1 Setting Boundary Condition Values Using a Fairing Function

1.1 Using β as a scaling parameter

Defining the nondimensional radius in the same way that SWIRL does:

$$\widetilde{r} = \frac{r}{r_T}$$

where r_T is the outer radius of the annulus.

The hub-to-tip ratio is defined as:

$$\sigma = \frac{r_H}{r_T} = \widetilde{r}_H$$

where $\widetilde{r}_H$ is the inner radius of the annular duct. The hub-to-tip ratio can also be zero indicating the duct is hollow.

A useful and similar parameter is introduced, β, where $0 \leq \beta \leq 1$

$$\beta = \frac{r - r_H}{r_T - r_H}$$

Dividing By r_T

$$\begin{aligned} \beta &= \frac{\frac{r}{r_T} - \frac{r_H}{r_T}}{\frac{r_T}{r_T} - \frac{r_H}{r_T}} \\ &= \frac{\widetilde{r} - \sigma}{1 - \sigma} \end{aligned}$$

Suppose a manufactured solution (MS) was decided. the goal is to change the boundary conditions of the manufactured solution in such way that allows us to adequately check the boundary conditions imposed on SWIRL. Defining the manufactured solution, $f_{MS}(\widetilde{r})$, where $\sigma \leq \widetilde{r} \leq 1$ and there are desired values of f at the boundaries desired values are going to be denoted as f_{minBC} and f_{maxBC}. The desired changes in f are defined as:

$$\begin{aligned} \Delta f_{minBC} &= f_{minBC} - f_{MS}(\widetilde{r} = \sigma) \\ \Delta f_{maxBC} &= f_{maxBC} - f_{MS}(\widetilde{r} = 1) \end{aligned}$$

We'd like to impose these changes smoothly on the manufactured solution function. To do this,the fairing functions, $A_{min}(\widetilde{r})$ and $A_{max}(\widetilde{r})$ where:

$$f_{BCsImposed}(\widetilde{r}) = f_{MS}(\widetilde{r}) + A_{min}(\widetilde{r})\Delta f_{minBC} + A_{max}(\widetilde{r})\Delta f_{maxBC}$$

Then, in order to set the condition at the appropriate boundary, the following conditions are set,

$$
\begin{aligned}
A_{min}(\widetilde{r} = \sigma) &= 1 \\
A_{min}(\widetilde{r} = 1) &= 0 \\
A_{max}(\widetilde{r} = 1) &= 1 \\
A_{max}(\widetilde{r} = \sigma) &= 0
\end{aligned}
$$

If $A_{min}(\widetilde{r})$ is defined as a function of $A_{max}(\widetilde{r})$ then only $A_{max}(\widetilde{r})$ needs to be defined, therefore

$$A_{min}(\widetilde{r}) = 1 - A_{max}(\widetilde{r})$$

It is also desirable to set the derivatives for the fairing function at the boundaries incase there are boundary conditions imposed on the derivatives of the fairing function.

$$
\begin{aligned}
\frac{\partial A_{max}}{\partial \widetilde{r}}|_{\widetilde{r}=\sigma} &= 0 \\
\frac{\partial A_{max}}{\partial \widetilde{r}}|_{\widetilde{r}=1} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial A_{min}}{\partial \widetilde{r}}|_{\widetilde{r}=\sigma} &= 0 \\
\frac{\partial A_{min}}{\partial \widetilde{r}}|_{\widetilde{r}=1} &= 0
\end{aligned}
$$

1.2 Minimum Boundary Fairing Function

Looking at A_{min} first, the polynomial is:

$$
\begin{aligned}
A_{min}(\beta) &= a + b\beta + c\beta^2 + d\beta^3 \\
A_{min}(\widetilde{r}) &= a + b\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) + c\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 + d\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3
\end{aligned}
$$

Taking the derivative,

$$
\begin{aligned}
A'_{min}(\widetilde{r}) &= b\left(\frac{1}{1-\sigma}\right) + 2c\left(\frac{1}{1-\sigma}\right)\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) + 3d\left(\frac{1}{1-\sigma}\right)\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 \\
A'_{min}(\beta) &= \left(\frac{1}{1-\sigma}\right)\left[b + 2c\beta + 3d\beta^2\right]
\end{aligned}
$$

Using the possible values of $\widetilde{r}$,

$$
\begin{aligned}
A_{min}(\sigma) &= a && = 1 \\
A_{min}(1) &= a + b + c + d && = 0 \\
A'_{min}(\sigma) &= b && = 0 \\
A'_{min}(1) &= b + 2c + 3d && = 0
\end{aligned}
$$

which has the solution,

$$
\begin{aligned}
a &= 1 \\
b &= 0 \\
c &= -3 \\
d &= 2
\end{aligned}
$$

giving the polynomial as:

$$
A_{min}(\widetilde{r}) = 1 - 3\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 + 2\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3
$$

