1 SWIRL code revisited

Going back to the original perturbation equations:

$$\begin{aligned}
\left(-i\omega + \frac{imV_\theta}{r} + ik_xV_x\right)v_r - \frac{2V_\theta}{r}v_\theta &= -\frac{1}{\overline{\rho}}\frac{\partial p}{\partial r} + \frac{V_\theta^2}{\overline{\rho}rA^2}p \\
\left(-i\omega + \frac{imV_\theta}{r} + ik_xV_x\right)v_\theta + \left(\frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r}\right)v_r &= -\frac{im}{\overline{\rho}r}p \\
\left(-i\omega + \frac{imV_\theta}{r} + ik_xV_x\right)v_x + \frac{\partial V_x}{\partial r}v_r &= -\frac{ik_x}{\overline{\rho}}p \\
\frac{1}{\overline{\rho}A^2}\left(-i\omega + \frac{imV_\theta}{r} + ik_xV_x\right)p + \frac{V_\theta^2}{A^2r}v_r + \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{im}{r}v_\theta + ik_xv_x &= 0
\end{aligned}$$

and defining a nondimensionalization based on the tip quantities:

$$\begin{aligned}
\omega &= \widehat{\omega}\frac{A_T}{r_T} \\
k_x &= \frac{\widehat{k}_x}{r_T} \\
\overline{\rho} &= \left(\widehat{\overline{\rho}}\right)\overline{\rho}_T \\
A &= \widetilde{A}A_T \\
V_x &= \widehat{V}_xA_T \\
V_\theta &= \widehat{V}_\theta A_T \\
v_x &= \widehat{v}_xA_T \\
v_r &= \widehat{v}_rA_T \\
v_\theta &= \widehat{v}_\theta A_T \\
p &= \widehat{p}\overline{\rho}_TA_T \\
r &= \widetilde{r}r_T \\
\frac{\partial}{\partial r} &= \frac{1}{r_T}\frac{\partial}{\partial \widetilde{r}}
\end{aligned}$$

Substituting in,

$$\left(-i\widehat{\omega} + \frac{im\widehat{V}_\theta}{\widetilde{r}} + i\widehat{k}_x\widehat{V}_x\right)\widehat{v}_r\frac{A_T^2}{r_T} - \frac{2\widehat{V}_\theta}{\widetilde{r}}\widehat{v}_\theta\frac{A_T^2}{r_T} = -\frac{1}{\widehat{\overline{\rho}}}\frac{\partial\widehat{p}}{\partial\widetilde{r}}\frac{A_T^2}{r_T} + \frac{\widehat{V}_\theta^2}{\widehat{\overline{\rho}}\widetilde{r}\widetilde{A}^2}\widehat{p}\frac{A_T^2}{r_T}$$

$$\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{v}_\theta\frac{A_T^2}{r_T}+\left(\frac{\widehat{V}_\theta}{\widetilde{r}}+\frac{\partial\widehat{V}_\theta}{\partial\widetilde{r}}\right)\widehat{v}_r\frac{A_T^2}{r_T}=-\frac{im}{\widehat{\widetilde{\rho}}\widetilde{r}}\widehat{p}\frac{A_T^2}{r_T}$$

$$\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{v}_x\frac{A_T^2}{r_T}+\frac{\partial\widehat{V}_x}{\partial\widetilde{r}}\widehat{v}_r\frac{A_T^2}{r_T}=-\frac{i\widehat{k}_x}{\widehat{\widetilde{\rho}}}\widehat{p}\frac{A_T^2}{r_T}$$

$$\frac{1}{\widehat{\widetilde{\rho}}\widetilde{A}^2}\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{p}\frac{A_T}{r_T}+\frac{\widehat{V}_\theta^2}{\widetilde{A}^2\widetilde{r}}\widehat{v}_r\frac{A_T}{r_T}+\frac{\partial\widehat{v}_r}{\partial\widetilde{r}}\frac{A_T}{r_T}+\frac{\widehat{v}_r}{\widetilde{r}}\frac{A_T}{r_T}+\frac{im}{\widetilde{r}}\widehat{v}_\theta\frac{A_T}{r_T}+i\widehat{k}_x\widehat{v}_x\frac{A_T}{r_T}=0$$

