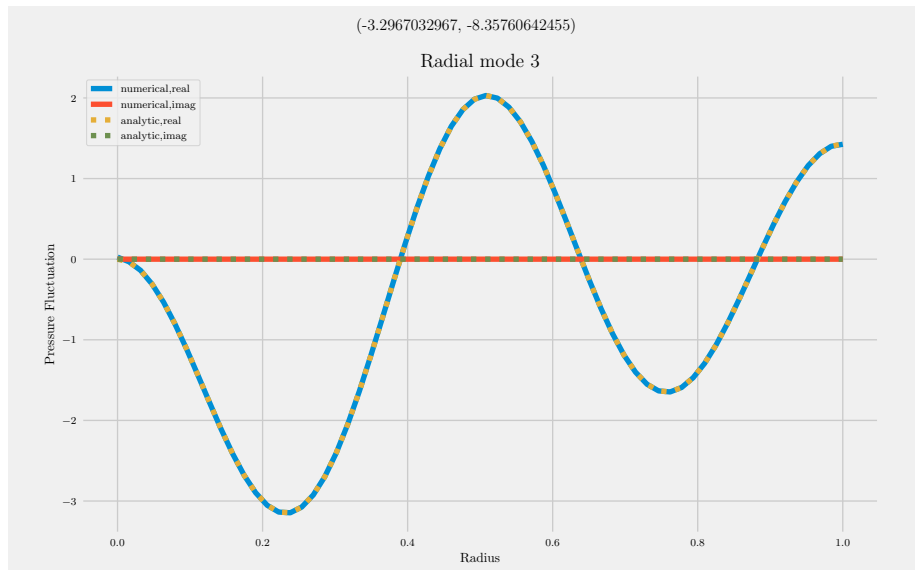


Figure 46: Fourth Order, Case number 0

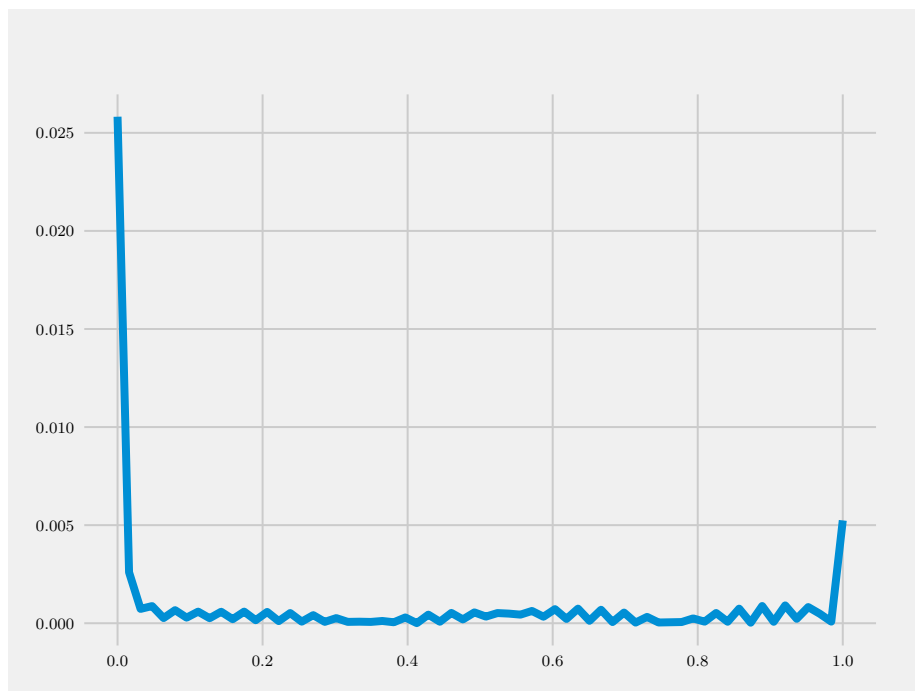


Figure 47: Normalized radial mode number 3 error, 0th identified mode

Fourth Order, Radial Mode 4

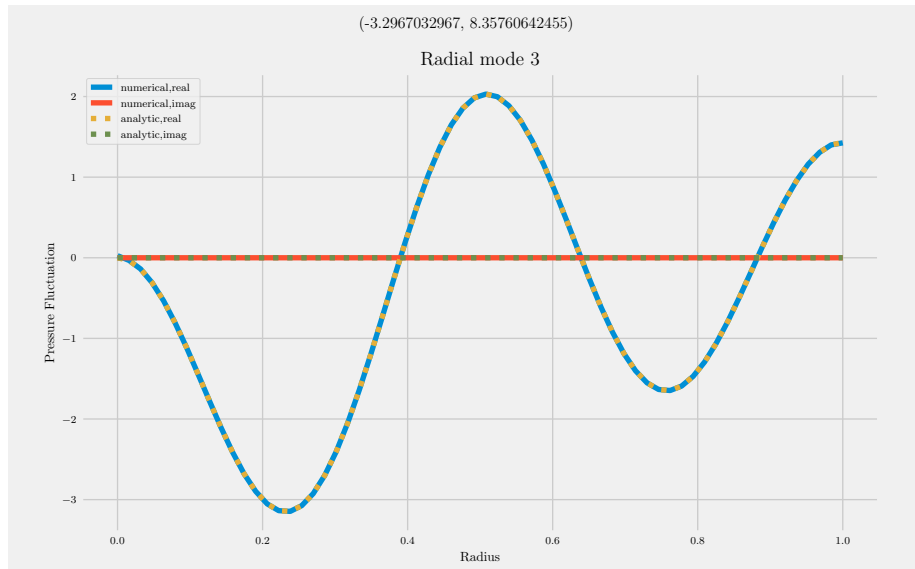


Figure 48: Fourth Order, Case number 1

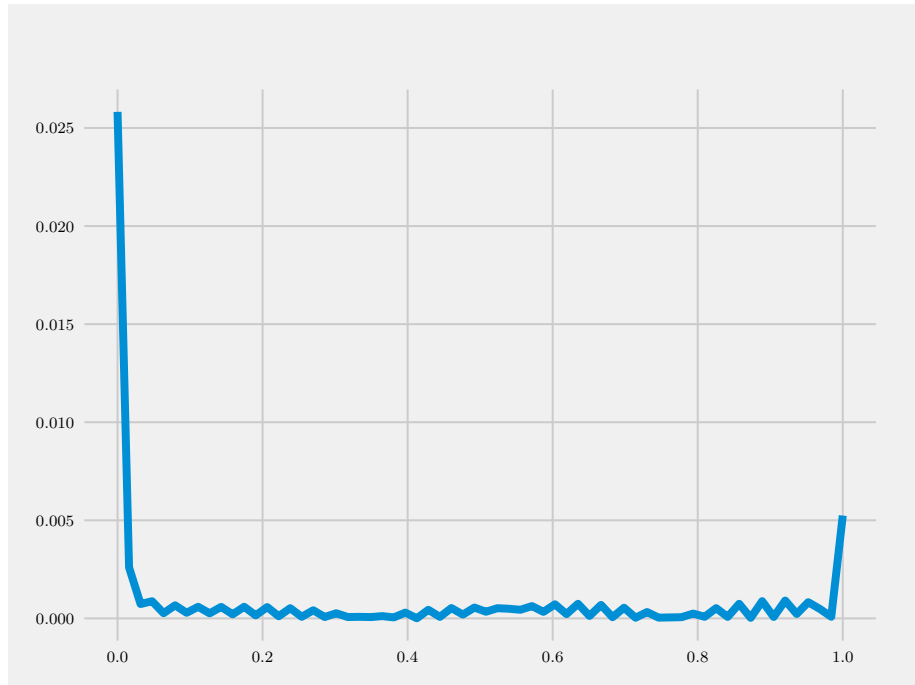


Figure 49: Normalized radial mode number 3 error, 1th identified mode

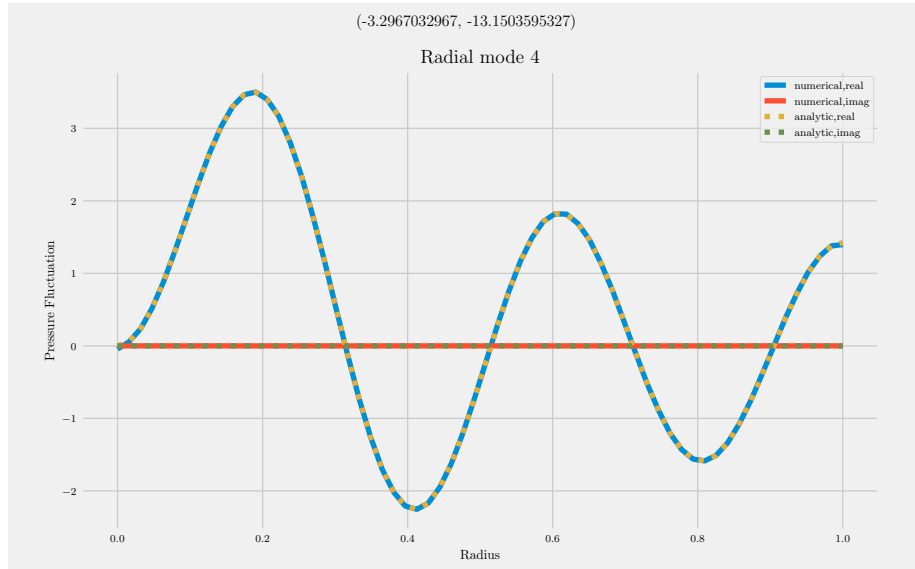


Figure 50: Fourth Order, Case number 0

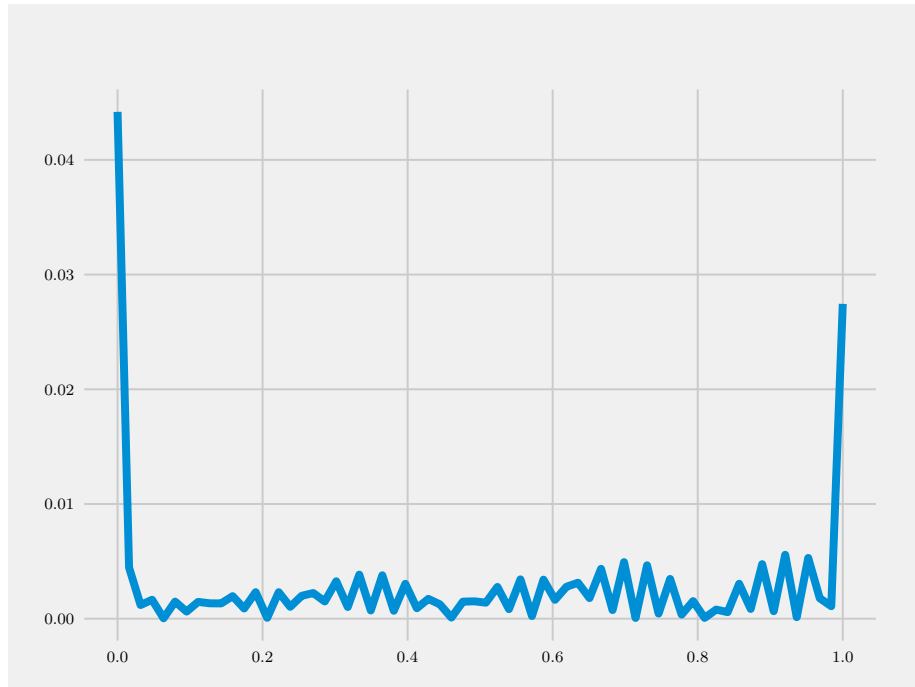


Figure 51: Normalized radial mode number 4 error, 0th identified mode

