Analytical Solution for Duct Mode Propagation in Uniform Flow

Swirl Validation

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This document shows the analytical duct mode solution as well as a numerical comparison.

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November 1, 2022

0.1 Introduction - Turbomachinery Noise

Turbomachinery noise generation occurs from pressure fluctuations from the series of fans within it's annular duct. While the jet that is produced from this stream of air freely radiates to the observer, the pressure fluctuations produced from the rotor may or may not propagate out of the inlet and exhaust and radiate to the observer. The production of this propagation can be characterized by standing waves referred to as modes, in particular, duct modes because the mode itself is dependent on the geometry of the column of air within the annular duct, as well as the speed of the flow moving through it

0.2 Duct Mode Theory

The pressure field within a duct is governed by the convective wave equation, a second order ODE as a function of radius.

The solution of the convective wave equation are eigenvalues and eigenvectors which may or may not correspond to acoustic disturbances fall into two groups. One group corresponding to the acoustics propagation and the other group corresponding to the convection speed of the flow. Both are modes that are a result from the pressure distribution from within the cylindrical domain.

0.2.1 Analytic Solution: Axial wavenumbers and Pressure Modes

Modes can be categorized based on the sign of the axial wavenumber and if it is complex in value. For example, for the uniform axial flow case, propagating modes are defined by axial wavenumbers, k_x, that have a real-part only, yielding the assumed fluctuation to resemble Euler's Formula ($e^{ik_x x}$). On the other hand, if the k_x is complex, then the mode will resemble an exponentially decaying function since the imaginary number cancels, leaving a minus sign in front of the axial wavenumber. These two distinctions are referred to as "cut-on" and "cut-off" in the field of ducted sound propagation. Furthermore, the sign of the imaginary part will change the direction of the mode's decay. If k_x is positive, the decay rate occurs in the negative direction. Conversely, if k_x is negative, the decay occurs in the positive direction. The axial wavenumber for uniform axial flow in a hollow duct is,

$$k_x = \frac{-M_x k \pm \sqrt{k^2 - (1 - M_x^2)J_{m,n}'^2}}{(1 - M_x^2)}. \tag{1}$$

where M_x is the axial Mach number, k is the temporal (referred to as reduced) frequency, and $J_{m,n}'$ is the derivative of the Bessel function of the first kind. The $\pm$ accounts for both upstream and downstream modes.

The condition for propagation is such that the axial wavenumber is larger than a "cut-off" value

$$k_{x,real} = \frac{\pm M_x k}{(M_x^2 - 1)}. \tag{2}$$

Every term that is being raised to the one half i.e. square rooted must be larger than zero to keep axial wavenumber from being imaginary. The mode will propagate or decay based on this condition.

0.2.2 Methods

Bessel's Function,

$$R(r) = AJ_0(k_r r) + BY_0(k_r r) \tag{3}$$

where J_0 and Y_0 are Bessel's functions of the first and second kind, and A and B are arbitrary constants. Both functions reduce as $k_r r$ gets large. The Bessel function of the second kind is unbounded as $k_r r$ goes to zero.

For a hard-walled duct, the radial velocity component is zero at the boundaries r_{min} and r_{max}.

$$\frac{dR(r)}{dr}\bigg|_{r=r_{max}} = \frac{dR(r)}{dr}\bigg|_{r=r_{min}} = 0 \tag{4}$$

In the case of a hollow duct, there is no minimum radius, therefore the wall boundary condition only applies at r_{max}.

Since the solution must be finite as $k_r r$ approaches zero, it can be observed that Y_0 approaches infinity. Since this would yield in a trivial solution, the coefficient $B = 0$, which reduces 3 to,

$$R(r) = AJ_0(k_r r) \tag{5}$$

Taking the derivative with respect to r yields,

$$\frac{dR(r)}{dr}\bigg|_{r} = AJ_0'(k_r r) = 0 \tag{6}$$

The boundary condition requires the Bessel function be zero at a hard wall. The terms inside of the Bessel function would then correspond to values along the domain, $k_r r$, which satisfy our equation. Let $k_r r = \alpha_{m,n}$ where, $alpha$ represents the zeros of the Bessel function, and m corresponds to the azimuthal mode order and n represents the radial mode order, i.e. the index for the number of zero crossings in the derivative of the Bessel function of the first kind.

Therefore,

$$\frac{dR}{dr}\bigg|_{r=r_{max}} = AJ_0'(k_r r_{max}) = 0 \tag{7}$$

$$(k_r r_{max}) = 0 \tag{8}$$

Recalling $\alpha_{m,n}$

$$(k_r r_{max}) = \alpha_{m,n} \tag{9}$$

$$k_r = \alpha_{m,n}/r_{max} \tag{10}$$

For annular ducts, r_{min} is no longer zero, therefore B cannot be removed since Y_0' has finite values as $k_r r$ increases.

$$\frac{dR}{dr}\bigg|_{r} = AJ_0'(k_r r) + BY_0'(k_r r) \tag{11}$$

Applying boundary conditions to both inner and outer walls at r_{min} and r_{max}

$$\frac{dR}{dr}\bigg|_{r=r_{min}} = AJ_0'(k_r r_{min}) + BY_0'(k_r r_{min}) \tag{12}$$

$$\frac{dR}{dr}\bigg|_{r=r_{max}} = AJ_0'(k_r r_{max}) + BY_0'(k_r r_{max}) \tag{13}$$

defining $\beta = \frac{B}{A}$,

$$\frac{dR}{dr}\bigg|_{r=r_{min}} = J_0'(k_r r_{min}) + \beta Y_0'(k_r r_{min}) \tag{14}$$

$$\frac{dR}{dr}\bigg|_{r=r_{max}} = J_0'(k_r r_{max}) + \beta Y_0'(k_r r_{max}) \qquad (15)$$

Defining,

$$k_r^{n+1} = k_r^n + \Delta k_r$$
$$\beta^{n+1} = \beta^n + \Delta\beta$$

And substituting into the boundary condition expressions,

$$\frac{dR}{dr}\bigg|_{r=r_{min}} = J_0'(k_r^{n+1} r_{min}) + \beta^{n+1} Y_0'(k_r^{n+1} r_{min}) \qquad (16)$$

$$\frac{dR}{dr}\bigg|_{r=r_{max}} = J_0'(k_r^{n+1} r_{max}) + \beta^{n+1} Y_0'(k_r^{n+1} r_{max}) \qquad (17)$$

$$\frac{dR}{dr}\bigg|_{r=r_{min}} = J_0'((k_r^n + \Delta k_r)\, r_{min}) + (\beta^n + \Delta\beta)\, Y_0'((k_r^n + \Delta k_r)\, r_{min}) \qquad (18)$$

0.3 Analytical Test Case

The results present a comparison of analytical and numerical solutions for a uniform flow, hard-wall, cylindrical duct for six grids. The physical parameters for this test case are reporte in Table 1. The grid points were doubled each iteration and a starting grid of 33 points was chosen. The first five radial modes were chosen in order to get a group of modes that were cut on and cut off. The case number's are used as identifiers for each mode propagating up or downstream. In the event that a spurious mode is identified, the case number will index it. The number of cut off modes was also constrained by setting a maximum value for the imaginary part of the axial wavenumber. While this magnitude is currently arbitrary, the value could be correlated to a desired decay rate for the corresponding mode. Each of the axial wavenumber's radial mode for each grid in also reported. The same test case was ran twice using second and fourth order central schemes for the radial derivatives. The approximate rate of convergence is also reported for both numerical schemes.

σ	0.0
k	10
m	2
M_x	0.3

Table 1: Validation test case parameters, Uniform Flow Hard-wall Cylindrical Duct
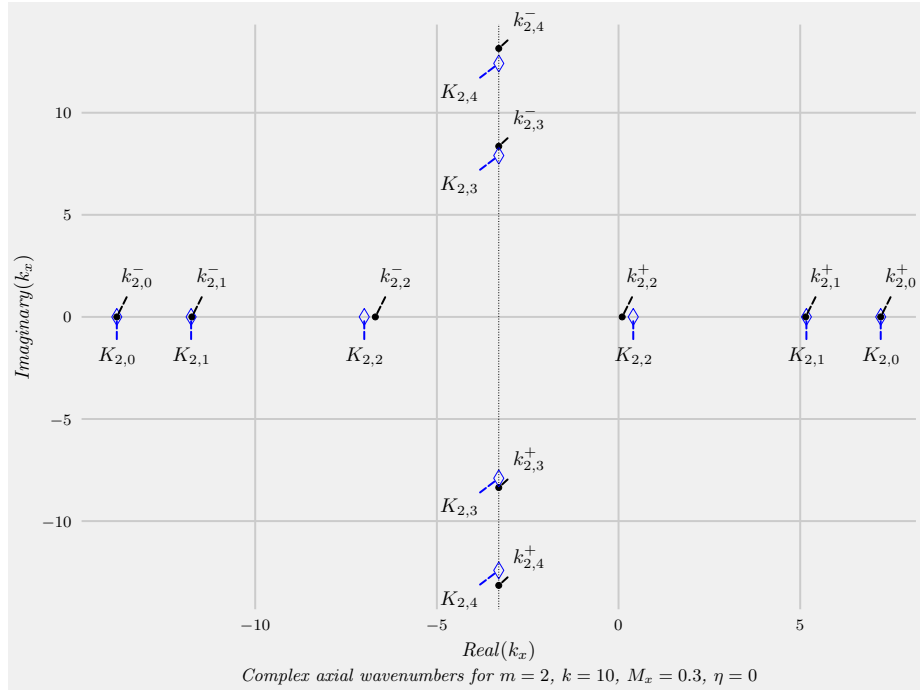
Figure 1: Second Order, Analytical Solution vs Numerical Approximation using 33 points

0.3.1 Axial Wavenumber
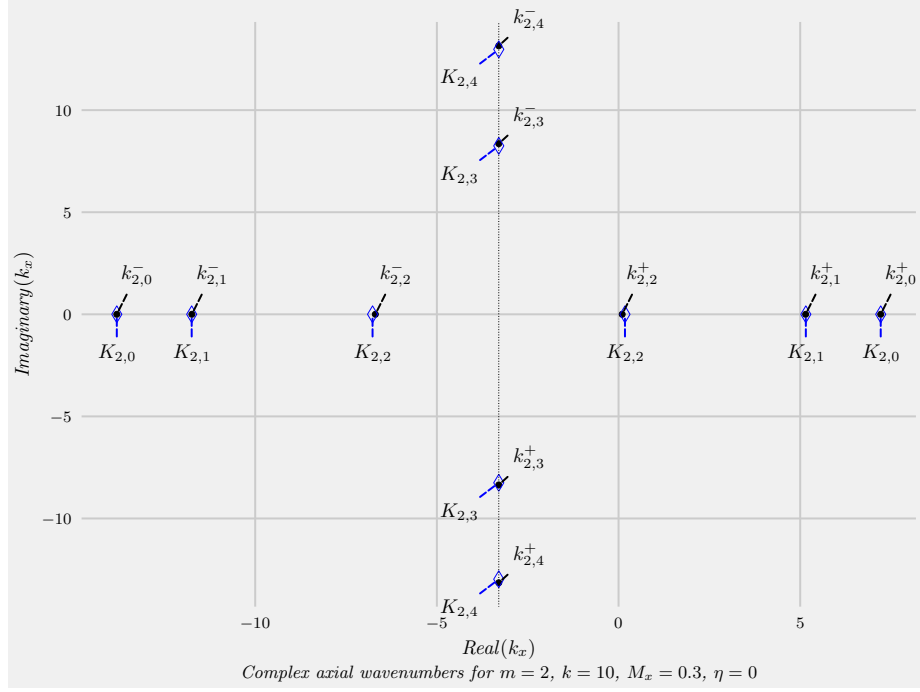
Figure 2: Second Order, Analytical Solution vs Numerical Approximation using 66 points
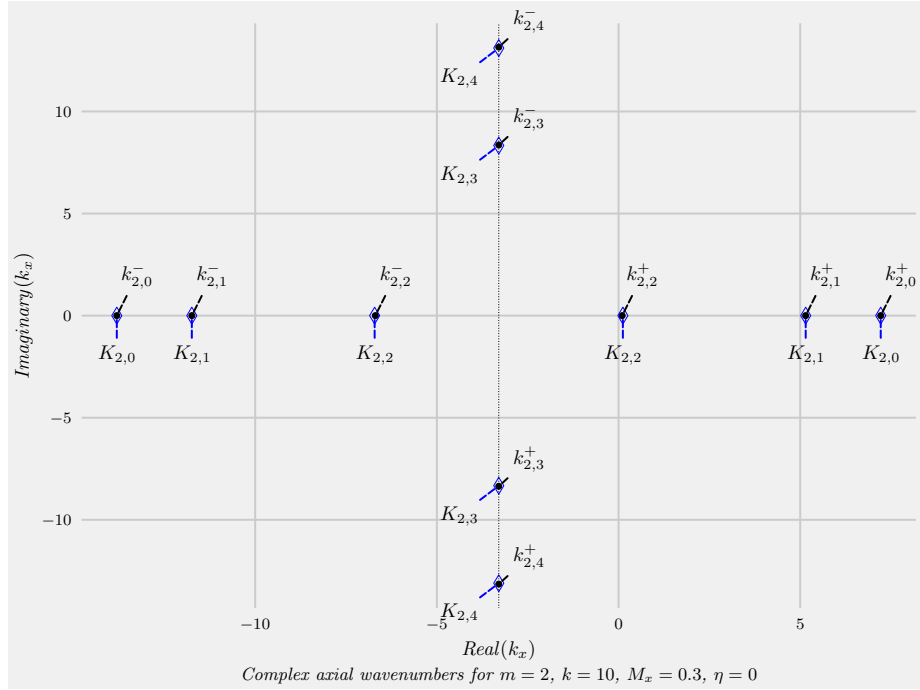
Figure 3: Second Order, Analytical Solution vs Numerical Approximation using 132 points
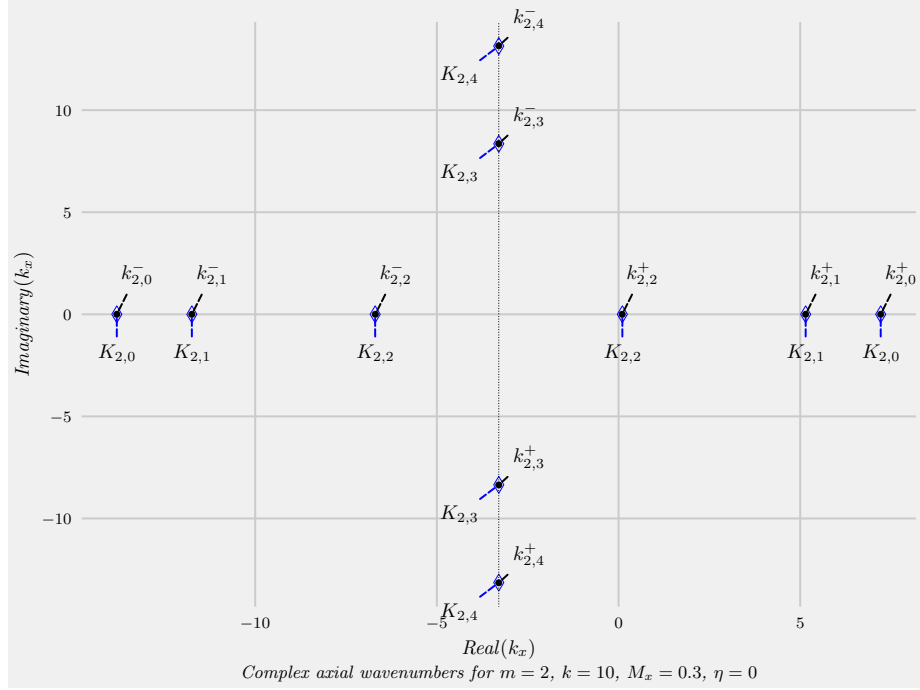
Figure 4: Second Order, Analytical Solution vs Numerical Approximation using 264 points
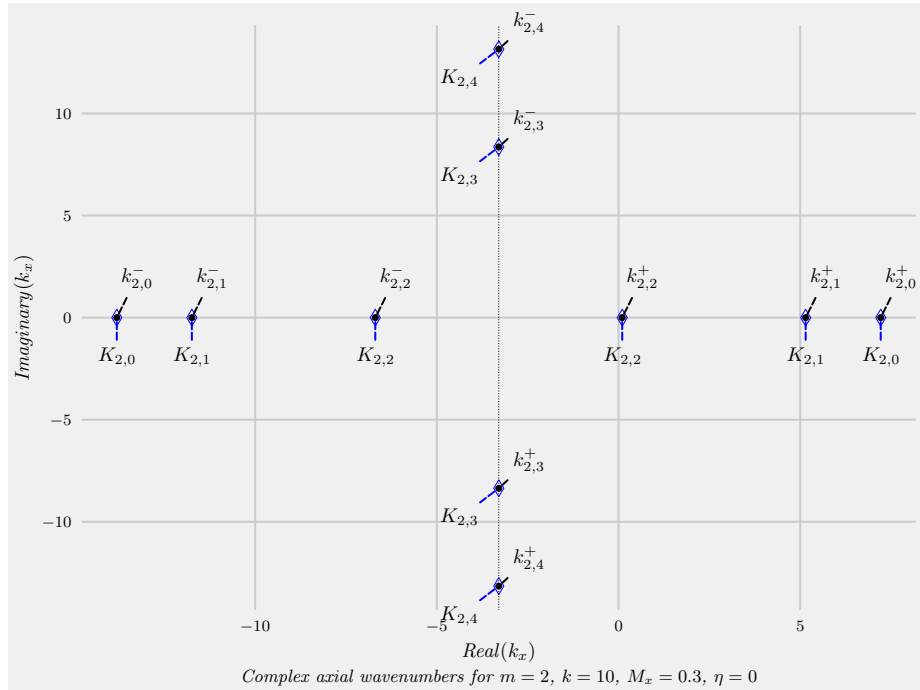
Figure 5: Second Order, Analytical Solution vs Numerical Approximation using 528 points
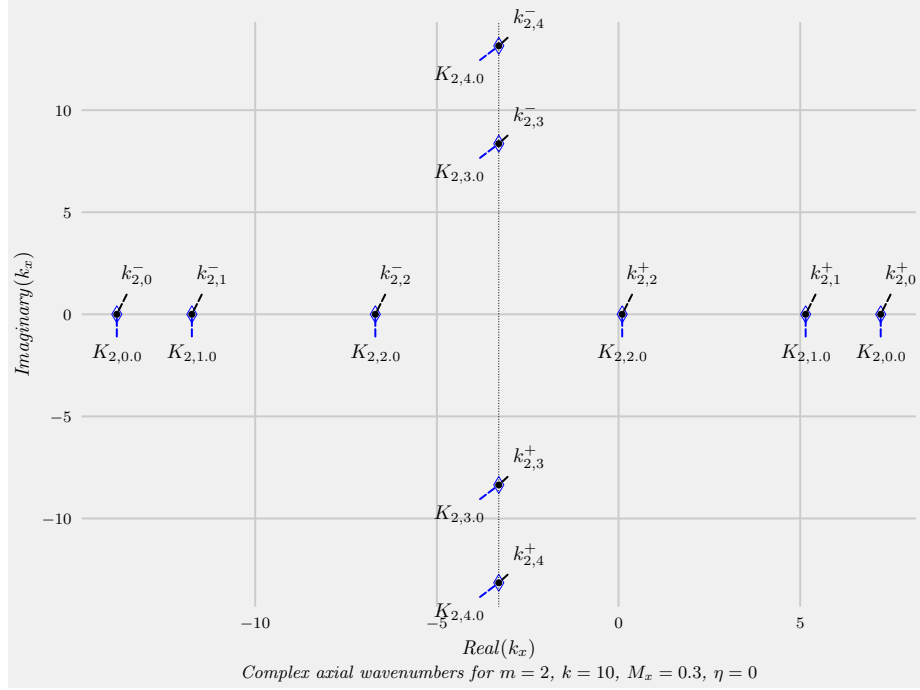
Figure 6: Second Order, Analytical Solution vs Numerical Approximation using 1056 points
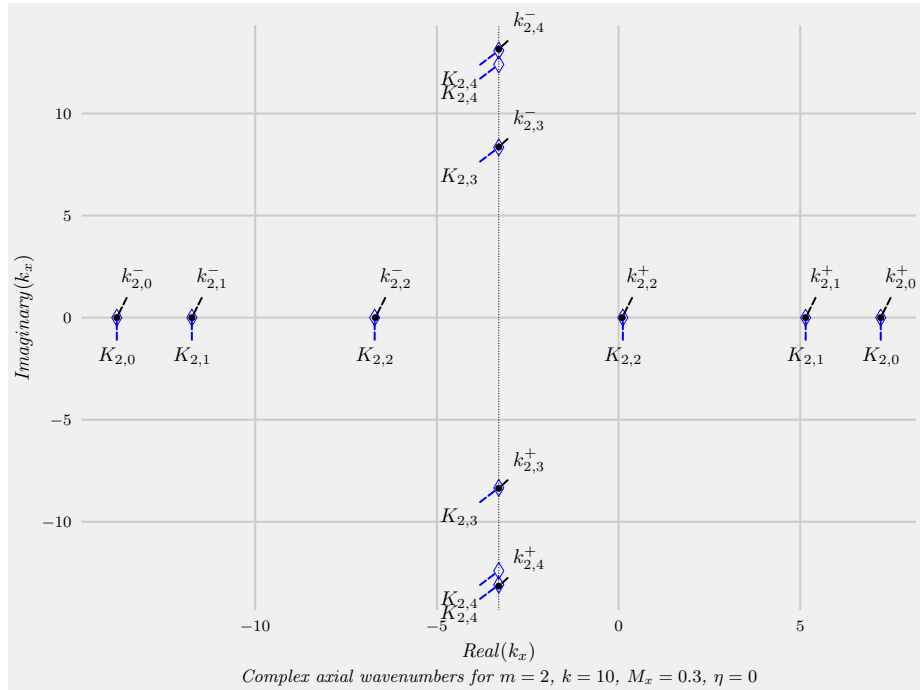
Figure 7: Fourth Order, Analytical Solution vs Numerical Approximation using 33 points

8
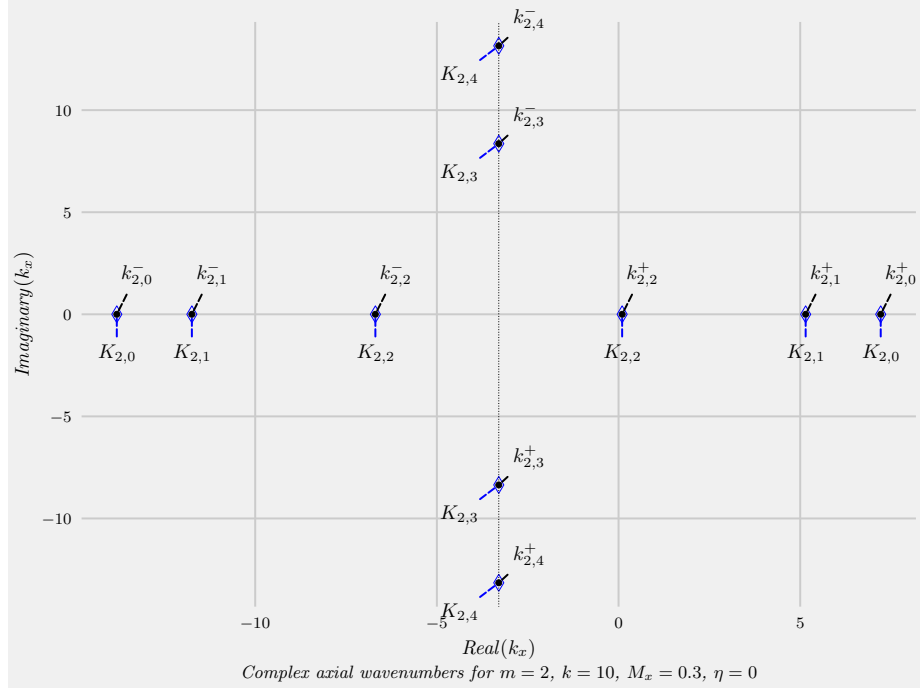
Figure 8: Fourth Order, Analytical Solution vs Numerical Approximation using 66 points
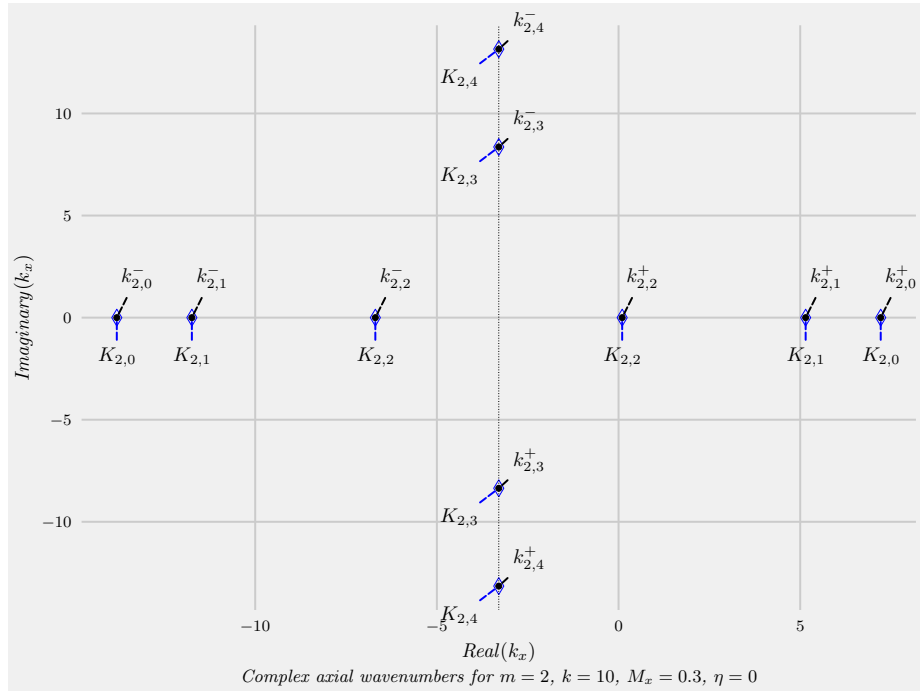
Figure 9: Fourth Order, Analytical Solution vs Numerical Approximation using 132 points

9
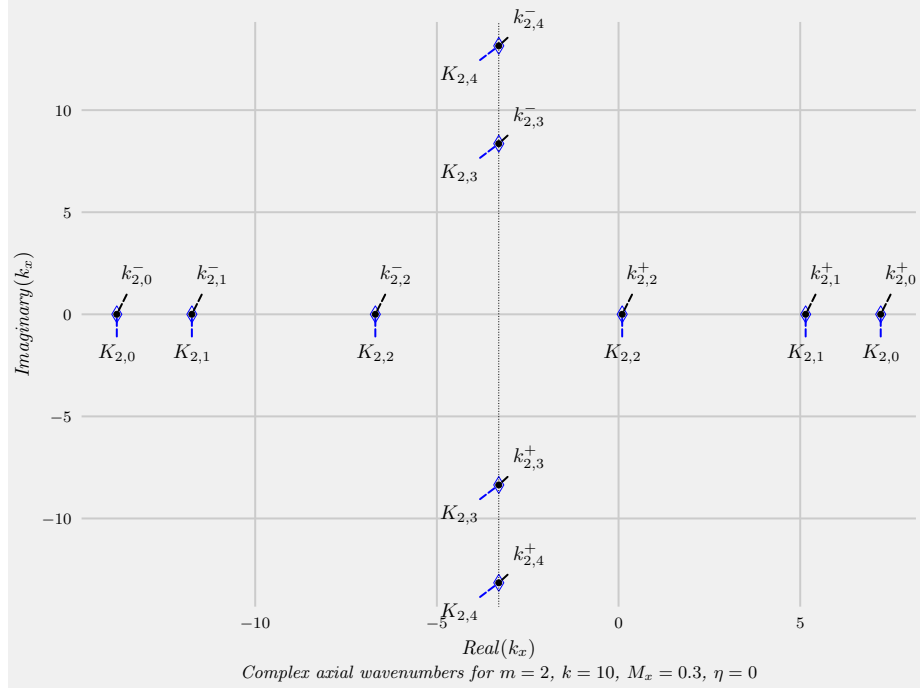
Figure 10: Fourth Order, Analytical Solution vs Numerical Approximation using 264 points
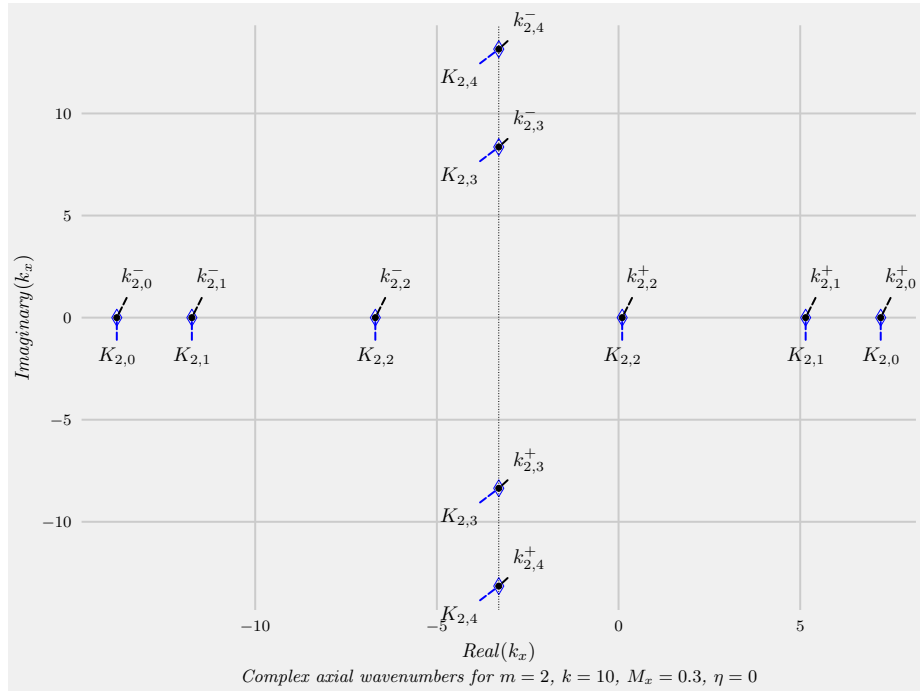
Figure 11: Fourth Order, Analytical Solution vs Numerical Approximation using 528 points
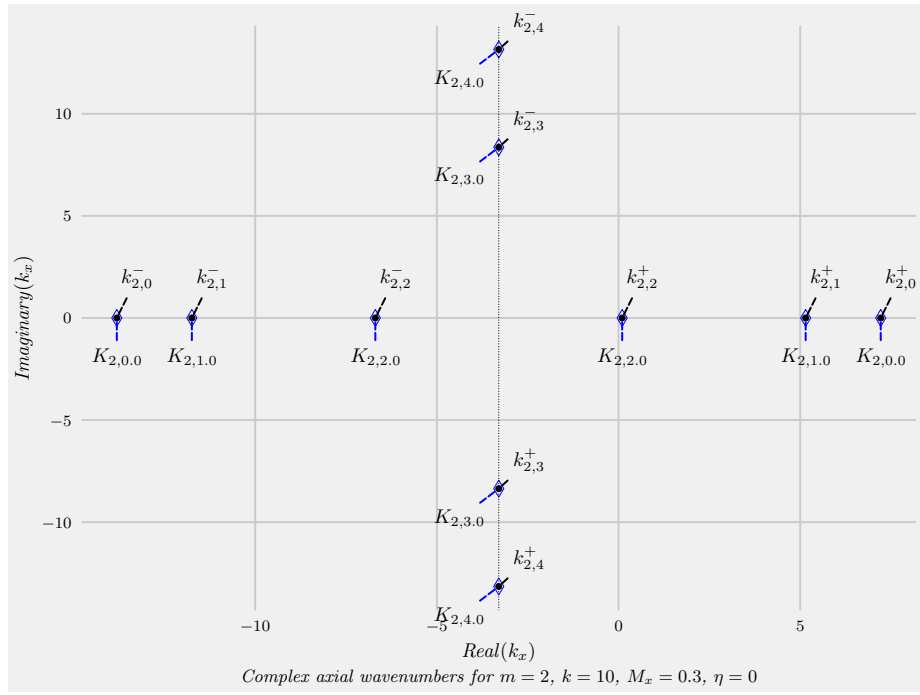
Figure 12: Fourth Order, Analytical Solution vs Numerical Approximation using 1056 points

11
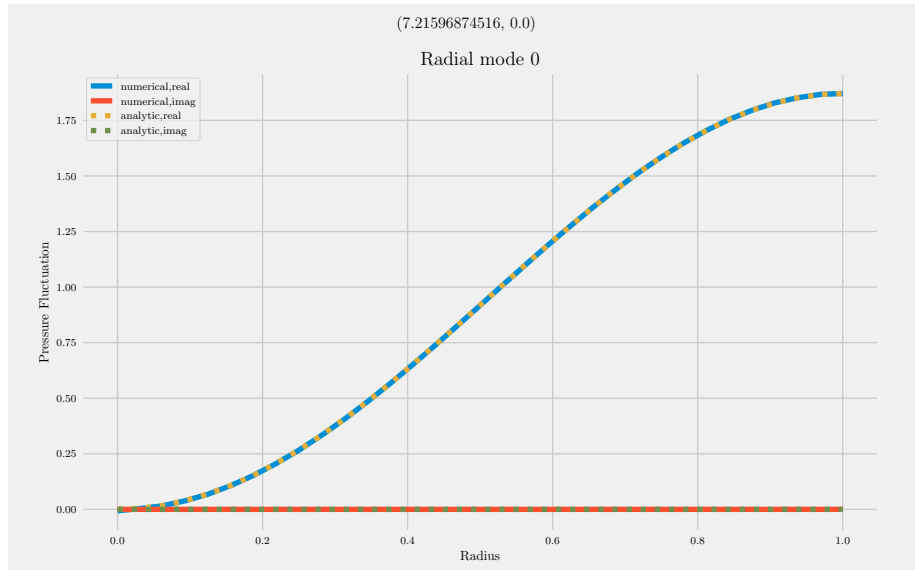
Figure 13: Second Order, Case number 0, 33points

0.3.2 Propagating Radial Modes

Second Order, Radial Mode 0
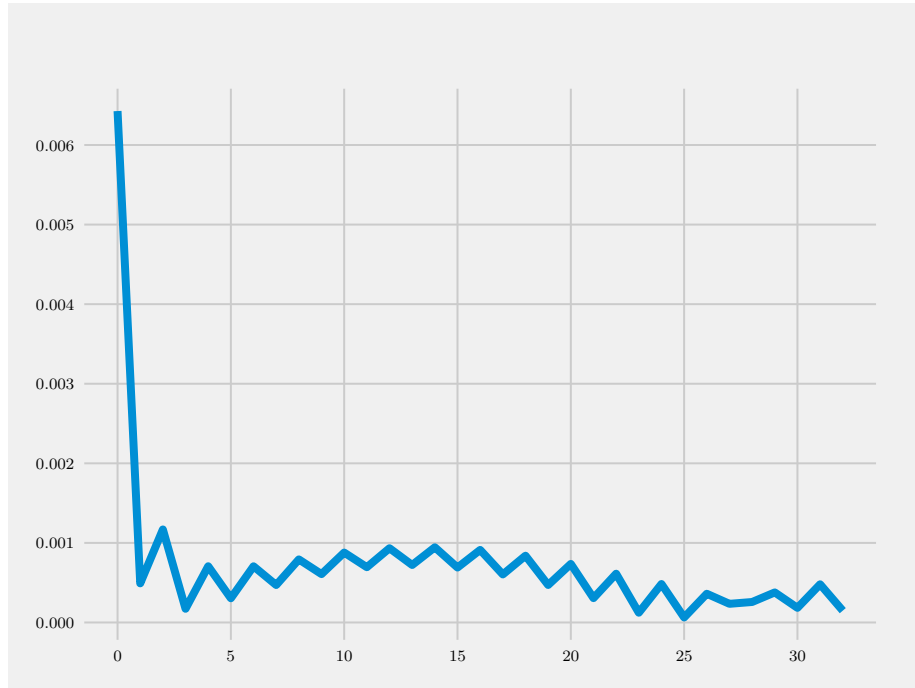
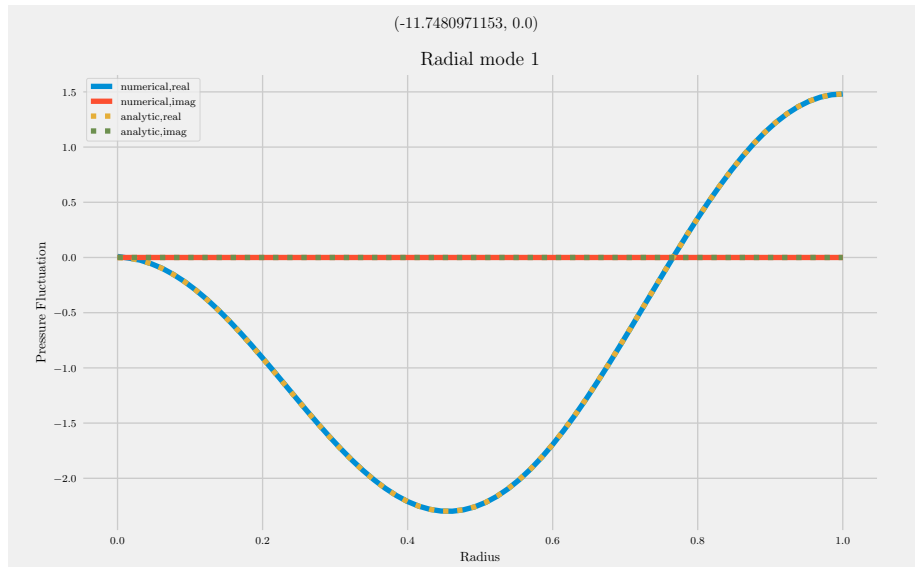Figure 14: Second Order, Case number 0, 33points
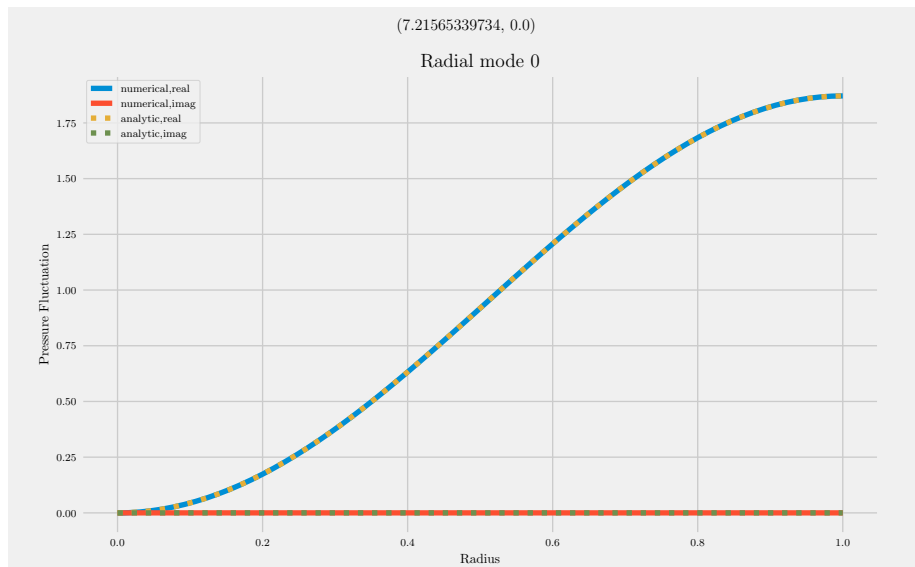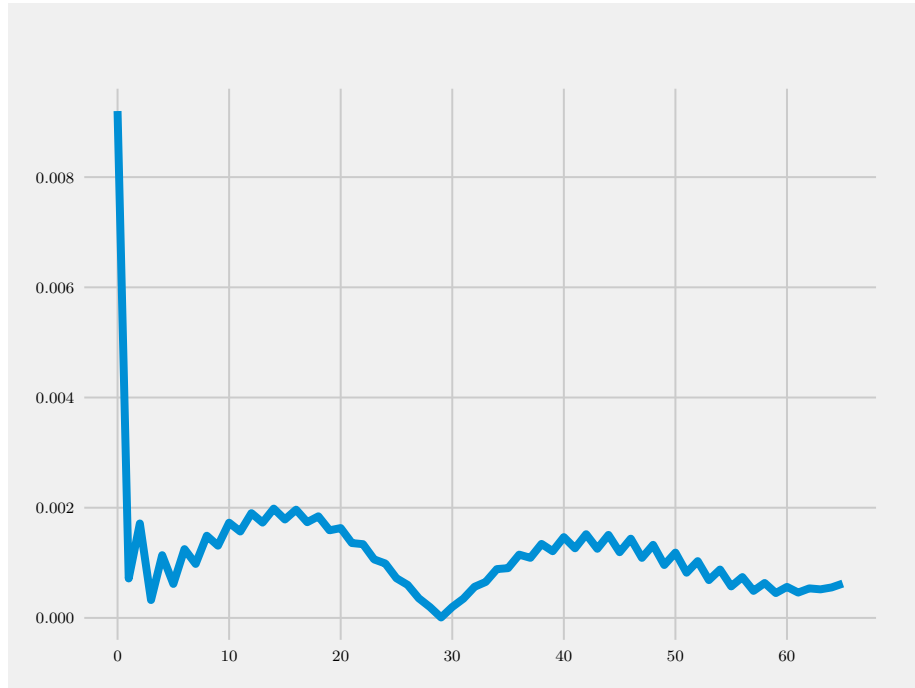

Figure 15: Second Order, Case number 0, 66points

13
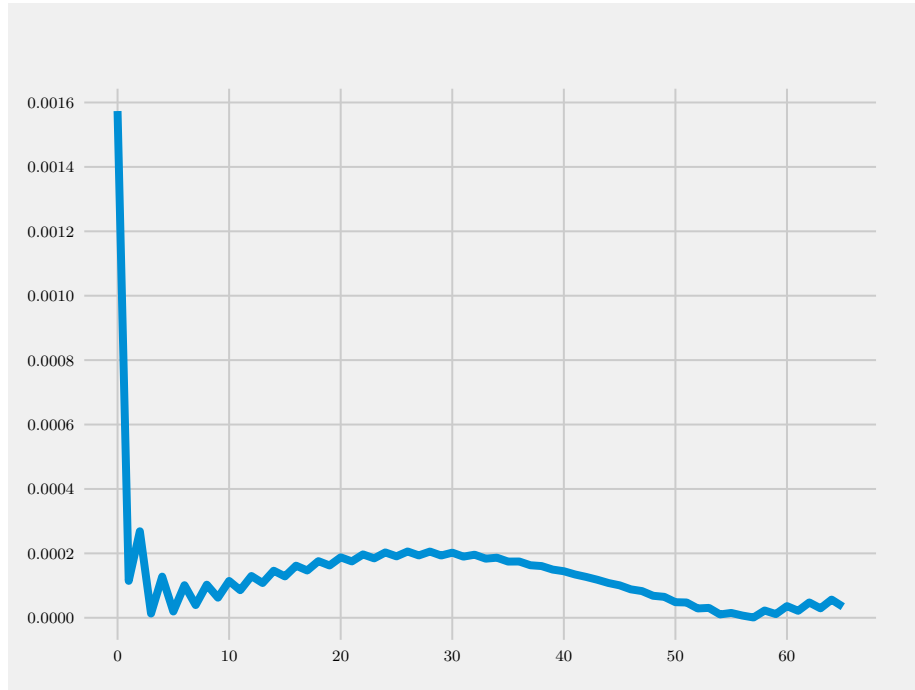
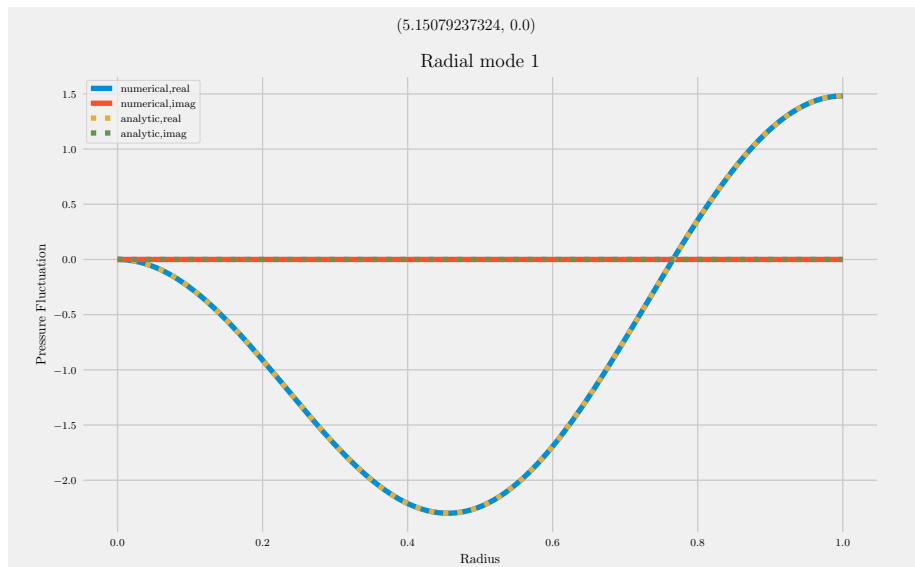Figure 16: Second Order, Case number 0, 66points
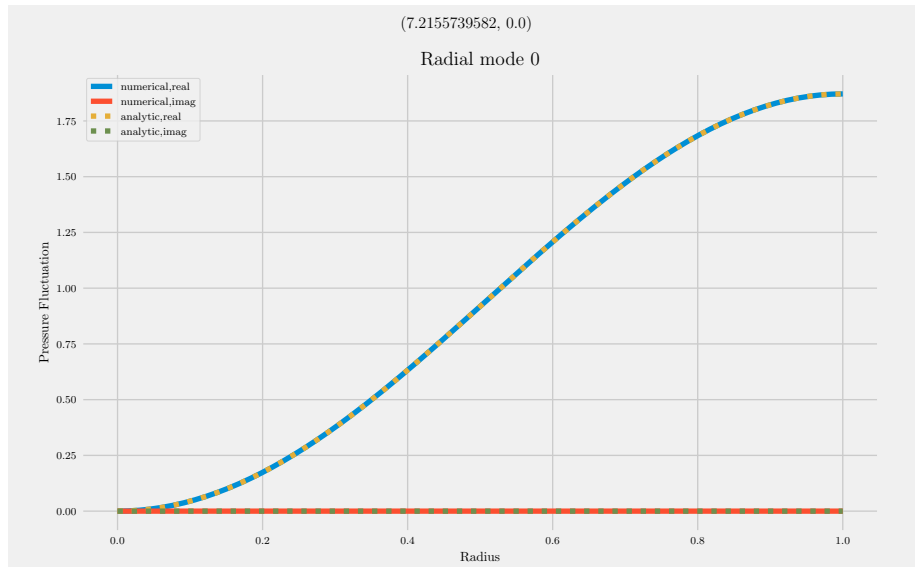
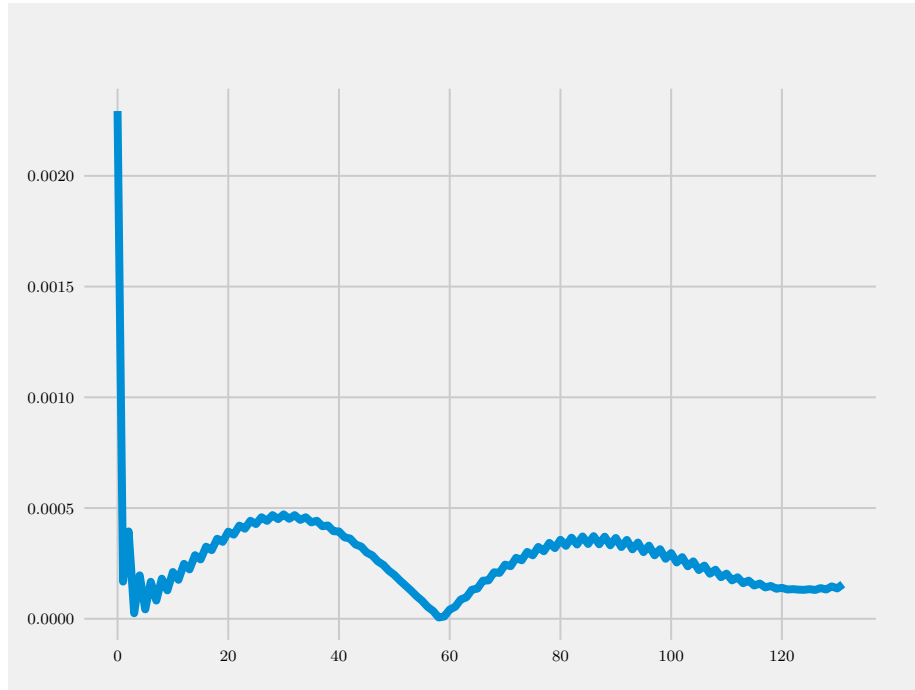Figure 17: Second Order, Case number 0, 132points
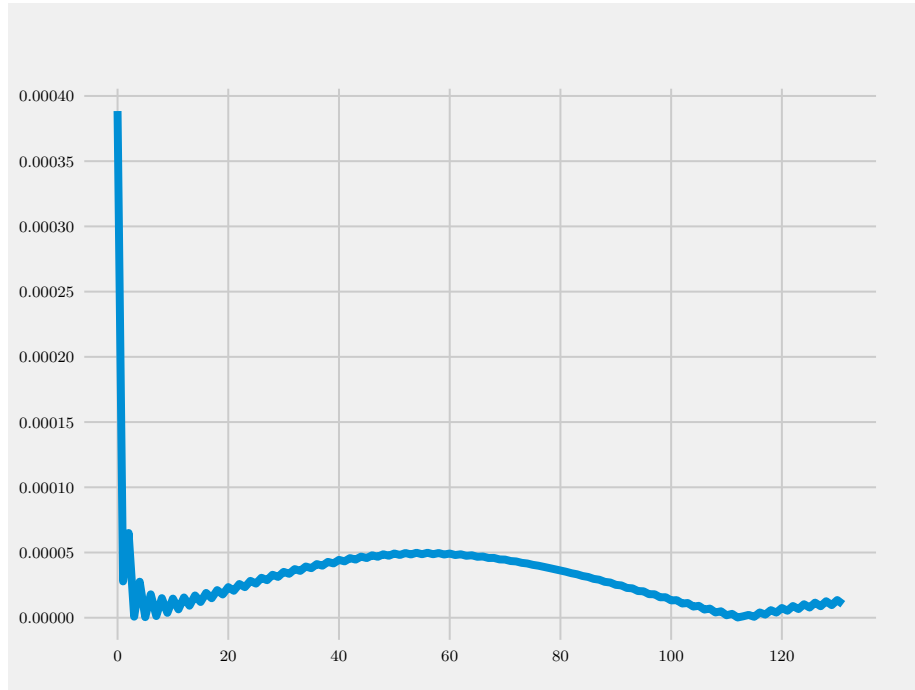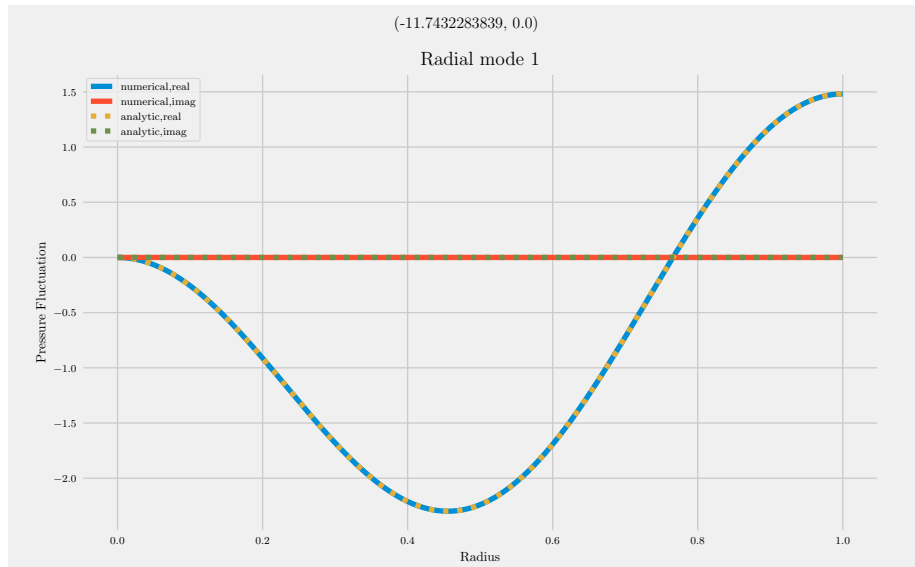
Figure 18: Second Order, Case number 0, 132points
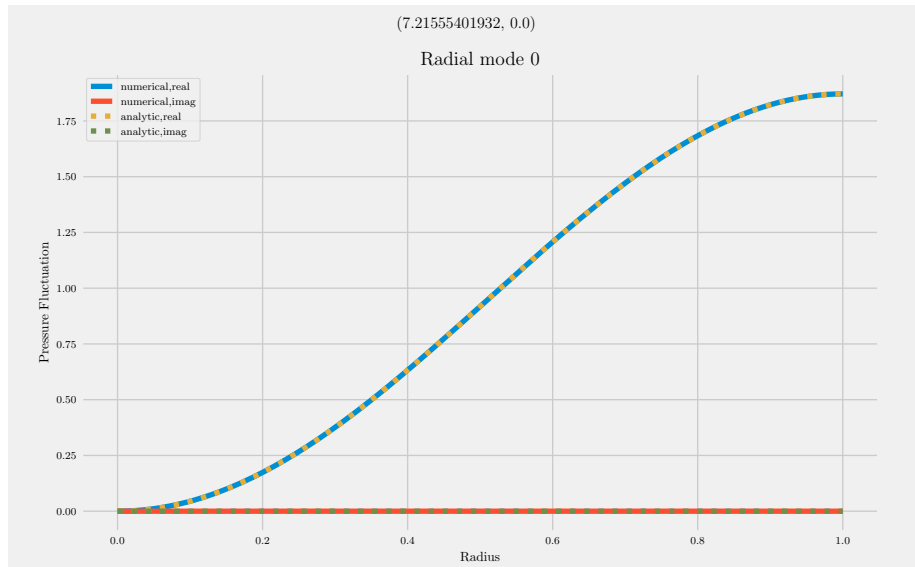
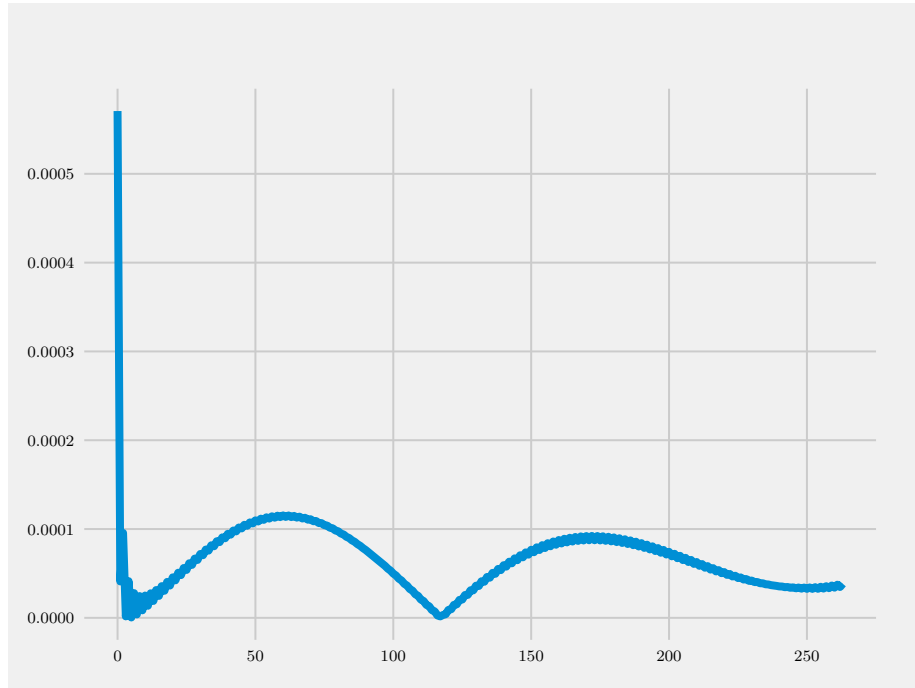
Figure 19: Second Order, Case number 0, 264points

