Verification and Validation of an Acoustic Mode Prediction Code for Turbomachinery Noise

Master's Thesis Defense

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Outline

Section 1

Steady Flow Aerodynamic Model

For an ideal polytropic gas with density, ρ, velocity, $\vec{V}$, and pressure, p , the Euler equations are,

$$\frac{D\rho}{Dt} = -\rho\nabla \cdot \vec{V} \tag{1}$$

$$\frac{D\vec{V}}{Dt} = -\frac{\nabla p}{\rho} + \vec{g} \tag{2}$$

$$\frac{Dp}{Dt} = -\gamma p\nabla \cdot \vec{V} \tag{3}$$

$$\tag{4}$$

where Equations (1, 2, 3) are the conservation of mass, momentum, and energy equations respectively.

If the flow is assumed to be asymmetric, then the radial velocity component is zero. With this considered, the velocity vector ,$\vec{V}$ in cylindrical coordinates become,

$$\vec{V}(r, \theta, x) = v_x(r)\hat{e}_x + v_\theta(r)\hat{e}_\theta \tag{5}$$

where $\hat{e}_x$ and $\hat{e}_\theta$ are unit vectors for the axial and tangential directions.The gradient operator ,∇ in cylindrical coordinates, is

$$\vec{\nabla} = \hat{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\hat{e}_\theta\frac{\partial}{\partial \theta} + \frac{\partial}{\partial z}\hat{e}_z = 0 \tag{6}$$

For a steady flow, $(\partial/\partial t = 0)$, the compressible flow equations can be further reduced to,

$$\nabla(\vec{V}\rho) = 0 \tag{7}$$

$$(\vec{V}\cdot\nabla)\vec{V} = 0 \tag{8}$$

$$\nabla S = 0 \tag{9}$$

where S represents the entropy in the mean flow.

Nonuniformities from swirling mean flow

If the mean flow contains a swirling component, i.e. a velocity vector in the tangential direction, the mean quantities, pressure , density are non-uniform, thus also changing the speed of sound. By integrating the radial momentum equation, an expression for the speed of sound was established to account for the resulting nonuniformities due to rotations in the flow.

$$p = \int_{r_{min}}^{r_{max}} \frac{\rho v_\theta^2}{r} dr \tag{10}$$

where r_{min} and r_{max} are the bounds of the radius. Since the flow is isentropic, the pressure is related to the speed of sound through $\nabla p = A^2 \nabla \rho$; which is used to compute ρ.

With the relationship $A^2 = \kappa p/\rho$, the speed of sound is found to be,

$$\tilde{A}(\tilde{r}) = exp\left[\left(\frac{1-\gamma}{2}\right)\int_{\tilde{r}}^{1}\frac{M_\theta}{\tilde{r}}\partial\tilde{r}\right]$$

Section 2

Unsteady Flow Aerodynamic Model

Goldstein demonstrated the linearized momentum and continuity PDE can be combined to derive the convective wave equation by taking the divergence of the momentum equation and taking the difference of the substantial derivative of the conservation of mass equation to yield,

$$\frac{1}{A^2}\frac{D^2\tilde{p}}{Dt^2} - \nabla^2\tilde{p} = 2\bar{\rho}\frac{dV_x}{dx}\frac{\partial\tilde{v}_r}{\partial x} \tag{11}$$

In the case of sheared flow, $dV_x/dx = 0$ the right hand side will be zero

$$\frac{1}{A^2}\left(\frac{\partial^2 \tilde{p}}{\partial t^2} + \vec{V} \cdot \vec{\nabla}(\tilde{p})\right) - \nabla^2 \tilde{p} = 0$$

Utilizing the relation, $\tilde{p} = p/\bar{\rho}A^2$,substituting the definitions for ∇ and ∇^2 and setting $\vec{V} = 0$ in cylindrical coordinates gives,

$$\frac{1}{A^2}\left(\frac{\partial^2 p}{\partial t^2}\right) - \left(\frac{\partial^2 p}{\partial t^2} + \frac{1}{\tilde{r}}\frac{\partial p}{\partial r} + \frac{1}{\tilde{r}^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial x^2}\right) = 0$$

The method of separation of variables requires an assumed solution as well as initial and boundary conditions.
For a partial differential equation, the assumed solution can be a linear combination of solutions to a system of ordinary differential equations that comprises the partial differential equation.
Since p is a function of four variables, the solution is assumed to be a linear combination of four solutions.
Each solution is assumed to be Euler's identity, a common ansant for linear partial differential equations and boundary conditions.
Defining,

$$p(x, r, \theta, t) = X(x)R(r)\Theta(\theta)T(t) \tag{12}$$

where,

$$X(x) = A_1 e^{ik_x x} + B_1 e^{-ik_x x}$$
$$\Theta(\theta) = A_2 e^{ik_\theta \theta} + B_2 e^{-ik_\theta \theta}$$
$$T(t) = A_3 e^{i\omega t} + B_3 e^{-i\omega t}$$