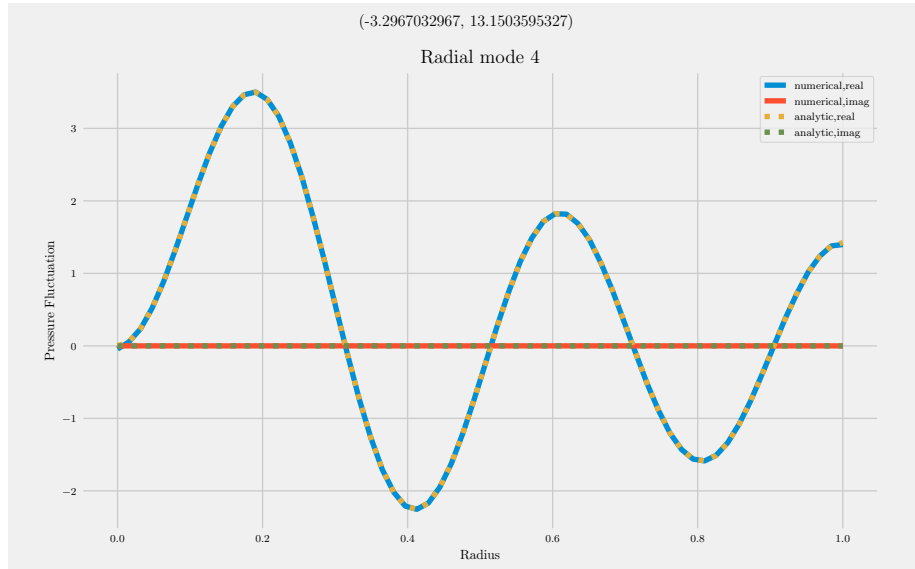


Figure 52: Fourth Order, Case number 1

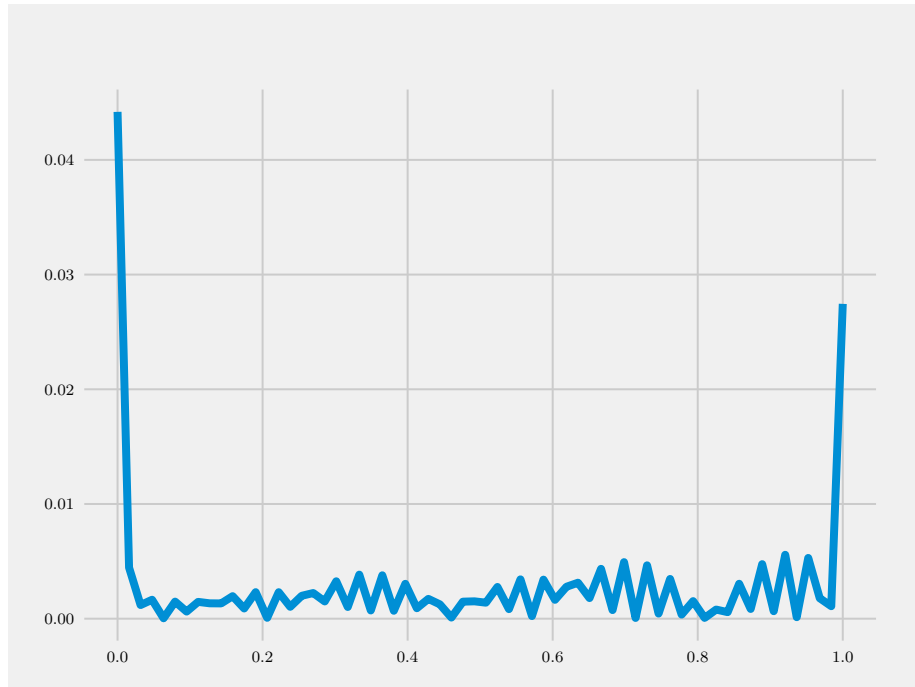


Figure 53: Normalized radial mode number 4 error, 1th identified mode

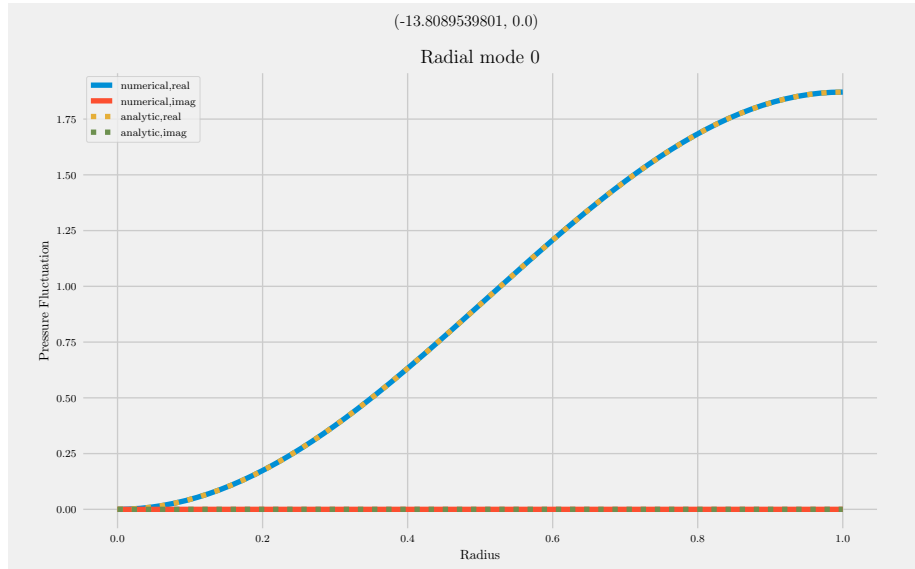


Figure 54: Fourth Order, Case number 0

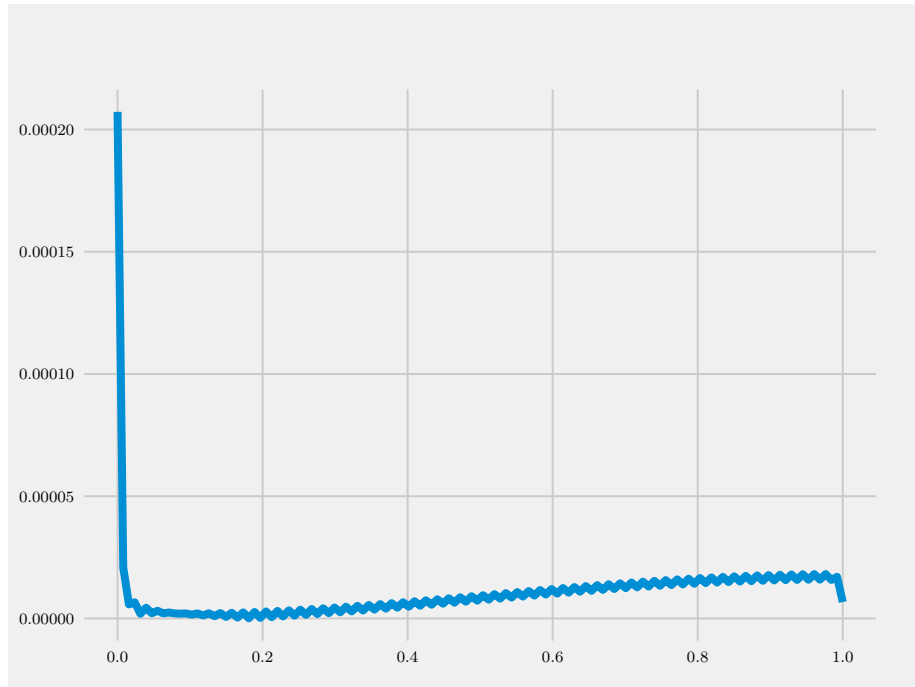


Figure 55: Normalized radial mode number 0 error, 0th identified mode

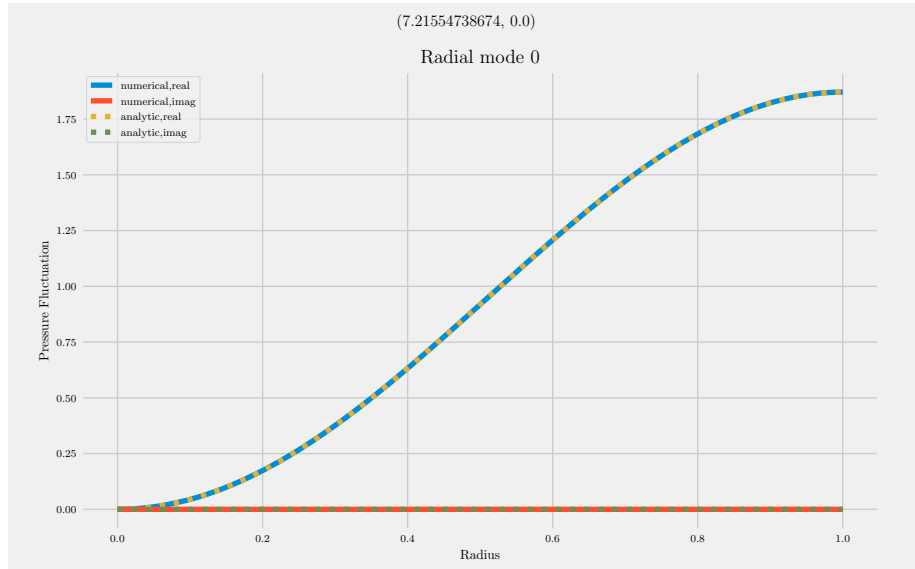


Figure 56: Fourth Order, Case number 1

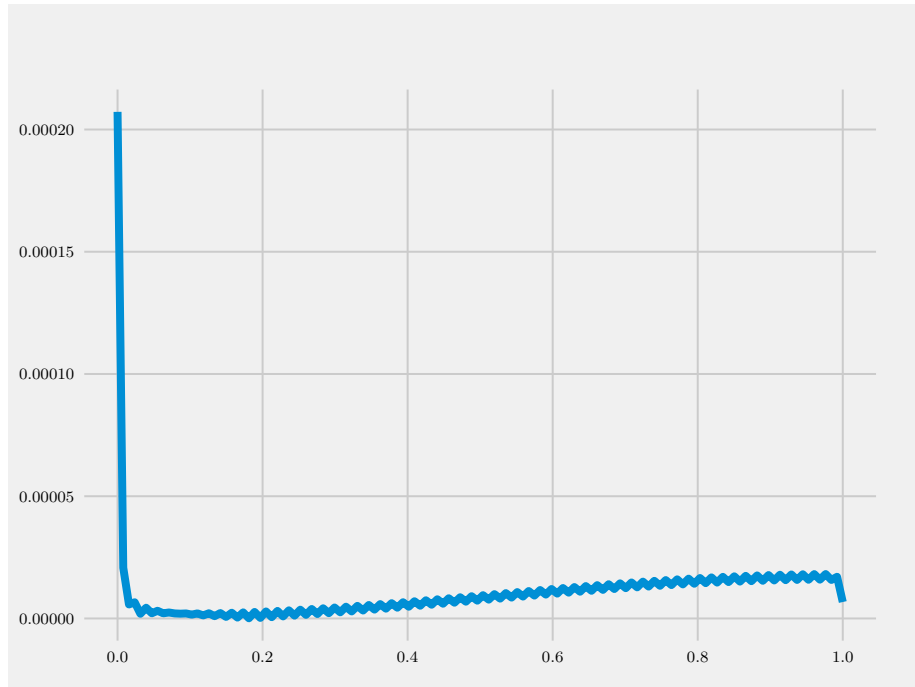


Figure 57: Normalized radial mode number 0 error, 1th identified mode

