A Thesis

entitled

Verification and Validation Method for

an Acoustic Mode Prediction Code for Turbomachinery Noise

by

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Submitted to the Graduate Faculty as partial fulfillment of the requirements for the

Masters of Science Degree in Mechanical Engineering

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An Abstract of

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Over the last 20 years, there has been an increase in computational fluid dynamic codes that have made numerical analysis more and more readily available, allowing turbomachine designers to create more novel designs. However, as airport noise limitations become more restrictive over time, reducing aircraft takeoff and landing noise remains a prominent issue in the aviation community. One popular method to reduce aircraft noise is using acoustic liners placed on the walls of the engine inlet and exhaust ducts. These liners are designed to reduce the amplitude of acoustic modes emanating from the bypass fan as they propagate through the engine. The SWIRL code is a frequency-domain linearized Euler equation solver that is designed to predict the effect of acoustic liners on acoustic modes propagating in realistic sheared and swirling mean flows, guiding the design of more efficient liner configurations. The purpose of this study is to validate SWIRL using the Method Of Manufactured Solutions (MMS). This study also investigated the effect of the integration and spatial differencing methods on the convergence for a given Manufactured Solution. In addition, the effect of boundary condition implementation was tested. The improved MMS convergence rates shown for these tests suggest that the revised SWIRL code provides more accurate solutions with less computational effort than the original formulation.

For my friends and family, who have always believed in my potential when I did not believe it myself.

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A .... mean flow speed of sound
A_T .... speed of sound at the duct radius
$\tilde{A}$ .... dimensionless speed of sound, $\frac{A}{A_T}$
D/Dt ...... material derivative, $\partial/\partial t + V \cdot \nabla$
D_N .... derivative matrix using N points
$\mathbf{e}_x$,$\mathbf{e}_\theta$ .... s

List of Abbreviations

CFD .... Computational Fluid Dynamics
GLE .... Gauss' law for electricity: $\nabla \cdot E = \frac{\rho}{\varepsilon_0} = 4\pi k\rho$
HHS .... Department of Health and Human Services
IaR .... I am root

Preface

Chapter 1

Background

1.1 Introduction

During the 1960s, the increased demand for commercialized aircraft transport introduced jet engines to support large cargo and passengers. Consequently, this rise in innovation resulted in high volume engine noise. After 1975, efforts to reduce aircraft noise eliminated the noise pollution for 90% of the population [1]. However, since the early 2000s, the advancement in noise reduction technologies has been moderately increasing, leaving a requirement for drastic improvement in aeroacoustic modeling and treatment strategies to compete with the demand for quiet subsonic flight. A turbomachine's general flow condition includes a series of axial, tangential, and radial velocity components that vary depending on the location of concern. The swirling flow between fan stages has been an area of interest due to the potential for acoustic treatment in a location previously avoided for its flow complexity, among other reasons.

In general, jet engine designers can model flow within a turbomachine with the Navier Stokes Equations, a set of partial differential equations that describe the mass, momentum and energy of a given viscous fluid. It is common in practice to utilize the Euler equations, a closely related set of PDEs that model inviscid fluid, as they provide an approximation for higher Reynold number flows where viscosity does not play a critical role. A popular approach to modeling sound propagation within a flow is to "linearize" the Euler equations, which decomposes the flow solution into a mean and fluctuating component (insert refs). Another method decomposes the flow into vortical and potential parts (ref Golubev & Atassi). In either case, this presents an initial value problem and for certain flows and domains, can obtain analytical solutions. Swirling flow has been a difficult problem to investigate in comparison to flows parallel to the wall domain of a duct [2]. . .

Chapter 2

Chapter 2: Literature Review

Chapter 3

Chapter 3: Theory

3.1 Divergence operations in new coordinate systems

The divergence, (∇), represents the operation of taking derivatives of a vector field. However, understanding the mathematical and physical representation of the divergence operator into new coordinate systems serves as a good prerequisite for the application of the Navier Stokes equations for the evaluation of aerodynamic models in unusual flow domains. Although there are many resources that will provide equations in varying coordinate systems, the derivation offers insight into the advantages and drawbacks of using a new reference frame for a flow domain. The divergence operator in Cartesian coordinates is,

$$\vec{\nabla} \equiv \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} = 0$$

The vectors, $\hat{e}_x, \hat{e}_y, \hat{e}_z$ (commonly denoted in literature as $\hat{\imath}$,$\hat{\jmath}$,$\hat{k}$) are the basis vectors of the Cartesian coordinate system. The vector hat ($\vec{}$) reminds us that divergence operation includes a scalar product of the basis vectors and the individual derivative terms themselves. These basis vectors *scale* with the derivatives d/dx d/dy d/dz in the direction of these basis vectors themselves. This implicitly captures the coordinate system and assumptions that corresponds to the basis vectors themselves.

To relate the basis vectors of the cylindrical coordinate system to the Cartesian coordinate system, we use the following relations,

$$
\begin{aligned}
r &= \sqrt{x^2+y^2} \\
\theta &= \tan^{-1}\left(\frac{y}{x}\right) \\
&= \cos^{-1}\left(\frac{x}{r}\right) \\
&= \sin^{-1}\left(\frac{y}{r}\right)
\end{aligned}
$$

Note that the equation above also establishes $x = r\cos\theta$ and $y = r\sin\theta$. The Cartesian basis vectors are related to the cylindrical basis vectors of the new coordinate system by,

$$
\begin{aligned}
\hat{e}_r &= \hat{e}_x \cos\theta + \hat{e}_y \sin\theta \\
\hat{e}_\theta &= -\hat{e}_x \sin\theta + \hat{e}_y \cos\theta \\
\hat{e}_z &= \hat{e}_z
\end{aligned}
$$

Defining these relationships, (they'll be useful later)

$$
\begin{aligned}
\frac{\partial \hat{e}_r}{\partial r} &= \frac{\partial \hat{e}_\theta}{\partial r} = \frac{\partial \hat{e}_z}{\partial r} = 0 \\
\frac{\partial \hat{e}_r}{\partial \theta} &= -\hat{e}_x \sin\theta + \hat{e}_y \cos\theta = \hat{e}_\theta \\
\frac{\partial \hat{e}_\theta}{\partial \theta} &= -\left(\hat{e}_x \cos\theta + \hat{e}_y \sin\theta\right) = -\hat{e}_r
\end{aligned}
$$

The multi-variable chain rule for differentiation is then used to express the Cartesian variables, $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, with respect to the cylindrical variable.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{dr}{dx} + \frac{\partial}{\partial \theta}\frac{d\theta}{dx} + \frac{\partial}{\partial z}\frac{dz}{dx}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{dr}{dy} + \frac{\partial}{\partial \theta}\frac{d\theta}{dy} + \frac{\partial}{\partial z}\frac{dz}{dy}$$

By,finding the derivatives of r & θ with respect to x and y, we can substitute terms in the Cartesian divergence definition. First, $\frac{dr}{dx}$ & $\frac{dr}{dy}$ is calculated,

$$\begin{aligned}
\frac{dr}{dx} &= \frac{d}{dx}\left(\left[x^2+y^2\right]^{1/2}\right) \\
&= \frac{1}{2}\left[x^2+y^2\right]^{-1/2}(2x) \\
&= \frac{x}{\sqrt{x^2+y^2}} \\
&= \frac{rcos\theta}{r} \\
&\boxed{\frac{dr}{dx} = cos\theta}
\end{aligned}$$

$$\begin{aligned}
\frac{dr}{dy} &= \frac{d}{dy}\left(\left[x^2+y^2\right]^{1/2}\right) \\
&= \frac{1}{2}\left[x^2+y^2\right]^{-1/2}(2y) \\
&= \frac{y}{\sqrt{x^2+y^2}} \\
&= \frac{rsin\theta}{r} \\
&\boxed{\frac{dr}{dy} = sin\theta}
\end{aligned}$$

Then, $\frac{d\theta}{dx}$ & $\frac{d\theta}{dy}$ is found.

$$\begin{aligned}
\frac{d\theta}{dx} &= \frac{d}{dx}\left(tan^{-1}\left(\frac{y}{x}\right)\right) \\
&= \frac{d}{du}tan^{-1}(u)\frac{d}{dx}\left(\frac{y}{x}\right) \\
&= \frac{1}{u^2+1}\frac{-y}{x^2} \\
&= -\frac{y}{y^2+x^2} \\
&\boxed{\frac{d\theta}{dx} = -\frac{sin\theta}{r}}
\end{aligned}$$

$$\begin{aligned}
\frac{d\theta}{dy} &= \frac{d}{dy}\left(tan^{-1}\left(\frac{y}{x}\right)\right) \\
&= \frac{d}{du}tan^{-1}(u)\frac{d}{dy}\left(\frac{y}{x}\right) \\
&= \frac{1}{u^2+1}\frac{1}{x} \\
&= \frac{x}{y^2+x^2} \\
&\boxed{\frac{d\theta}{dy} = \frac{cos\theta}{r}}
\end{aligned}$$

Through substitution back into the chain rule expansion,

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial r}\cos\theta - \frac{\partial}{\partial\theta}\frac{1}{r}\sin\theta \\
\frac{\partial}{\partial y} &= \frac{\partial}{\partial r}\sin\theta + \frac{\partial}{\partial\theta}\frac{1}{r}\cos\theta
\end{aligned}$$

We can now convert our divergence operator, $\vec{\nabla}$

$$\begin{aligned}\vec{\nabla} &= \frac{\partial}{\partial x}\hat{e}_x + \frac{\partial}{\partial y}\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z = 0 \\ &= \left(\frac{\partial}{\partial r}\cos\theta - \frac{\partial}{\partial \theta}\frac{1}{r}\sin\theta\right)\hat{e}_x + \left(\frac{\partial}{\partial r}\sin\theta + \frac{\partial}{\partial \theta}\frac{1}{r}\cos\theta\right)\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z = 0\end{aligned}$$

Rearranging like terms (containing cylindrical derivative variables), and factoring out $1/r$

$$\begin{aligned}\vec{\nabla} &= \left(\frac{\partial}{\partial r}\cos\theta - \frac{\partial}{\partial \theta}\frac{1}{r}\sin\theta\right)\hat{e}_x + \left(\frac{\partial}{\partial r}\sin\theta + \frac{\partial}{\partial \theta}\frac{1}{r}\cos\theta\right)\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z = 0 \\ &= \left(\hat{e}_x\cos\theta + \hat{e}_y\sin\theta\right)\frac{\partial}{\partial r} + \frac{1}{r}\left(\hat{e}_y\cos\theta - \hat{e}_x\sin\theta\right)\frac{\partial}{\partial \theta} + \frac{\partial}{\partial z}\hat{e}_z = 0\end{aligned}$$

Recalling the definitions for $\hat{e}_r$ and $\hat{e}_\theta$, we can use these expressions to rewrite ∇ in polar coordinates

$$\vec{\nabla} = \hat{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\hat{e}_\theta\frac{\partial}{\partial \theta} + \frac{\partial}{\partial z}\hat{e}_z = 0$$

$$\frac{DV}{dt} = \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma}$$

$$\boldsymbol{\sigma} = -p\boldsymbol{I_3} + \tau$$

where $[\boldsymbol{I_3}]$ is a 3 by 3 identity matrix and τ is the shear stress tensor. The velocity vector for a three dimensional flow.

$$\vec{V} = v_r(r,\theta,x,t)\hat{e}_r + v_\theta(r,\theta,x,t)\hat{e}_\theta + v_x(r,\theta,x,t)\hat{e}_x \tag{3.1}$$

In Kousen's work, a velocity vector is written as a function of radius, and the radial velocity component is neglected.

$$\vec{V} = v_\theta(r)\hat{e}_\theta + v_x(r)\hat{e}_x \tag{3.2}$$

We will go with the first definition and cancel out the radial velocity later on.

$$\frac{DV}{dt} = \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial r}\frac{dr}{dt} + \frac{\partial \vec{V}}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \vec{V}}{\partial x}\frac{dx}{dt}$$

