

Analytical Solution for Duct Mode Propagation in Uniform Flow

Swirl Validation

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This document shows the analytical duct mode solution as well as a numerical comparison.

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0.1 Introduction - Turbomachinery Noise

Turbomachinery noise generation occurs from pressure fluctuations from the series of fans within it's annular duct. While the jet that is produced from this stream of air freely radiates to the observer, the pressure fluctuations produced from the rotor may or may not propagate out of the inlet and exhaust and radiate to the observer. The production of this propagation can be characterized by standing waves referred to as modes, in particular, duct modes because the mode itself is dependent on the geometry of the column of air within the annular duct, as well as the speed of the flow moving through it

0.2 Duct Mode Theory

The pressure field within a duct is governed by the convective wave equation, a second order ODE as a function of radius.

The solution of the convective wave equation are eigenvalues and eigenvectors which may or may not correspond to acoustic disturbances fall into two groups. One group corresponding to the acoustics propagation and the other group corresponding to the convection speed of the flow. Both are modes that are a result from the pressure distribution from within the cylindrical domain.

0.2.1 Analytic Solution: Axial wavenumbers and Pressure Modes

Modes can be categorized based on the sign of the axial wavenumber and if it is complex in value. For example, for the uniform axial flow case, propagating modes are defined by axial wavenumbers, k_x , that have a real-part only, yielding the assumed fluctuation to resemble Euler's Formula ($e^{ik_x x}$). On the other hand, if the k_x is complex, then the mode will resemble an exponentially decaying function since the imaginary number cancels, leaving a minus sign in front of the axial wavenumber. These two distinctions are referred to as "cut-on" and "cut-off" in the field of ducted sound propagation. Furthermore, the sign of the imaginary part will change the direction of the mode's decay. If k_x is positive, the decay rate occurs in the negative direction. Conversely, if k_x is negative, the decay occurs in the positive direction. The axial wavenumber for uniform axial flow in a hollow duct is,

$$k_x = \frac{-M_x k \pm \sqrt{k^2 - (1 - M_x^2)J_{m,n}'^2}}{(1 - M_x^2)}. \quad (1)$$

where M_x is the axial Mach number, k is the temporal (referred to as reduced) frequency, and $J_{m,n}'$ is the derivative of the Bessel function of the first kind. The \pm accounts for both upstream and downstream modes.

The condition for propagation is such that the axial wavenumber is larger than a "cut-off" value

$$k_{x,real} = \frac{\pm M_x k}{(M_x^2 - 1)}. \quad (2)$$

Every term that is being raised to the one half i.e. square rooted must be larger than zero to keep axial wavenumber from being imaginary. The mode will propagate or decay based on this condition.

0.2.2 Methods

Bessel's Function,

$$R(r) = AJ_0(k_r r) + BY_0(k_r r) \quad (3)$$

where J_0 and Y_0 are Bessel's functions of the first and second kind, and A and B are arbitrary constants. Both functions reduce as $k_r r$ gets large. The Bessel function of the second kind is unbounded as $k_r r$ goes to zero.

For a hard-walled duct, the radial velocity component is zero at the boundaries r_{min} and r_{max} .

$$\left. \frac{dR(r)}{dr} \right|_{r=r_{max}} = \left. \frac{dR(r)}{dr} \right|_{r=r_{min}} = 0 \quad (4)$$

In the case of a hollow duct, there is no minimum radius, therefore the wall boundary condition only applies at r_{max} .

Since the solution must be finite as $k_r r$ approaches zero, it can be observed that Y_0 approaches infinity. Since this would yield in a trivial solution, the coefficient $B = 0$, which reduces 3 to,

$$R(r) = AJ_0(k_r r) \quad (5)$$

Taking the derivative with respect to r yields,

$$\left. \frac{dR(r)}{dr} \right|_r = AJ'_0(k_r r) = 0 \quad (6)$$

The boundary condition requires the Bessel function be zero at a hard wall. The terms inside of the Bessel function would then correspond to values along the domain, $k_r r$, which satisfy our equation. Let $k_r r = \alpha_{m,n}$ where, α represents the zeros of the Bessel function, and m corresponds to the azimuthal mode order and n represents the radial mode order, i.e. the index for the number of zero crossings in the derivative of the Bessel function of the first kind.

Therefore,

$$\left. \frac{dR}{dr} \right|_{r=r_{max}} = AJ'_0(k_r r_{max}) = 0 \quad (7)$$

$$(k_r r_{max}) = 0 \quad (8)$$

Recalling $\alpha_{m,n}$

$$(k_r r_{max}) = \alpha_{m,n} \quad (9)$$

$$k_r = \alpha_{m,n}/r_{max} \quad (10)$$

For annular ducts, r_{min} is no longer zero, therefore B cannot be removed since Y'_0 has finite values as $k_r r$ increases.

$$\left. \frac{dR}{dr} \right|_r = AJ'_0(k_r r) + BY'_0(k_r r) \quad (11)$$

Applying boundary conditions to both inner and outer walls at r_{min} and r_{max}

$$\left. \frac{dR}{dr} \right|_{r=r_{min}} = AJ'_0(k_r r_{min}) + BY'_0(k_r r_{min}) \quad (12)$$

$$\left. \frac{dR}{dr} \right|_{r=r_{max}} = AJ'_0(k_r r_{max}) + BY'_0(k_r r_{max}) \quad (13)$$

defining $\beta = \frac{B}{A}$,

$$\left. \frac{dR}{dr} \right|_{r=r_{min}} = J'_0(k_r r_{min}) + \beta Y'_0(k_r r_{min}) \quad (14)$$

$$\left. \frac{dR}{dr} \right|_{r=r_{max}} = J'_0(k_r r_{max}) + \beta Y'_0(k_r r_{max}) \quad (15)$$

Defining,

$$\begin{aligned} k_r^{n+1} &= k_r^n + \Delta k_r \\ \beta^{n+1} &= \beta^n + \Delta \beta \end{aligned}$$

And substituting into the boundary condition expressions,

$$\left. \frac{dR}{dr} \right|_{r=r_{min}} = J'_0(k_r^{n+1} r_{min}) + \beta^{n+1} Y'_0(k_r^{n+1} r_{min}) \quad (16)$$

$$\left. \frac{dR}{dr} \right|_{r=r_{max}} = J'_0(k_r^{n+1} r_{max}) + \beta^{n+1} Y'_0(k_r^{n+1} r_{max}) \quad (17)$$

0.3 Analytical Test Case

The results present a comparison of analytical and numerical solutions for a uniform flow, hard-wall, cylindrical duct for six grids. The physical parameters for this test case are reported in Table 1. The grid points were doubled each iteration and a starting grid of 33 points was chosen. The first five radial modes were chosen in order to get a group of modes that were cut on and cut off. The case number's are used as identifiers for each mode propagating up or downstream. In the event that a spurious mode is identified, the case number will index it. The number of cut off modes was also constrained by setting a maximum value for the imaginary part of the axial wavenumber. While this magnitude is currently arbitrary, the value could be correlated to a desired decay rate for the corresponding mode. Each of the axial wavenumber's radial mode for each grid is also reported. The same test case was ran twice using second and fourth order central schemes for the radial derivatives. The approximate rate of convergence is also reported for both numerical schemes.

σ	<i>0.0</i>
k	<i>10</i>
m	<i>2</i>
M_x	<i>0.3</i>

Table 1: Validation test case parameters, Uniform Flow Hard-wall CylindricalDuct

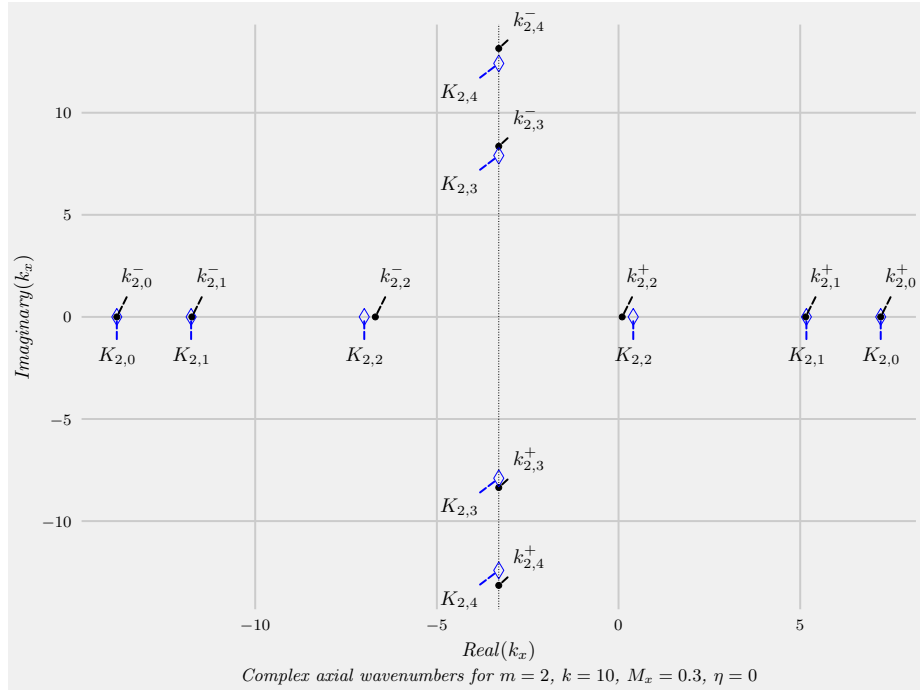


Figure 1: Second Order, Analytical Solution vs Numerical Approximation using 33 points

0.3.1 Axial Wavenumber

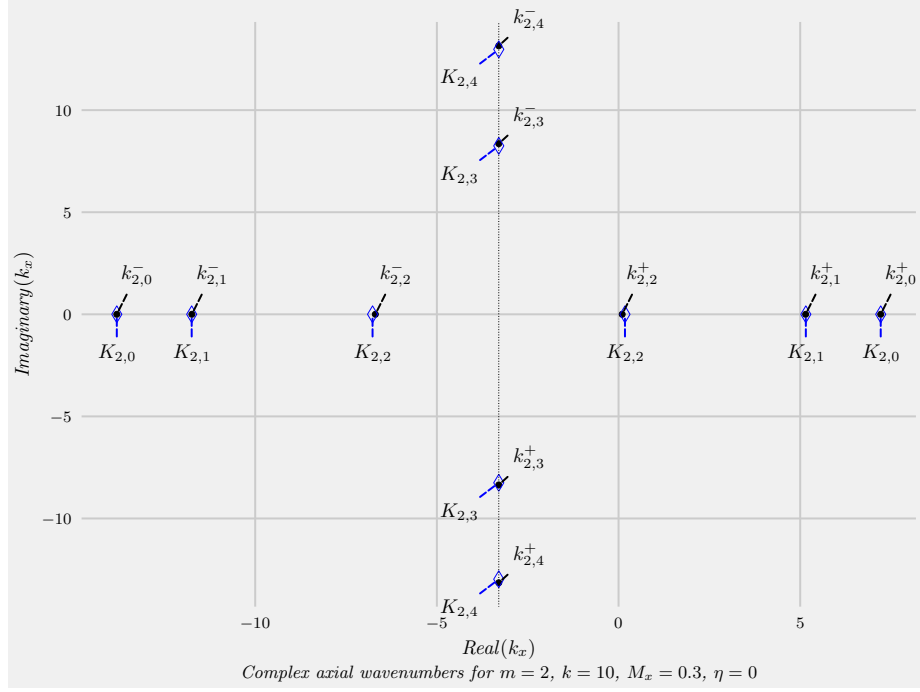


Figure 2: Second Order, Analytical Solution vs Numerical Approximation using 66 points