The next step is to rewrite the wave equation in terms of X, R, Θ, and T. To further simplify the result, each term is divided by p.

$$\frac{1}{A^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{1}{R}\frac{\partial R}{\partial r} + \frac{1}{r^2}\frac{1}{\Theta}\frac{\partial \Theta}{\partial \theta} + \frac{1}{X}\frac{\partial^2 X}{\partial x^2} \tag{13}$$

$$\frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2}+\frac{1}{r}\frac{\partial R}{\partial r}\right)-\frac{k_\theta^2}{r^2}-k_x^2+k^2=0 \tag{14}$$

The remaining terms are manipulated to follow the same form as *Bessel's Differntial Equation* ,

$$x^2\frac{d^2 y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \tag{15}$$

The general solution to Bessel's differential equation is a linear combination of the Bessel functions of the first kind, $J_n(k_r r)$ and of the second kind, $Y_n(k_r r)$ [**?**]. The subscript n refers to the order of Bessel's equation.

$$R(r)=(AJ_n(k_r r)+BY_n(k_r r)) \tag{16}$$

where the coefficients A and B are found after applying radial boundary conditions.

By rearranging Equation (14), a comparison can be made to Equation (15) to show that the two equations are of the same form.
The first step is to revisit the radial derivatives that have not been addressed. As was done for the other derivative terms, the radial derivatives will also be set equal to a separation constant, $-k_r^2$.

$$\underbrace{\frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) - \frac{k_\theta^2}{r^2}}_{-k_r^2} - k_x^2 + k^2 = 0 \tag{17}$$

The reader may be curious as to why the tangential separation constant k_θ is included within the definition of the radial separation constant.
Recall the ODE for the tangential direction,

$$\frac{\partial \Theta}{\partial \theta} \frac{1}{\Theta} = -k_\theta^2$$
$$\frac{\partial \Theta}{\partial \theta} \frac{1}{\Theta} + \Theta k_\theta^2 = 0$$

where the solution is more or less,

$$\Theta(\theta) = e^{ik_\theta \theta}$$

In order to have non trivial, sensible solutions, the value of $\Theta(0)$ and $\Theta(2\pi)$ need to be the same, and this needs to be true for any multiple of 2π for a fixed r. Taking Θ to be one, a unit circle, it can be shown that the domain is only going to be an integer multiple.
Therefore, there is an implied periodic azimuthal boundary condition, i.e. $0 < \theta \leq 2\pi$ and $k_\theta = m$.

There is a unique treatment for the radial derivatives.

$$-k_r^2 = \frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) - \frac{m^2}{r^2}$$

To further simplify, the chain rule is used to do a change of variables, $x = k_r r$

$$\begin{aligned}\frac{\partial R}{\partial r} &= \frac{dR}{dx}\frac{dx}{dr} \\ &= \frac{dR}{dx}\frac{d}{dr}(k_r r) \\ &= \frac{dR}{dx}k_r\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 R}{\partial r^2} &= \frac{d^2 R}{dx^2}\left(\frac{dx}{dr}\right)^2 + \frac{dR}{dr}\frac{d^2 x}{dr^2} \\
&= \frac{d^2 R}{dx^2}\frac{d}{dr}k_r^2 + k_r\frac{d^2 r}{dr^2} \\
&= \frac{d^2 R}{dx^2}\frac{d}{dr}k_r^2
\end{aligned}
$$

Substituting this into Equation (14),

$$
\left(\frac{d^2 R}{dx^2}k_r^2 + \frac{1}{r}\frac{d^2 R}{dx^2}k_r\right) + \left(k_r^2 - \frac{m^2}{r^2}\right)R \tag{18}
$$

Dividing Equation 18 by k_r^2,

$$\left(\frac{d^2R}{dx^2} + \frac{1}{k_r r}\frac{d^2R}{dx^2}\right) + \left(1 - \frac{m^2}{k_r^2 r^2}\right) R \tag{19}$$

$$\left(\frac{d^2R}{dx^2} + \frac{1}{x^2}\frac{d^2R}{dx^2}\right) + \left(1 - \frac{m^2}{x^2}\right) R \tag{20}$$