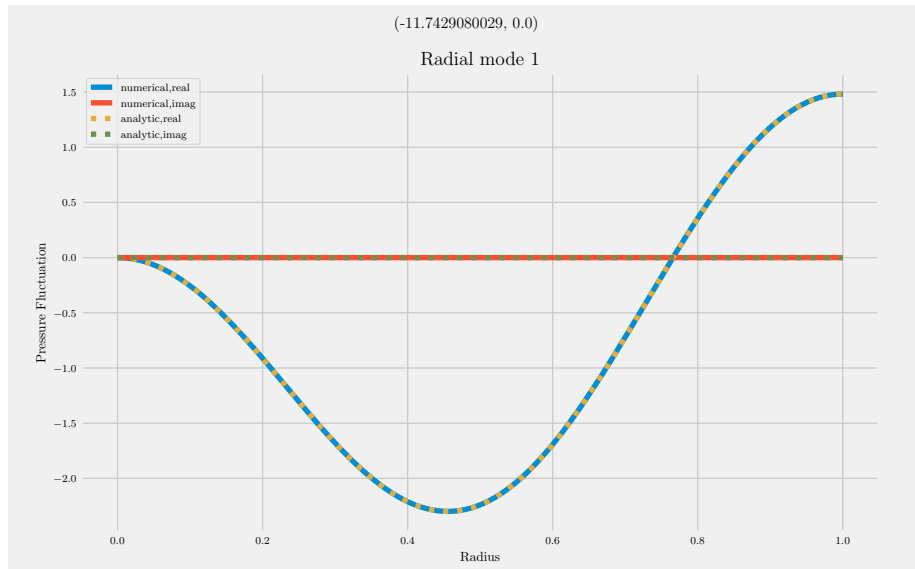


Figure 58: Fourth Order, Case number 0

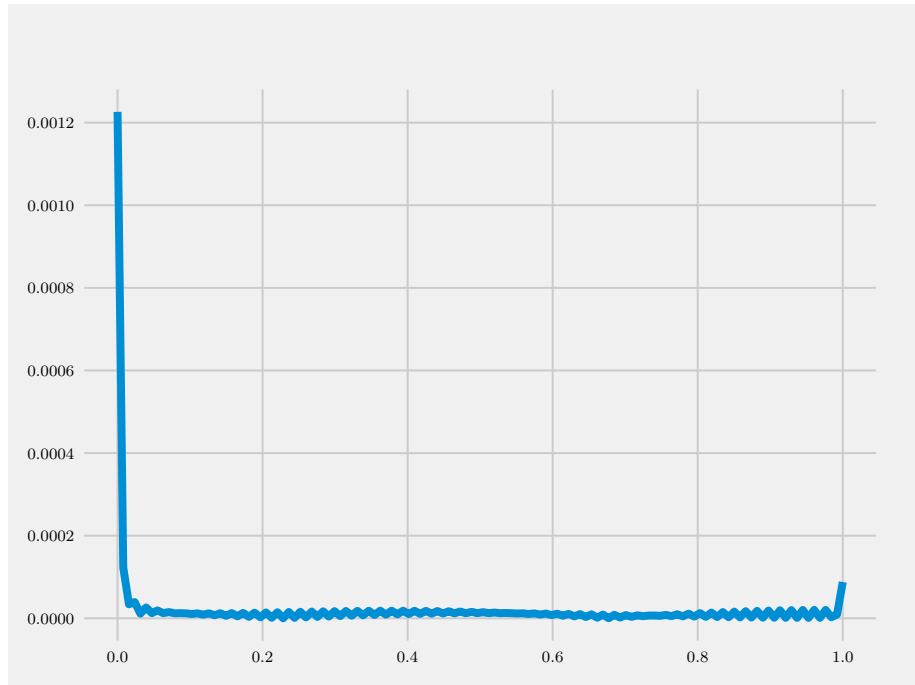


Figure 59: Normalized radial mode number 1 error, 0th identified mode

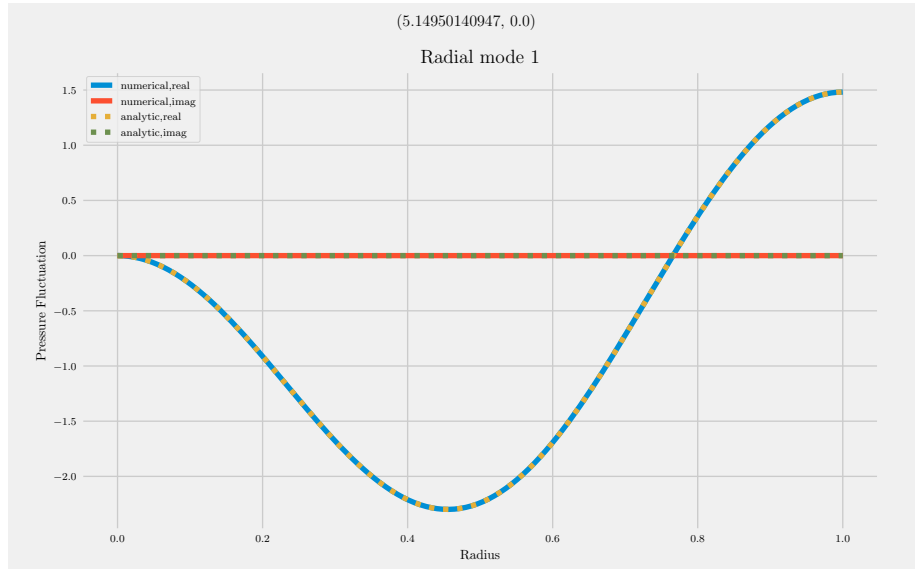


Figure 60: Fourth Order, Case number 1

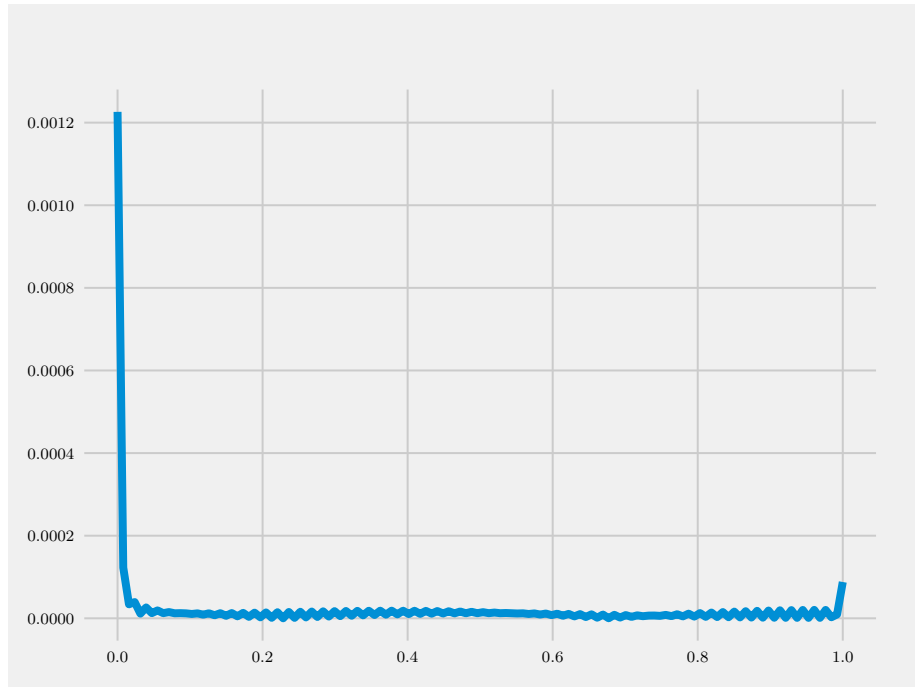


Figure 61: Normalized radial mode number 1 error, 1th identified mode

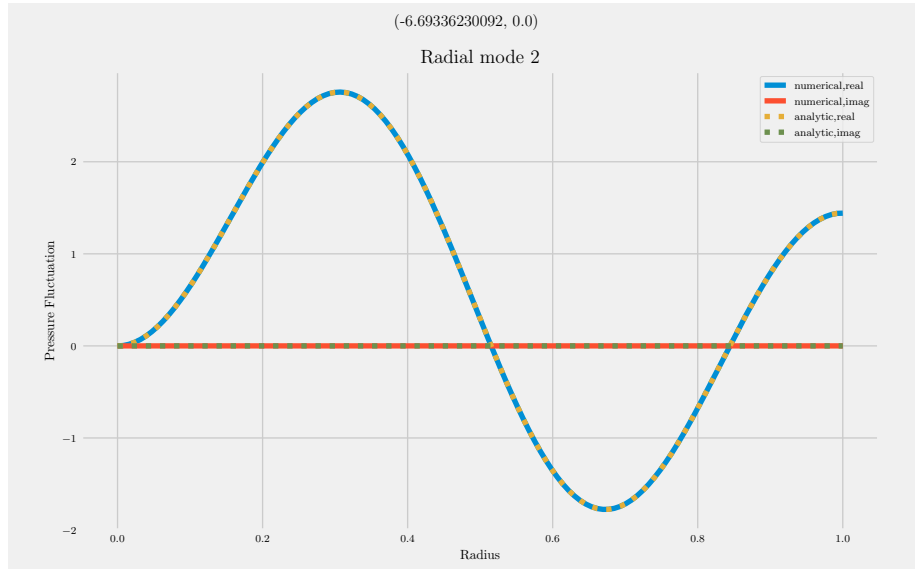


Figure 62: Fourth Order, Case number 0

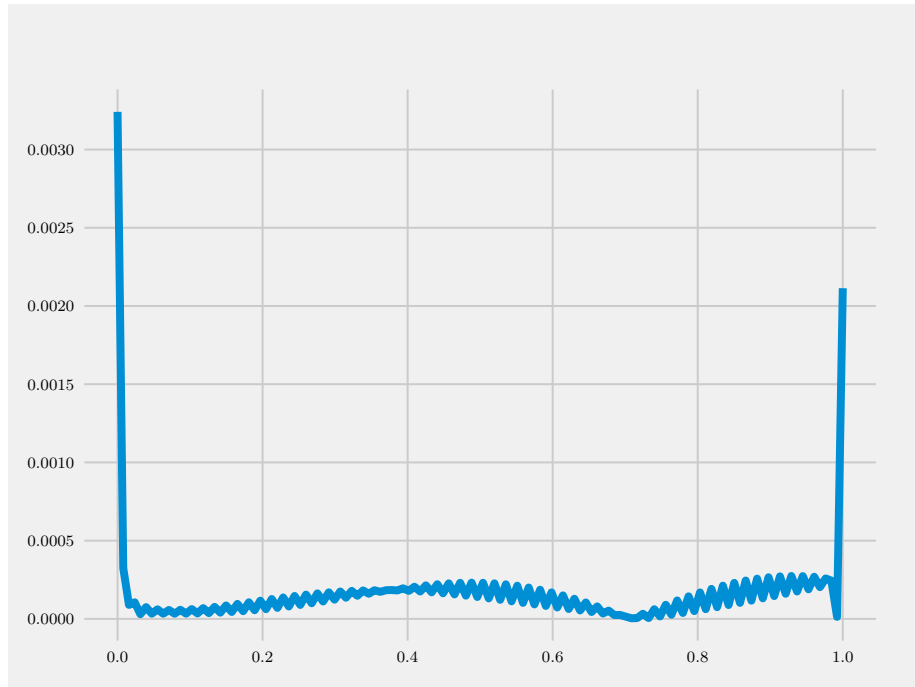


Figure 63: Normalized radial mode number 2 error, 0th identified mode