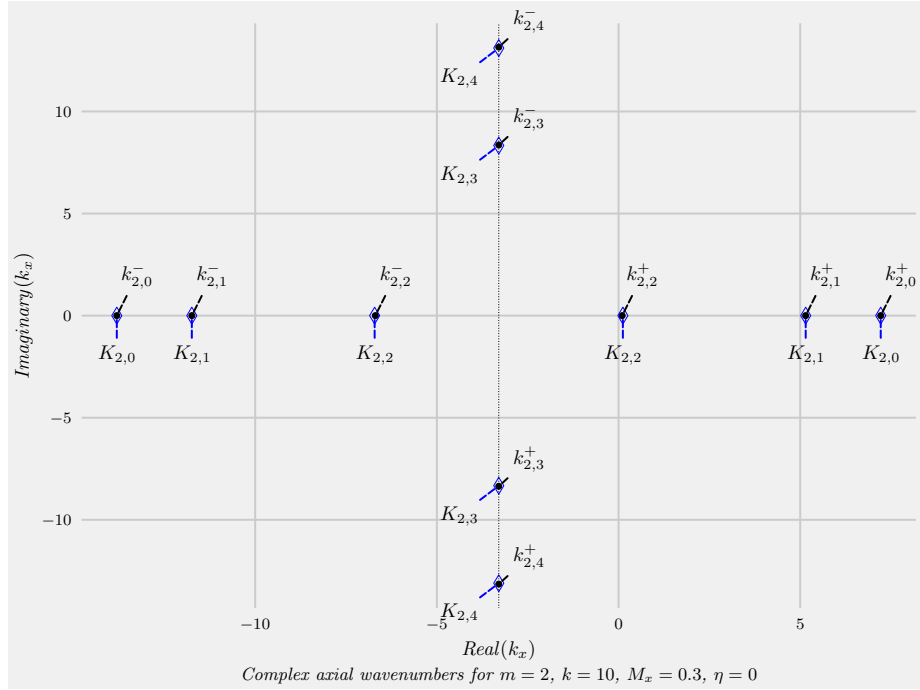


Figure 3: Second Order, Analytical Solution vs Numerical Approximation using 132 points

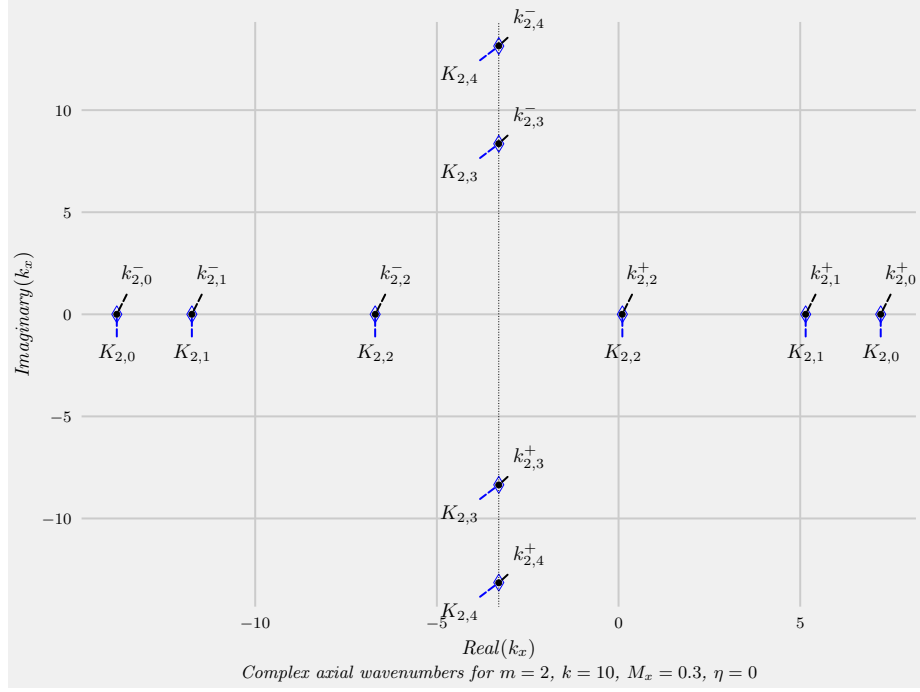


Figure 4: Second Order, Analytical Solution vs Numerical Approximation using 264 points

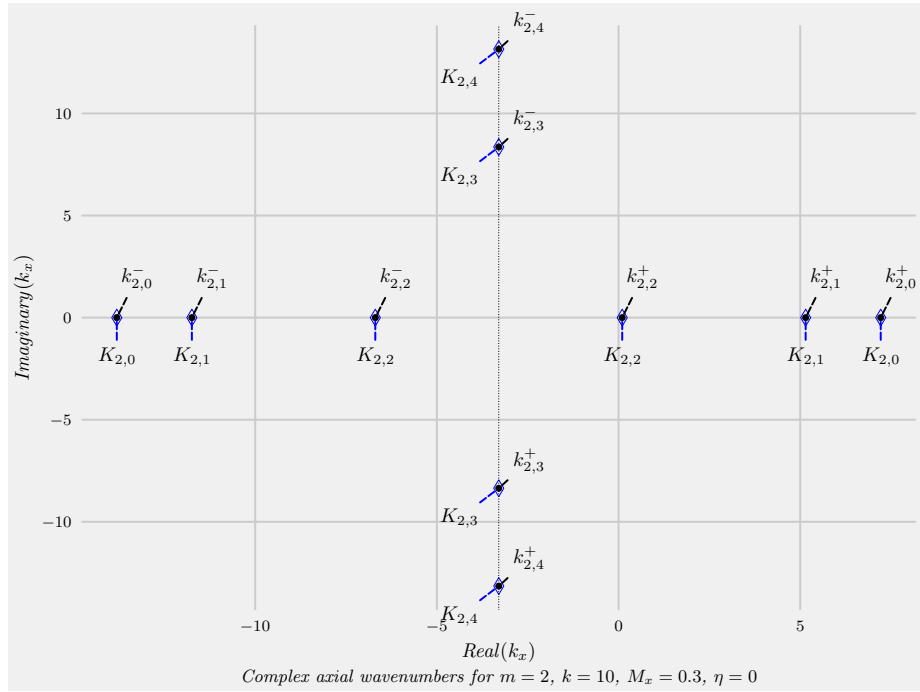


Figure 5: Second Order, Analytical Solution vs Numerical Approximation using 528 points

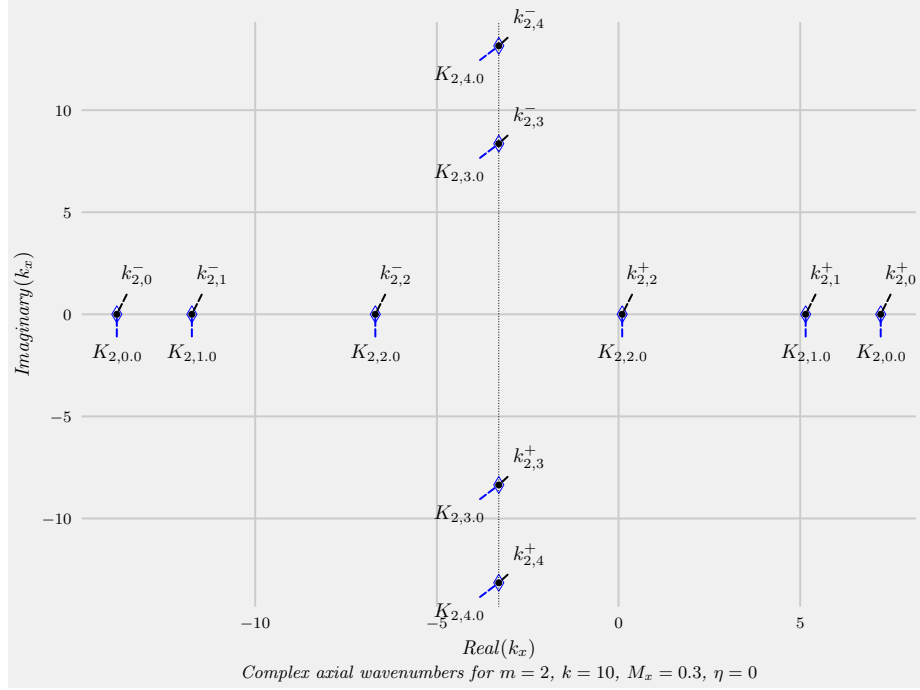


Figure 6: Second Order, Analytical Solution vs Numerical Approximation using 1056 points

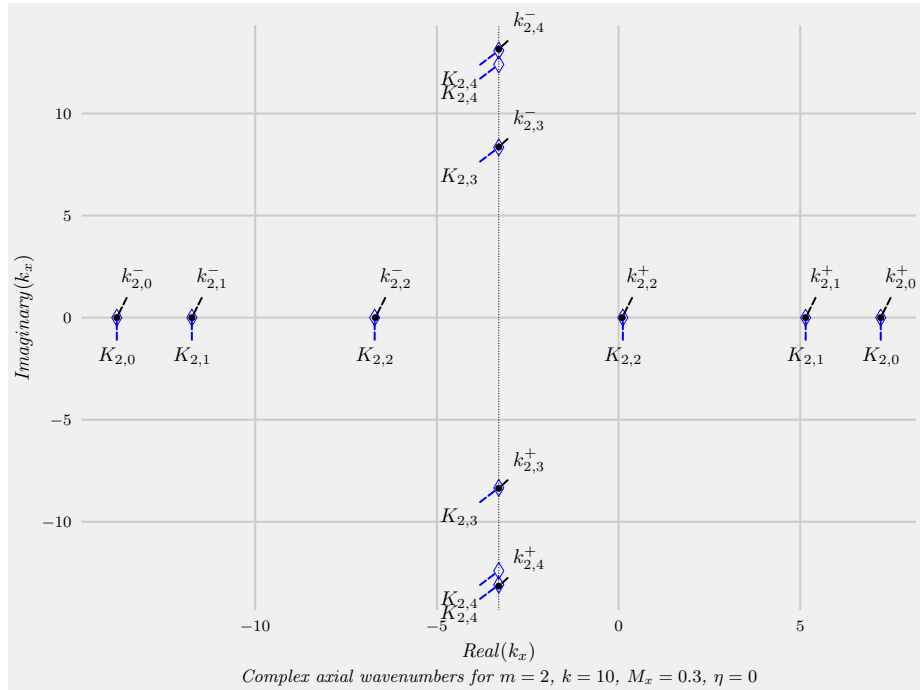


Figure 7: Fourth Order, Analytical Solution vs Numerical Approximation using 33 points

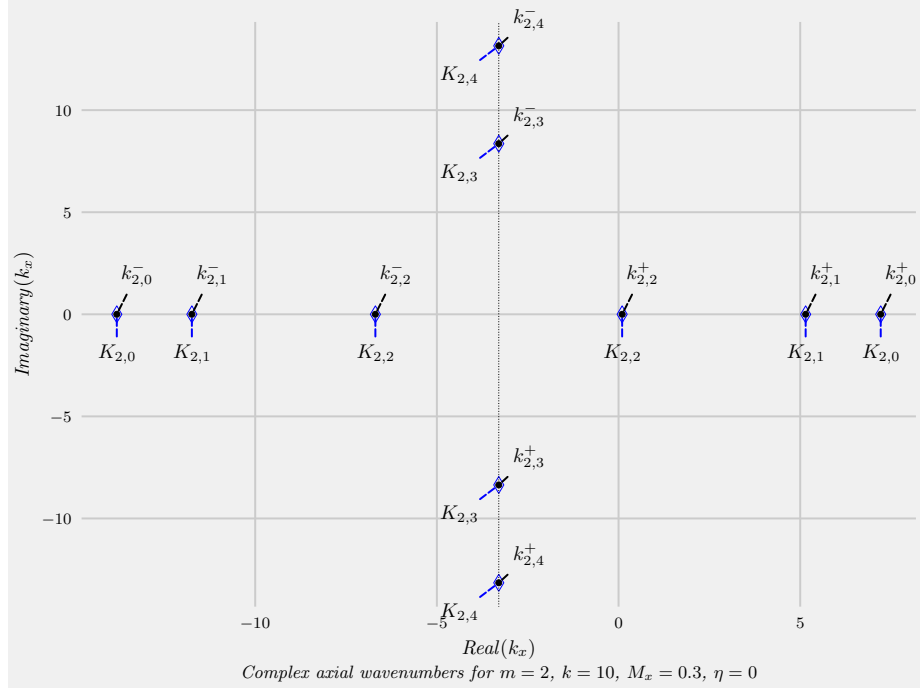


Figure 8: Fourth Order, Analytical Solution vs Numerical Approximation using 66 points