Starting with the first term,

$$\begin{aligned}
\frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} &= \frac{\partial}{\partial t}\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_x\hat{e}_x\right) * 1 \\
&= \frac{\partial v_r}{\partial t}\hat{e}_r + \cancel{\frac{\partial \hat{e}_r}{\partial t}}v_r + \frac{\partial v_\theta}{\partial t}\hat{e}_\theta + \cancel{\frac{\partial \hat{e}_\theta}{\partial t}}v_\theta + \frac{\partial v_x}{\partial t}\hat{e}_x + \cancel{\frac{\partial \hat{e}_x}{\partial t}}v_x \\
&\boxed{\frac{\partial \vec{V}}{\partial t} = \frac{\partial v_r}{\partial t}\hat{e}_r + \frac{\partial v_\theta}{\partial t}\hat{e}_\theta + \frac{\partial v_x}{\partial t}\hat{e}_x}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \vec{V}}{\partial r}\frac{dr}{dt} &= \frac{\partial \vec{V}}{\partial r}v_r \\
&= \frac{\partial}{\partial r}\left[v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_x\hat{e}_x\right]v_r \\
&= \left(\frac{\partial v_r}{\partial r}\hat{e}_r + \cancel{\frac{\partial \hat{e}_r}{\partial r}}v_r + \frac{\partial v_\theta}{\partial r}\hat{e}_\theta + \cancel{\frac{\partial \hat{e}_\theta}{\partial r}}v_\theta + \frac{\partial v_x}{\partial r}\hat{e}_x + \cancel{\frac{\partial \hat{e}_x}{\partial r}}v_\theta\right)v_r \\
&\boxed{\frac{\partial \vec{V}}{\partial r}\frac{dr}{dt} = \left[\frac{\partial v_r}{\partial r}\hat{e}_r + \frac{\partial v_\theta}{\partial r}\hat{e}_\theta + \frac{\partial v_x}{\partial r}\hat{e}_x\right]v_r}
\end{aligned}
$$

Recalling that arc length is $ds = rd\theta$, and angular velocity is $d\theta/dt = v_\theta/r$

$$
\begin{aligned}
\frac{\partial \vec{V}}{\partial \theta}\frac{d\theta}{dt} &= \frac{\partial \vec{V}}{\partial \theta}\frac{v_\theta}{r} \\
&= \frac{\partial}{\partial \theta}\left[v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_x\hat{e}_x\right]\frac{v_\theta}{r} \\
&= \left[\frac{\partial v_r}{\partial \theta}\hat{e}_r + \underbrace{\frac{\partial \hat{e}_r}{\partial \theta}}_{\hat{e}_\theta}v_r + \frac{\partial v_\theta}{\partial \theta}\hat{e}_\theta + \underbrace{\frac{\partial \hat{e}_\theta}{\partial \theta}}_{-\hat{e}_r}v_\theta + \frac{\partial v_x}{\partial \theta}\hat{e}_x\right]\frac{v_\theta}{r} \\
&\boxed{\frac{\partial \vec{V}}{\partial \theta}\frac{dr}{dt} = \left[\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\hat{e}_r + \left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\hat{e}_\theta + \frac{\partial v_x}{\partial \theta}\hat{e}_x\right]\frac{v_\theta}{r}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \vec{V}}{\partial x}\frac{dx}{dt} &= \frac{\partial \vec{V}}{\partial x} v_x \\
&= \frac{\partial}{\partial x}\left[v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_x \hat{e}_x\right] v_r \\
&= \left(\frac{\partial v_r}{\partial x}\hat{e}_r + \cancel{\frac{\partial \hat{e}_r}{\partial x}} v_r + \frac{\partial v_\theta}{\partial x}\hat{e}_\theta + \cancel{\frac{\partial \hat{e}_\theta}{\partial x}} v_\theta + \frac{\partial v_x}{\partial r}\hat{e}_x + \cancel{\frac{\partial \hat{e}_x}{\partial x}} v_\theta\right) v_x \\
&\boxed{\frac{\partial \vec{V}}{\partial x}\frac{dx}{dt} = \left[\frac{\partial v_r}{\partial x}\hat{e}_r + \frac{\partial v_\theta}{\partial x}\hat{e}_\theta + \frac{\partial v_x}{\partial x}\hat{e}_x\right] v_x}
\end{aligned}
$$

Putting these terms together,

$$
\begin{aligned}
\frac{DV}{dt} = &\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta^2}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} + v_x\frac{\partial v_r}{\partial x}\right]\hat{e}_r + \\
&\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_x\frac{\partial v_\theta}{\partial x}\right]\hat{e}_\theta + \\
&\left[\frac{\partial v_x}{\partial t} + v_r\frac{\partial v_x}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_x}{\partial \theta} + v_x\frac{\partial v_x}{\partial x}\right]\hat{e}_x
\end{aligned}
$$

If we neglect viscosity on the right hand side, we will arrive at the linearized Euler equations

$$
\begin{aligned}
\nabla\sigma &= -\nabla p[\mathbf{I_3}] \\
&= -\frac{1}{\rho}\left\{\begin{matrix} \frac{\partial p}{\partial r} & 0 & 0 \\ 0 & \frac{1}{r}\frac{\partial p}{\partial \theta} & 0 \\ 0 & 0 & \frac{\partial p}{\partial x} \end{matrix}\right\}
\end{aligned}
$$

3.1.1 Setting up SWIRL's Aerodynamic Model

The Euler Equations in Cylindrical Form are,

$$\frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} + \frac{v_\theta}{r}\frac{\partial \rho}{\partial \theta} + v_x \frac{\partial \rho}{\partial \theta} + \rho\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_x}{\partial x}\right) = 0$$

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_x \frac{\partial v_r}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_x \frac{\partial v_\theta}{\partial x} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

$$\frac{\partial v_x}{\partial t} + v_r \frac{\partial v_x}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_x}{\partial \theta} + v_x \frac{\partial v_x}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial p}{\partial t} + v_r \frac{\partial p}{\partial r} + \frac{v_\theta}{r}\frac{\partial p}{\partial \theta} + v_x \frac{\partial p}{\partial \theta} + \gamma p\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{v_\theta}{\partial \theta} + \frac{\partial v_x}{\partial x}\right) = 0$$

SWIRL utilizes the following assumptions to simplify the aerodynamic model

- No flow in the radial direction. Consequentially, the flow is axisymmetric along the downstream direction.
- No surface or body forces
- Isentropic conditions

For steady flow, the continuity, momentum and entropy equations are

$$\nabla(\vec{V}\bar{\rho}) = 0$$

$$(\vec{V} \cdot \nabla)\vec{V}$$

$$\nabla S = 0$$

If we neglect radial velocity, the velocity vector in cylindrical coordinates are

$$\vec{V}(r, \theta, x) = V_x(r)\hat{e}_x + V_\theta(r)\hat{e}_\theta$$

3.2 Applying model to various flows

Kousen studied three specific flow configuration.

- axial shear flow
- solid body swirl
- free vortex swirl

3.2.1 Axial Shear Flow

In Kousen's paper, axial sheared flows through a constant area duct was also investigated.he only effect on the velocity gradient occurs along the x axis. All other primitive variables (pressure and density which is $\propto$ speed of sound) are constant. As a result, the only changes that occur are in the x direction. This implies that $\partial/\partial\theta = 0$. For the conservation of mass,

$$\nabla(\vec{V}\bar{\rho}) = \left(\underbrace{\cancel{\frac{\partial(\bar{\rho}v_r)}{\partial r}}}_{v_r=0} + \underbrace{\cancel{\frac{1}{r}\frac{\partial\bar{\rho}v_\theta}{\partial\theta}}}_{\frac{\partial}{\partial\theta}} + \frac{\partial\bar{\rho}v_x}{\partial x} \right) = \frac{\partial\bar{\rho}v_x}{\partial x}$$

Conservation of Momentum in the radial direction becomes:

$$(\vec{V}\cdot\nabla)\vec{V} = \cancel{v_r\frac{\partial v_r}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}} - \frac{v_\theta^2}{r} + \cancel{v_x\frac{\partial v_r}{\partial x}} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{v_\theta^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{\rho v_\theta^2}{r} = \frac{\partial p}{\partial r}$$

θ direction

$$(\vec{V}\cdot\nabla)\vec{V} = \cancel{v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r v_\theta}{r}} + v_x\frac{\partial v_\theta}{\partial x} = \cancel{-\frac{1}{\rho r}\frac{\partial p}{\partial\theta}}$$

Dividing v_x to the other side,

$$\frac{\partial v_\theta}{\partial x} = 0$$

Similarly for the x direction,

$$\frac{\partial v_x}{\partial x} = 0$$

In regards to the entropy equation, having an isentropic flow, $\nabla S = 0$ implies $a^2 = \frac{\nabla \bar{p}}{\nabla \bar{\rho}}$

3.3 Accounting for solid body swirl

If the flow contains a swirling component, then the primitive variables are nonuniform through the flow, and mean flow assumptions are not valid. To account to this, we integrate the momentum equation in the radial direction with respect to the radius.

Equation (2.5) in [**?**] is

$$P = \int_{\tilde{r}}^{1} \frac{\bar{\rho} V_\theta^2}{\tilde{r}} d\tilde{r}$$

where $\tilde{r}$ is the radius dimensional radius normalized by the tip diameter $r_t = r_{max}$

To show the work, we will start with the dimensional form of the equation,

$$\frac{\bar{\rho} v_\theta^2}{r} = \frac{\partial p}{\partial r}$$

Applying separation of variables

$$\int_{r}^{r_{max}} \frac{\bar{\rho} v_\theta^2}{r} \partial r = -\int_{P(r)}^{P(r_{max})} \partial p$$

Since $\tilde{r} = r/r_{max}$

$$r = \tilde{r} r_{max}$$

taking total derivatives(applying chain rule)

$$dr = d(\tilde{r} r_{max}) = d(\tilde{r}) r_{max}$$

Substituting these back in and evaluating the right hand side,

$$\int_{\tilde{r}}^{1} \frac{\bar{\rho} v_\theta^2}{\tilde{r}} \partial \tilde{r} = P(1) - P(\tilde{r})$$

For reference the minimum value of $\tilde{r}$ is

$$\sigma = \frac{r_{max}}{r_{min}}$$

For

$$\frac{\partial a^2}{\partial r} = \frac{\partial}{\partial r} \left(\frac{\gamma P}{\rho} \right)$$

Using the quotient rule, we can extract the definition of the speed of sound.

$$\begin{aligned}
&= \frac{\partial P}{\partial r} \frac{\gamma \bar{\rho}}{\bar{\rho}^2} - \left(\frac{\gamma P}{\bar{\rho}^2} \right) \frac{\partial \bar{\rho}}{\partial r} \\
&= \frac{\partial P}{\partial r} \frac{\gamma}{\bar{\rho}} - \left(\frac{a^2}{\bar{\rho}} \right) \frac{\partial \bar{\rho}}{\partial r} \\
\text{Using } \partial P / a^2 = \partial \rho \rightarrow &= \frac{\partial P}{\partial r} \frac{\gamma}{\bar{\rho}} - \left(\frac{1}{\bar{\rho}} \right) \frac{\partial \bar{P}}{\partial r} \\
\frac{\partial a^2}{\partial r} &= \frac{\partial P}{\partial r} \frac{\gamma - 1}{\bar{\rho}} \\
\text{or..} \frac{\bar{\rho}}{\gamma - 1} \frac{\partial a^2}{\partial r} &= \frac{\partial P}{\partial r}
\end{aligned}$$

Going back to the radial momentum equation, and rearranging the

$$\frac{\bar{\rho} v_\theta^2}{r} = \frac{\partial P}{\partial r}$$

$$\frac{\cancel{\bar{\rho}} v_\theta^2}{r} = \frac{\cancel{\bar{\rho}}}{\gamma - 1} \frac{\partial a^2}{\partial r}$$

$$\frac{v_\theta^2}{r} (\gamma - 1) = \frac{\partial a^2}{\partial r}$$

$$\text{Dividing both sides by } a^2 \rightarrow \frac{M_\theta}{r} (\gamma - 1) = \frac{\partial a^2}{\partial r} \frac{1}{a^2}$$

$$\text{Integrating both sides } \int_r^{r_{max}} \frac{M_\theta}{r} (\gamma - 1)\, \partial r = \int_{a^2(r)}^{a^2(r_{max})} \frac{1}{a^2} \partial a^2$$

$$\int_r^{r_{max}} \frac{M_\theta^2}{r} (\gamma - 1)\, \partial r = ln(a^2(r_{max})) - ln(a^2(r))$$

$$\int_r^{r_{max}} \frac{M_\theta^2}{r} (\gamma - 1)\, \partial r = ln\left(\frac{a^2(r_{max})}{a^2(r)}\right)$$

Defining non dimensional speed of sound $\tilde{a} = \frac{a(r)}{a(r_{max})}$

$$\int_r^{r_{max}} \frac{M_\theta}{r} (\gamma - 1)\, \partial r = ln\left(\frac{1}{\tilde{a}^2}\right)$$

$$= -2ln(\tilde{a})$$

$$\tilde{a}(r) = exp\left[-\int_r^{r_{max}} \frac{M_\theta}{r} \frac{(\gamma - 1)}{2} \partial r\right]$$

$$\text{replacing r with } \tilde{r} \rightarrow \tilde{a}(r) = exp\left[-\int_r^{r_{max}} \frac{M_\theta}{r} \frac{(\gamma - 1)}{2} \partial r\right]$$

$$\tilde{a}(\tilde{r}) = exp\left[\left(\frac{1 - \gamma}{2}\right) \int_{\tilde{r}}^1 \frac{M_\theta}{\tilde{r}} \partial \tilde{r}\right]$$