15
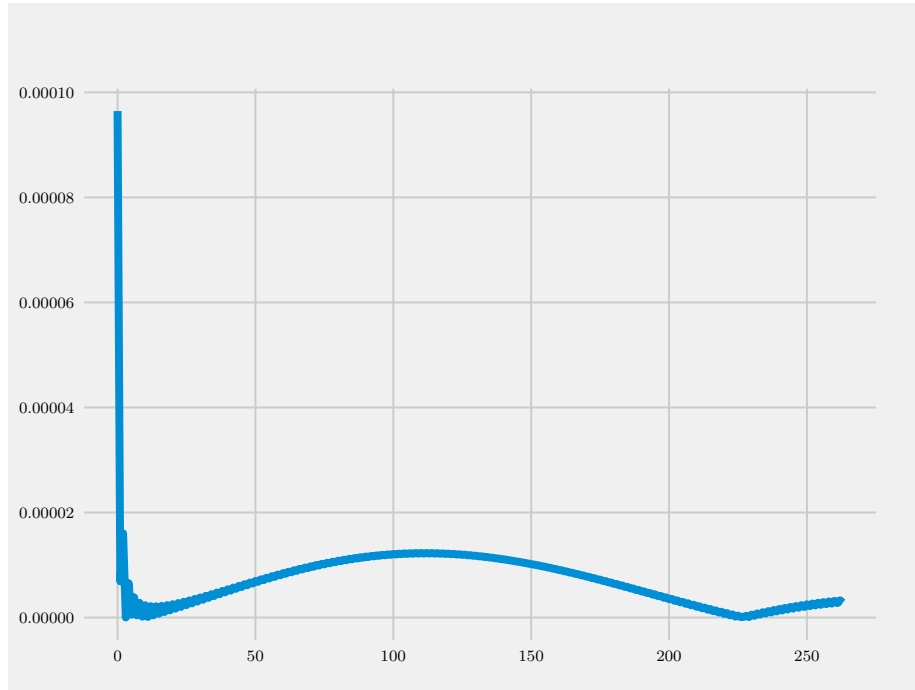
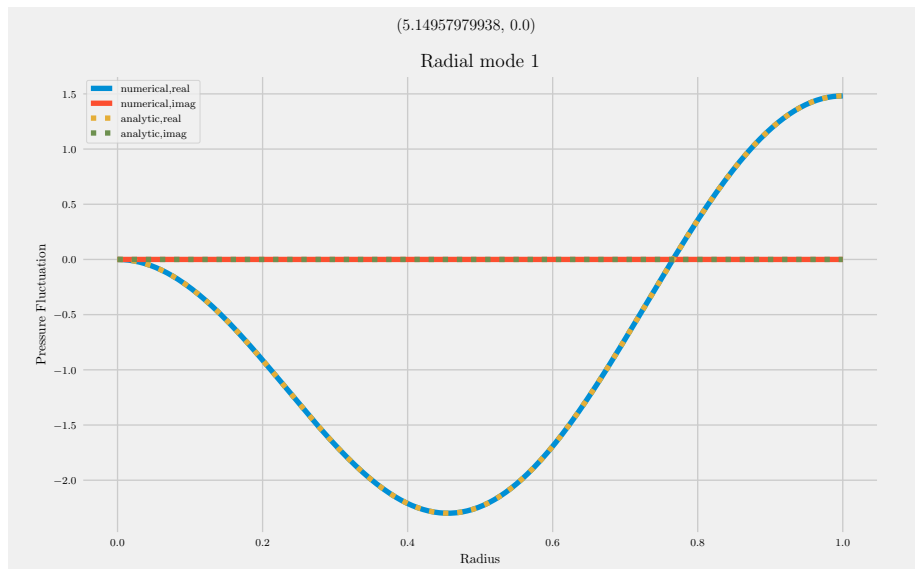Figure 20: Second Order, Case number 0, 264points
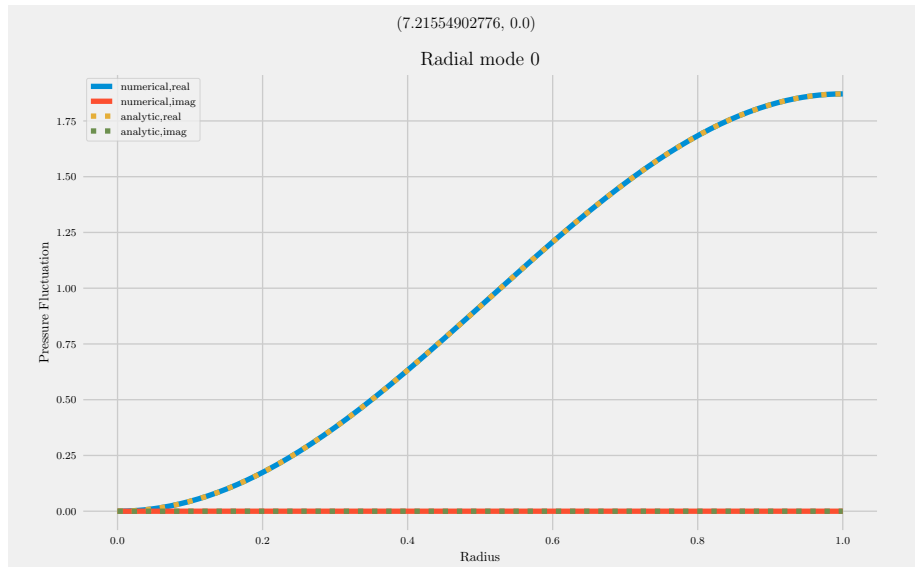
Figure 21: Second Order, Case number 0, 528points
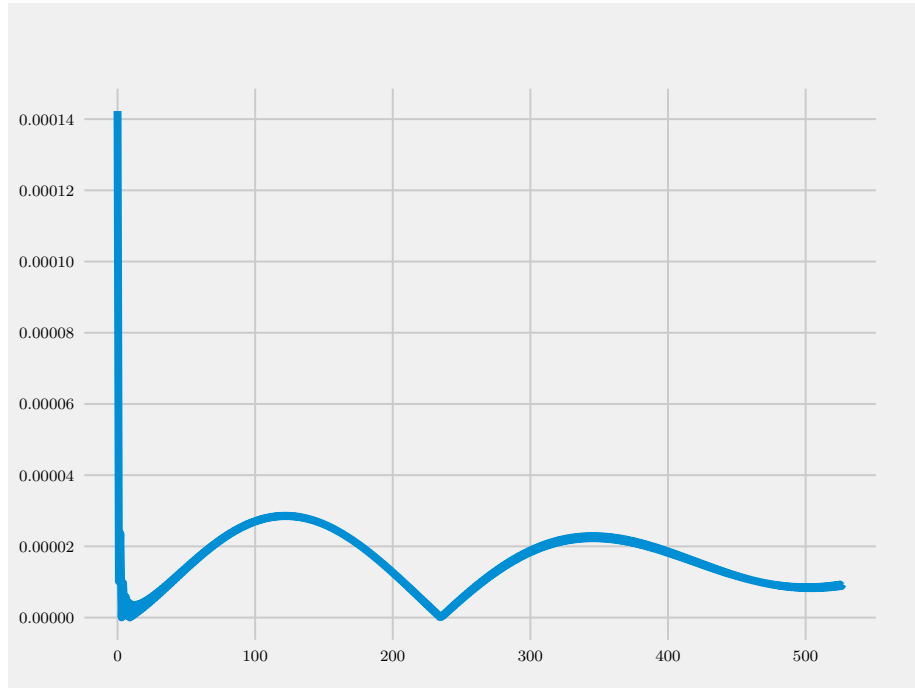
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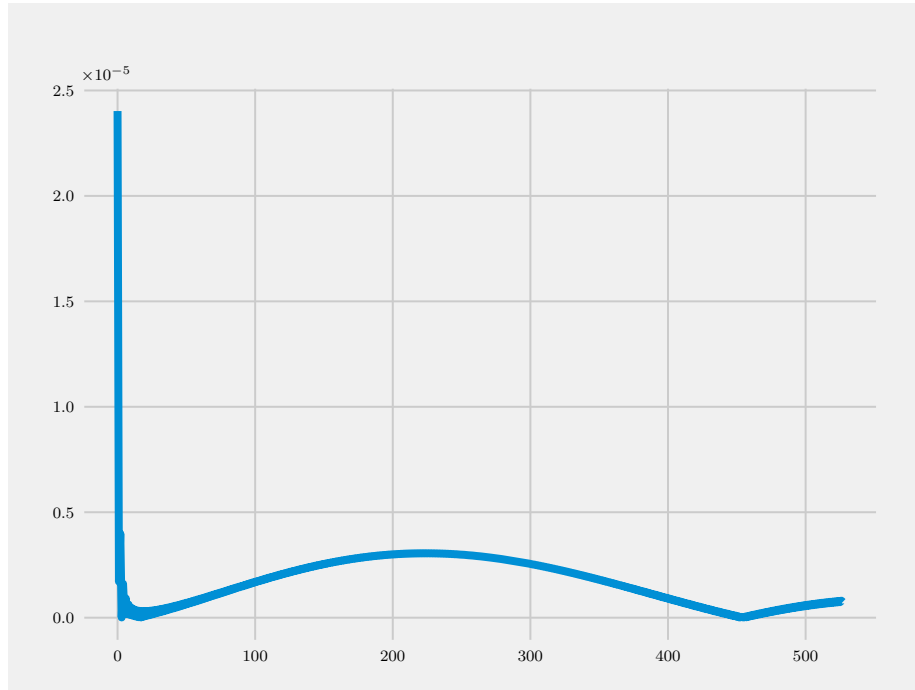
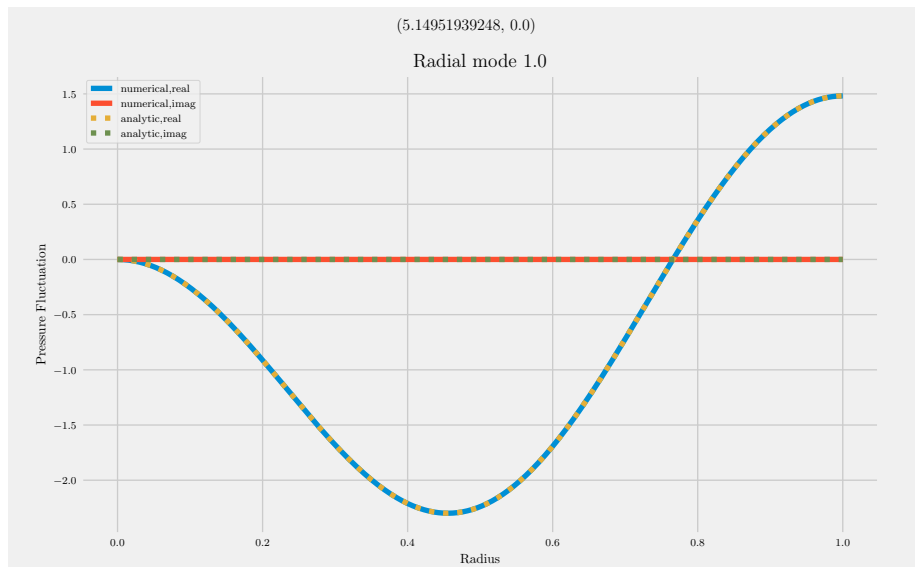Figure 22: Second Order, Case number 0, 528points
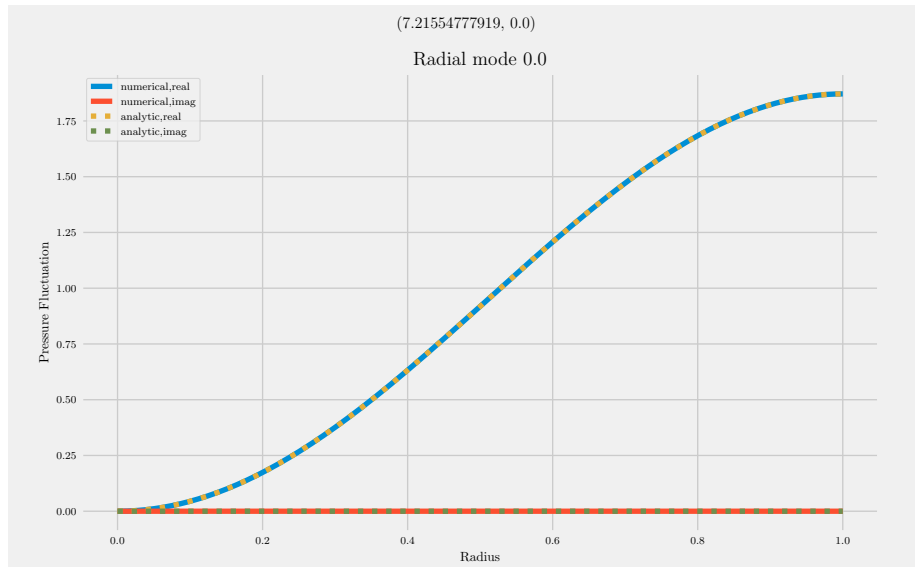
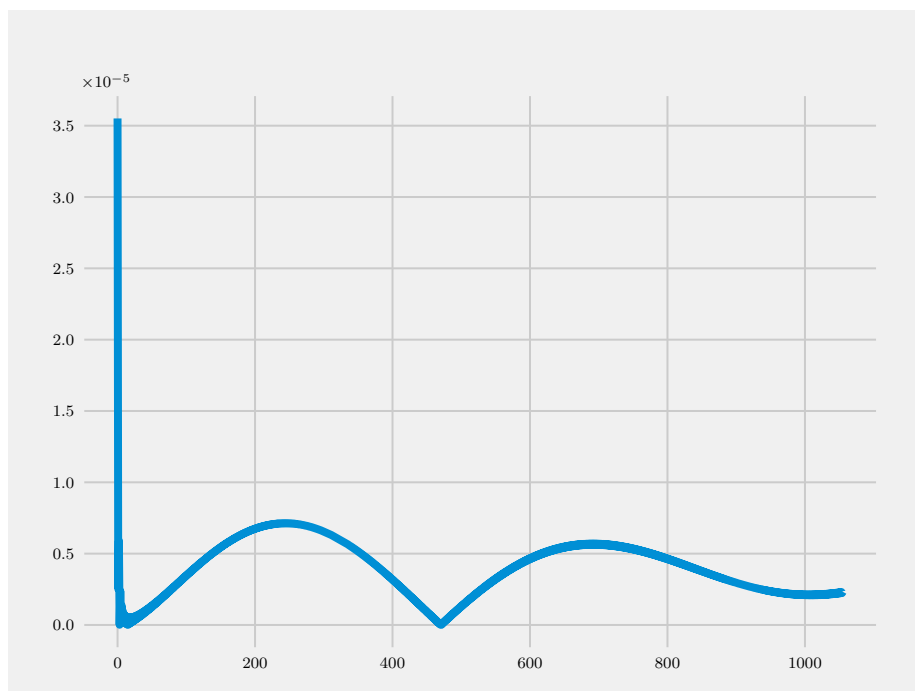Figure 23: Second Order, Case number 0, 1056points

17
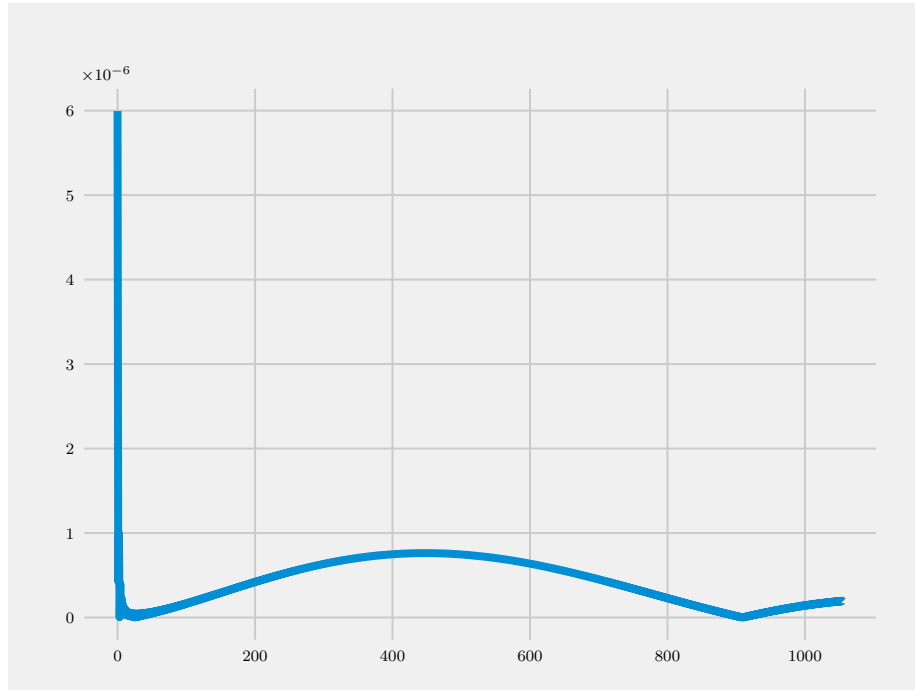
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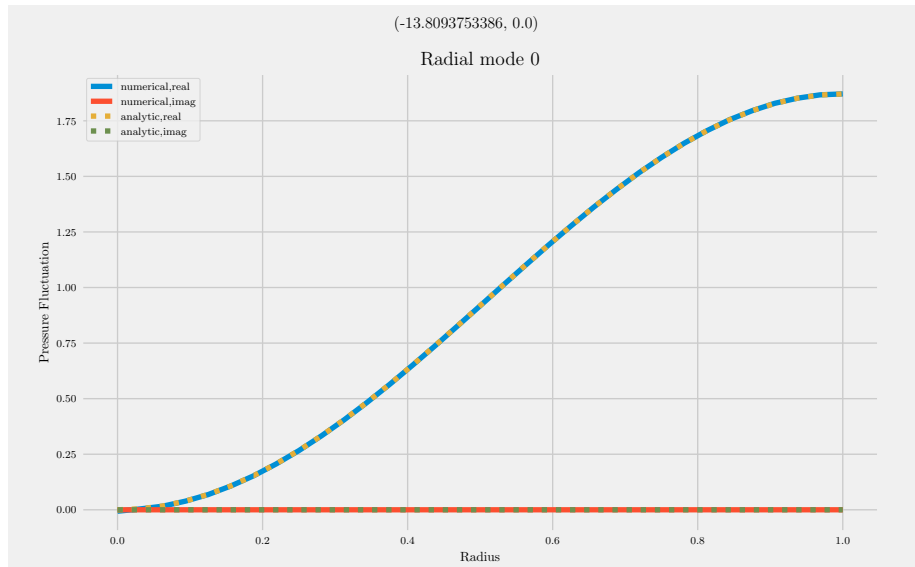
Figure 25: Second Order, Case number 1, 33points
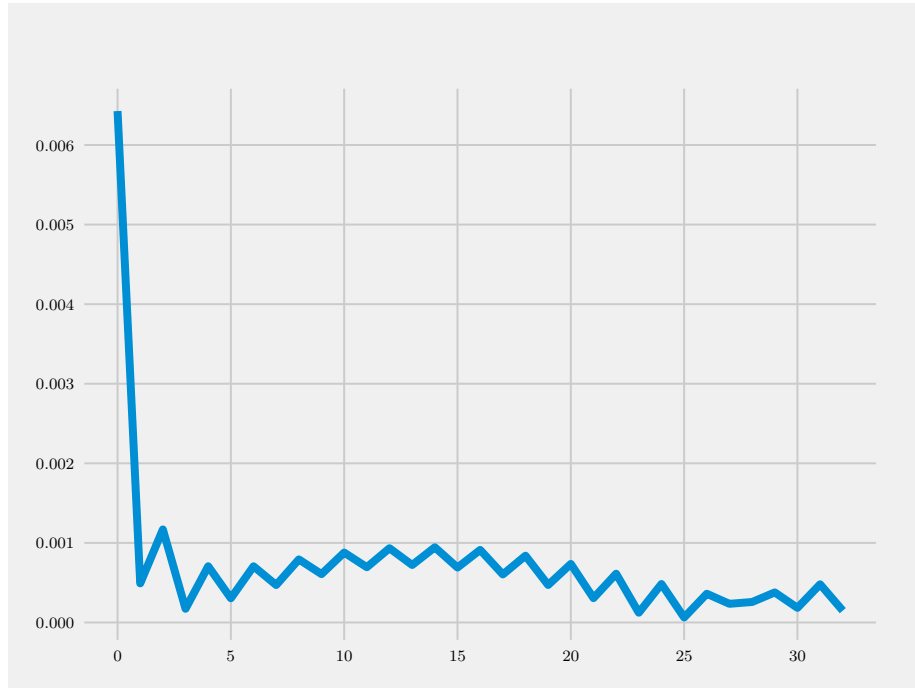
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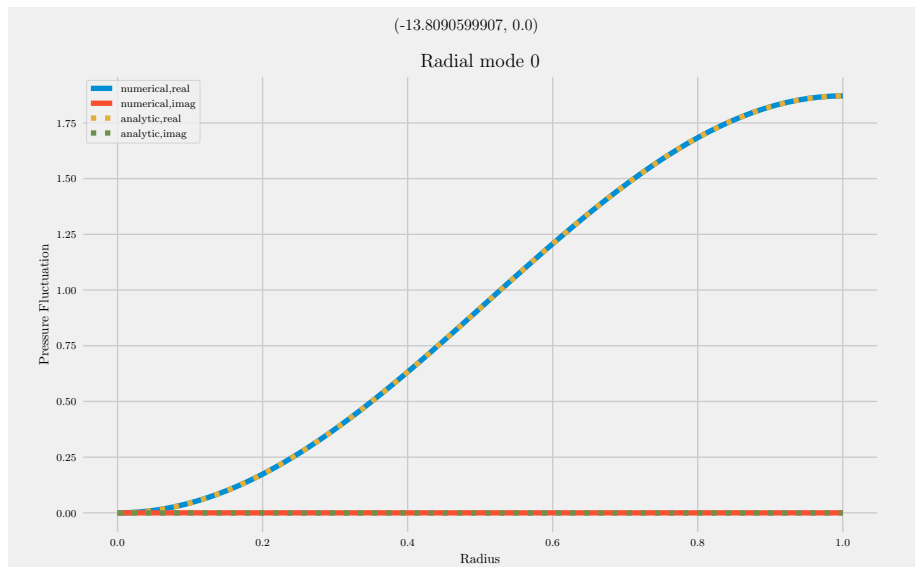
Figure 27: Second Order, Case number 1, 66points

19
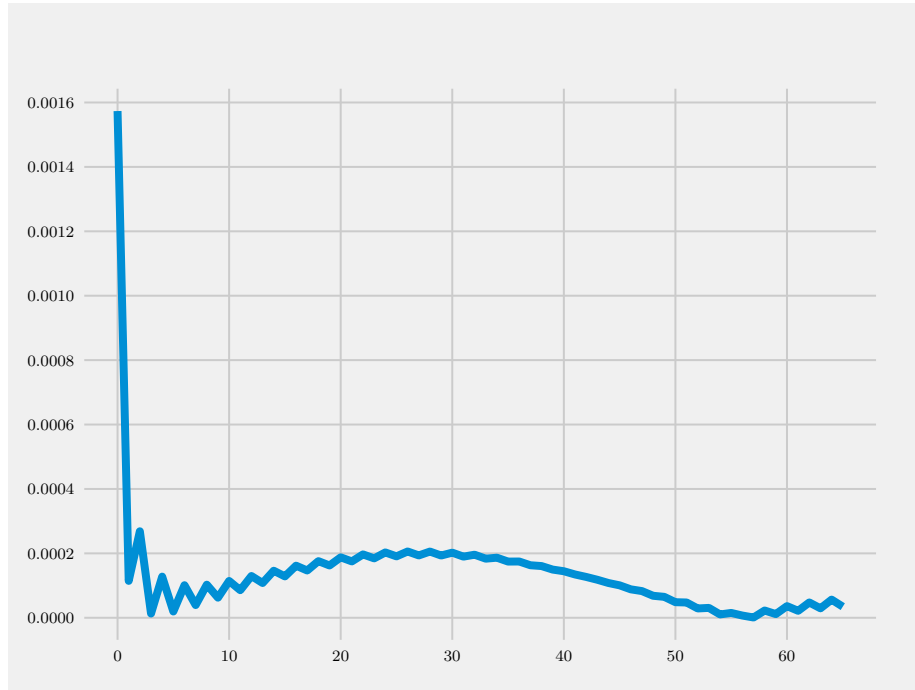
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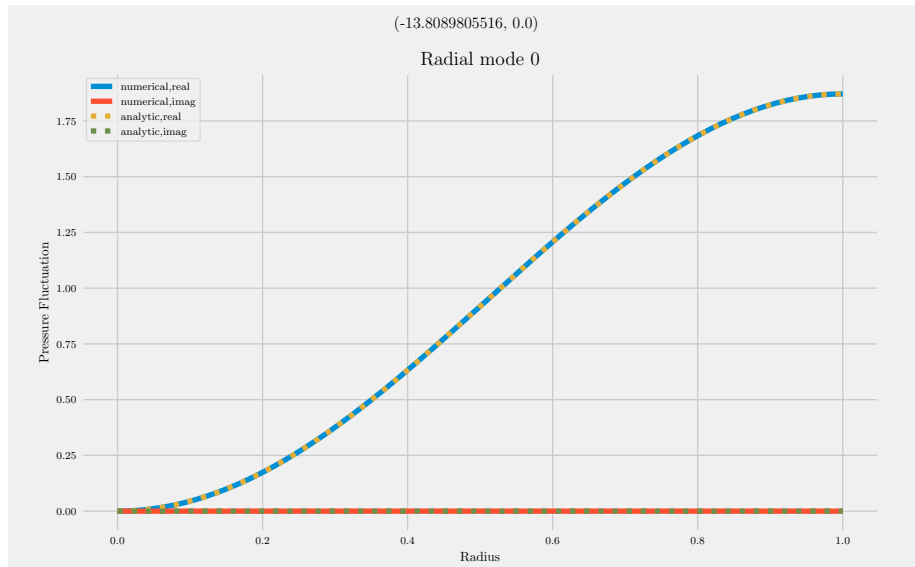
Figure 29: Second Order, Case number 1, 132points

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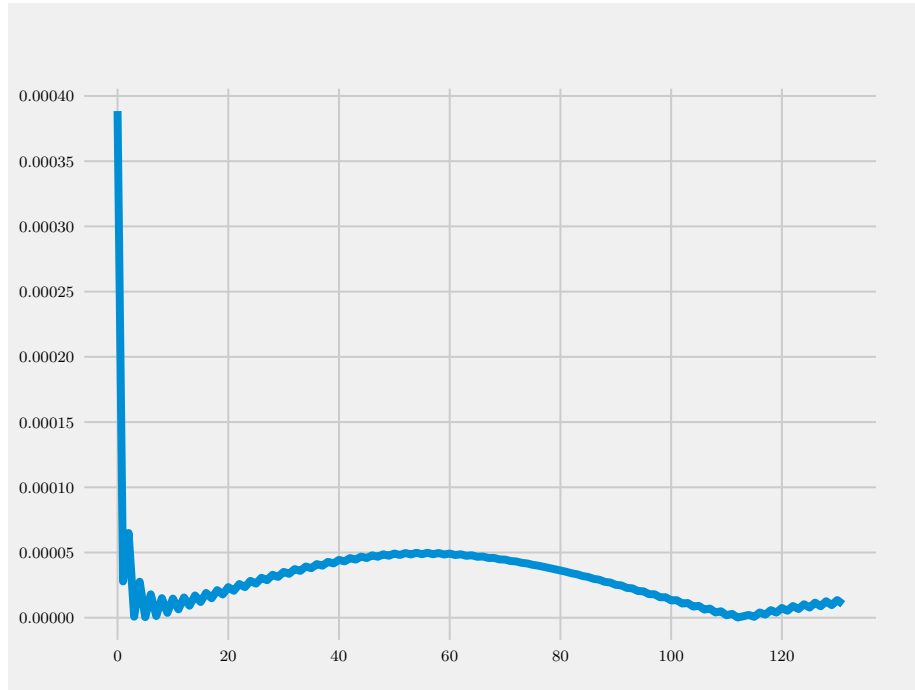
Figure 30: Second Order, Case number 1, 132points
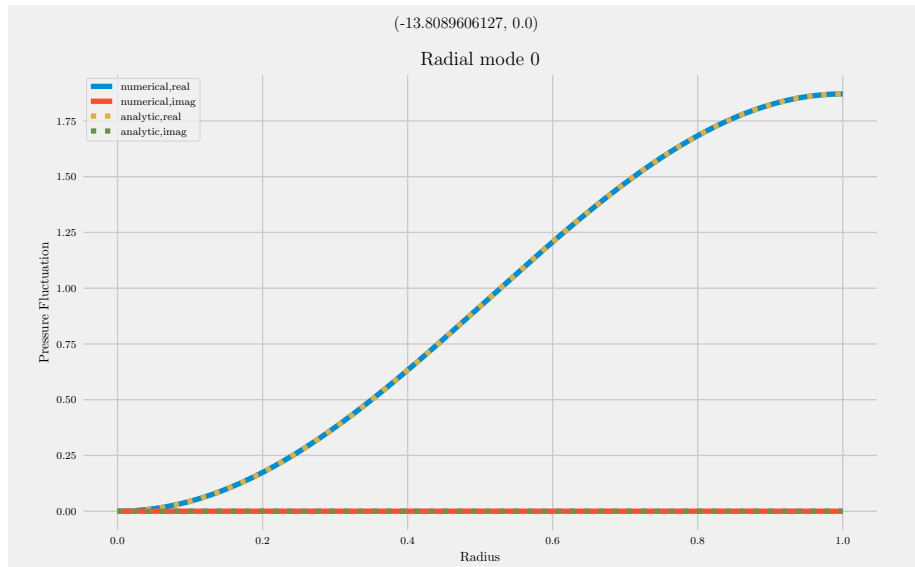
Figure 31: Second Order, Case number 1, 264points

21
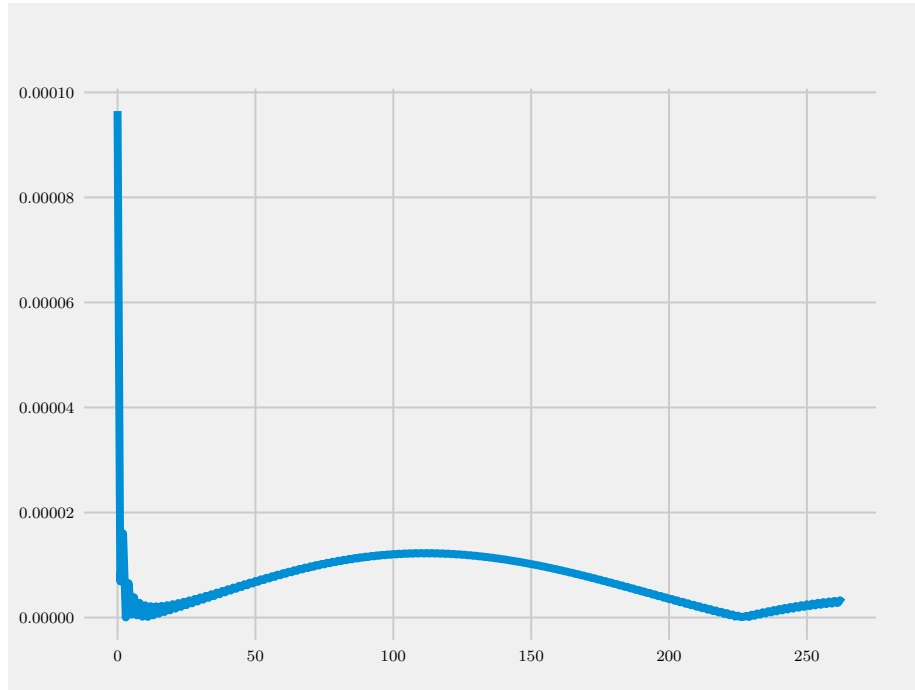
Figure 32: Second Order, Case number 1, 264points
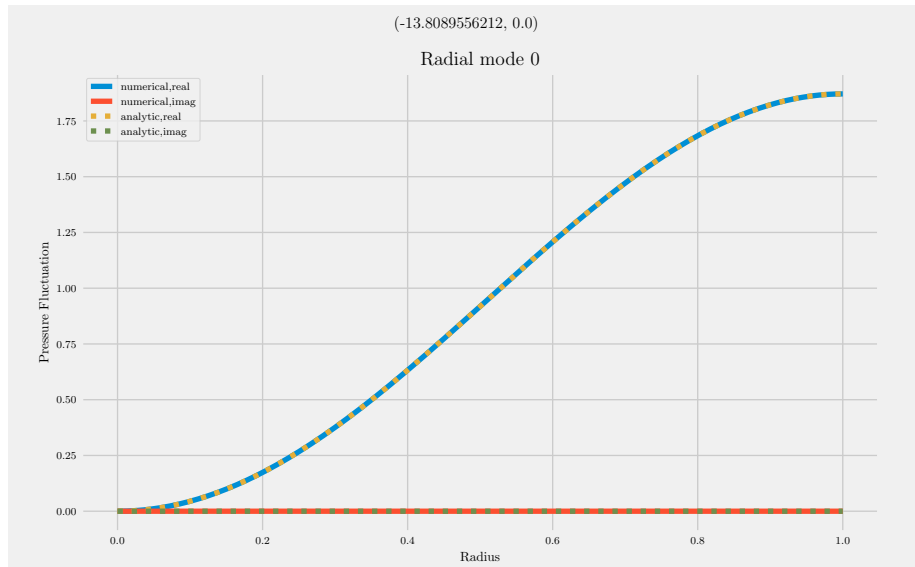
Figure 33: Second Order, Case number 1, 528points
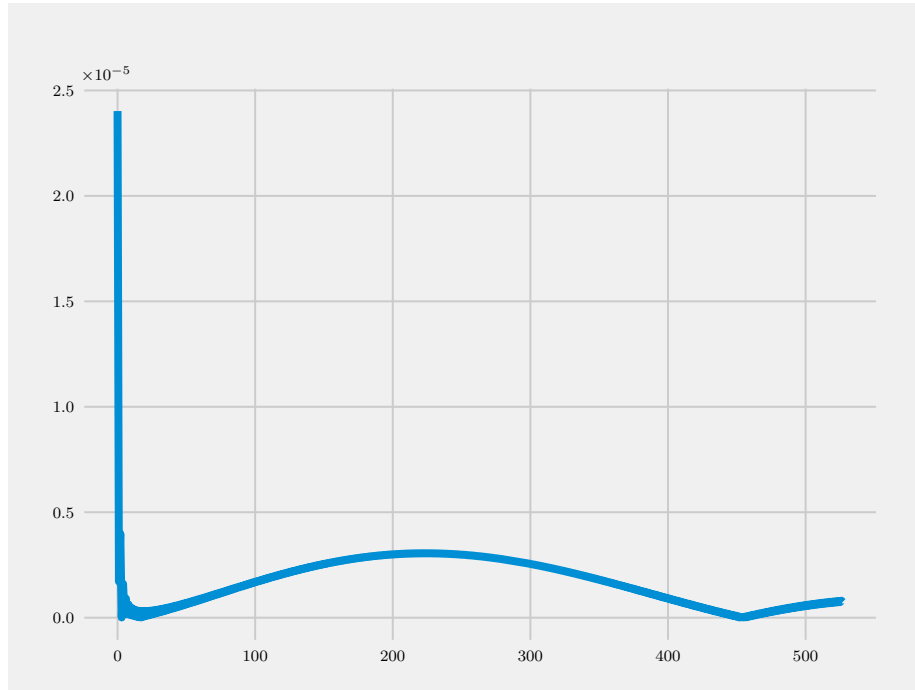
Figure 34: Second Order, Case number 1, 528points
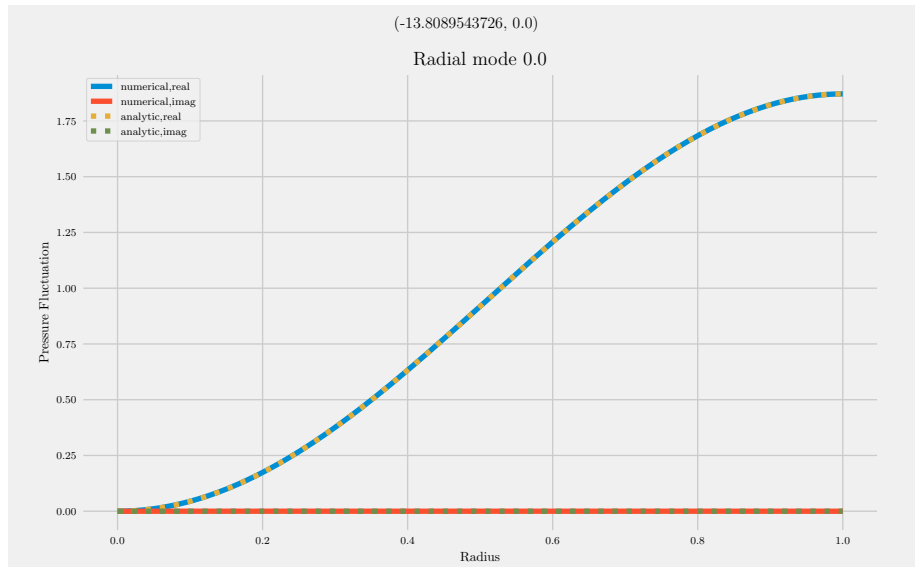
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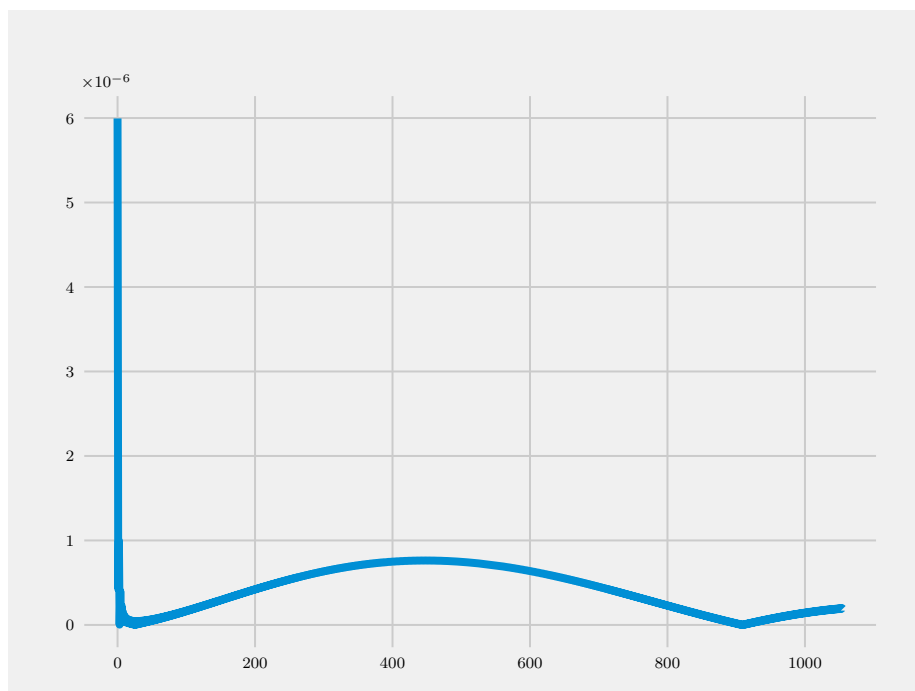
Figure 36: Second Order, Case number 1, 1056points
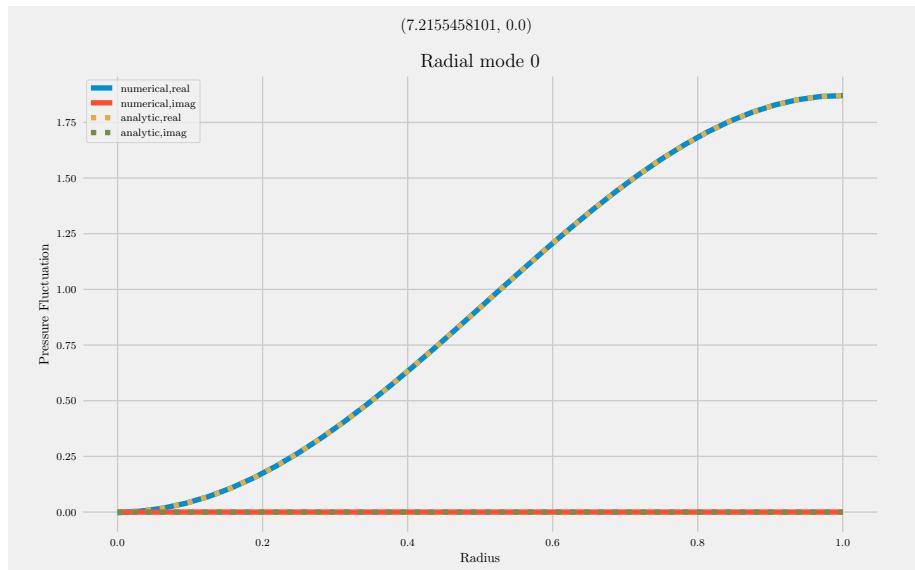
Figure 37: Fourth Order, Case number 0, 33points

Fourth Order, Radial Mode 0
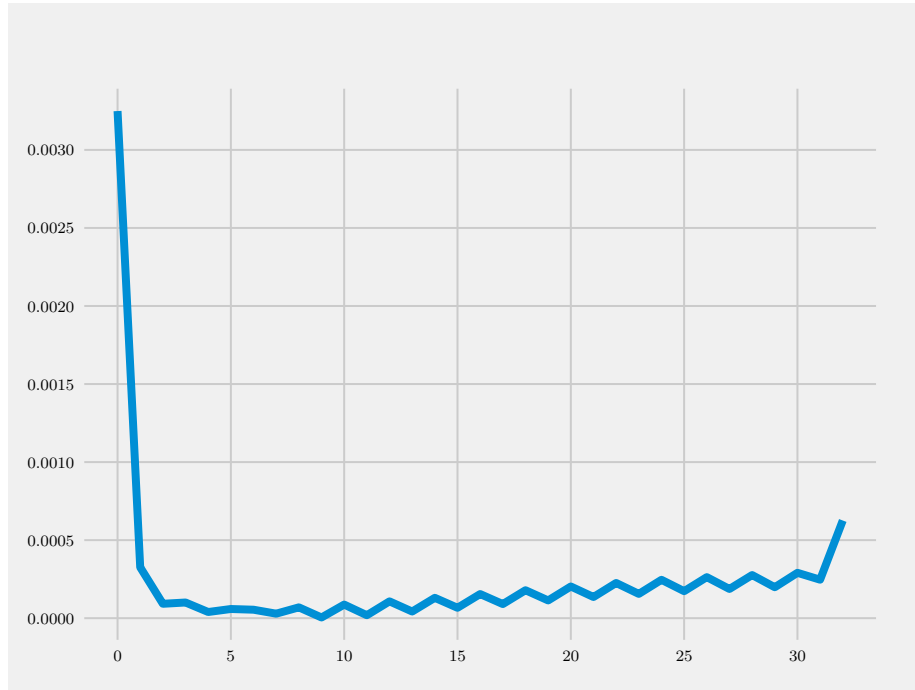
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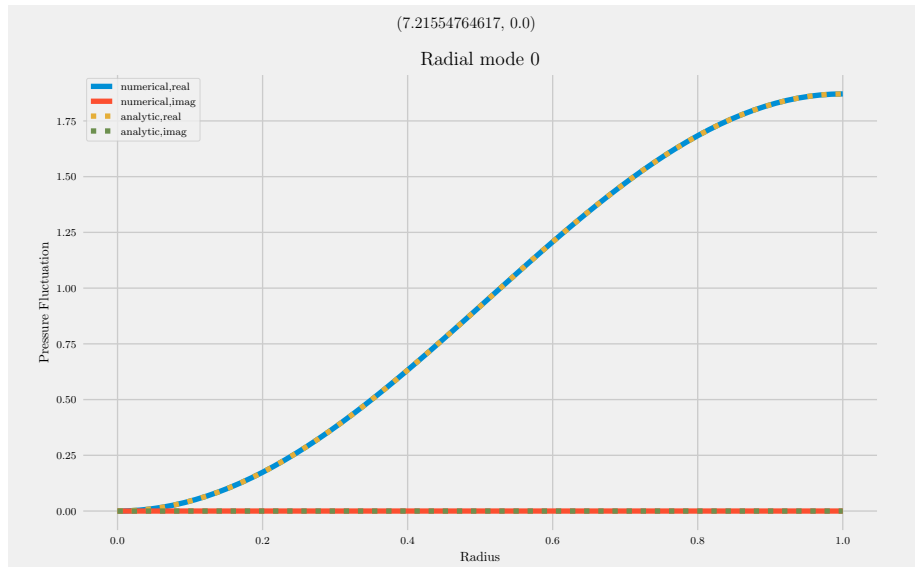
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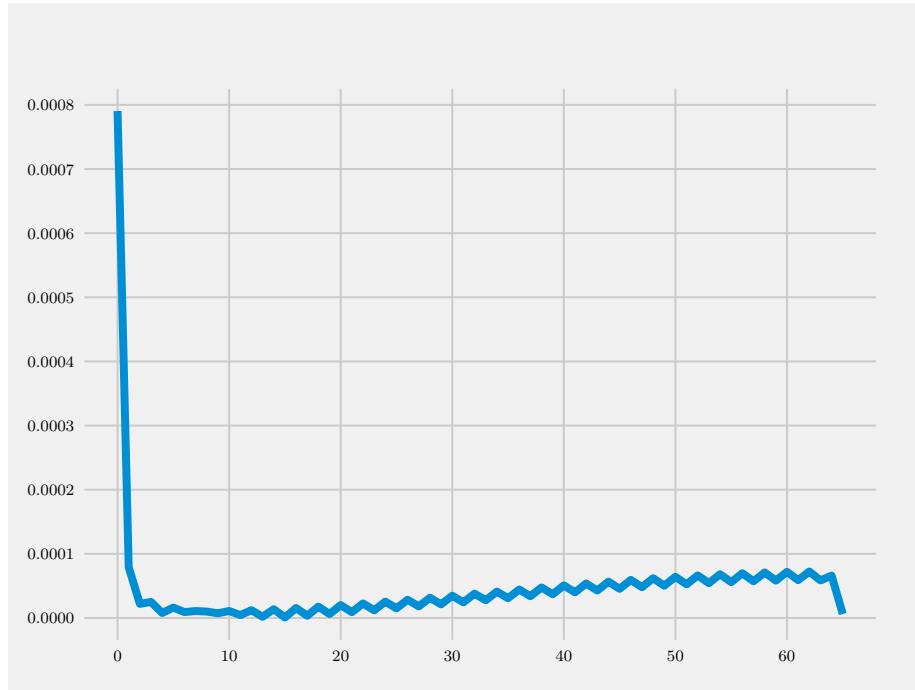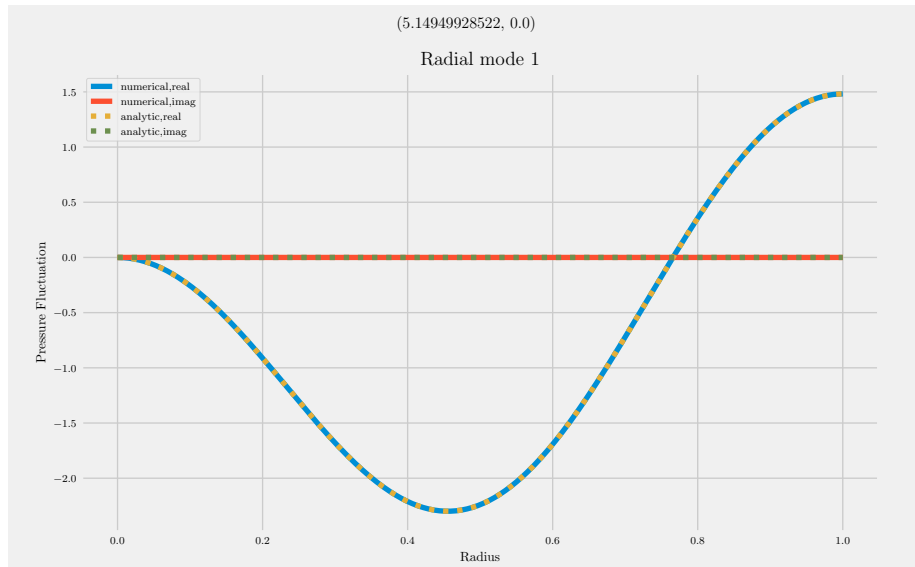
Figure 40: Fourth Order, Case number 0, 66points
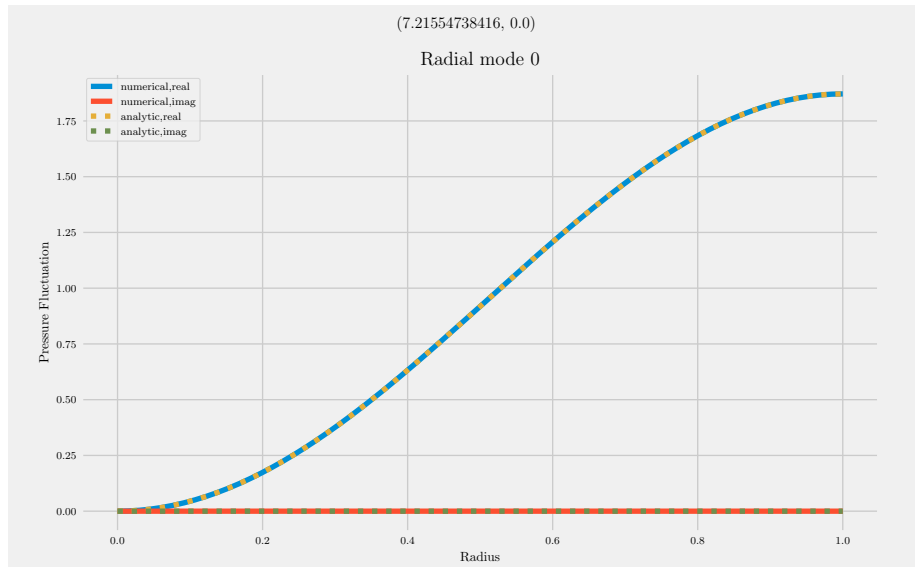
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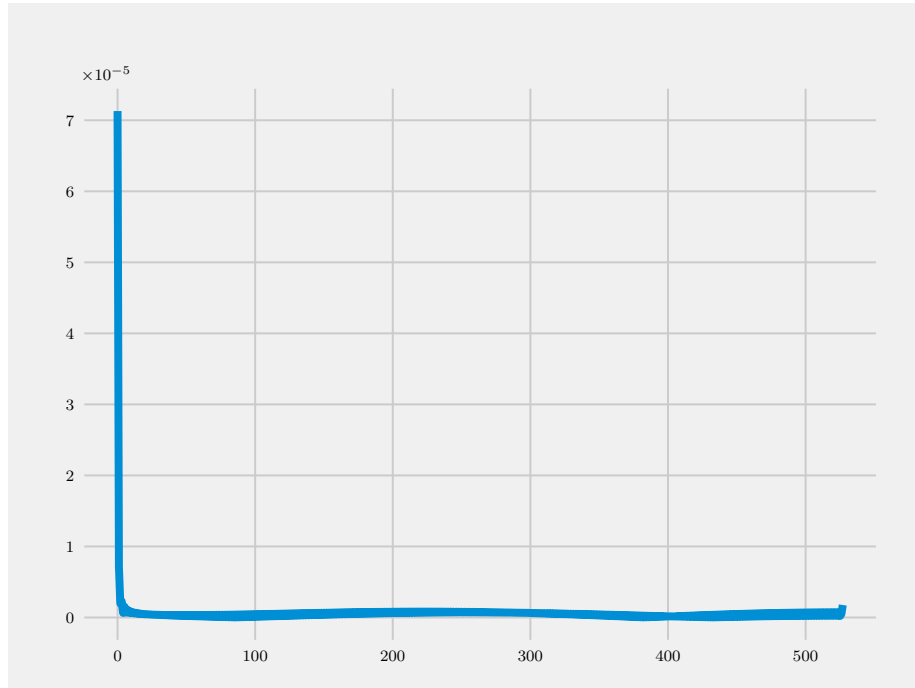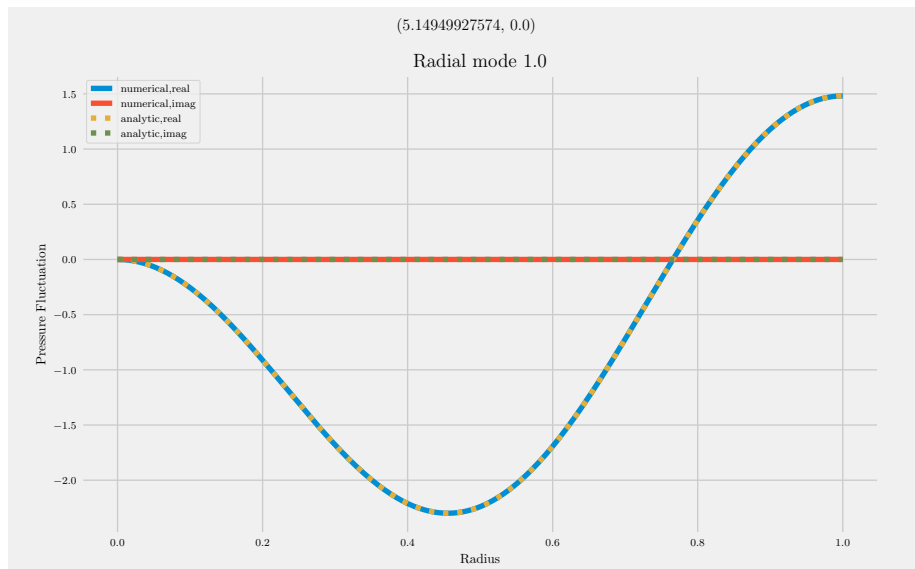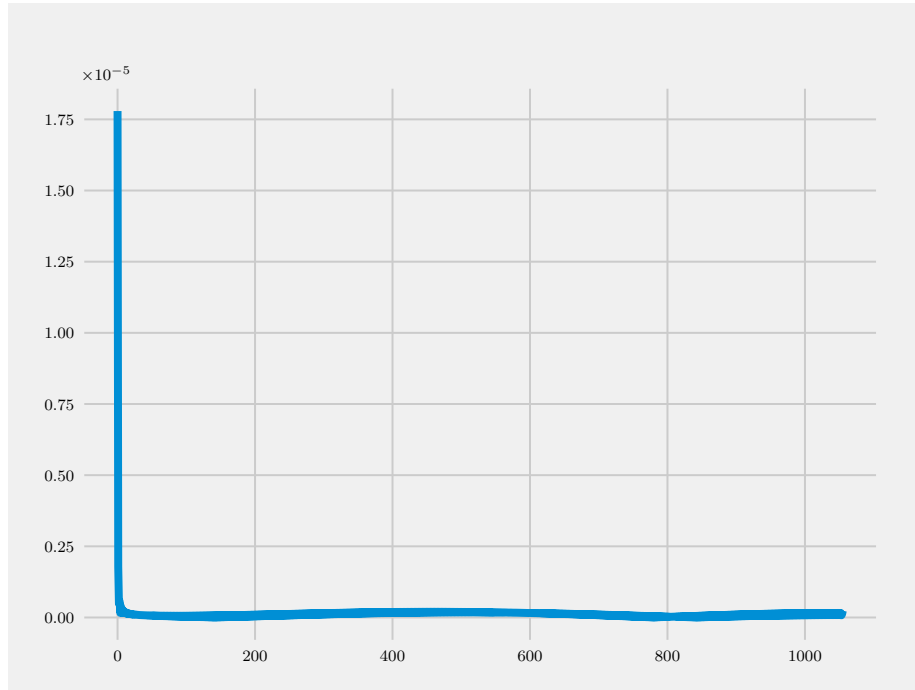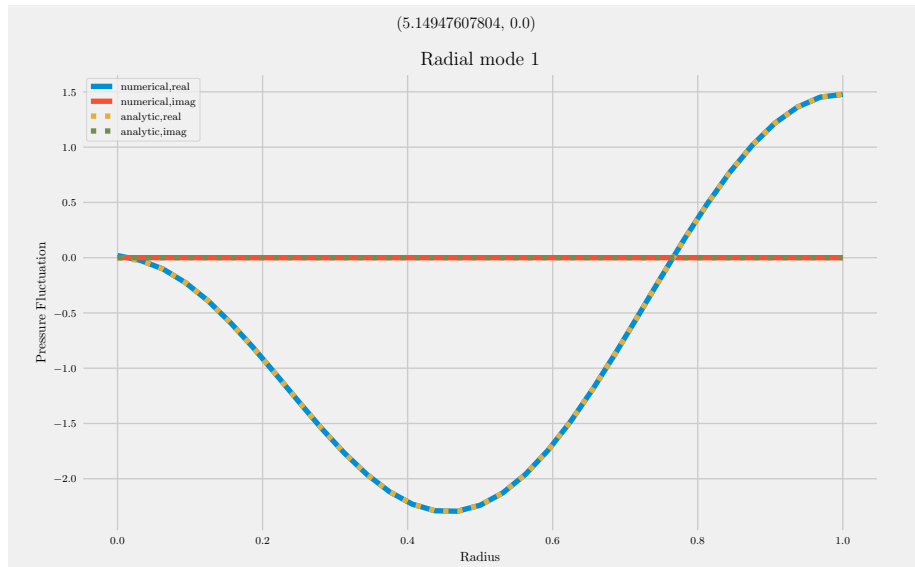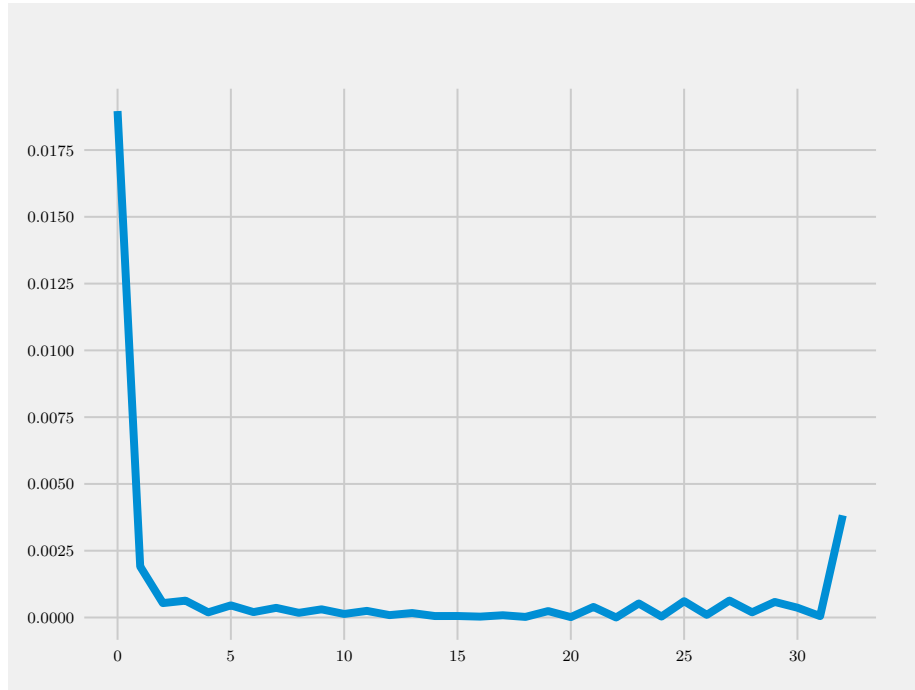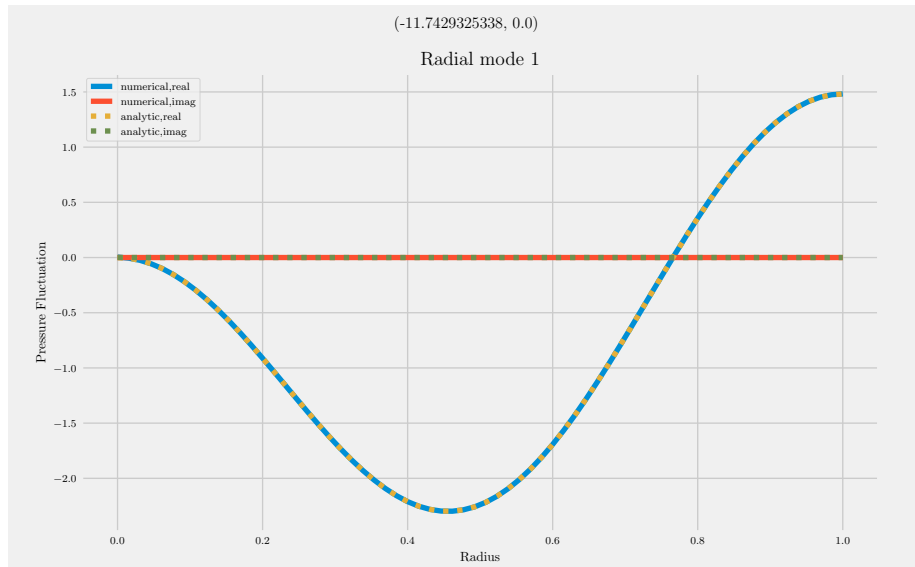
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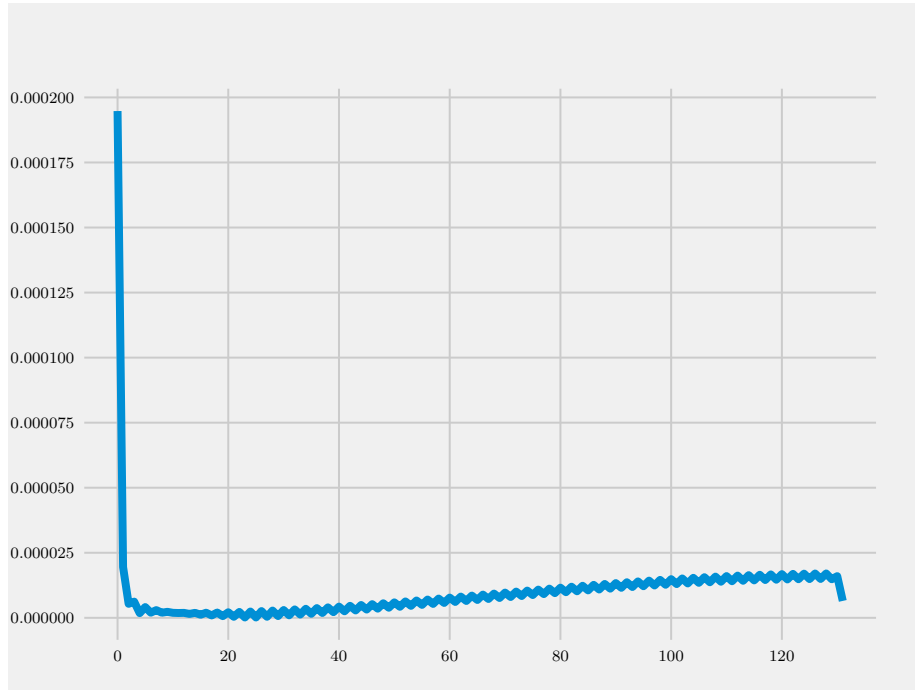
Figure 42: Fourth Order, Case number 0, 132points
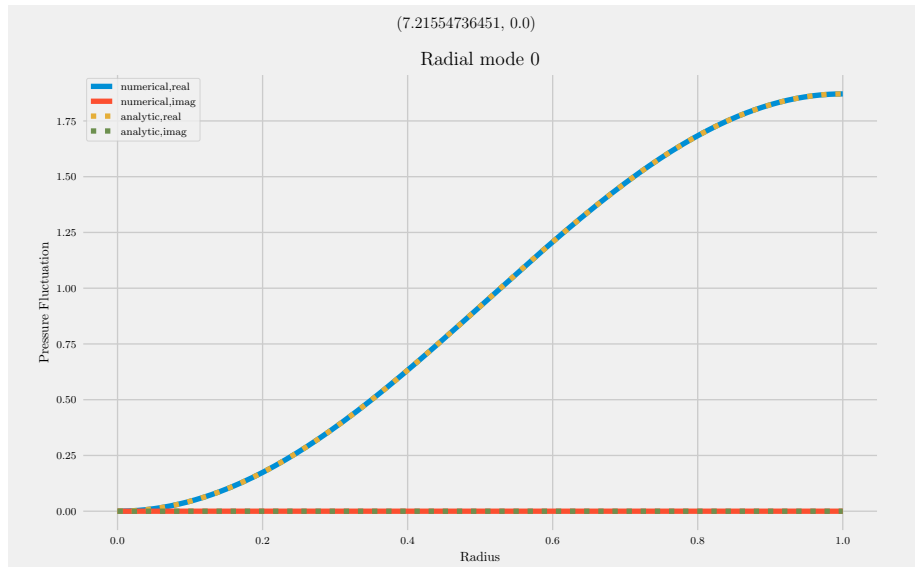
Figure 43: Fourth Order, Case number 0, 264points