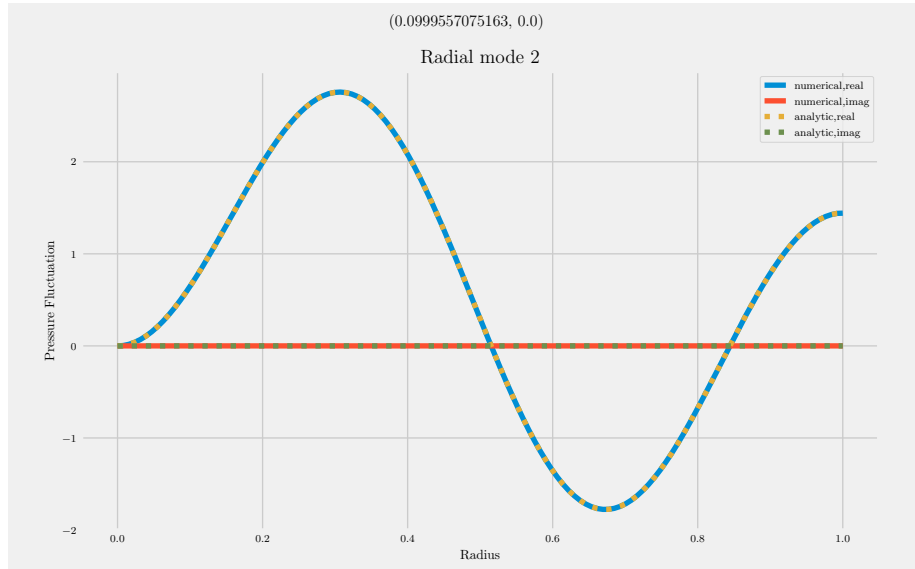


Figure 64: Fourth Order, Case number 1

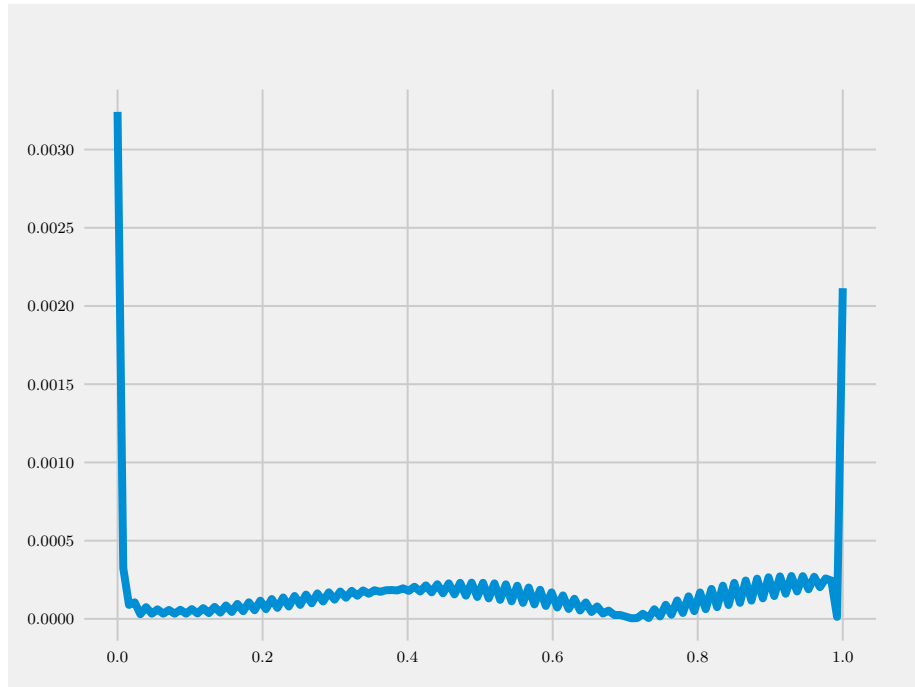


Figure 65: Normalized radial mode number 2 error, 1th identified mode

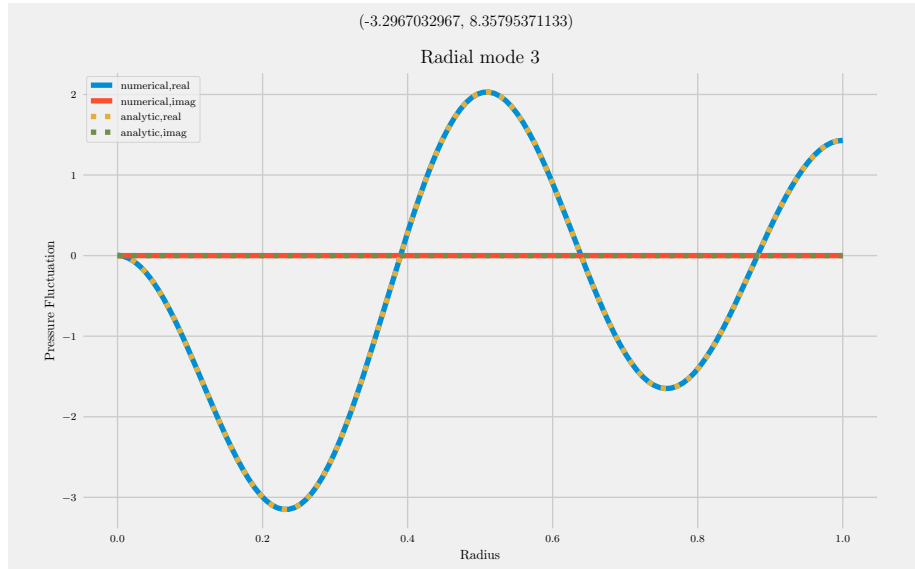


Figure 66: Fourth Order, Case number 0

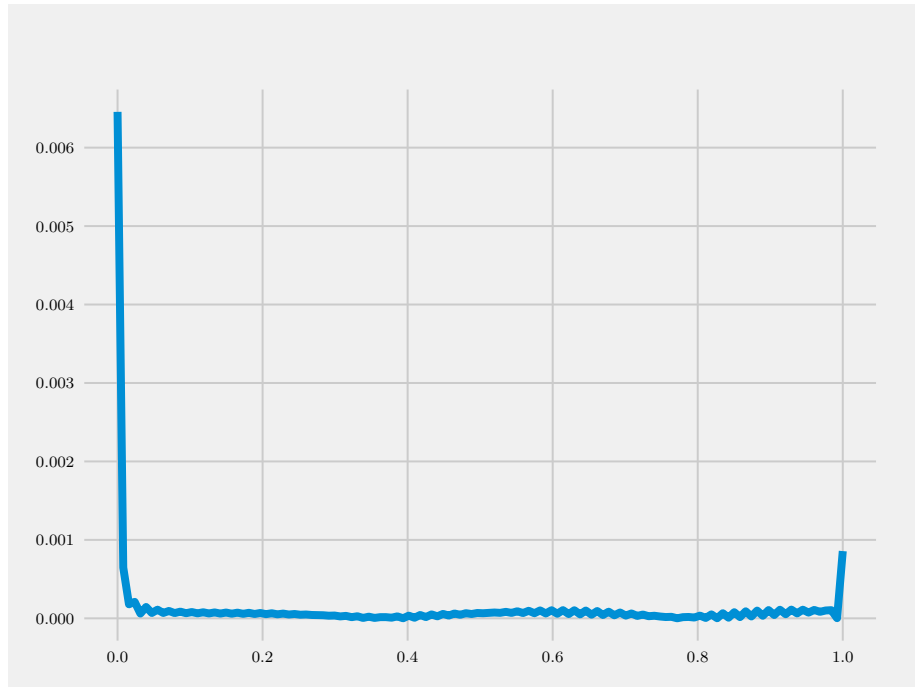


Figure 67: Normalized radial mode number 3 error, 0th identified mode

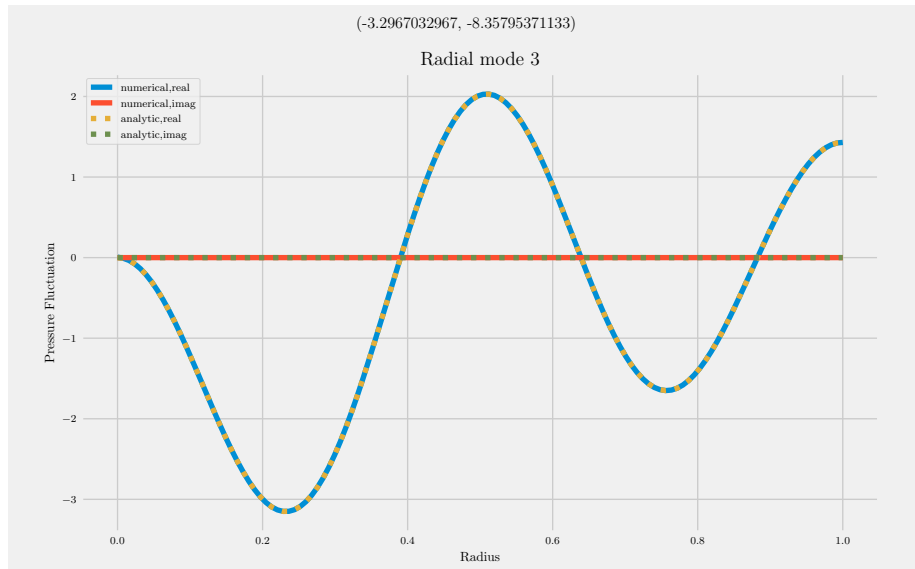


Figure 68: Fourth Order, Case number 1

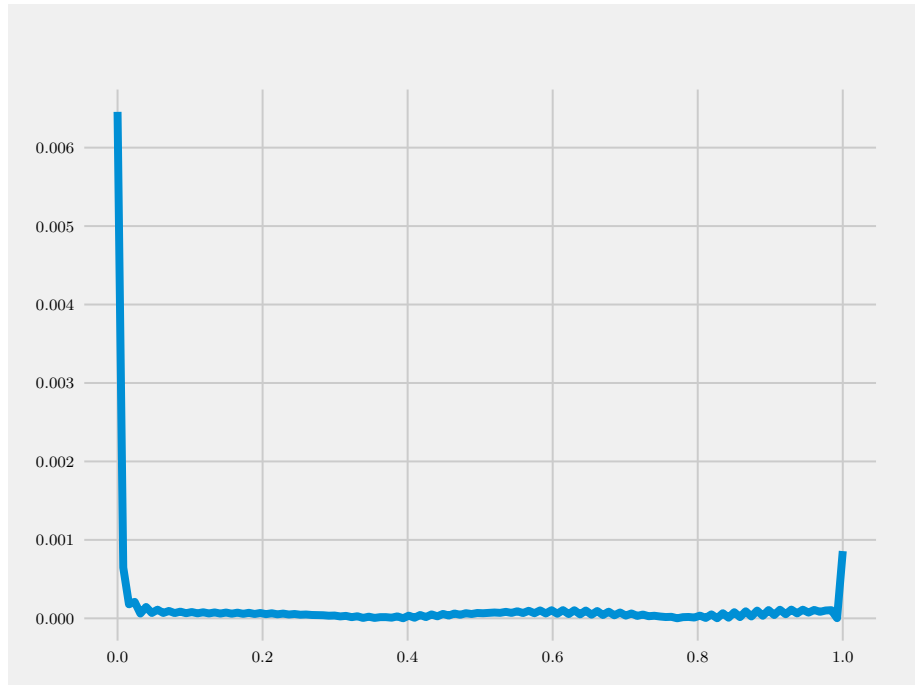


Figure 69: Normalized radial mode number 3 error, 1th identified mode