3.3.1 Linearizing the governing equations

3.3.1.1 Linearizing Conservation of Mass

To linearize the Euler equations, we substitute each flow variable with its equivalent mean and perturbation components. Note that the mean term is only a function of space whereas the perturbation component is a dependent on both space and time (functional dependence is not explicity written with each variable). Assuming that we can divide the variable into a known laminar flow solution to the Navier-Stokes equations and a small amplitude perturbation solution:

$$v_r = V_r(x) + v_r' \tag{3.3}$$

$$v_\theta = V_\theta + v_\theta' \tag{3.4}$$

$$v_x = V_x + v_x' \tag{3.5}$$

$$p = \bar{p} + p' \tag{3.6}$$

$$\rho = \bar{\rho} + \rho' \tag{3.7}$$

Starting with continuity,

$$\frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} + \frac{v_\theta}{r}\frac{\partial \rho}{\partial \theta} + v_x \frac{\partial \rho}{\partial \theta} + \rho\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_x}{\partial x}\right) = 0$$

$$\frac{\partial \bar{\rho} + \rho'}{\partial t} + (V_r + v_r')\frac{\partial \bar{\rho} + \rho'}{\partial r} + \frac{V_\theta + v_\theta'}{r}\frac{\partial \bar{\rho} + \rho'}{\partial \theta} + (V_x + v_x')\frac{\partial \bar{\rho} + \rho'}{\partial \theta} +$$

$$(\bar{\rho} + \rho')\left(\frac{1}{r}\frac{\partial (r(V_r + v_r'))}{\partial r} + \frac{1}{r}\frac{\partial V_\theta + v_\theta'}{\partial \theta} + \frac{\partial (V_x + v_x')}{\partial x}\right) = 0$$

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \rho'}{\partial t} +$$

$$V_r\frac{\partial \bar{\rho}}{\partial r} + v_r'\frac{\partial \bar{\rho}}{\partial r} + V_r\frac{\partial \rho'}{\partial r} + v_r'\frac{\partial \rho'}{\partial r} +$$

$$\frac{1}{r}\left(V_\theta\frac{\partial \bar{\rho}}{\partial \theta} + v_\theta'\frac{\partial \bar{\rho}}{\partial \theta} + V_\theta\frac{\partial \rho'}{\partial \theta} + v_\theta'\frac{\partial \rho'}{\partial \theta}\right) +$$

$$V_x\frac{\partial \bar{\rho}}{\partial x} + v_x'\frac{\partial \bar{\rho}}{\partial x} + V_x\frac{\partial \rho'}{\partial x} + v_x'\frac{\partial \rho'}{\partial x} +$$

$$\bar{\rho}\left(\frac{1}{r}\left(\frac{\partial (rV_r)}{\partial r} + \frac{\partial (r(v_r')}{\partial r} + \frac{\partial V_\theta}{\partial \theta} + \frac{\partial v_\theta'}{\partial \theta}\right) + \frac{\partial V_x}{\partial x} + \frac{\partial v_x'}{\partial x}\right) +$$

$$\rho'\left(\frac{1}{r}\left(\frac{\partial (rV_r)}{\partial r} + \frac{\partial (r(v_r')}{\partial r} + \frac{\partial V_\theta}{\partial \theta} + \frac{\partial v_\theta'}{\partial \theta}\right) + \frac{\partial V_x}{\partial x} + \frac{\partial v_x'}{\partial x}\right) = 0$$

Important things to note

- The small disturbances are infinitesimal (thus linearized)
- Neglect second order terms.
- The continuity equation is comprised of mean velocity components. This is subtracted off in each of the governing equations

Blue will be used for terms that are removed after subtracting in the original continuity equation, green will be used to cancel higher(2nd) order terms.Red will be used if we take the radial velocity to be zero.

$$
\begin{aligned}
&= \cancel{\frac{\partial \bar{\rho}}{\partial t}} + \frac{\partial \rho'}{\partial t} + \\
&\cancel{V_r \frac{\partial \bar{\rho}}{\partial r}} + \cancel{v'_r \frac{\partial \bar{\rho}}{\partial r}} + \cancel{V_r \frac{\partial \rho'}{\partial r}} + \cancel{v'_r \frac{\partial \rho'}{\partial r}} + \\
&\frac{1}{r}\left(\cancel{V_\theta \frac{\partial \bar{\rho}}{\partial \theta}} + \cancel{v'_\theta \frac{\partial \bar{\rho}}{\partial \theta}} + V_\theta \frac{\partial \rho'}{\partial \theta} + \cancel{v'_\theta \frac{\partial \rho'}{\partial \theta}} \right) + \\
&\cancel{V_x \frac{\partial \bar{\rho}}{\partial x}} + \cancel{v'_x \frac{\partial \bar{\rho}}{\partial x}} + V_x \frac{\partial \rho'}{\partial \theta} + \cancel{v'_x \frac{\partial \rho'}{\partial x}} + \\
&\bar{\rho}\left(\frac{1}{r}\left(\cancel{\frac{\partial (r V_r)}{\partial r}} + \frac{\partial (r(v'_r)}{\partial r} + \cancel{\frac{\partial V_\theta}{\partial \theta}} + \frac{\partial v'_\theta}{\partial \theta} \right) + \cancel{\frac{\partial V_x}{\partial x}} + \frac{\partial v'_x}{\partial x} \right) + \\
&\rho'\left(\frac{1}{r}\left(\cancel{\frac{\partial (r V_r)}{\partial r}} + \cancel{\frac{\partial (r(v'_r)}{\partial r}} + \cancel{\frac{\partial V_\theta}{\partial \theta}} + \cancel{\frac{\partial v'_\theta}{\partial \theta}} \right) + \cancel{\frac{\partial V_x}{\partial x}} + \cancel{\frac{\partial v'_x}{\partial x}} = 0 \right)
\end{aligned}
$$

$$
\boxed{\frac{\partial \rho'}{\partial t} + \frac{V_\theta}{r}\frac{\partial \rho'}{\partial \theta} + V_x \frac{\partial \rho'}{\partial x} + \bar{\rho}\left(\frac{1}{r}\left(\frac{\partial r v'_r}{\partial r} + \frac{\partial v'_\theta}{\partial \theta} \right) + \frac{\partial v'_x}{\partial x} \right) = 0}
$$

3.3.1.2 Linearizing the Conservation of Momentum in the r direction

Starting with the mean momentum equation

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_x\frac{\partial v_r}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

Looking at the left hand side first

$$\begin{aligned}
\frac{\partial (V_r + v_r')}{\partial t} + (V_r + v_r')\frac{\partial (V_r + v_r')}{\partial r} + \frac{V_\theta + v_\theta'}{r}\frac{\partial (V_r + v_r')}{\partial \theta} - \frac{(V_\theta + v_\theta')^2}{r} + (V_x + v_x')\frac{\partial (V_r + v_r')}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} \\
\cancel{\frac{\partial V_r}{\partial t}} + \frac{\partial v_r'}{\partial t} + & \\
\cancel{V_r\frac{\partial V_r}{\partial r}} + \cancel{v_r'\frac{\partial V_r}{\partial r}} + \cancel{V_r\frac{\partial v_r'}{\partial r}} + \cancel{v_r'\frac{\partial v_r'}{\partial r}} + & \\
\frac{1}{r}\left(\cancel{V_\theta\frac{\partial V_r}{\partial \theta}} + \cancel{v_\theta'\frac{\partial V_r}{\partial \theta}} + V_\theta\frac{\partial v_r'}{\partial \theta} + \cancel{v_\theta'\frac{\partial v_r'}{\partial \theta}}\right) - & \\
\frac{1}{r}\left(\cancel{V_\theta^2} + 2V_\theta v_\theta' + \cancel{v_\theta'^2}\right) + & \\
\cancel{V_x\frac{\partial V_r}{\partial x}} + \cancel{v_x'\frac{\partial V_r}{\partial x}} + V_x\frac{\partial v_r'}{\partial x} + \cancel{v_x'\frac{\partial v_r'}{\partial x}} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} \\
\frac{\partial v_r'}{\partial t} + V_r\frac{\partial v_r'}{\partial r} + \frac{V_\theta}{r}\frac{\partial v_r'}{\partial \theta} - \frac{2V_\theta v_\theta'}{r} + V_x\frac{\partial v_r'}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial r}
\end{aligned}$$

Now looking at the right side, Expanding the $1/\rho$ using a Taylor series approximation

$$
\begin{aligned}
\frac{1}{\bar\rho+\rho'} &= \frac{1}{\bar\rho} + \left(\frac{1}{\bar\rho+\rho'} - \frac{1}{\rho}\right) \\
&= \frac{1}{\bar\rho} + \left(\frac{\bar\rho}{\bar\rho(\bar\rho+\rho')} - \frac{1}{\rho}\frac{\bar\rho+\rho'}{\bar\rho+\rho'}\right) \\
&= \frac{1}{\bar\rho} - \left(\frac{\bar\rho-\bar\rho+\rho'}{\bar\rho(\bar\rho+\rho')}\right) \\
&= \frac{1}{\bar\rho} - \frac{\rho'}{\bar\rho} \qquad \underbrace{\left(\frac{1}{\bar\rho+\rho}\right)}_{\text{This is what we're solving for!}} \\
&= \frac{1}{\bar\rho} - \frac{\rho'}{\bar\rho}\underbrace{\left[\frac{1}{\bar\rho} + \left(\frac{1}{\bar\rho+\rho'} - \frac{1}{\rho}\right)\right]}_{\text{This is from step 1}} \\
&= \frac{1}{\bar\rho} - \frac{\rho'}{\bar\rho^2} + \qquad \underbrace{\left[\left(\frac{\rho'}{\bar\rho}\right)^2 \frac{1}{\bar\rho+\rho'}\right]}_{\text{These are higher order terms that will go to }\infty}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\rho}\frac{\partial p}{\partial r} &= \left(-\frac{1}{\bar\rho} + \frac{\rho'}{\bar\rho^2}\right)\left(\frac{\partial\bar p + p'}{\partial r}\right) \\
\frac{1}{\rho}\frac{\partial p}{\partial r} &= -\cancel{\frac{1}{\bar\rho}\frac{\partial\bar p}{\partial r}} - \frac{1}{\bar\rho}\frac{\partial p'}{\partial r} + \frac{\rho'}{\bar\rho^2}\frac{\partial\bar p}{\partial r} + \cancel{\frac{\rho'}{\bar\rho^2}\frac{\partial p'}{\partial r}} \\
\frac{1}{\rho}\frac{\partial p}{\partial r} &= -\frac{1}{\bar\rho}\frac{\partial p'}{\partial r} + \frac{\rho'}{\bar\rho^2}\frac{\partial\bar p}{\partial r}
\end{aligned}
$$

$$
\boxed{\frac{\partial v_r'}{\partial t} + \frac{V_\theta}{r}\frac{\partial v_r'}{\partial\theta} - \frac{2V_\theta v_\theta'}{r} + V_x\frac{\partial v_r'}{\partial x} = \frac{1}{\bar\rho}\frac{\partial p'}{\partial r} + \frac{\rho'}{\bar\rho^2}\frac{\partial\bar p}{\partial r}}
$$

3.3.1.3 Linearizing the Conservation of Momentum in the θ direction

Starting with the mean momentum equation

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_x \frac{\partial v_\theta}{\partial x} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

Looking at the left hand side first

$$\begin{aligned}
&\frac{\partial (V_\theta + v'_\theta)}{\partial t} + (V_r + v'_r)\frac{\partial (V_\theta + v'_\theta)}{\partial r} + \\
&\frac{V_\theta + v'_\theta}{r}\frac{\partial (V_\theta + v'_\theta)}{\partial \theta} + \frac{(V_r + v'_r)(V_\theta + v'_\theta)}{r} + (V_x + v'_x)\frac{\partial (V_\theta + v'_\theta)}{\partial x} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \\
&\cancel{\frac{\partial V_\theta}{\partial t}} + \frac{\partial v'_\theta}{\partial t} + \\
&\cancel{V_r \frac{\partial V_\theta}{\partial r}} + \underbrace{v'_r \frac{\partial V_\theta}{\partial r}}_{v'_r = 0} + \cancel{V_r \frac{\partial v'_\theta}{\partial r}} + \cancel{v'_r \frac{\partial v'_\theta}{\partial r}} + \\
&\frac{1}{r}\left(\cancel{V_\theta \frac{\partial V_\theta}{\partial \theta}} + \cancel{v'_\theta \frac{\partial V_\theta}{\partial \theta}} + V_\theta \frac{\partial v'_\theta}{\partial \theta} + \cancel{v'_\theta \frac{\partial v'_\theta}{\partial \theta}}\right) + \\
&\frac{1}{r}\left(\cancel{V_r V_\theta} + v'_r V_\theta + \cancel{V_r v'_\theta} + \cancel{v'_r v'_\theta}\right) + \\
&\cancel{V_x \frac{\partial V_\theta}{\partial x}} + \cancel{v'_x \frac{\partial V_\theta}{\partial x}} + V_x \frac{\partial v'_\theta}{\partial x} + \cancel{v'_x \frac{\partial v'_\theta}{\partial x}} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \\
&\frac{\partial v'_\theta}{\partial t} + v'_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial v'_\theta}{\partial \theta} + \frac{v'_r V_\theta}{r} + V_x \frac{\partial v'_r}{\partial x} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}
\end{aligned}$$