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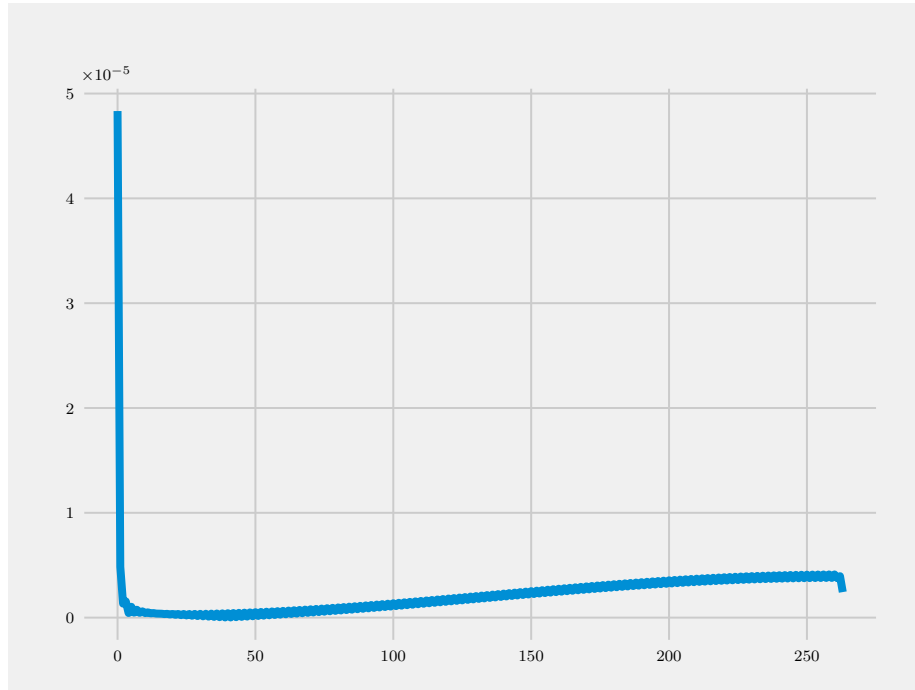
Figure 44: Fourth Order, Case number 0, 264points
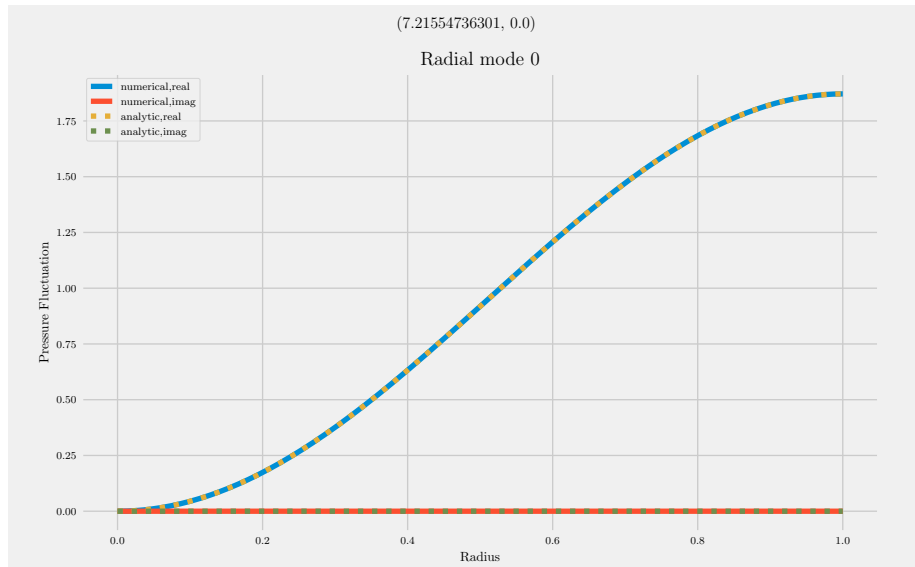
Figure 45: Fourth Order, Case number 0, 528points

29
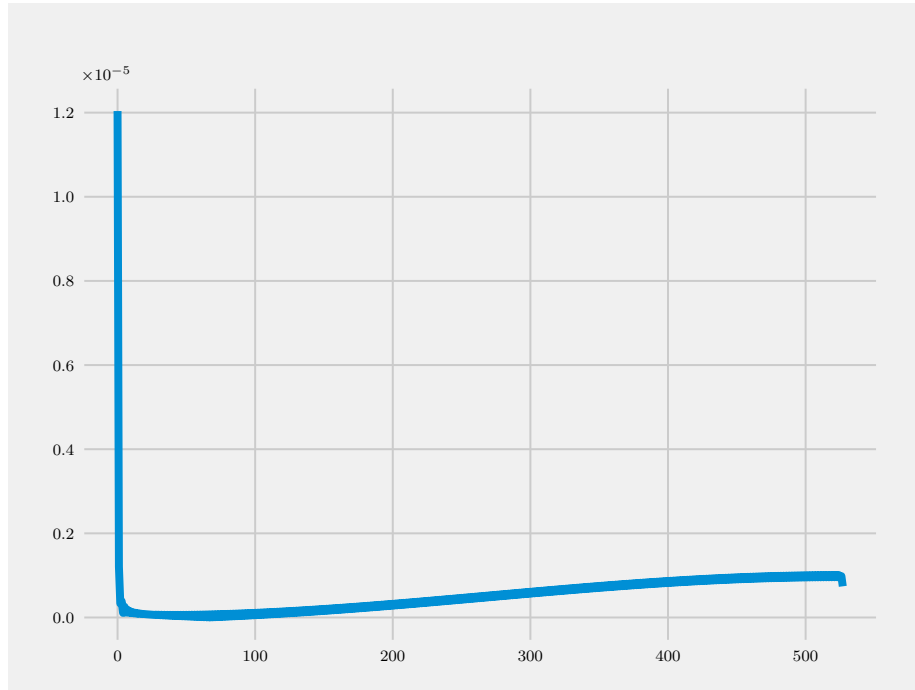
Figure 46: Fourth Order, Case number 0, 528points
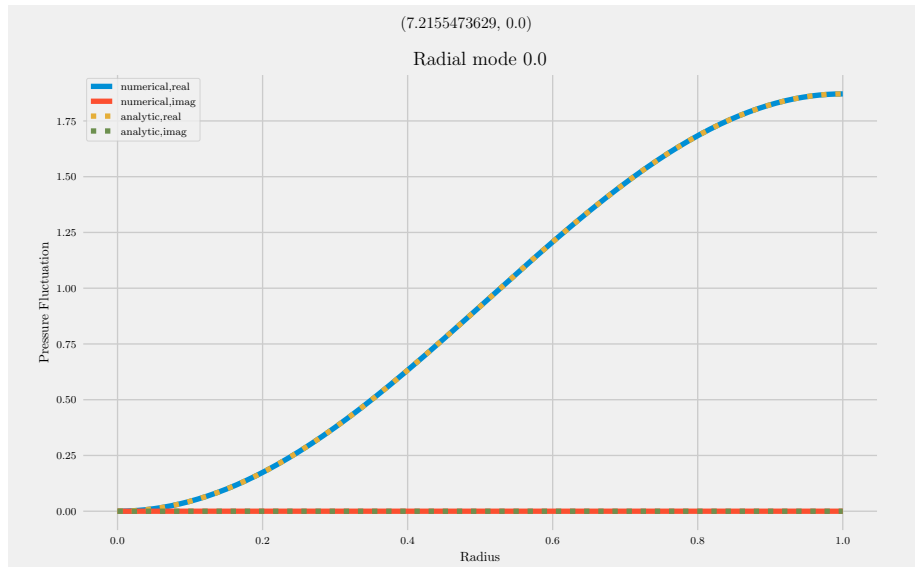
Figure 47: Fourth Order, Case number 0, 1056points
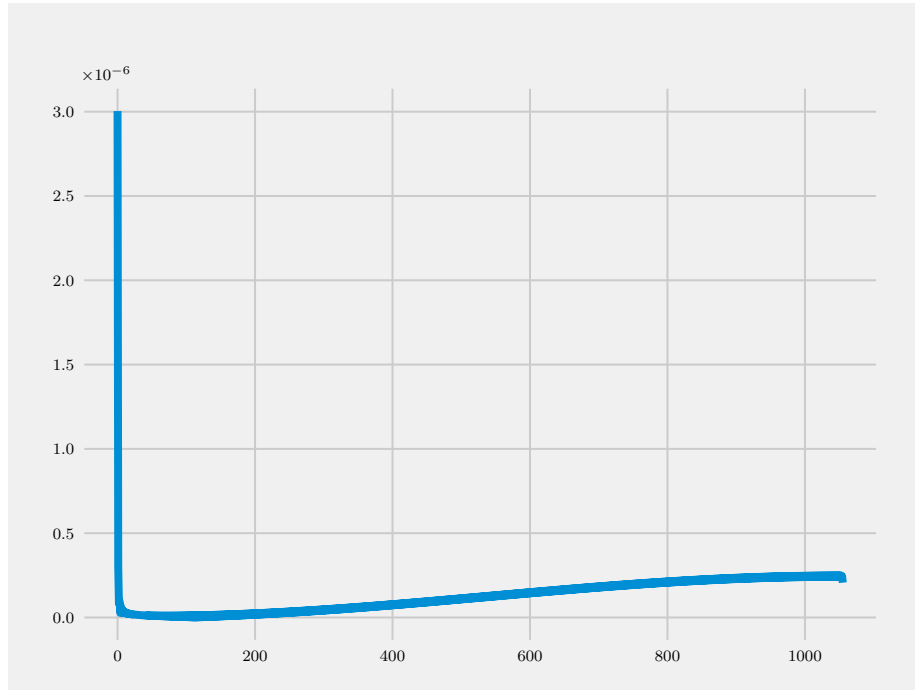
Figure 48: Fourth Order, Case number 0, 1056points
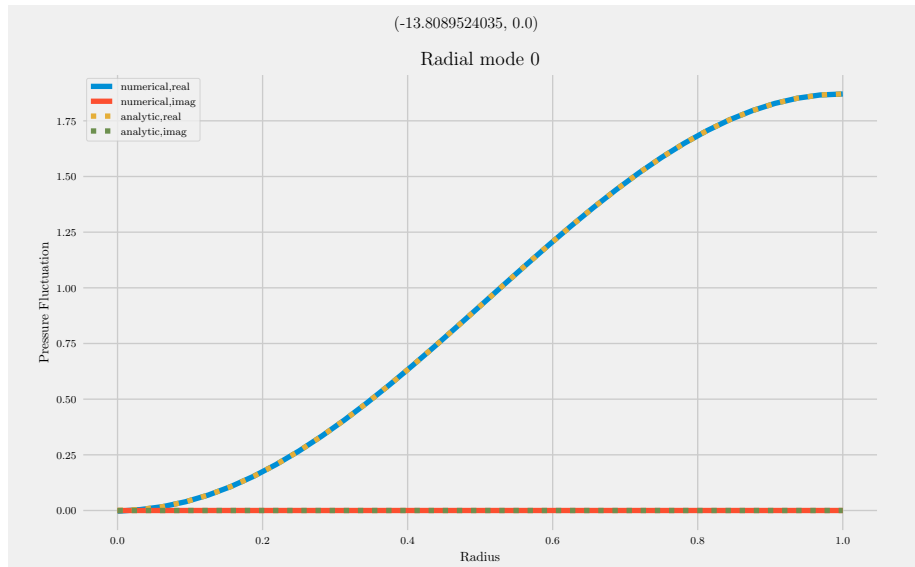
Figure 49: Fourth Order, Case number 1, 33points
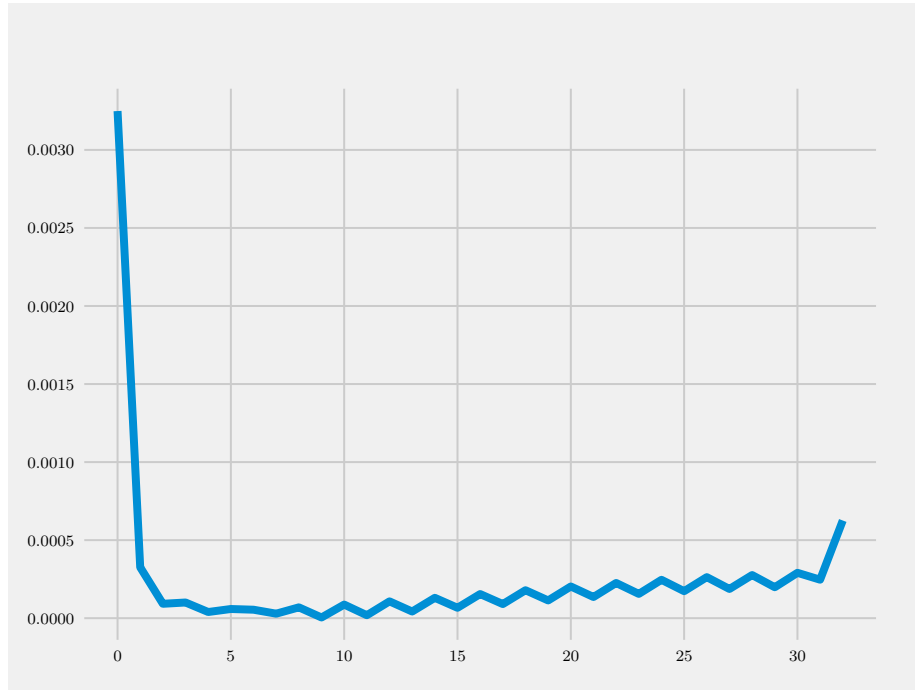
Figure 50: Fourth Order, Case number 1, 33points
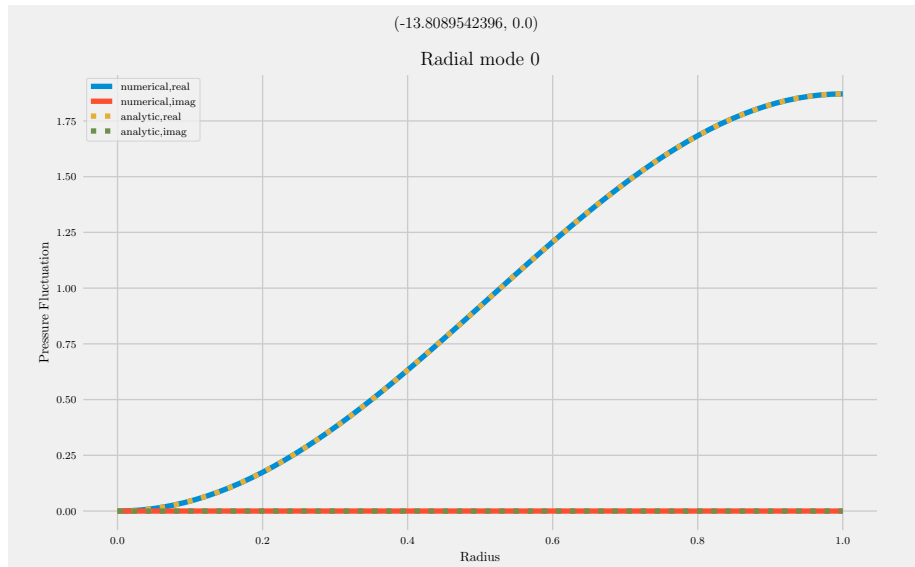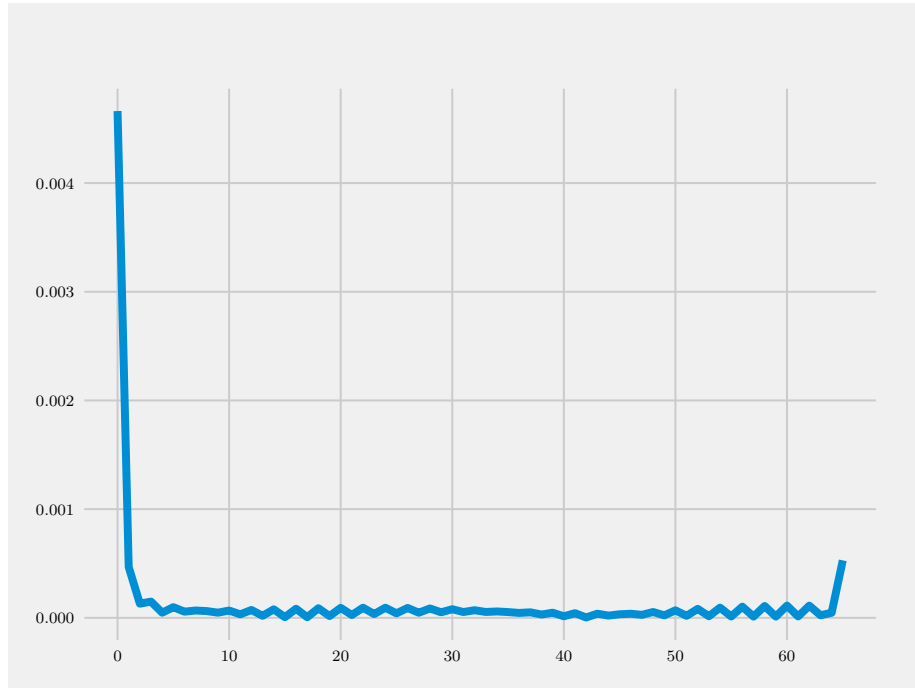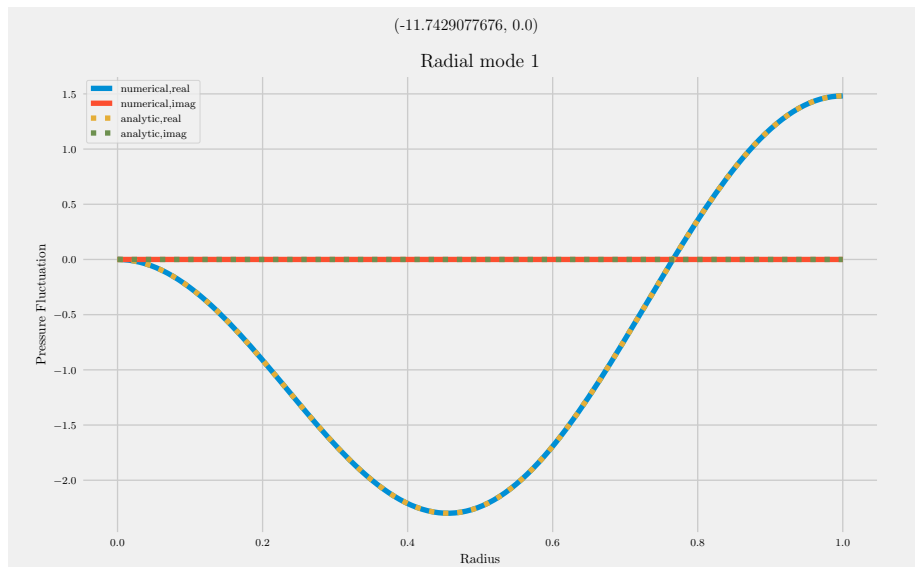

Figure 51: Fourth Order, Case number 1, 66points
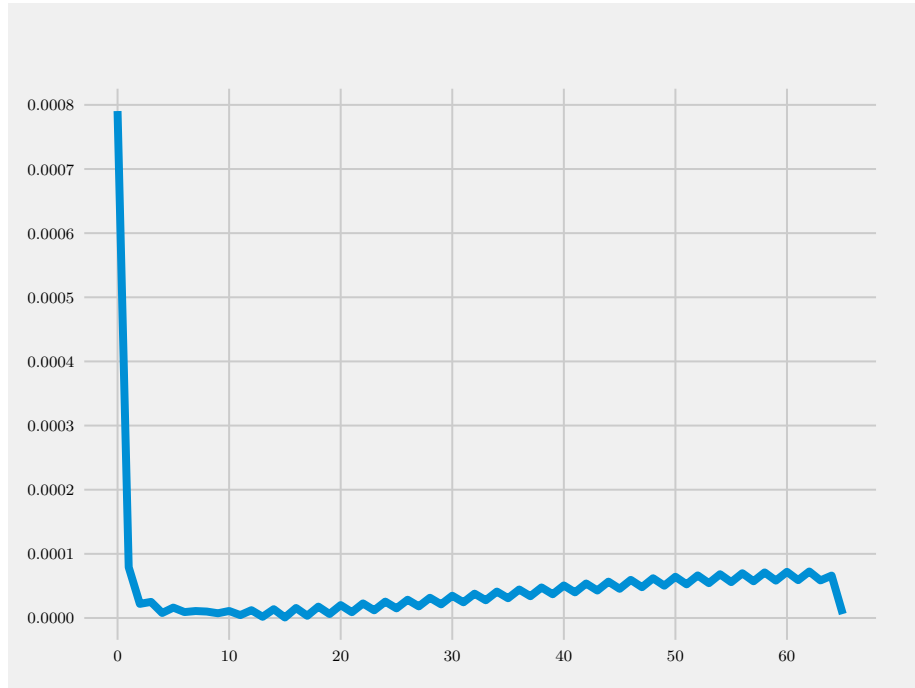
Figure 52: Fourth Order, Case number 1, 66points
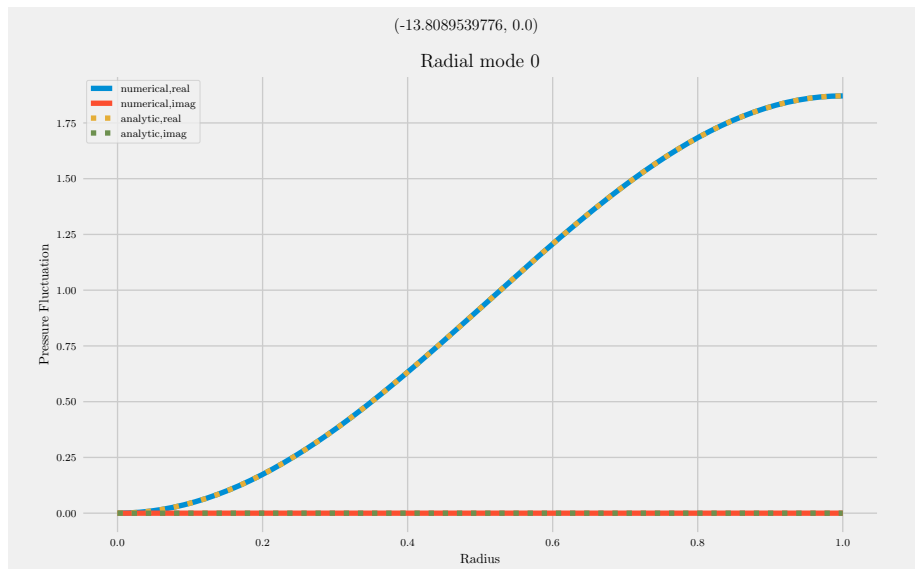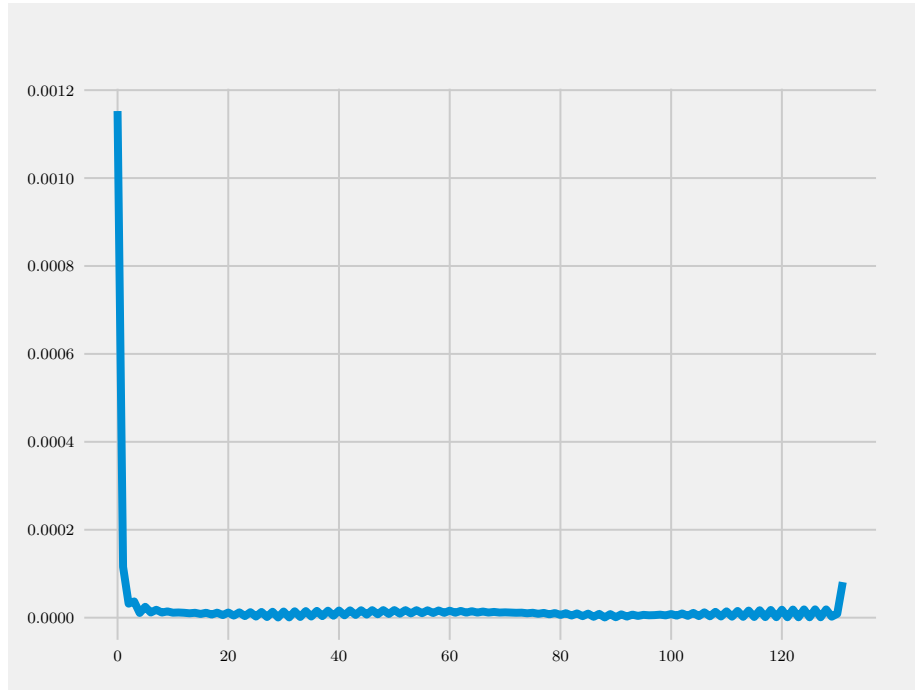

Figure 53: Fourth Order, Case number 1, 132points
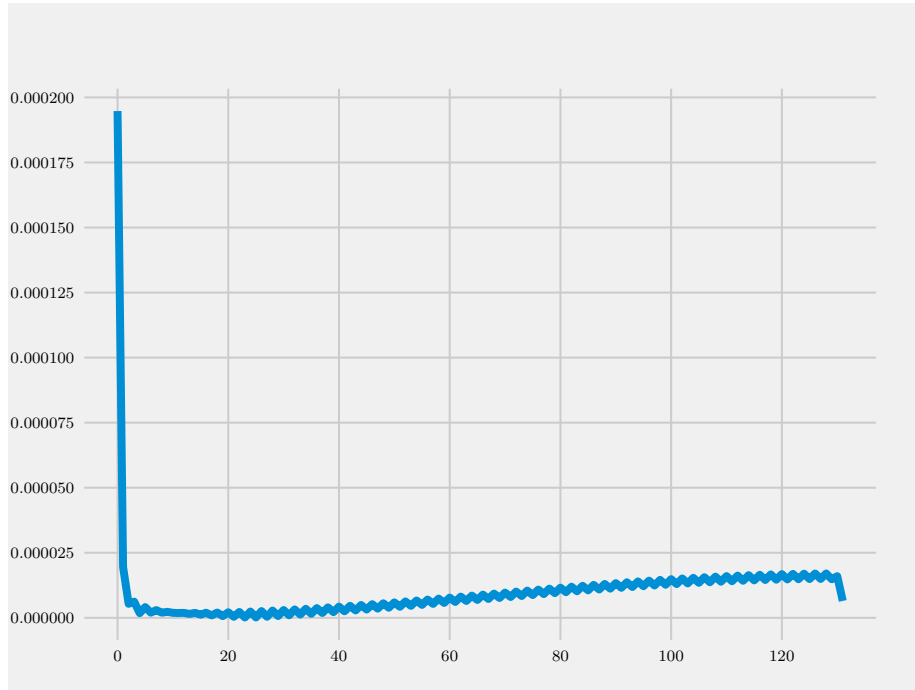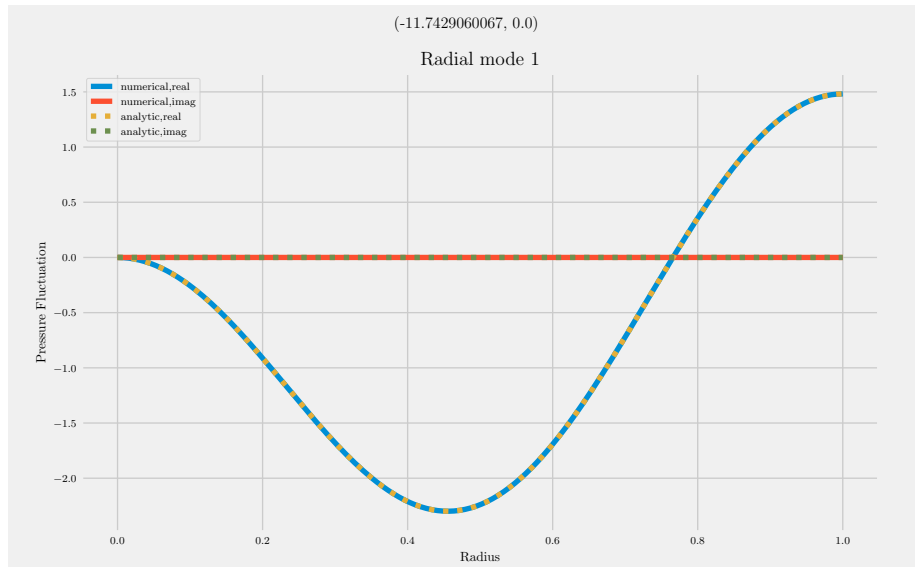
Figure 54: Fourth Order, Case number 1, 132points
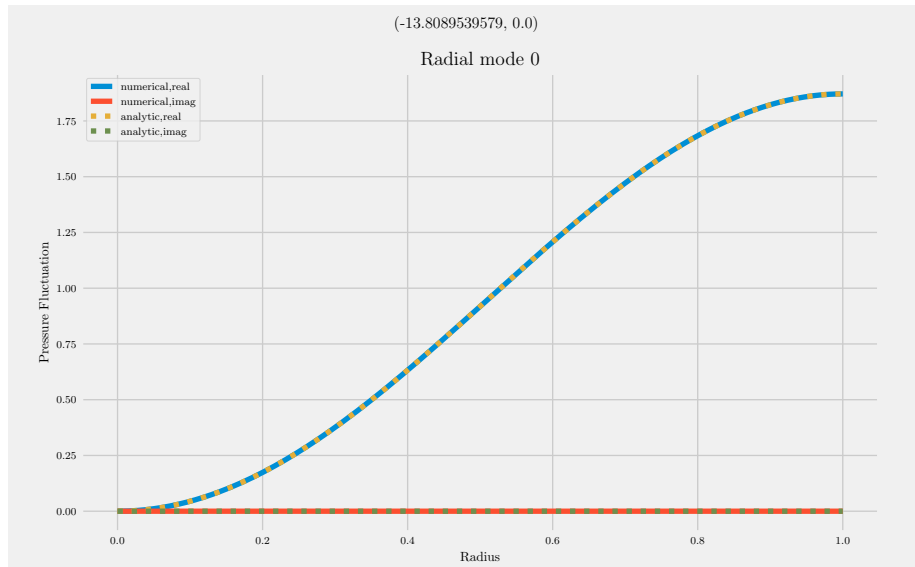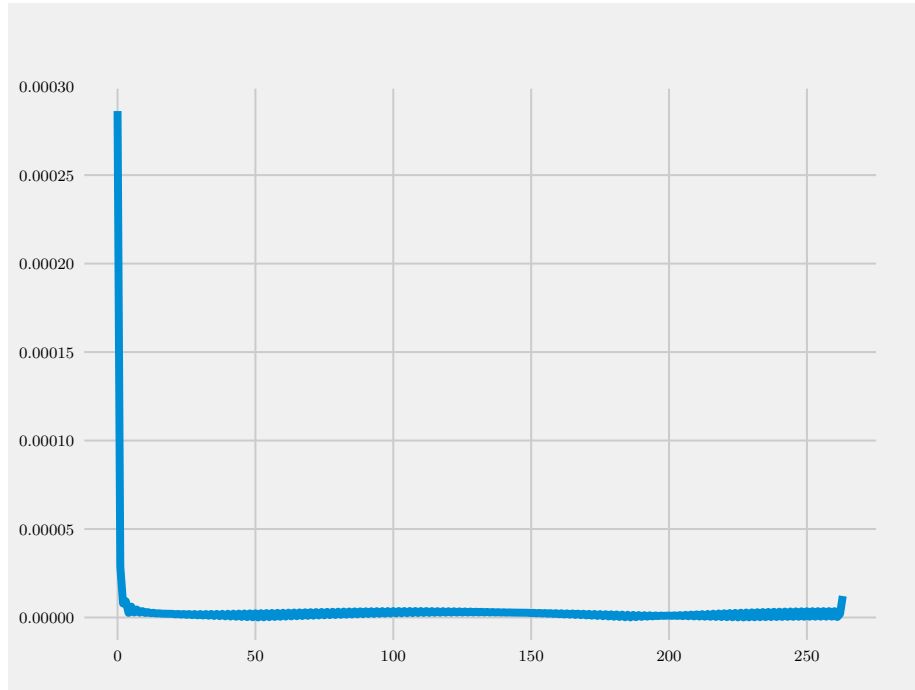

Figure 55: Fourth Order, Case number 1, 264points
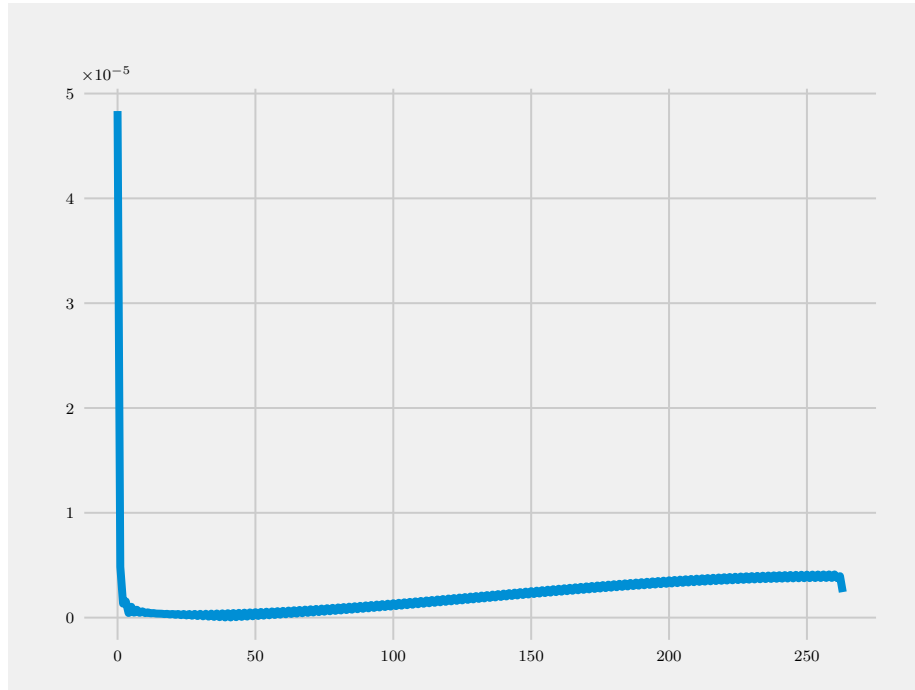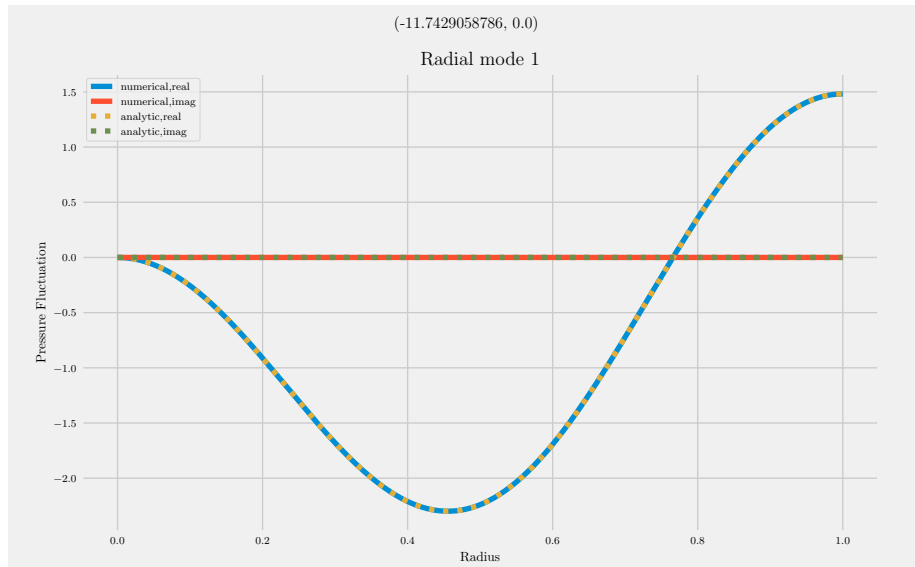
Figure 56: Fourth Order, Case number 1, 264points
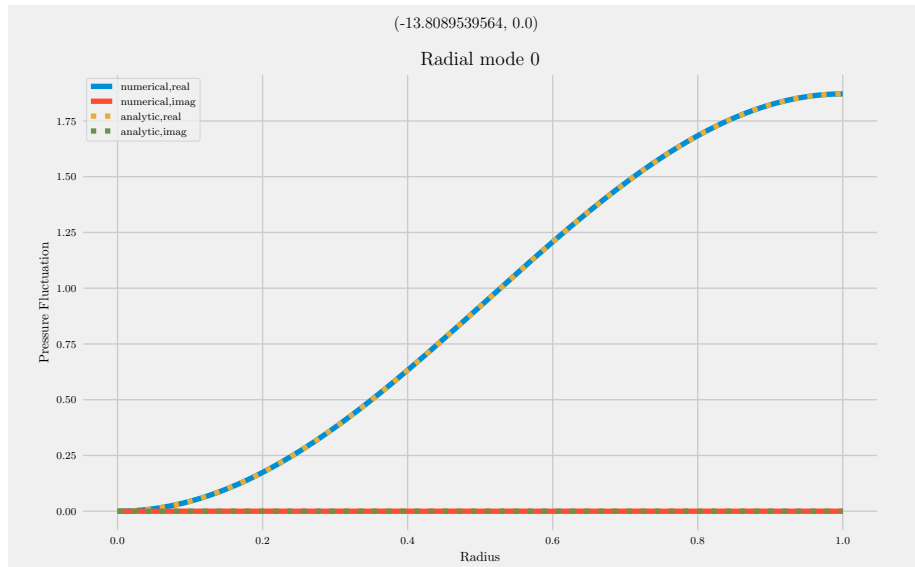
Figure 57: Fourth Order, Case number 1, 528points
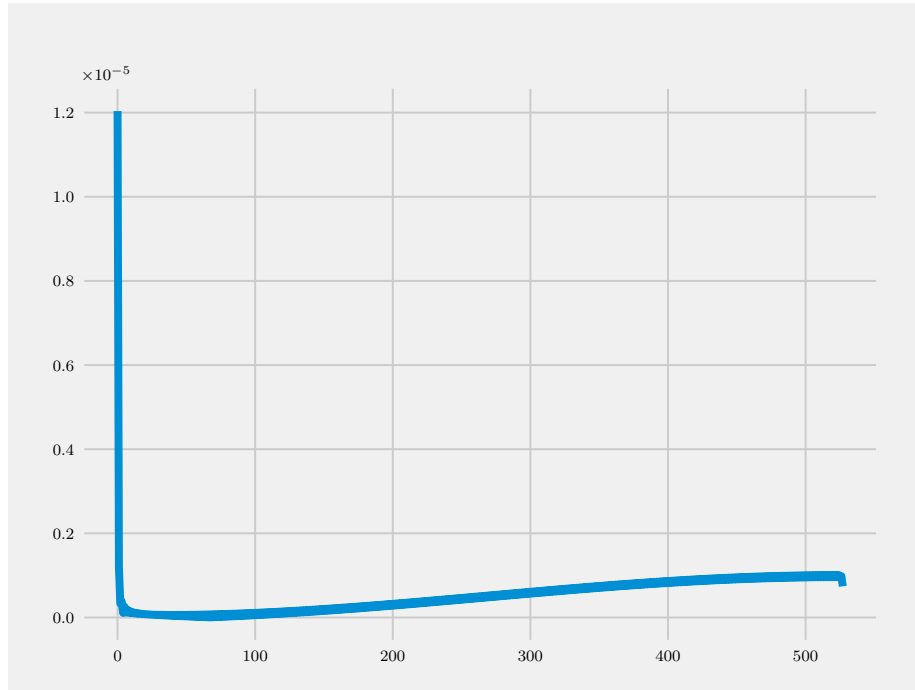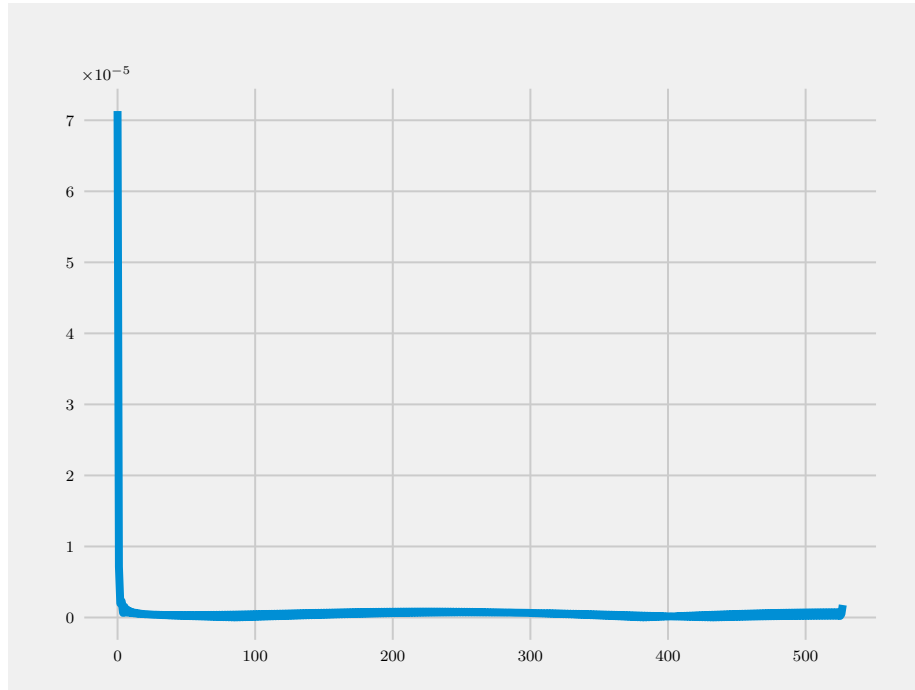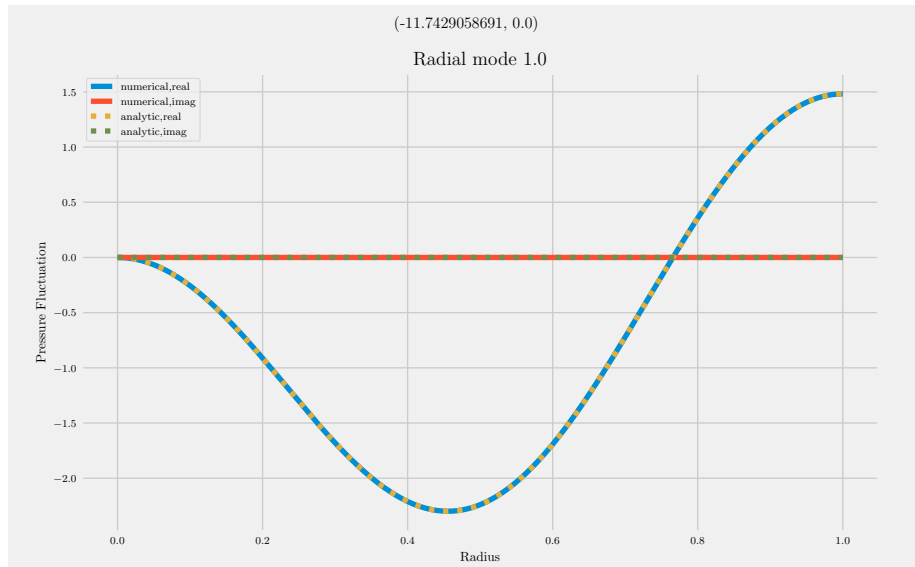
Figure 58: Fourth Order, Case number 1, 528points
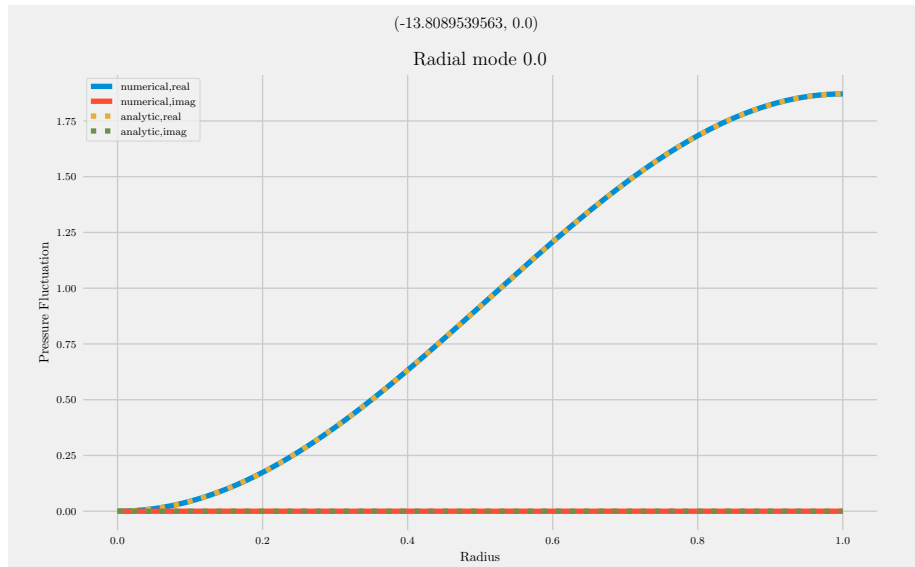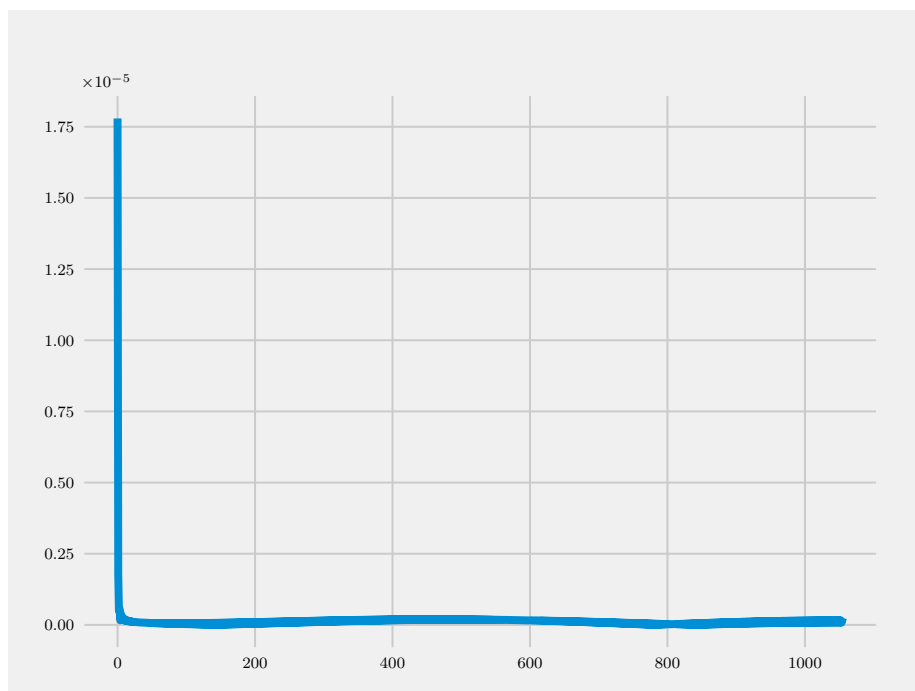

Figure 59: Fourth Order, Case number 1, 1056points
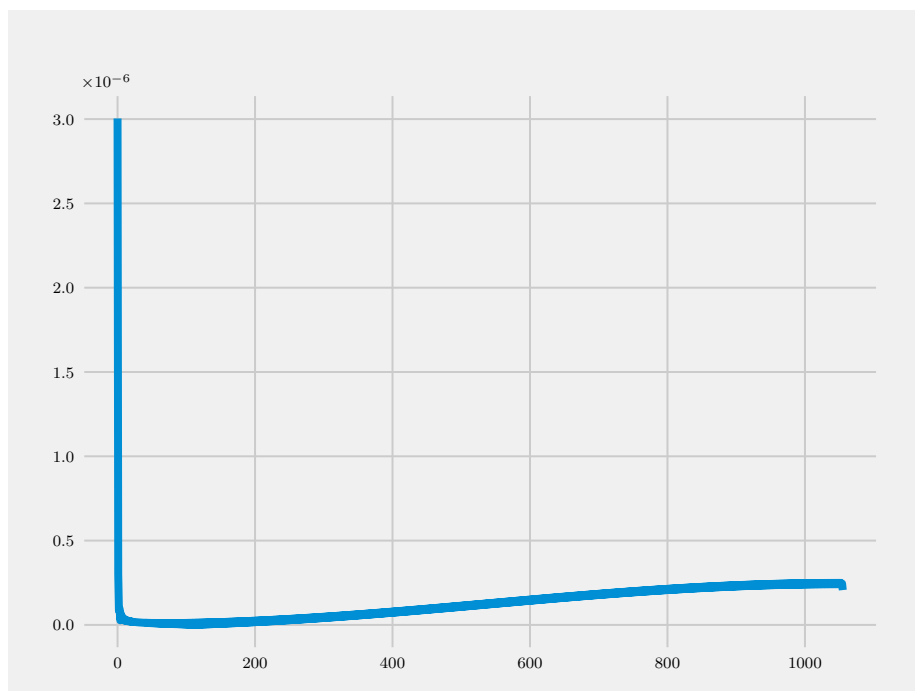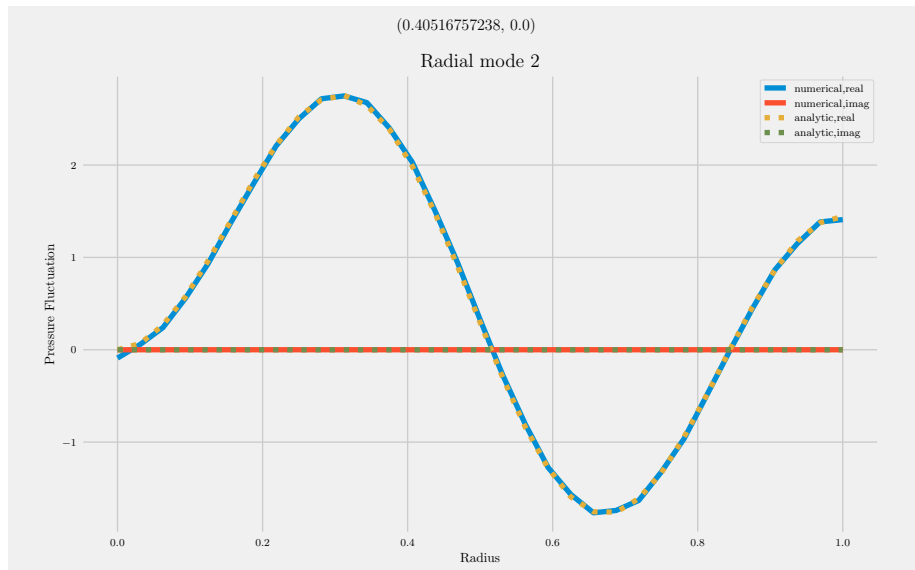
Figure 60: Fourth Order, Case number 1, 1056points
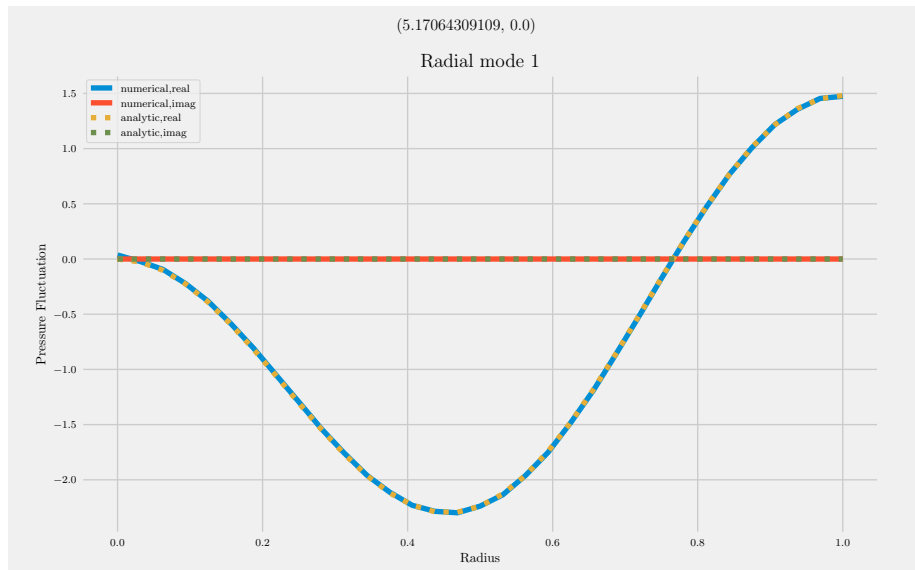
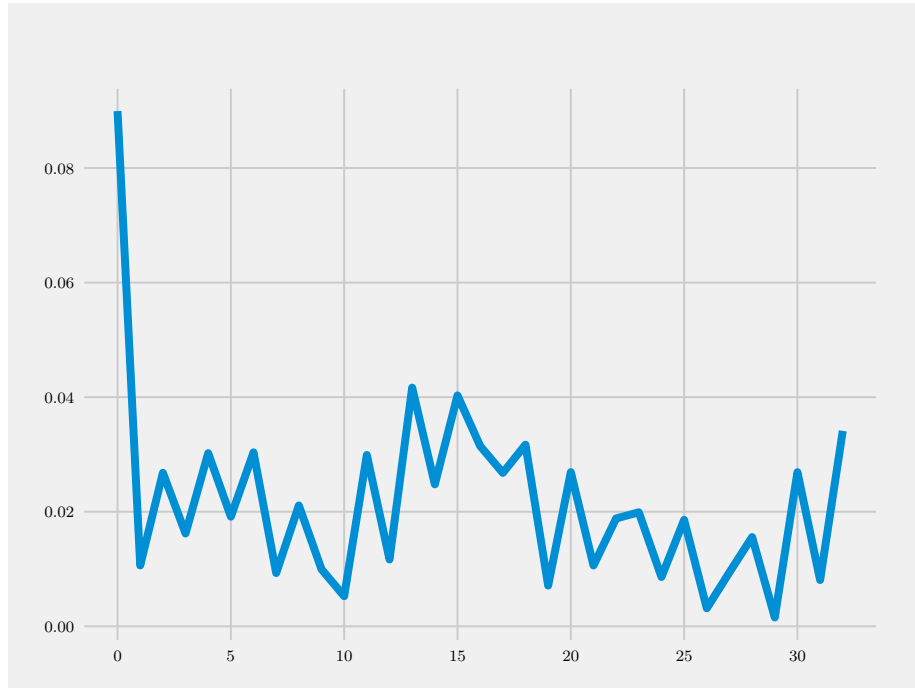Figure 61: Second Order, Case number 0, 33points

Second Order, Radial Mode 1
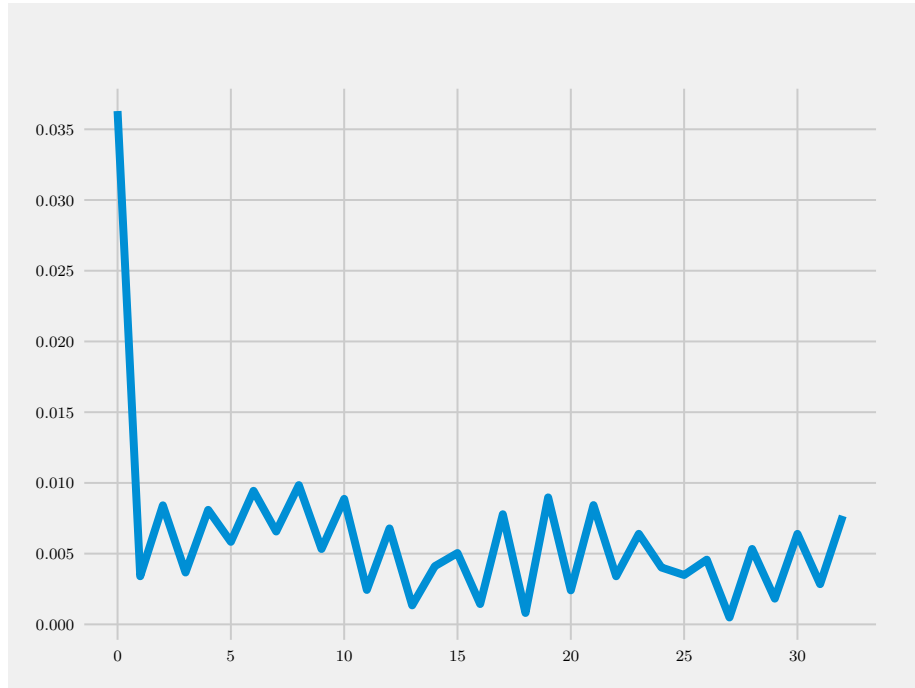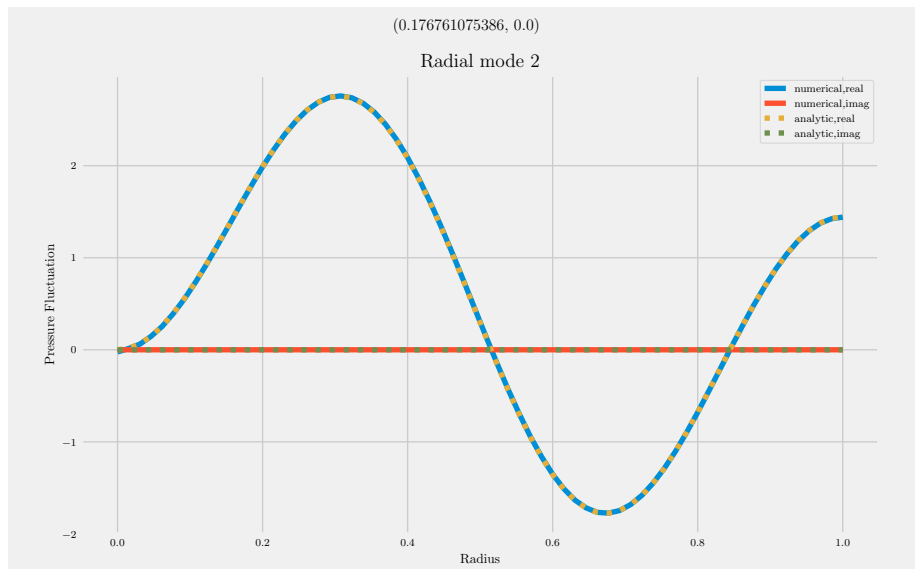
Figure 62: Second Order, Case number 0, 33points
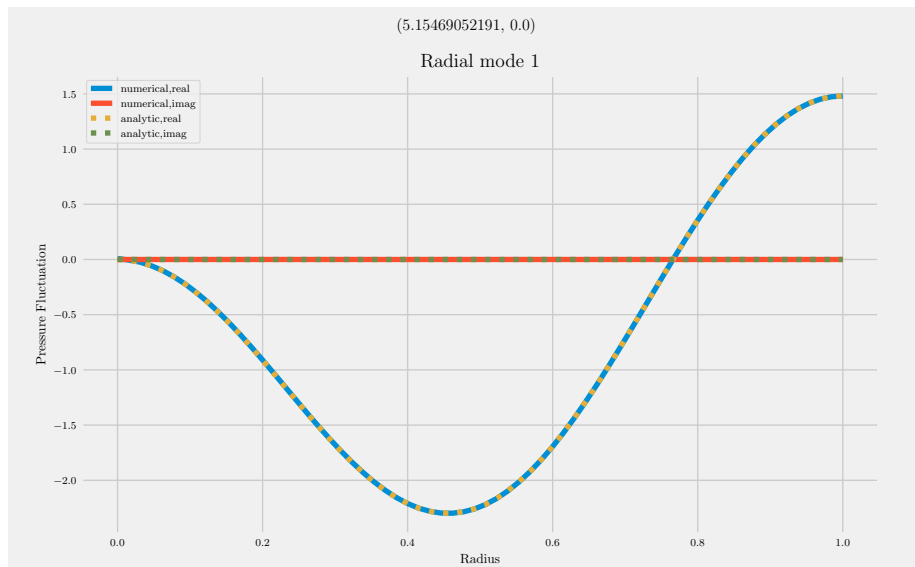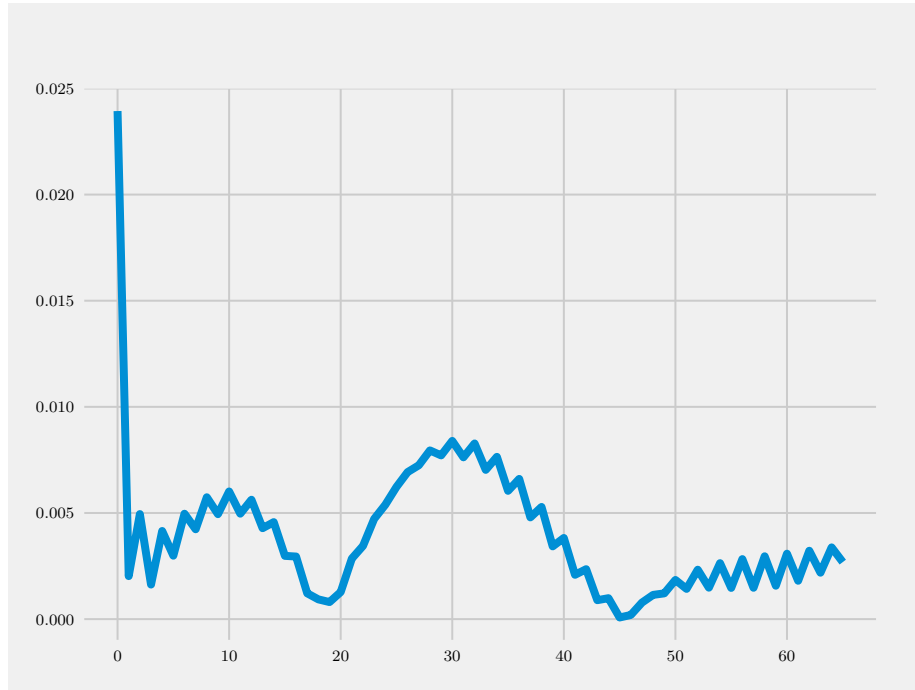