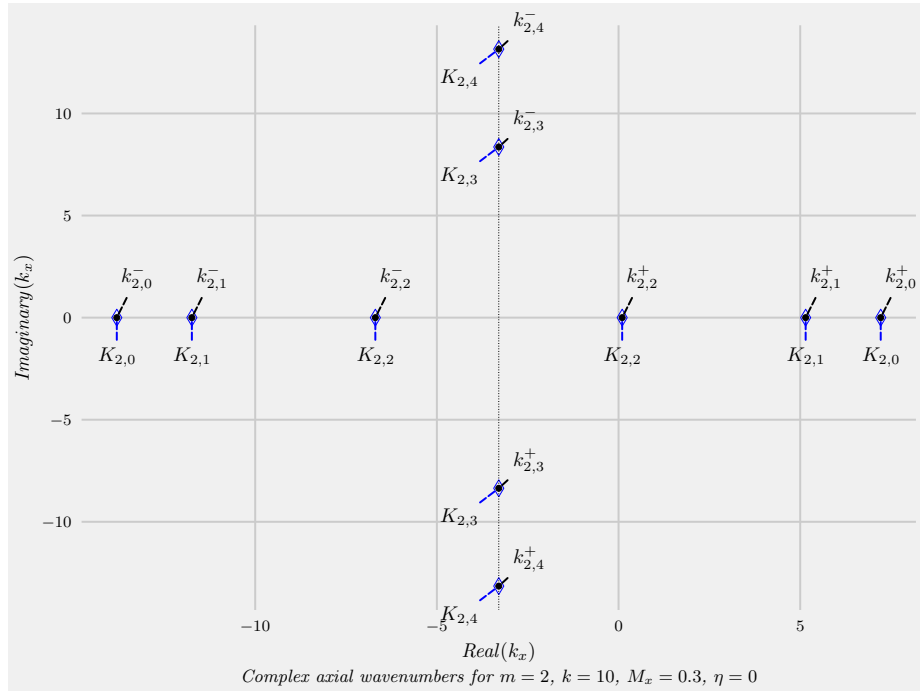


Figure 9: Fourth Order, Analytical Solution vs Numerical Approximation using 132 points

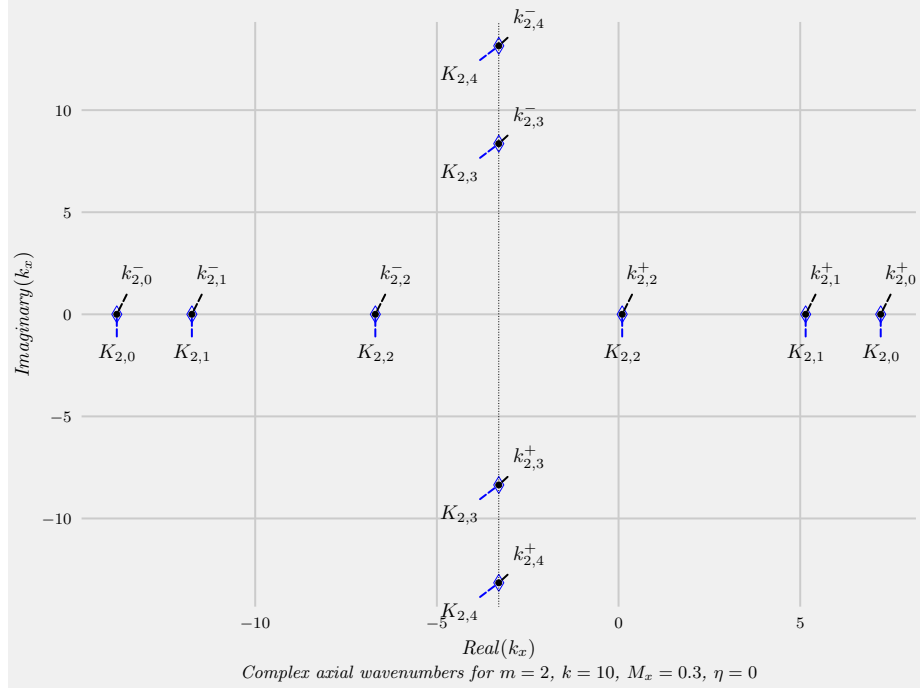


Figure 10: Fourth Order, Analytical Solution vs Numerical Approximation using 264 points

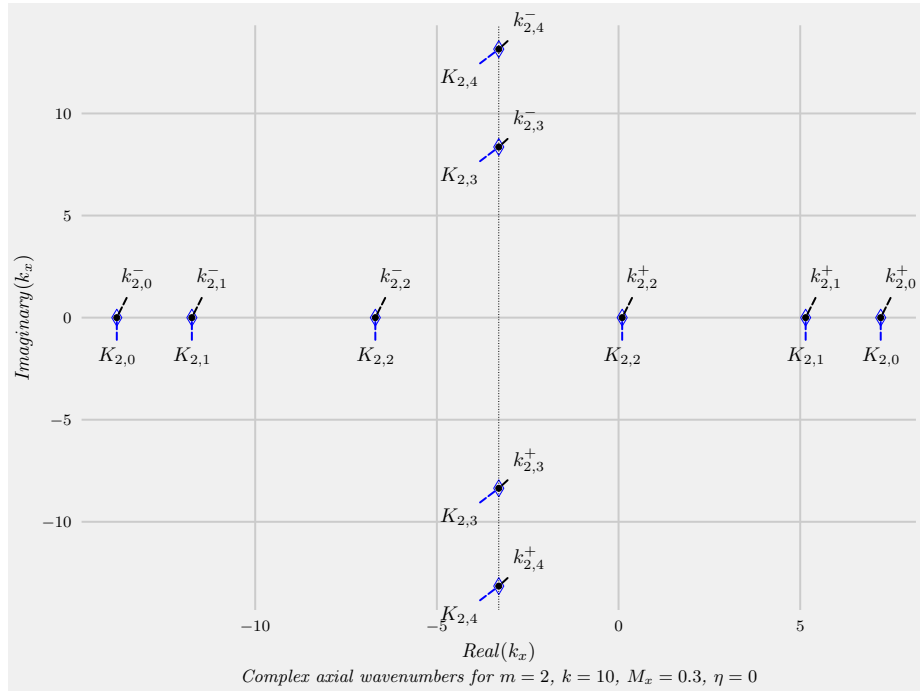
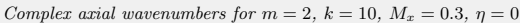


Figure 11: Fourth Order, Analytical Solution vs Numerical Approximation using 528 points