Now looking at the right side, Expanding the $1/\rho$ using a Taylor series approximation

$$\begin{aligned}
-\frac{1}{\rho r}\frac{\partial p}{\partial \theta} &= \left(-\frac{1}{\bar{\rho}} + \frac{\rho'}{\bar{\rho}^2}\right)\left(\frac{\partial \bar{p} + p'}{\partial \theta}\right) \\
\frac{1}{\rho r}\frac{\partial p}{\partial \theta} &= -\cancel{\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial \theta}} - \frac{1}{\bar{\rho} r}\frac{\partial p'}{\partial \theta} + \cancel{\frac{\rho'}{\bar{\rho}^2 r}\frac{\partial \bar{p}}{\partial \theta}} + \cancel{\frac{\rho'}{\bar{\rho}^2 r}\frac{\partial p'}{\partial \theta}} \\
\frac{1}{\rho r}\frac{\partial p}{\partial \theta} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial \theta}
\end{aligned}$$

$$\boxed{\frac{\partial v'_\theta}{\partial t} + v'_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial v'_\theta}{\partial \theta} + \frac{v'_r V_\theta}{r} + V_x \frac{\partial v'_r}{\partial x} = -\frac{1}{\bar{\rho} r} \frac{\partial p'}{\partial \theta}}$$

3.3.1.4 Linearizing the Conservation of Momentum in the x direction

Starting with the mean momentum equation

$$\frac{\partial v_x}{\partial t} + v_r \frac{\partial v_x}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_x}{\partial \theta} + v_x \frac{\partial v_x}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial (V_x + v_x')}{\partial t} + (V_r + v_r')\frac{\partial (V_x + v_x')}{\partial r} + \frac{V_\theta + v_\theta'}{r}\frac{\partial (V_x + v_x')}{\partial \theta} + (V_x + v_x')\frac{\partial (V_x + v_x')}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\cancel{\frac{\partial V_x}{\partial t}} + \frac{\partial v_x'}{\partial t} +$$

$$\cancel{V_r\frac{\partial V_x}{\partial r}} + v_r'\frac{\partial V_x}{\partial r} + \cancel{V_r\frac{\partial v_x'}{\partial r}} + \cancel{v_r'\frac{\partial v_x'}{\partial r}} +$$

$$\frac{1}{r}\left(\cancel{V_\theta\frac{\partial V_x}{\partial \theta}} + \cancel{v_\theta'\frac{\partial V_x}{\partial \theta}} + V_\theta\frac{\partial v_x'}{\partial \theta} + \cancel{v_\theta'\frac{\partial v_x'}{\partial \theta}}\right) +$$

$$\cancel{V_x\frac{\partial V_x}{\partial x}} + \cancel{v_x'\frac{\partial V_x}{\partial x}} + V_x\frac{\partial v_x'}{\partial x} + \cancel{v_x'\frac{\partial v_x'}{\partial x}} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\boxed{\frac{\partial v_x'}{\partial t} + v_r'\frac{\partial V_x}{\partial r} + \frac{V_\theta}{r}\frac{\partial v_x'}{\partial \theta} + V_x\frac{\partial v_x'}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}}$$

$$\rightarrow -\frac{1}{\rho}\frac{\partial p}{\partial x} = \left(-\frac{1}{\bar{\rho}} + \frac{\rho'}{\bar{\rho}^2}\right)\left(\frac{\partial \bar{p} + p'}{\partial x}\right)$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\cancel{\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x}} - \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} + \cancel{\frac{\rho'}{\bar{\rho}^2 r}\frac{\partial \bar{p}}{\partial x}} + \cancel{\frac{\rho'}{\bar{\rho}^2 r}\frac{\partial p'}{\partial x}}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}$$

$$\boxed{\frac{\partial v_x'}{\partial t} + v_r'\frac{\partial V_x}{\partial r} + \frac{V_\theta}{r}\frac{\partial v_x'}{\partial \theta} + V_x\frac{\partial v_x'}{\partial x} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}}$$

3.3.1.5 Linearizing the Energy Equation

$$\frac{\partial p}{\partial t} + v_r\frac{\partial p}{\partial r} + \frac{v_\theta}{r}\frac{\partial p}{\partial x} + v_x\frac{\partial p}{\partial x} + \gamma p\left(\frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_x}{\partial x}\right) = 0$$

$$\begin{aligned}\frac{\partial(\bar{P}+P')}{\partial t} + (V_r+v_r')\frac{\partial(\bar{P}+P')}{\partial r} + \frac{(V_\theta+v_\theta')}{r}\frac{\partial(\bar{P}+p')}{\partial x} + (V_x+v_x')\frac{\partial(\bar{P}+P')}{\partial x} + ...\\ \gamma(\bar{P}+P')\left(\frac{1}{r}\frac{\partial r(V_r+v_r')}{\partial r} + \frac{1}{r}\frac{\partial(V_\theta+v_\theta')}{\partial \theta} + \frac{\partial(V_x+v_x')}{\partial x}\right) = 0\end{aligned}$$

$$\begin{aligned}\frac{\partial \bar{P}}{\partial t} + \frac{\partial P'}{\partial t} +\\ V_r\frac{\partial \bar{P}}{\partial r} + V_r\frac{\partial P'}{\partial r} + V_r'\frac{\partial \bar{P}}{\partial r} + V_r'\frac{\partial P'}{\partial r} +\\ \frac{V_\theta}{r}\frac{\partial \bar{P}}{\partial x} + \frac{V_\theta}{r}\frac{\partial P'}{\partial x} + \frac{v_\theta'}{r}\frac{\partial \bar{P}}{\partial x} + \frac{v_\theta'}{r}\frac{\partial P'}{\partial x} +\\ V_x\frac{\partial \bar{P}}{\partial x} + V_x\frac{\partial P'}{\partial x} + v_x'\frac{\partial \bar{P}}{\partial x} + v_x'\frac{\partial P'}{\partial x} +\\ \gamma\bar{P}\left(\frac{1}{r}\frac{\partial r V_r}{\partial r} + \frac{1}{r}\frac{\partial r v_r'}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial v_\theta'}{\partial \theta} + \frac{\partial V_x}{\partial x} + \frac{\partial v_x'}{\partial x}\right)\\ \gamma P'\left(\frac{1}{r}\frac{\partial r V_r}{\partial r} + \frac{1}{r}\frac{\partial r v_r'}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial v_\theta'}{\partial \theta} + \frac{\partial V_x}{\partial x} + \frac{\partial v_x'}{\partial x}\right)\end{aligned}$$

$$\boxed{\frac{\partial p'}{\partial t} + v_r'\frac{\partial P}{\partial r} + \frac{V_\theta}{r}\frac{\partial p'}{\partial \theta} + V_x\frac{\partial p'}{x} + \gamma P\left(\frac{1}{r}\frac{\partial(r v_r')}{\partial r} + \frac{1}{r}\frac{\partial v_\theta'}{\partial \theta} + \frac{\partial v_x'}{\partial x}\right) = 0}$$

The linearized Euler equations are,

$$
\begin{aligned}
\frac{\partial \rho'}{\partial t} + \frac{V_\theta}{r}\frac{\partial \rho'}{\partial \theta} + V_x\frac{\partial \rho'}{\partial x} + \bar{\rho}\left(\frac{1}{r}\left(\frac{\partial r v_r'}{\partial r} + \frac{\partial v_\theta'}{\partial \theta}\right) + \frac{\partial v_x'}{\partial x}\right) &= 0\\
\frac{\partial v_r'}{\partial t} + \frac{V_\theta}{r}\frac{\partial v_\theta'}{\partial \theta} - \frac{2V_\theta v_\theta'}{r} + V_x\frac{\partial v_r'}{\partial x} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial r} + \frac{\rho'}{\bar{\rho}^2}\frac{\partial \bar{p}}{\partial r}\\
\frac{\partial v_\theta'}{\partial t} + v_r'\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial v_\theta'}{\partial \theta} + \frac{v_r' V_\theta}{r} + V_x\frac{\partial v_r'}{\partial x} &= -\frac{1}{\bar{\rho} r}\frac{\partial p'}{\partial \theta}\\
\frac{\partial v_x'}{\partial t} + v_r'\frac{\partial V_x}{\partial r} + \frac{V_\theta}{r}\frac{\partial v_x'}{\partial \theta} + V_x\frac{\partial v_x'}{\partial x} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}\\
\frac{\partial p'}{\partial t} + v_r'\frac{\partial P}{\partial r} + \frac{V_\theta}{r}\frac{\partial p'}{\partial \theta} + V_x\frac{\partial p'}{x} + \gamma P\left(\frac{1}{r}\frac{\partial (r v_r')}{\partial r} + \frac{1}{r}\frac{\partial v_\theta'}{\partial \theta} + \frac{\partial v_x'}{\partial x}\right) &= 0
\end{aligned}
$$

Recalling:

$$
\frac{\partial P}{\partial r} = \frac{\bar{\rho} V_\theta^2}{r}
$$

$$
\gamma P = \bar{\rho} A^2
$$

$$
\rho' = \frac{1}{A^2} p'
$$

We can rearrange the equations to reflect Equations 2.33-2.36. Note that the momentum equation in the θ and x directions remain unchanged. The term $\frac{\partial (r v_r')}{\partial r} = \frac{\partial (r)}{\partial r} v_r' + \frac{\partial v_r'}{\partial r} r$ in the Energy equation

$$
\begin{aligned}
\frac{1}{\bar{\rho} A^2}\left(\frac{\partial p'}{\partial t} + \frac{V_\theta}{r}\frac{\partial p'}{\partial \theta} + V_x\frac{\partial p'}{\partial x}\right) + \frac{V_\theta^2}{A^2 r} v_r' + \frac{\partial v_r'}{\partial r} + \frac{v_r'}{r} + \frac{1}{r}\frac{\partial v_\theta'}{\partial \theta} + \frac{\partial v_x'}{\partial x} &= 0\\
\frac{\partial v_r'}{\partial t} + \frac{V_\theta}{r}\frac{\partial v_r'}{\partial \theta} - \frac{2V_\theta v_\theta'}{r} + V_x\frac{\partial v_r'}{\partial x} &= \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial r} + \frac{V_\theta}{\bar{\rho} r A^2} p'\\
\frac{\partial v_\theta'}{\partial t} + v_r'\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial v_\theta'}{\partial \theta} + \frac{v_r' V_\theta}{r} + V_x\frac{\partial v_\theta'}{\partial x} &= -\frac{1}{\bar{\rho} r}\frac{\partial p'}{\partial \theta}\\
\frac{\partial v_x'}{\partial t} + v_r'\frac{\partial V_x}{\partial r} + \frac{V_\theta}{r}\frac{\partial v_x'}{\partial \theta} + V_x\frac{\partial v_x'}{\partial x} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}
\end{aligned}
$$

3.4 Substituting Pertubation Variables

One key assumption is that the perturbation quantites, $\tilde{p}$, $\tilde{v}_r$,$\tilde{v}_\theta$, and $\tilde{v}_x$, are all exponential and that they are solely a function of radius,

$$v_r' = v_r(r)e^{i(k_x x+m\theta-\omega t)}$$

$$v_\theta' = v_\theta(r)e^{i(k_x x+m\theta-\omega t)}$$

$$v_x' = v_x(r)e^{i(k_x x+m\theta-\omega t)}$$

$$p' = p(r)e^{i(k_x x+m\theta-\omega t)}$$

Substituting the quantities into the linearized equations will give us the final governing equations. Starting with the Conservation of Momentum in the r direction,

$$\frac{\partial v_r'}{\partial t} + \frac{V_\theta}{r}\frac{\partial v_r'}{\partial \theta} - \frac{2V_\theta}{r}v_r' + V_x\frac{\partial v_r'}{\partial x} = \frac{\partial p'}{\partial r} + \frac{\rho'}{\bar{\rho}^2}\frac{\partial P}{\partial r}$$

Looking at the left hand side (LHS) of the equation, the derivatives are:

$$\frac{\partial v_r'}{\partial t} = \underbrace{\frac{\partial v_r(r)}{\partial t}}_{0} e^{i(k_x x + m\theta - \omega t)} + v_r(r)\left(-i\omega e^{i(k_x x + m\theta - \omega t)}\right)$$

$$\frac{\partial v_r'}{\partial \theta} = \underbrace{\frac{\partial v_r(r)}{\partial \theta}}_{0} e^{i(k_x x + m\theta - \omega t)} + v_r(r)\left(ime^{i(k_x x + m\theta - \omega t)}\right)$$

$$\frac{\partial v_r'}{\partial x} = \underbrace{\frac{\partial v_r(r)}{\partial x}}_{0} e^{i(k_x x + m\theta - \omega t)} + v_r(r)\left(ik_x e^{i(k_x x + m\theta - \omega t)}\right)$$