1.3 Max boundary polynomial

Following the same procedure for A_{max} gives

$$A_{min}(\widetilde{r}) = 3\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 - 2\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3$$

1.4 Corrected function

The corrected function is then,

$$\begin{aligned}
f_{BCsImposed}(\widetilde{r}) = f_{MS}(\widetilde{r}) & \qquad + A_{min}\Delta f_{minBC} + A_{max}\Delta f_{maxBC} \\
= f_{MS}(\widetilde{r}) & \qquad + \left(1 - 3\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 + 2\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3\right)[\Delta f_{minBC}] \\
& \qquad + \left(3\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 - 2\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3\right)[\Delta f_{maxBC}] \\
f_{BCsImposed}(\beta) = f_{MS}(\beta) & \quad + \Delta f_{minBC} + \left(3\beta^2 - 2\beta^3\right)[\Delta f_{maxBC} - \Delta f_{minBC}]
\end{aligned}$$

Note that we're carrying the correction throughout the domain, as opposed to limiting the correction at a certain distance away from the boundary. The application of this correction ensures that there is no discontinuous derivatives inside the domain; as suggested in Roach's MMS guidelines (insert ref)

What is meant by "just because A_{min} and its first derivative go to zero doesn't mean that the second derivatives"

1.5 Symbolic Sanity Checks

We want to ensure that $f_{BCsImposed}$ has the desired boundary conditions, $f_{minBC/maxBC}$ instead of the original boundary values that come along for the ride in the manufactured solutions, $f_{MS}(\widetilde{r} = \sigma/1)$. In another iteration of this method, we will be changing the derivative values, so let's check the values of $\frac{\partial f_{BCsImposed}}{\partial \widetilde{r}}$ to make sure those aren't effected unintentionally.

Symbolic Sanity Check 1

The modified manufactured solution, $f_{BCsImposed}$ with the fairing functions A_{min} and A_{max} substituted in is,

$$f_{BCsImposed}(\widetilde{r}) = \left(3\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 - 2\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3\right)\left[\Delta f_{maxBC}\right].$$

Further simplification yields,

$$\begin{aligned} f_{BCsImposed}(\widetilde{r}=\sigma) &= \left(f_{MS}(\widetilde{r}=\sigma) + \Delta f_{minBC} + \left(3\left(\frac{\sigma-\sigma}{1-\sigma}\right)^2 - 2\left(\frac{\sigma-\sigma}{1-\sigma}\right)^3 - \right)\left[\Delta f_{maxBC} - \Delta f_{minBC}\right]\right) \\ &= f_{MS}(\widetilde{r}=\sigma) + \Delta f_{minBC} \\ &= f_{MS}(\widetilde{r}=\sigma) + (f_{minBC} - f_{MS}(\widetilde{r}=\sigma)) \\ &= f_{minBC} \end{aligned}$$

$$\begin{aligned} f_{BCsImposed}(\widetilde{r}=1) &= \left(f_{MS}(\widetilde{r}=1) + \Delta f_{minBC} + \left(3\left(\frac{1-\sigma}{1-\sigma}\right)^2 - 2\left(\frac{1-\sigma}{1-\sigma}\right)^3 - \right)\left[\Delta f_{maxBC} - \Delta f_{minBC}\right]\right) \\ &= f_{MS}(\widetilde{r}=1) + \Delta f_{maxBC} \\ &= f_{MS}(\widetilde{r}=1) + (f_{maxBC} - f_{MS}(\widetilde{r}=1)) \\ &= f_{maxBC} \end{aligned}$$

$$\frac{\partial}{\partial \widetilde{r}}\left(f_{BCsImposed}(\widetilde{r}) = \left(3\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 - 2\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3\right)\left[\Delta f_{maxBC}\right]\right)$$

$$\frac{\partial f_{MS}}{\partial \widetilde{r}} + \left(\frac{6}{1-\sigma}\right)\left(\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) - \left(\frac{r-\sigma}{1-\sigma}\right)^2\right)\left(\Delta f_{maxBC} - \Delta f_{minBC}\right)$$

At $\widetilde{r} = \sigma$, the derivative is:

$$\frac{\partial f_{MS}}{\partial \widetilde{r}}|_\sigma$$
$$\frac{\partial f_{MS}}{\partial \widetilde{r}}|_1$$

2 Setting the BC Derivative Values using a Fairing Function

The desired changes in f' are defined as:

$$\Delta f'_{minBC} = f'_{minBC} - f'_{MS}(\widetilde{r} = \sigma)$$
$$\Delta f'_{maxBC} = f'_{maxBC} - f'_{MS}(\widetilde{r} = 1)$$