1056 points

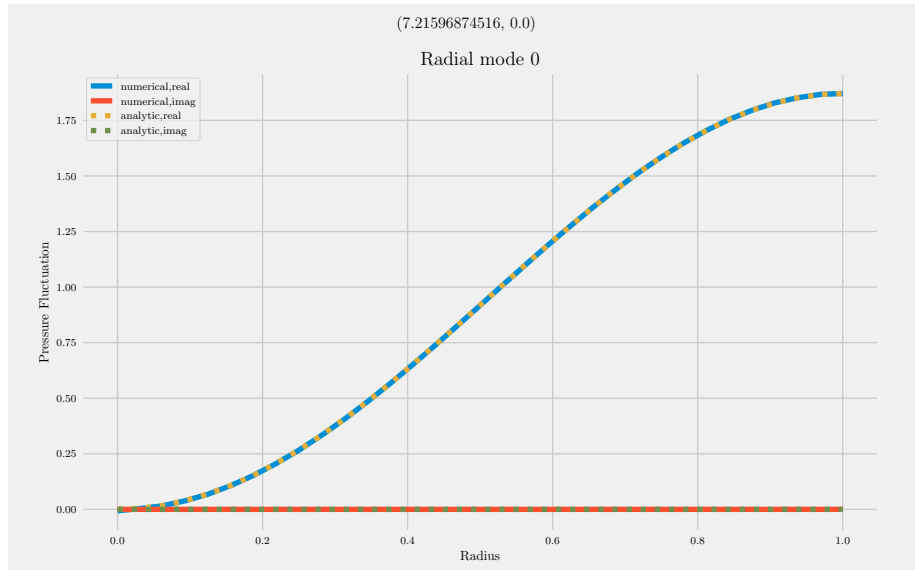


Figure 13: Second Order, Case number 0, 33points

0.3.2 Propagating Radial Modes

Second Order, Radial Mode 0

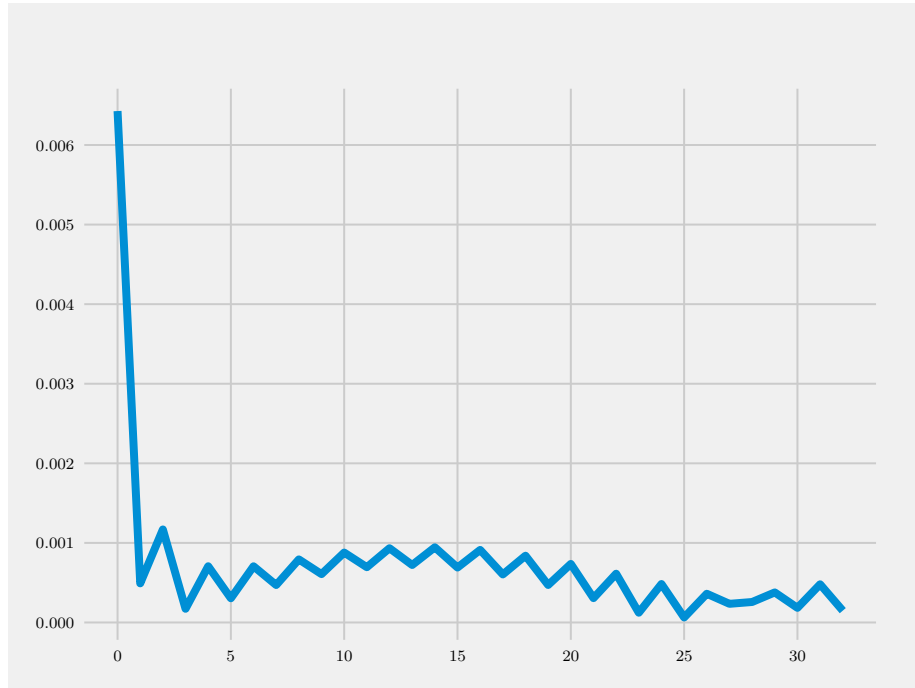


Figure 14: Second Order, Case number 0, 33points

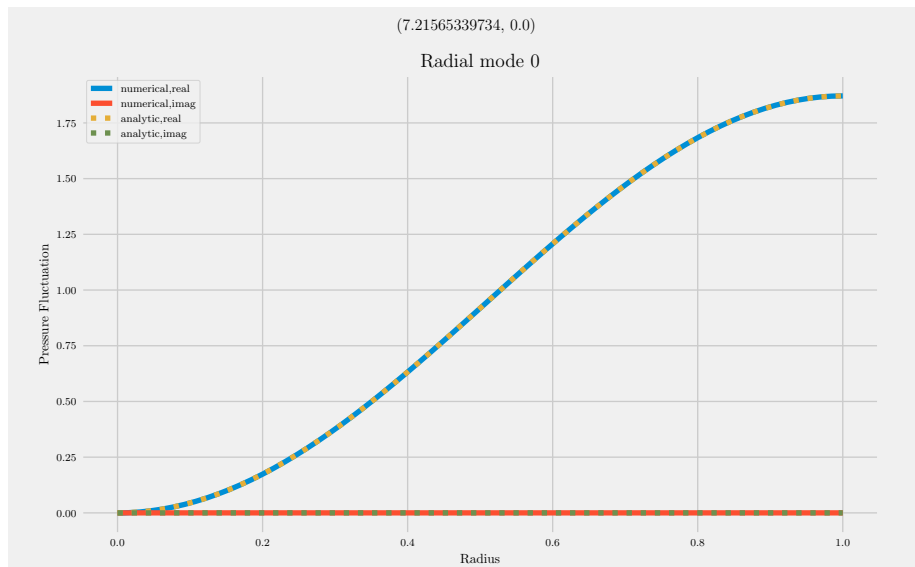


Figure 15: Second Order, Case number 0, 66points

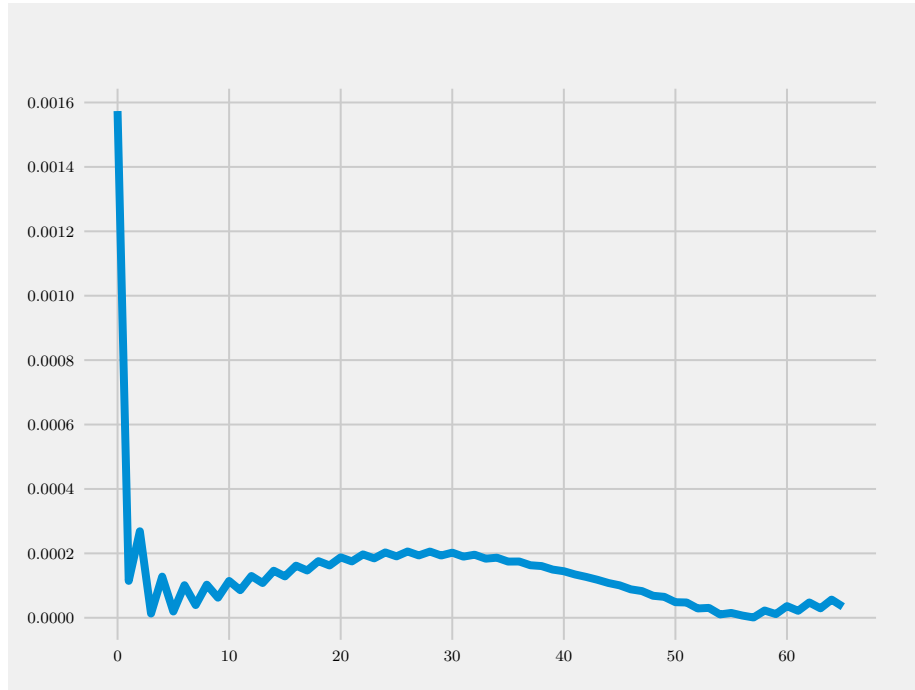


Figure 16: Second Order, Case number 0, 66points

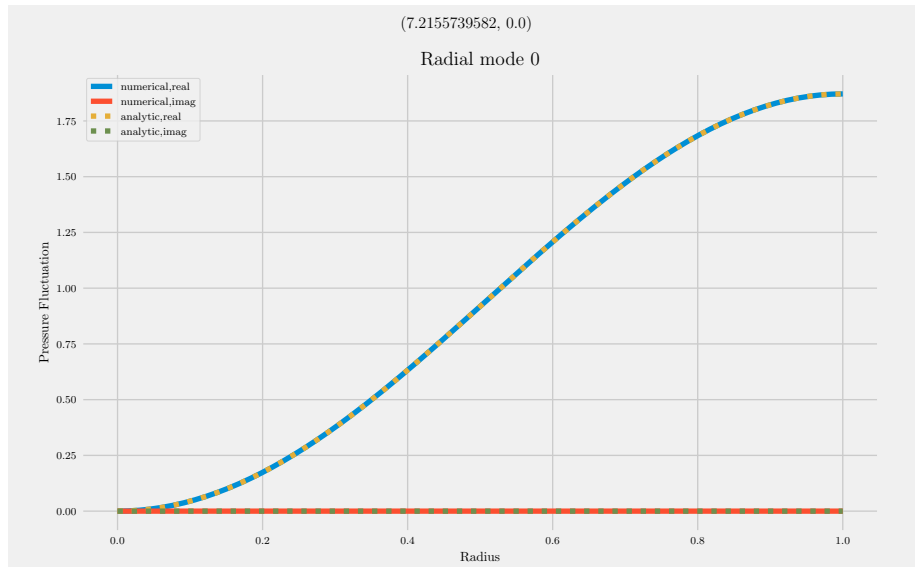


Figure 17: Second Order, Case number 0, 132points

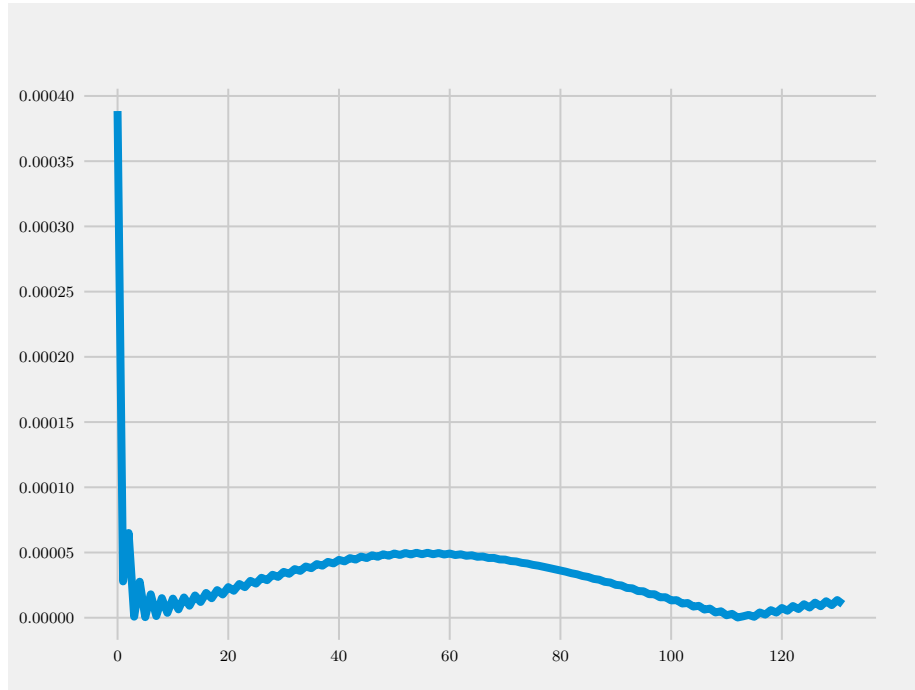


Figure 18: Second Order, Case number 0, 132points

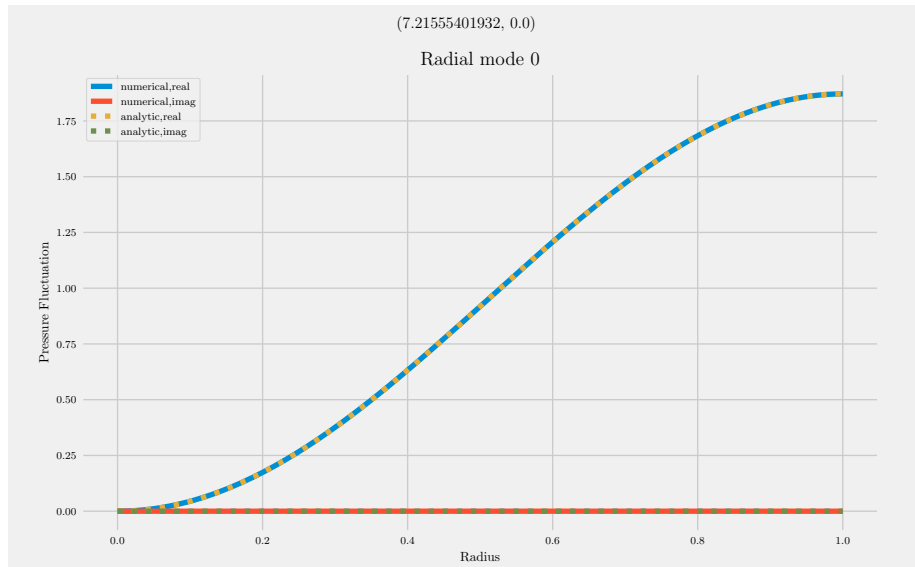


Figure 19: Second Order, Case number 0, 264points