Figure 63: Second Order, Case number 0, 66points

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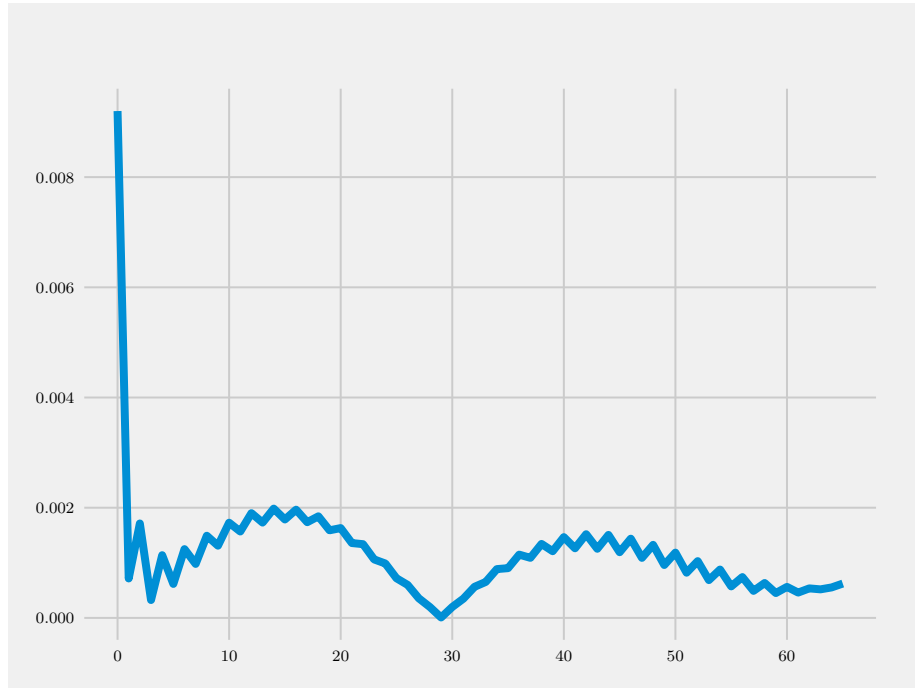
Figure 64: Second Order, Case number 0, 66points
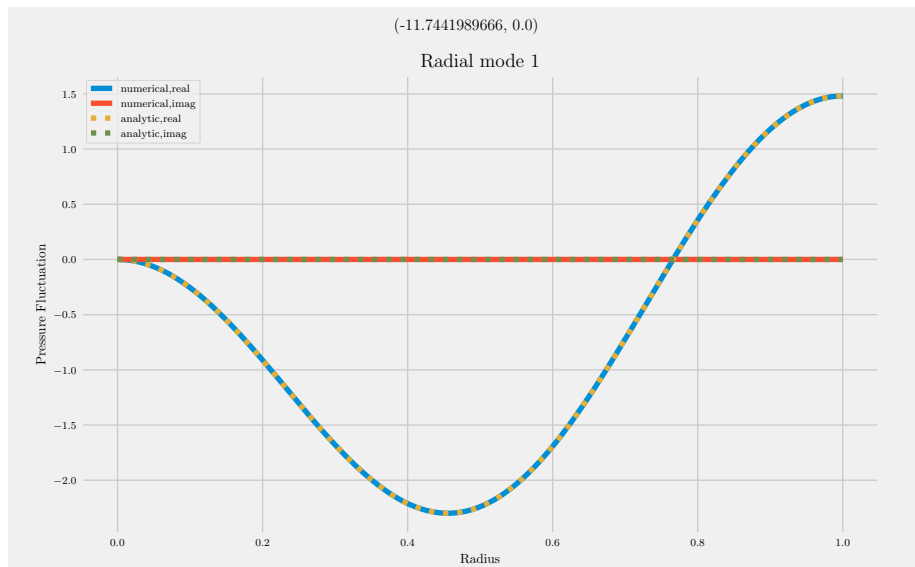
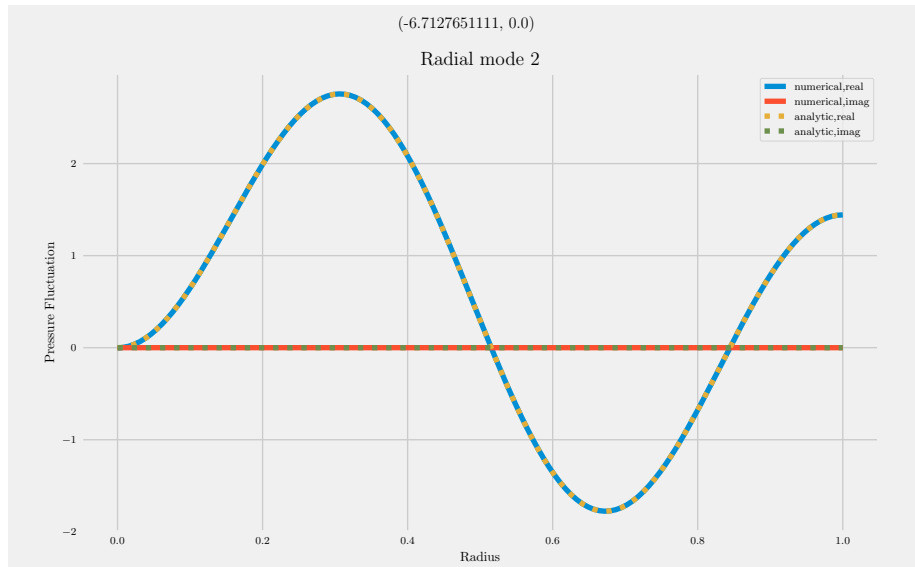

Figure 65: Second Order, Case number 0, 132points
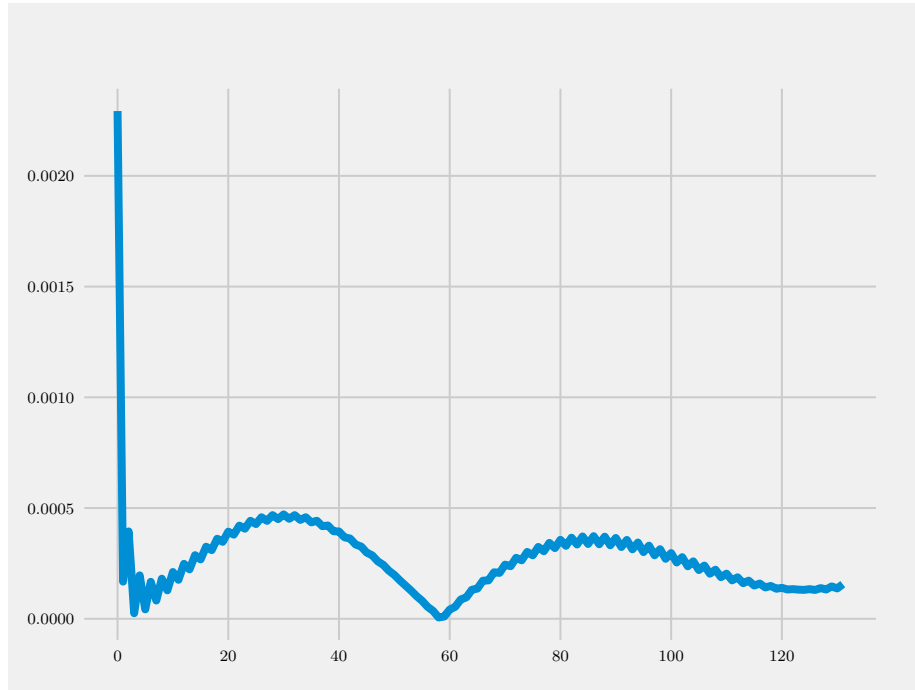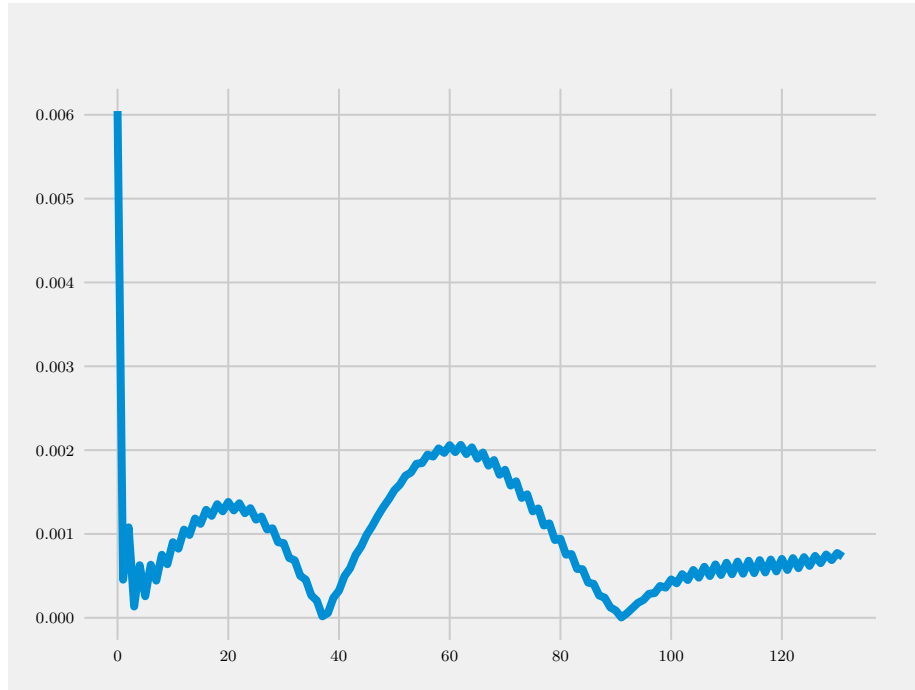
Figure 66: Second Order, Case number 0, 132points
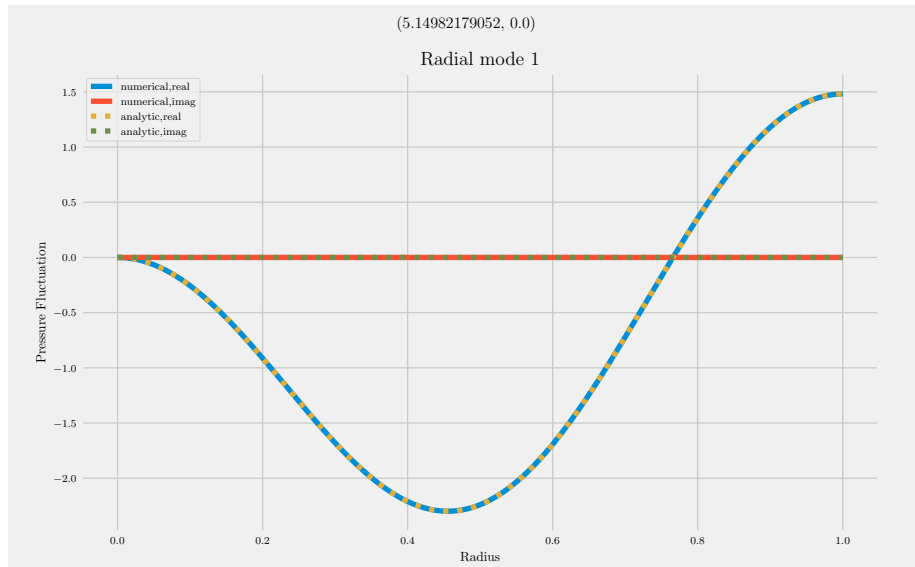
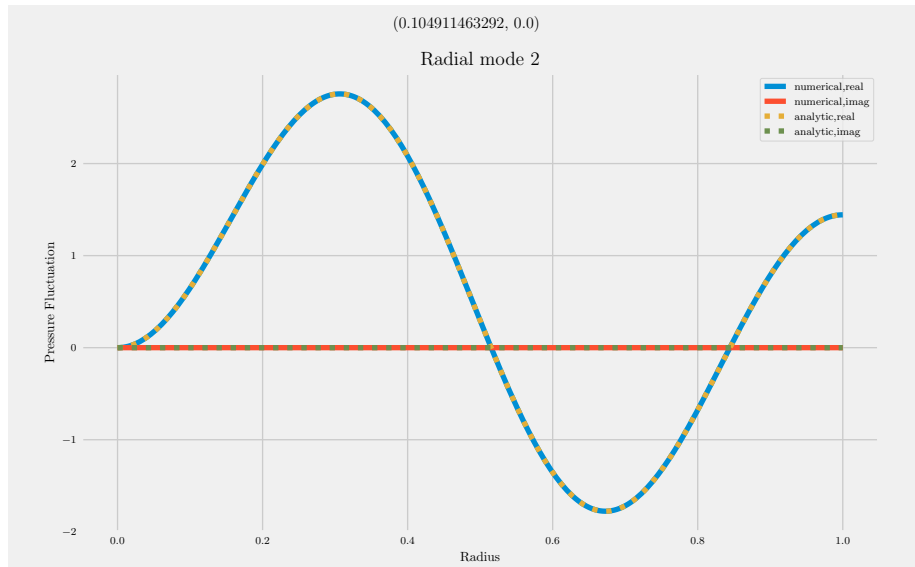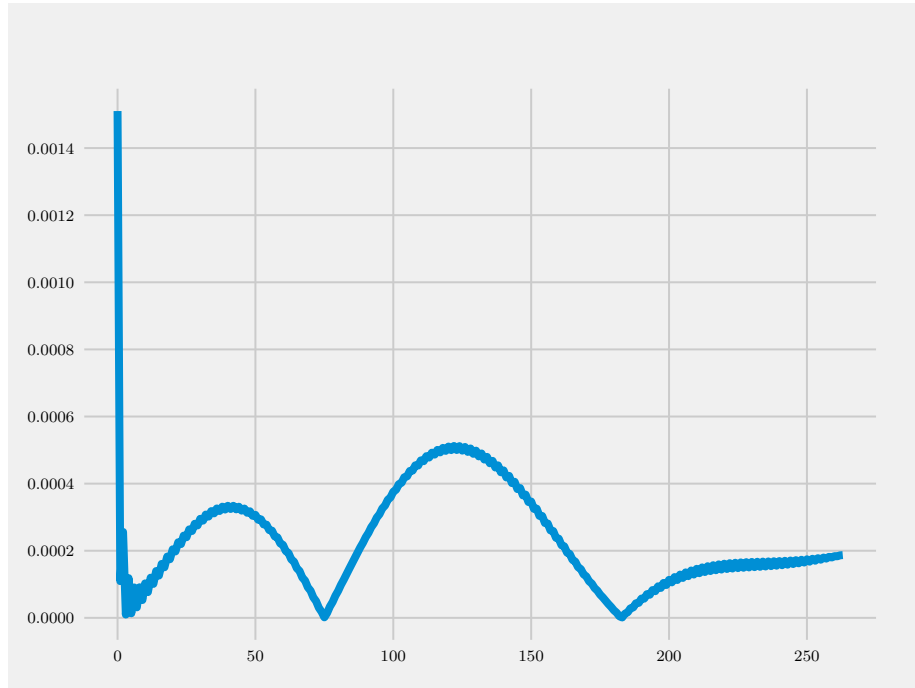
Figure 67: Second Order, Case number 0, 264points
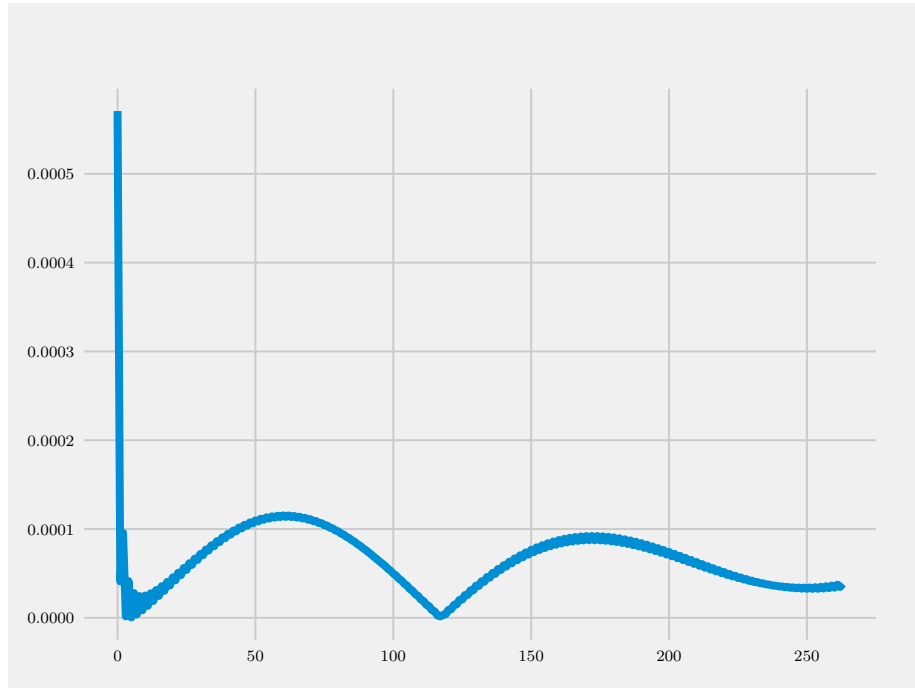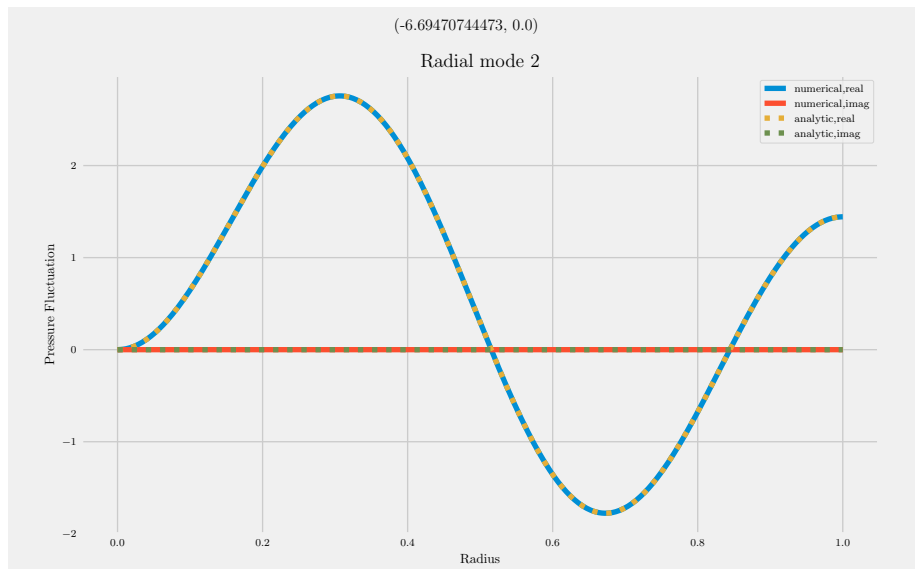
Figure 68: Second Order, Case number 0, 264points
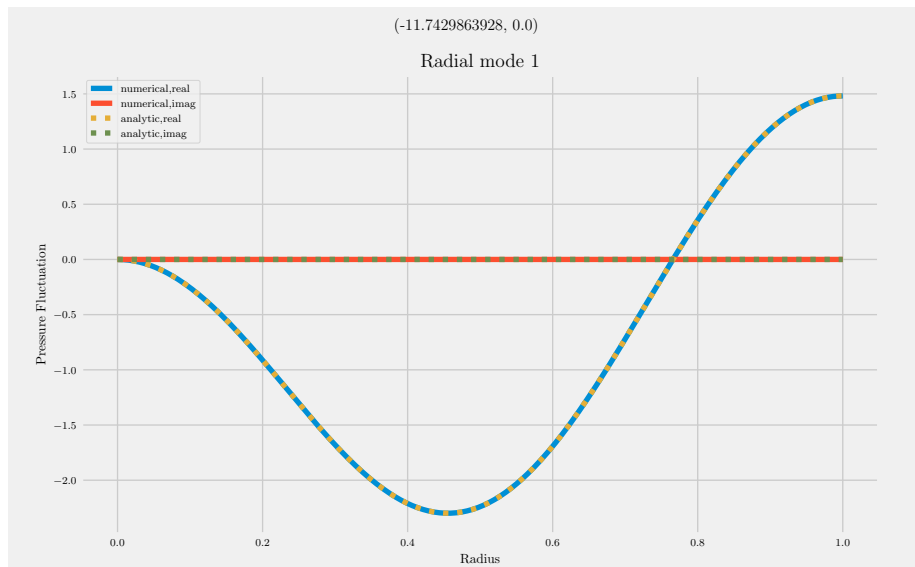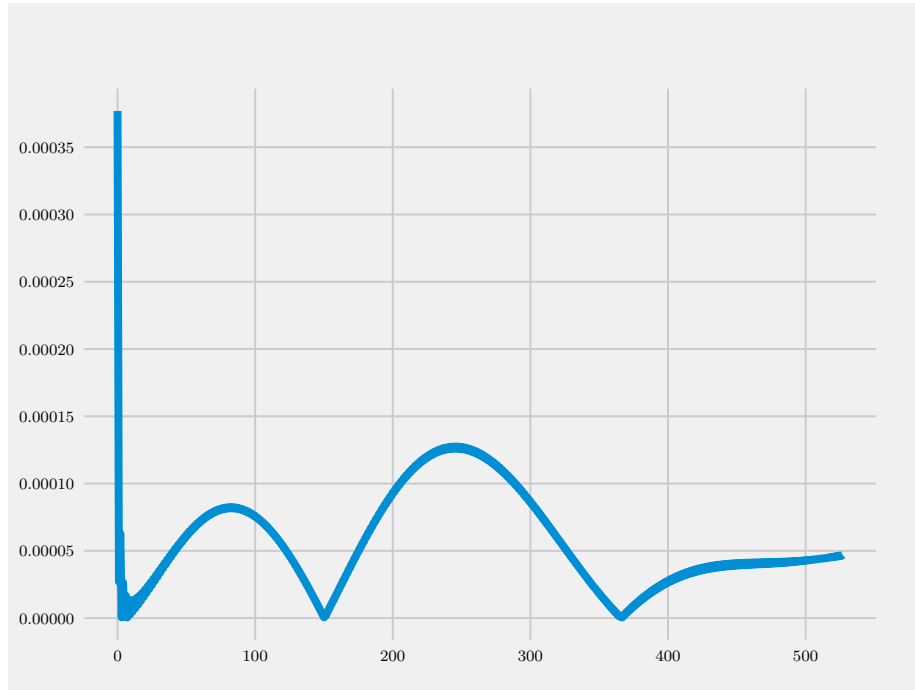

Figure 69: Second Order, Case number 0, 528points
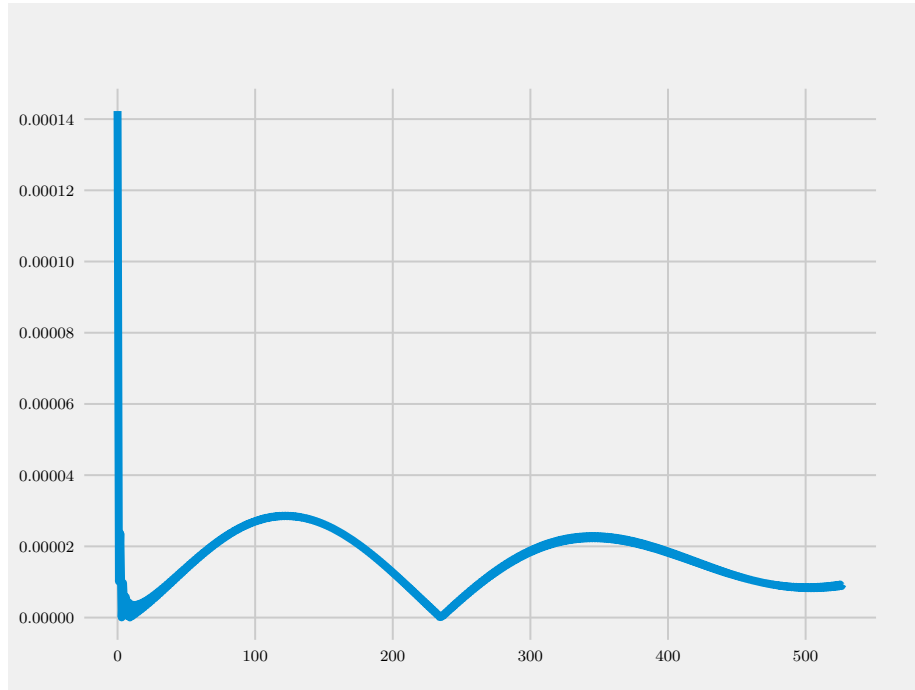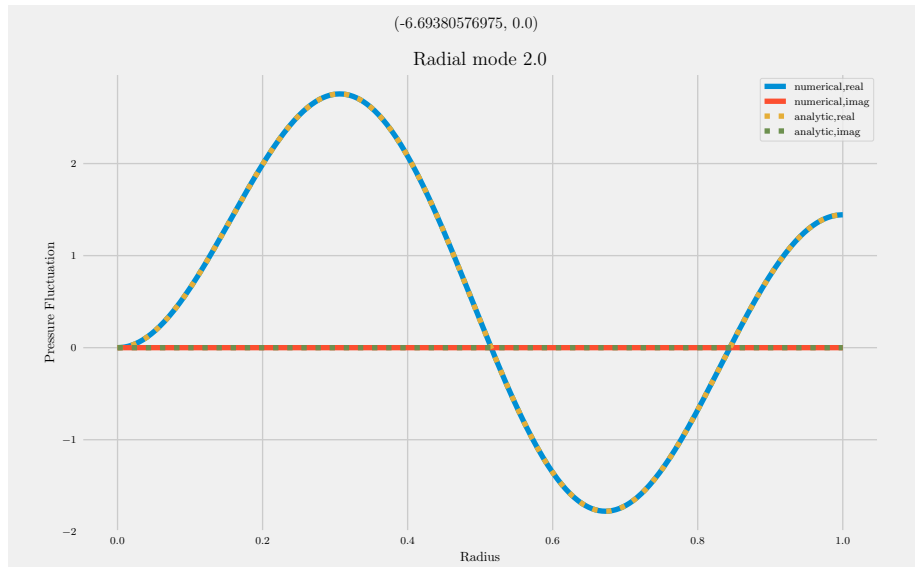
Figure 70: Second Order, Case number 0, 528points
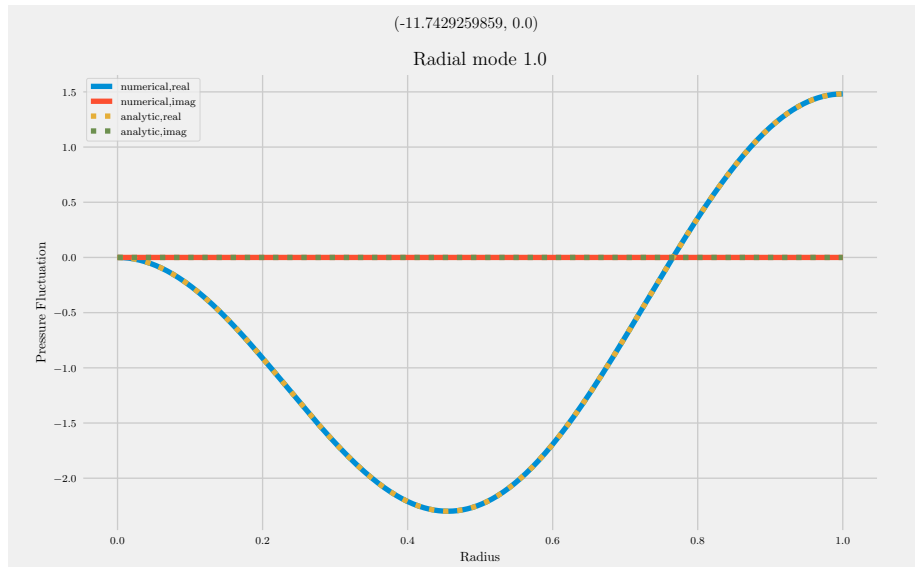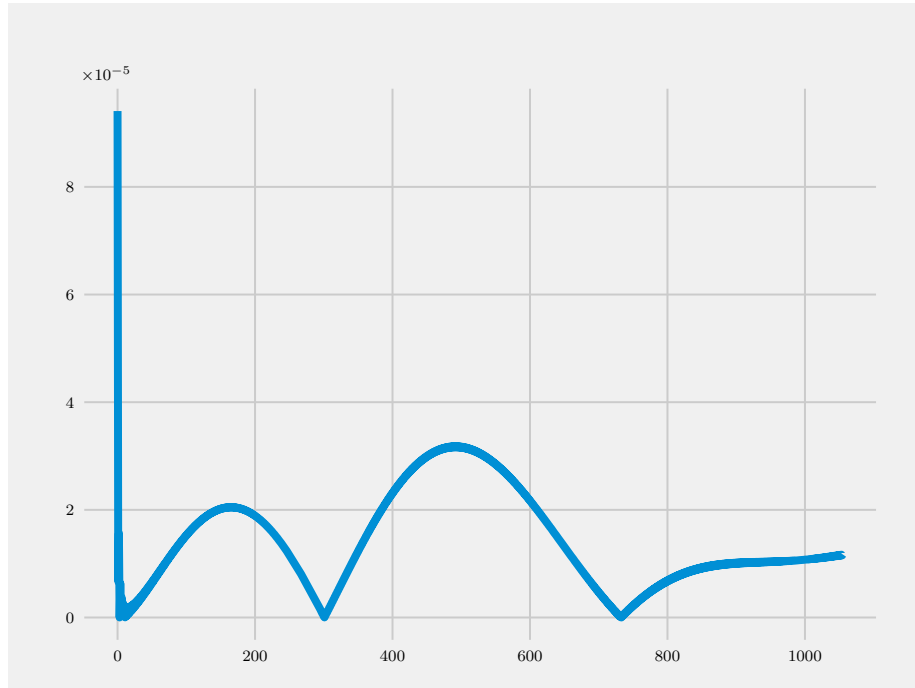

Figure 71: Second Order, Case number 0, 1056points
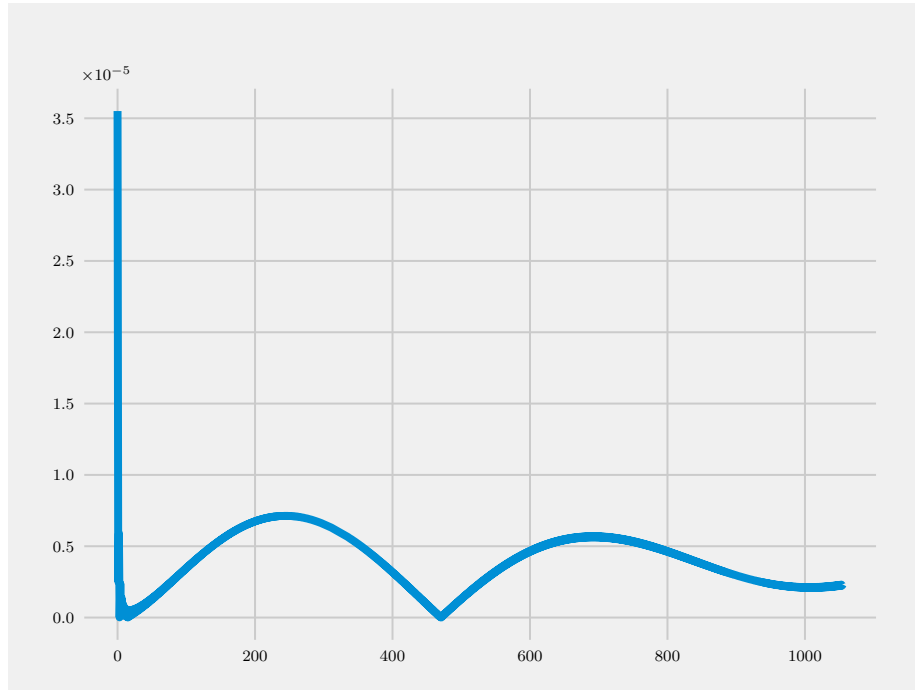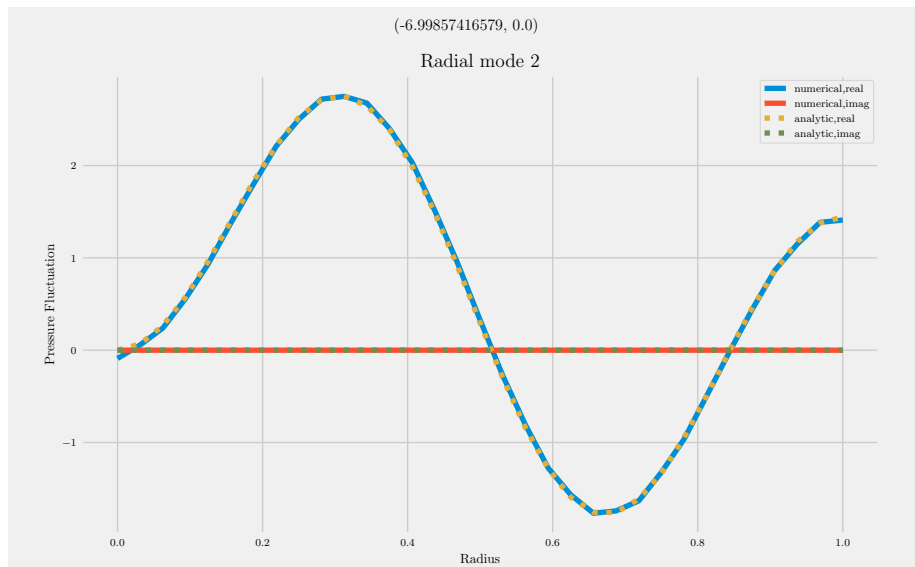
Figure 72: Second Order, Case number 0, 1056points
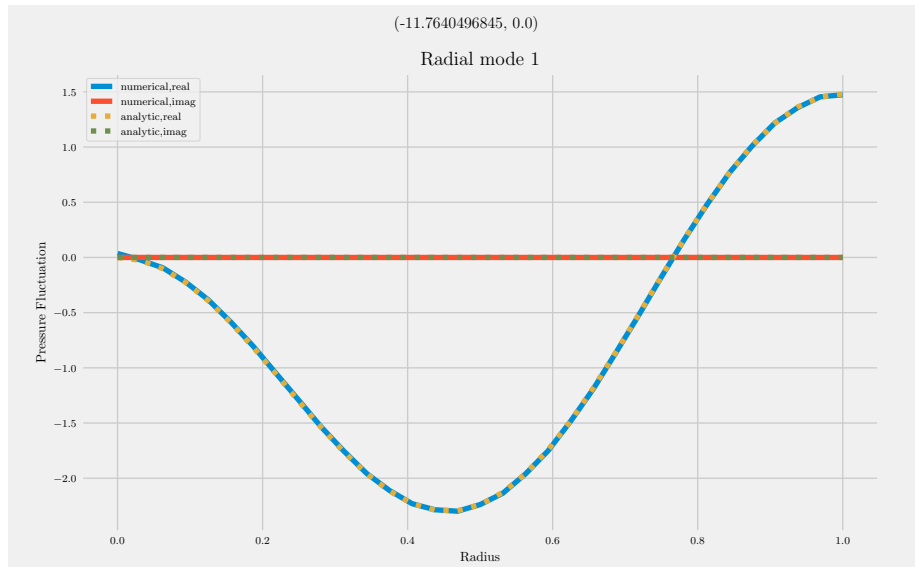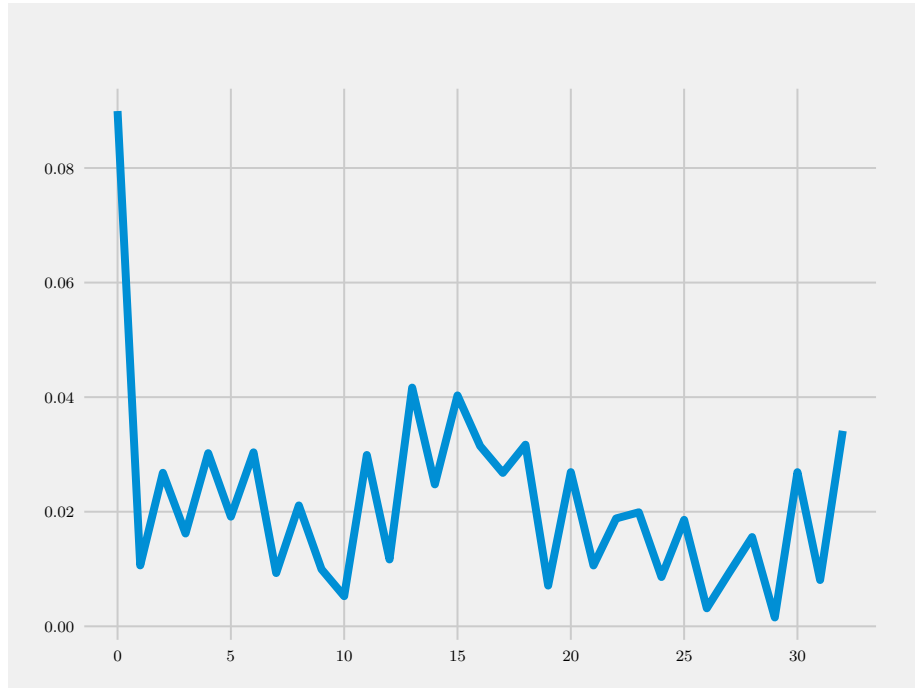

Figure 73: Second Order, Case number 1, 33points
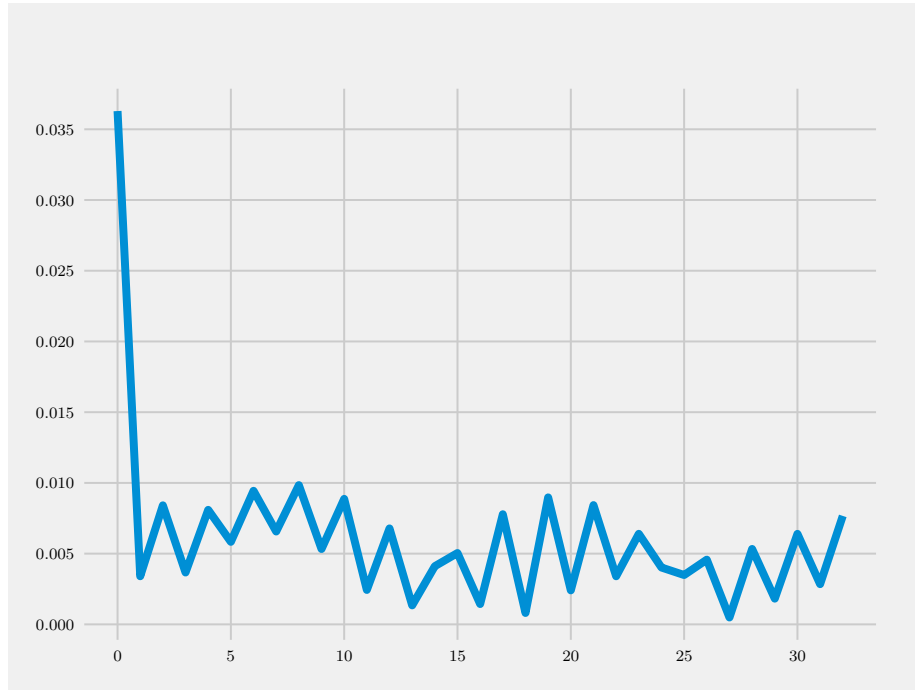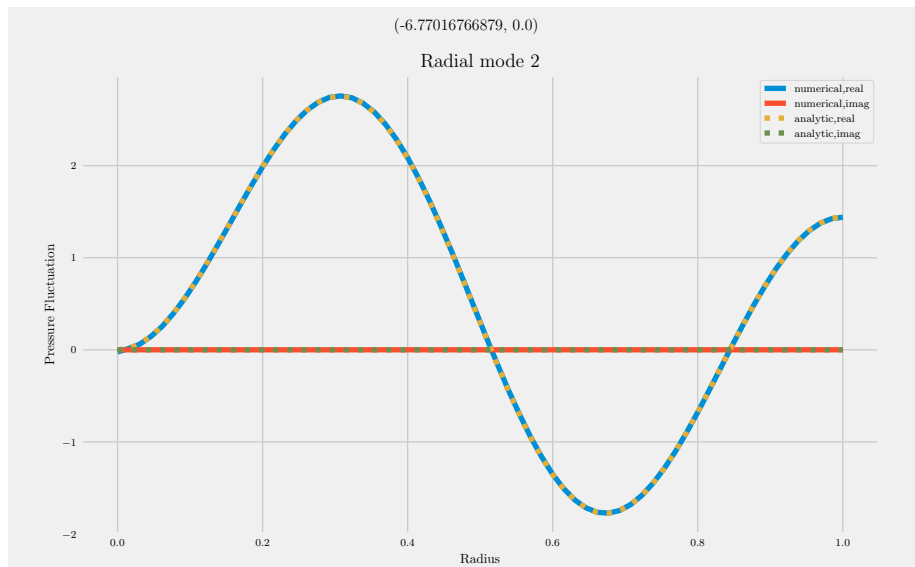
Figure 74: Second Order, Case number 1, 33points
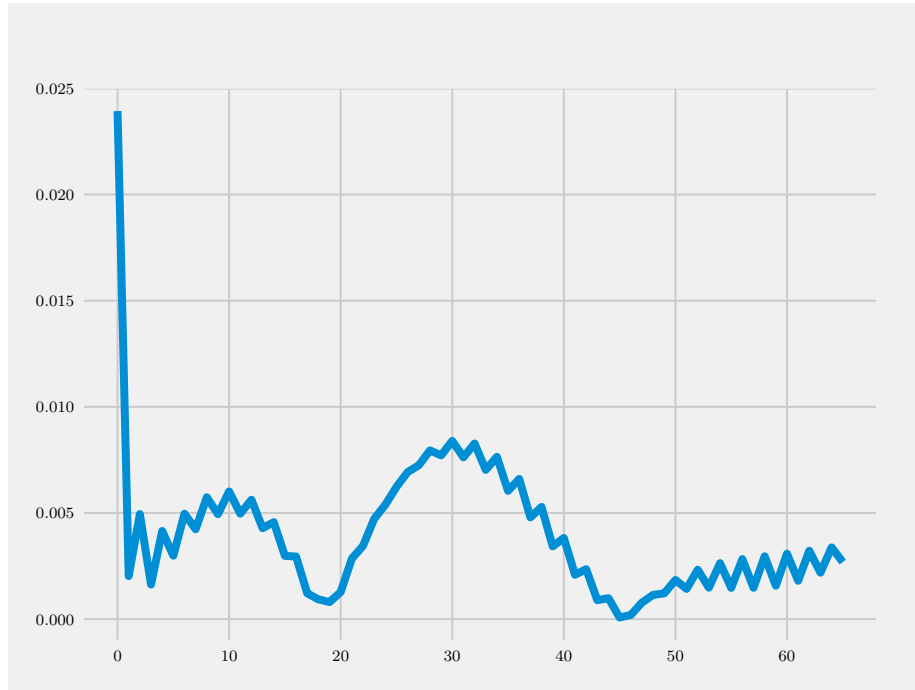

Figure 75: Second Order, Case number 1, 66points

Figure 76: Second Order, Case number 1, 66points

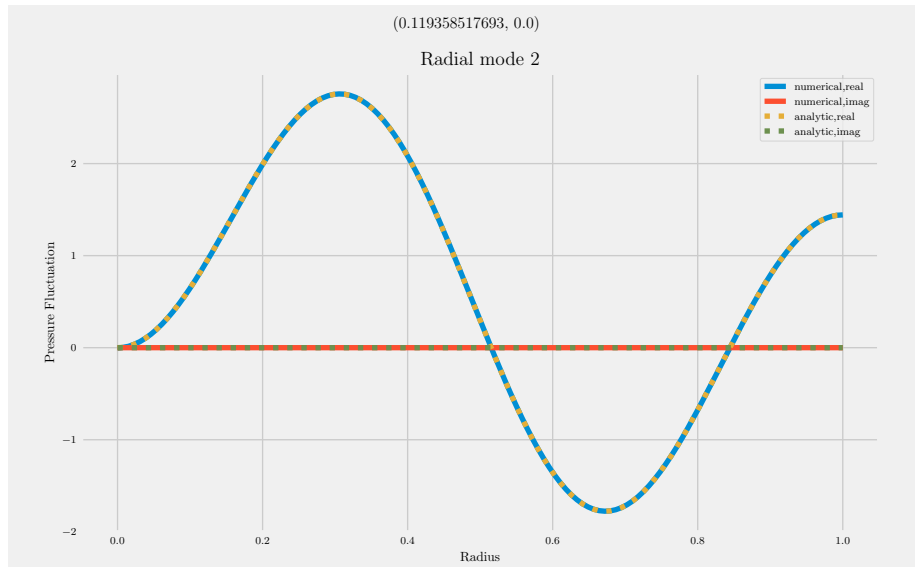
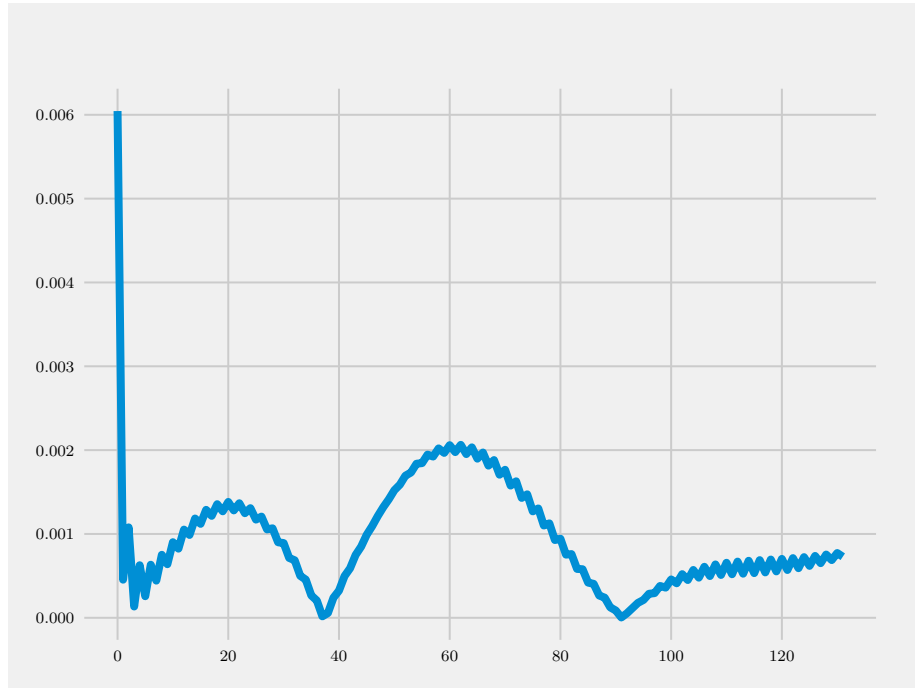Figure 77: Second Order, Case number 1, 132points

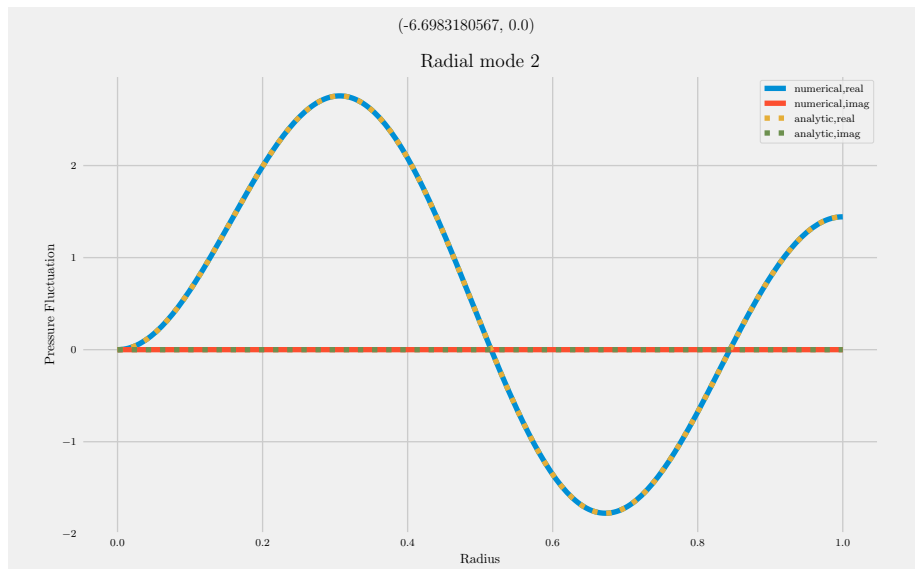Figure 78: Second Order, Case number 1, 132points
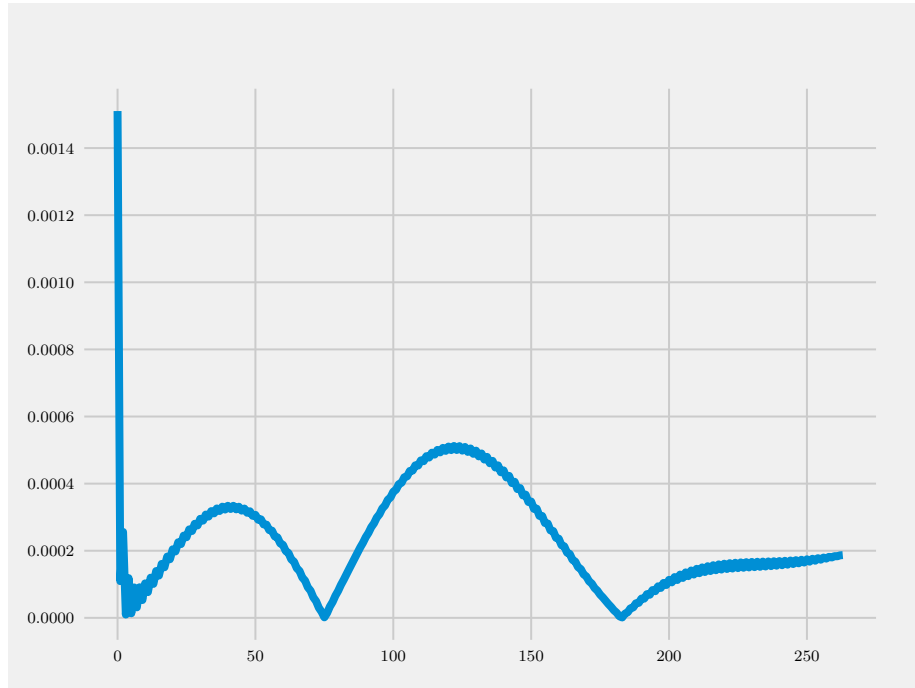

Figure 79: Second Order, Case number 1, 264points
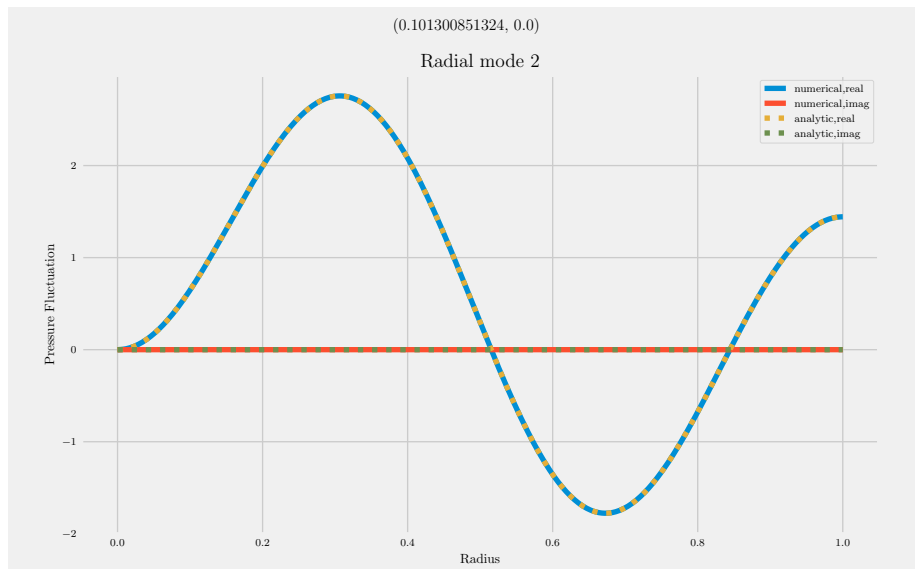
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Figure 80: Second Order, Case number 1, 264points

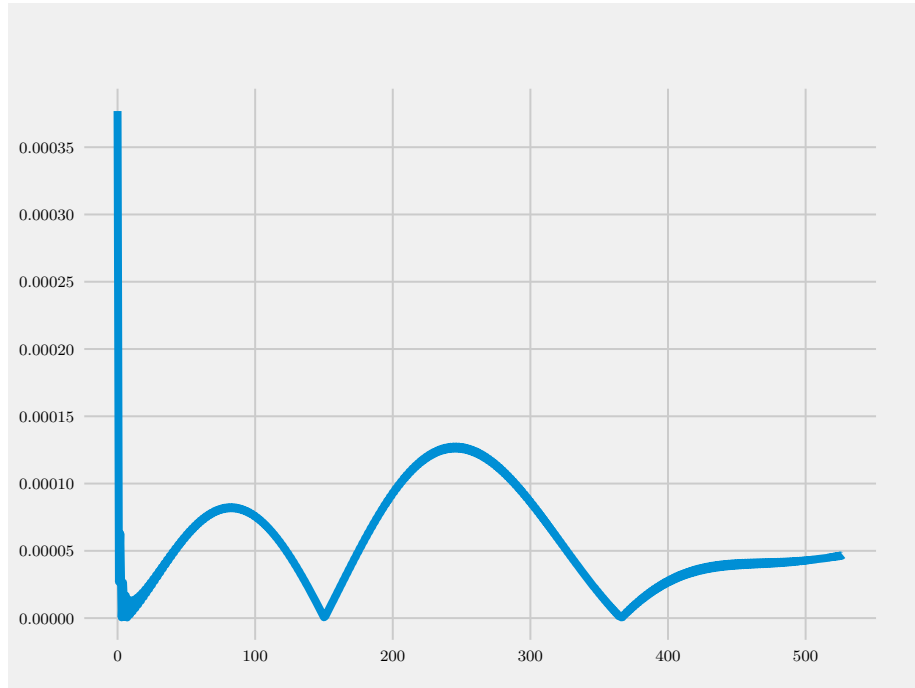Figure 81: Second Order, Case number 1, 528points
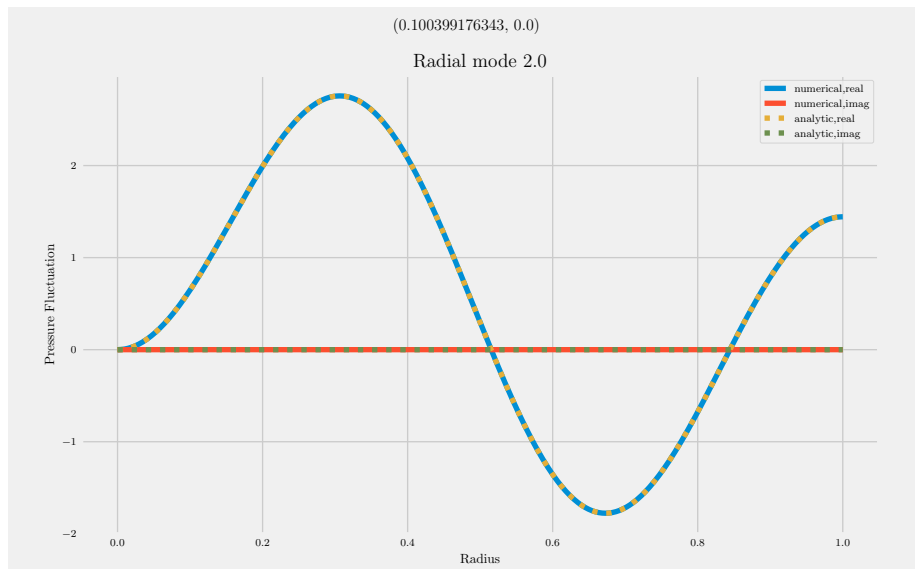
Figure 82: Second Order, Case number 1, 528points
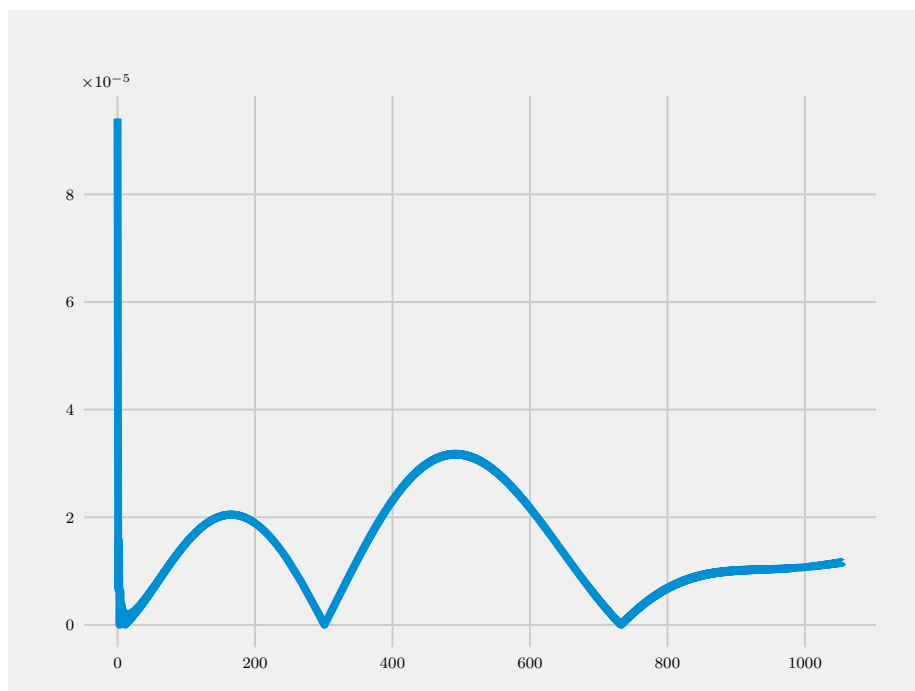