We'd like to impose these changes smoothly on the manufactured solution function. To do this,the fairing functions, $B_{min}(\widetilde{r})$ and $B_{max}(\widetilde{r})$ where:

$$f_{BCsImposed}(\widetilde{r}) = f_{MS}(\widetilde{r}) + B_{min}(\widetilde{r})\Delta f'_{minBC} + B_{max}(\widetilde{r})\Delta f'_{maxBC}$$

Then, in order to set the condition at the appropriate boundary, the following conditions are set, not the values of B at the boundaries are unaffected,

$$\begin{aligned}
B_{min}(\widetilde{r} = \sigma) &= 0 \\
B_{min}(\widetilde{r} = 1) &= 0 \\
B_{max}(\widetilde{r} = 1) &= 0 \\
B_{max}(\widetilde{r} = \sigma) &= 0 B'_{min}(\widetilde{r} = \sigma) && = 1 \\
B'_{min}(\widetilde{r} = 1) &= 0 \\
B'_{max}(\widetilde{r} = 1) &= 1 \\
B'_{max}(\widetilde{r} = \sigma) &= 0
\end{aligned}$$

2.1 Min boundary derivative polynomial

The polynomial is of the form:

$$\begin{aligned}
B_{min}(\beta) &= a + b\beta + c\beta^2 + d\beta^3 \\
B_{min}(\widetilde{r}) &= a + b\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right) + c\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right)^2 + d\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right)^3
\end{aligned}$$

Taking the derivative,

$$\begin{aligned}
B'_{min}(\widetilde{r}) &= b\left(\frac{1}{1-\sigma}\right) + 2c\left(\frac{1}{1-\sigma}\right)\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right) + 3d\left(\frac{1}{1-\sigma}\right)\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right)^2 \\
B'_{min}(\beta) &= \left(\frac{1}{1-\sigma}\right)\left[b + 2c\beta + 3d\beta^2\right]
\end{aligned}$$

Applying the four constraints gives:

$$\begin{aligned}
a &= 0 \\
b &= (1 - \sigma) \\
a + b + c + d &= 0 \\
2 + 2c + 3d &= 0
\end{aligned}$$

$$
\begin{aligned}
c + d &= -b \\
2c + 3d &= -b
\end{aligned}
$$

$$
\begin{aligned}
c &= -2b \\
d &= b
\end{aligned}
$$

and the min boundary derivative polynomial is:

$$
\begin{aligned}
B_{min}\left(\widetilde{r}\right) &= b\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) - 2b\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 + b\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3 \\
&= \left(1-\sigma\right)\left(\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) - 2\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 + \left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3\right)
\end{aligned}
$$

2.2 Polynomial function, max boundary derivative

The polynomial is of the form:

The polynomial is of the form:

$$
\begin{aligned}
B_{max}\left(\beta\right) &= a + b\beta + c\beta^2 + d\beta^3 \\
B_{max}\left(\widetilde{r}\right) &= a + b\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) + c\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 + d\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3
\end{aligned}
$$

which has the derivative,

$$
\begin{aligned}
B'_{max}\left(\widetilde{r}\right) &= b\left(\frac{1}{1-\sigma}\right) + 2c\left(\frac{1}{1-\sigma}\right)\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) + 3d\left(\frac{1}{1-\sigma}\right)\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 \\
B'_{max}\left(\beta\right) &= \left(\frac{1}{1-\sigma}\right)\left[b + 2c\beta + 3d\beta^2\right]
\end{aligned}
$$

Applying the four constraints gives:

$$
\begin{aligned}
a &= 0 \\
b &= 0 \\
a + b + c + d &= 0 \\
b + 2c + 3d &= \left(1-\sigma\right)
\end{aligned}
$$

working this out:

$$
\begin{aligned}
c + d &= 0 \\
2c + 3d &= \left(1-\sigma\right)
\end{aligned}
$$

gives

$$
c = -\left(1-\sigma\right)d \qquad = \left(1-\sigma\right)
$$

and the max boundary derivative polynomial is:

$$B_{max}\left(\widetilde{r}\right) = (1-\sigma)\left(-\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 + \left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3\right)$$

2.3 Putting it together

The corrected function is then:

$$f_{BCsImposed}\left(\widetilde{r}\right) = f_{MS} + B_{min}\left(\widetilde{r}\right)\Delta f'_{minBC} + B_{max}\left(\widetilde{r}\right)\Delta f'_{maxBC}$$

$$= f_{MS} +$$

$$(1-\sigma)\left(\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) - \left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2\right)\Delta f'_{minBC} +$$

$$(1-\sigma)\left(-\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 + \left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3\right)\left(\Delta f'_{minBC} + \Delta f'_{maxBC}\right)$$

Now here we will do sanity checks for the derivative boundaries.