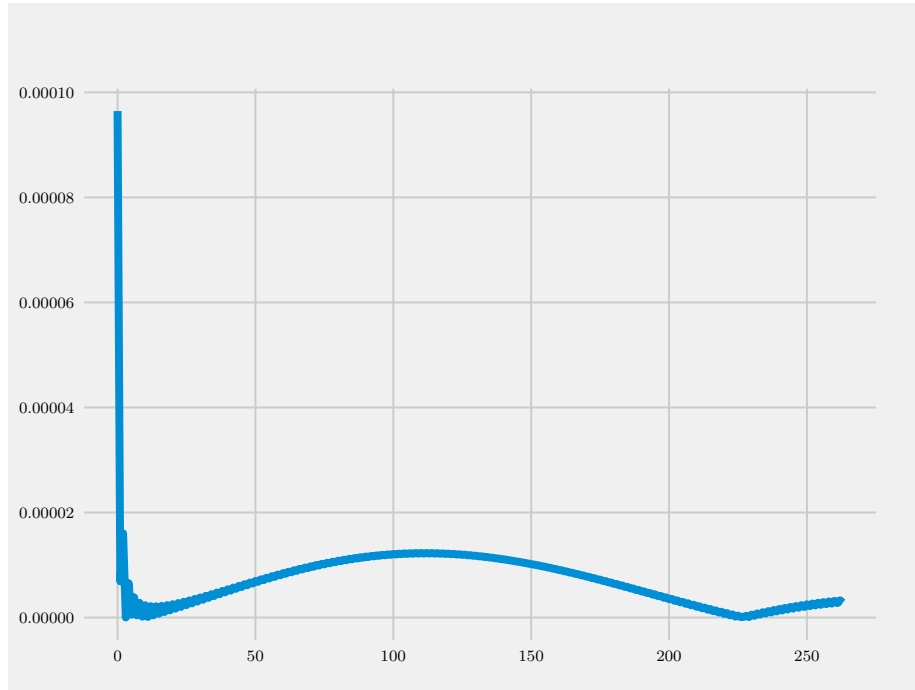


Figure 20: Second Order, Case number 0, 264points

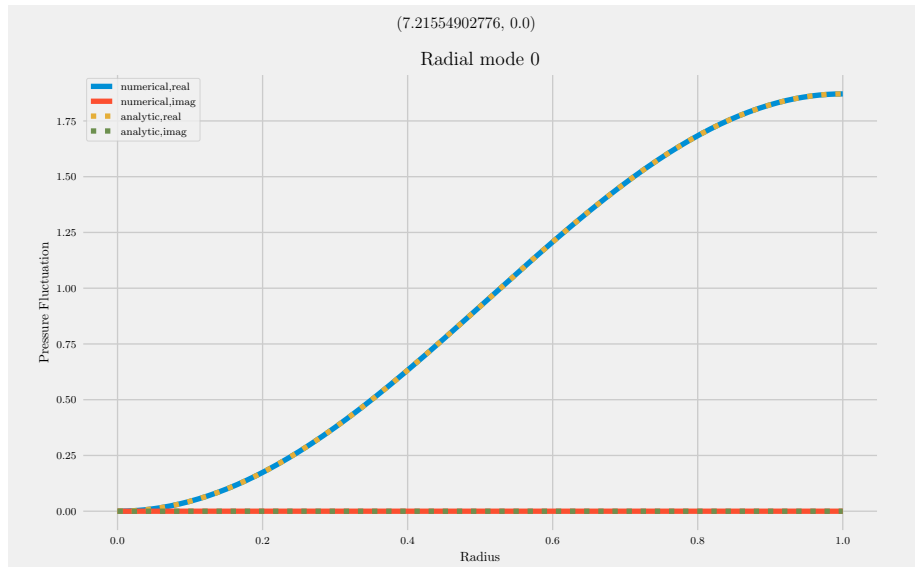


Figure 21: Second Order, Case number 0, 528points

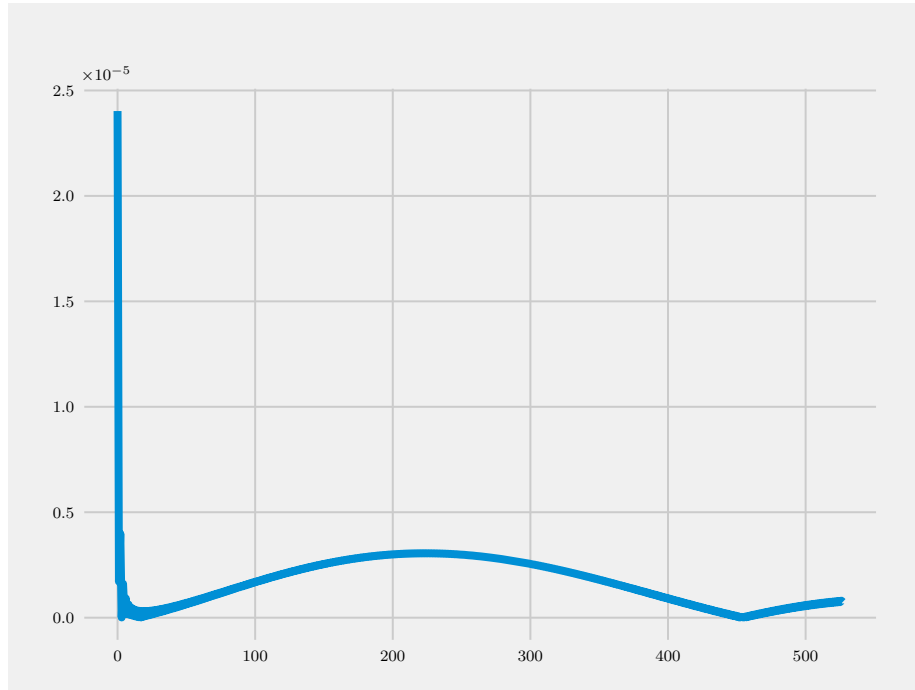


Figure 22: Second Order, Case number 0, 528points

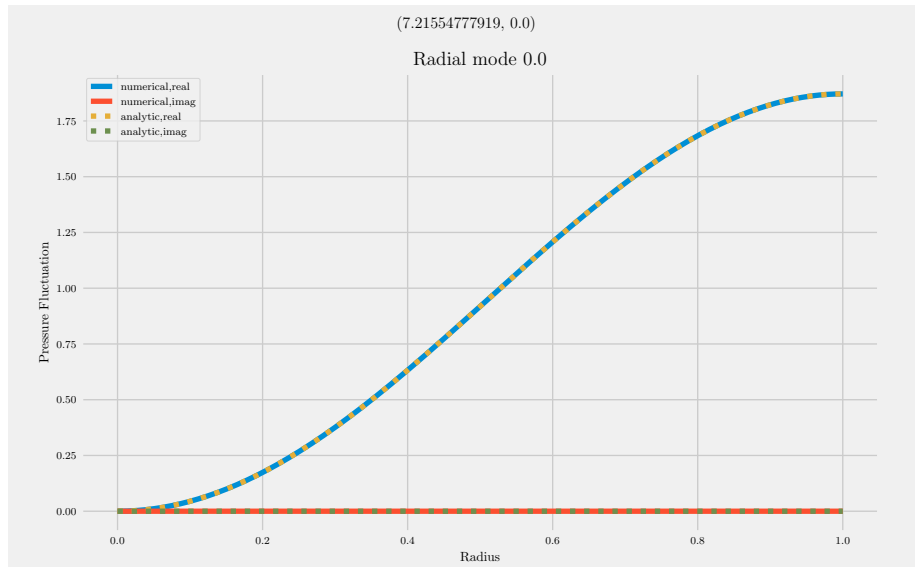


Figure 23: Second Order, Case number 0, 1056points

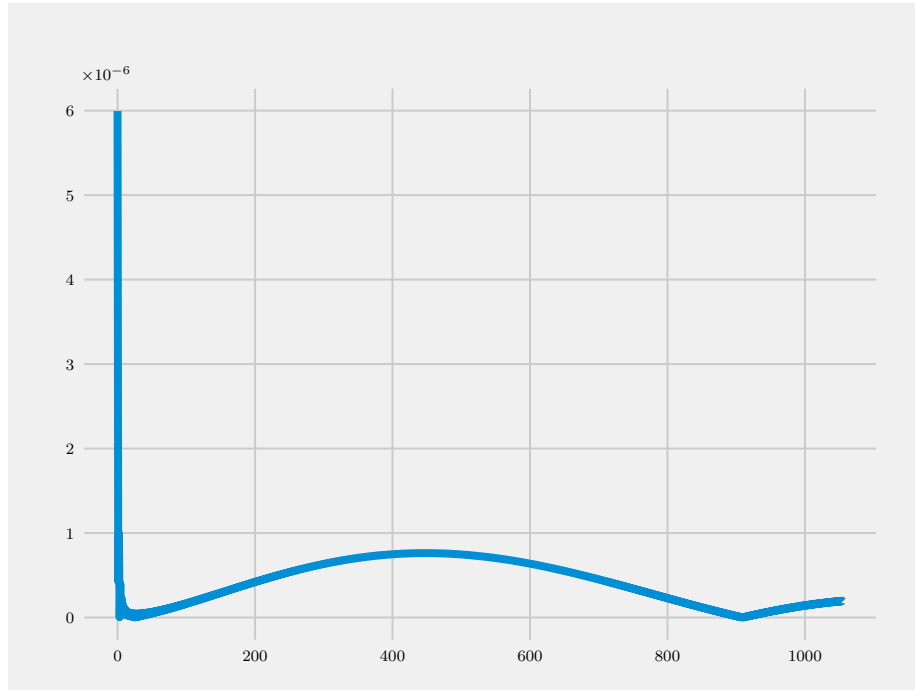


Figure 24: Second Order, Case number 0, 1056points

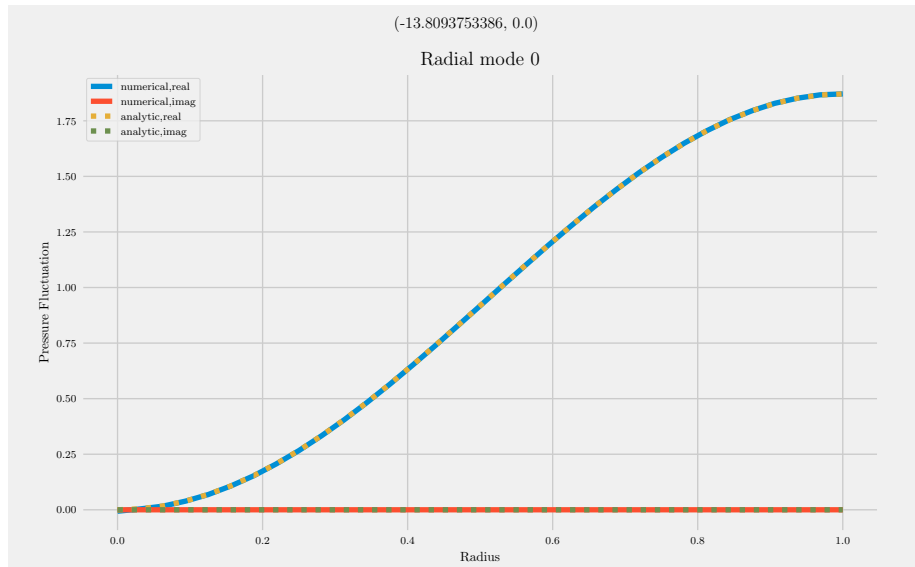


Figure 25: Second Order, Case number 1, 33points

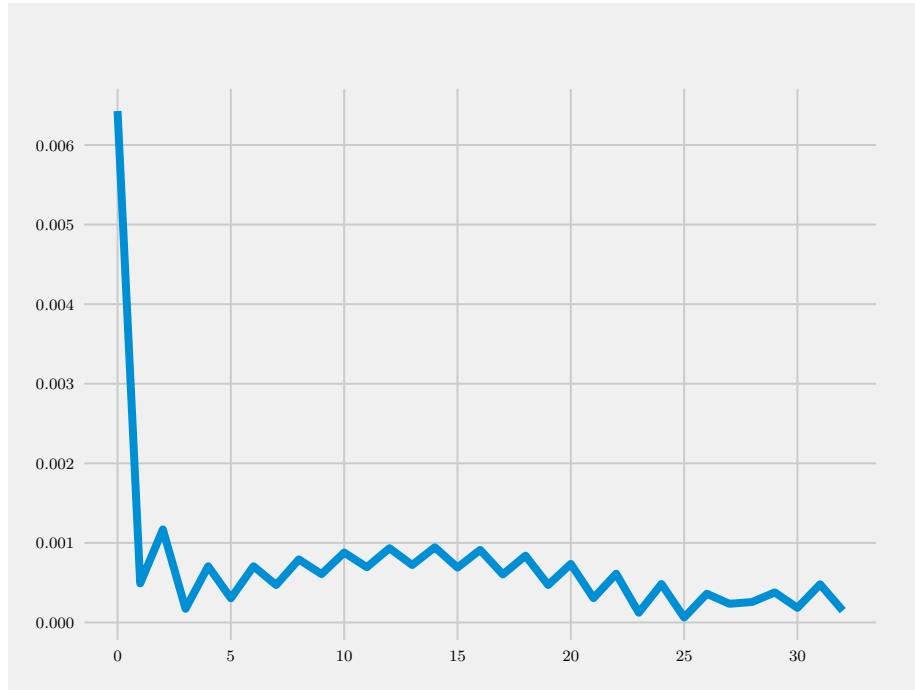


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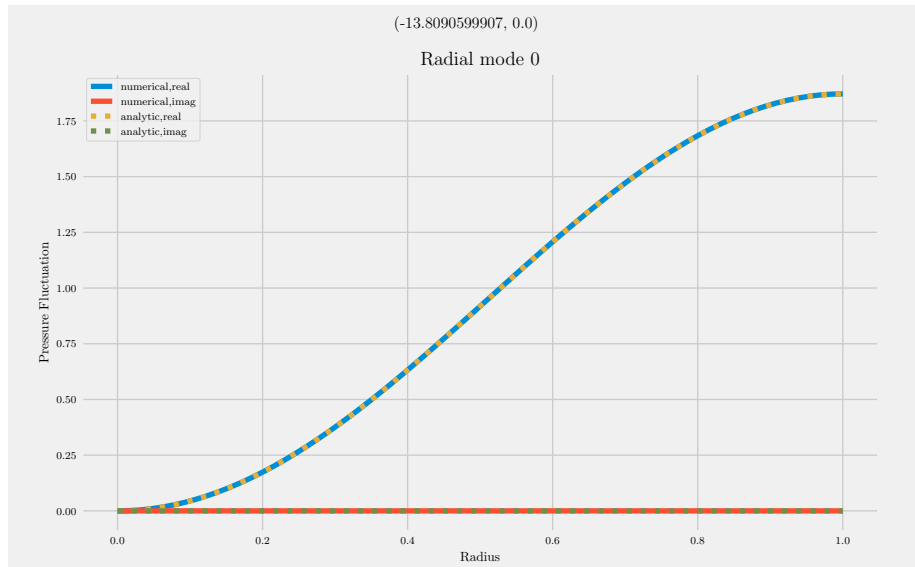