Simplifying the equation and rearranging gives:

$$\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{v}_r-\frac{2\widehat{V}_\theta}{\widetilde{r}}\widehat{v}_\theta=-\frac{1}{\widehat{\widetilde{\rho}}}\frac{\partial\widehat{p}}{\partial\widetilde{r}}+\frac{\widehat{V}_\theta^2}{\widehat{\widetilde{\rho}}\widetilde{r}\widetilde{A}^2}\widehat{p}$$

$$\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{v}_\theta+\left(\frac{\widehat{V}_\theta}{\widetilde{r}}+\frac{\partial\widehat{V}_\theta}{\partial\widetilde{r}}\right)\widehat{v}_r=-\frac{im}{\widehat{\widetilde{\rho}}\widetilde{r}}\widehat{p}$$

$$\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{v}_x+\frac{\partial\widehat{V}_x}{\partial\widetilde{r}}\widehat{v}_r=-\frac{i\widehat{k}_x}{\widehat{\widetilde{\rho}}}\widehat{p}$$

$$\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{p}+\widehat{\widetilde{\rho}}\widetilde{A}^2\left(\frac{\widehat{V}_\theta^2}{\widetilde{A}^2\widetilde{r}}\widehat{v}_r+\frac{\partial\widehat{v}_r}{\partial\widetilde{r}}+\frac{\widehat{v}_r}{\widetilde{r}}+\frac{im}{\widetilde{r}}\widehat{v}_\theta+i\widehat{k}_x\widehat{v}_x\right)=0$$

1.1 Some basic definitions

In cylindrical coordinates,

$$\begin{aligned}
\vec{\nabla}\cdot\vec{V} &= \frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_x}{\partial x}\\
&= \frac{\partial V_r}{\partial r}+\frac{V_r}{r}+\frac{\partial v_r'}{\partial r}+\frac{v_r'}{r}+\frac{1}{r}\frac{\partial v_\theta'}{\partial\theta}+\frac{\partial v_x'}{\partial x}\\
&= \frac{\partial v_r'}{\partial r}+\frac{v_r'}{r}+\frac{1}{r}\frac{\partial v_\theta'}{\partial\theta}+\frac{\partial v_x'}{\partial x}\\
&= \frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{im}{r}v_\theta+ik_xv_x
\end{aligned}$$

$$\vec{\nabla}\times\vec{V} = \begin{pmatrix} \left(\frac{1}{r}\frac{\partial V_x}{\partial\theta}-\frac{\partial V_\theta}{\partial x}\right)\vec{e}_r \\ +\left(\frac{\partial V_r}{\partial x}-\frac{\partial V_x}{\partial r}\right)\vec{e}_\theta \\ +\left(\frac{V_\theta}{r}+\frac{\partial V_\theta}{\partial r}-\frac{1}{r}\frac{\partial V_r}{\partial\theta}\right)\vec{e}_x \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{im}{r}v_x - ik_x v_\theta\right)\vec{e}_r \\ +\left(ik_x v_r - \frac{\partial v_x}{\partial r}\right)\vec{e}_\theta \\ +\left(\frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{im}{r}v_r\right)\vec{e}_x \end{pmatrix}$$

$$\vec{\nabla}\phi = \begin{pmatrix} \frac{\partial \phi}{\partial r}\vec{e}_r \\ +\frac{1}{r}\frac{\partial \phi}{\partial \theta}\vec{e}_\theta \\ +\frac{\partial \phi}{\partial z}\vec{e}_z \end{pmatrix}$$

Let's define:

$$\begin{aligned} \phi &= \kappa(r)\, e^{i(k_x x + m\theta - \omega t)} \\ \vec{v}_\phi &= \vec{\nabla}\phi \\ &= \begin{pmatrix} \frac{\partial \kappa}{\partial r} e^{i(k_x x + m\theta - \omega t)}\vec{e}_r \\ +\frac{im}{r}\kappa e^{i(k_x x + m\theta - \omega t)}\vec{e}_\theta \\ +ik_x \kappa e^{i(k_x x + m\theta - \omega t)}\vec{e}_x \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial \kappa}{\partial r} e^{i(k_x x + m\theta - \omega t)}\vec{e}_r \\ +\frac{im\kappa}{r} e^{i(k_x x + m\theta - \omega t)}\vec{e}_\theta \\ +ik_x \kappa e^{i(k_x x + m\theta - \omega t)}\vec{e}_x \end{pmatrix} \end{aligned}$$

and

$$\vec{\psi} = \begin{pmatrix} \alpha(r)\, e^{i(k_x x + m\theta - \omega t)}\vec{e}_r \\ +\widehat{\beta}(r)\, e^{i(k_x x + m\theta - \omega t)}\vec{e}_\theta \\ +\tau(r)\, e^{i(k_x x + m\theta - \omega t)}\vec{e}_x \end{pmatrix}$$