Similarly for the right hand side (RHS),

$$\frac{\partial p'}{\partial r} = \frac{\partial P(r)}{\partial r}e^{i(k_x x + m\theta - \omega t)} + P(r)\underbrace{\frac{\partial}{\partial r}e^{i(k_x x + m\theta - \omega t)}}_{0}$$

Recalling $p'/\rho' = A^2 \rightarrow \rho' = \frac{1}{A^2}p'$ $\frac{\partial \bar{P}}{\partial r} = \frac{\bar{\rho}v_\theta^2}{r}$

$$\frac{\rho'}{\bar{\rho}^2}\frac{\partial \bar{P}}{\partial r} = \frac{V_\theta^2}{A^2}\frac{1}{\bar{\rho}r}p'$$

After substituting and canceling common terms,

$$v_r\left(-i\omega\cancel{e^{i(k_x x + m\theta - \omega t)}}\right) + \frac{V_\theta}{r}v_r\left(im\cancel{e^{i(k_x x + m\theta - \omega t)}}\right) - \frac{2V_\theta}{r}v_r\cancel{e^{i(k_x x + m\theta - \omega t)}} + V_x\left(v_r\left(ik_x\cancel{e^{i(k_x x + m\theta - \omega t)}}\right)\right)$$
$$= \left(-\frac{1}{\bar{\rho}}\frac{\partial P(r)}{\partial r}\cancel{e^{i(k_x x + m\theta - \omega t)}} + \frac{V_\theta^2}{A^2}\frac{1}{\bar{\rho}r}P(r)\cancel{e^{i(k_x x + m\theta - \omega t)}}\right)$$

$$\boxed{\left(-i\omega + \frac{imV_\theta}{r} + ik_xV_x\right)v_r - \frac{2V_\theta}{r}v_\theta = -\frac{1}{\bar{\rho}}\frac{\partial P}{\partial r} + \frac{V_{\theta^2}}{A^2}\frac{1}{\bar{\rho}r}p}$$

Continuing with conservation of momentum in the θ direction,

$$\frac{\partial v'_\theta}{\partial t} + v'_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial v'_\theta}{\partial \theta} + \frac{v'_rV_\theta}{r} + V_x\frac{\partial v'_\theta}{\partial x} = -\frac{1}{\bar{\rho}r}\frac{\partial p'}{\partial \theta}$$

$$\frac{\partial v'_\theta}{\partial t} = \underbrace{\frac{\partial v_\theta(r)}{\partial t}}_{0}e^{i(k_xx+m\theta-\omega t)} + v_\theta(r)\left(-i\omega e^{i(k_xx+m\theta-\omega t)}\right)$$

$$\frac{\partial v'_\theta}{\partial \theta} = \underbrace{\frac{\partial v'_\theta(r)}{\partial \theta}}_{0}e^{i(k_xx+m\theta-\omega t)} + v_\theta(r)\left(ime^{i(k_xx+m\theta-\omega t)}\right)$$

$$\frac{\partial v'_\theta}{\partial x} = \underbrace{\frac{\partial v_\theta(r)}{\partial x}}_{0}e^{i(k_xx+m\theta-\omega t)} + v_\theta(r)\left(ik_xe^{i(k_xx+m\theta-\omega t)}\right)$$

$$\frac{\partial p'}{\partial \theta} = \underbrace{\frac{\partial P(r)}{\partial \theta}}_{0}e^{i(k_xx+m\theta-\omega t)} + P(r)\underbrace{\frac{\partial}{\partial \theta}e^{i(k_xx+m\theta-\omega t)}}_{me^{i(k_xx+m\theta-\omega t)}}$$

After substituting and canceling common terms

$$\boxed{\left(-i\omega + \frac{imV_\theta}{r} + ik_xV_x\right)v_\theta + \left(\frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r}\right)v_\theta = -\frac{m}{\bar{\rho}r}p}$$

Next, the conservation of momentum in the x direction,

$$\frac{\partial v'_x}{\partial t} + v'_r\frac{\partial V_x}{\partial r} + \frac{V_\theta}{r}\frac{\partial v'_x}{\partial \theta} + V_x\frac{\partial v'_x}{\partial x} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v'_x}{\partial t} = \underbrace{\frac{\partial v_x(r)}{\partial t}}_{0}e^{i(k_xx+m\theta-\omega t)} + v_x(r)\left(-i\omega e^{i(k_xx+m\theta-\omega t)}\right)$$

$$\frac{\partial v_x'}{\partial \theta} = \underbrace{\frac{\partial v_x(r)}{\partial \theta}}_{0} e^{i(k_x x + m\theta - \omega t)} + v_x(r) \left(ime^{i(k_x x + m\theta - \omega t)} \right)$$

$$\frac{\partial v_x'}{\partial x} = \frac{\partial v_x(r)}{\partial x} e^{i(k_x x + m\theta - \omega t)} + v_x(r) \left(ik_x e^{i(k_x x + m\theta - \omega t)} \right)$$

$$\frac{\partial p'}{\partial x} = 0 + ik_x p e^{i(k_x x + m\theta - \omega t)}$$

$$\boxed{\left(-i\omega + \frac{imV_\theta}{r} + ik_x V_x \right) v_x + \frac{\partial V_x}{\partial r} v_r = -\frac{ik_x}{\bar{\rho}} p}$$

Continuing with the Conservation of Energy,

$$\frac{1}{\bar{\rho} A^2} \left(\frac{\partial p'}{\partial t} + \frac{V_\theta}{r} \frac{\partial p'}{\partial \theta} + V_x \frac{\partial p'}{\partial x} \right) + \frac{V_\theta^2}{A^2 r} v_r' + \frac{\partial v_r'}{\partial r} + \frac{1}{r} \frac{\partial v_\theta'}{\partial \theta} + \frac{\partial v_x'}{\partial x} = 0$$

Left hand side (LHS) derivatives:

$$\frac{\partial p'}{\partial t} = \underbrace{\frac{\partial p(r)}{\partial t}}_{0} e^{i(k_x x + m\theta - \omega t)} + p(r) \left(-i\omega e^{i(k_x x + m\theta - \omega t)} \right)$$

$$\frac{\partial p'}{\partial \theta} = \underbrace{\frac{\partial p(r)}{\partial \theta}}_{0} e^{i(k_x x + m\theta - \omega t)} + p(r) \left(ime^{i(k_x x + m\theta - \omega t)} \right)$$

$$\frac{\partial p'}{\partial x} = \underbrace{\frac{\partial p(r)}{\partial x}}_{0} e^{i(k_x x + m\theta - \omega t)} + p(r) \left(ik_x e^{i(k_x x + m\theta - \omega t)} \right)$$

$$\frac{\partial v_r'}{\partial r} = \frac{\partial v_r(r)}{\partial r} e^{i(k_x x + m\theta - \omega t)} + v_r(r) \underbrace{\frac{\partial}{\partial r} \left(e^{i(k_x x + m\theta - \omega t)} \right)}_{0}$$

$$\frac{\partial v_\theta'}{\partial \theta} = \frac{\partial v_\theta(r)}{\partial \theta} e^{i(k_x x + m\theta - \omega t)} + v_\theta(r) \left(ime^{i(k_x x + m\theta - \omega t)} \right)$$

$$\frac{\partial v_x'}{\partial x} = \underbrace{\frac{\partial v_x(r)}{\partial x}}_{0} e^{i(k_x x + m\theta - \omega t)} + v_x(r)\left(ik_x e^{i(k_x x + m\theta - \omega t)}\right)$$

After substituting and canceling common terms,

$$\frac{1}{\bar{\rho}A^2}\left(p(r)\left(-i\omega e^{i(k_x x + m\theta - \omega t)}\right) + \frac{V_\theta}{r}p(r)\left(ime^{i(k_x x + m\theta - \omega t)}\right) + V_x p(r)\left(ik_x e^{i(k_x x + m\theta - \omega t)}\right)\right) + \frac{V_\theta^2}{A^2 r}v_r' + \frac{\partial v_r(r)}{\partial r}e^{i(k_x x + m\theta - \omega t)} + \frac{1}{r}\left(v_\theta(r)\left(ime^{i(k_x x + m\theta - \omega t)}\right)\right) + v_x(r)\left(ik_x e^{i(k_x x + m\theta - \omega t)}\right) = 0$$

$$\boxed{\frac{1}{\bar{\rho}A^2}\left(-i\omega + \frac{imV_\theta}{r} + ik_x V_x +\right)p(r) + \frac{V_\theta^2}{A^2 r}v_r + \frac{v_r}{r} + \frac{\partial v_r(r)}{\partial r} + \frac{im}{r}v_\theta(r) + ik_x v_x(r) = 0}$$

The Linearized Euler equations now become

$$\begin{aligned}
r\text{-direction: } & i\left(-\omega + \frac{m}{r} + k_x V_x\right)v_r - \frac{2\bar{v}_\theta}{r}v_\theta = -\frac{1}{\bar{\rho}}\frac{\partial P}{\partial r} + \frac{V_{\theta^2}}{A^2}\frac{1}{\bar{\rho}r}p \\
\theta\text{-direction: } & i\left(-\omega + \frac{m}{r} + k_x V_x\right)v_\theta + \left(\frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r}\right)v_\theta = -\frac{m}{\bar{\rho}r}p \\
x\text{-direction: } & i\left(-\omega + \frac{mV_\theta}{r} + k_x Vx\right)v_x + \frac{\partial V_x}{\partial r}v_r = -\frac{ik_x}{\bar{\rho}}p \\
\text{Energy: } & \frac{1}{\bar{\rho}A^2}\left(-i\omega + \frac{imV_\theta}{r} + ik_x V_x +\right)p(r) + \frac{V_\theta^2}{A^2 r}v_r + \frac{v_r}{r} + \frac{\partial v_r(r)}{\partial r} + \frac{im}{r}v_\theta(r) + ik_x v_x(r) \quad = 0
\end{aligned}$$

3.5 Non-Dimensionalization

Defining

$$r_T = r_{max}$$

$$A_T = A(r_{max})$$

$$k = \frac{\omega r_T}{A_T}$$

$$\bar{\gamma} = k_x r_T$$

$$\tilde{r} = \frac{r}{r_T}$$

$$\frac{\partial}{\partial r} = \frac{\partial \tilde{r}}{\partial r}\frac{\partial}{\partial \tilde{r}} = \frac{1}{r_T}\frac{\partial}{\partial \tilde{r}}$$

$$V_\theta = M_\theta A$$

$$V_x = M_x A$$

$$\tilde{A} = \frac{A}{A_T}$$

$$v_x = \tilde{v}_x A$$

$$v_r = \tilde{v}_r A$$

$$v_\theta = \tilde{v}_\theta A$$

$$p = \tilde{p}\bar{\rho}A^2$$

$$r\text{-direction: } i\left(-\omega+\frac{mV_\theta}{r}+k_xV_x\right)v_r-\frac{2\bar{v}_\theta}{r}v_\theta=-\frac{1}{\bar{\rho}}\frac{\partial P}{\partial r}+\frac{V_{\theta^2}}{A^2}\frac{1}{\bar{\rho}r}p$$

$$\theta\text{-direction: } i\left(-\omega+\frac{m}{r}+k_xV_x\right)v_\theta+\left(\frac{V_\theta}{r}+\frac{\partial V_\theta}{\partial r}\right)v_\theta=-\frac{m}{\bar{\rho}r}p$$

$$x\text{-direction: } i\left(-\omega+\frac{mV_\theta}{r}+k_xVx\right)v_x+\frac{\partial V_x}{\partial r}v_r=-\frac{ik_x}{\bar{\rho}}p$$

$$\text{Energy: } \frac{1}{\bar{\rho}A^2}i\left(-\omega+\frac{mV_\theta}{r}+k_xV_x\right)p(r)+\frac{V_\theta^2}{A^2r}v_r+\frac{v_r}{r}+\frac{\partial v_r(r)}{\partial r}+\frac{im}{r}v_\theta(r)+ik_xv_x(r)=0$$

Substituting the non dimensional quantities, and noting r_T and A^2 in each term, the radial momentum equation becomes,

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{v}_r-\frac{2M_\theta\tilde{v}_\theta}{\tilde{r}}=-\frac{1}{\bar{\rho}A^2}\frac{\partial\tilde{p}\bar{\rho}A^2}{\partial\tilde{r}}+M_\theta\frac{\tilde{p}}{\tilde{r}}}$$

Similarly for the $\theta$and x directions:

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{v}_\theta+\left(\frac{M_\theta}{\tilde{r}}+\frac{1}{A}\frac{\partial M_\theta A}{\partial\tilde{r}}\right)\tilde{v}_r=\frac{im}{\tilde{r}}\tilde{P}}$$

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{v}_x+\frac{1}{A}\frac{\partial M_xA}{\partial\tilde{r}}\tilde{v}_r=-i\bar{\gamma}\tilde{P}}$$

and for the Energy equation:

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{p}+\frac{M_\theta^2}{\tilde{r}}\tilde{v}_r+\frac{1}{A}\frac{\partial(\tilde{v}_rA)}{\partial\tilde{r}}+\frac{\tilde{v}_r}{\tilde{r}}+\frac{im}{\tilde{r}}\tilde{v}_\theta+i\bar{\gamma}\tilde{v}_x=0}$$

Expanding mean derivatives (using product rule) $\frac{\partial\tilde{p}\bar{\rho}A^2}{\partial\tilde{r}}$

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{v}_r-\frac{2M_\theta\tilde{v}_\theta}{\tilde{r}}=-\left(\frac{\partial\tilde{p}}{\partial\tilde{r}}+\frac{\tilde{p}}{\bar{\rho}A^2}\frac{\partial\bar{\rho}A^2}{\partial\tilde{r}}\right)+M_\theta\frac{\tilde{p}}{\tilde{r}}}$$

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{v}_\theta+\left(\frac{M_\theta}{\tilde{r}}+\frac{1}{A}\frac{\partial M_\theta A}{\partial\tilde{r}}\right)\tilde{v}_r=\frac{im}{\tilde{r}}\tilde{P}}$$

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{v}_x+\frac{1}{A}\frac{\partial M_x A}{\partial\tilde{r}}\tilde{v}_r=-i\bar{\gamma}\tilde{P}}$$

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{p}+\frac{M_\theta^2}{\tilde{r}}\tilde{v}_r+\frac{\partial\tilde{v}_r}{\partial\tilde{r}}+\frac{1}{A}\frac{\partial A}{\partial\tilde{r}}v_r+\frac{\tilde{v}_r}{\tilde{r}}+\frac{im}{\tilde{r}}\tilde{v}_\theta+i\bar{\gamma}\tilde{v}_x=0}$$

$$\frac{1}{A}\frac{\partial A}{\partial \tilde{r}}$$

Recall, $\partial/\partial r = (1/r_T)(\partial/\partial\tilde{r}$,

$$\begin{aligned}\frac{1}{A}\frac{\partial A}{\partial \tilde{r}} &= r_T\left(\frac{1}{A}\frac{\partial A}{\partial r}\right)\\ &= \frac{r_T}{A^2}\left(A\frac{\partial A}{\partial r}\right)\end{aligned}$$

Using the trick, $\frac{\partial}{\partial r}\left(\frac{A^2}{2}\right) = A\frac{\partial A}{\partial r}$

$$\begin{aligned}&= \frac{r_T}{A^2}\left(\frac{\partial}{\partial r}\left(\frac{A^2}{2}\right)\right)\\ &= \frac{r_T}{2A^2}\frac{\partial A^2}{\partial r}\end{aligned}$$

Using the definition derived earlier (Needs eqn reference) $\frac{\partial A^2}{\partial r} = \frac{\gamma-1}{2}\frac{v_\theta^2}{r}$

$$\begin{aligned}&= \frac{r_T}{2A^2}\frac{\gamma-1}{2}\frac{v_\theta^2}{r}\\ &= \frac{\gamma-1}{2}\frac{M_\theta^2}{\tilde{r}}\end{aligned}$$

$$\begin{aligned}\frac{\partial \rho A^2}{\partial \tilde{r}} &= \gamma\frac{\partial p}{\partial \tilde{r}}\\ &= \gamma\frac{\rho v_\theta^2}{\tilde{r}}\\ &= r_T\gamma\frac{\rho v_\theta^2}{r}\end{aligned}$$

$$\frac{1}{\rho A^2}\frac{\partial \rho A^2}{\partial r} = \frac{\gamma M_\theta^2}{\tilde{r}}$$

Substituting in yields ,

$$\boxed{i\left[-\frac{k}{\tilde{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma}M_x\right]\tilde{v}_r - \frac{2M_\theta\tilde{v}_\theta}{\tilde{r}} = -\frac{\partial\tilde{p}}{\partial\tilde{r}} - (\gamma-1)\frac{\gamma M_\theta}{\tilde{r}}\tilde{p}}$$

$$\boxed{i\left[-\frac{k}{\tilde{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma}M_x\right]\tilde{v}_\theta + \left(\frac{M_\theta}{\tilde{r}} + \frac{1}{A}\frac{\partial M_\theta A}{\partial\tilde{r}}\right)\tilde{v}_r = \frac{im}{\tilde{r}}\tilde{p}}$$

$$\boxed{i\left[-\frac{k}{\tilde{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma}M_x\right]\tilde{v}_x + \frac{1}{A}\frac{\partial M_x A}{\partial\tilde{r}}\tilde{v}_r = -i\bar{\gamma}\tilde{p}}$$

$$\boxed{i\left[-\frac{k}{\tilde{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma}M_x\right]\tilde{p} + \frac{M_\theta^2}{\tilde{r}}\tilde{v}_r + \frac{\partial\tilde{v}_r}{\partial\tilde{r}} + \frac{1}{A}\frac{\partial A}{\partial\tilde{r}}v_r + \frac{\tilde{v}_r}{\tilde{r}} + \frac{im}{\tilde{r}}\tilde{v}_\theta + i\bar{\gamma}\tilde{v}_x = 0}$$

Defining, $\lambda = -i\bar{\gamma}$

and defining

$$\{\bar{x}\} = \begin{Bmatrix} \tilde{v}_r \\ \tilde{v}_\theta \\ \tilde{v}_x \\ \tilde{p} \end{Bmatrix}$$

The governing equations can be written in the form of $[A]x - \lambda[B]x$

$$\begin{bmatrix} -i\left(\frac{k}{\bar{A}} - \frac{mM_\theta}{\tilde{r}}\right) - \lambda M_x & -\frac{2M_\theta}{\tilde{r}} & 0 & \frac{\partial}{\partial \tilde{r}} + \frac{\gamma-1}{\tilde{r}} \\ \frac{M_\theta}{\tilde{r}} + \frac{\partial M_\theta}{\partial \tilde{r}} + \left(\frac{\gamma-1}{2}\right)\frac{M_\theta^3}{\tilde{r}} & -i\left(\frac{k}{\bar{A}} - \frac{mM_\theta}{\tilde{r}}\right) - \lambda M_x & 0 & \frac{im}{\tilde{r}} \\ \frac{\partial M_x}{\partial \tilde{r}} + \left(\frac{\gamma-1}{2}\frac{M_x M_\theta^2}{\tilde{r}}\right) & 0 & -i\left(\frac{k}{\bar{A}} - \frac{mM_\theta}{\tilde{r}}\right) - \lambda M_x & -\lambda \\ \frac{\partial}{\partial \tilde{r}} + \frac{\gamma+1}{2}\frac{M_\theta^2}{\tilde{r}} + \frac{1}{\tilde{r}} & \frac{im}{\tilde{r}} & -\lambda & -i\left(\frac{k}{\bar{A}} - \frac{mM_\theta}{\tilde{r}}\right) - \lambda M_x \end{bmatrix} \bar{x} =$$

Chapter 4

Chapter 4: Numerical Models

4.1 Numerical Integration

4.2 Introduction

The Method of Manufactured Solutions (MMS) is a process for generating an analytical solution for a code that provides the numerical solution for a given domain. The goal of MMS is to establish a manufactured solution that can be used to establish the accuracy of the code within question. For this study, SWIRL, a code used to calculate the radial modes within an infinitely long duct is being validated through code verification. SWIRL accepts a given mean flow and uses numerical integration to obtain the speed of sound. The integration technique is found to be the composite trapezoidal rule through asymptotic error analysis.

For SWIRL, the absolute bare minimum requirement is to define the corresponding flow components for the domain of interest. SWIRL assumes no flow in the radial direction, leaving only two other components, axial and tangential for a 3D cylindrical domain. Since SWIRL is also non dimensionalized, the mean flow components are defined using the Mach number. SWIRL uses the tangential mach number to obtain the speed of sound using numerical integration. The speed of sound is then used to find the rest of the primative variables for the given flow.

4.3 Methods

SWIRL is a linearized Euler equations of motion code that calculates the axial wavenumber and radial mode shapes from small unsteady disturbances in a mean flow. The mean flow varies along the axial and tangetial directons as a function of radius. The flow domain can either be a circular or annular duct, with or without acoustic liner. SWIRL was originally written by Kousen [insert ref].

The SWIRL code requires two mean flow parameters as a function of radius, M_x , and M_θ. Afterwards, the speed of sound, $\widetilde{A}$ is calculated by integrating M_θ with respect to r. To verify that SWIRL is handling and returning the accompanying mean flow parameters, the error between the mean flow input and output variables are computed. Since the trapezoidal rule is used to numerically integrate M_θ, the discretization error and order of accuracy is computed. Since finite differencing schemes are to be used on the result of this integration, it is crucial to accompany the integration with methods of equal or less order of accuracy. This will be determined by applying another MMS on the eigenproblem which will also have an order of accuracy.

4.3.1 Theory

To relate the speed of sound to a given flow, the radial momentum equation is used. If the flow contains a swirling component,then the primitive variables are nonuniform through the flow, and mean flow assumptions are not valid.

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} - \frac{v_\theta^2}{r} v_x \frac{\partial v_r}{\partial x} = \frac{1}{\rho}\frac{\partial P}{\partial r}$$

To account to for this, the radial momentum is simplified by assuming the flow is steady, the flow has no radial component. In addition, the viscous and body forces are neglected. Then the radial pressure derivative term is set equal to the dynamic pressure term. Seperation of variables is applied.

$$\frac{v_\theta^2}{r} = \frac{1}{\rho}\frac{\partial P}{\partial r}$$
$$P = \int_r^{r_{max}} \frac{\rho V_\theta^2}{r}$$

To show the work, we will start with the dimensional form of the equation and differentiate both sides. Applying separation of variables,

$$\int_r^{r_{max}} \frac{\bar{\rho} v_\theta^2}{r} \partial r = - \int_{P(r)}^{P(r_{max})} \partial p.$$

Since $\tilde{r} = r/r_{max}$,

$$r = \tilde{r} r_{max}.$$

Taking total derivatives (i.e. applying chain rule),

$$dr = d(\tilde{r} r_{max}) = d(\tilde{r}) r_{max},$$

Substituting these back in and evaluating the right hand side,

$$\int_{\tilde{r}}^{1} \frac{\bar{\rho} v_\theta^2}{\tilde{r}} \partial \tilde{r} = P(1) - P(\tilde{r})$$

For reference the minimum value of $\tilde{r}$ is,

$$\sigma = \frac{r_{max}}{r_{min}}$$

For the radial derivative, the definition of the speed of sound is utilized,

$$\frac{\partial A^2}{\partial r} = \frac{\partial}{\partial r} \left(\frac{\gamma P}{\rho} \right).$$

Using the quotient rule, the definition of the speed of sound is extracted,

$$\begin{aligned} &= \frac{\partial P}{\partial r} \frac{\gamma \bar{\rho}}{\bar{\rho}^2} - \left(\frac{\gamma P}{\bar{\rho}^2} \right) \frac{\partial \bar{\rho}}{\partial r} \\ &= \frac{\partial P}{\partial r} \frac{\gamma}{\bar{\rho}} - \left(\frac{A^2}{\bar{\rho}} \right) \frac{\partial \bar{\rho}}{\partial r} \end{aligned}$$

Using isentropic condition $\partial P / A^2 = \partial \rho$,

$$\begin{aligned} &= \frac{\partial P}{\partial r} \frac{\gamma}{\bar{\rho}} - \left(\frac{1}{\bar{\rho}} \right) \frac{\partial P}{\partial r} \\ \frac{\partial A^2}{\partial r} &= \frac{\partial P}{\partial r} \frac{\gamma - 1}{\bar{\rho}} \end{aligned}$$

$$\frac{\bar{\rho}}{\gamma - 1} \frac{\partial A^2}{\partial r} = \frac{\partial P}{\partial r}$$

Going back to the radial momentum equation, and rearranging the terms will simplify the expression. The following terms are defined to start the nondimensionalization.

$$
\begin{aligned}
M_\theta &= \frac{V_\theta}{A} \\
\widetilde{r} &= \frac{r}{r_{max}} \\
\widetilde{A} &= \frac{A}{A_{r,max}} \\
A &= \widetilde{A} A_{r,max} \\
r &= \widetilde{r} r_{max} \\
\frac{\partial}{\partial r} &= \frac{\partial \widetilde{r}}{\partial r} \frac{\partial}{\partial \widetilde{r}} \\
&= \frac{1}{r_{max}} \frac{\partial}{\partial \widetilde{r}}
\end{aligned}
$$

Dividing by A,

$$\frac{M_\theta^2}{r}(\gamma - 1) = \frac{\partial A^2}{\partial r} \frac{1}{A^2}$$

Now there is two options, either find the derivative of $\bar{A}$ or the integral of M_θ with respect to r.