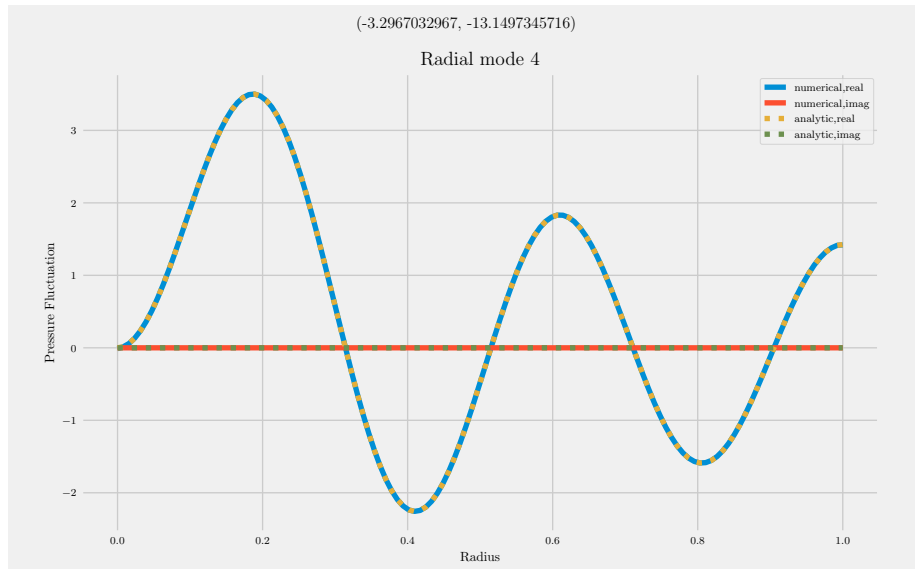


Figure 70: Fourth Order, Case number 0

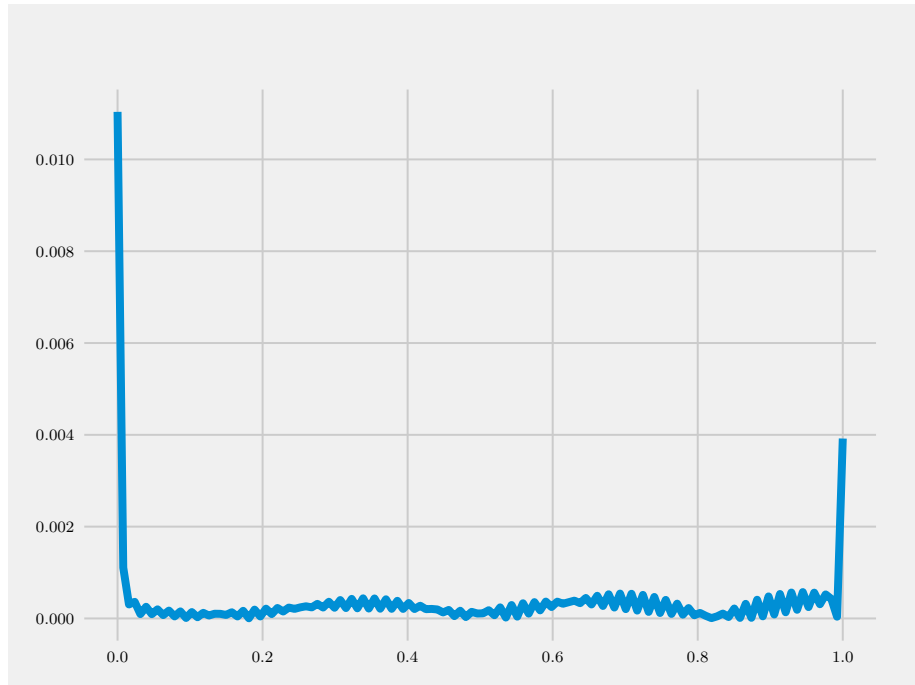


Figure 71: Normalized radial mode number 4 error, 0th identified mode

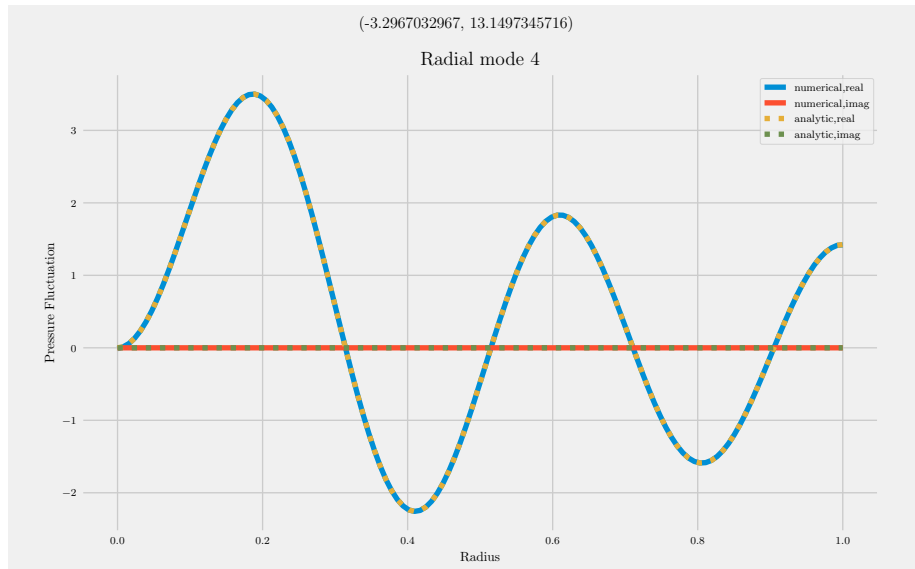


Figure 72: Fourth Order, Case number 1

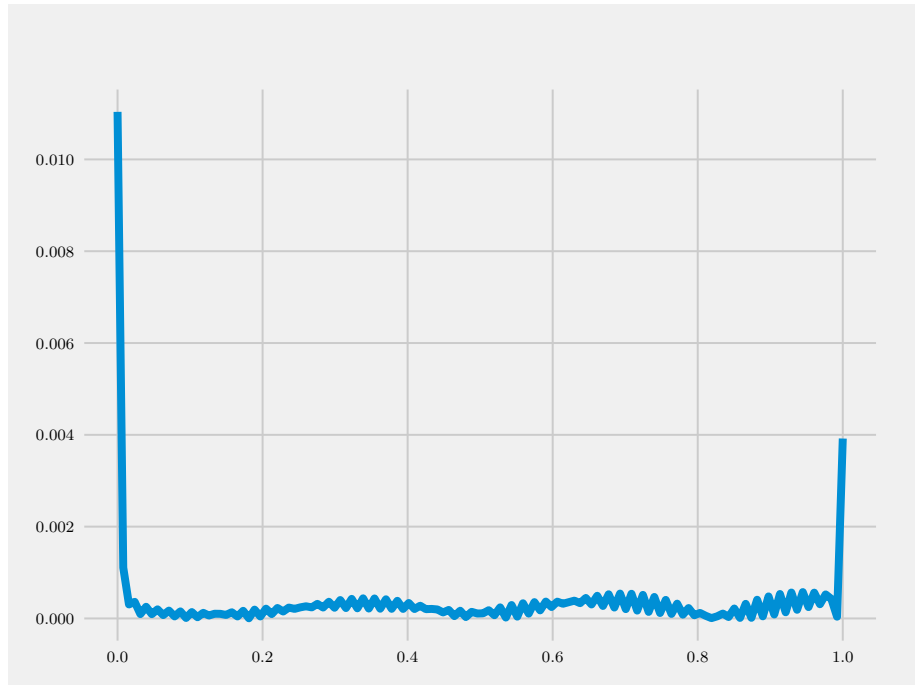


Figure 73: Normalized radial mode number 4 error, 1th identified mode

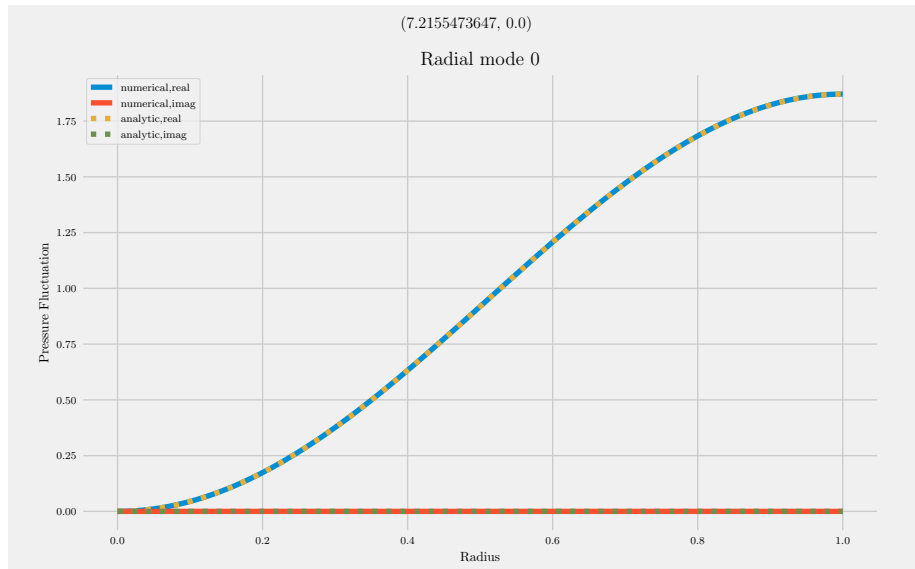


Figure 74: Fourth Order, Case number 0

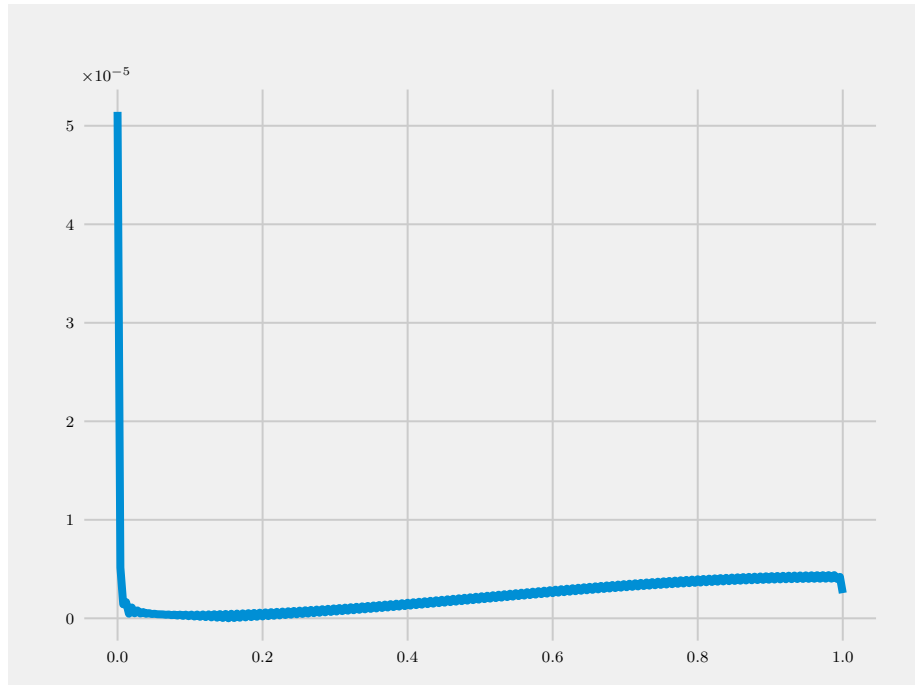


Figure 75: Normalized radial mode number 0 error, 0th identified mode