$$\begin{aligned} \vec{v}_\psi &= \vec{\nabla} \times \vec{\psi} \\ &= \begin{pmatrix} \left(\frac{im}{r}\tau - ik_x \widehat{\beta}\right) e^{i(k_x x + m\theta - \omega t)}\vec{e}_r \\ +\left(ik_x \alpha - \frac{\partial \tau}{\partial r}\right) e^{i(k_x x + m\theta - \omega t)}\vec{e}_\theta \\ +\left(\frac{\widehat{\beta}}{r} + \frac{\partial \widehat{\beta}}{\partial r} - \frac{im}{r}\alpha\right) e^{i(k_x x + m\theta - \omega t)}\vec{e}_x \end{pmatrix} \end{aligned}$$

Defining the perturbation velocities as:

$$
\begin{array}{rcl}
\vec{v} & = & \vec{v}_\phi + \vec{v}_\psi \\
 & = & \left(\begin{array}{r}
\left(\frac{\partial \kappa}{\partial r} + \frac{im}{r}\tau - ik_x\widehat{\beta} \right) e^{i(k_x x + m\theta - \omega t)} \vec{e}_r \\
+ \left(\frac{im}{r}\kappa + ik_x\alpha - \frac{\partial \tau}{\partial r} \right) e^{i(k_x x + m\theta - \omega t)} \vec{e}_\theta \\
+ \left(\frac{\widehat{\beta}}{r} + \frac{\partial \widehat{\beta}}{\partial r} - \frac{im}{r}\alpha + ik_x\kappa \right) e^{i(k_x x + m\theta - \omega t)} \vec{e}_x
\end{array} \right)
\end{array}
$$

The perturbation pressure is defined as:

$$
\widehat{p} \quad = \quad p\,(r)\, e^{i(k_x x + m\theta - \omega t)}
$$

1.2 Trying it out

Nondimensionalizing and substituting into the perturbation equations gives:

$$
\left(\begin{array}{r}
\left(-i\widehat{\omega} + \frac{im\widehat{V}_\theta}{\widetilde{r}} + i\widehat{k}_x\widehat{V}_x \right) \left(\frac{\partial \widehat{\kappa}}{\partial \widetilde{r}} + \frac{im}{\widetilde{r}}\widehat{\tau} - i\widehat{k}_x\widehat{\beta} \right) \\
- \frac{2\widehat{V}_\theta}{\widetilde{r}} \left(\frac{im}{\widetilde{r}}\widehat{\kappa} + i\widehat{k}_x\widehat{\alpha} - \frac{\partial \widehat{\tau}}{\partial \widetilde{r}} \right)
\end{array} \right) = -\frac{1}{\widehat{\widehat{\rho}}}\frac{\partial \widehat{p}}{\partial \widetilde{r}} + \frac{\widehat{V}_\theta^2}{\widehat{\widehat{\rho}}\widetilde{r}\widetilde{A}^2}\widehat{p}
$$

$$
\left(\begin{array}{r}
\left(-i\widehat{\omega} + \frac{im\widehat{V}_\theta}{\widetilde{r}} + i\widehat{k}_x\widehat{V}_x \right) \left(\frac{im}{\widetilde{r}}\widehat{\kappa} + i\widehat{k}_x\widehat{\alpha} - \frac{\partial \widehat{\tau}}{\partial \widetilde{r}} \right) \\
+ \left(\frac{\widehat{V}_\theta}{\widetilde{r}} + \frac{\partial \widehat{V}_\theta}{\partial \widetilde{r}} \right) \left(\frac{\partial \widehat{\kappa}}{\partial \widetilde{r}} + \frac{im}{\widetilde{r}}\widehat{\tau} - i\widehat{k}_x\widehat{\beta} \right)
\end{array} \right) = -\frac{im}{\widehat{\widehat{\rho}}\widetilde{r}}\widehat{p}
$$