Figure 83: Second Order, Case number 1, 1056points

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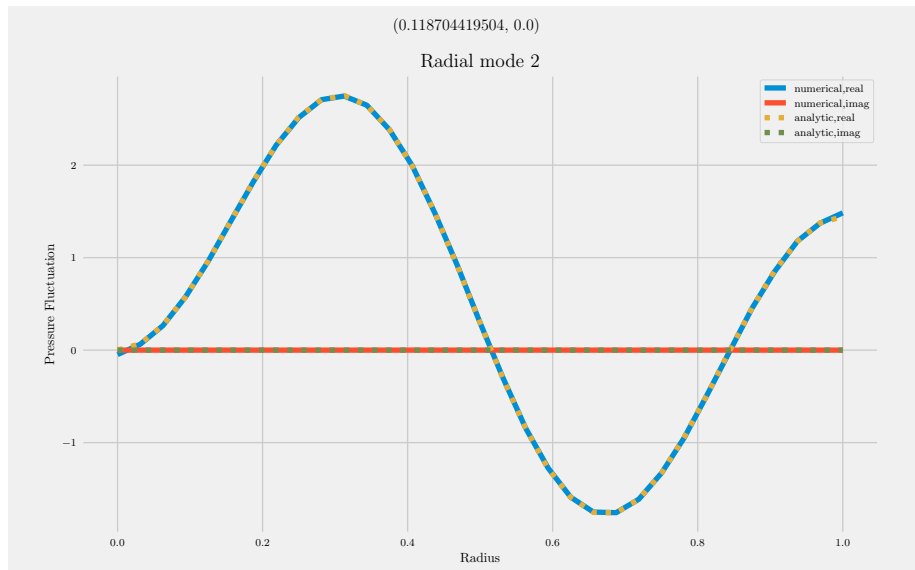Figure 84: Second Order, Case number 1, 1056points
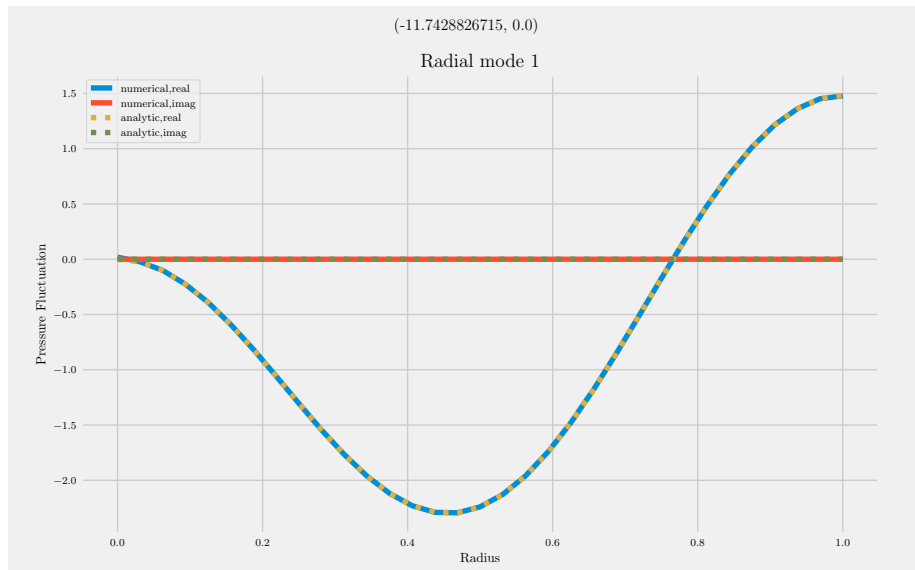
Figure 85: Fourth Order, Case number 0, 33points
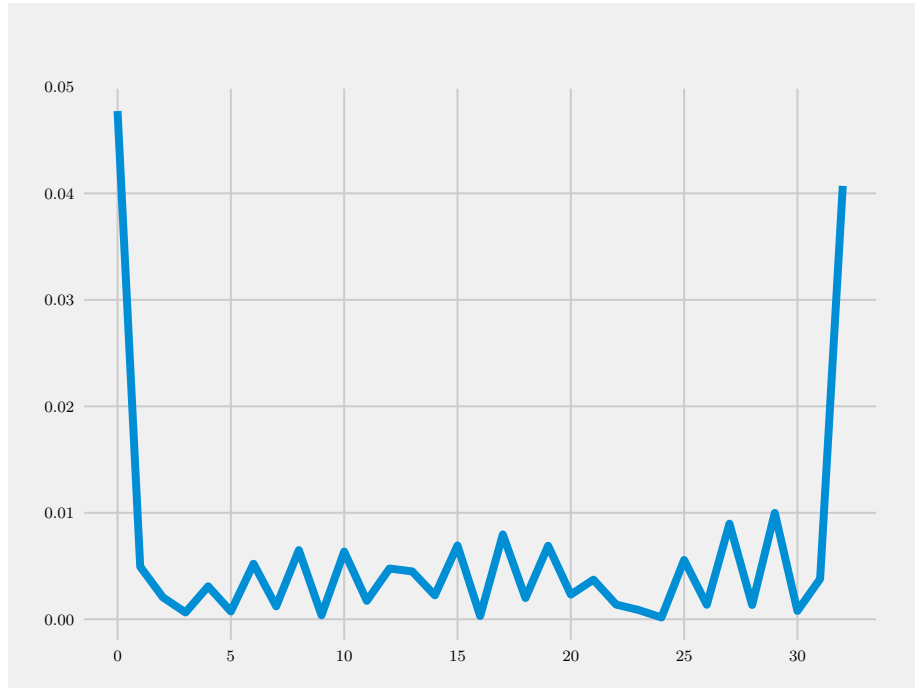
Fourth Order, Radial Mode 1
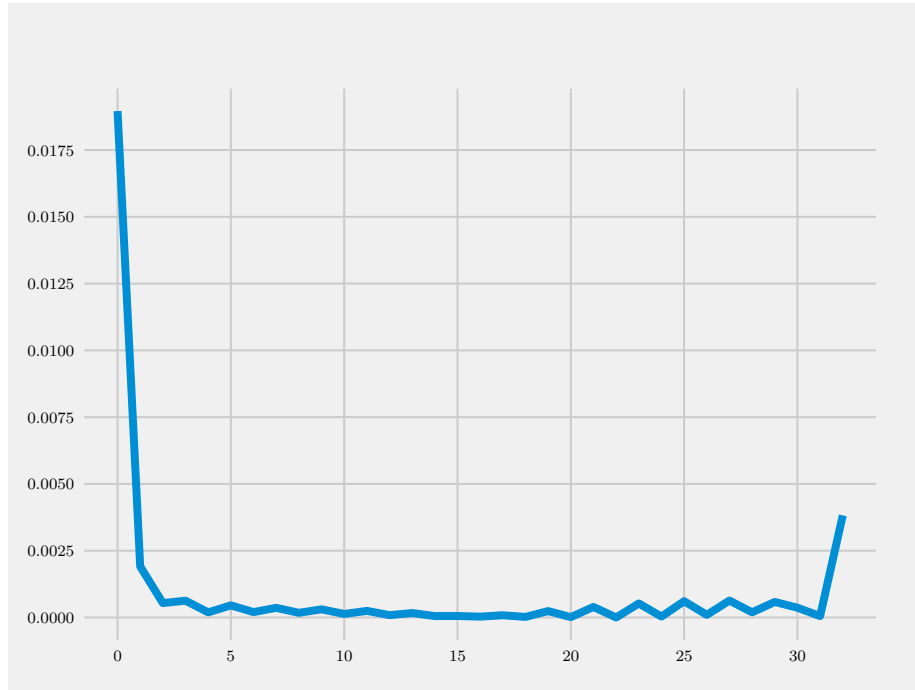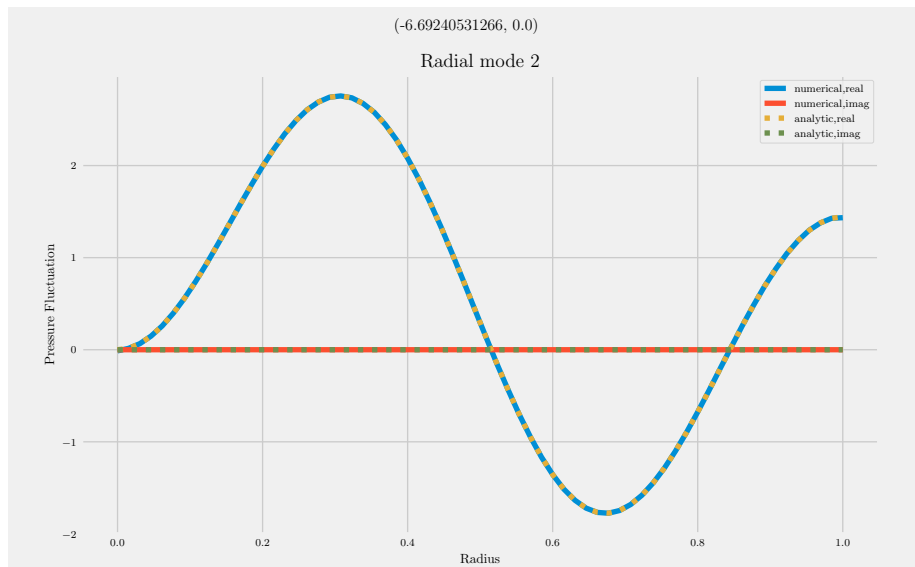
Figure 86: Fourth Order, Case number 0, 33points
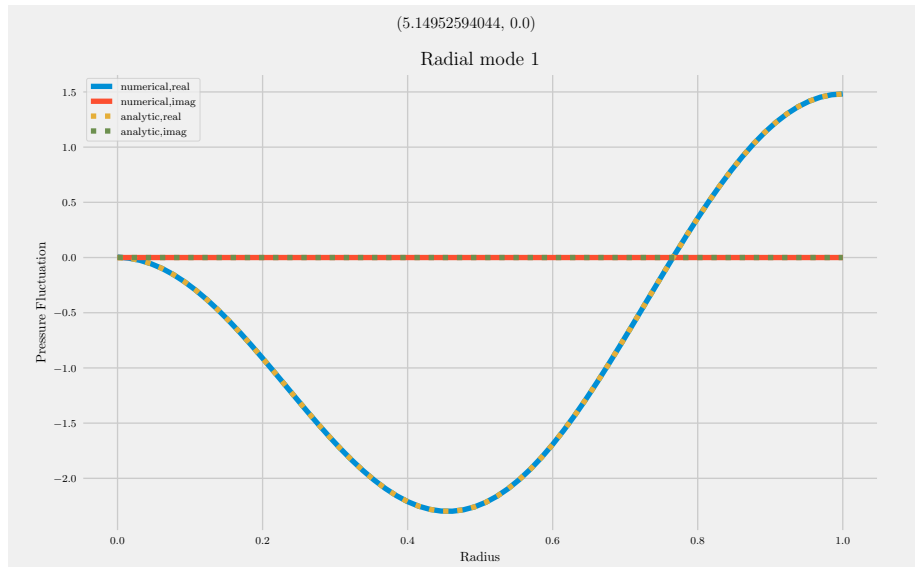
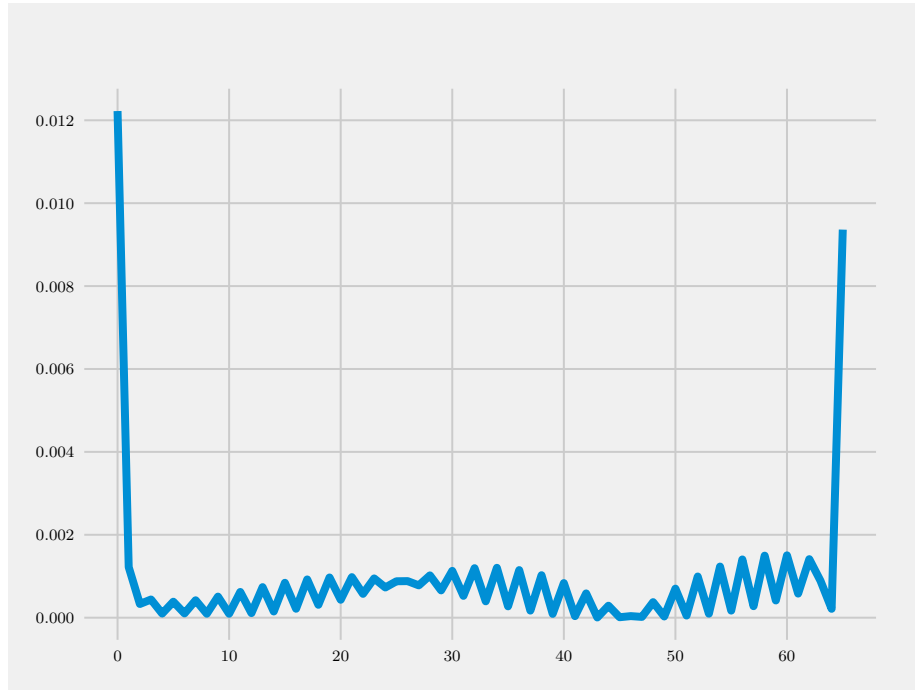Figure 87: Fourth Order, Case number 0, 66points
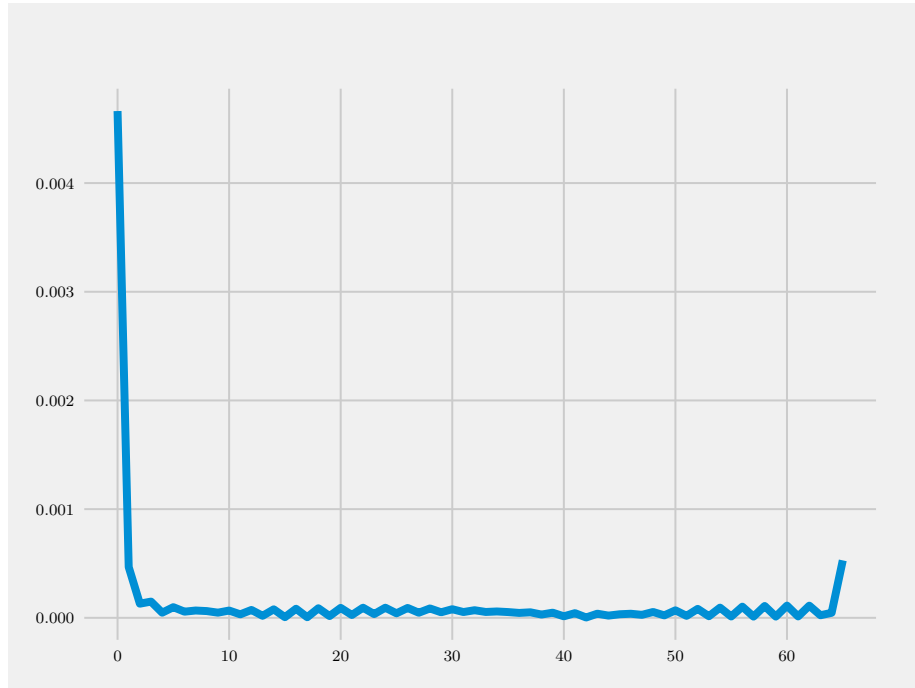
Figure 88: Fourth Order, Case number 0, 66points
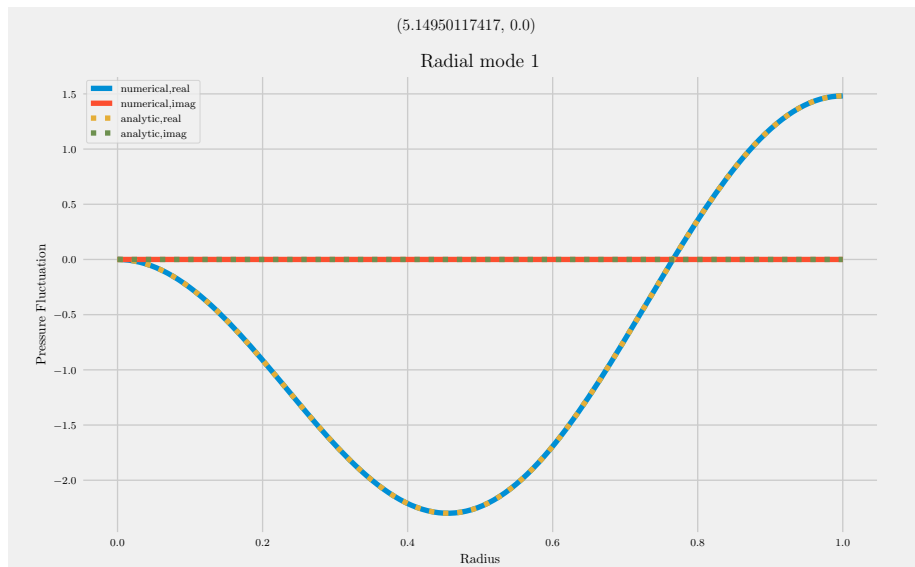
Figure 89: Fourth Order, Case number 0, 132points
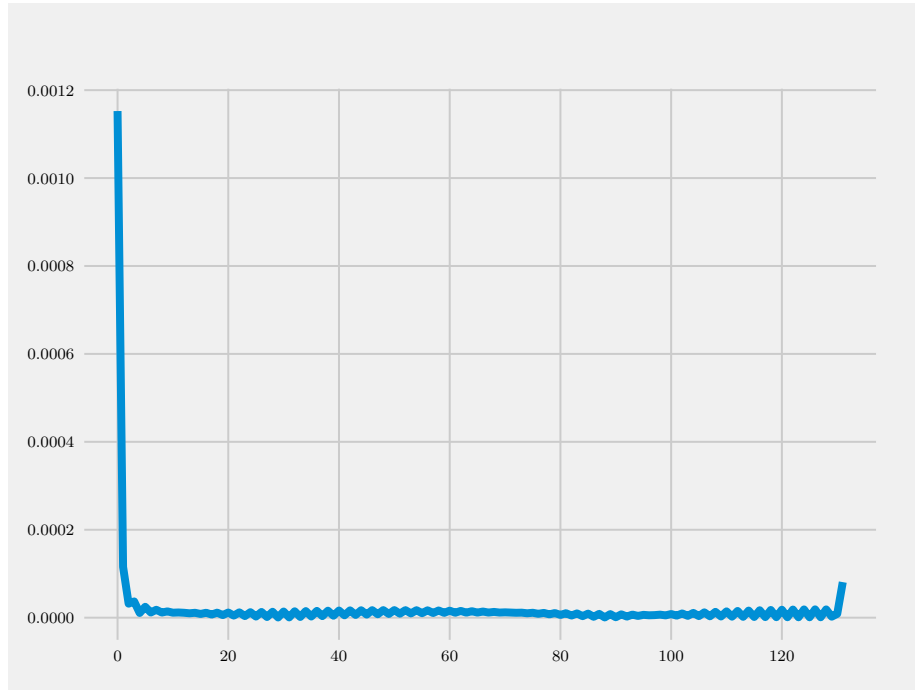
Figure 90: Fourth Order, Case number 0, 132points
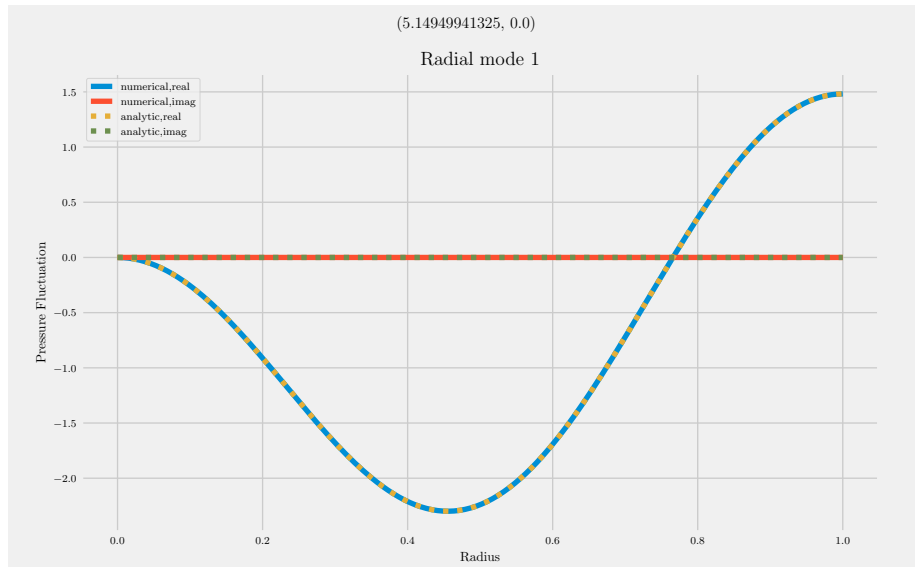
Figure 91: Fourth Order, Case number 0, 264points

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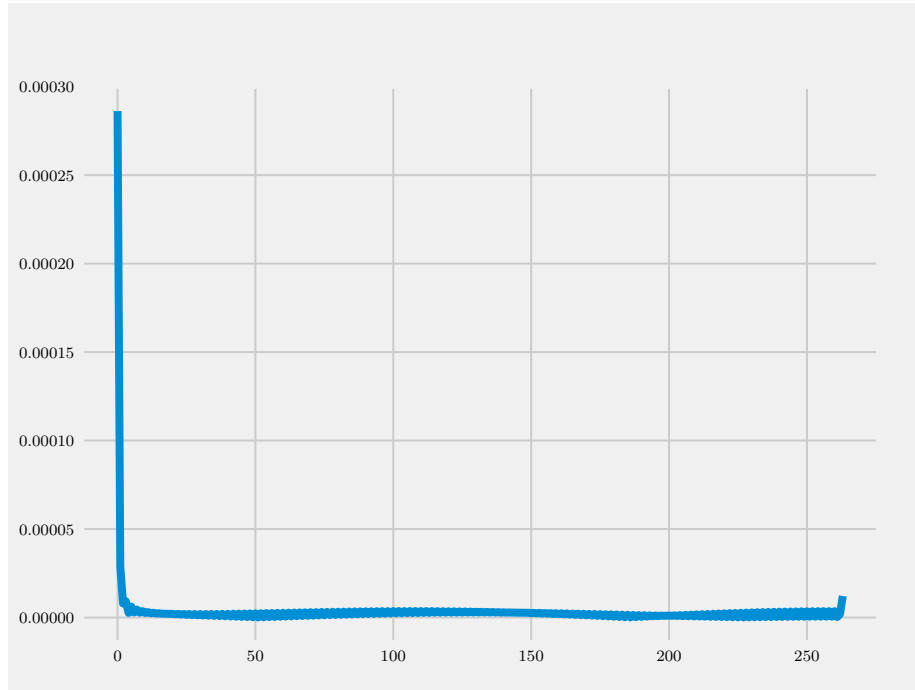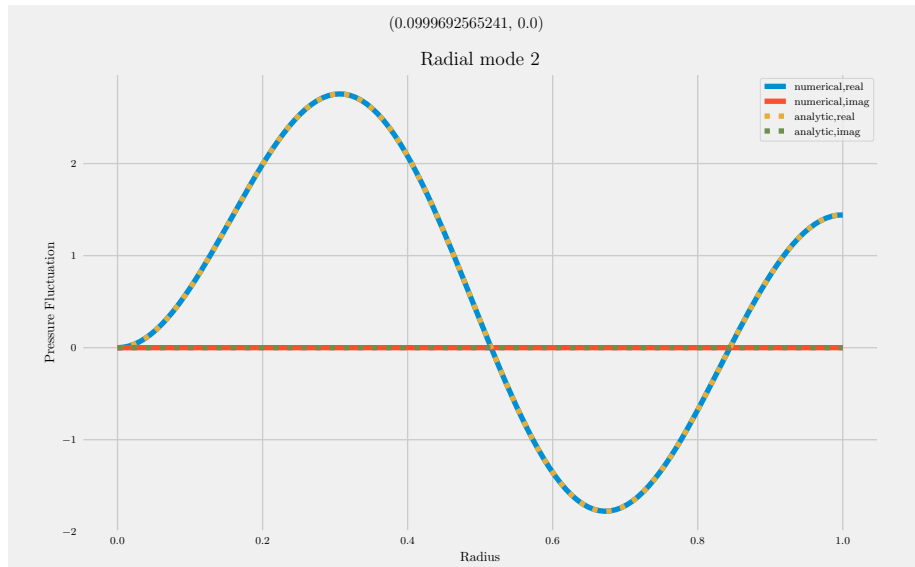
Figure 92: Fourth Order, Case number 0, 264points

Figure 93: Fourth Order, Case number 0, 528points

Figure 94: Fourth Order, Case number 0, 528points

Figure 95: Fourth Order, Case number 0, 1056points

Figure 96: Fourth Order, Case number 0, 1056points

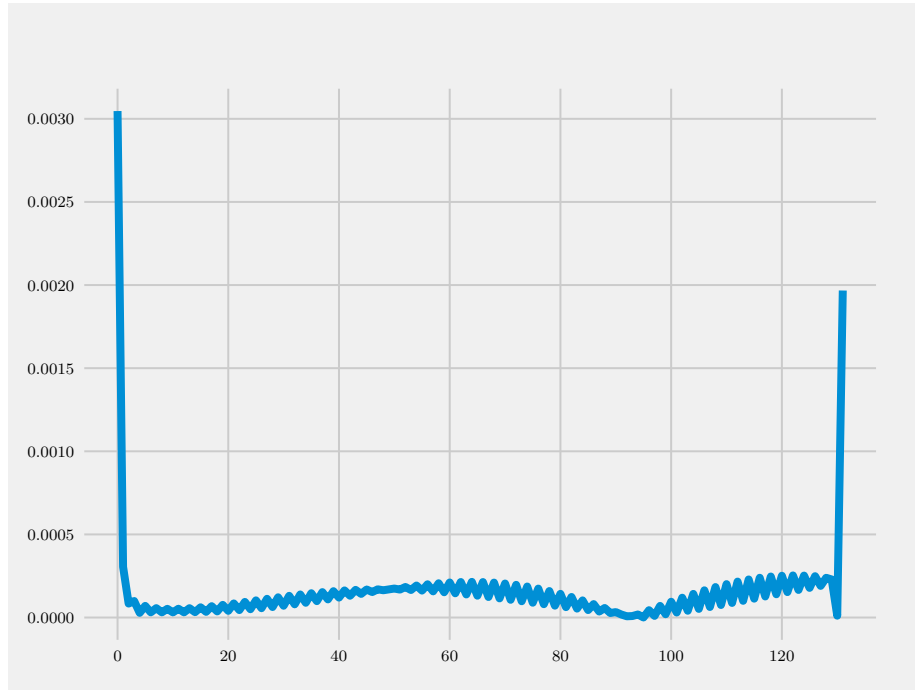Figure 97: Fourth Order, Case number 1, 33points

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Figure 98: Fourth Order, Case number 1, 33points

Figure 99: Fourth Order, Case number 1, 66points

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Figure 100: Fourth Order, Case number 1, 66points

Figure 101: Fourth Order, Case number 1, 132points

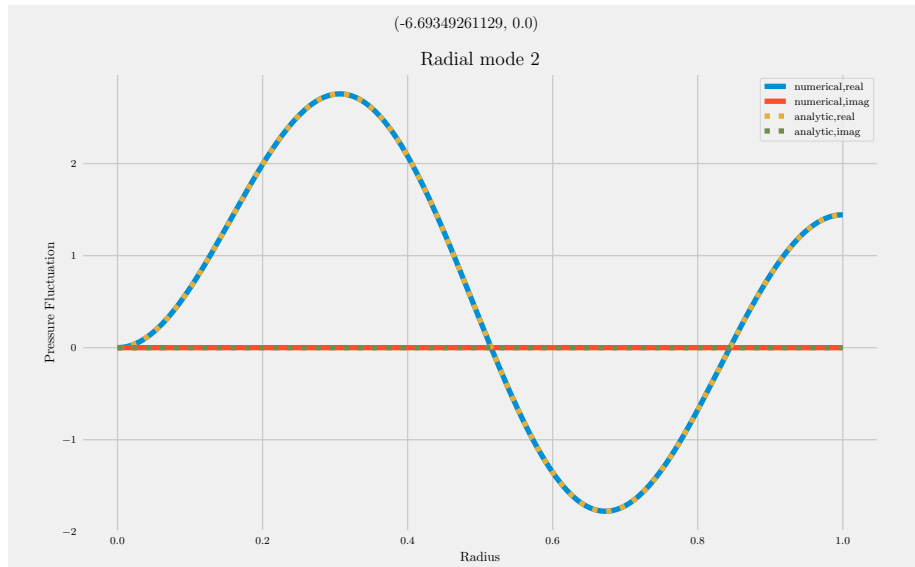Figure 102: Fourth Order, Case number 1, 132points

Figure 103: Fourth Order, Case number 1, 264points
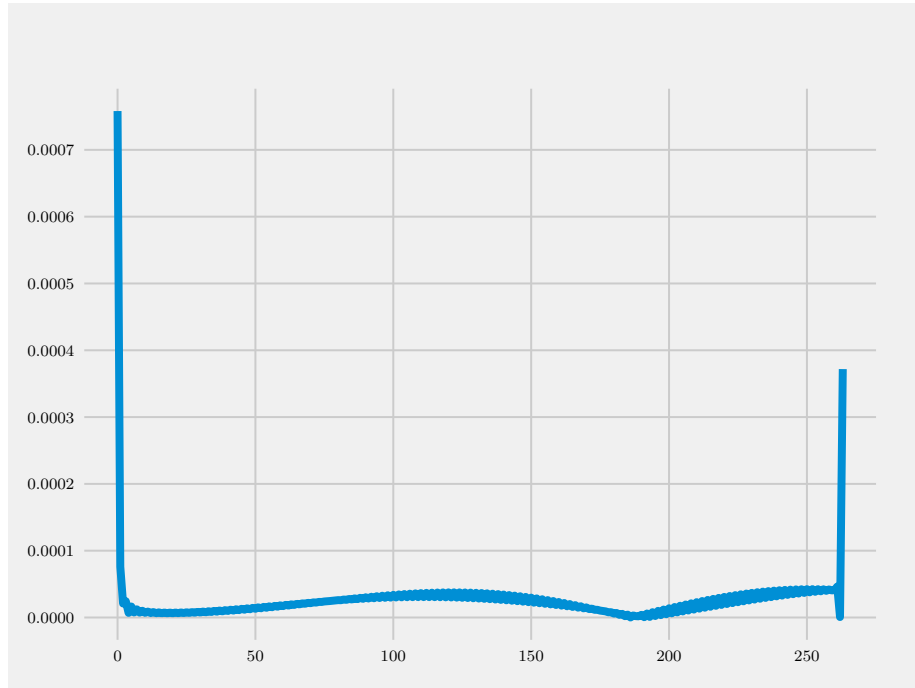
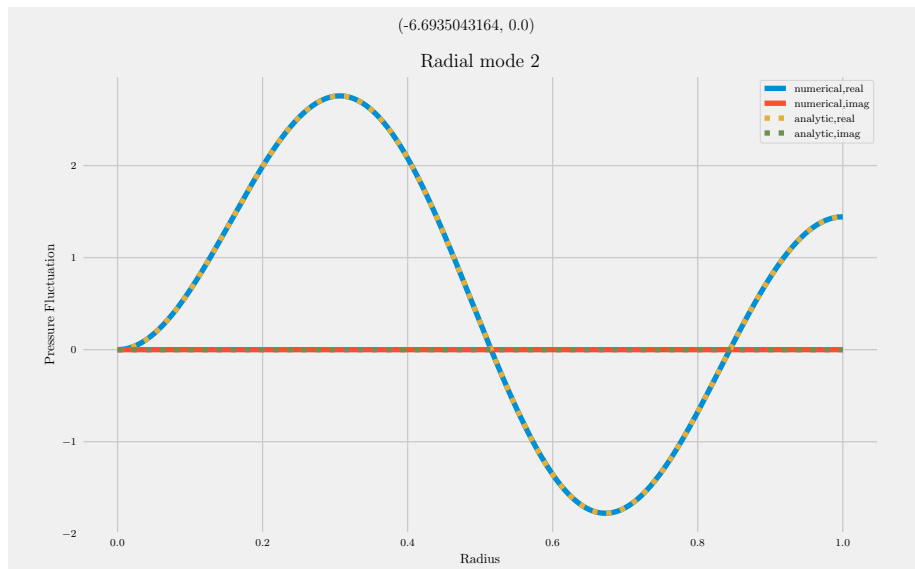Figure 104: Fourth Order, Case number 1, 264points
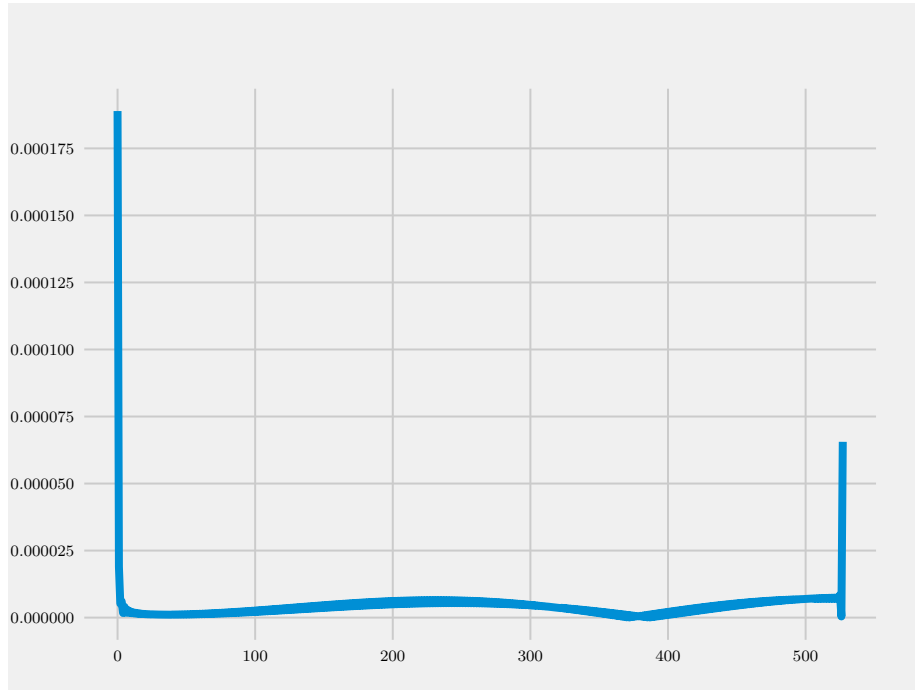

Figure 105: Fourth Order, Case number 1, 528points

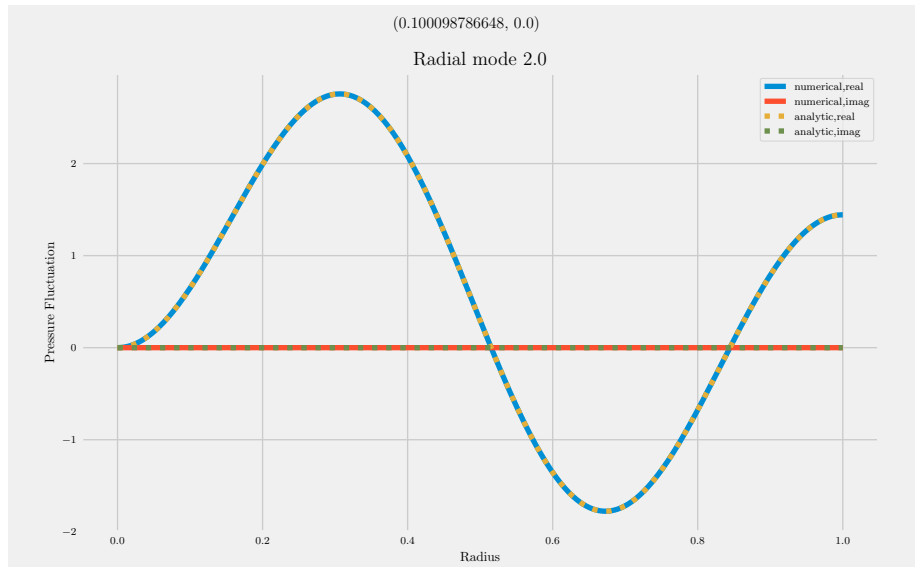Figure 106: Fourth Order, Case number 1, 528points

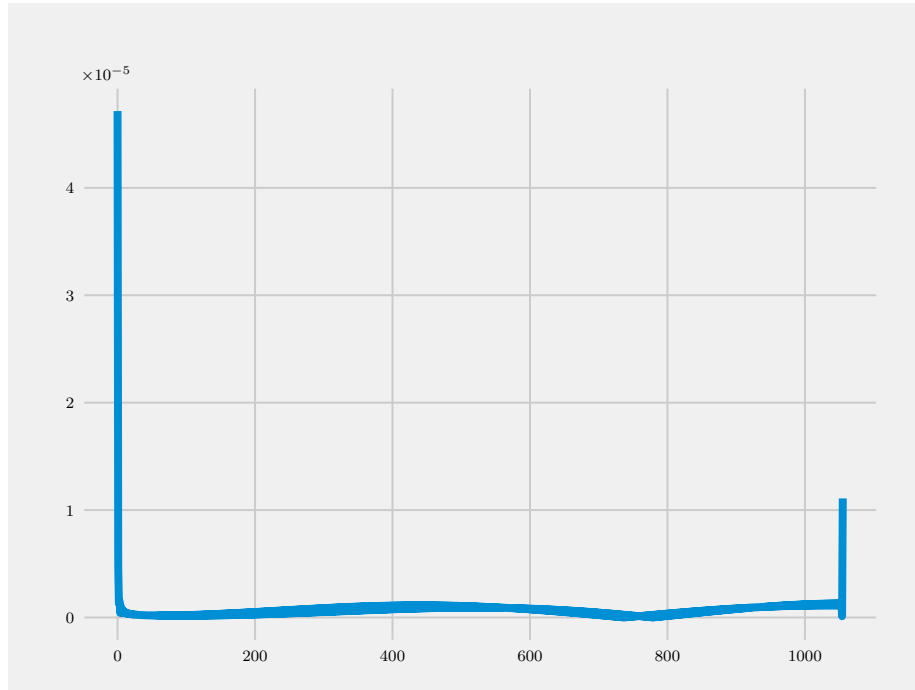Figure 107: Fourth Order, Case number 1, 1056points

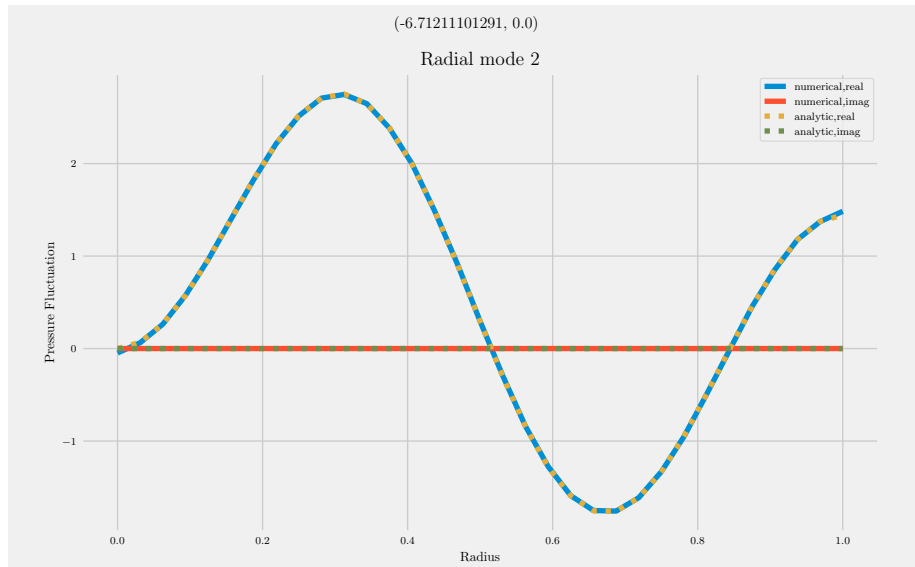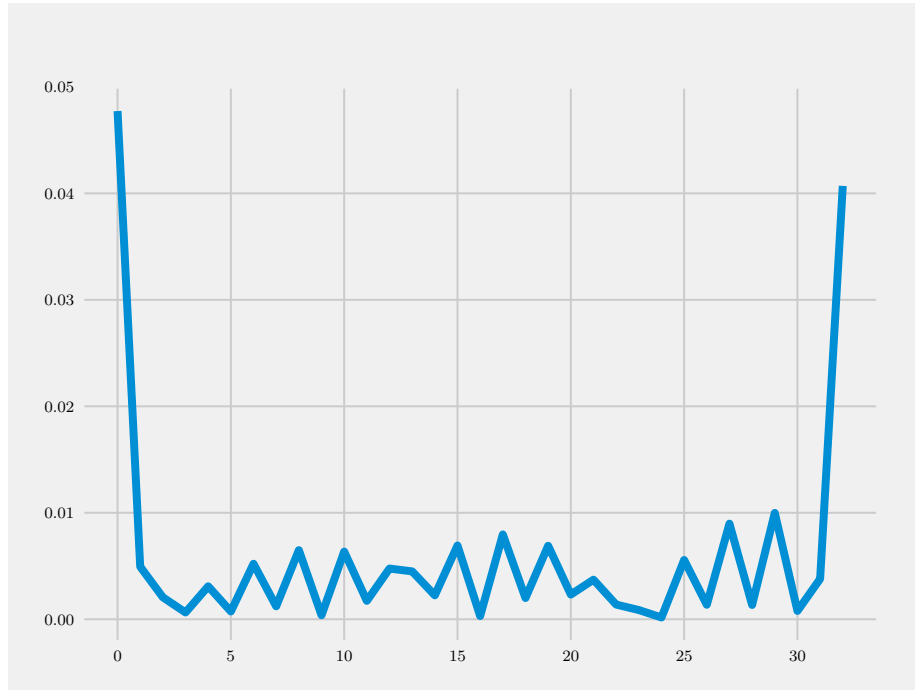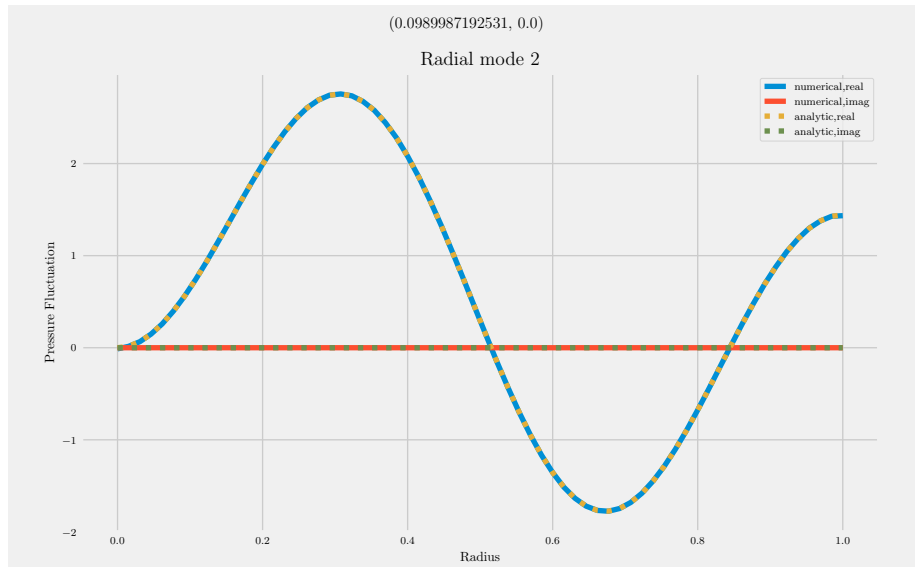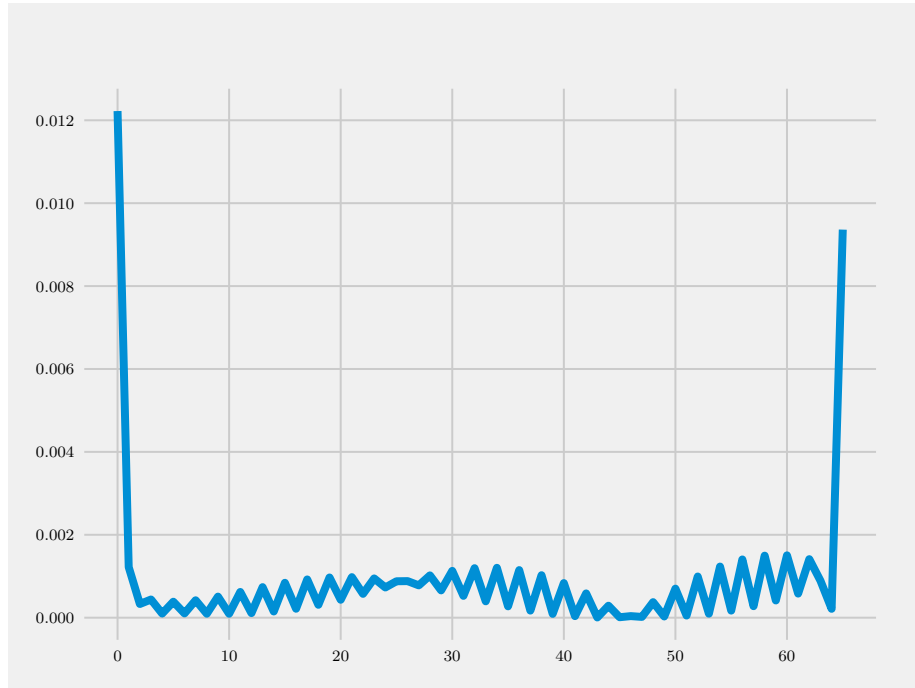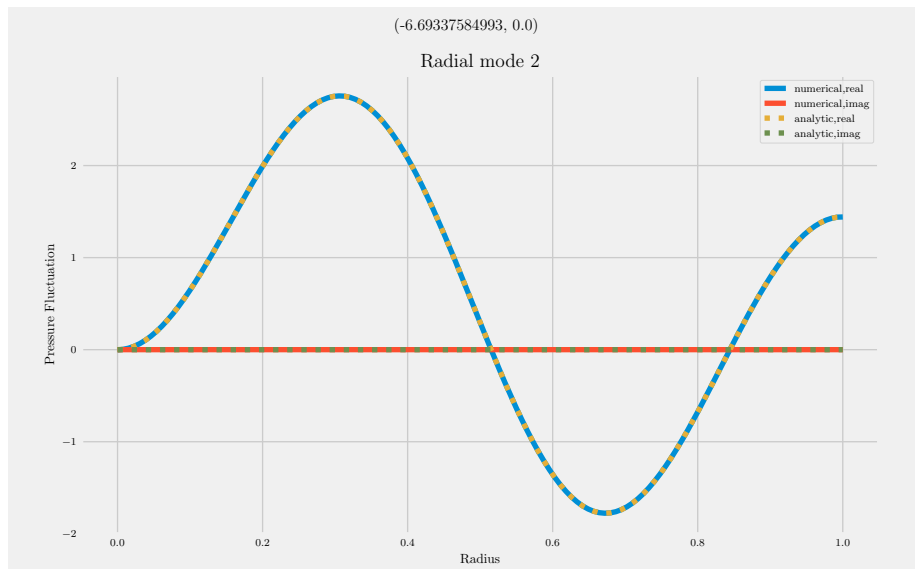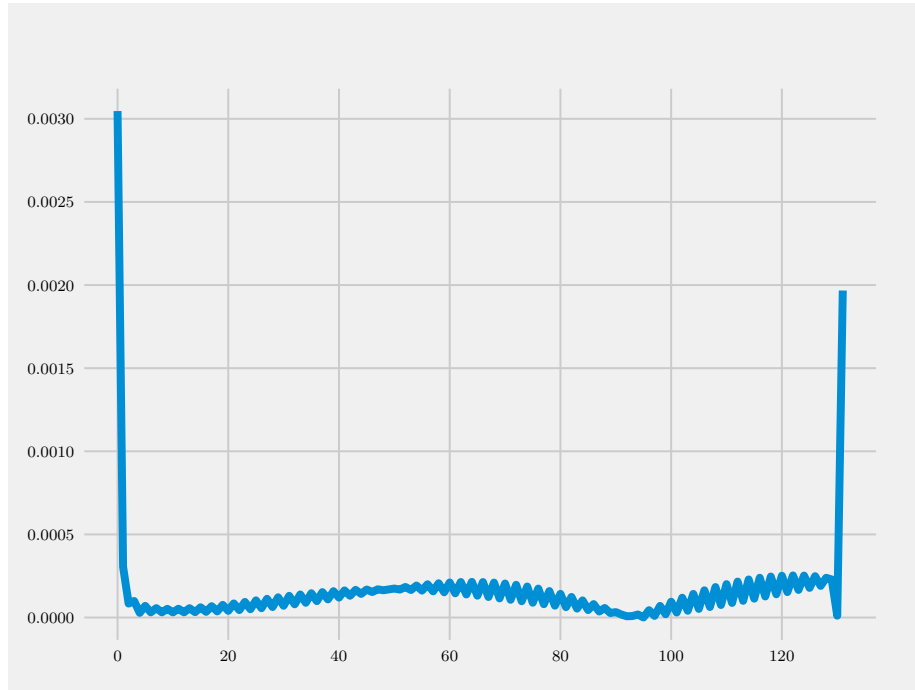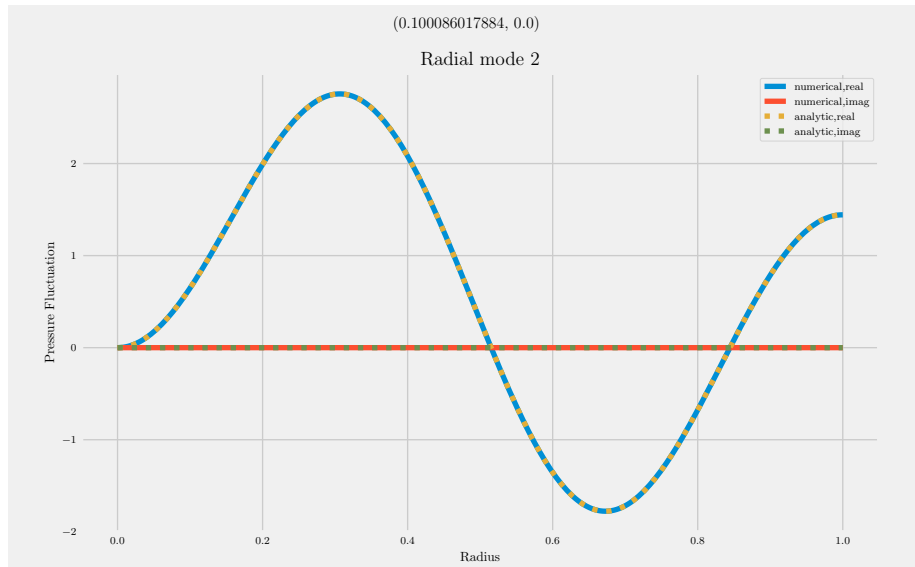Figure 108: Fourth Order, Case number 1, 1056points

Figure 109: Second Order, Case number 0, 33points

Second Order, Radial Mode 2

Figure 110: Second Order, Case number 0, 33points

Figure 111: Second Order, Case number 0, 66points

Figure 112: Second Order, Case number 0, 66points

Figure 113: Second Order, Case number 0, 132points

Figure 114: Second Order, Case number 0, 132points

Figure 115: Second Order, Case number 0, 264points

Figure 116: Second Order, Case number 0, 264points

Figure 117: Second Order, Case number 0, 528points

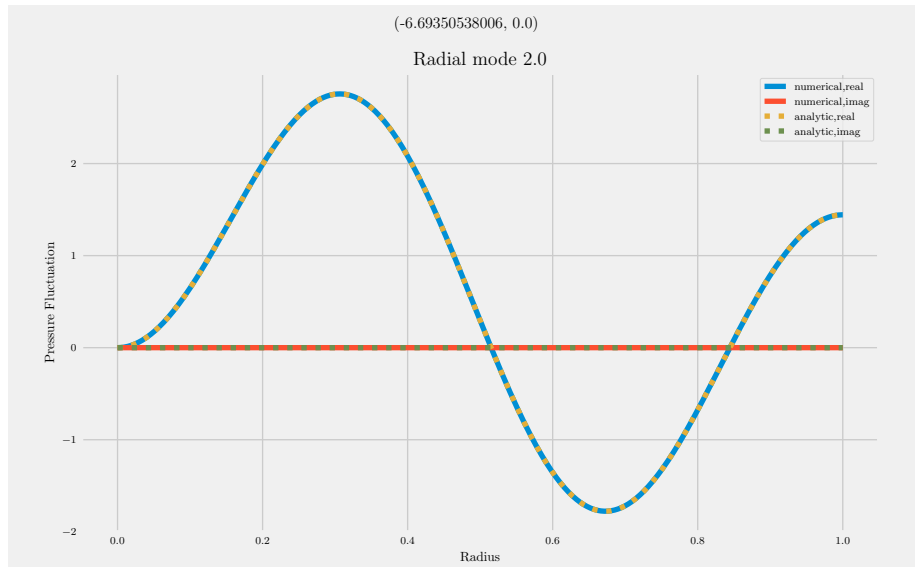Figure 118: Second Order, Case number 0, 528points

Figure 119: Second Order, Case number 0, 1056points

Figure 120: Second Order, Case number 0, 1056points

Figure 121: Second Order, Case number 1, 33points

Figure 122: Second Order, Case number 1, 33points

Figure 123: Second Order, Case number 1, 66points

Figure 124: Second Order, Case number 1, 66points

Figure 125: Second Order, Case number 1, 132points
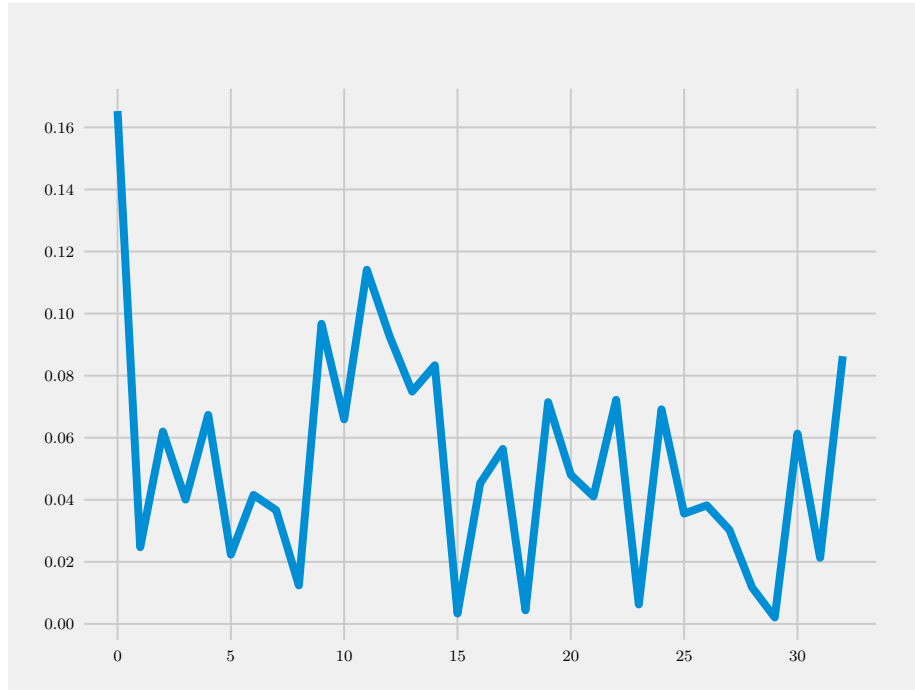
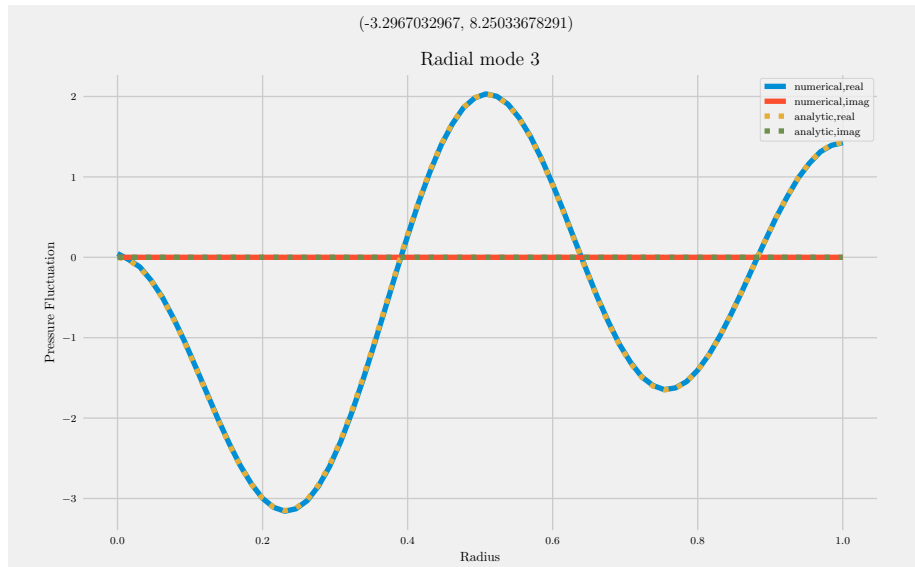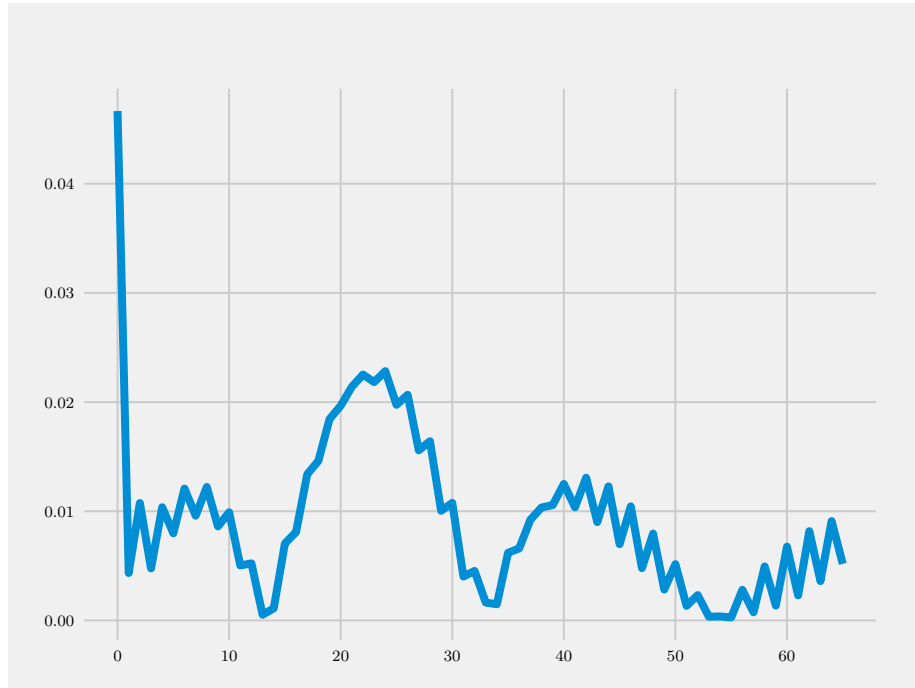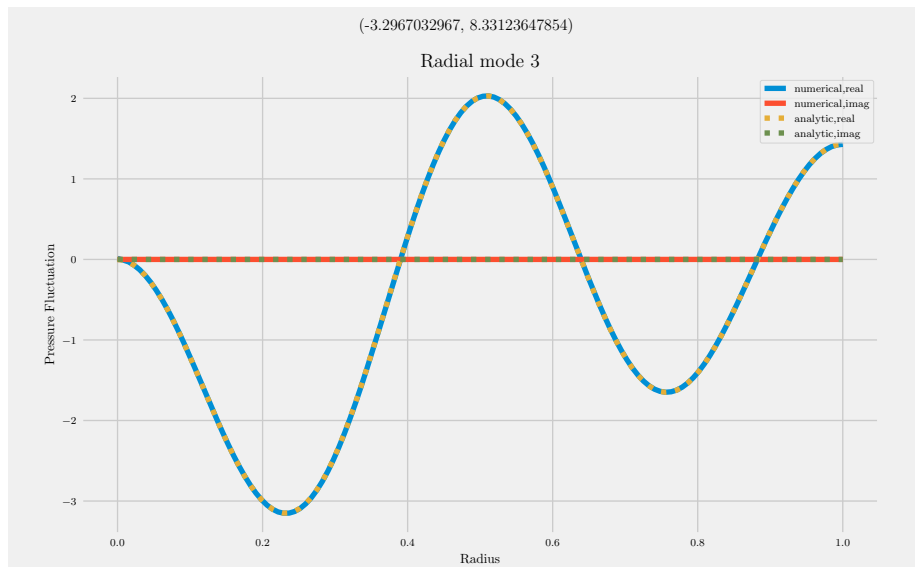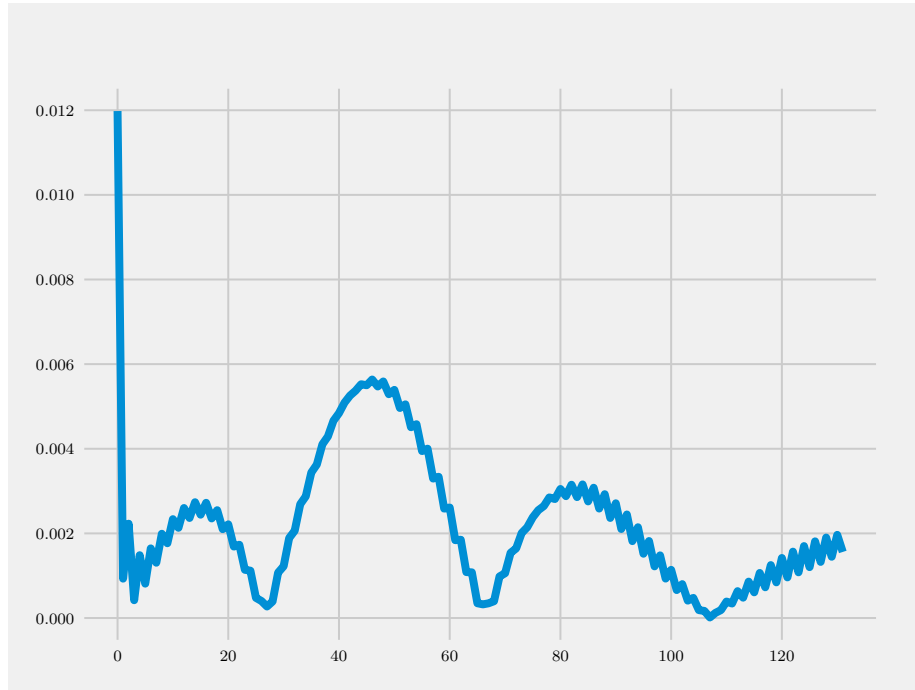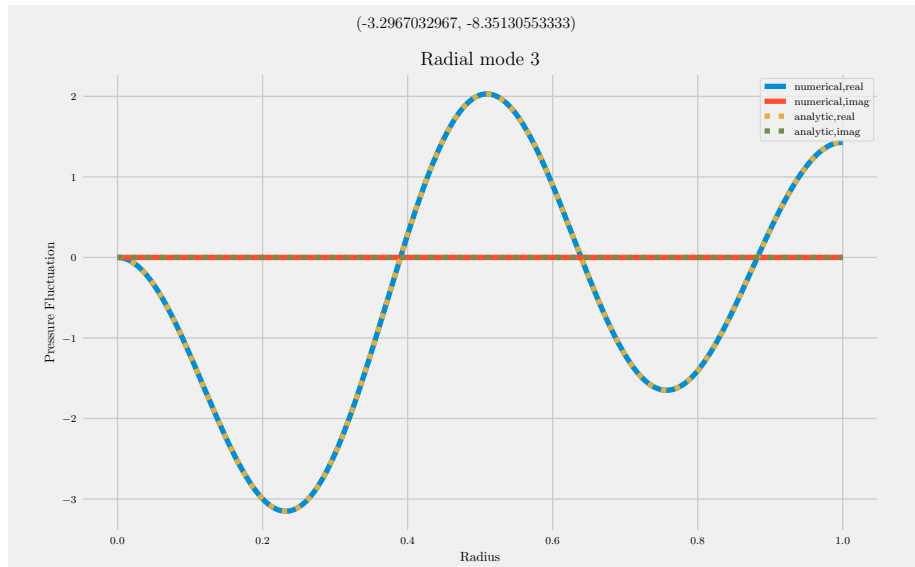Figure 126: Second Order, Case number 1, 132points