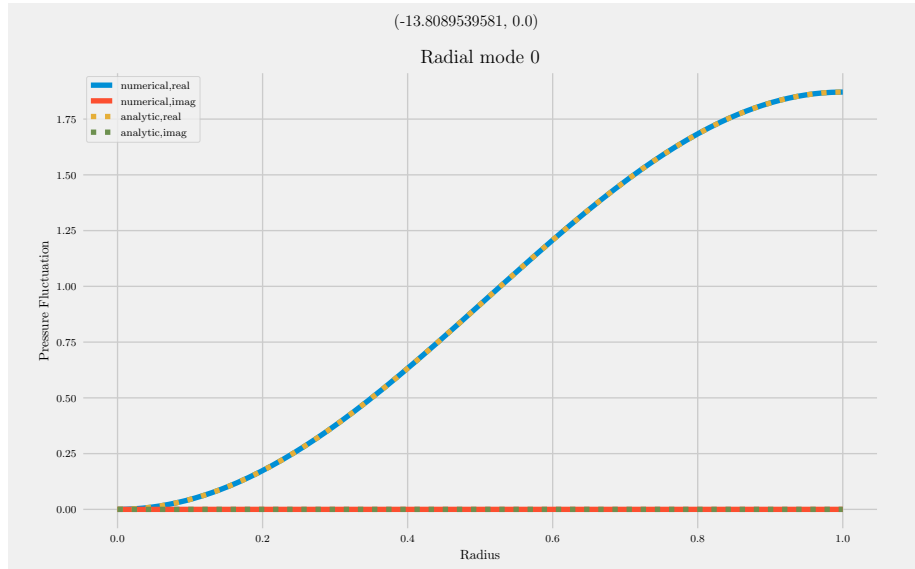


Figure 76: Fourth Order, Case number 1

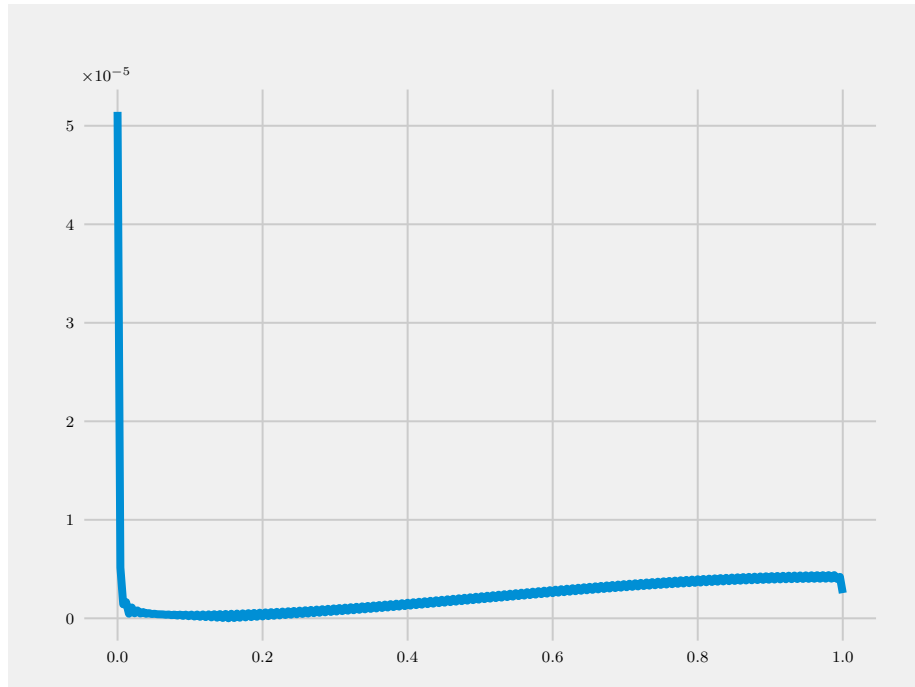


Figure 77: Normalized radial mode number 0 error, 1th identified mode

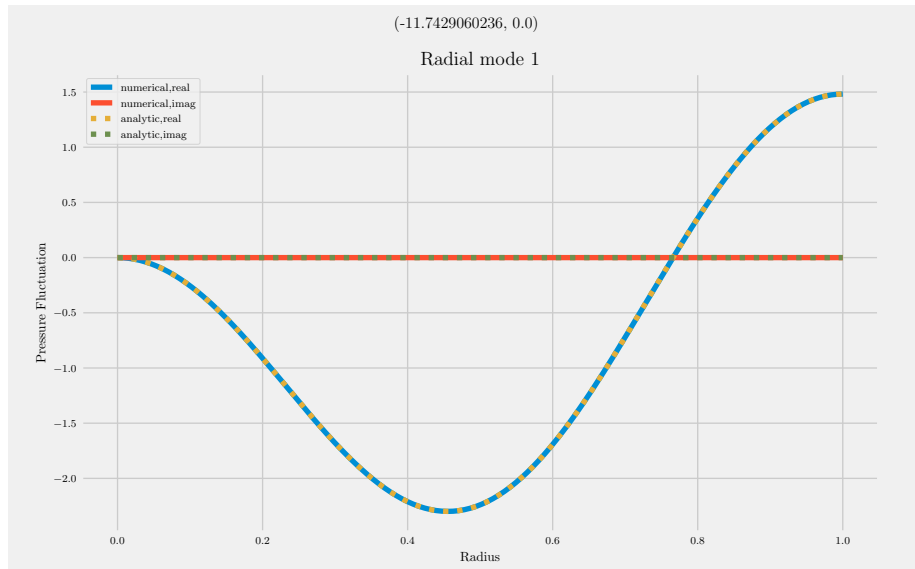


Figure 78: Fourth Order, Case number 0

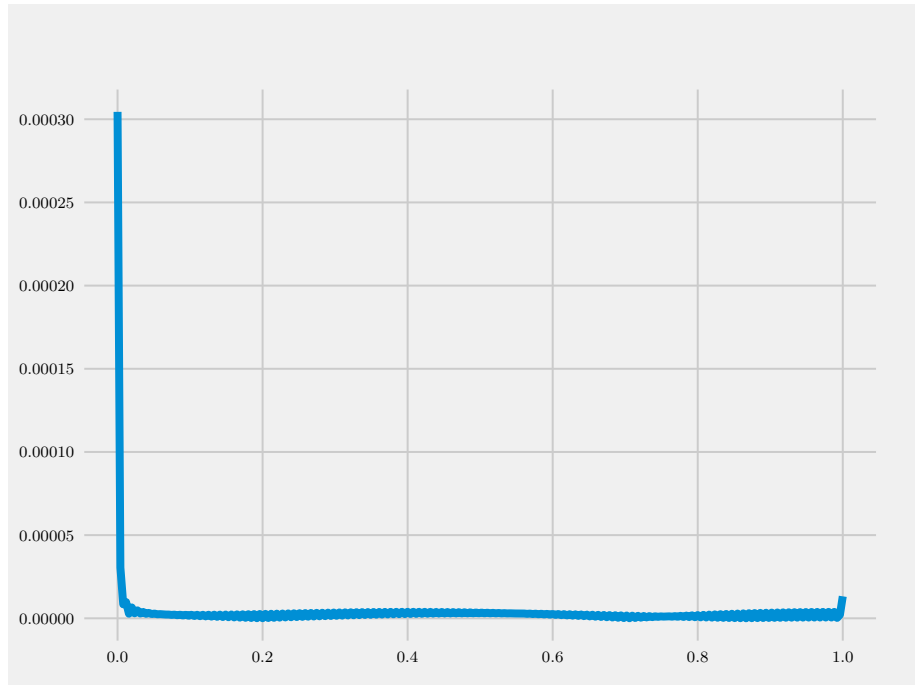


Figure 79: Normalized radial mode number 1 error, 0th identified mode

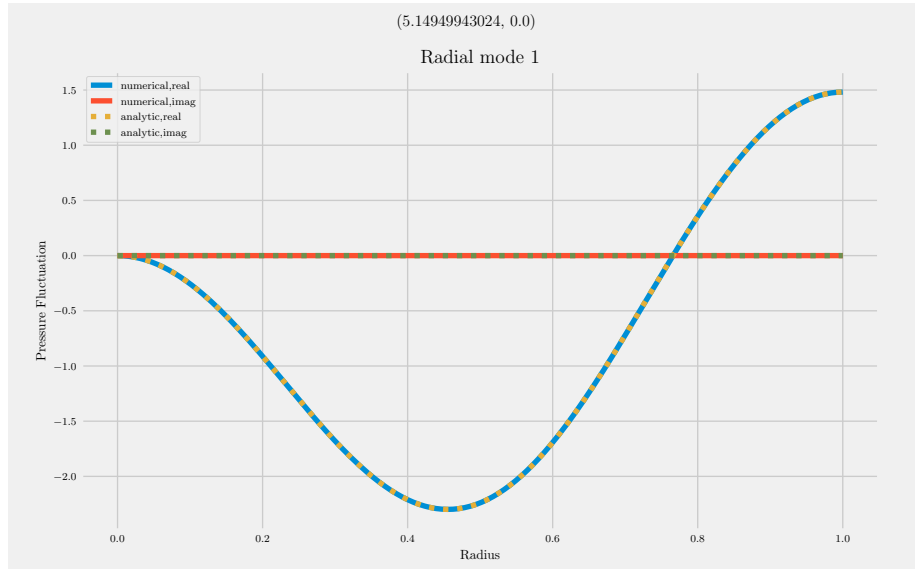


Figure 80: Fourth Order, Case number 1

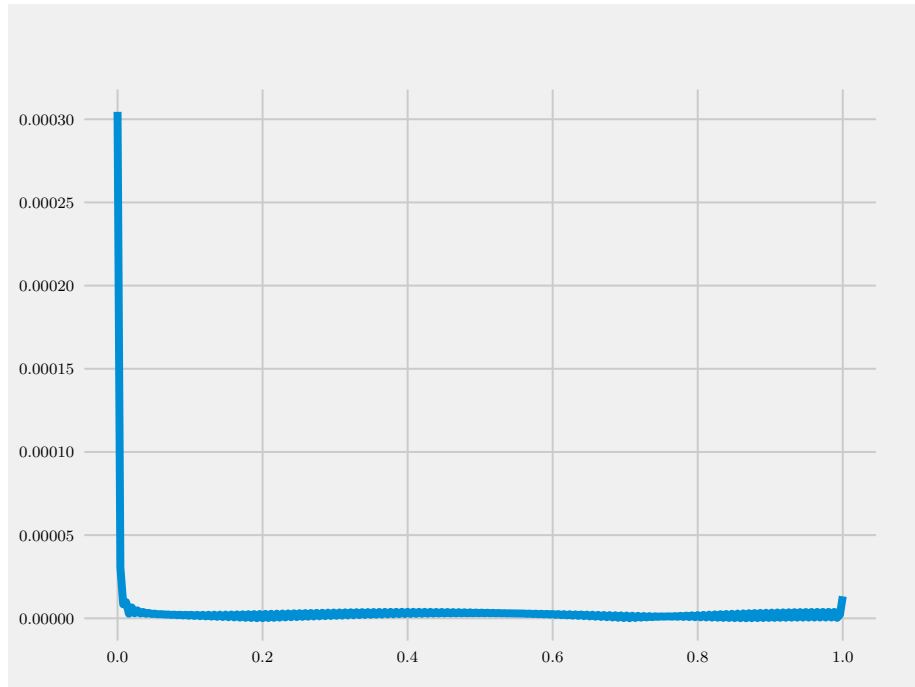


Figure 81: Normalized radial mode number 1 error, 1th identified mode