$$
\left(\begin{array}{r}
\left(-i\widehat{\omega} + \frac{im\widehat{V}_\theta}{\widetilde{r}} + i\widehat{k}_x\widehat{V}_x \right) \left(\frac{\widehat{\beta}}{\widetilde{r}} + \frac{\partial \widehat{\beta}}{\partial \widetilde{r}} - \frac{im}{\widetilde{r}}\widehat{\alpha} + i\widehat{k}_x\widehat{\kappa} \right) \\
+ \frac{\partial \widehat{V}_x}{\partial \widetilde{r}} \left(\frac{\partial \widehat{\kappa}}{\partial \widetilde{r}} + \frac{im}{\widetilde{r}}\widehat{\tau} - i\widehat{k}_x\widehat{\beta} \right)
\end{array} \right) = -\frac{i\widehat{k}_x}{\widehat{\widehat{\rho}}}\widehat{p}
$$

$$
\left(-i\widehat{\omega} + \frac{im\widehat{V}_\theta}{\widetilde{r}} + i\widehat{k}_x\widehat{V}_x \right) \widehat{p} + \widehat{\widehat{\rho}}\widetilde{A}^2 \left(\begin{array}{r}
\frac{\widehat{V}_\theta^2}{\widetilde{A}^2\widetilde{r}} \left(\frac{\partial \widehat{\kappa}}{\partial \widetilde{r}} + \frac{im}{\widetilde{r}}\widehat{\tau} - i\widehat{k}_x\widehat{\beta} \right) \\
+ \frac{\partial}{\partial \widetilde{r}} \left(\frac{\partial \widehat{\kappa}}{\partial \widetilde{r}} + \frac{im}{\widetilde{r}}\widehat{\tau} - i\widehat{k}_x\widehat{\beta} \right) \\
+ \frac{1}{\widetilde{r}} \left(\frac{\partial \widehat{\kappa}}{\partial \widetilde{r}} + \frac{im}{\widetilde{r}}\widehat{\tau} - i\widehat{k}_x\widehat{\beta} \right) \\
+ \frac{im}{\widetilde{r}} \left(\frac{im}{\widetilde{r}}\widehat{\kappa} + i\widehat{k}_x\widehat{\alpha} - \frac{\partial \widehat{\tau}}{\partial \widetilde{r}} \right) \\
+ i\widehat{k}_x \left(\frac{\widehat{\beta}}{\widetilde{r}} + \frac{\partial \widehat{\beta}}{\partial \widetilde{r}} - \frac{im}{\widetilde{r}}\widehat{\alpha} + i\widehat{k}_x\widehat{\kappa} \right)
\end{array} \right) = 0
$$

1.2.1 Pressure equation

Working on the pressure equation gives:

$$\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{p}+\widehat{\overline{\rho}}\widetilde{A}^2\left(\begin{array}{r}\frac{\widehat{V}_\theta^2}{\widetilde{A}^2\widetilde{r}}\left(\frac{\partial\widehat{\kappa}}{\partial\widetilde{r}}+\frac{im}{\widetilde{r}}\widehat{\tau}-i\widehat{k}_x\widehat{\beta}\right)\\+\left(\frac{\partial^2\widehat{\kappa}}{\partial\widetilde{r}^2}-\frac{im}{\widetilde{r}^2}\widehat{\tau}+\frac{im}{\widetilde{r}}\frac{\partial\widehat{\tau}}{\partial\widetilde{r}}-i\widehat{k}_x\frac{\partial\widehat{\beta}}{\partial\widetilde{r}}\right)\\+\left(\frac{1}{\widetilde{r}}\frac{\partial\widehat{\kappa}}{\partial\widetilde{r}}+\frac{im}{\widetilde{r}^2}\widehat{\tau}-\frac{i\widehat{k}_x}{\widetilde{r}}\widehat{\beta}\right)\\+\left(\left(\frac{im}{\widetilde{r}}\right)^2\widehat{\kappa}+\frac{im}{\widetilde{r}}i\widehat{k}_x\widehat{\alpha}-\frac{im}{\widetilde{r}}\frac{\partial\widehat{\tau}}{\partial\widetilde{r}}\right)\\+\left(i\widehat{k}_x\frac{\widehat{\beta}}{\widetilde{r}}+i\widehat{k}_x\frac{\partial\widehat{\beta}}{\partial\widetilde{r}}-\frac{im}{\widetilde{r}}i\widehat{k}_x\widehat{\alpha}+\left(i\widehat{k}_x\right)^2\widehat{\kappa}\right)\end{array}\right)=0$$