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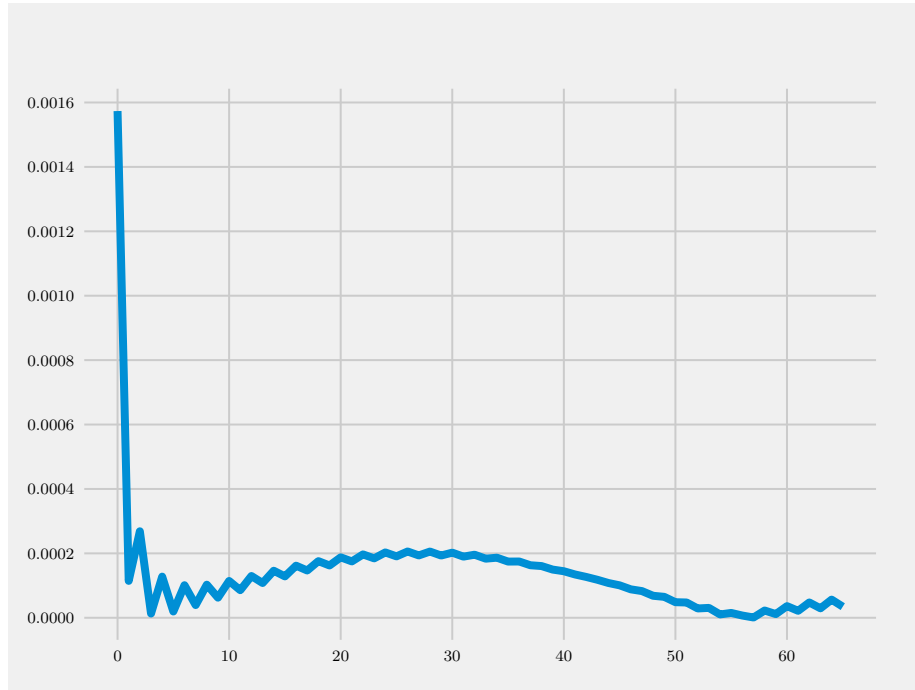