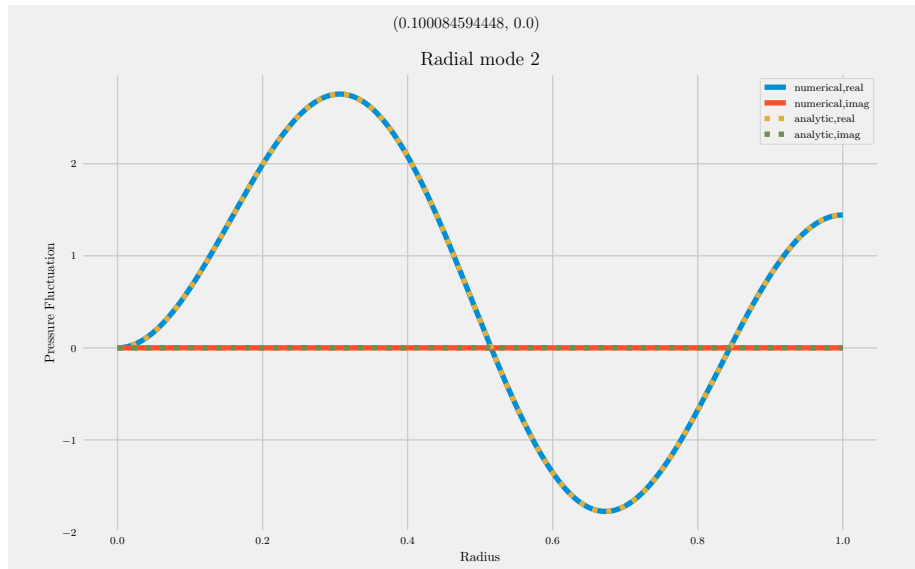


Figure 82: Fourth Order, Case number 0

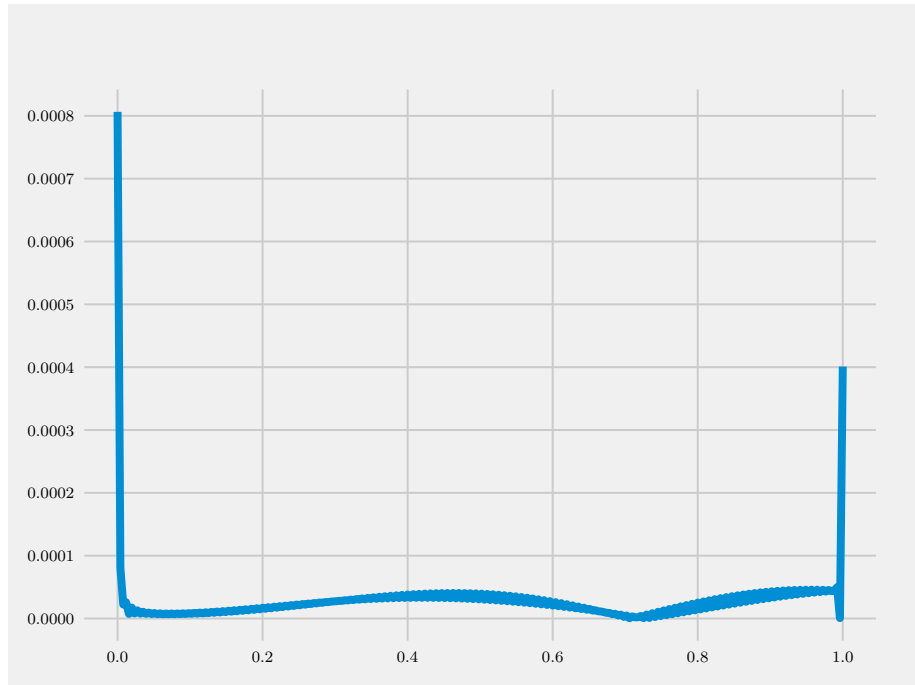


Figure 83: Normalized radial mode number 2 error, 0th identified mode

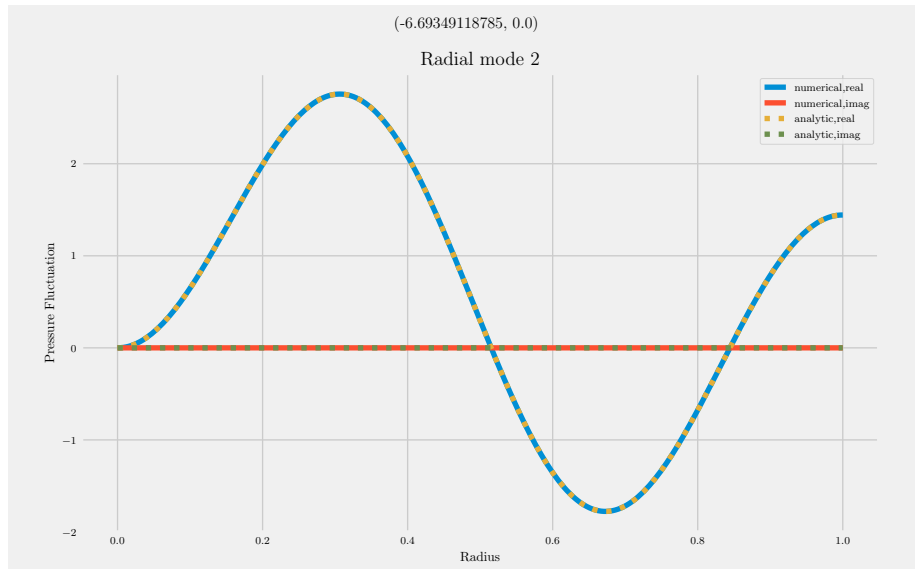


Figure 84: Fourth Order, Case number 1

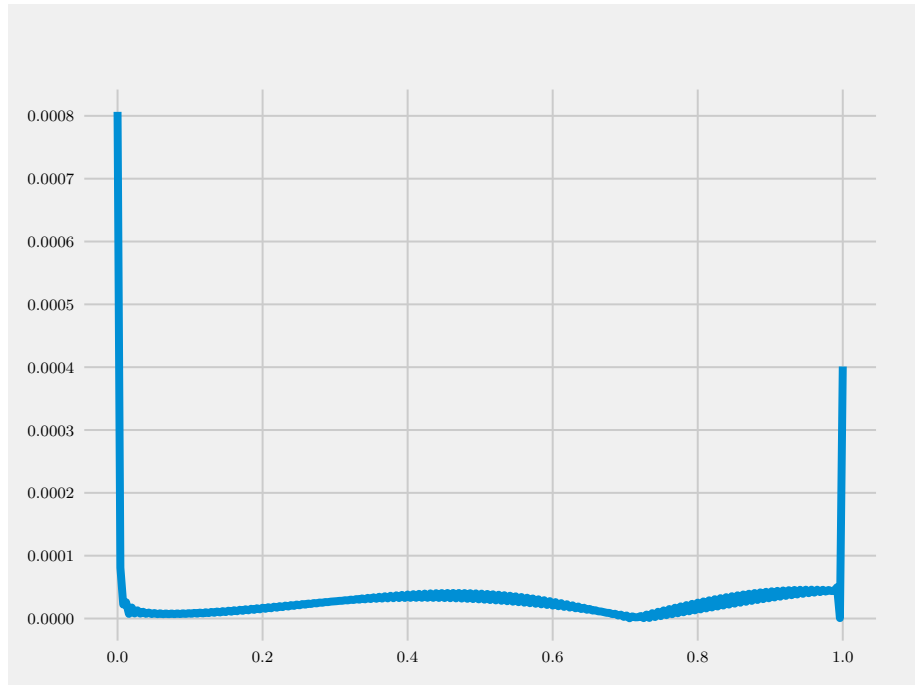


Figure 85: Normalized radial mode number 2 error, 1th identified mode

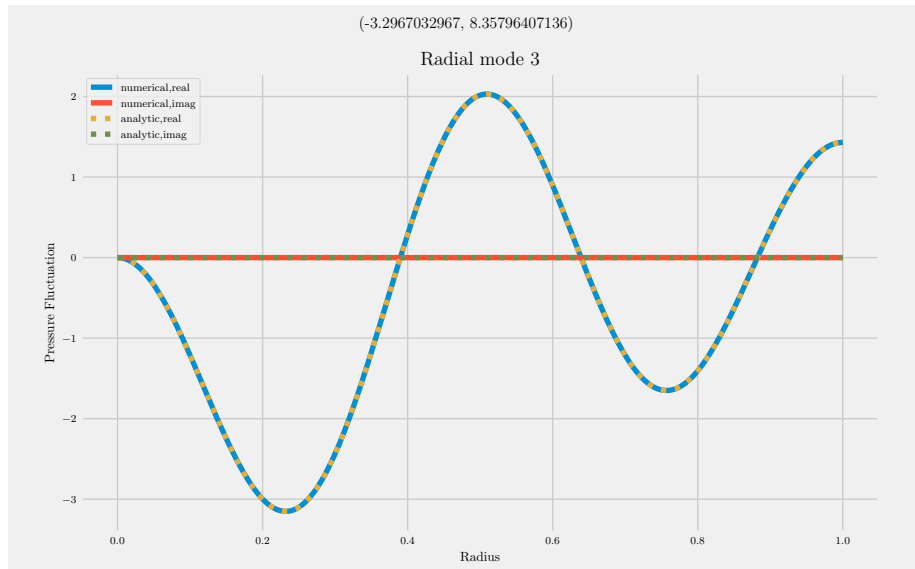


Figure 86: Fourth Order, Case number 0

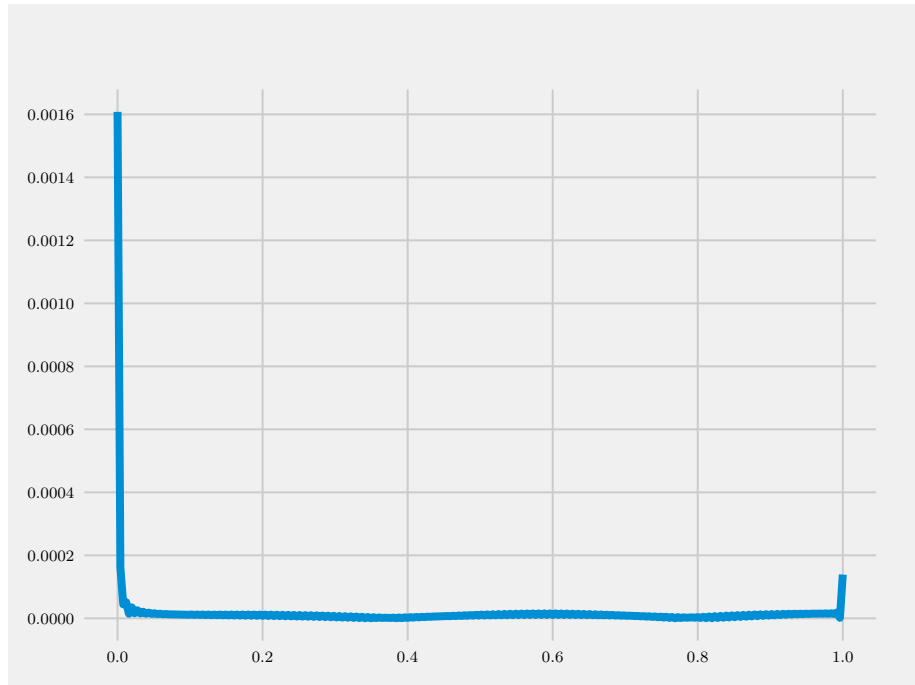


Figure 87: Normalized radial mode number 3 error, 0th identified mode