$$\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{p}+\widehat{\overline{\rho}}\widetilde{A}^2\left(\begin{array}{r}\frac{\widehat{V}_\theta^2}{\widetilde{A}^2\widetilde{r}}\left(\frac{\partial\widehat{\kappa}}{\partial\widetilde{r}}+\frac{im}{\widetilde{r}}\widehat{\tau}-i\widehat{k}_x\widehat{\beta}\right)\\+\left(\frac{\partial^2\widehat{\kappa}}{\partial\widetilde{r}^2}+\frac{1}{\widetilde{r}}\frac{\partial\widehat{\kappa}}{\partial\widetilde{r}}+\left(\frac{im}{\widetilde{r}}\right)^2\widehat{\kappa}+\left(i\widehat{k}_x\right)^2\widehat{\kappa}\right)\end{array}\right)=0$$

$$\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{p}+\left(\begin{array}{r}\frac{\overline{\rho}\widehat{V}_\theta^2}{\widetilde{r}}\left(\frac{\partial\widehat{\kappa}}{\partial\widetilde{r}}+\frac{im}{\widetilde{r}}\widehat{\tau}-i\widehat{k}_x\widehat{\beta}\right)\\+\widehat{\overline{\rho}}\widetilde{A}^2\left(\frac{\partial^2\widehat{\kappa}}{\partial\widetilde{r}^2}+\frac{1}{\widetilde{r}}\frac{\partial\widehat{\kappa}}{\partial\widetilde{r}}+\left(\frac{im}{\widetilde{r}}\right)^2\widehat{\kappa}+\left(i\widehat{k}_x\right)^2\widehat{\kappa}\right)\end{array}\right)=0$$

$$\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{p}+\left(\begin{array}{r}\frac{\overline{\rho}\widehat{V}_\theta^2}{\widetilde{r}}\widehat{v}_r\\+\widehat{\overline{\rho}}\widetilde{A}^2\left(\frac{\partial^2\widehat{\kappa}}{\partial\widetilde{r}^2}+\frac{1}{\widetilde{r}}\frac{\partial\widehat{\kappa}}{\partial\widetilde{r}}+\left(\frac{im}{\widetilde{r}}\right)^2\widehat{\kappa}+\left(i\widehat{k}_x\right)^2\widehat{\kappa}\right)\end{array}\right)=0$$

1.2.2 Vorticity equations: Radial vorticity

The radial component of vorticity is:

$$\begin{aligned}\omega_r &= \frac{im}{r}v_x-ik_xv_\theta\\&=\left(\frac{im}{r}\left(\frac{\widehat{\beta}}{\widetilde{r}}+\frac{\partial\widehat{\beta}}{\partial\widetilde{r}}-\frac{im}{\widetilde{r}}\widehat{\alpha}+i\widehat{k}_x\widehat{\kappa}\right)-ik_x\left(\frac{im}{\widetilde{r}}\widehat{\kappa}+i\widehat{k}_x\widehat{\alpha}-\frac{\partial\widehat{\tau}}{\partial\widetilde{r}}\right)\right)e^{i(k_xx+m\theta-\omega t)}\\&=\left(\left(\widehat{k}_x^2+\frac{m^2}{\widetilde{r}^2}\right)\widehat{\alpha}+\frac{im}{r}\left(\frac{\widehat{\beta}}{\widetilde{r}}+\frac{\partial\widehat{\beta}}{\partial\widetilde{r}}\right)+i\widehat{k}_x\frac{\partial\widehat{\tau}}{\partial\widetilde{r}}\right)e^{i(k_xx+m\theta-\omega t)}\end{aligned}$$

The radial vorticity amplitude is defined as:

$$\widehat{\omega}_r = \left(\widehat{k}_x^2+\frac{m^2}{\widetilde{r}^2}\right)\widehat{\alpha}+\frac{im}{r}\left(\frac{\widehat{\beta}}{\widetilde{r}}+\frac{\partial\widehat{\beta}}{\partial\widetilde{r}}\right)+i\widehat{k}_x\frac{\partial\widehat{\tau}}{\partial\widetilde{r}}$$