Multiplying Equation (20) by x^2 gives,

$$\frac{d^2R}{dr^2}x^2 + \frac{dR}{dr}x + \left(x^2 - m^2\right)R \tag{21}$$

which matches the form of Bessel's equation

Therefore, the solution goes from this, to this,

$$y(x) = AJ_n(x) + BY_n(x) \tag{22}$$

to this,

$$R(r) = AJ_n(k_r r) + BY_n(k_r r) \tag{23}$$

$$\begin{aligned}
\frac{\partial p}{\partial r}|_{r=r_{min}} = \frac{\partial p}{\partial r}|_{r=r_{max}} = 0 \rightarrow \frac{\partial}{\partial r}\left(X\Theta TR\right) = 0 \\
X\Theta T\frac{\partial R}{\partial r} = 0 \\
\frac{\partial R}{\partial r} = 0
\end{aligned}$$

where,

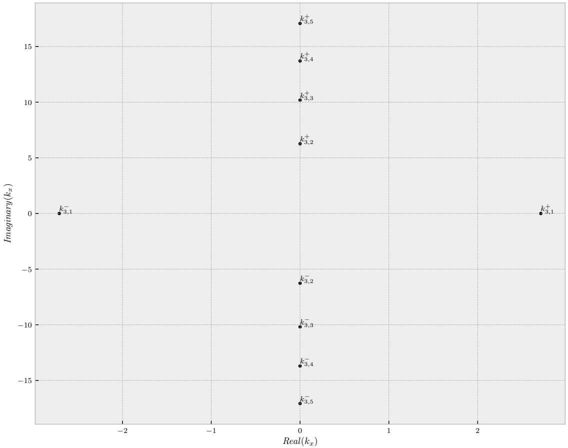
$$\frac{\partial R}{\partial r}|_{r_{min}} = AJ_n'(k_r r_{min}) + BY_n'(k_r r_{min}) = 0 \rightarrow B = -A\frac{J_n'(k_r r_{min})}{Y_n'(k_r r_{min})}$$

$$
\begin{aligned}
\frac{\partial R}{\partial r} &= AJ'_n(k_r r_{max}) + BY'_n(k_r r_{max}) = 0 \\
&= AJ'_n(k_r r_{max}) - A\frac{J'_n(k_r r_{min})}{Y'_n(k_r r_{min})} Y'_n(k_r r_{max}) = 0 \\
&= \frac{J'_n(k_r r_{min})}{J'_n(k_r r_{max})} - \frac{Y'_n(k_r r_{min})}{Y'_n(k_r r_{max})} = 0
\end{aligned}
$$

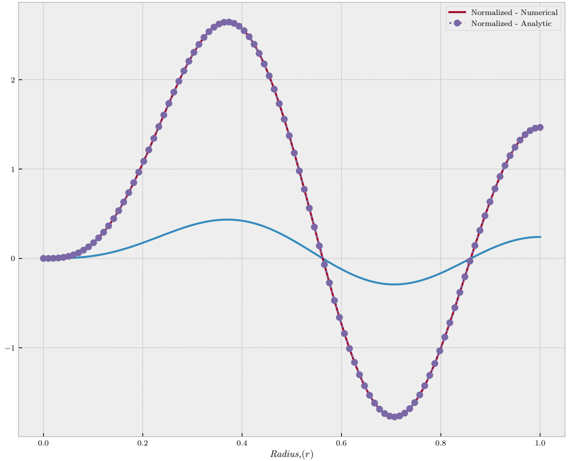
where $k_r r$ are the zero crossings for the derivatives of the Bessel functions of the first and second kind.

In summary, the wave equation for no flow in a hollow duct with hard walls is obtained from Equation (17).