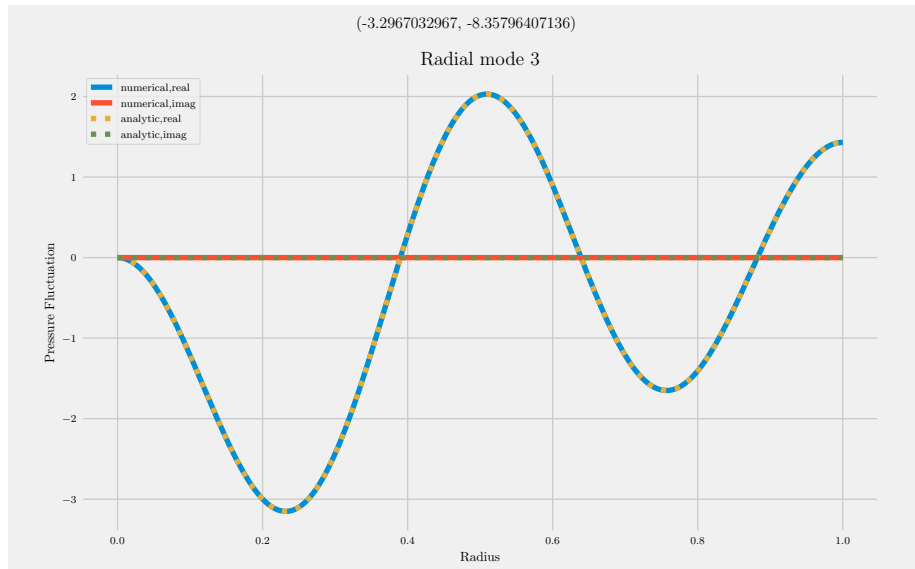


Figure 88: Fourth Order, Case number 1

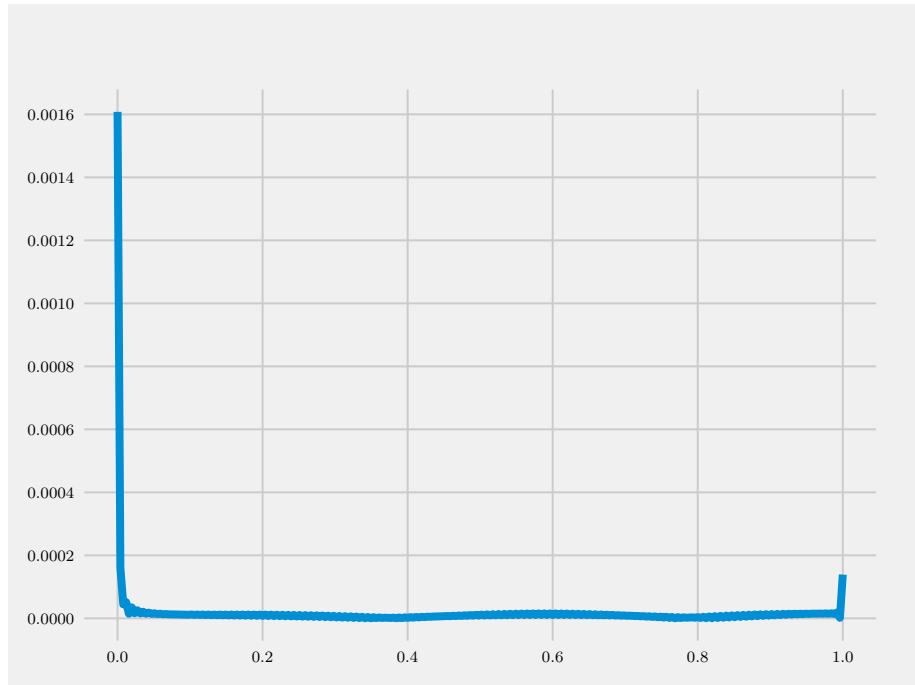


Figure 89: Normalized radial mode number 3 error, 1th identified mode

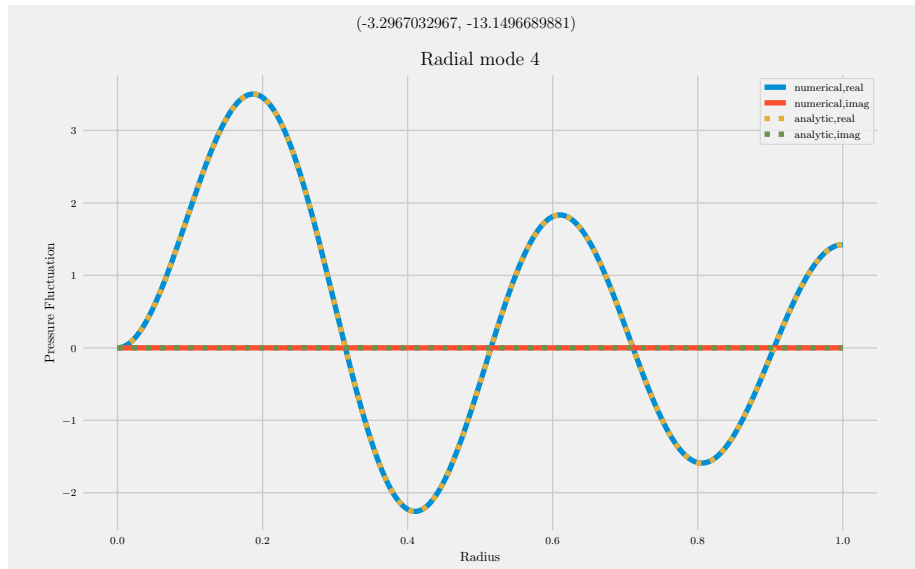


Figure 90: Fourth Order, Case number 0

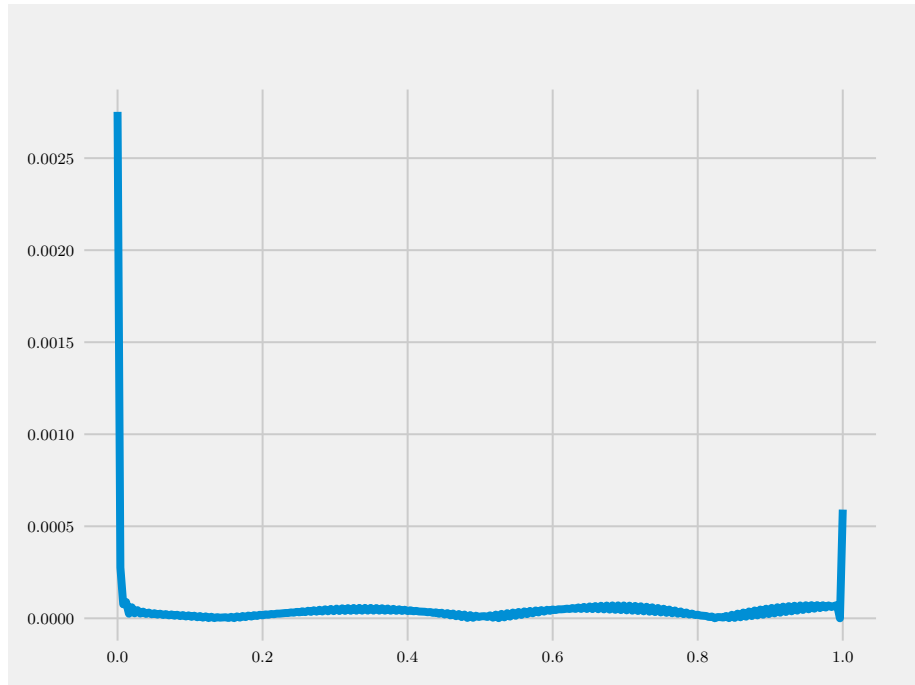


Figure 91: Normalized radial mode number 4 error, 0th identified mode

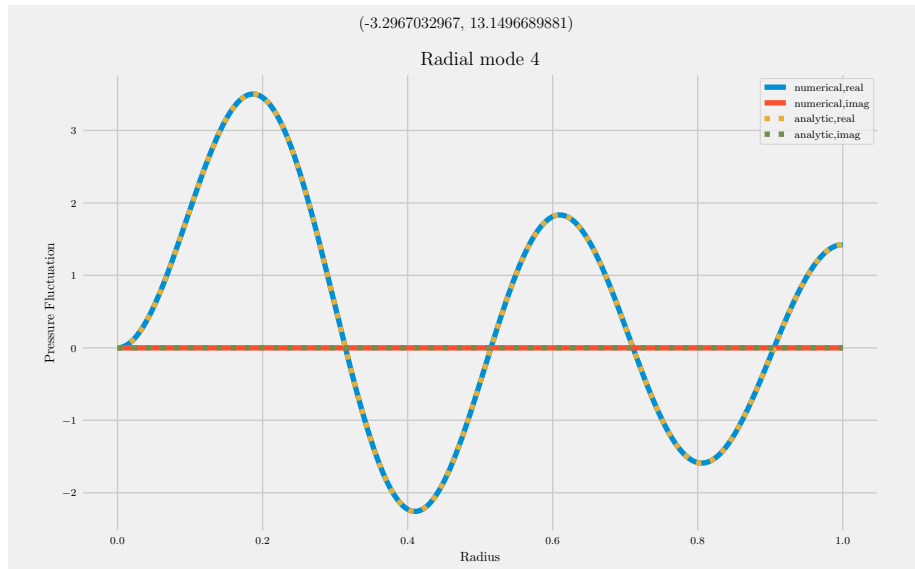


Figure 92: Fourth Order, Case number 1

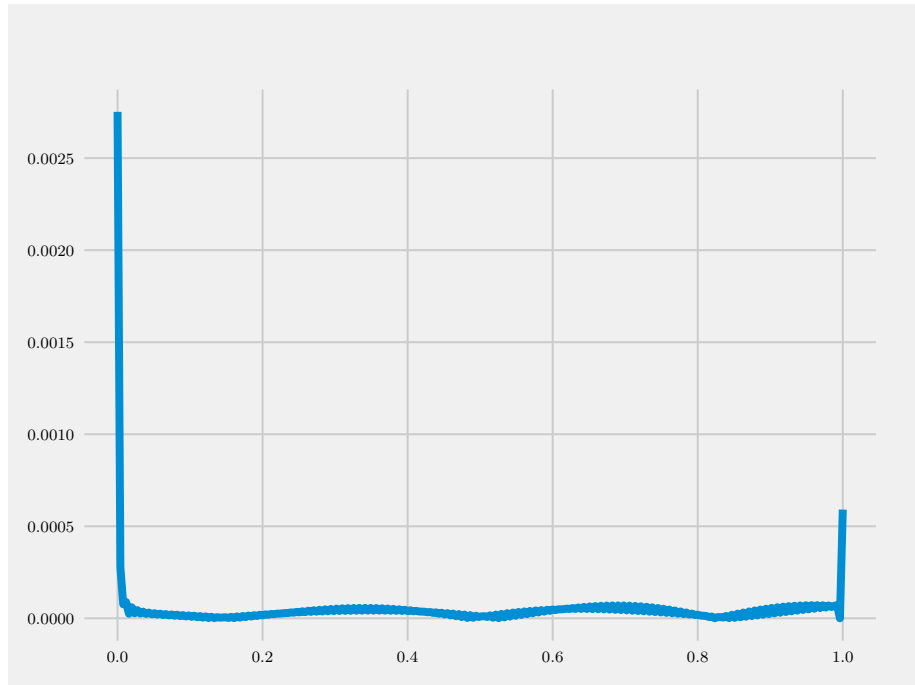


Figure 93: Normalized radial mode number 4 error, 1th identified mode

0.4 Discussion