The radial vorticity equation is then

$$
\begin{pmatrix}
\frac{im}{\widetilde{r}}\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\left(\frac{\widehat{\beta}}{\widetilde{r}}+\frac{\partial\widehat{\beta}}{\partial\widetilde{r}}-\frac{im}{\widetilde{r}}\widehat{\alpha}+i\widehat{k}_x\widehat{\kappa}\right) \\
+\frac{im}{\widetilde{r}}\frac{\partial\widehat{V}_x}{\partial\widetilde{r}}\left(\frac{\partial\widehat{\kappa}}{\partial\widetilde{r}}+\frac{im}{\widetilde{r}}\widehat{\tau}-i\widehat{k}_x\widehat{\beta}\right) \\
-i\widehat{k}_x\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\left(\frac{im}{\widetilde{r}}\widehat{\kappa}+i\widehat{k}_x\widehat{\alpha}-\frac{\partial\widehat{\tau}}{\partial\widetilde{r}}\right) \\
-i\widehat{k}_x\left(\frac{\widehat{V}_\theta}{\widetilde{r}}+\frac{\partial\widehat{V}_\theta}{\partial\widetilde{r}}\right)\left(\frac{\partial\widehat{\kappa}}{\partial\widetilde{r}}+\frac{im}{\widetilde{r}}\widehat{\tau}-i\widehat{k}_x\widehat{\beta}\right)
\end{pmatrix} = -\frac{i\widehat{k}_x}{\widehat{\overline{\rho}}}\frac{im}{\widetilde{r}}\widehat{p}+i\widehat{k}_x\frac{im}{\widehat{\overline{\rho}}\widetilde{r}}\widehat{p}
$$

$$
\begin{pmatrix}
\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{\omega}_r \\
+\left(\frac{im}{\widetilde{r}}\frac{\partial\widehat{V}_x}{\partial\widetilde{r}}-i\widehat{k}_x\left(\frac{\widehat{V}_\theta}{\widetilde{r}}+\frac{\partial\widehat{V}_\theta}{\partial\widetilde{r}}\right)\right)\left(\frac{\partial\widehat{\kappa}}{\partial\widetilde{r}}+\frac{im}{\widetilde{r}}\widehat{\tau}-i\widehat{k}_x\widehat{\beta}\right)
\end{pmatrix} = 0
$$

$$
\begin{pmatrix}
\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{\omega}_r \\
+\left(\frac{im}{\widetilde{r}}\frac{\partial\widehat{V}_x}{\partial\widetilde{r}}-i\widehat{k}_x\left(\frac{\widehat{V}_\theta}{\widetilde{r}}+\frac{\partial\widehat{V}_\theta}{\partial\widetilde{r}}\right)\right)\widehat{v}_r
\end{pmatrix} = 0
$$

1.2.3 Azimuthal vorticity

The azimuthal vorticity is:

$$
\begin{aligned}
\widehat{\omega}_\theta &= i\widehat{k}_x\widehat{v}_r-\frac{\partial\widehat{v}_x}{\partial\widetilde{r}} \\
&= \begin{pmatrix}
i\widehat{k}_x\left(\frac{\partial\widehat{\kappa}}{\partial\widetilde{r}}+\frac{im}{\widetilde{r}}\widehat{\tau}-i\widehat{k}_x\widehat{\beta}\right) \\
-\frac{\partial}{\partial\widetilde{r}}\left(\frac{\widehat{\beta}}{r}+\frac{\partial\widehat{\beta}}{\partial r}-\frac{im}{\widetilde{r}}\widehat{\alpha}+i\widehat{k}_x\widehat{\kappa}\right)
\end{pmatrix} \\
&= \begin{pmatrix}
i\widehat{k}_x\left(\frac{im}{r}\tau-ik_x\widehat{\beta}\right) \\
-\frac{\partial}{\partial\widetilde{r}}\left(\frac{\widehat{\beta}}{r}+\frac{\partial\widehat{\beta}}{\partial r}-\frac{im}{\widetilde{r}}\widehat{\alpha}\right)
\end{pmatrix}
\end{aligned}
$$

The azimuthal vorticity equation is:

$$
\begin{pmatrix}
i\widehat{k}_x\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{v}_r-i\widehat{k}_x\frac{2\widehat{V}_\theta}{\widetilde{r}}\widehat{v}_\theta \\
-\frac{\partial}{\partial\widetilde{r}}\left(\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{v}_x\right)-\frac{\partial}{\partial\widetilde{r}}\left(\frac{\partial\widehat{V}_x}{\partial\widetilde{r}}\widehat{v}_r\right)
\end{pmatrix} = \begin{pmatrix}
-i\widehat{k}_x\frac{1}{\widehat{\overline{\rho}}}\frac{\partial\widehat{p}}{\partial\widetilde{r}}+i\widehat{k}_x\frac{\widehat{V}_\theta^2}{\widetilde{\overline{\rho}}\widetilde{r}\widetilde{A}^2}\widehat{p} \\
+\frac{\partial}{\partial\widetilde{r}}\left(\frac{i\widehat{k}_x}{\widehat{\overline{\rho}}}\widehat{p}\right)
\end{pmatrix}
$$