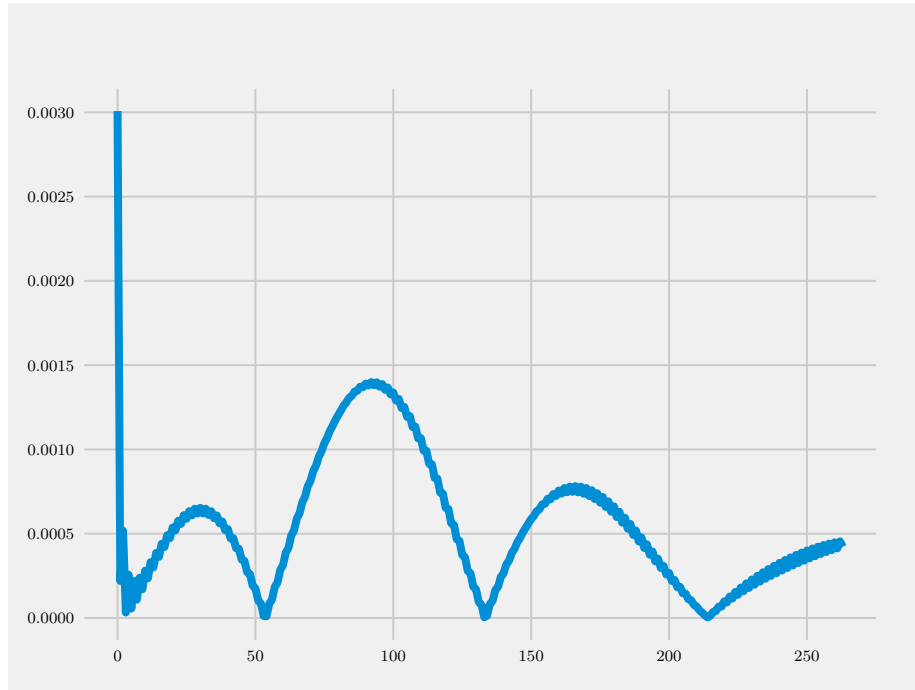Figure 127: Second Order, Case number 1, 264points

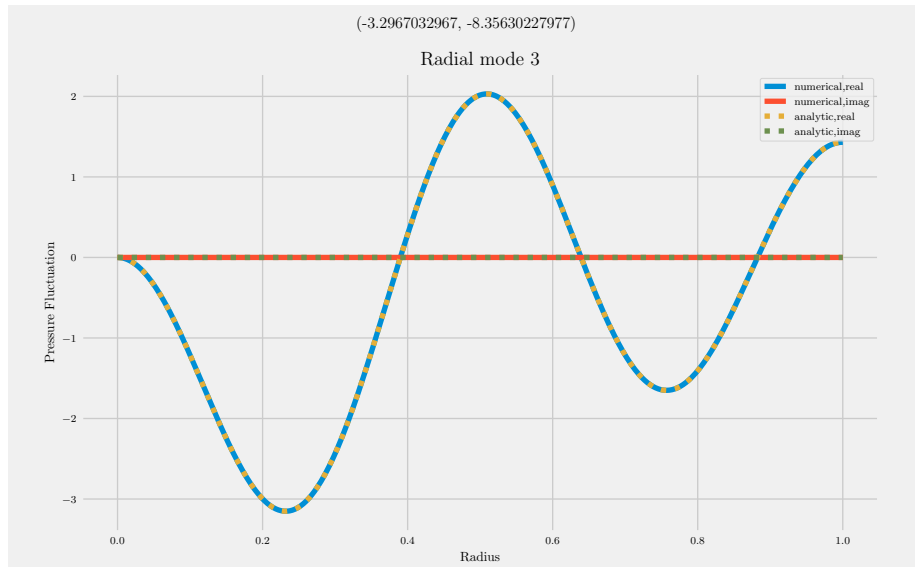Figure 128: Second Order, Case number 1, 264points

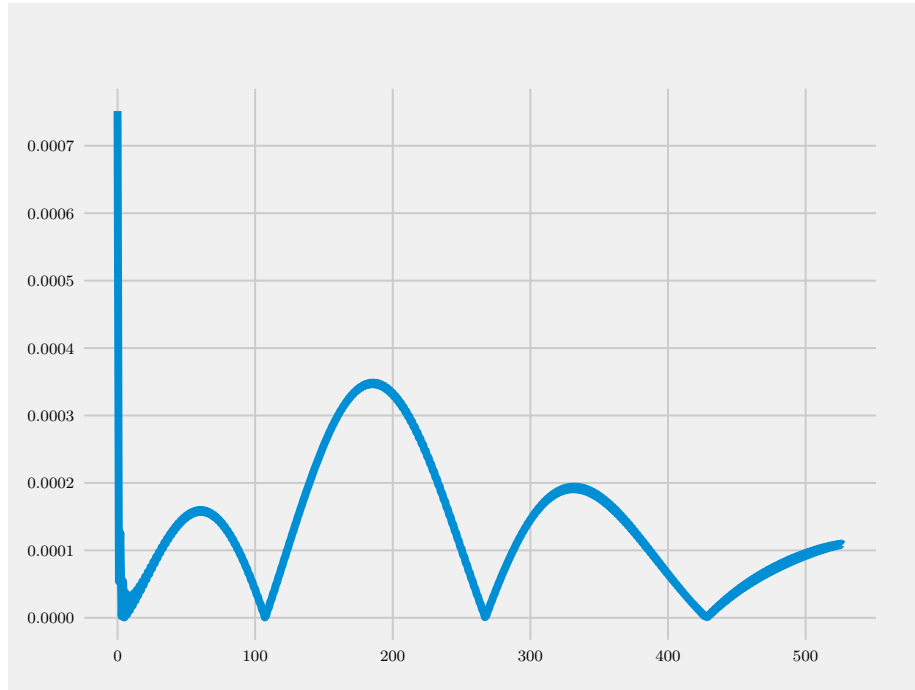Figure 129: Second Order, Case number 1, 528points
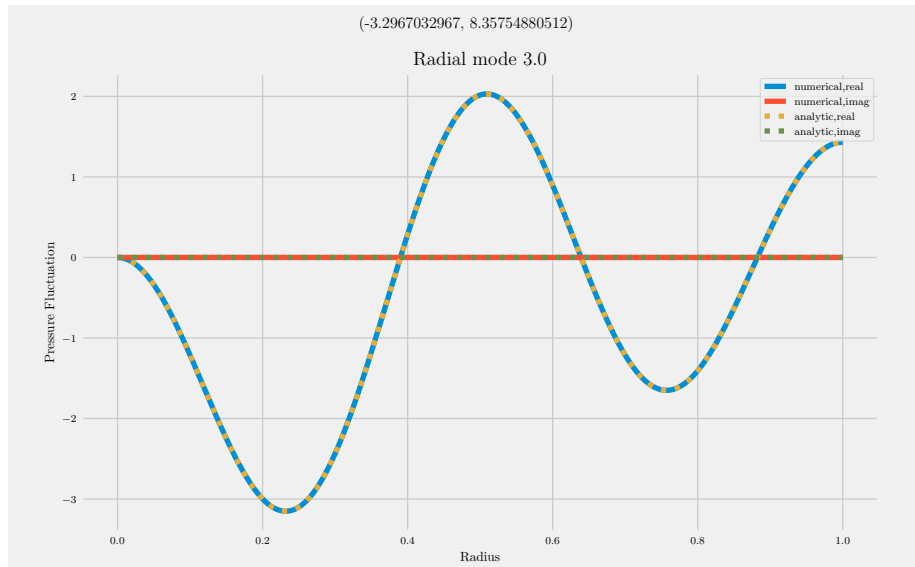
74

Figure 130: Second Order, Case number 1, 528points

Figure 131: Second Order, Case number 1, 1056points
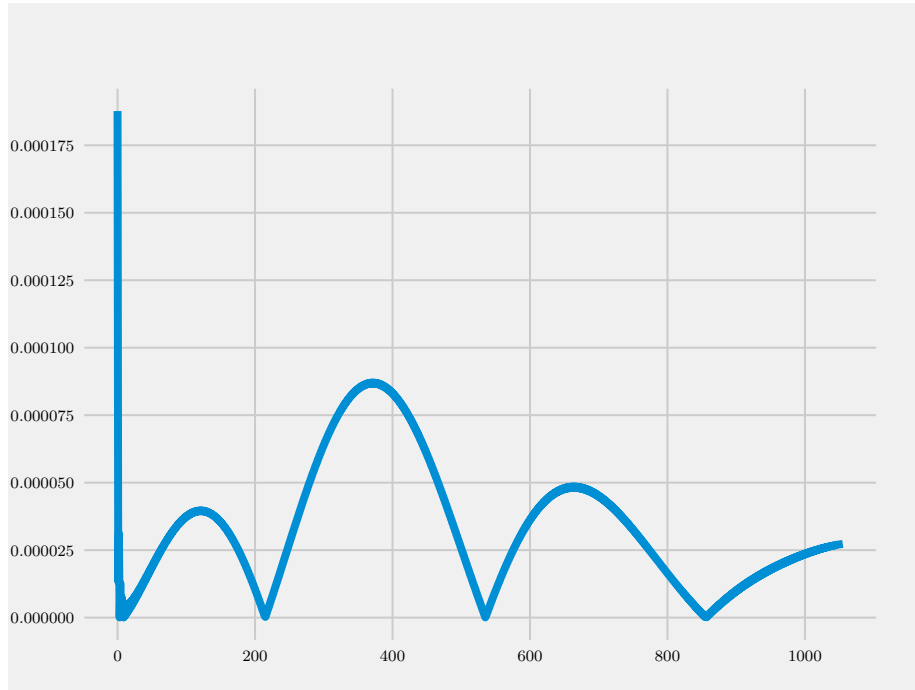
Figure 132: Second Order, Case number 1, 1056points

Figure 133: Fourth Order, Case number 0, 33points

Fourth Order, Radial Mode 2

Figure 134: Fourth Order, Case number 0, 33points

Figure 135: Fourth Order, Case number 0, 66points

Figure 136: Fourth Order, Case number 0, 66points

Figure 137: Fourth Order, Case number 0, 132points

Figure 138: Fourth Order, Case number 0, 132points

Figure 139: Fourth Order, Case number 0, 264points

Figure 140: Fourth Order, Case number 0, 264points

Figure 141: Fourth Order, Case number 0, 528points

Figure 142: Fourth Order, Case number 0, 528points

Figure 143: Fourth Order, Case number 0, 1056points

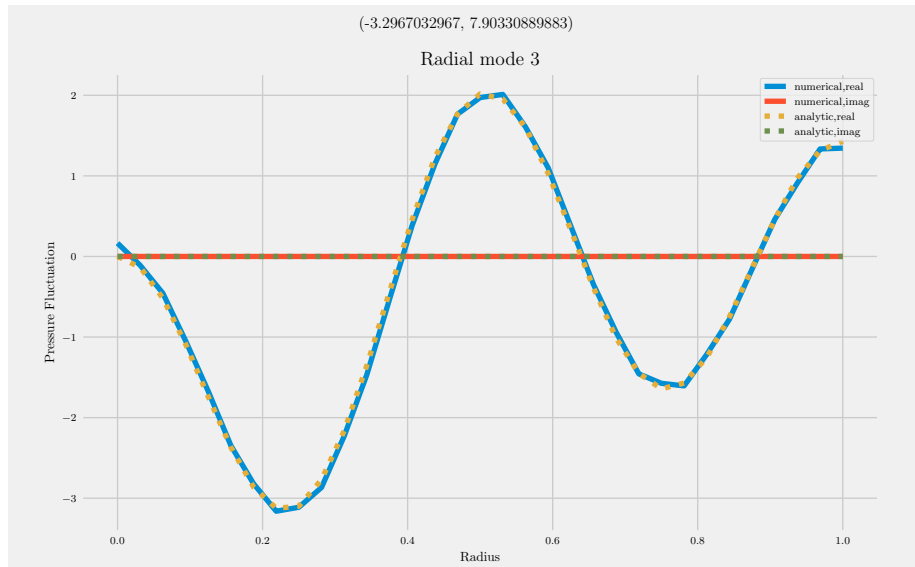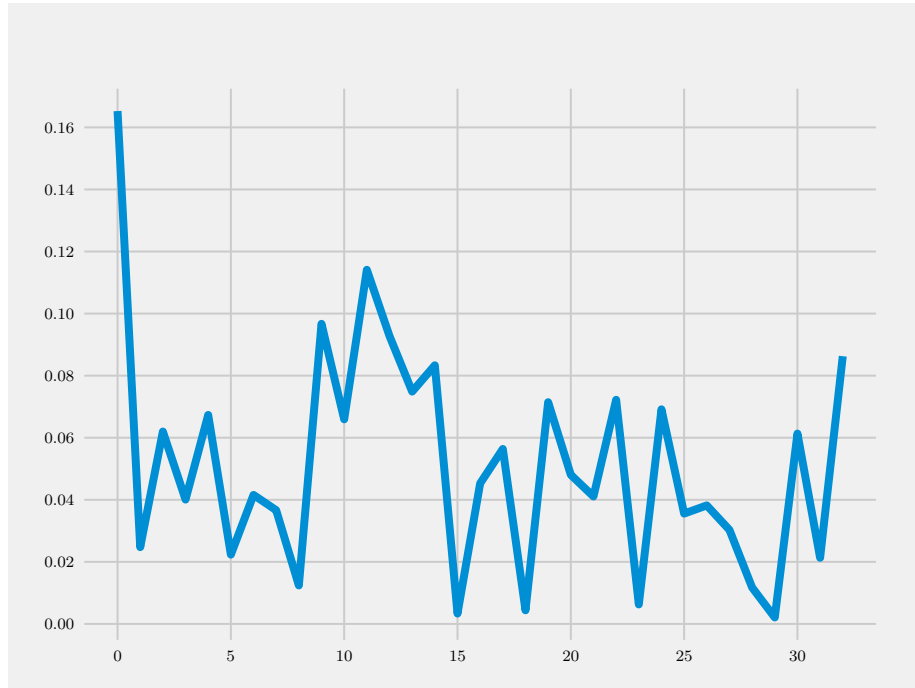Figure 144: Fourth Order, Case number 0, 1056points

Figure 145: Fourth Order, Case number 1, 33points

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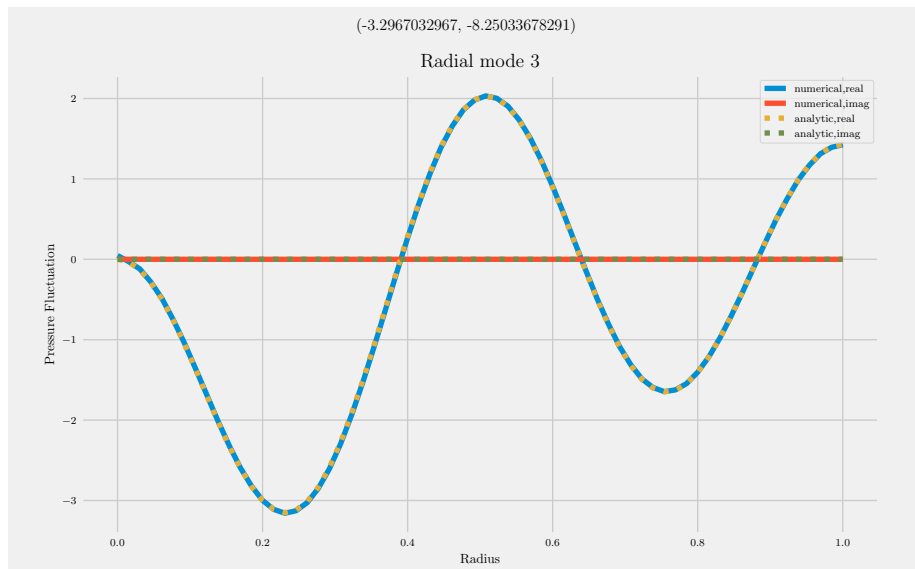
Figure 146: Fourth Order, Case number 1, 33points

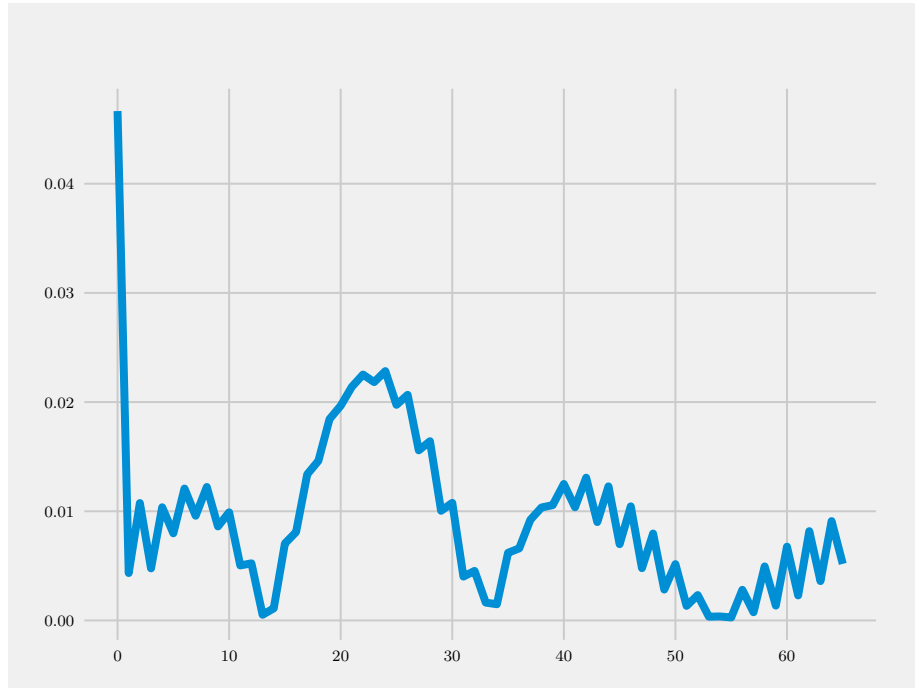Figure 147: Fourth Order, Case number 1, 66points

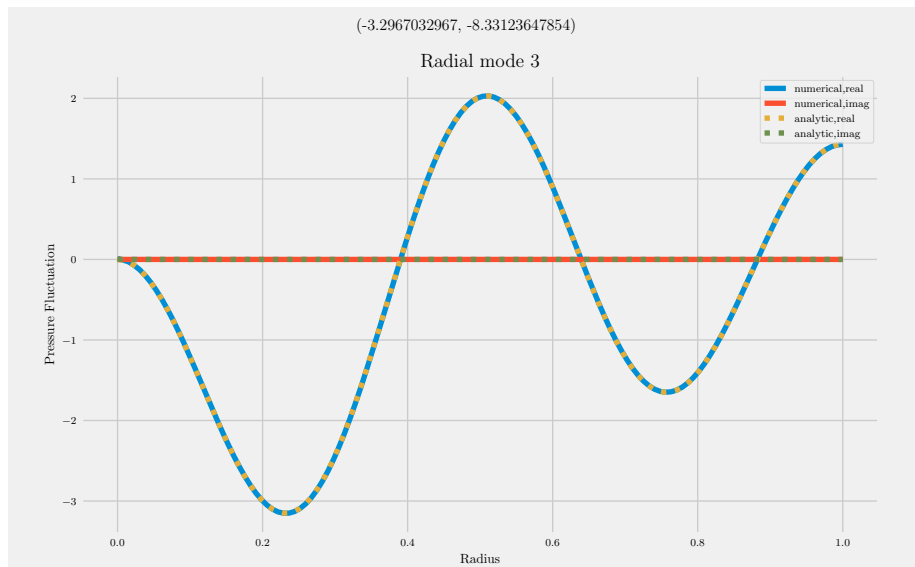Figure 148: Fourth Order, Case number 1, 66points

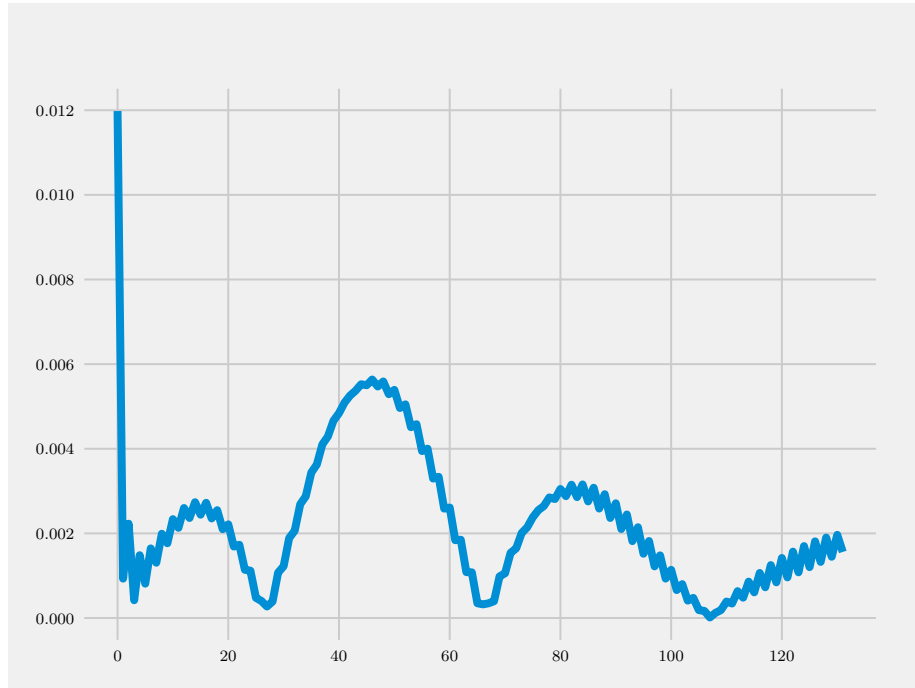Figure 149: Fourth Order, Case number 1, 132points

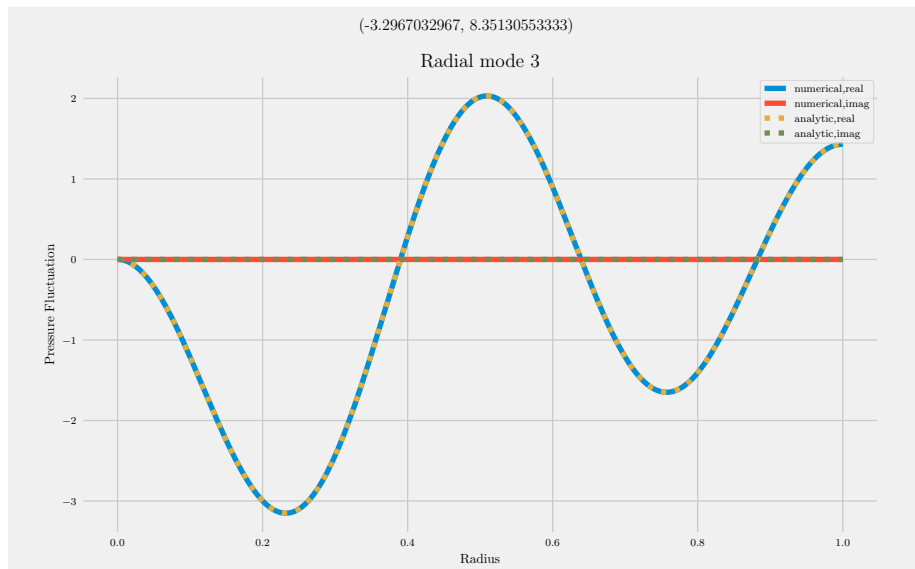Figure 150: Fourth Order, Case number 1, 132points

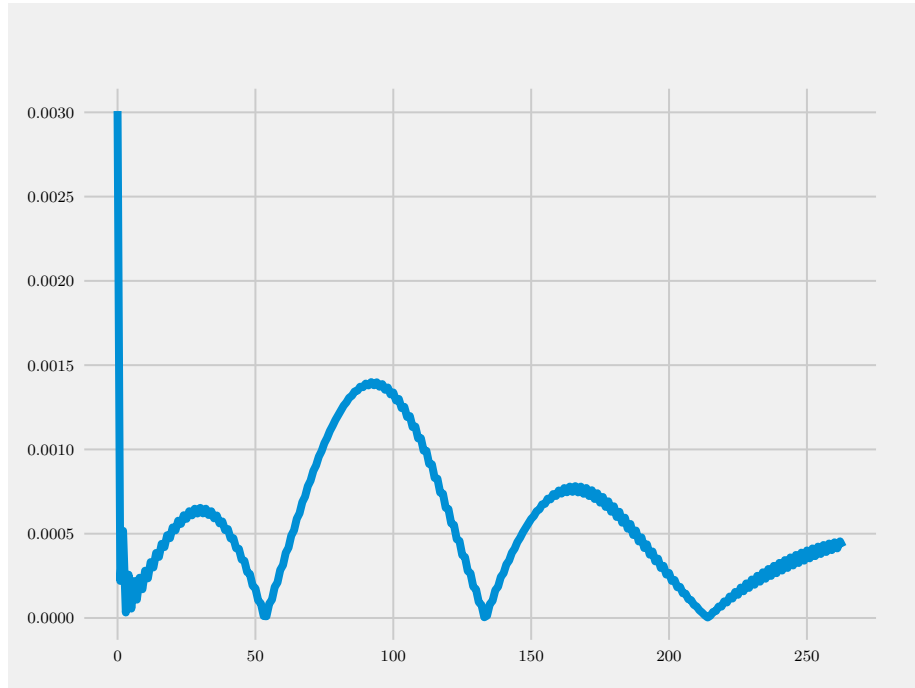Figure 151: Fourth Order, Case number 1, 264points
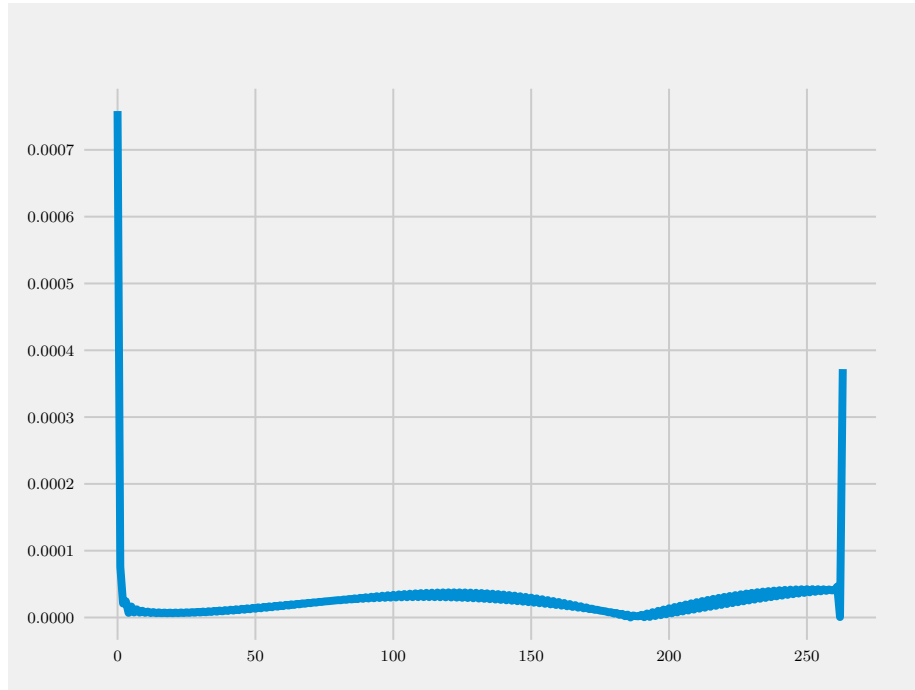
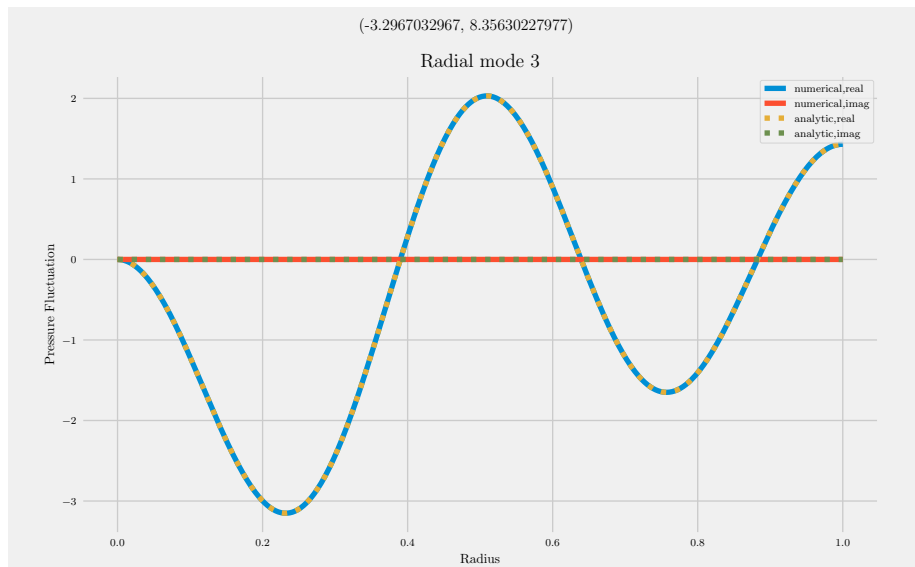Figure 152: Fourth Order, Case number 1, 264points
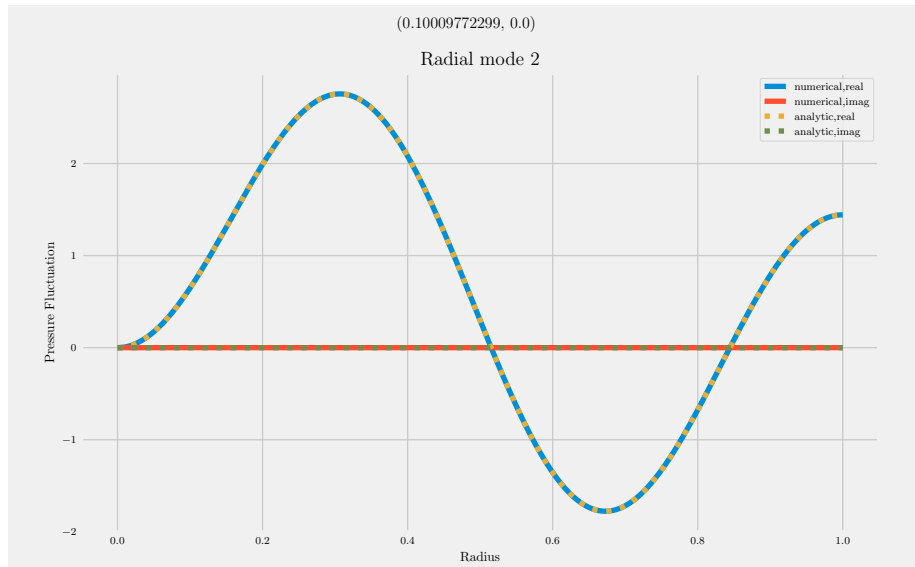
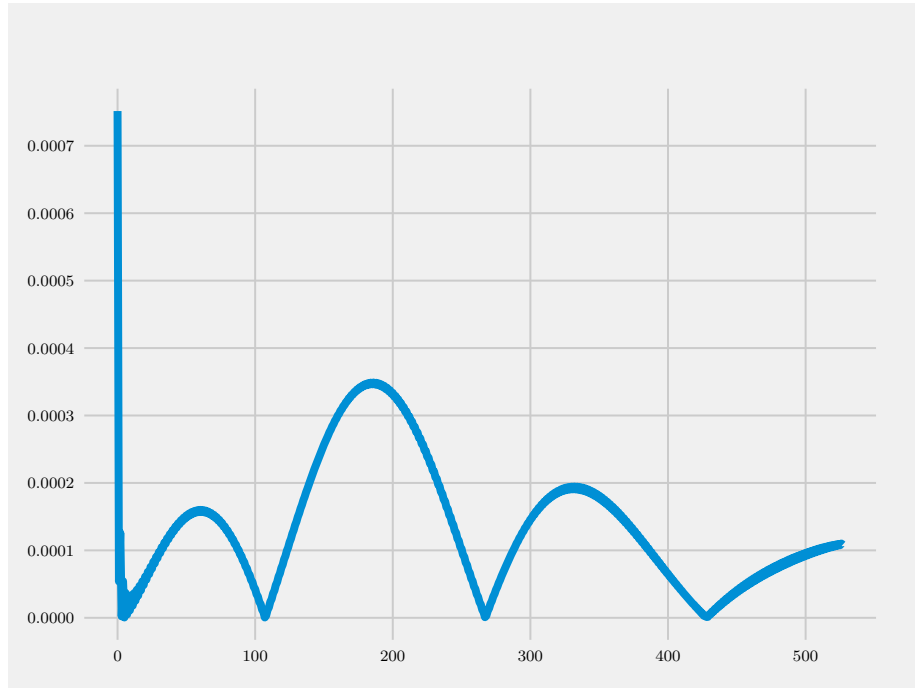Figure 153: Fourth Order, Case number 1, 528points
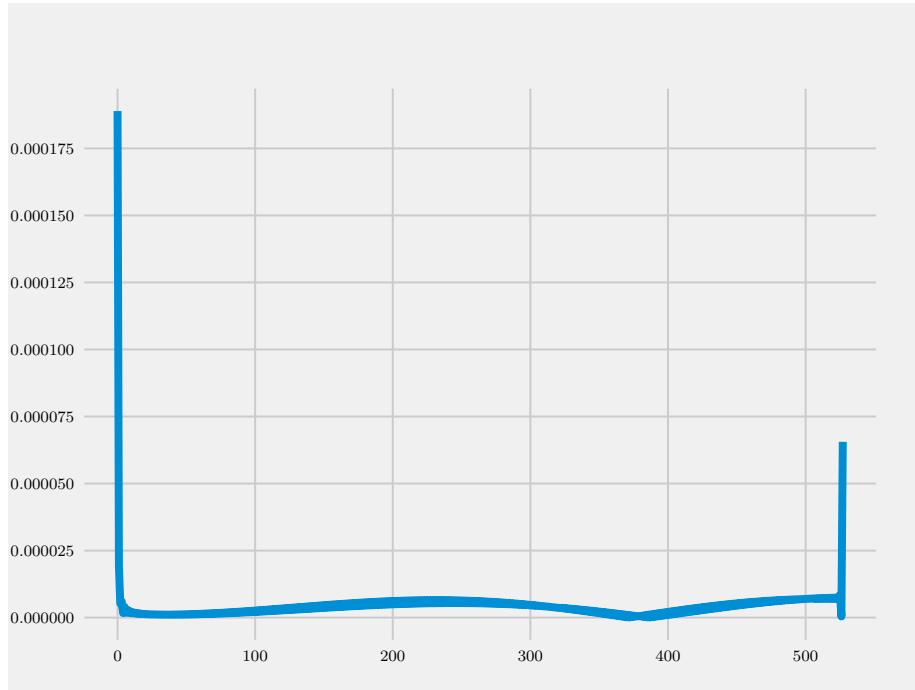
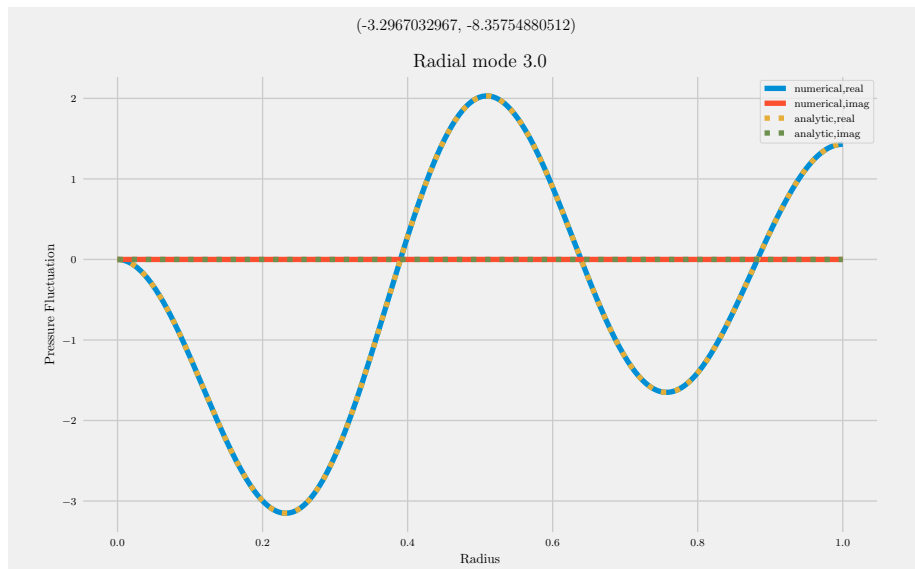Figure 154: Fourth Order, Case number 1, 528points

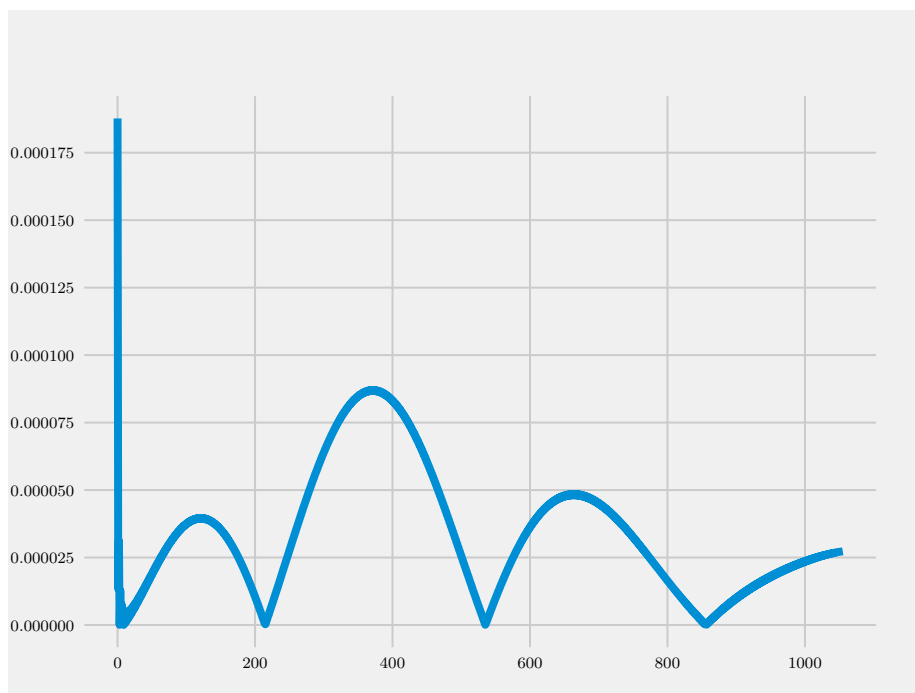Figure 155: Fourth Order, Case number 1, 1056points

88
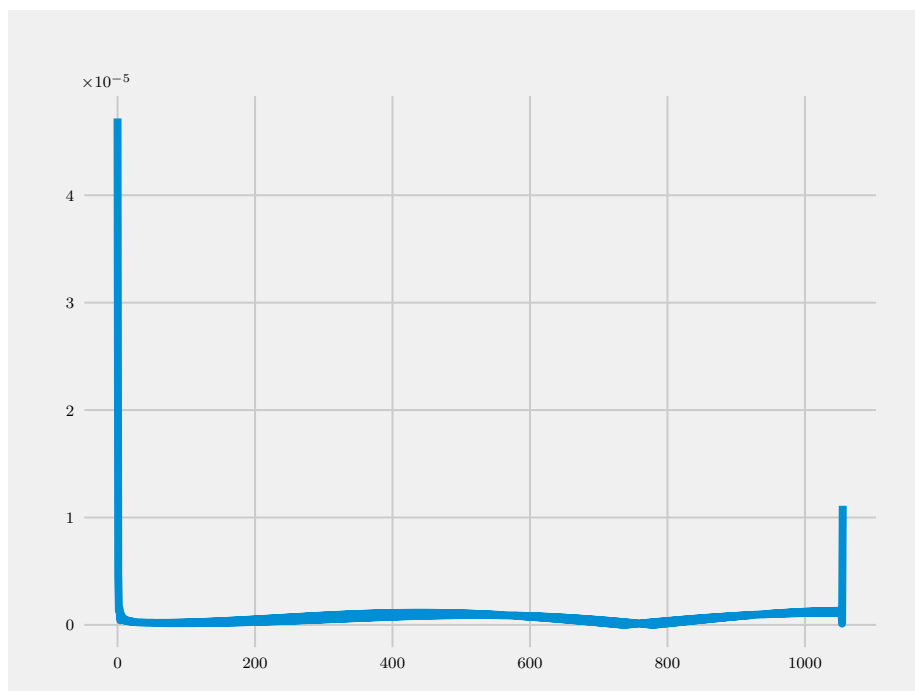
Figure 156: Fourth Order, Case number 1, 1056points
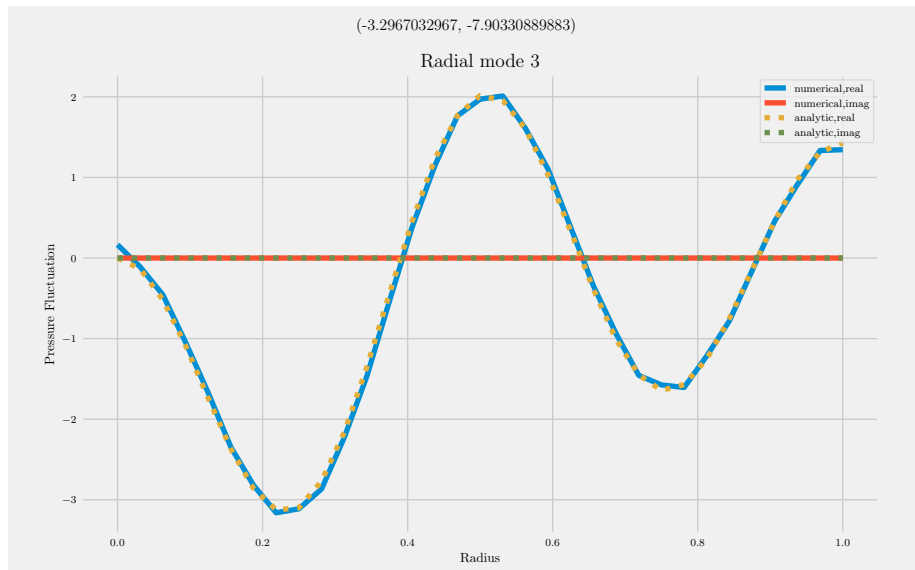
Figure 157: Second Order, Case number 0, 33points

Second Order, Radial Mode 3

Figure 158: Second Order, Case number 0, 33points

Figure 159: Second Order, Case number 0, 66points
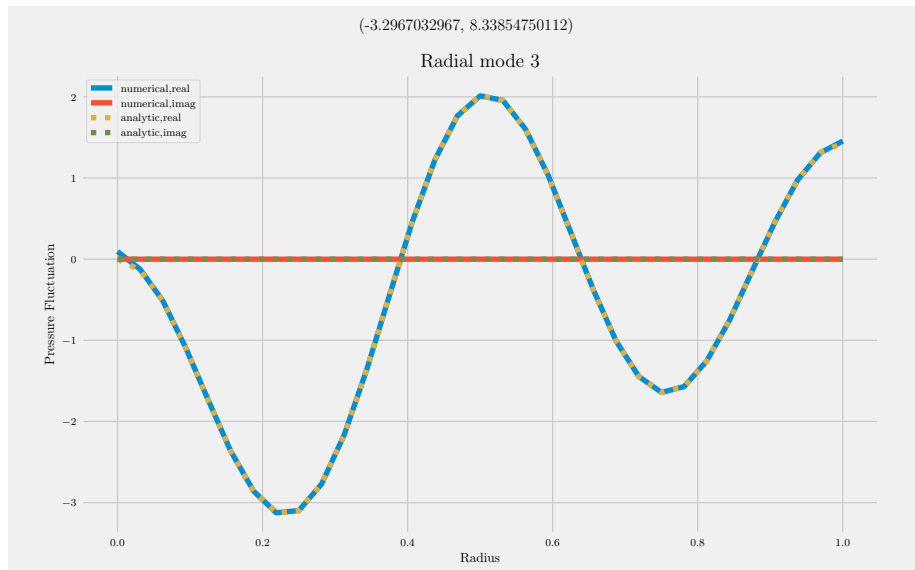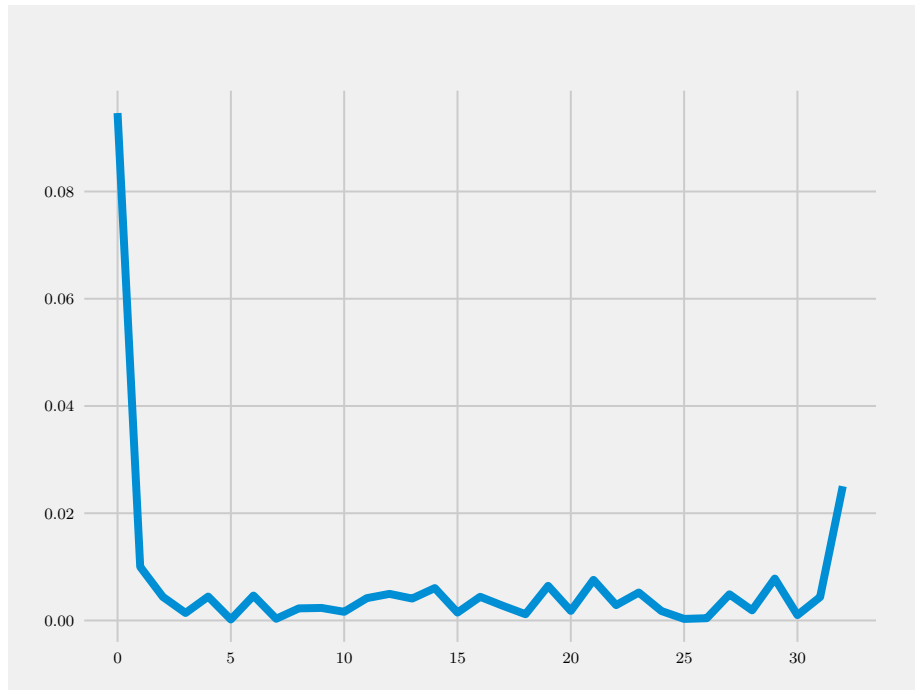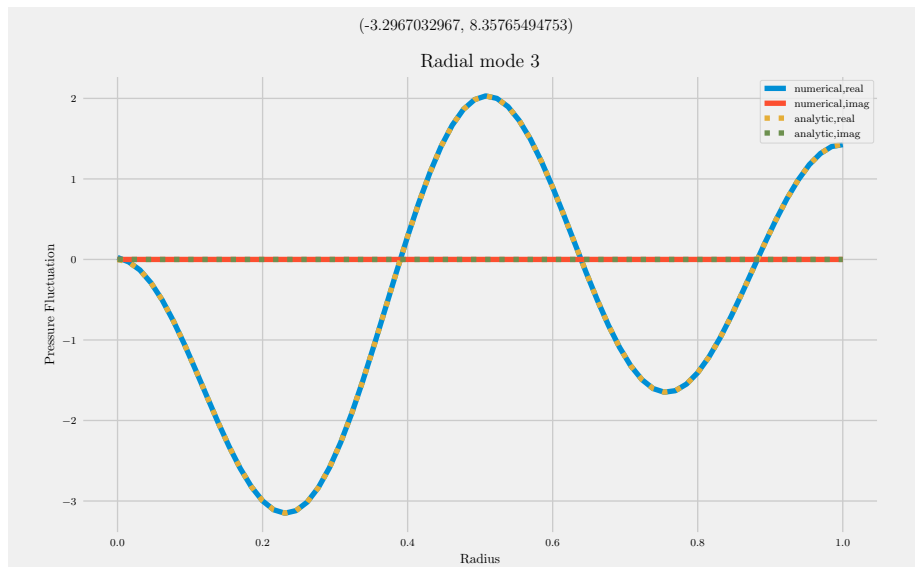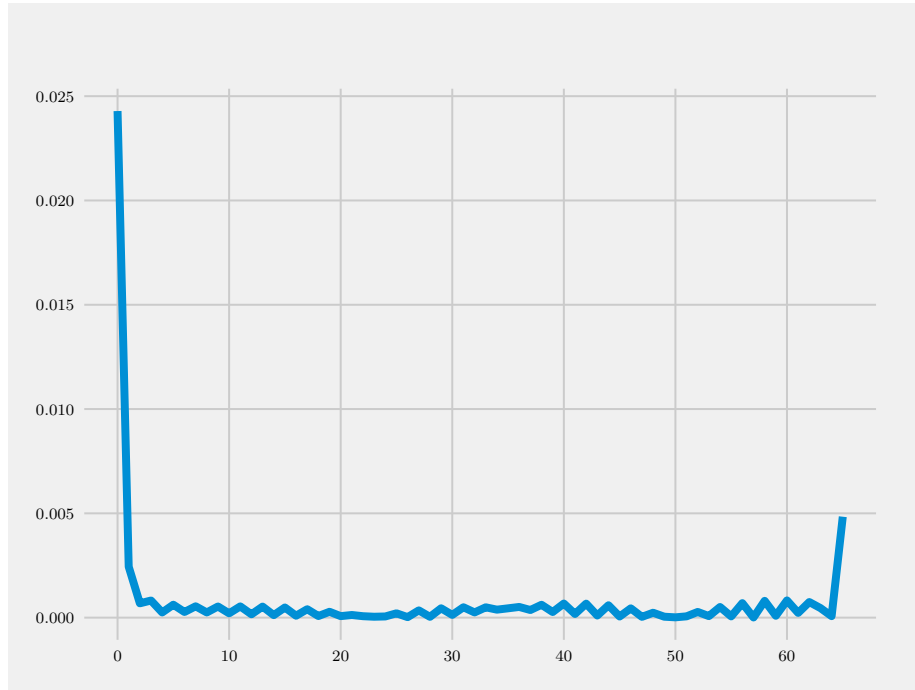
91

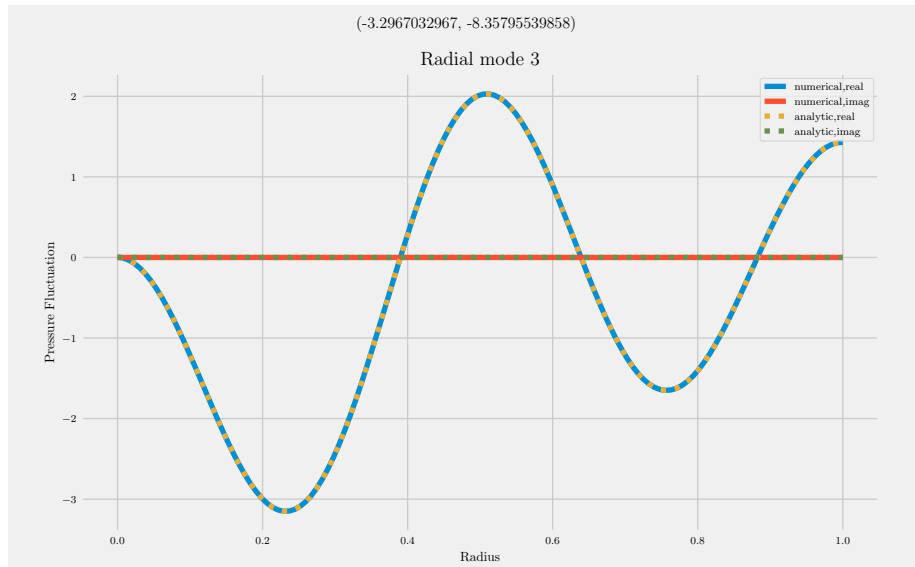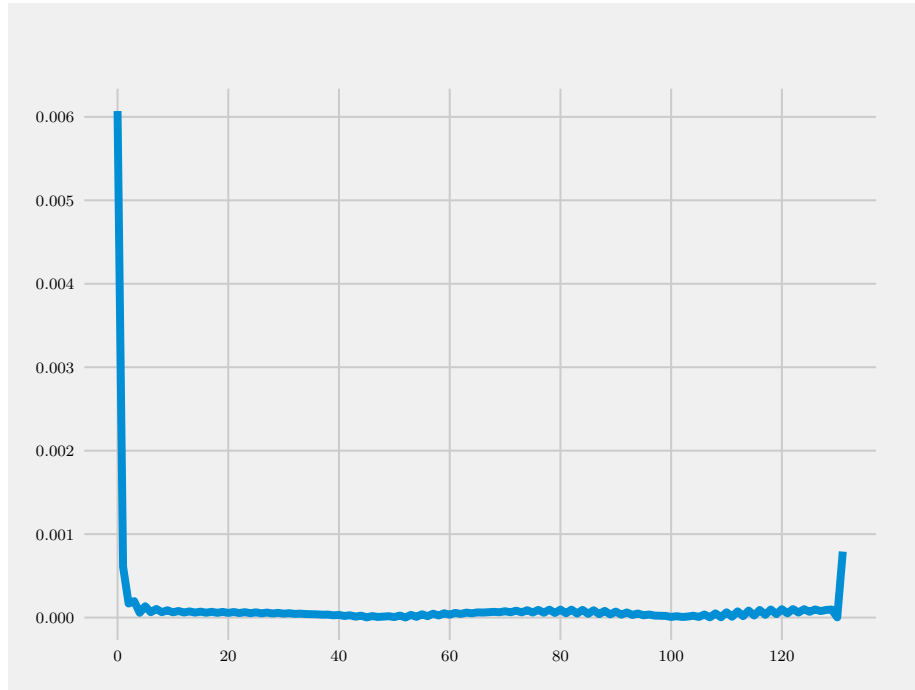Figure 160: Second Order, Case number 0, 66points

Figure 161: Second Order, Case number 0, 132points

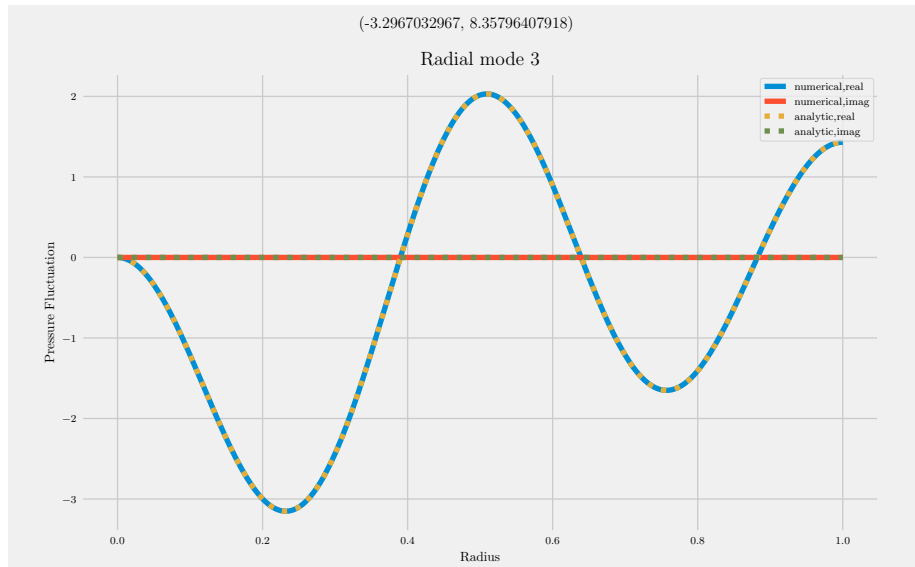Figure 162: Second Order, Case number 0, 132points

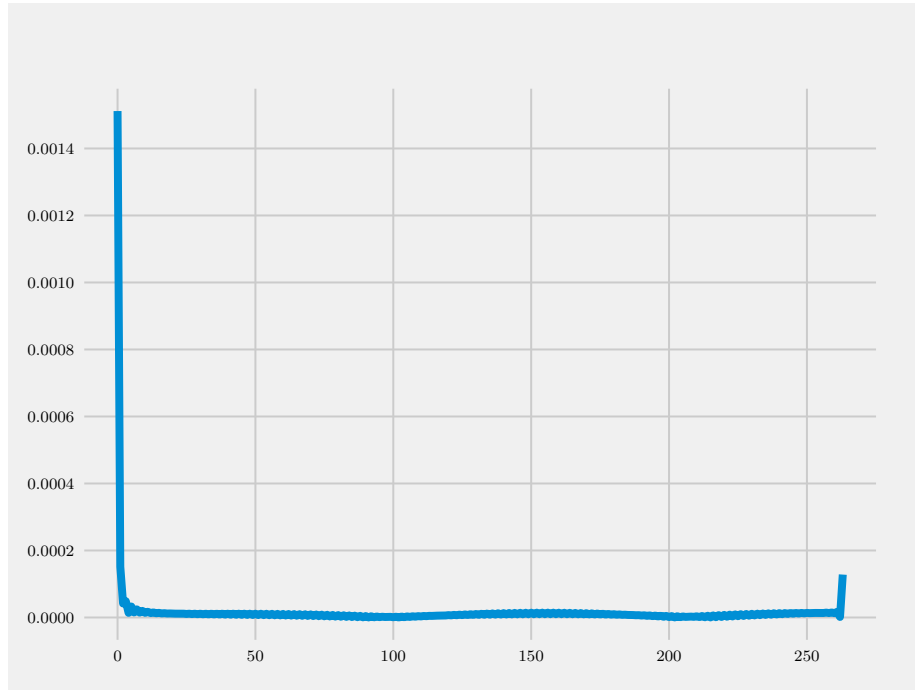Figure 163: Second Order, Case number 0, 264points

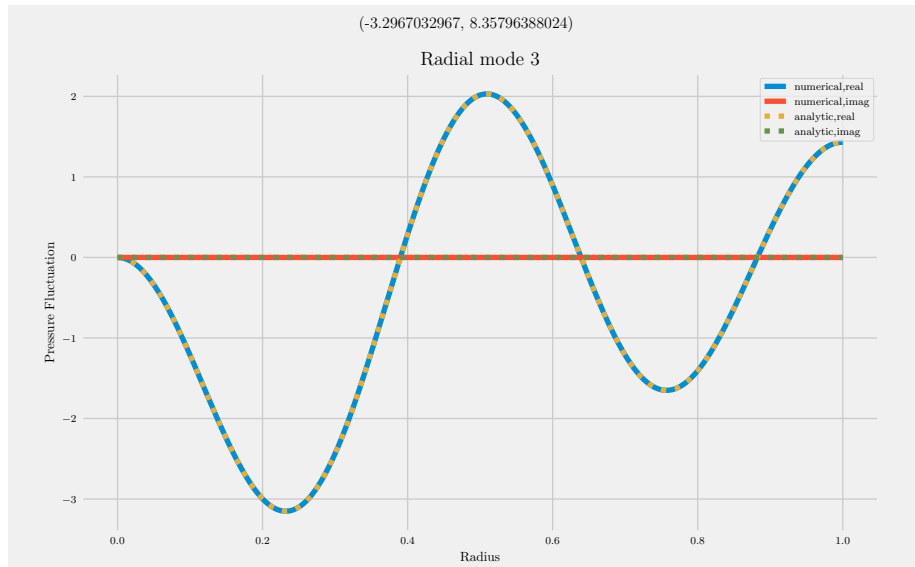Figure 164: Second Order, Case number 0, 264points