1. Defining non dimensional speed of sound $\tilde{A} = \frac{A(r)}{A(r_{max})}$

$$\int_{r}^{r_{max}} \frac{M_\theta}{r} (\gamma - 1)\, \partial r = ln\left(\frac{1}{\tilde{A}^2}\right)$$
$$= -2ln(\tilde{A})$$
$$\tilde{A}(r) = exp\left[-\int_{r}^{r_{max}} \frac{M_\theta}{r}\frac{(\gamma-1)}{2}\partial r\right]$$
$$\text{replacing r with } \tilde{r} \rightarrow \tilde{A}(r) = exp\left[-\int_{r}^{r_{max}} \frac{M_\theta}{r}\frac{(\gamma-1)}{2}\partial r\right]$$
$$\tilde{A}(\tilde{r}) = exp\left[\left(\frac{1-\gamma}{2}\right)\int_{\tilde{r}}^{1} \frac{M_\theta}{\tilde{r}}\partial \tilde{r}\right]$$

2. Or we can differentiate

Solving for M_θ ,

$$M_\theta^2 = \frac{\partial A^2}{\partial r}\frac{r}{A^2\,(\gamma-1)}$$

Nondimensionalizing and substituting,

$$M_\theta^2\frac{(\gamma-1)}{\widetilde{r}r_{max}} = \frac{1}{(\widetilde{A}A_{r,max})^2}\frac{A_{r,max}^2}{r_{max}}\frac{\partial\widetilde{A}^2}{\partial\widetilde{r}}$$
$$M_\theta^2\frac{(\gamma-1)}{\widetilde{r}} = \frac{1}{\widetilde{A}^2}\frac{\partial\widetilde{A}^2}{\partial\widetilde{r}}$$
$$M_\theta = \sqrt{\frac{\widetilde{r}}{(\gamma-1)\widetilde{A}^2}\frac{\partial\widetilde{A}^2}{\partial\widetilde{r}}} \tag{4.1}$$

4.3.2 Calculation of Observed Order-of-Accuracy

The numerical scheme used to perform the integration of the tangential velocity will have a theoretical order-of-accuracy. To find the theoretical order-of-accuracy, the discretization error must first be defined. The error, ϵ, is a function of id spacing, Δr

$$\epsilon = \epsilon(\Delta r)$$

The discretization error in the solution should be proportional to $(\Delta r)^{\alpha}$ where $\alpha > 0$ is the theoretical order for the computational method. The error for each grid is expressed as

$$\epsilon_{M_\theta}(\Delta r) = |M_{\theta,analytic} - M_{\theta,calc}|$$

where $M_{\theta,analytic}$ is the tangential mach number that is defined from the speed of sound we also defined and the $M_{\theta,calc}$ is the result from SWIRL. The Δr is to indicate that this is a discretization error for a specific grid spacing. Applying the same concept to to the speed of sound,

If we define this error on various grid sizes and compute ϵ for each grid, the observed order of accuracy can be estimated and compared to the theoretical order of accuracy. For instance, if the numerical soution is second-order accurate and the error is converging to a value, the L2 norm of the error will decrease by a factor of 4 for every halving of the grid cell size.

Since the input variables should remain unchanged (except from minor changes from the Akima interpolation), the error for the axial and tangential mach number should be zero. As for the speed of sound, since we are using an analytic expression for the tangential mach number, we know what the theoretical result would be from the numerical integration technique as shown above. Similarly we define the discretization error for the speed of sound.

$$\epsilon_A(\Delta r) = |A_{analytic} - A_{calc}|$$

For a perfect answer, we expect ϵ to be zero. Since a Taylor series can be used to derive the numerical schemes, we know that the truncation of higher order terms is what indicates the error we expect from using a scheme that is constructed with such truncated Taylor series.

The error at each grid point j is expected to satisfy the following,

$$0 = |A_{analytic}(r_j) - A_{calc}(r_j)|$$

$$\widetilde{A}_{analytic}(r_j) = \widetilde{A}_{calc}(r_j) + (\Delta r)^{\alpha}\beta(r_j) + H.O.T$$

where the value of $\beta(r_j)$ does not change with grid spacing, and α is the asymptotic order of accuracy of the method. It is important to note that the numerical method recovers the original equations as the grid spacing approached zero. It is important to note that β represents the first derivative of the Taylor Series. Subtracting $A_{analytic}$ from both sides gives,

$$A_{calc}(r_j) - A_{analytic}(r_j) = A_{analytic}(r_j) - A_{analytic}(r_j) + \beta(r_j)(\Delta r)^{\alpha}$$

$$\epsilon_A(r_j)(\Delta r) = \beta(r_j)(\Delta r)^{\alpha}$$

To estimate the order of accuracy of the accuracy, we define the global errors by calculating the L2 Norm of the error which is denoted as $\hat{\epsilon}_A$

$$\hat{\epsilon}_A = \sqrt{\frac{1}{N}\sum_{j=1}^{N}\epsilon(r_j)^2}$$

$$\hat{\beta}_A(r_j) = \sqrt{\frac{1}{N}\sum_{j=1}^{N}\beta(r_j)^2}$$

As the grid density increases, $\hat{\beta}$ should asymptote to a constant value. Given two grid densities, Δr and $\sigma\Delta r$, and assuming that the leading error term is much larger than any other error term,

$$\begin{aligned}
\hat{\epsilon}_{grid1} &= \hat{\epsilon}(\Delta r) = \hat{\beta}(\Delta r)^{\alpha} \\
\hat{\epsilon}_{grid2} &= \hat{\epsilon}(\sigma\Delta r) = \hat{\beta}(\sigma\Delta r)^{\alpha} \\
&= \hat{\beta}(\Delta r)^{\alpha}\sigma^{\alpha}
\end{aligned}$$

The ratio of two errors is given by,

$$\begin{aligned}
\frac{\hat{\epsilon}_{grid2}}{\hat{\epsilon}_{grid1}} &= \frac{\hat{\beta}(\Delta r)^{\alpha}}{\hat{\beta}(\Delta r)^{\alpha}}\sigma^{\alpha} \\
&= \sigma^{\alpha}
\end{aligned}$$

Thus, α,the asymptotic rate of convergence is computed as follows

$$\alpha = \frac{\ln\frac{\hat{\epsilon}_{grid2}}{\hat{\epsilon}_{grid1}}}{\ln(\sigma)}$$

Defining for a doubling of grid points , Similarly for the eigenvalue problem,

$$[A]x = \lambda[B]x$$

4.4 Fairing Functions

Goal: How can we modify a manufactured solution such that the endpoints are suitable for comparison against a codes boundary condition implementation

4.5 Setting Boundary Condition Values Using a Fairing Function

4.5.1 Using β as a scaling parameter

Defining the nondimensional radius in the same way that SWIRL does:

$$\widetilde{r} = \frac{r}{r_T}$$

where r_T is the outer radius of the annulus.

The hub-to-tip ratio is defined as:

$$\sigma = \frac{r_H}{r_T} = \widetilde{r}_H$$

where $\widetilde{r}_H$ is the inner radius of the annular duct. The hub-to-tip ratio can also be zero indicating the duct is hollow.

A useful and similar parameter is introduced, β, where $0 \leq \beta \leq 1$

$$\beta = \frac{r - r_H}{r_T - r_H}$$

Dividing By r_T

$$\beta = \frac{\frac{r}{r_T} - \frac{r_H}{r_T}}{\frac{r_T}{r_T} - \frac{r_H}{r_T}}$$
$$= \frac{\widetilde{r} - \widetilde{r}_H}{1 - \sigma}$$

Suppose a manufactured solution f_{MS} with boundaries $f_{MS}(r = \sigma)$ and $f_{MS}(\widetilde{r} = 1)$ is the specified analytical solution. The goal is to change the boundary conditions of the manufactured solution in such way that allows us to adequately check the boundary conditions imposed on SWIRL. Defining the manufactured solution, $f_{MS}(\widetilde{r})$, where $\sigma \leq \widetilde{r} \leq 1$ and there are desired values of f at the boundaries desired values are going to be denoted as f_{minBC} and f_{maxBC}. The desired changes in f are defined as:

$$\Delta f_{minBC} = f_{minBC} - f_{MS}(\widetilde{r} = \sigma)$$
$$\Delta f_{maxBC} = f_{maxBC} - f_{MS}(\widetilde{r} = 1)$$

We'd like to impose these changes smoothly on the manufactured solution function. To do this,the fairing functions, $A_{min}(\widetilde{r})$ and $A_{max}(\widetilde{r})$ where:

$$f_{BCsImposed}(\widetilde{r}) = f_{MS}(\widetilde{r}) + A_{min}(\widetilde{r})\Delta f_{minBC} + A_{max}(\widetilde{r})\Delta f_{maxBC}$$

Then, in order to set the condition at the appropriate boundary, the following conditions are set,

$$A_{min}(\widetilde{r} = \sigma) = 1$$

$$A_{min}(\widetilde{r} = 1) = 0$$

$$A_{max}(\widetilde{r} = 1) = 1$$

$$A_{max}(\widetilde{r} = \sigma) = 0$$

If $A_{min}(\widetilde{r})$ is defined as a function of $A_{max}(\widetilde{r})$ then only $A_{max}(\widetilde{r})$ needs to be defined, therefore

$$A_{min}(\widetilde{r}) = 1 - A_{max}(\widetilde{r})$$

It is also desirable to set the derivatives for the fairing function at the boundaries incase there are boundary conditions imposed on the derivatives of the fairing function.

$$\frac{\partial A_{max}}{\partial \widetilde{r}}|_{\widetilde{r}=\sigma} = 0$$
$$\frac{\partial A_{max}}{\partial \widetilde{r}}|_{\widetilde{r}=1} = 0$$

$$\frac{\partial A_{min}}{\partial \widetilde{r}}|_{\widetilde{r}=\sigma} = 0$$
$$\frac{\partial A_{min}}{\partial \widetilde{r}}|_{\widetilde{r}=1} = 0$$

4.5.2 Minimum Boundary Fairing Function

Looking at A_{min} first, the polynomial is:

$$
\begin{aligned}
A_{min}(\beta) &= a + b\beta + c\beta^2 + d\beta^3 \\
A_{min}(\widetilde{r}) &= a + b\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) + c\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 + d\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3
\end{aligned}
$$

Taking the derivative,

$$
\begin{aligned}
A'_{min}(\widetilde{r}) &= b\left(\frac{1}{1-\sigma}\right) + 2c\left(\frac{1}{1-\sigma}\right)\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) + 3d\left(\frac{1}{1-\sigma}\right)\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 \\
A'_{min}(\beta) &= \left(\frac{1}{1-\sigma}\right)\left[b + 2c\beta + 3d\beta^2\right]
\end{aligned}
$$

Now we will use the conditons mentioned earlier as constraints to this system of equations Using the possible values of $\widetilde{r}$,

$$
\begin{aligned}
A_{min}(\sigma) &= a && = 1 \\
A_{min}(1) &= a + b + c + d && = 0 \\
A'_{min}(\sigma) &= b && = 0 \\
A'_{min}(1) &= b + 2c + 3d && = 0
\end{aligned}
$$

which has the solution,

$$a = 1$$

$$b = 0$$

$$c = -3$$

$$d = 2$$

giving the polynomial as:

$$A_{min}(\widetilde{r}) = 1 - 3\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right)^2 + 2\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right)^3$$

4.5.3 Max boundary polynomial

Following the same procedure for A_{max} gives

$$A_{min}(\widetilde{r}) = 3\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right)^2 - 2\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right)^3$$

4.5.4 Corrected function

The corrected function is then,

$$\begin{aligned}
f_{BCsImposed}(\widetilde{r}) = f_{MS}(\widetilde{r}) & \qquad + A_{min}\Delta f_{minBC} + A_{max}\Delta f_{maxBC} \\
= f_{MS}(\widetilde{r}) & \qquad + \left(1 - 3\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right)^2 + 2\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right)^3\right)[\Delta f_{minBC}] \\
& \qquad + \left(3\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right)^2 - 2\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right)^3\right)[\Delta f_{maxBC}] \\
f_{BCsImposed}(\beta) = f_{MS}(\beta) & \qquad + \Delta f_{minBC} + \left(3\beta^2 - 2\beta^3\right)[\Delta f_{maxBC} - \Delta f_{minBC}]
\end{aligned}$$

Note that we're carrying the correction throughout the domain, as opposed to limiting the correction at a certain distance away from the boundary. The application of this correction ensures that there is no discontinuous derivatives inside the domain; as suggested in Roach's MMS guidelines (insert ref)

What is meant by "just because A_{min} and its first derivative go to zero doesn't mean that the second derivatives"

4.5.5 Symbolic Sanity Checks

We want to ensure that $f_{BCsImposed}$ has the desired boundary conditions, $f_{minBC/maxBC}$ instead of the original boundary values that come along for the ride in the manufactured solutions, $f_{MS}(\widetilde{r} = \sigma/1)$. In another iteration of this method, we will be changing the derivative values, so let's check the values of $\frac{\partial f_{BCsImposed}}{\partial \widetilde{r}}$ to make sure those aren't effected unintentionally.