$$
\begin{pmatrix}
i\widehat{k}_x\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{v}_r \\
-i\widehat{k}_x\frac{2\widehat{V}_\theta}{\widetilde{r}}\widehat{v}_\theta \\
-\left(\frac{im}{\widetilde{r}}\frac{\partial\widehat{V}_\theta}{\partial\widetilde{r}}-\frac{im\widehat{V}_\theta}{\widetilde{r}^2}+i\widehat{k}_x\frac{\partial\widehat{V}_x}{\partial\widetilde{r}}\right)\widehat{v}_x \\
-\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\frac{\partial\widehat{v}_x}{\partial\widetilde{r}} \\
-\frac{\partial^2\widehat{V}_x}{\partial\widetilde{r}^2}\widehat{v}_r-\frac{\partial\widehat{V}_x}{\partial\widetilde{r}}\frac{\partial\widehat{v}_r}{\partial\widetilde{r}}
\end{pmatrix}
=
\begin{pmatrix}
-i\widehat{k}_x\frac{1}{\widehat{\bar{\rho}}}\frac{\partial\widehat{p}}{\partial\widetilde{r}}+i\widehat{k}_x\frac{\widehat{V}_\theta^2}{\widehat{\bar{\rho}}\widetilde{r}\widetilde{A}^2}\widehat{p} \\
-\frac{\partial\widehat{\bar{\rho}}}{\partial\widetilde{r}}\frac{i\widehat{k}_x}{\widehat{\bar{\rho}}^2}\widehat{p} \\
+\frac{i\widehat{k}_x}{\widehat{\bar{\rho}}}\frac{\partial\widehat{p}}{\partial\widetilde{r}}
\end{pmatrix}
$$

$$
\begin{pmatrix}
i\widehat{k}_x\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{v}_r \\
-i\widehat{k}_x\frac{2\widehat{V}_\theta}{\widetilde{r}}\widehat{v}_\theta \\
-\left(\frac{im}{\widetilde{r}}\frac{\partial\widehat{V}_\theta}{\partial\widetilde{r}}-\frac{im\widehat{V}_\theta}{\widetilde{r}^2}+i\widehat{k}_x\frac{\partial\widehat{V}_x}{\partial\widetilde{r}}\right)\widehat{v}_x \\
-\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\frac{\partial\widehat{v}_x}{\partial\widetilde{r}} \\
-\frac{\partial^2\widehat{V}_x}{\partial\widetilde{r}^2}\widehat{v}_r-\frac{\partial\widehat{V}_x}{\partial\widetilde{r}}\frac{\partial\widehat{v}_r}{\partial\widetilde{r}}
\end{pmatrix}
=
\begin{pmatrix}
i\widehat{k}_x\left(\frac{\widehat{V}_\theta^2}{\widehat{\bar{\rho}}\widetilde{r}\widetilde{A}^2}-\frac{\partial\widehat{\bar{\rho}}}{\partial\widetilde{r}}\frac{1}{\widehat{\bar{\rho}}^2}\right)\widehat{p}
\end{pmatrix}
$$

$$
\begin{pmatrix}
i\widehat{k}_x\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{v}_r \\
-i\widehat{k}_x\frac{2\widehat{V}_\theta}{\widetilde{r}}\widehat{v}_\theta \\
-\left(\frac{im}{\widetilde{r}}\frac{\partial\widehat{V}_\theta}{\partial\widetilde{r}}-\frac{im\widehat{V}_\theta}{\widetilde{r}^2}+i\widehat{k}_x\frac{\partial\widehat{V}_x}{\partial\widetilde{r}}\right)\widehat{v}_x \\
-\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\frac{\partial\widehat{v}_x}{\partial\widetilde{r}} \\
-\frac{\partial^2\widehat{V}_x}{\partial\widetilde{r}^2}\widehat{v}_r-\frac{\partial\widehat{V}_x}{\partial\widetilde{r}}\frac{\partial\widehat{v}_r}{\partial\widetilde{r}}
\end{pmatrix}
=
\begin{pmatrix}
i\widehat{k}_x\left(\frac{\widehat{\bar{\rho}}\widehat{V}_\theta^2}{\widetilde{r}}\frac{1}{\widehat{\bar{\rho}}^2\widetilde{A}^2}-\frac{\partial\widehat{\bar{\rho}}}{\partial\widetilde{r}}\frac{1}{\widehat{\bar{\rho}}^2}\right)\widehat{p}
\end{pmatrix}
$$