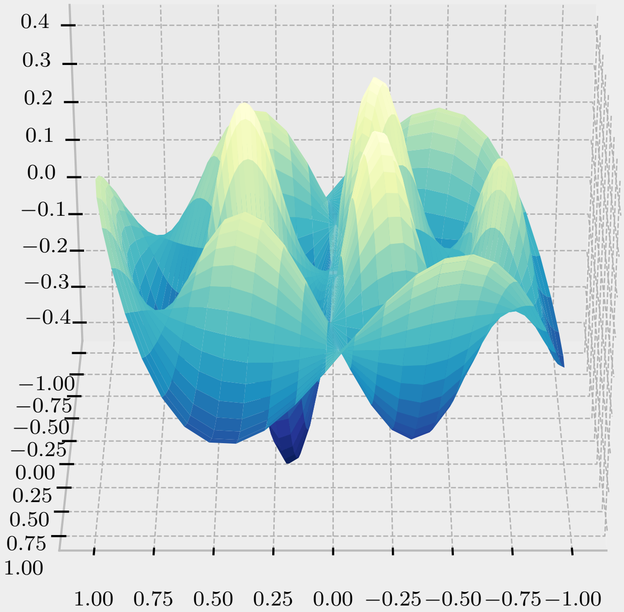
$$
k^2 = k_r^2 + k_x^2 \tag{24}
$$



Complex axial wavenumbers for $m = 0$, $k = 5$, $M_x = 0$ $\eta = 0$



Analytical Propagating Mode for $m = 3, k_r = 3, k = 5, M_x = 0, \eta = 0$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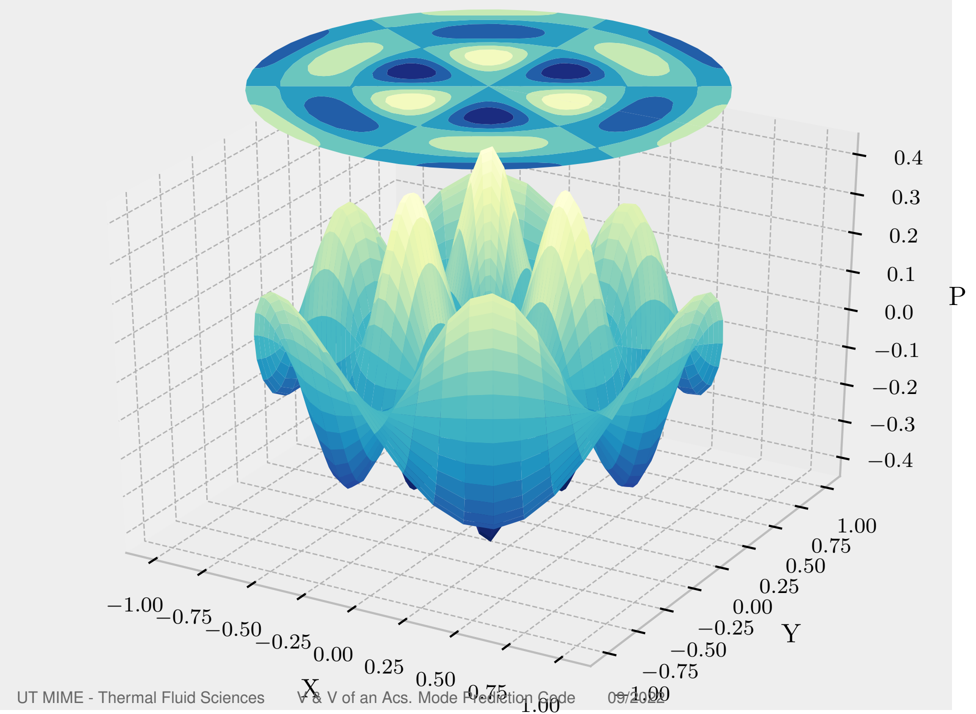
1.00 0.75 0.50 0.25 0.00 −0.25 −0.50 −0.75 −1.00



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0.50
0.75
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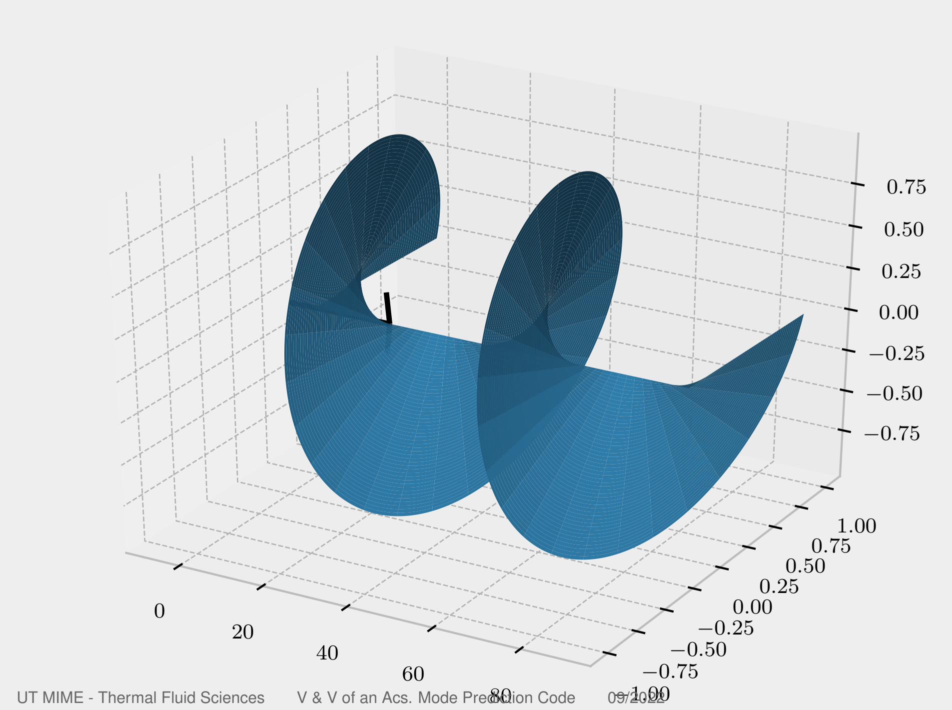
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−0.50
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−1.00
0
20
40
60
80