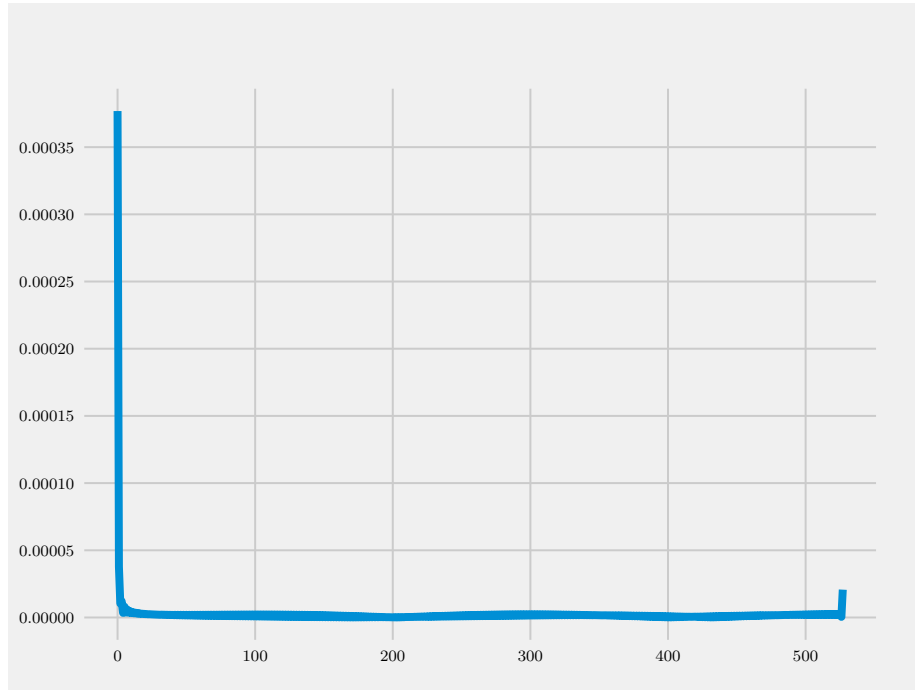Figure 165: Second Order, Case number 0, 528points

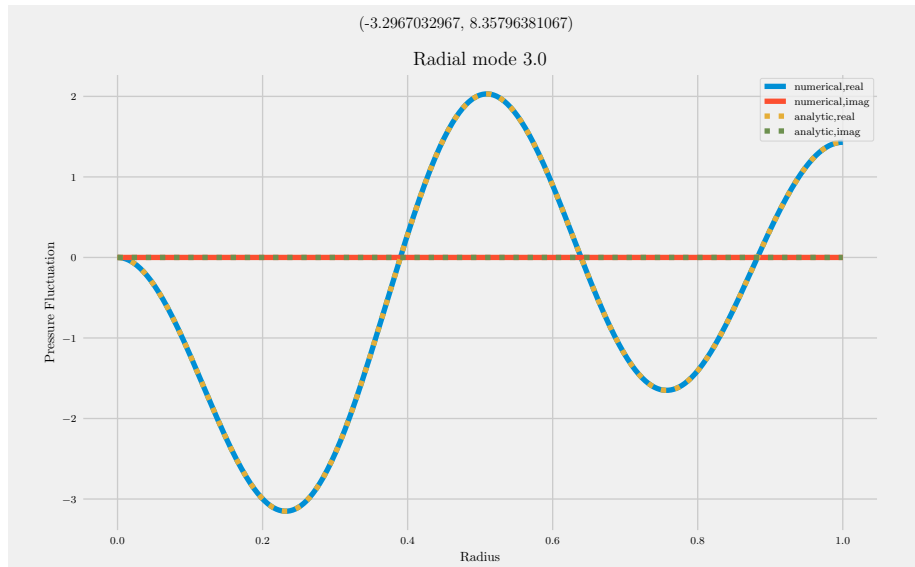Figure 166: Second Order, Case number 0, 528points

Figure 167: Second Order, Case number 0, 1056points

Figure 168: Second Order, Case number 0, 1056points

Figure 169: Second Order, Case number 1, 33points

Figure 170: Second Order, Case number 1, 33points

Figure 171: Second Order, Case number 1, 66points

Figure 172: Second Order, Case number 1, 66points

Figure 173: Second Order, Case number 1, 132points

Figure 174: Second Order, Case number 1, 132points

Figure 175: Second Order, Case number 1, 264points

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Figure 176: Second Order, Case number 1, 264points

Figure 177: Second Order, Case number 1, 528points

Figure 178: Second Order, Case number 1, 528points

Figure 179: Second Order, Case number 1, 1056points

Figure 180: Second Order, Case number 1, 1056points
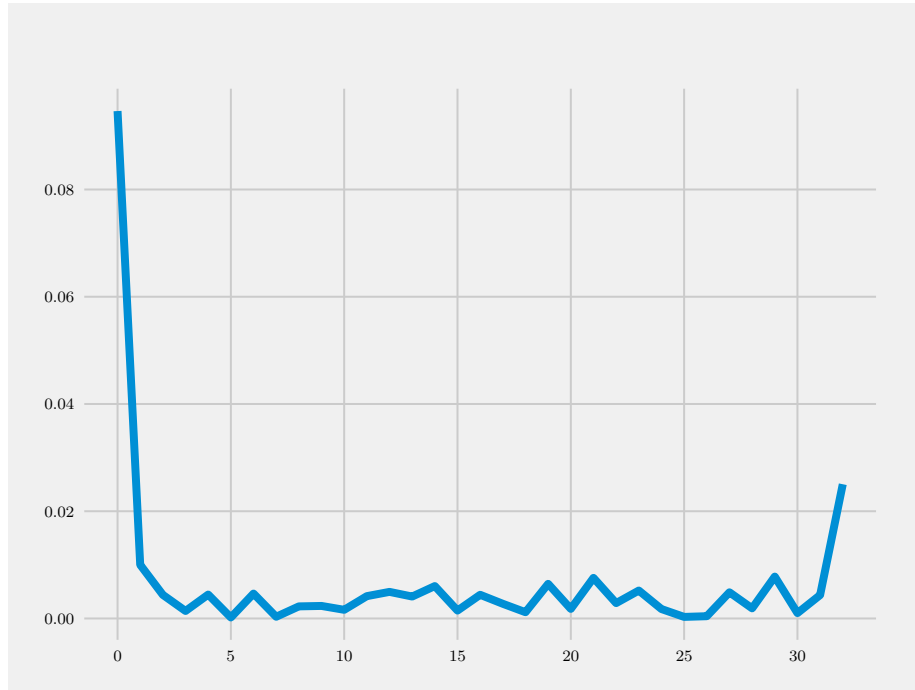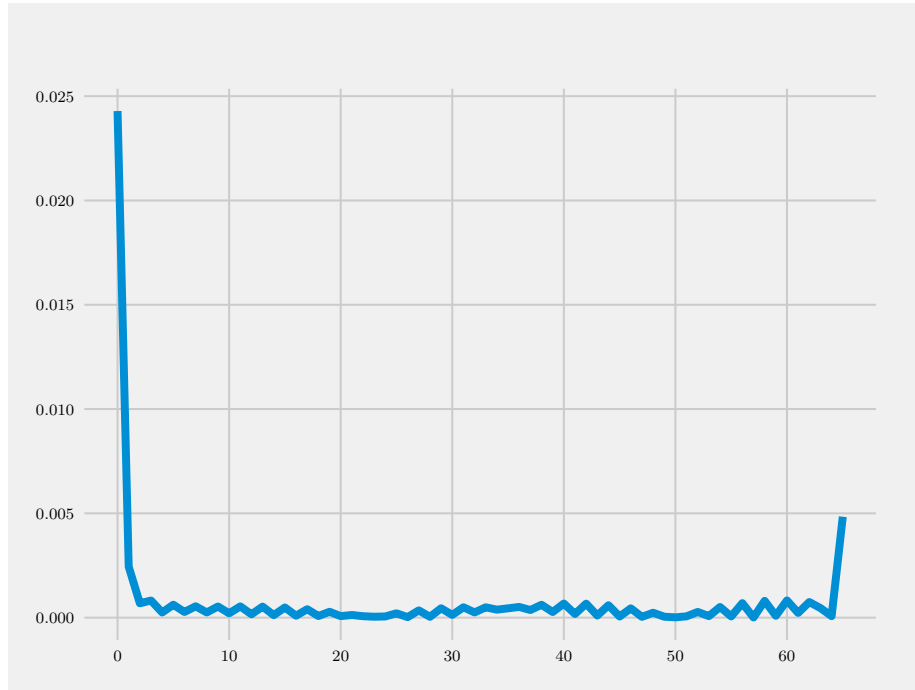
Figure 181: Fourth Order, Case number 0, 33points

Fourth Order, Radial Mode 3

Figure 182: Fourth Order, Case number 0, 33points

Figure 183: Fourth Order, Case number 0, 66points

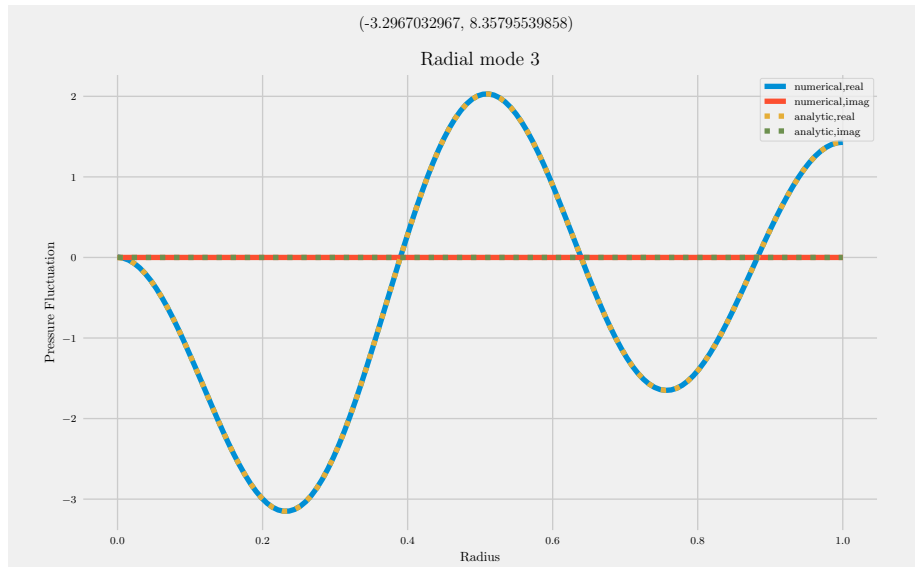Figure 184: Fourth Order, Case number 0, 66points

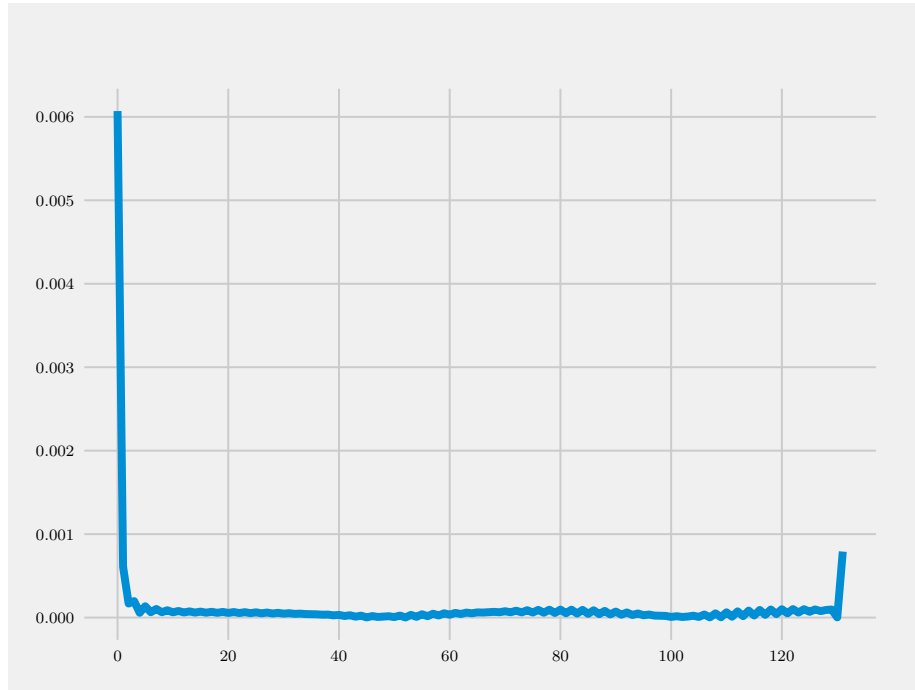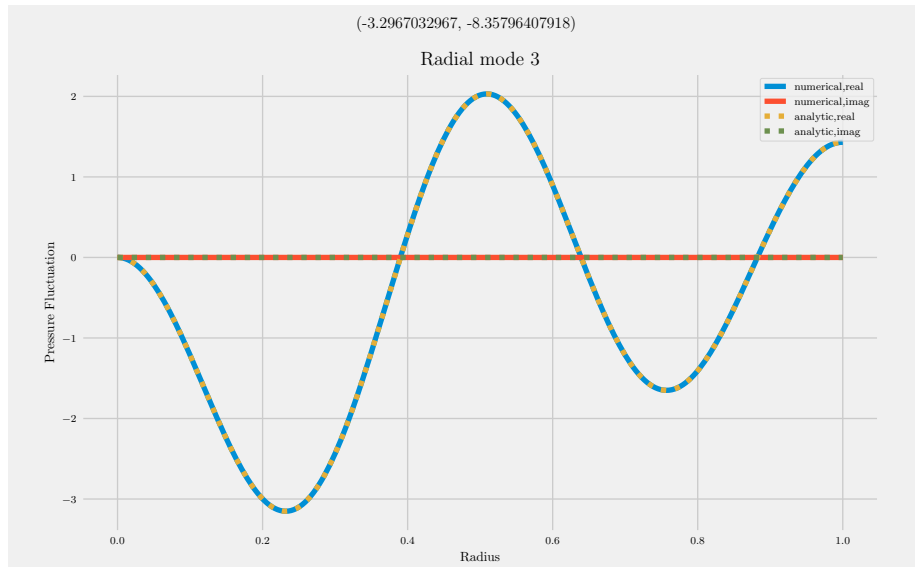Figure 185: Fourth Order, Case number 0, 132points

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Figure 186: Fourth Order, Case number 0, 132points

Figure 187: Fourth Order, Case number 0, 264points

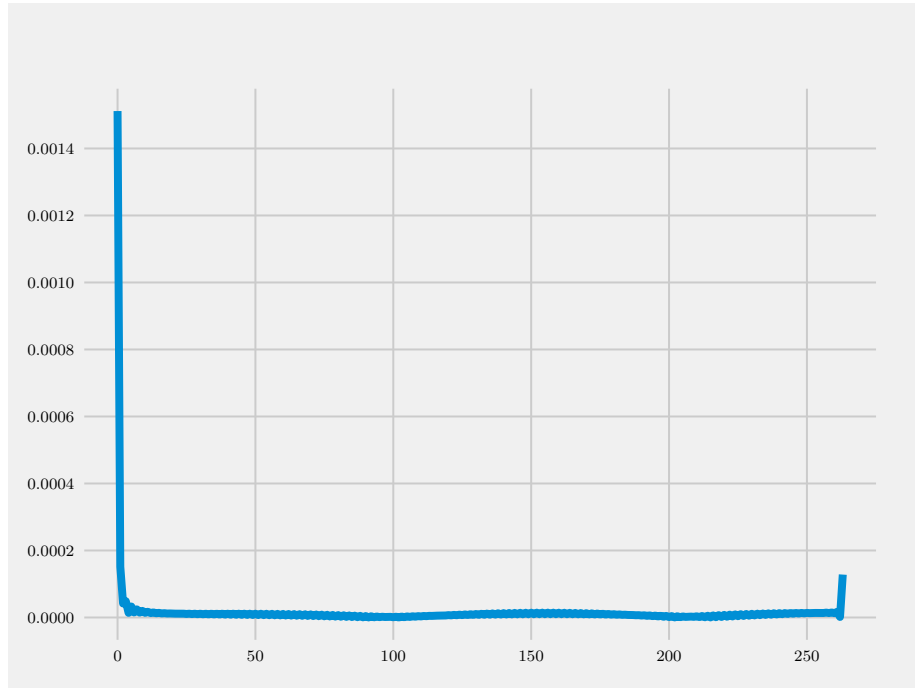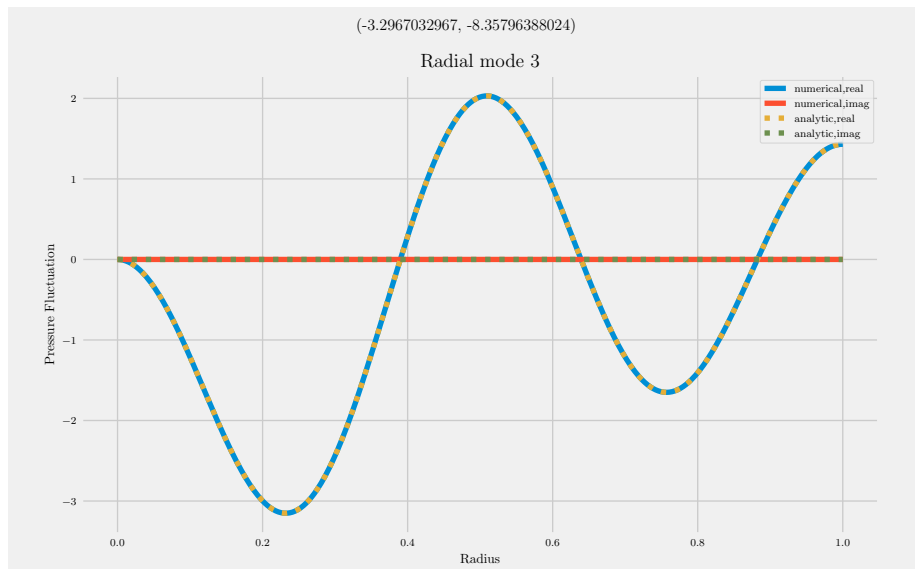Figure 188: Fourth Order, Case number 0, 264points

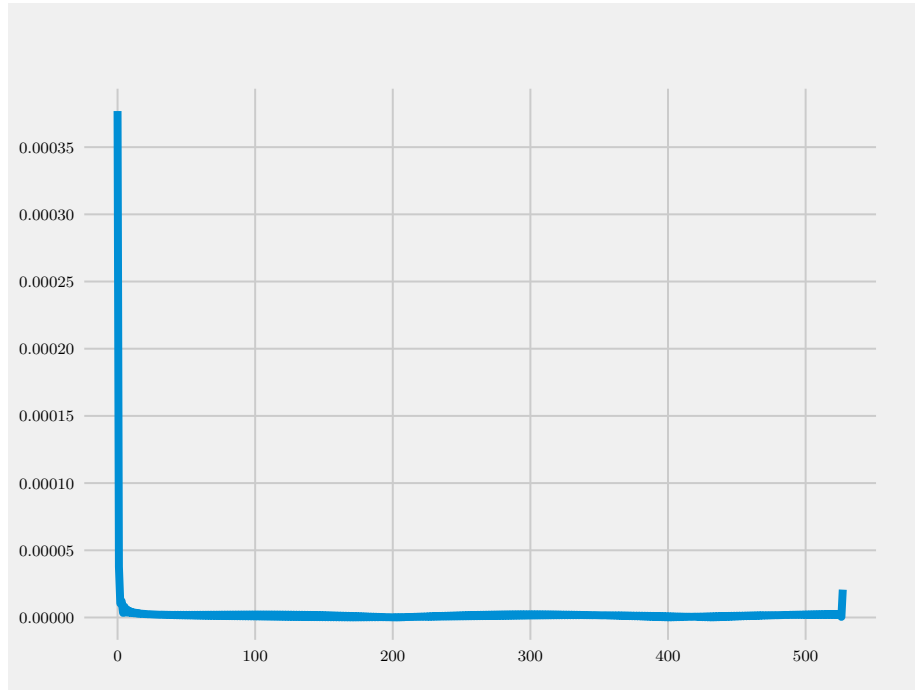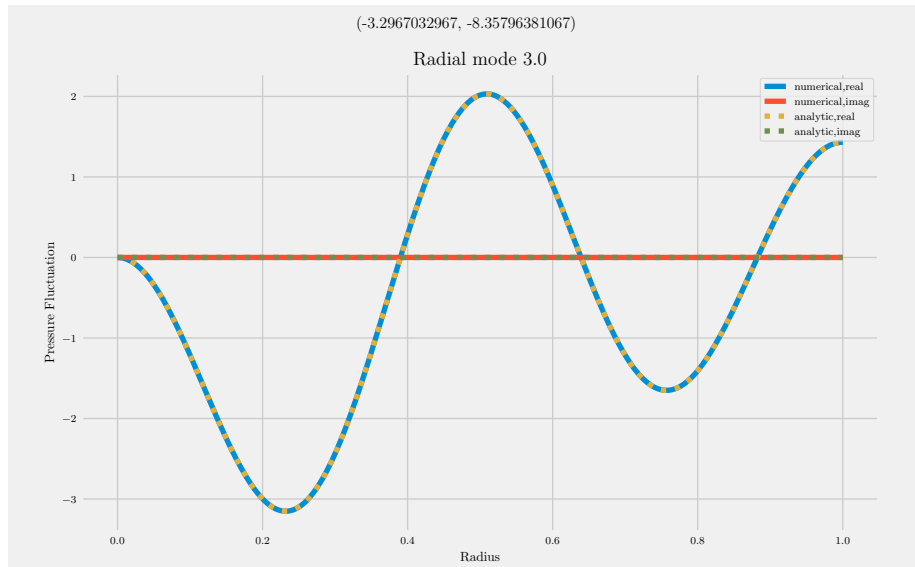Figure 189: Fourth Order, Case number 0, 528points

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Figure 190: Fourth Order, Case number 0, 528points

Figure 191: Fourth Order, Case number 0, 1056points
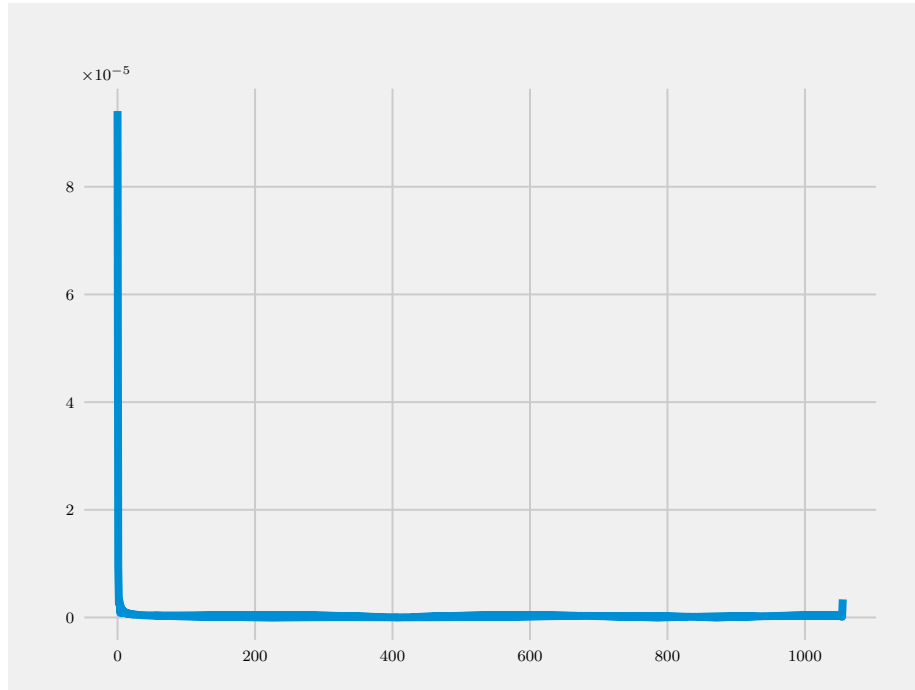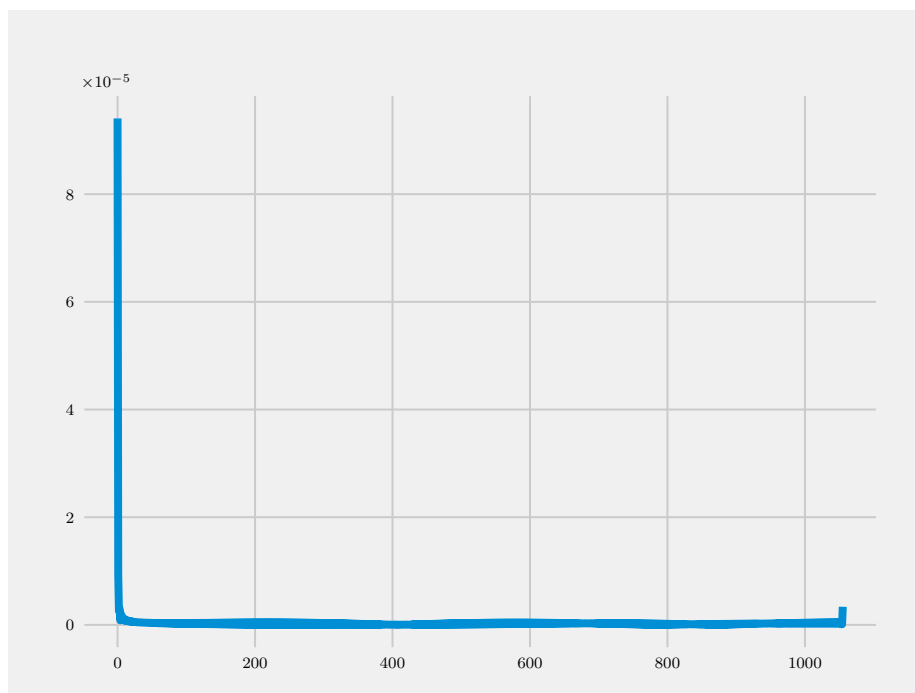
Figure 192: Fourth Order, Case number 0, 1056points
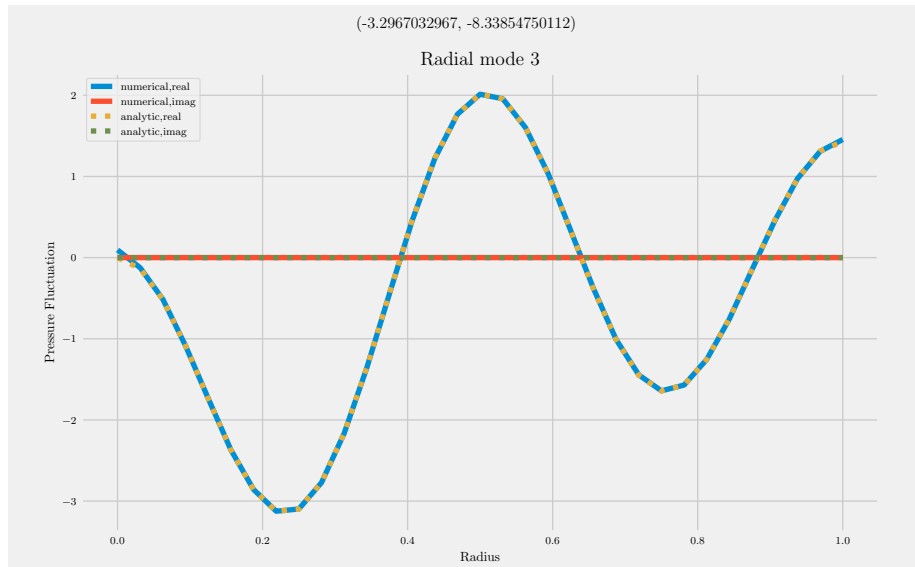
Figure 193: Fourth Order, Case number 1, 33points

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Figure 194: Fourth Order, Case number 1, 33points
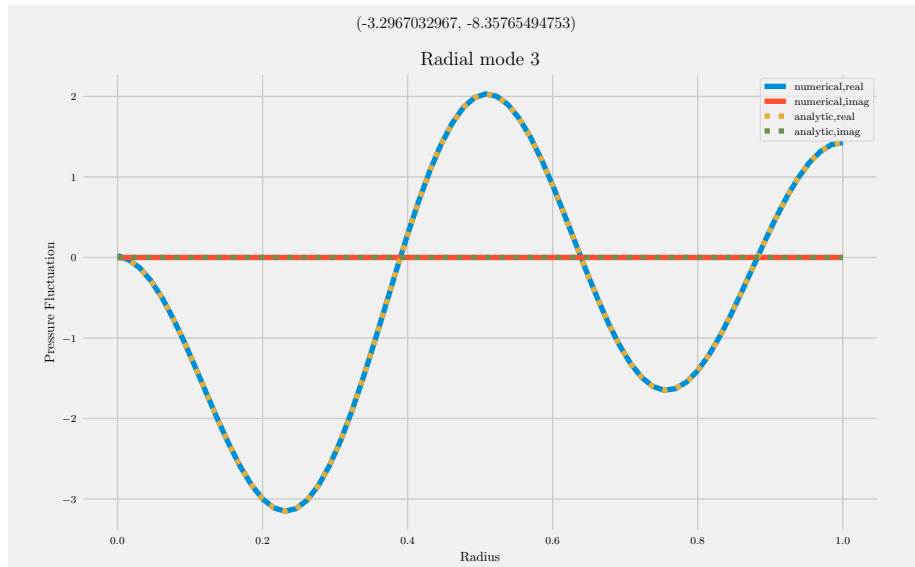
Figure 195: Fourth Order, Case number 1, 66points

Figure 196: Fourth Order, Case number 1, 66points

Figure 197: Fourth Order, Case number 1, 132points

Figure 198: Fourth Order, Case number 1, 132points

Figure 199: Fourth Order, Case number 1, 264points

Figure 200: Fourth Order, Case number 1, 264points

Figure 201: Fourth Order, Case number 1, 528points

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Figure 202: Fourth Order, Case number 1, 528points

Figure 203: Fourth Order, Case number 1, 1056points

Figure 204: Fourth Order, Case number 1, 1056points
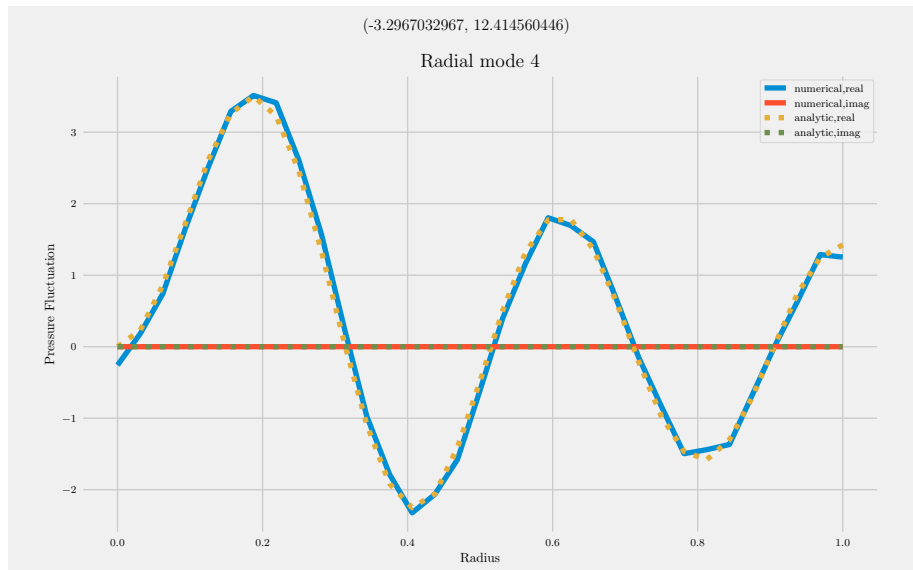
Figure 205: Second Order, Case number 0, 33points

Second Order, Radial Mode 4
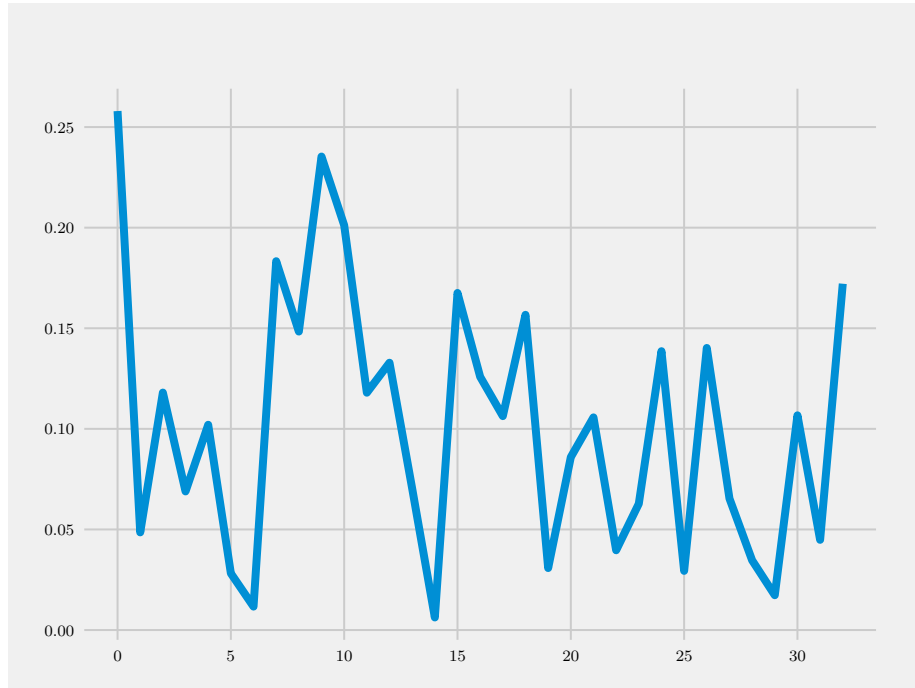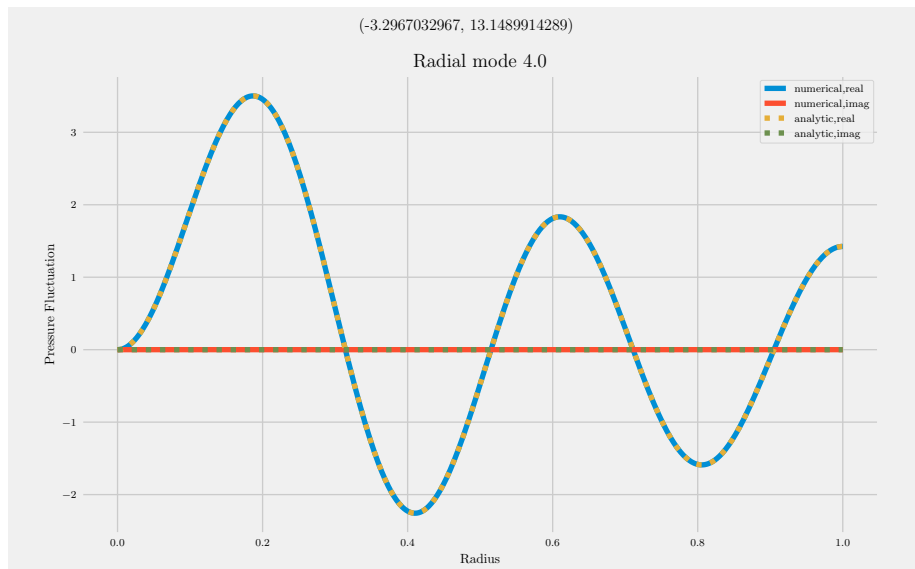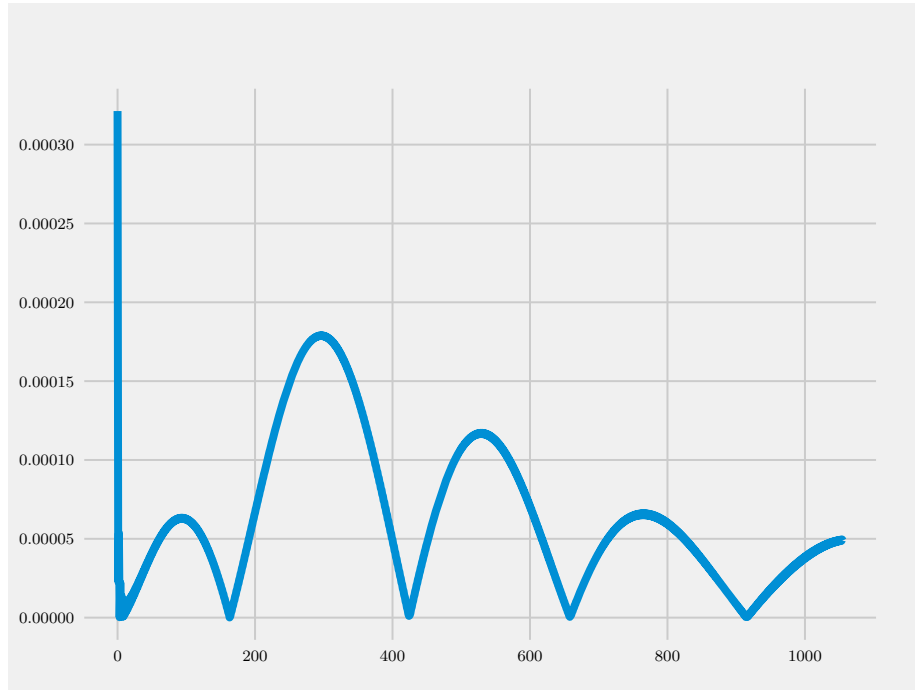
Figure 206: Second Order, Case number 0, 33points
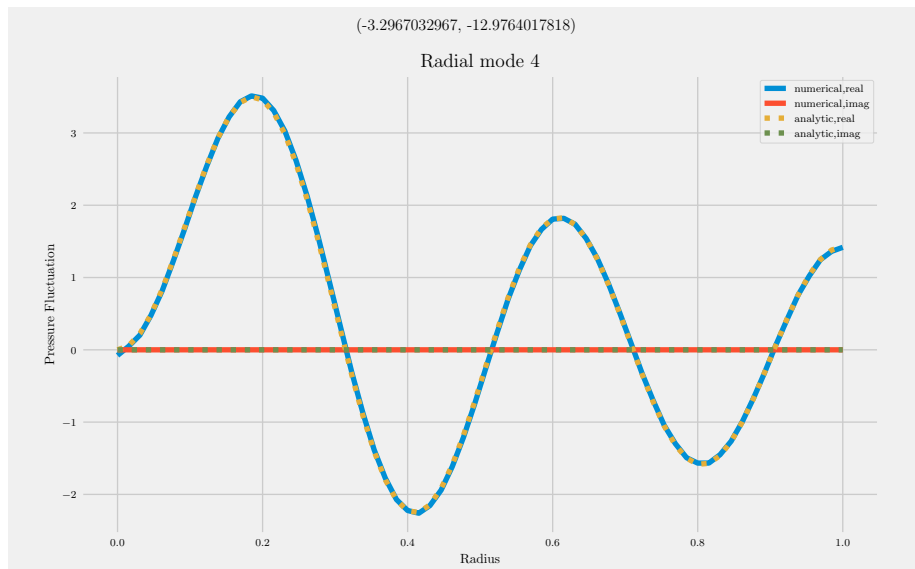
Figure 207: Second Order, Case number 0, 66points

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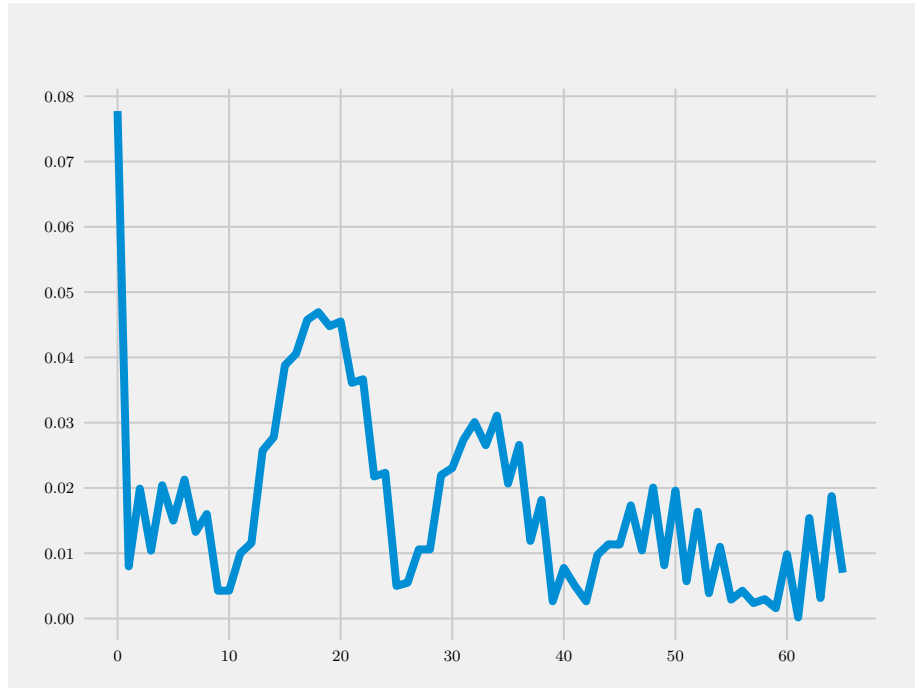
Figure 208: Second Order, Case number 0, 66points
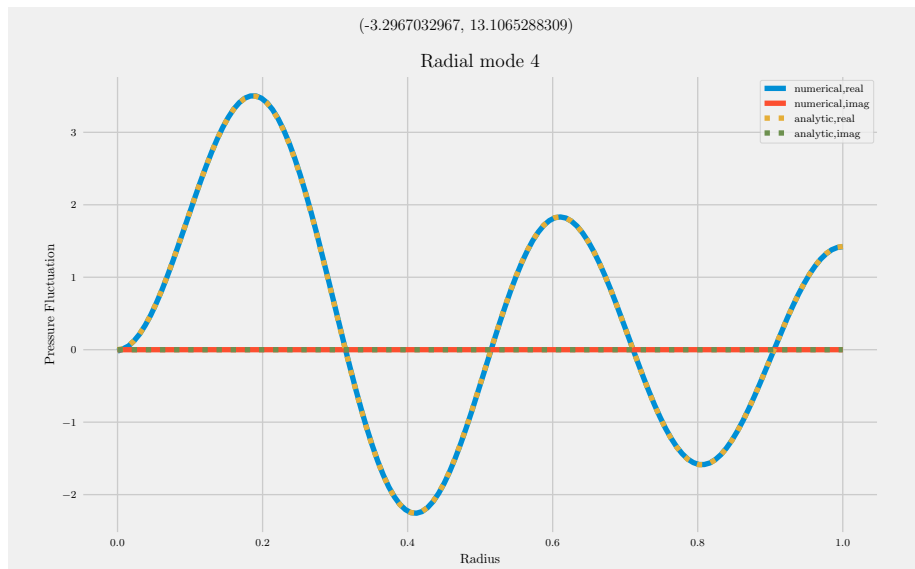
Figure 209: Second Order, Case number 0, 132points

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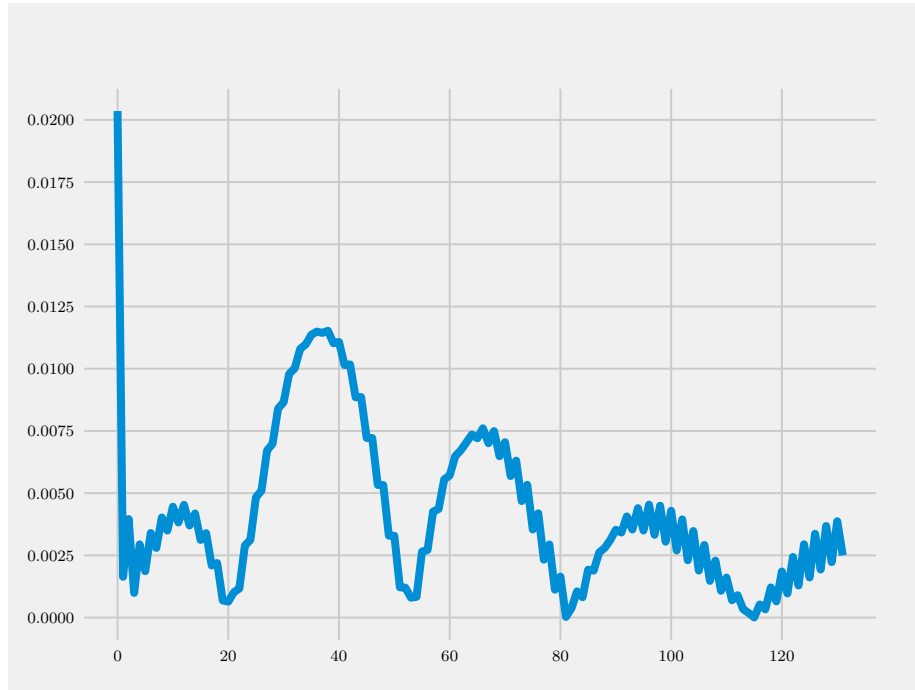
Figure 210: Second Order, Case number 0, 132points
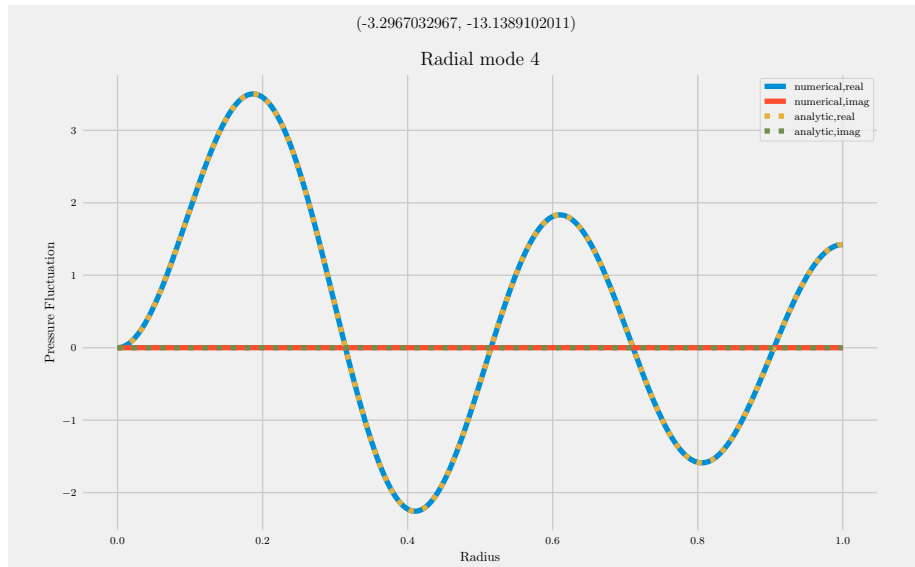
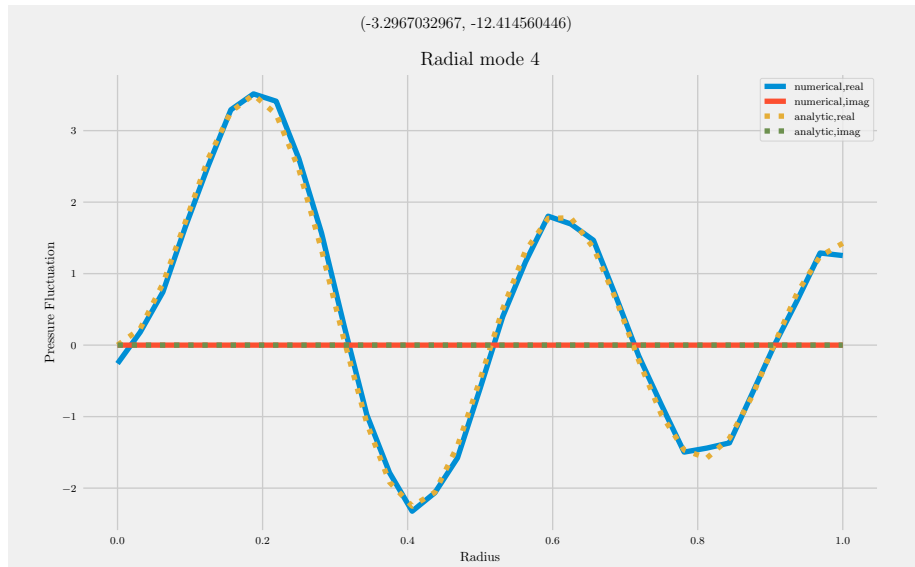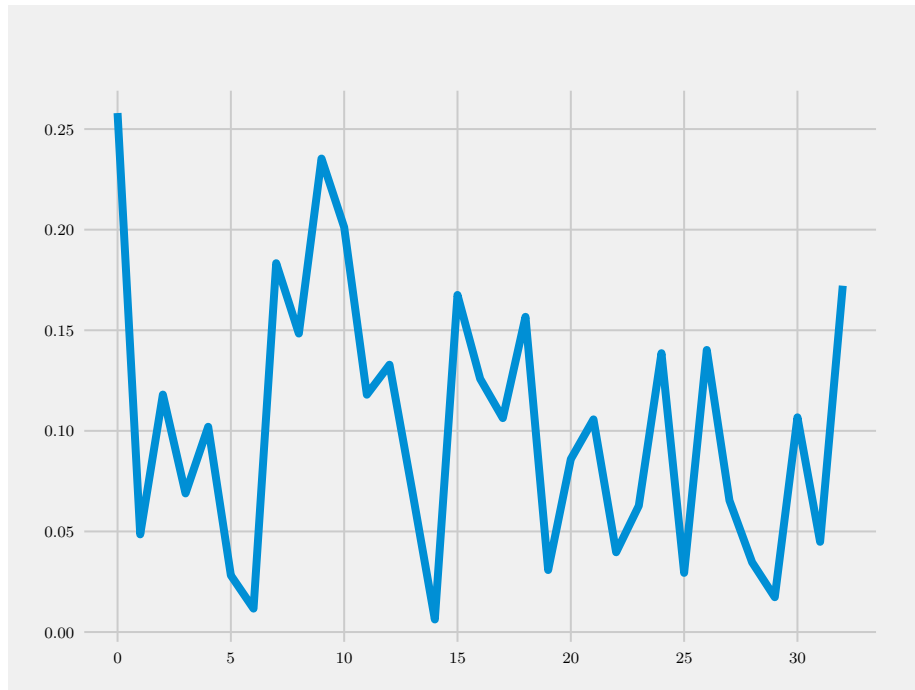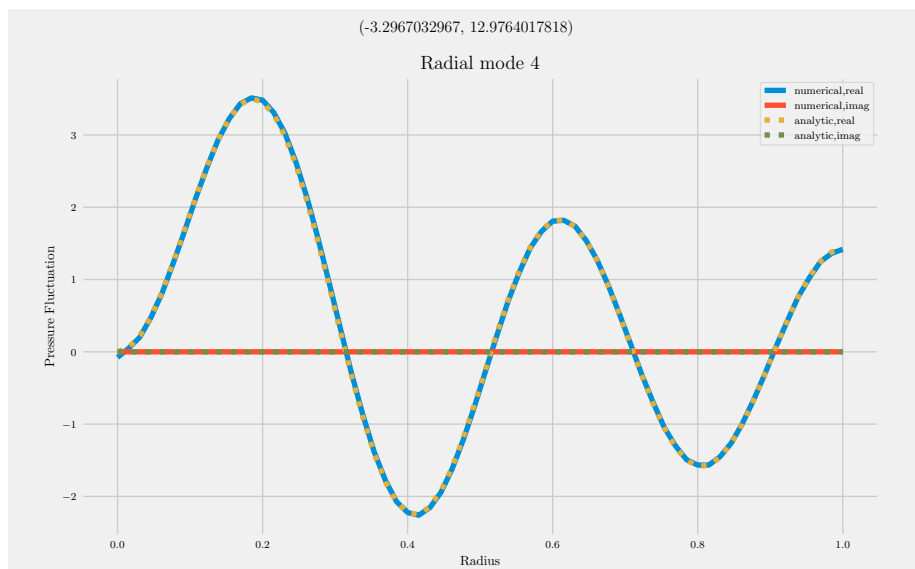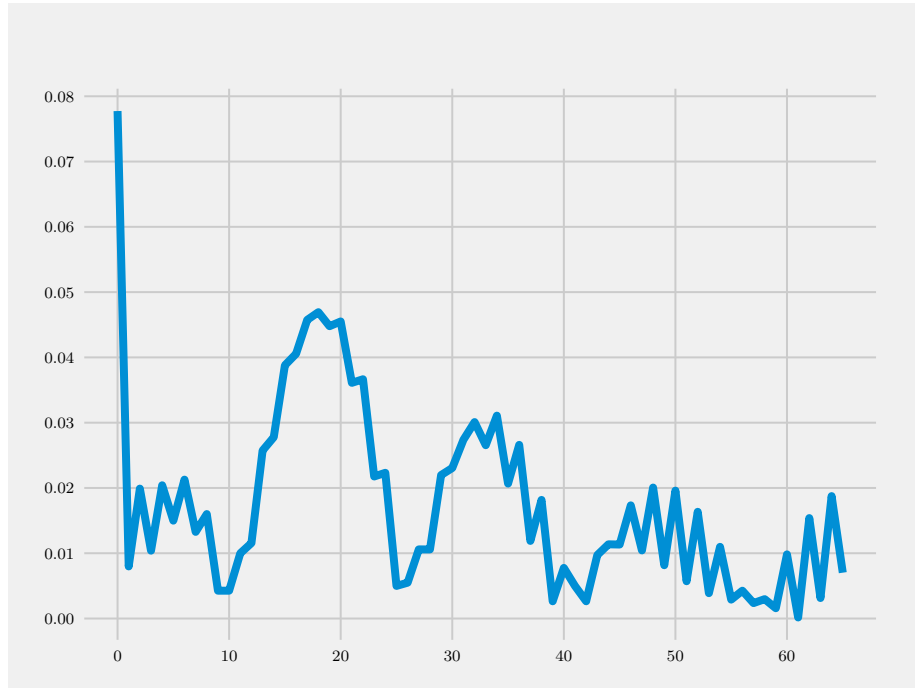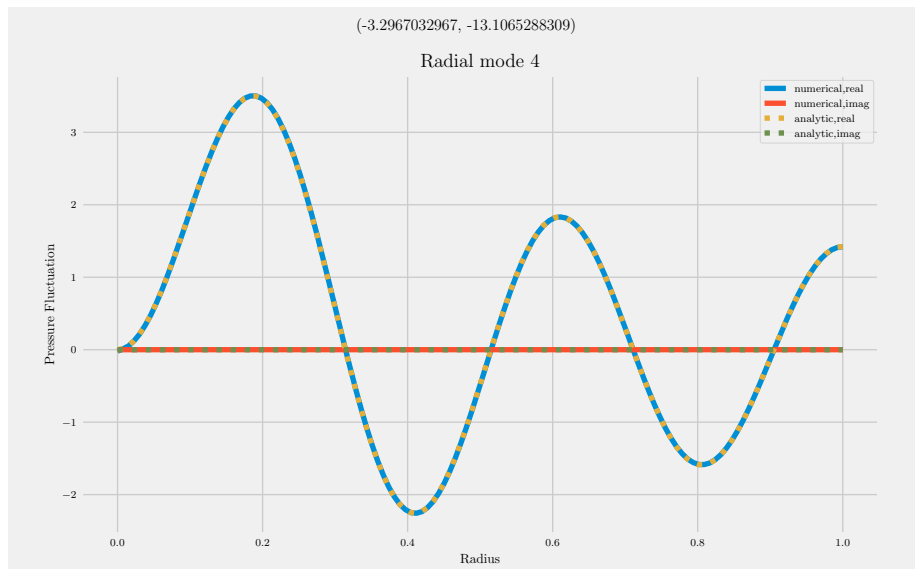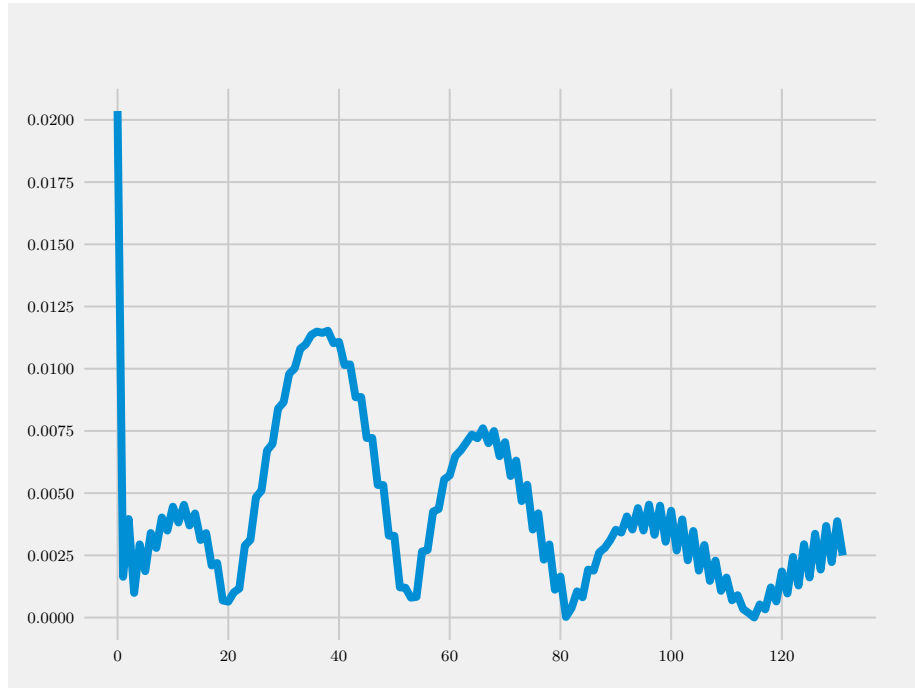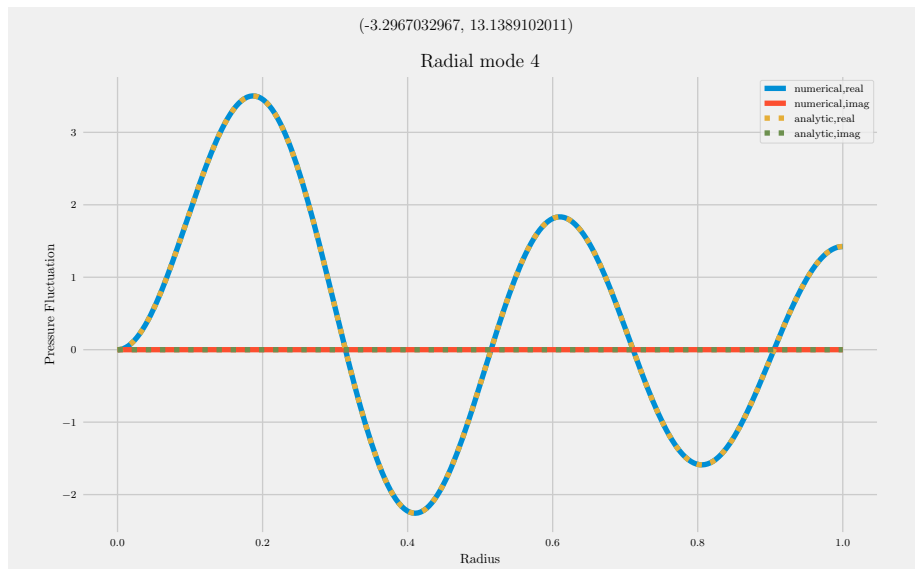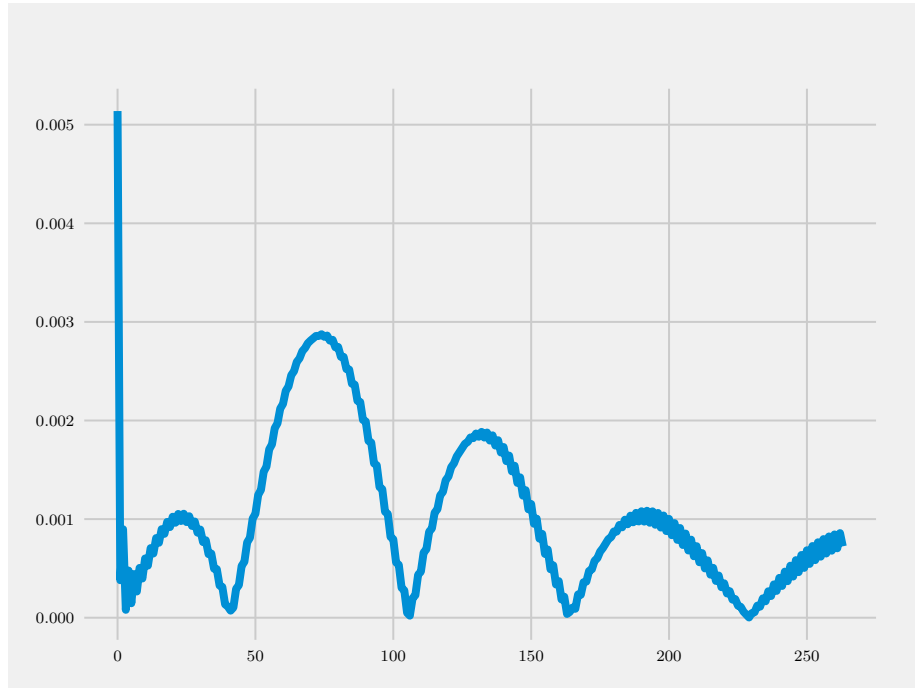Figure 211: Second Order, Case number 0, 264points