$$
\begin{pmatrix}
i\widehat{k}_x\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\widehat{v}_r \\
-i\widehat{k}_x\frac{2\widehat{V}_\theta}{\widetilde{r}}\widehat{v}_\theta \\
-\left(\frac{im}{\widetilde{r}}\frac{\partial\widehat{V}_\theta}{\partial\widetilde{r}}-\frac{im\widehat{V}_\theta}{\widetilde{r}^2}+i\widehat{k}_x\frac{\partial\widehat{V}_x}{\partial\widetilde{r}}\right)\widehat{v}_x \\
-\left(-i\widehat{\omega}+\frac{im\widehat{V}_\theta}{\widetilde{r}}+i\widehat{k}_x\widehat{V}_x\right)\frac{\partial\widehat{v}_x}{\partial\widetilde{r}} \\
-\frac{\partial^2\widehat{V}_x}{\partial\widetilde{r}^2}\widehat{v}_r-\frac{\partial\widehat{V}_x}{\partial\widetilde{r}}\frac{\partial\widehat{v}_r}{\partial\widetilde{r}}
\end{pmatrix}
=
\begin{pmatrix}
i\widehat{k}_x\left(\frac{\partial\widehat{P}}{\partial\widetilde{r}}\frac{1}{\widehat{\bar{\rho}}^2\widetilde{A}^2}-\frac{\partial\widehat{\bar{\rho}}}{\partial\widetilde{r}}\frac{1}{\widehat{\bar{\rho}}^2}\right)\widehat{p}
\end{pmatrix}
$$

$$
\begin{pmatrix}
i\widehat{k}_x \left(-i\widehat{\omega} + \frac{im\widehat{V}_\theta}{\widetilde{r}} + i\widehat{k}_x \widehat{V}_x \right) \widehat{v}_r \\
-i\widehat{k}_x \frac{2\widehat{V}_\theta}{\widetilde{r}} \widehat{v}_\theta \\
-\left(\frac{im}{\widetilde{r}} \frac{\partial \widehat{V}_\theta}{\partial \widetilde{r}} - \frac{im\widehat{V}_\theta}{\widetilde{r}^2} + i\widehat{k}_x \frac{\partial \widehat{V}_x}{\partial \widetilde{r}} \right) \widehat{v}_x \\
-\left(-i\widehat{\omega} + \frac{im\widehat{V}_\theta}{\widetilde{r}} + i\widehat{k}_x \widehat{V}_x \right) \frac{\partial \widehat{v}_x}{\partial \widetilde{r}} \\
-\frac{\partial^2 \widehat{V}_x}{\partial \widetilde{r}^2} \widehat{v}_r - \frac{\partial \widehat{V}_x}{\partial \widetilde{r}} \frac{\partial \widehat{v}_r}{\partial \widetilde{r}}
\end{pmatrix}
= \begin{pmatrix} i\widehat{k}_x \left(\widetilde{A}^2 \frac{\partial \widehat{\overline{\rho}}}{\partial \widetilde{r}} \frac{1}{\widehat{\overline{\rho}}^2 \widetilde{A}^2} - \frac{\partial \widehat{\overline{\rho}}}{\partial \widetilde{r}} \frac{1}{\widehat{\overline{\rho}}^2} \right) \widehat{p} \end{pmatrix}
$$

$$
\begin{pmatrix}
\left(-i\widehat{\omega} + \frac{im\widehat{V}_\theta}{\widetilde{r}} + i\widehat{k}_x \widehat{V}_x \right) \widehat{\omega}_\theta \\
-i\widehat{k}_x \frac{2\widehat{V}_\theta}{\widetilde{r}} \widehat{v}_\theta \\
-\left(\frac{im}{\widetilde{r}} \left(\frac{\partial \widehat{V}_\theta}{\partial \widetilde{r}} - \frac{\widehat{V}_\theta}{\widetilde{r}} \right) + i\widehat{k}_x \left(\frac{\partial \widehat{V}_x}{\partial \widetilde{r}} \right) \right) \widehat{v}_x \\
-\frac{\partial^2 \widehat{V}_x}{\partial \widetilde{r}^2} \widehat{v}_r - \frac{\partial \widehat{V}_x}{\partial \widetilde{r}} \frac{\partial \widehat{v}_r}{\partial \widetilde{r}}
\end{pmatrix} = 0
$$