Figure 28: Second Order, Case number 1, 66points

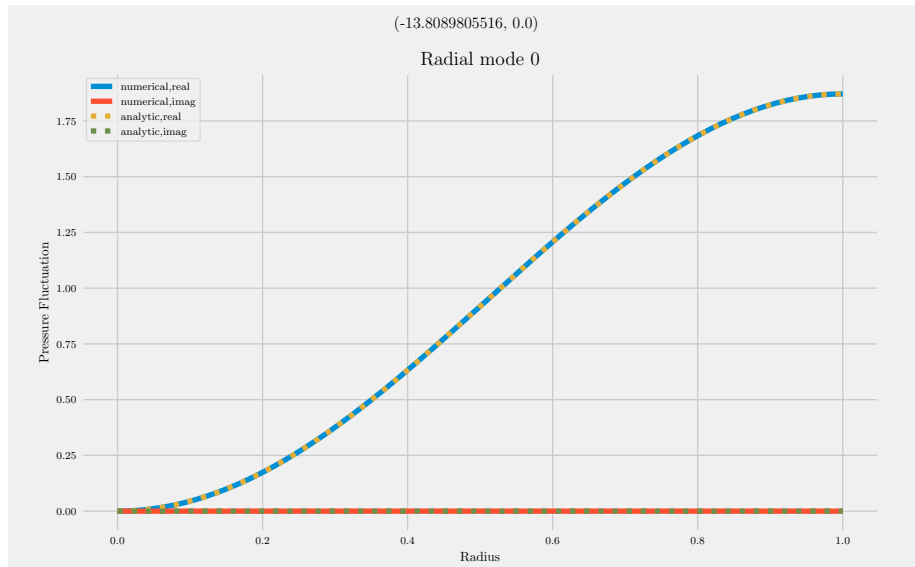


Figure 29: Second Order, Case number 1, 132points

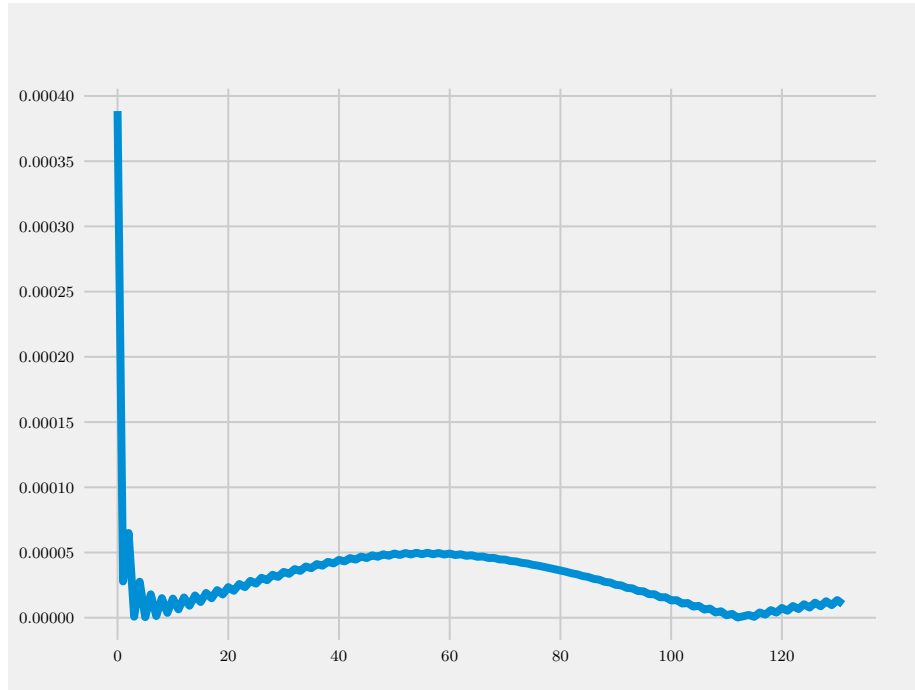


Figure 30: Second Order, Case number 1, 132points

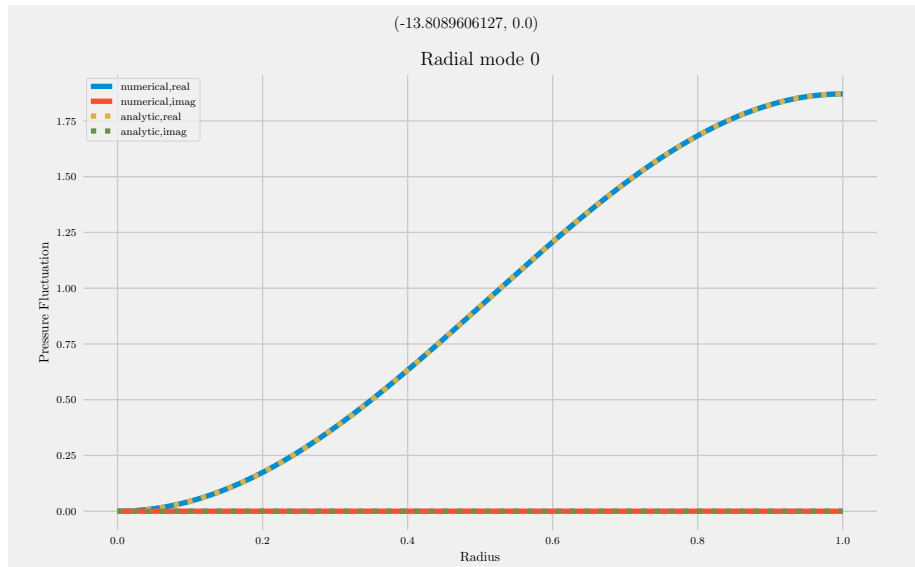


Figure 31: Second Order, Case number 1, 264points

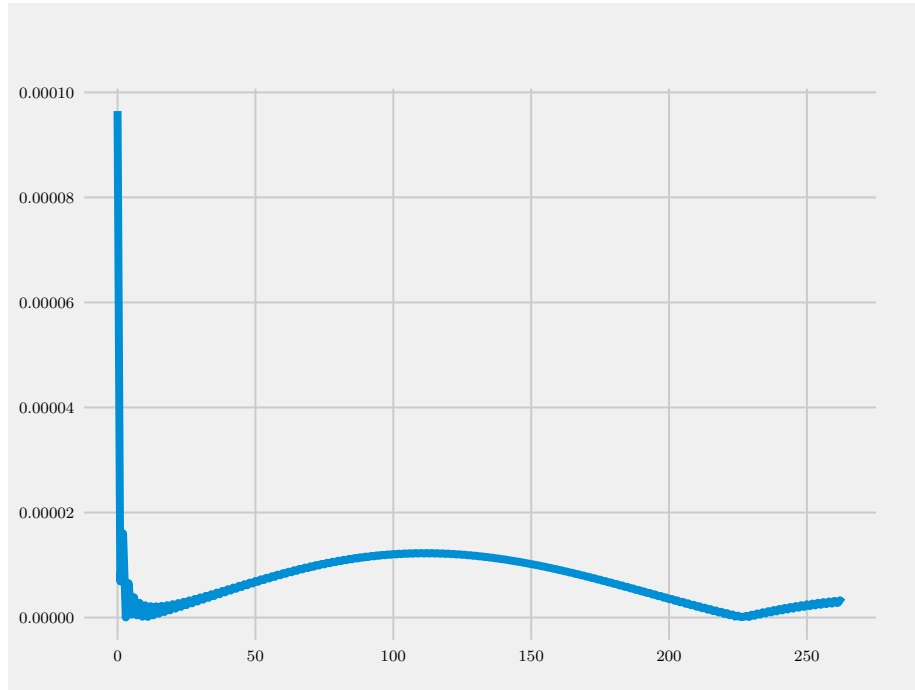


Figure 32: Second Order, Case number 1, 264points

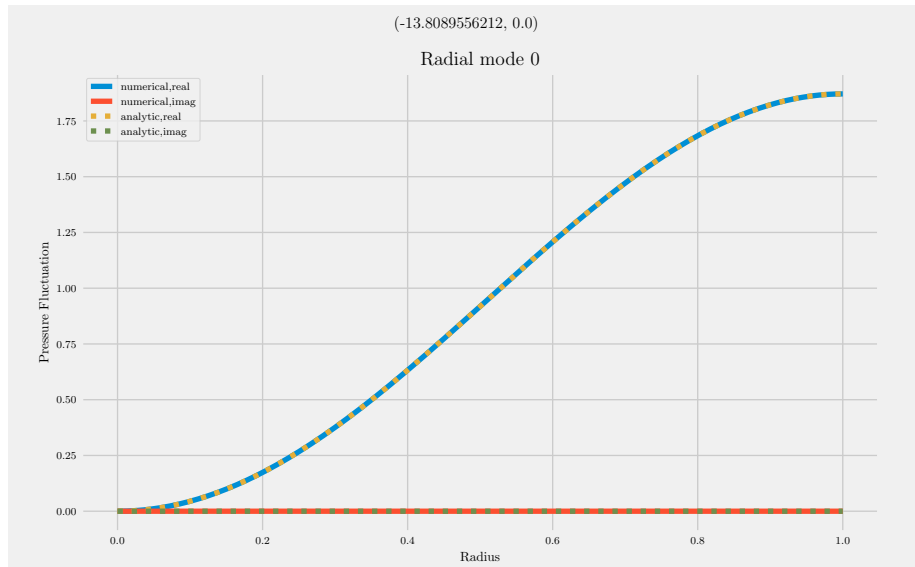


Figure 33: Second Order, Case number 1, 528points

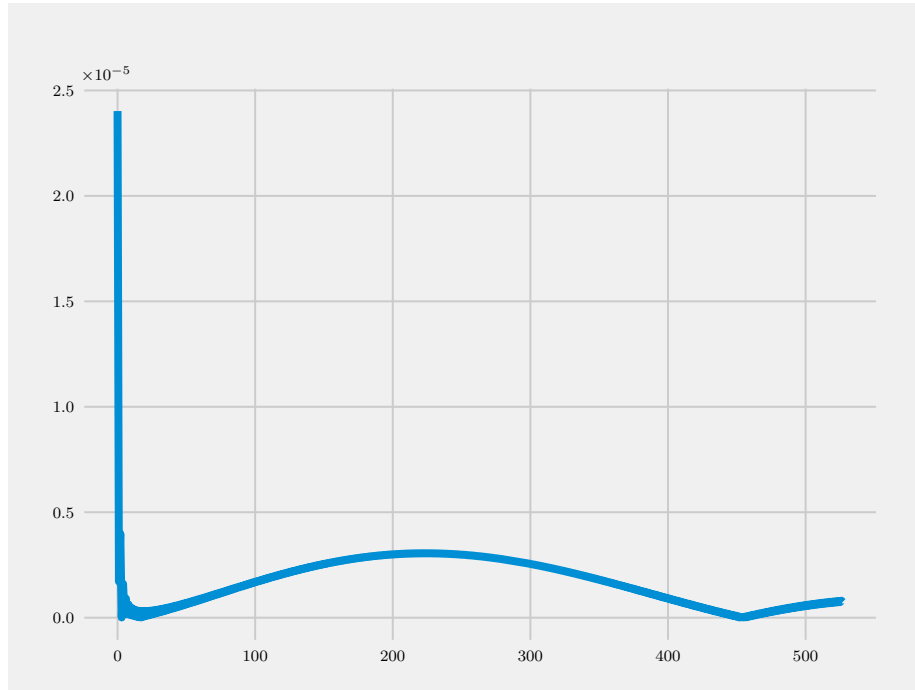


Figure 34: Second Order, Case number 1, 528points

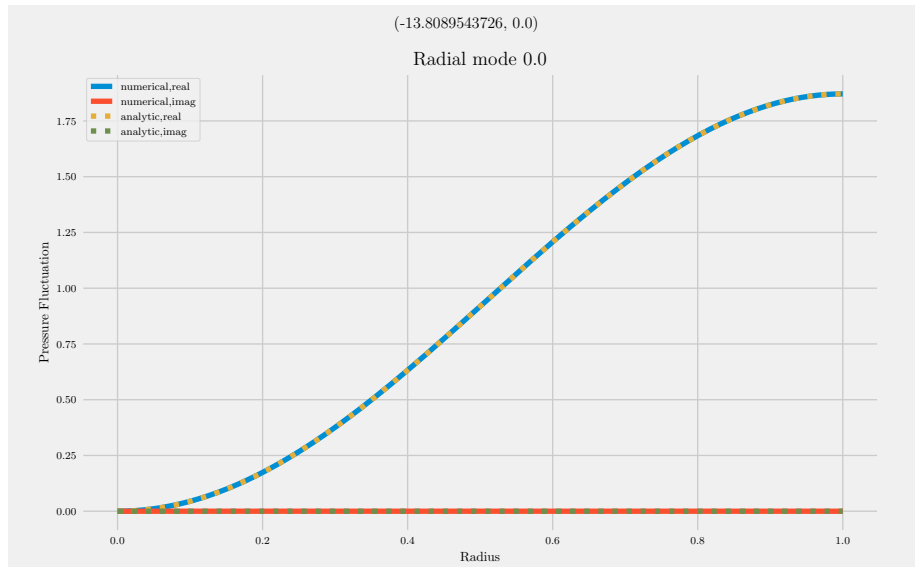


Figure 35: Second Order, Case number 1, 1056points

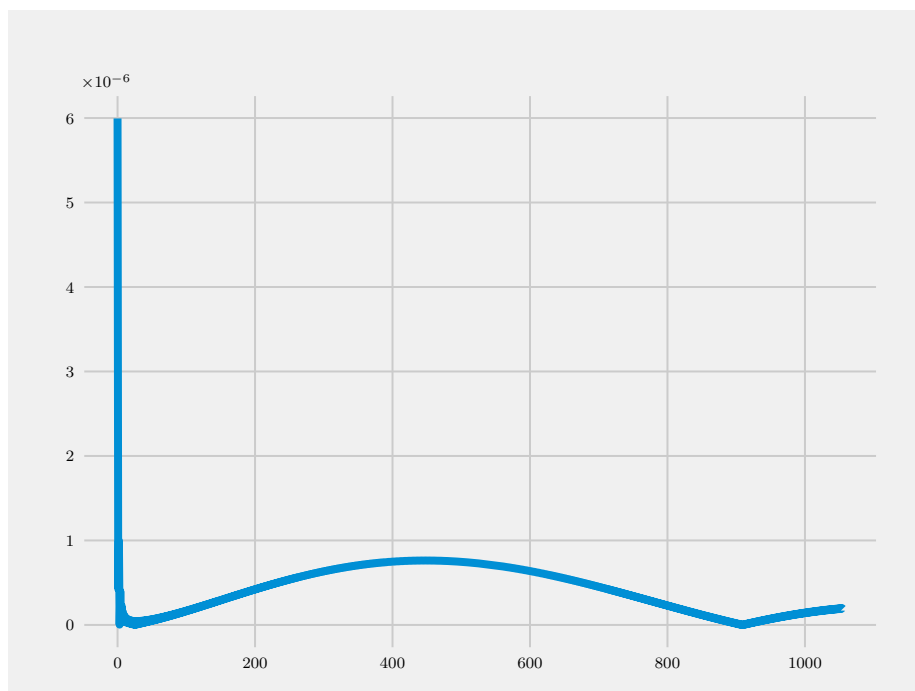


Figure 36: Second Order, Case number 1, 1056points

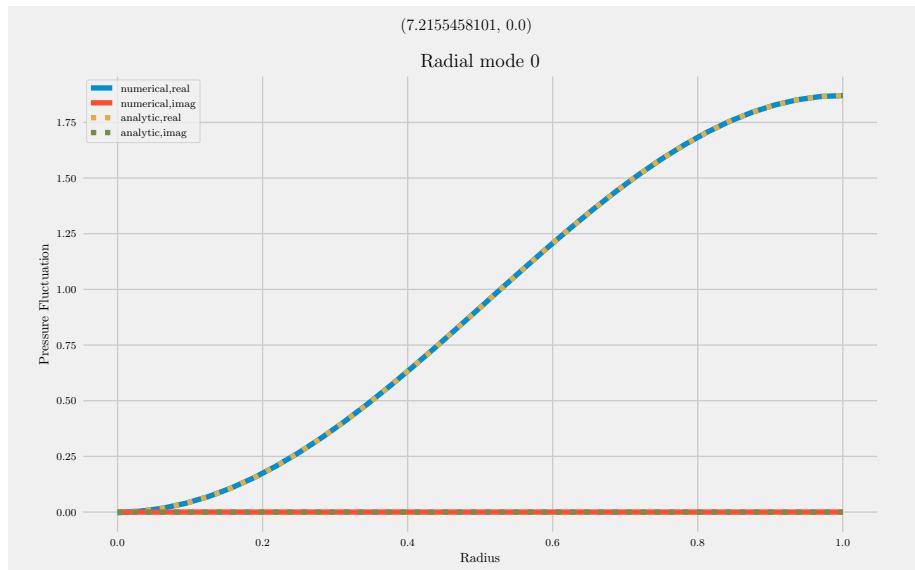