Symbolic Sanity Check 1

The modified manufactured solution, $f_{BCsImposed}$ with the fairing functions A_{min} and A_{max} substituted in is,

$$f_{BCsImposed}(\widetilde{r}) = \left(3\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 - 2\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3\right)\left[\Delta f_{maxBC}\right].$$

Further simplification yields,

$$\begin{aligned} f_{BCsImposed}(\widetilde{r}=\sigma) &= \left(f_{MS}(\widetilde{r}=\sigma) + \Delta f_{minBC} + \left(3\left(\frac{\sigma-\sigma}{1-\sigma}\right)^2 - 2\left(\frac{\sigma-\sigma}{1-\sigma}\right)^3 - \right)\left[\Delta f_{maxBC} - \Delta f_{n}\right.\right. \\ &= f_{MS}(\widetilde{r}=\sigma) + \Delta f_{minBC} \\ &= f_{MS}(\widetilde{r}=\sigma) + (f_{minBC} - f_{MS}(\widetilde{r}=\sigma)) \\ &= f_{minBC} \end{aligned}$$

$$
\begin{aligned}
f_{BCsImposed}(\widetilde{r}=1) &= \left(f_{MS}(\widetilde{r}=1) + \Delta f_{minBC} + \left(3\left(\frac{1-\sigma}{1-\sigma}\right)^2 - 2\left(\frac{1-\sigma}{1-\sigma}\right)^3 - \right) [\Delta f_{maxBC} - \Delta f_{mi} \right. \\
&= f_{MS}(\widetilde{r}=1) + \Delta f_{maxBC} \\
&= f_{MS}(\widetilde{r}=1) + (f_{maxBC} - f_{MS}(\widetilde{r}=1)) \\
&= f_{maxBC}
\end{aligned}
$$

$$
\frac{\partial}{\partial \widetilde{r}} \left(f_{BCsImposed}(\widetilde{r}) = \left(3\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 - 2\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3 \right) [\Delta f_{maxBC}] \right)
$$

$$
\frac{\partial f_{MS}}{\partial \widetilde{r}} + \left(\frac{6}{1-\sigma}\right) \left(\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) - \left(\frac{r-\sigma}{1-\sigma}\right)^2 \right) (\Delta f_{maxBC} - \Delta f_{minBC})
$$

At $\widetilde{r} = \sigma$, the derivative is:

$$
\begin{gathered}
\frac{\partial f_{MS}}{\partial \widetilde{r}}|_{\sigma} \\
\frac{\partial f_{MS}}{\partial \widetilde{r}}|_{1}
\end{gathered}
$$

4.5.6 Min boundary derivative polynomial

The polynomial is of the form:

$$
\begin{aligned}
B_{min}\left(\beta\right) &= a + b\beta + c\beta^2 + d\beta^3 \\
B_{min}\left(\widetilde{r}\right) &= a + b\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) + c\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 + d\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3
\end{aligned}
$$

Taking the derivative,

$$B'_{min}\left(\widetilde{r}\right) = b\left(\frac{1}{1-\sigma}\right) + 2c\left(\frac{1}{1-\sigma}\right)\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) + 3d\left(\frac{1}{1-\sigma}\right)\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2$$
$$B'_{min}\left(\beta\right) = \left(\frac{1}{1-\sigma}\right)\left[b + 2c\beta + 3d\beta^2\right]$$

Applying the four constraints gives:

$$\begin{aligned} a &= 0 \\ b &= (1-\sigma) \\ a+b+c+d &= 0 \\ 2+2c+3d &= 0 \end{aligned}$$

$$\begin{aligned} c+d &= -b \\ 2c+3d &= -b \end{aligned}$$

$$\begin{aligned} c &= -2b \\ d &= b \end{aligned}$$

and the min boundary derivative polynomial is:

$$\begin{aligned} B_{min}\left(\widetilde{r}\right) &= b\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) - 2b\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 + b\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3 \\ &= (1-\sigma)\left(\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) - 2\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 + \left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3\right) \end{aligned}$$

4.5.7 Polynomial function, max boundary derivative

The polynomial is of the form:

The polynomial is of the form:

$$B_{max}\left(\beta\right) = a + b\beta + c\beta^2 + d\beta^3$$
$$B_{max}\left(\widetilde{r}\right) = a + b\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right) + c\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right)^2 + d\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right)^3$$

which has the derivative,

$$B'_{max}\left(\widetilde{r}\right) = b\left(\frac{1}{1 - \sigma}\right) + 2c\left(\frac{1}{1 - \sigma}\right)\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right) + 3d\left(\frac{1}{1 - \sigma}\right)\left(\frac{\widetilde{r} - \sigma}{1 - \sigma}\right)^2$$
$$B'_{max}\left(\beta\right) = \left(\frac{1}{1 - \sigma}\right)\left[b + 2c\beta + 3d\beta^2\right]$$

Applying the four constraints gives:

$$a = 0$$
$$b = 0$$
$$a + b + c + d = 0$$
$$b + 2c + 3d = (1 - \sigma)$$

working this out:

$$c + d = 0$$
$$2c + 3d = (1 - \sigma)$$

gives

$$c = -\left(1-\sigma\right)d \qquad\qquad = \left(1-\sigma\right)$$

and the max boundary derivative polynomial is:

$$B_{max}\left(\widetilde{r}\right) = \left(1-\sigma\right)\left(-\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 + \left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3\right)$$

4.5.8 Putting it together

The corrected function is then:

$$f_{BCsImposed}\left(\widetilde{r}\right) = f_{MS} + B_{min}\left(\widetilde{r}\right)\Delta f'_{minBC} + B_{max}\left(\widetilde{r}\right)\Delta f'_{maxBC}$$

$$= f_{MS} +$$
$$\left(1-\sigma\right)\left(\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right) - \left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2\right)\Delta f'_{minBC} +$$
$$\left(1-\sigma\right)\left(-\left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^2 + \left(\frac{\widetilde{r}-\sigma}{1-\sigma}\right)^3\right)\left(\Delta f'_{minBC} + \Delta f'_{maxBC}\right)$$

Chapter 5

Results and Discusssion

References

[1] Department Of Transportation Federal Aviation Administration. Aviation environmental and energy policy statement - july 2012, 2012.

[2] A. J. COOPER and N. PEAKE. Propagation of unsteady disturbances in a slowly varying duct with mean swirling flow. *Journal of Fluid Mechanics*, 445:207–234, 10 2001.

Appendix A

Theory Appendix

Appendix B

Chapter 3 Appendix

B.1 Appendix A: Speed of Sound

Sound wave is a pressure disturbance that moves with at a speed a By applying a rectangular control volume around this pressure wave, we can apply our conservation equations. We are assuming that these properities are increasing by a small increment. This is why each variable is added by a infinitesimally small term.

Recalling the conservation of mass (continuity equation), $\dot{m} = constant$

$$\dot{m}_{left} = \dot{m}_{right}$$

Recalling he definition of density, $\rho = m/\bar{V}$ and rewriting $\bar{V} = uA$ (check the units)

$$\rho a \cancel{A} = (\rho + d\rho)(a + da)\cancel{A}$$

Futher expanding gives,

$$\rho a = (\rho a + \rho da + a d\rho + da d\rho)$$

We say that $dad\rho$ is so small, we can assume it is zero. This is often referred to as "neglecting higher order terms (H.O.T)". The expression then becomes

$$\frac{da}{a} = -\frac{d\rho}{\rho}$$

For the momentum equation $P + \rho u^2 = constant$

$$P + \rho a^2 = P + dP + (\rho + d\rho)(a + da)(a + da)$$

But we just said that $\rho a = (\rho + d\rho)(a + da)$

$$P + \rho a^2 = P + dP + \rho a(a + da)$$

$$dP + \rho a da = a$$

Multiplying the second term by a and divide by a, this is essentially multiplying the second term by one.

$$dP + \rho a^2 \frac{da}{a} = a$$

recalling the relation $\frac{da}{a} = -\frac{d\rho}{\rho}$

$$dp - a^2 d\rho = 0$$

$$a^2 = \frac{dp}{d\rho}$$

Since a sound wave is a very weak wave, when it travels through a medium, it only increases the pressure and density, etc. slightly. The effect of this is that friction and heat transfer can be neglected. Since friction cannot be undone, we call this an irreversible process. Whenever there is no transfer of heat, it is called this adiabatic. Thus, the propagation of sound is an adiabatic, reversible process, otherwise called isentropic. Isentropic implies no increase in entropy, which is *not* true in the presence of shock waves.

In the case of a thermally perfect gas, we can say $P = \rho RT$

For a callorically perfect gas we can say $pv^\gamma = constant$, where v is volume per unit mass, or specific volume

Differentiating and recalling that $v = 1/\rho$

$$a = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\gamma RT}$$

$$dm = (\rho u)_2 - (\rho u)_1$$

$$D(mV) = \left(\rho u^2 + P\right)_2$$

For Steady flow,

$$\cancel{dm} = (\rho u)_2 - (\rho u)_1$$

$$(\rho u)_1 = (\rho u)_2$$

Similarly, for the Momentum equation,

$$\left(\rho u^2 + P\right)_1 = \left(\rho u^2 + P\right)_2$$

Let us change the coordinate system motion for the traveling wave be independent of time, and thus corresponds to *steady state* wave propagation.

$$u_1 = \bar{u} + a - \frac{1}{2}\partial u$$

$$u_2 = \bar{u} + a + \frac{1}{2}\partial u$$

where, $\bar{u}$ is the average flow velocity and a is the wave speed.

Substituting this back into the conservation of mass

$$(\rho u)_1 = (\rho u)_2$$

$$\left(\rho - \frac{1}{2}\partial\rho\right)\left(\bar{u} - a - \frac{1}{2}\partial u\right) = \left(\rho + \frac{1}{2}\partial\rho\right)\left(\bar{u} - a + \frac{1}{2}\partial u\right)$$

Further expanding

$$\cancel{\rho\bar{u} - \rho a} + \frac{1}{2}\left(-\rho\partial u - \bar{u} - \bar{u}\partial\rho + a\partial\rho\right) + \cancel{\frac{1}{4}\partial\rho\partial u} = \cancel{\rho\bar{u} - \rho a} + \frac{1}{2}\left(\rho\partial u + \bar{u} - \bar{u}\partial\rho - a\partial\rho\right) + \cancel{\frac{1}{4}\partial\rho\partial u}$$

$$\frac{1}{2}\left(-\rho\partial u - \bar{u}\partial\rho + a\partial\rho\right) = \frac{1}{2}\left(\rho\partial u + \bar{u}\partial\rho - a\partial\rho\right)$$

$$\rho\partial u + u\partial\rho - a\partial\rho = 0$$

$$\rho\partial u + \left(u - a\right)\partial\rho = 0$$

Momentum Equation

$$\left(\rho u^2 + P\right)_1 = \left(\rho u^2 + P\right)_2$$

B.2 Appendix B: Isentropic Waves

$$dU = dW + dQ$$

For adiabatic, reversible processes, the work done by a system with constant pressure and a change in volume is $-pdV$ and the change in heat energy is zero. Hence,

$$dU = -pdV$$

.

The change in enthalpy of such a system can be found by taking the derivative of its expression for a thermodynamic process

$$H = U + pV$$

$$dH = dU + pdV + vdP$$

$$dH = -pdV + pdV + vdP$$

$$dH = vdP$$

The specific heats at constant pressure and constant volume

$$\left(\frac{\partial U}{\partial T}\right)_v = C_v$$

$$\left(\frac{\partial H}{\partial T}\right)_p = C_p$$

$$\gamma = \frac{C_p}{C_v} = \frac{dH}{dU} = -\frac{VdP}{pdV} = -\frac{V}{dV}\frac{dP}{p}$$

Integrating both sides

$$\gamma\frac{dV}{V} = \frac{-dP}{P} \rightarrow \gamma\int\frac{1}{V}dV = -\int\frac{1}{P}dP$$

$$\gamma ln(V) + ln(P) = C$$

Using log rules

$$ln(V^\gamma) + ln(P) = C$$

$$ln(pV^\gamma) = C$$

$$pV^\gamma = e^C = C$$

$$\frac{p}{\rho^\gamma}$$

Appendix C

Insert the Heading to Appendix B

[Insert text to Appendix B (if appendix is needed)]