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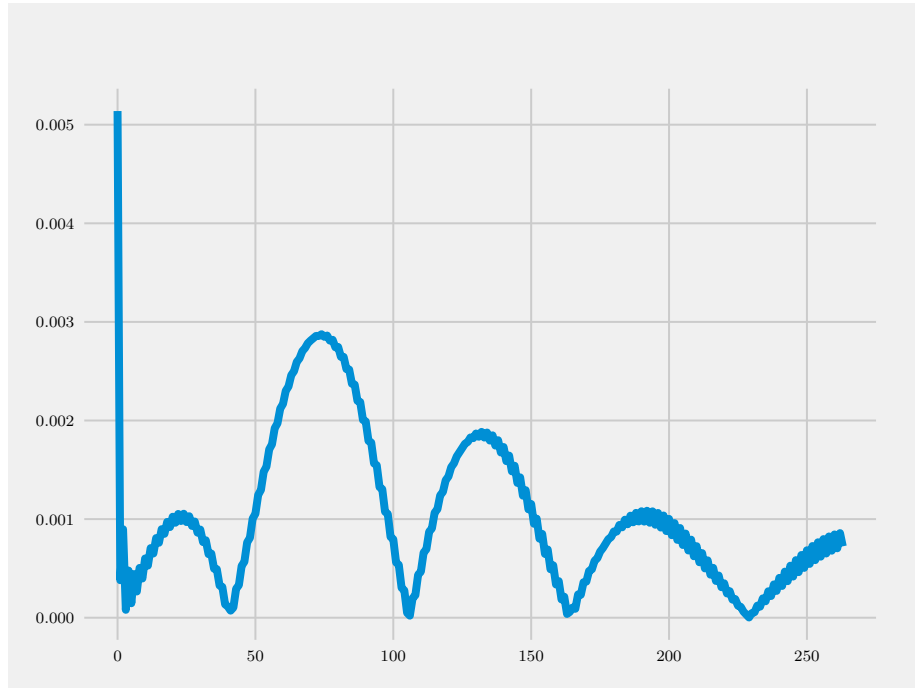
Figure 212: Second Order, Case number 0, 264points
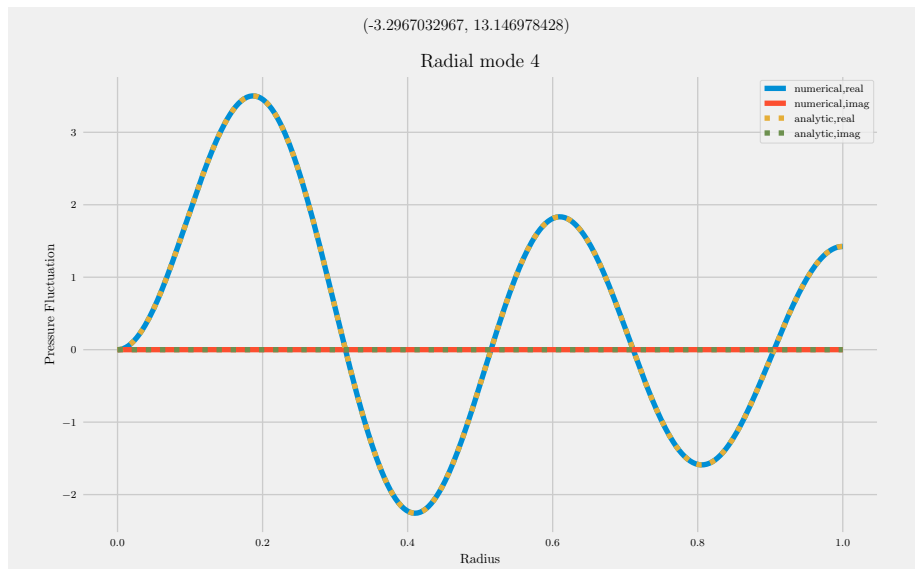
Figure 213: Second Order, Case number 0, 528points
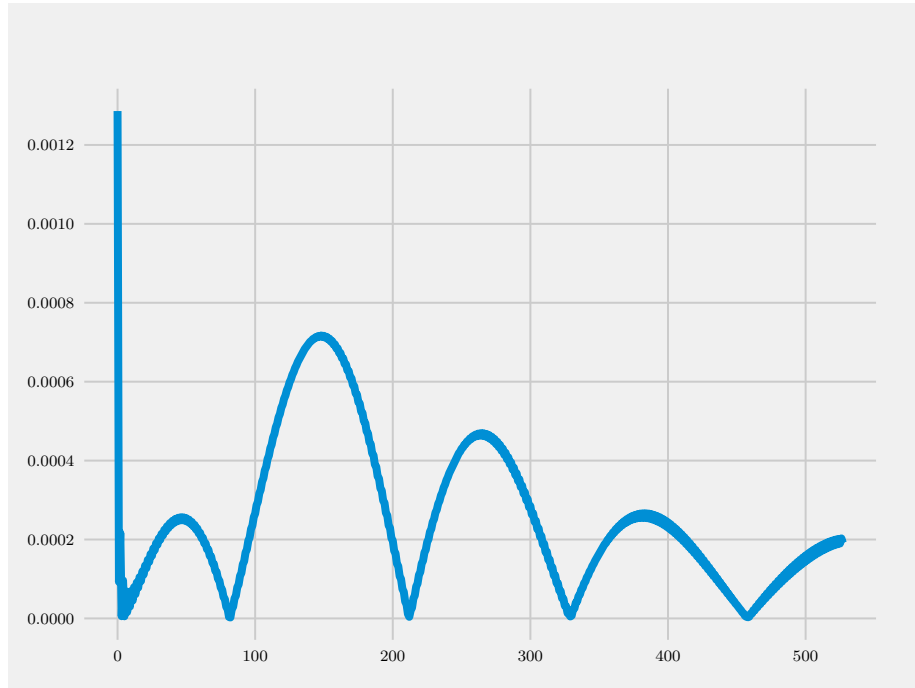
Figure 214: Second Order, Case number 0, 528points

Figure 215: Second Order, Case number 0, 1056points

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Figure 216: Second Order, Case number 0, 1056points

Figure 217: Second Order, Case number 1, 33points

Figure 218: Second Order, Case number 1, 33points

Figure 219: Second Order, Case number 1, 66points

Figure 220: Second Order, Case number 1, 66points

Figure 221: Second Order, Case number 1, 132points

Figure 222: Second Order, Case number 1, 132points

Figure 223: Second Order, Case number 1, 264points
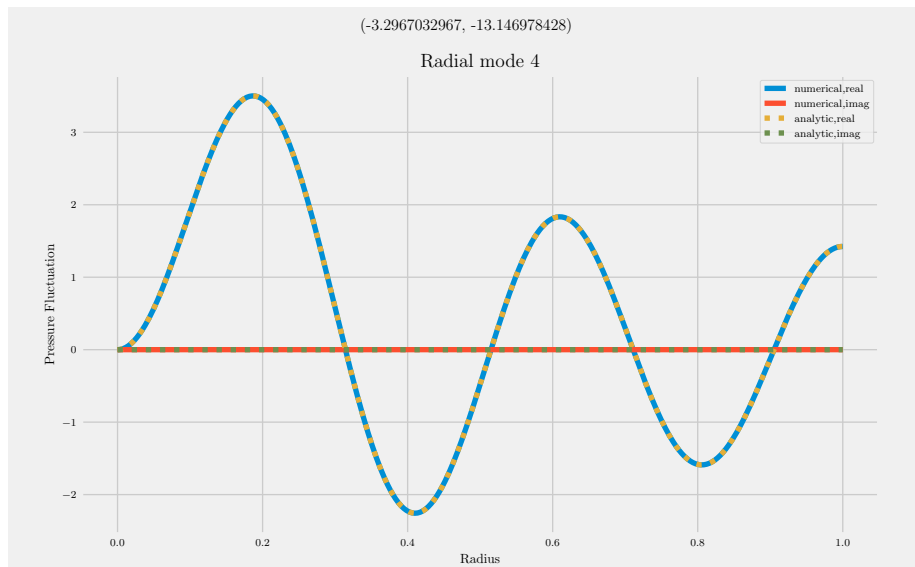
Figure 224: Second Order, Case number 1, 264points
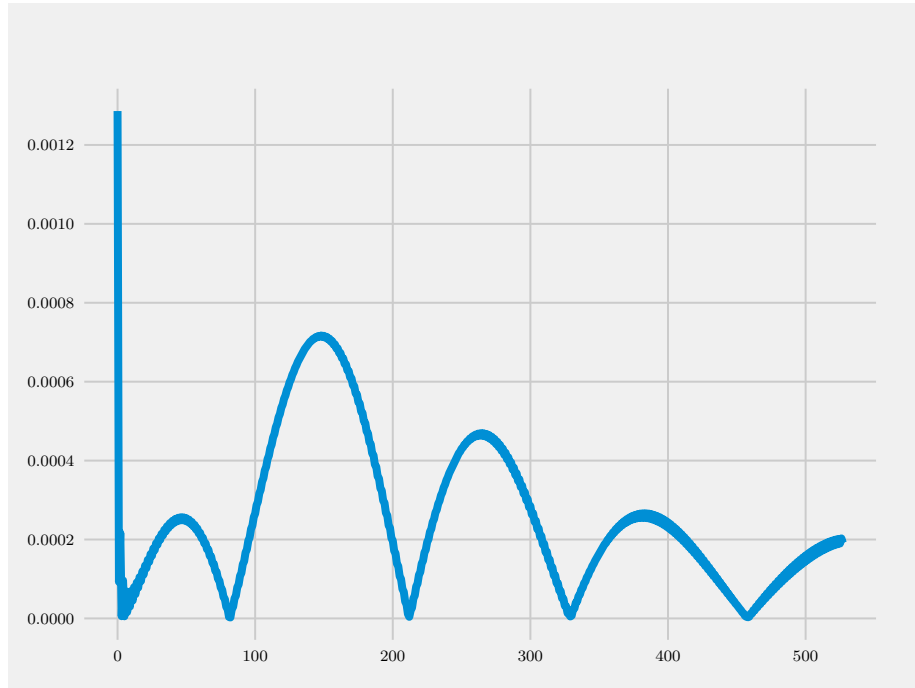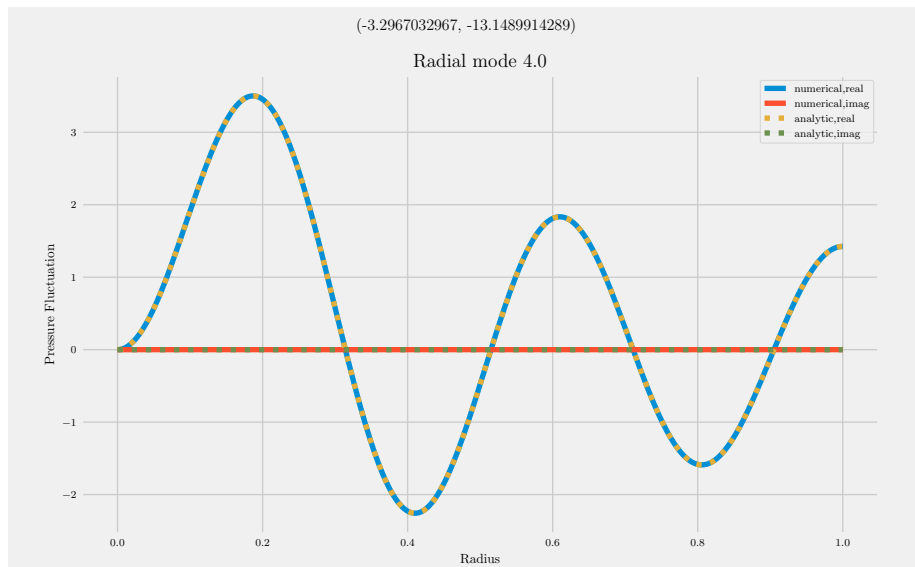

Figure 225: Second Order, Case number 1, 528points

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Figure 226: Second Order, Case number 1, 528points
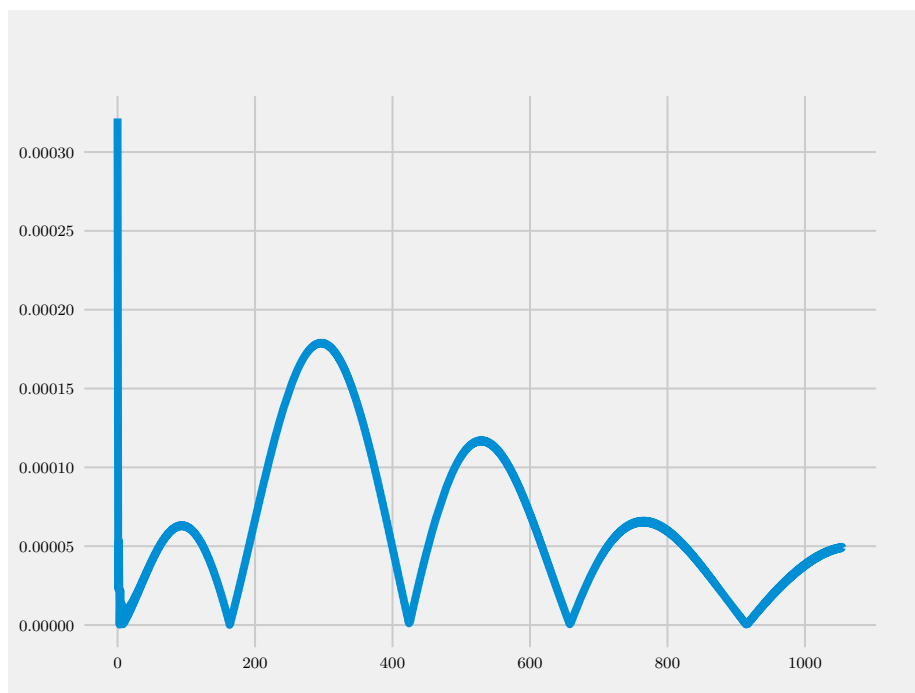

Figure 227: Second Order, Case number 1, 1056points

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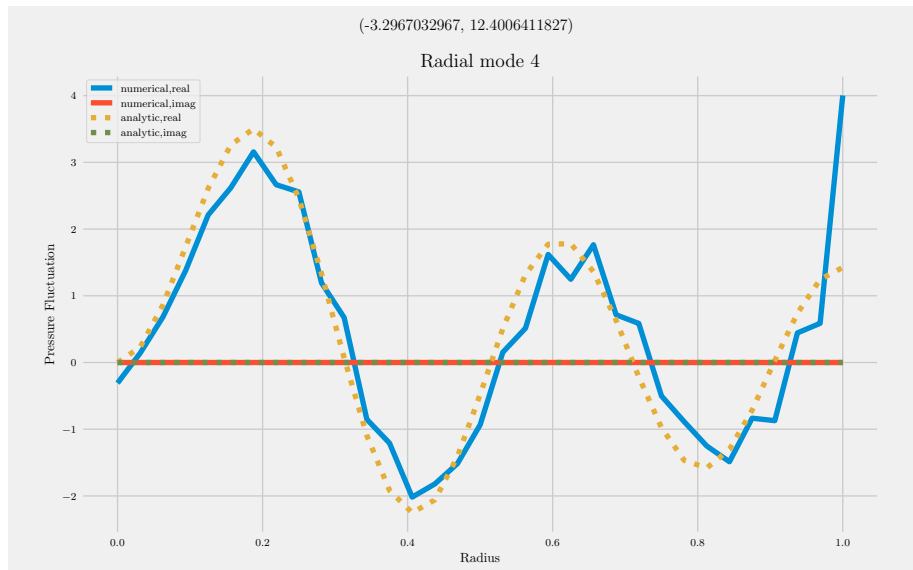Figure 228: Second Order, Case number 1, 1056points

Figure 229: Fourth Order, Case number 0, 33points
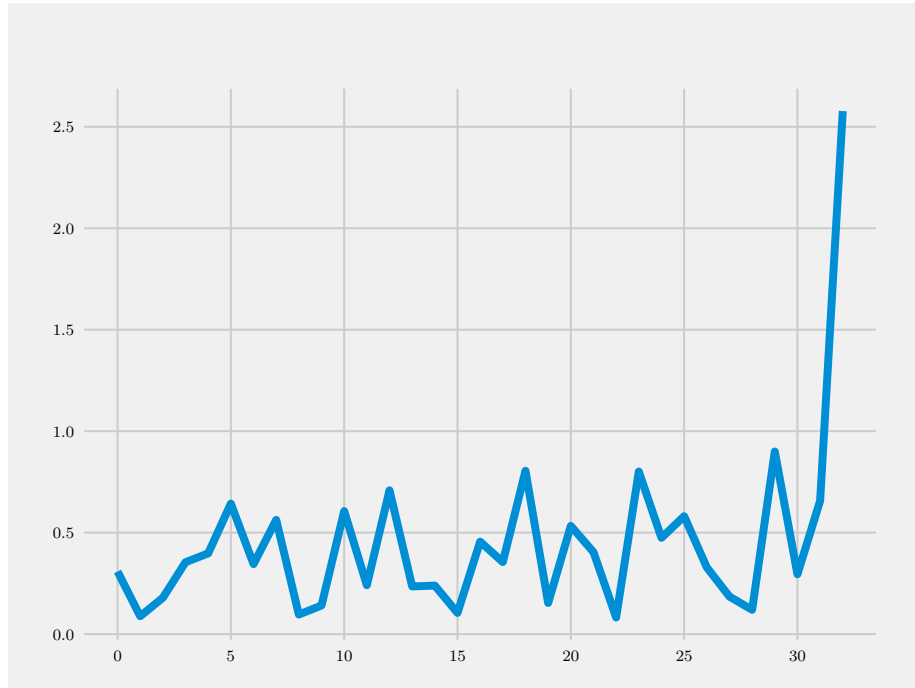
Fourth Order, Radial Mode 4
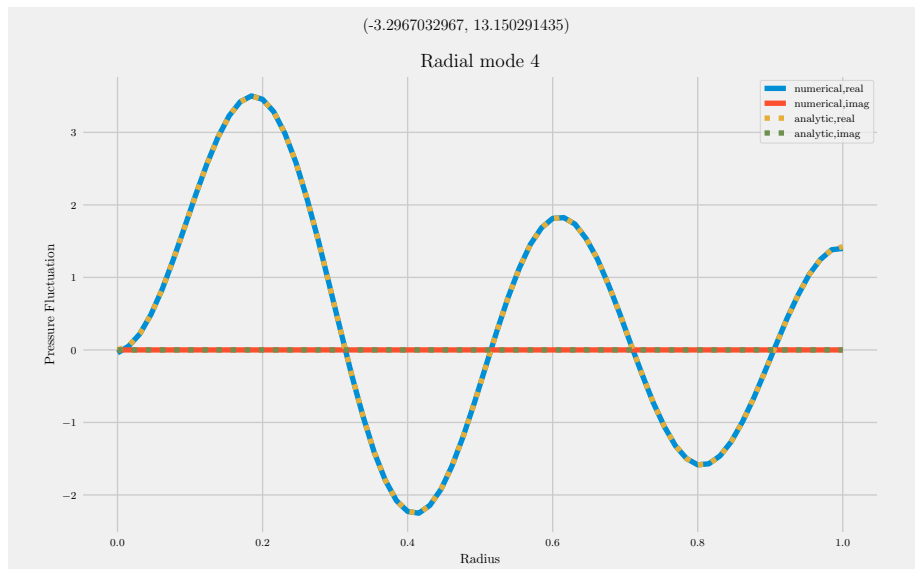
Figure 230: Fourth Order, Case number 0, 33points
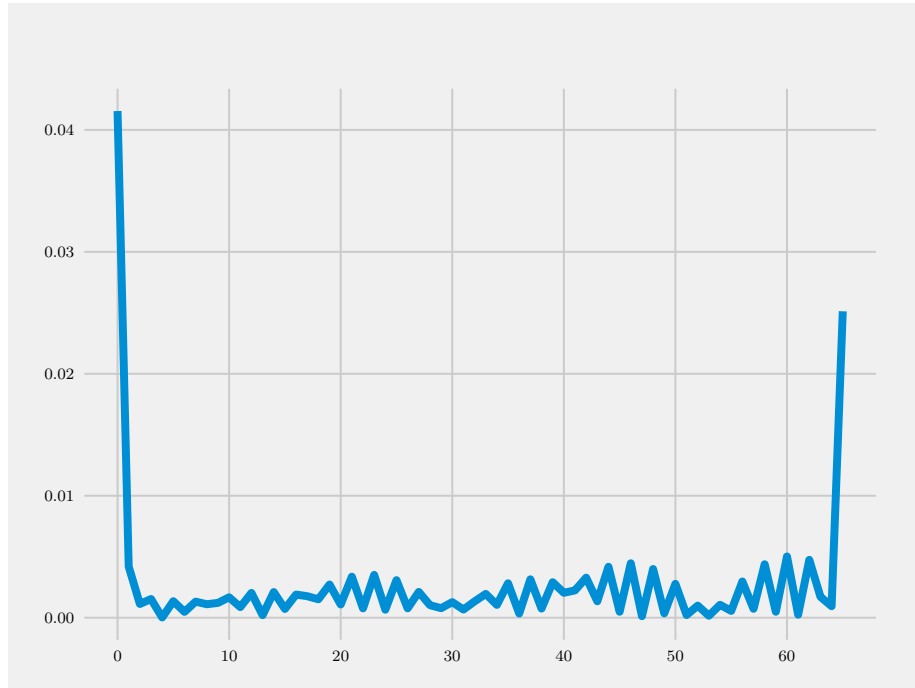

Figure 231: Fourth Order, Case number 0, 66points

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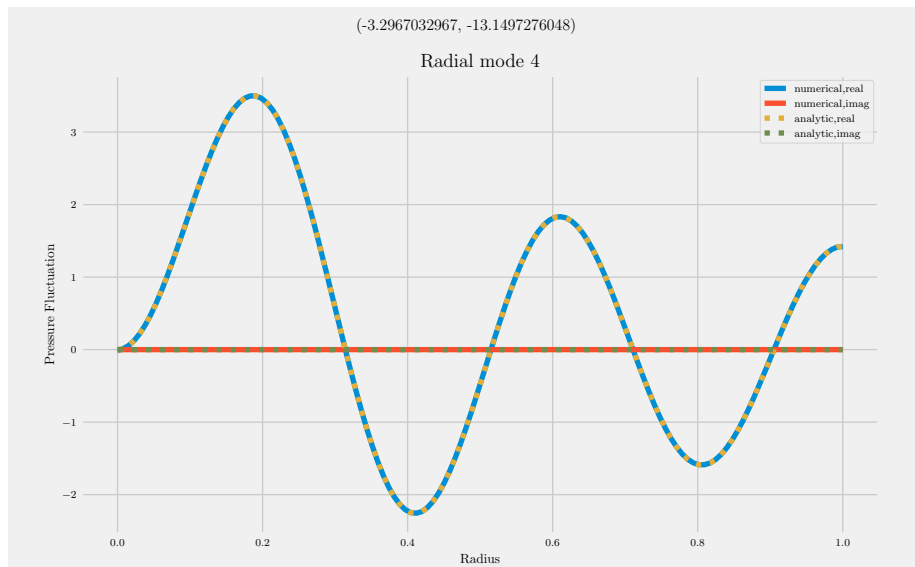
Figure 232: Fourth Order, Case number 0, 66points

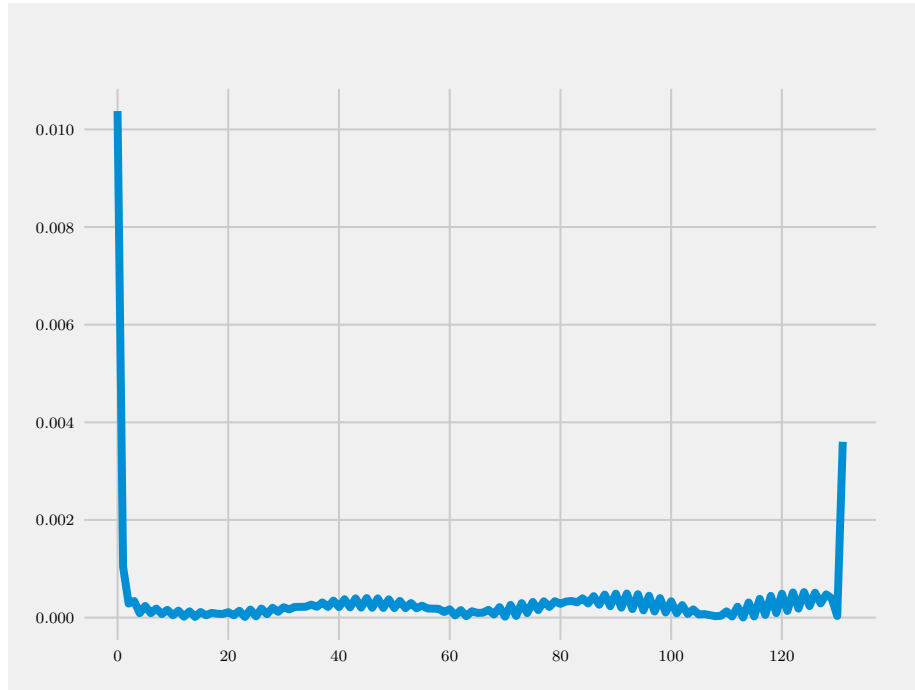Figure 233: Fourth Order, Case number 0, 132points

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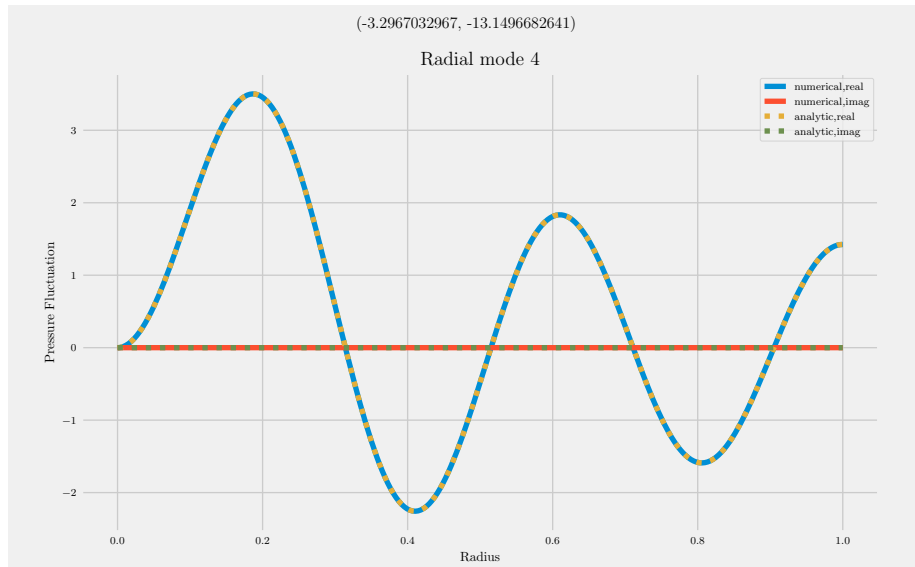Figure 234: Fourth Order, Case number 0, 132points
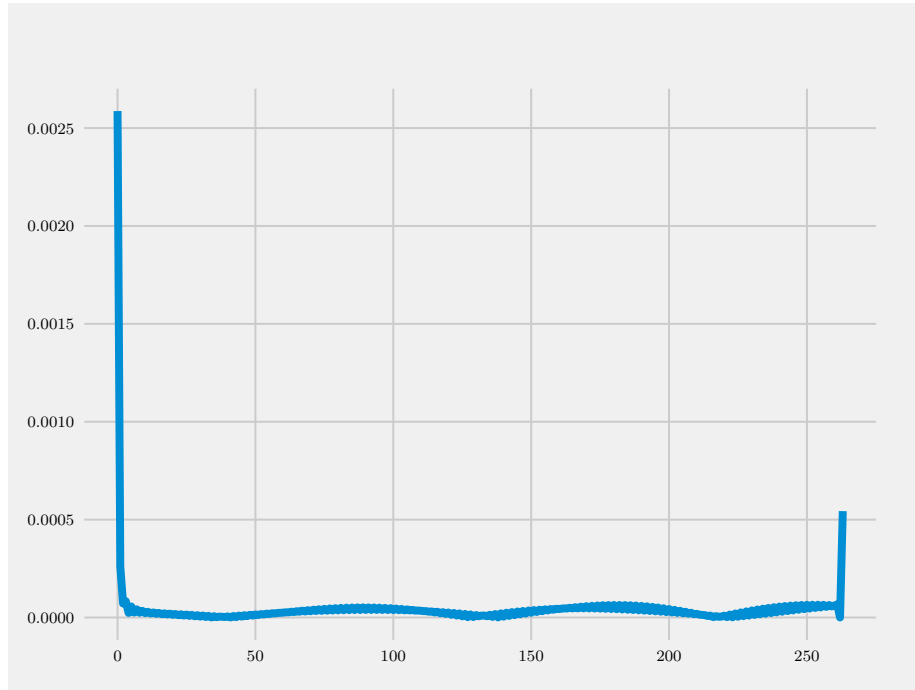

Figure 235: Fourth Order, Case number 0, 264points
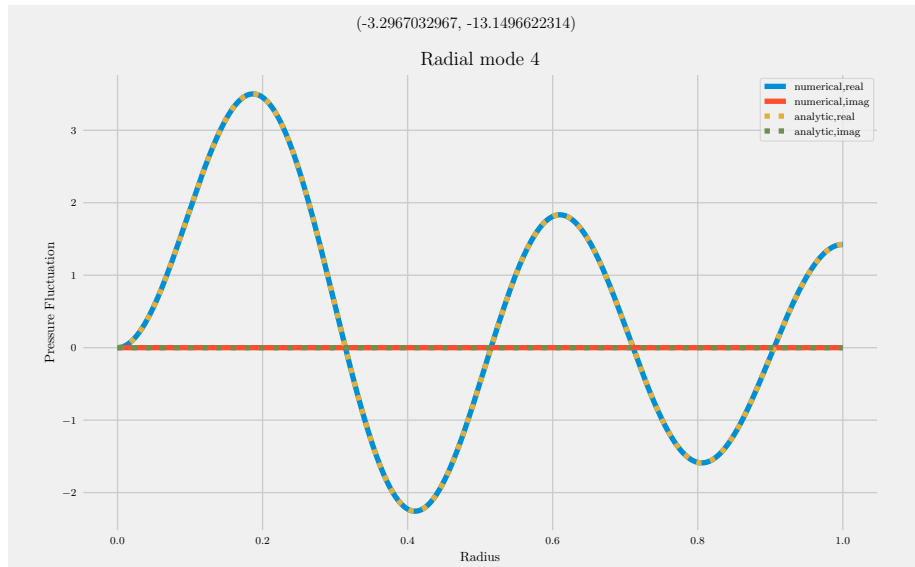
Figure 236: Fourth Order, Case number 0, 264points
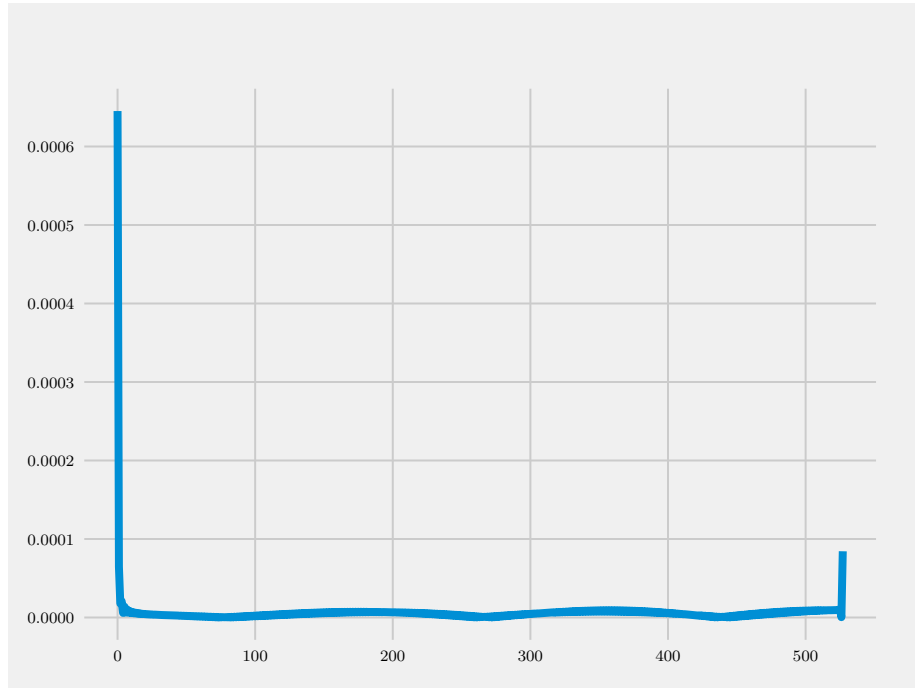

Figure 237: Fourth Order, Case number 0, 528points

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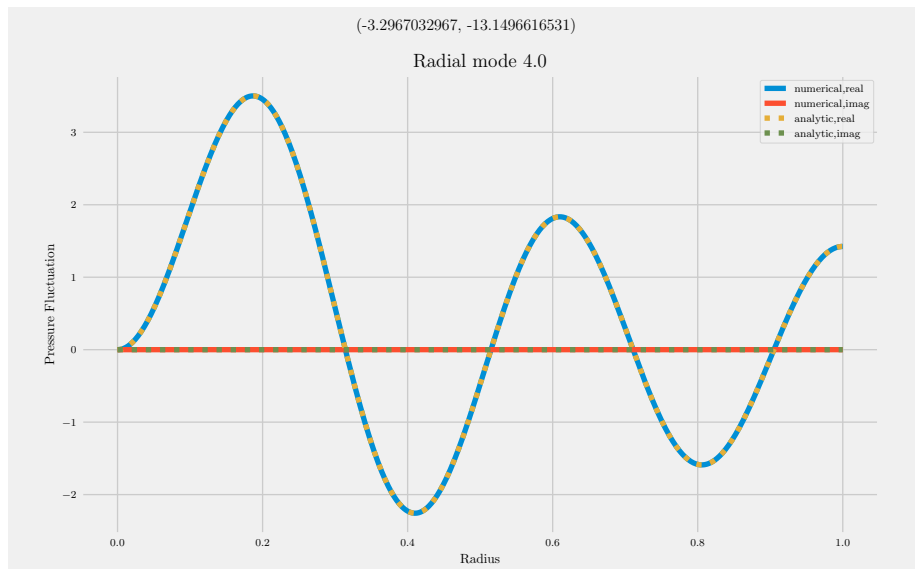Figure 238: Fourth Order, Case number 0, 528points

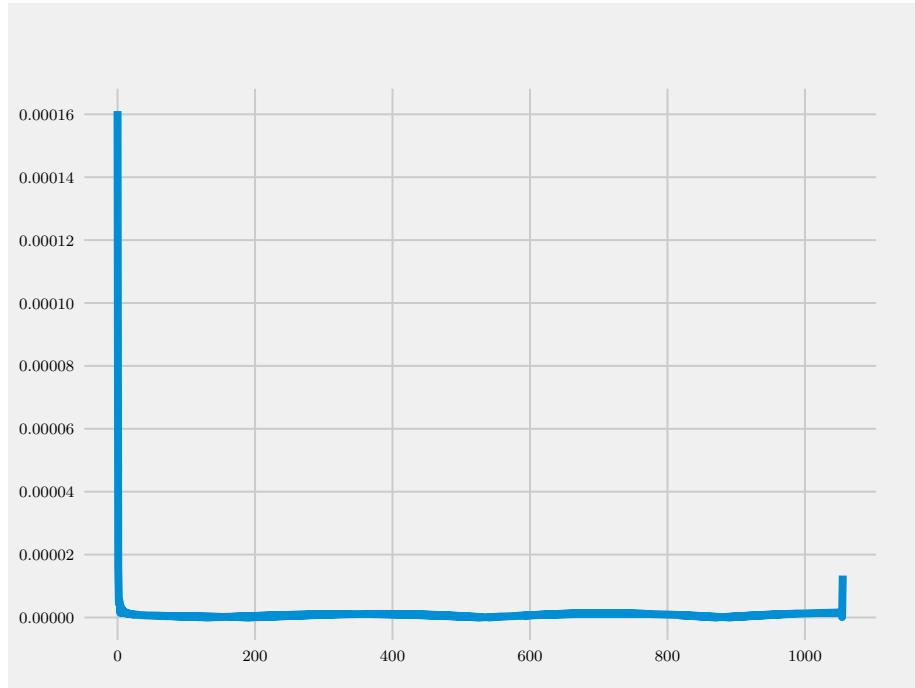Figure 239: Fourth Order, Case number 0, 1056points

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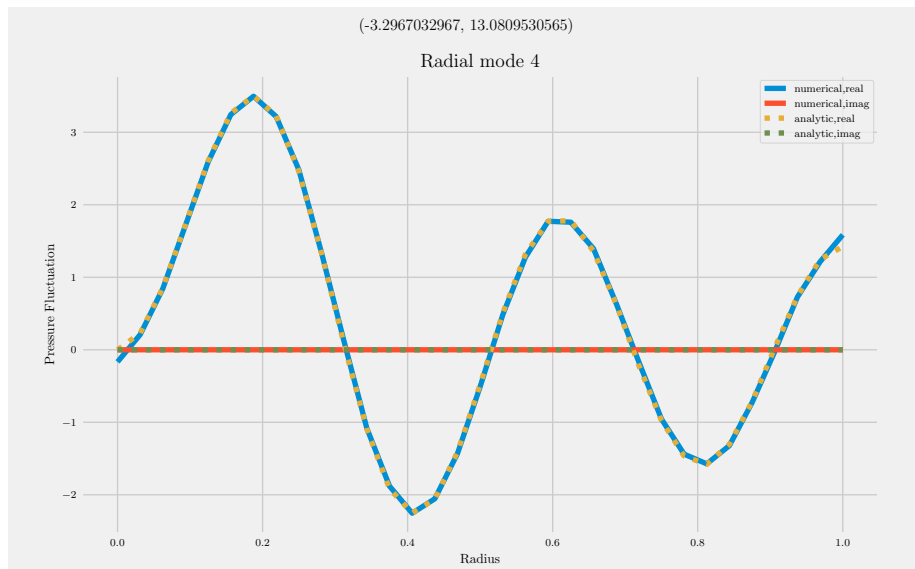Figure 240: Fourth Order, Case number 0, 1056points

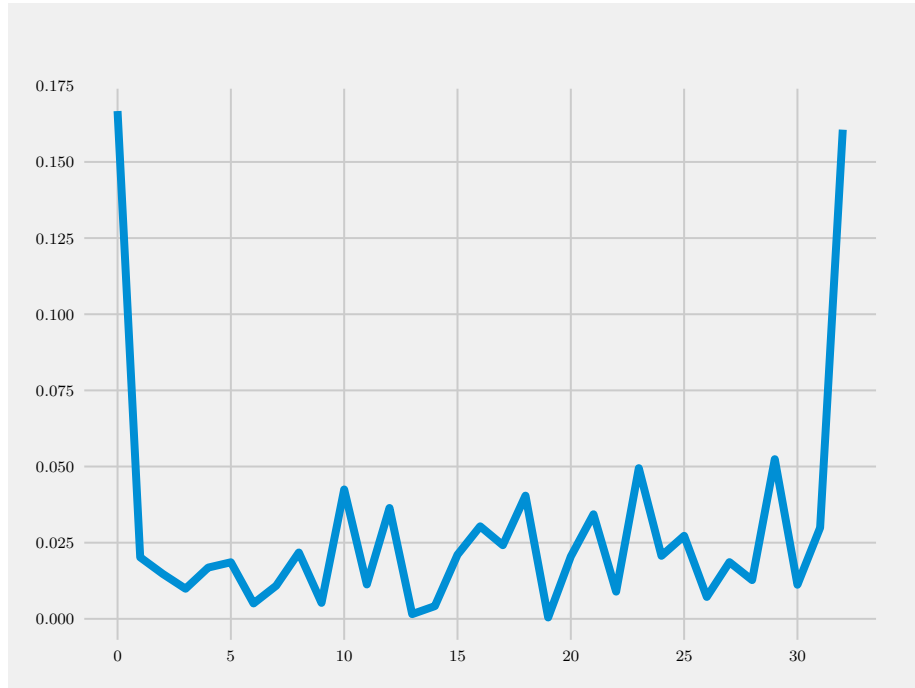Figure 241: Fourth Order, Case number 1, 33points

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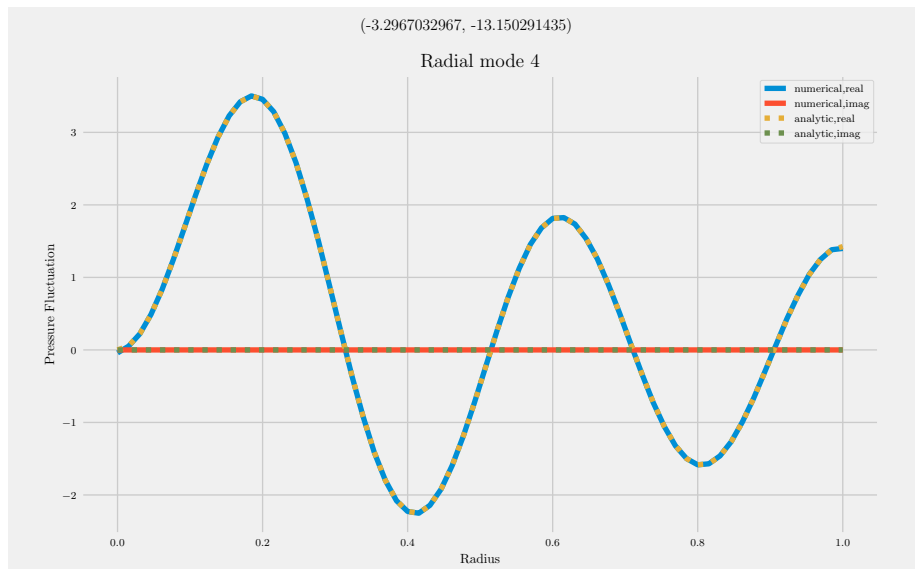Figure 242: Fourth Order, Case number 1, 33points

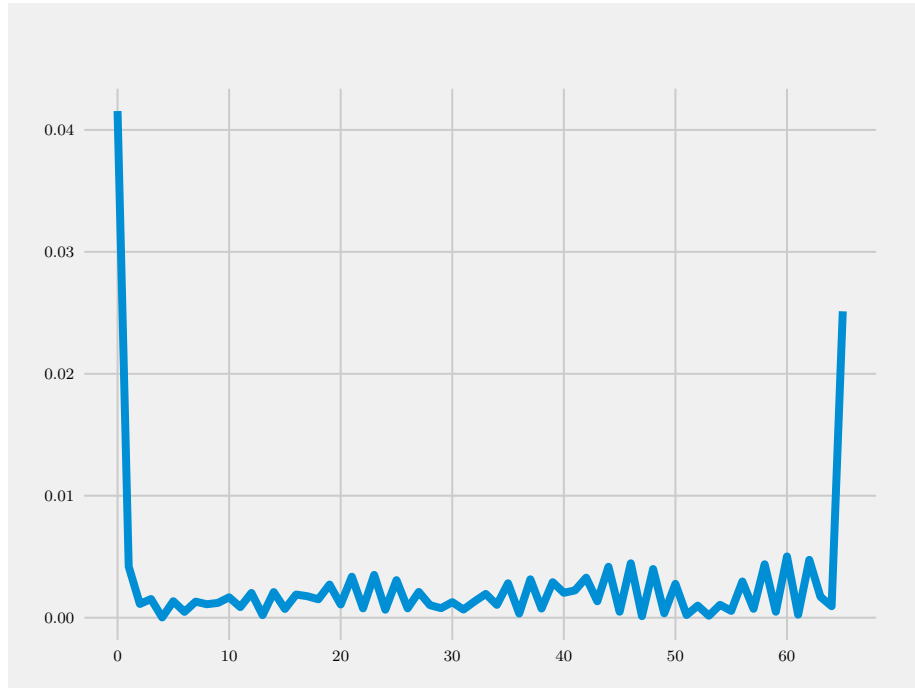Figure 243: Fourth Order, Case number 1, 66points

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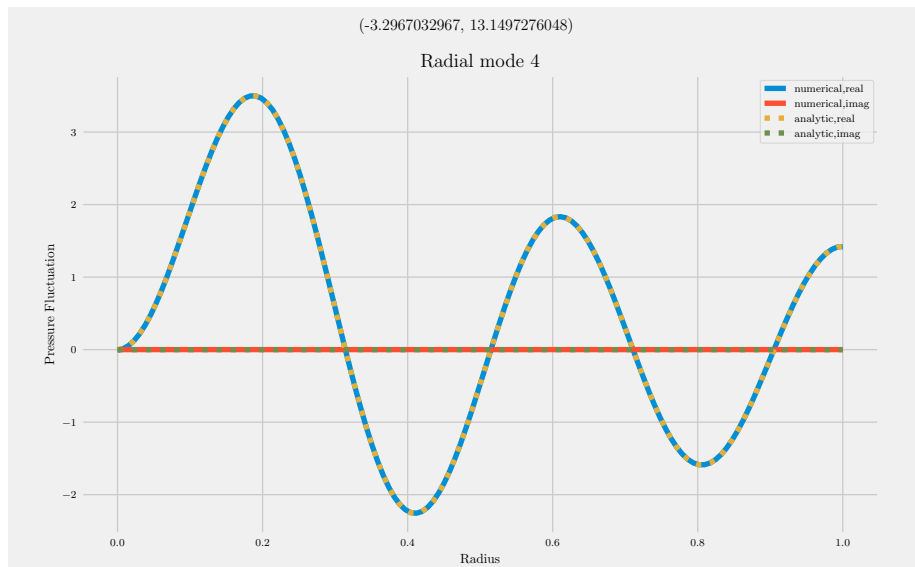Figure 244: Fourth Order, Case number 1, 66points

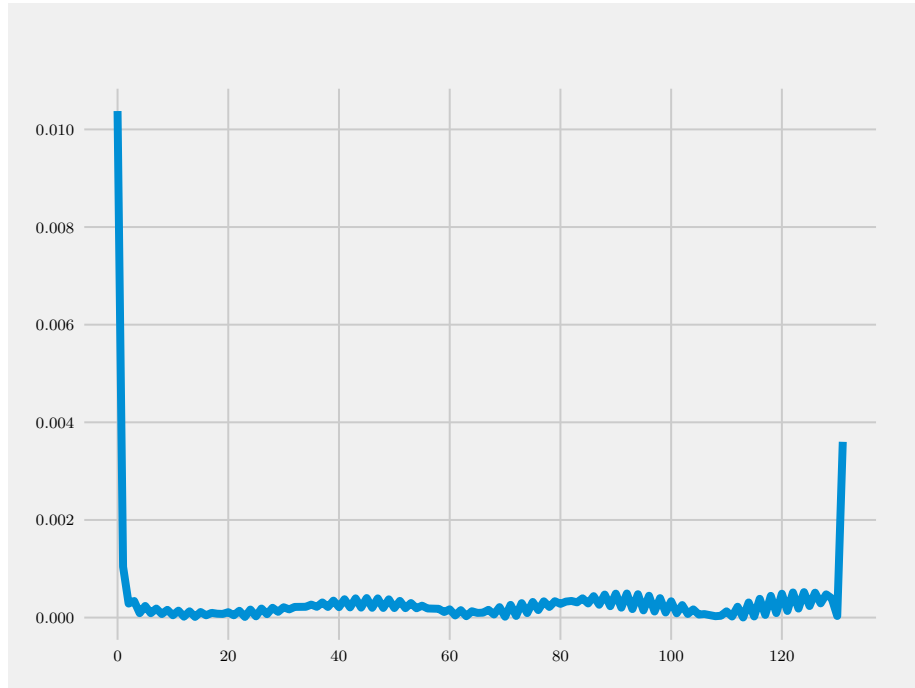Figure 245: Fourth Order, Case number 1, 132points

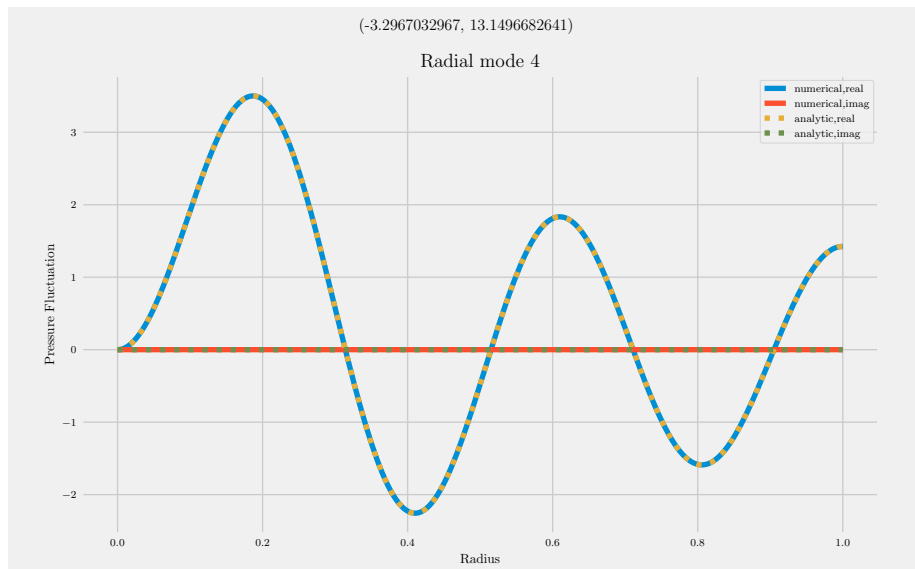Figure 246: Fourth Order, Case number 1, 132points
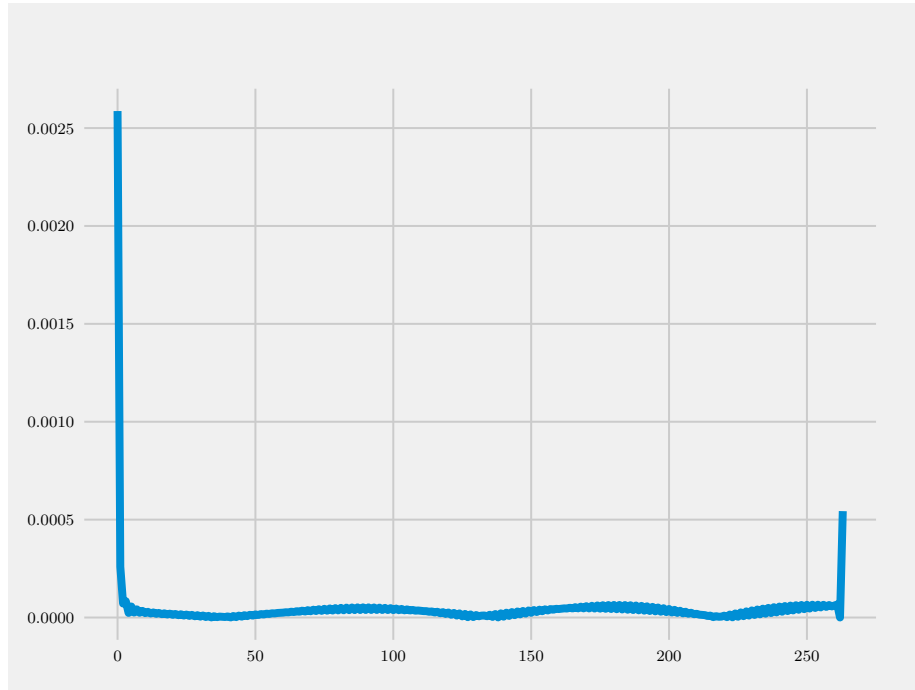

Figure 247: Fourth Order, Case number 1, 264points

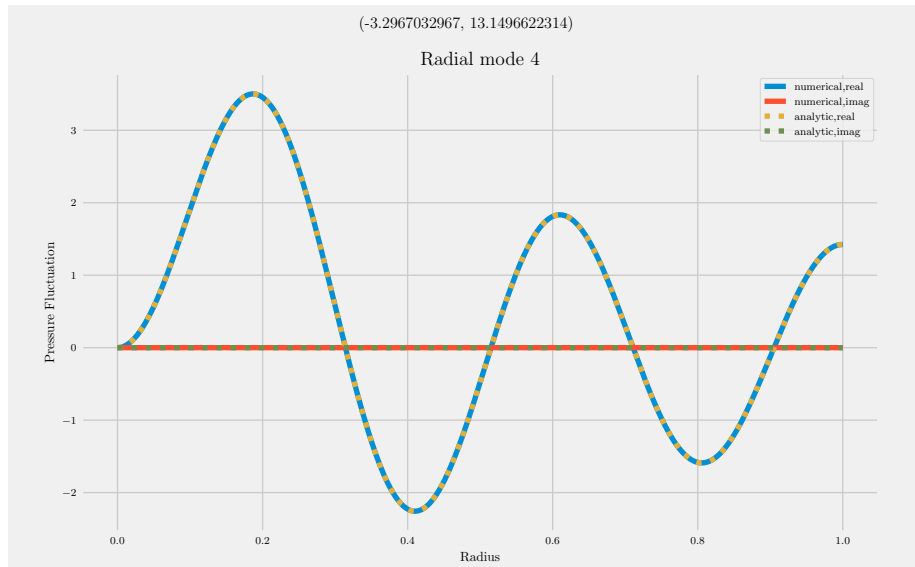Figure 248: Fourth Order, Case number 1, 264points
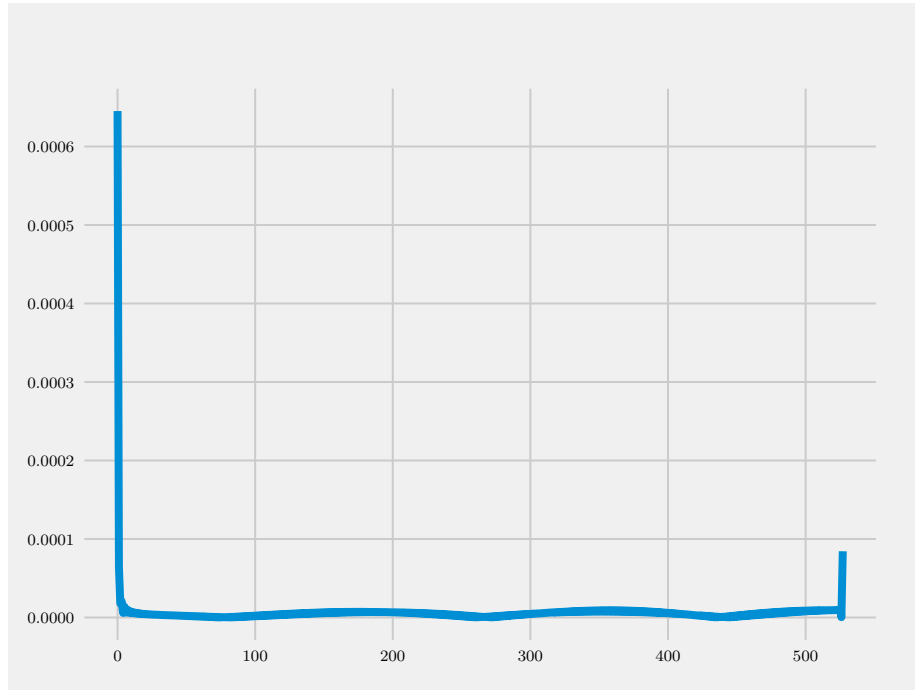

Figure 249: Fourth Order, Case number 1, 528points
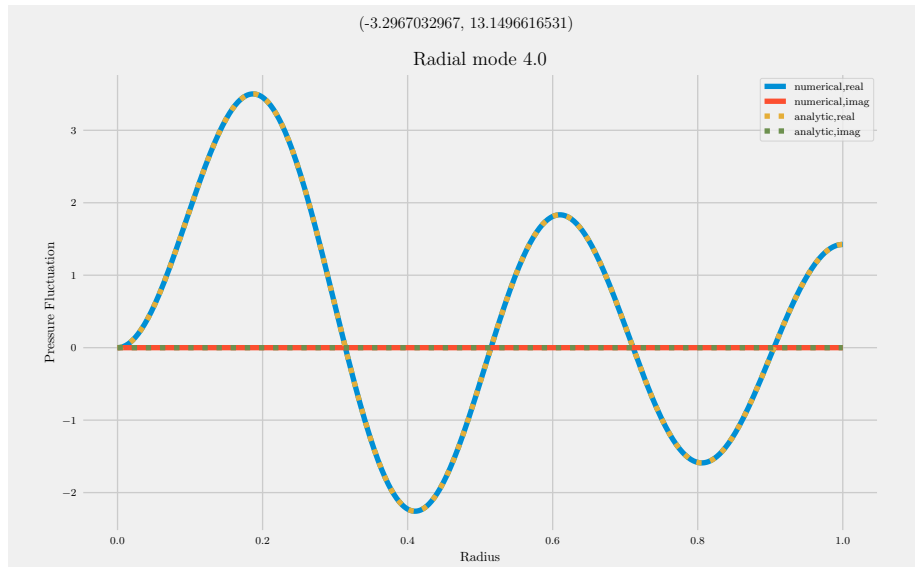
Figure 250: Fourth Order, Case number 1, 528points
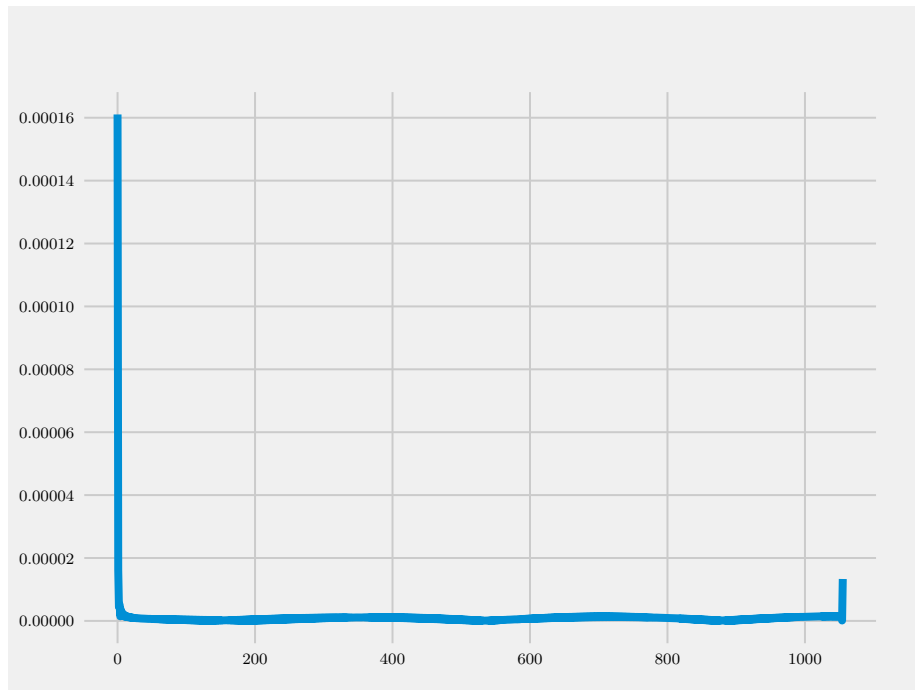

Figure 251: Fourth Order, Case number 1, 1056points

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Figure 252: Fourth Order, Case number 1, 1056points

0.4 Discussion