Figure 37: Fourth Order, Case number 0, 33points

Fourth Order, Radial Mode 0

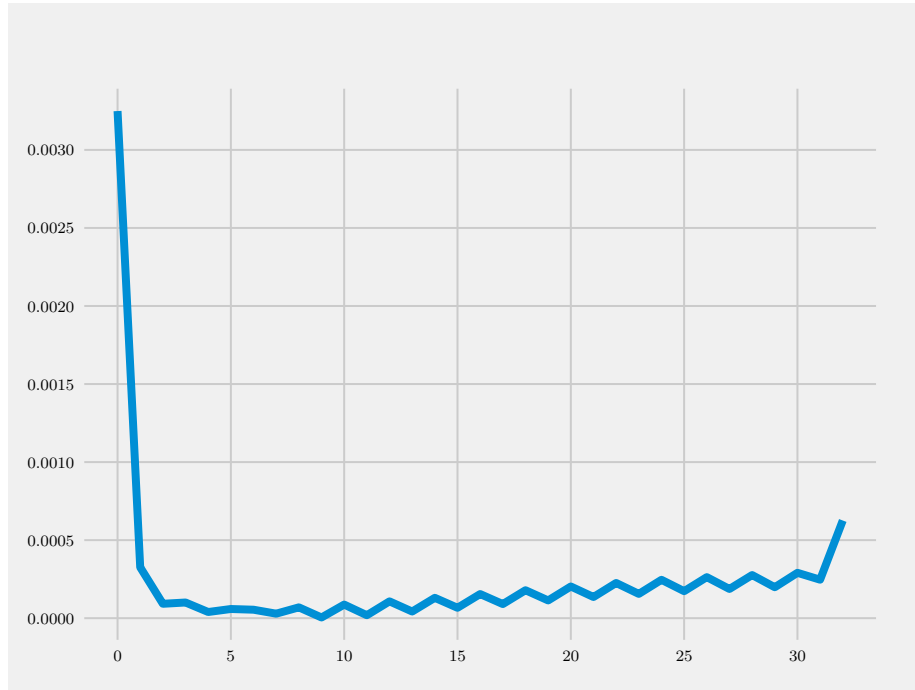


Figure 38: Fourth Order, Case number 0, 33points

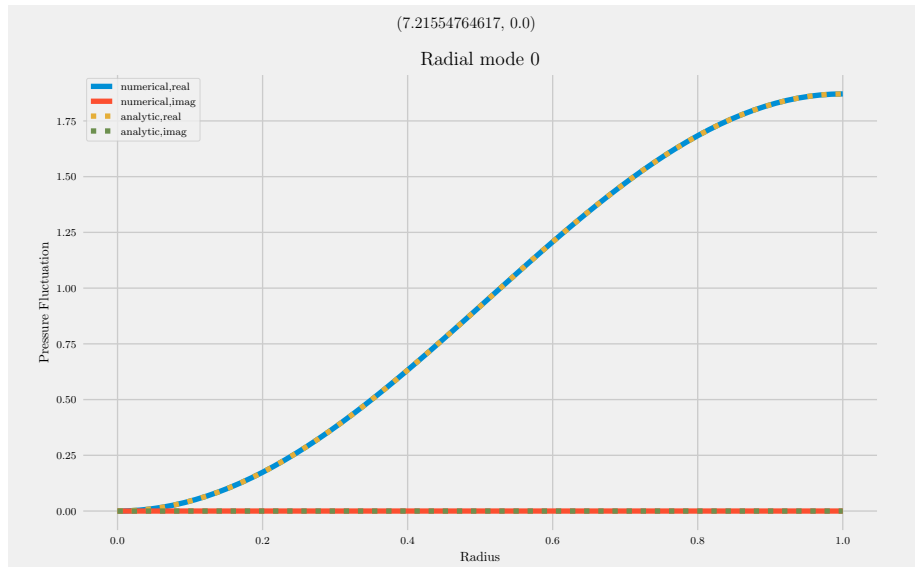


Figure 39: Fourth Order, Case number 0, 66points

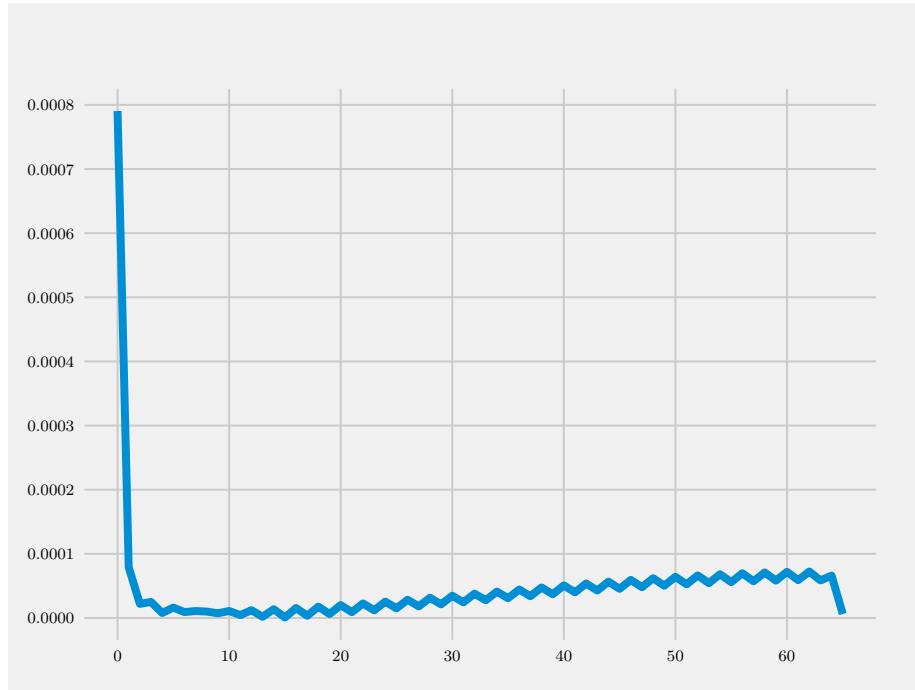


Figure 40: Fourth Order, Case number 0, 66points

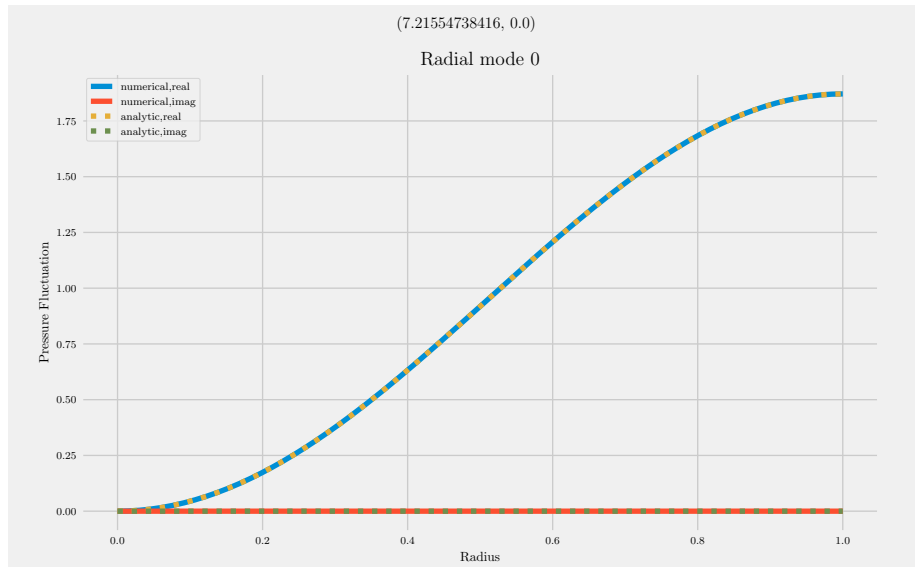


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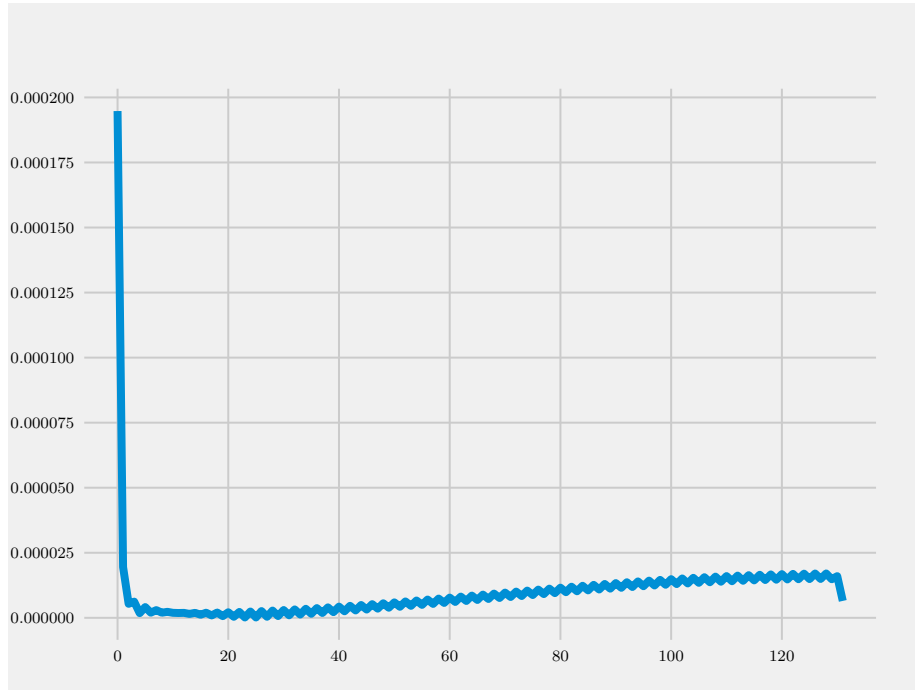


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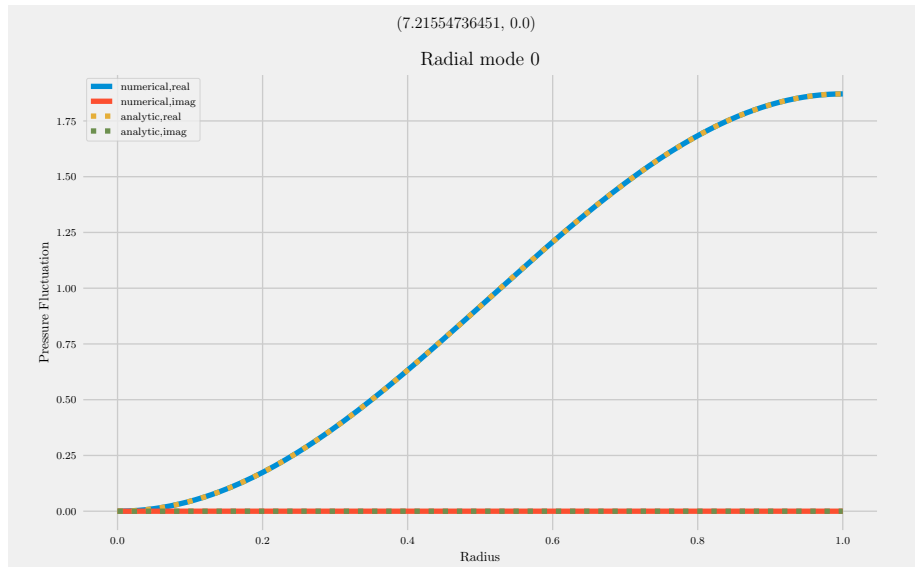


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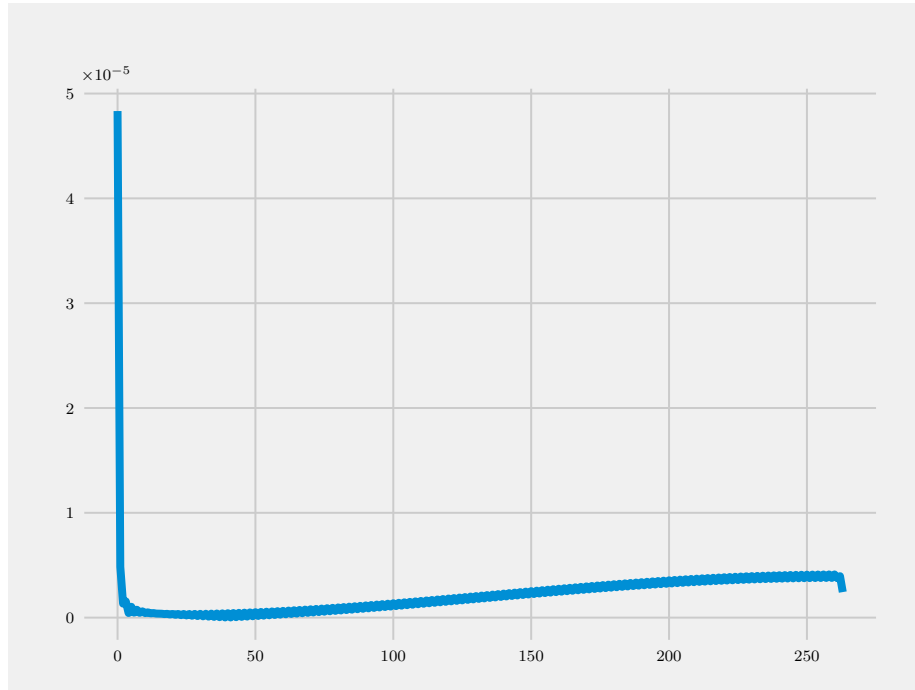


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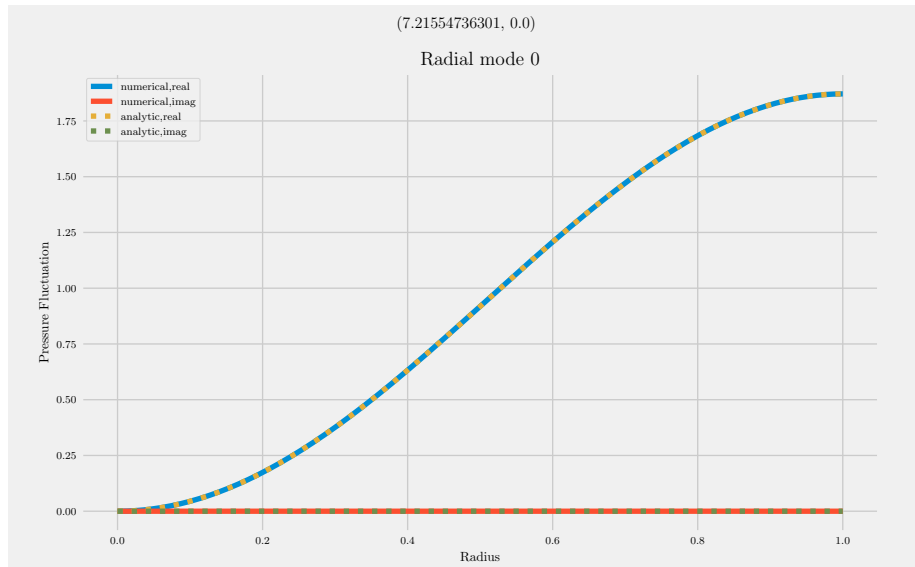


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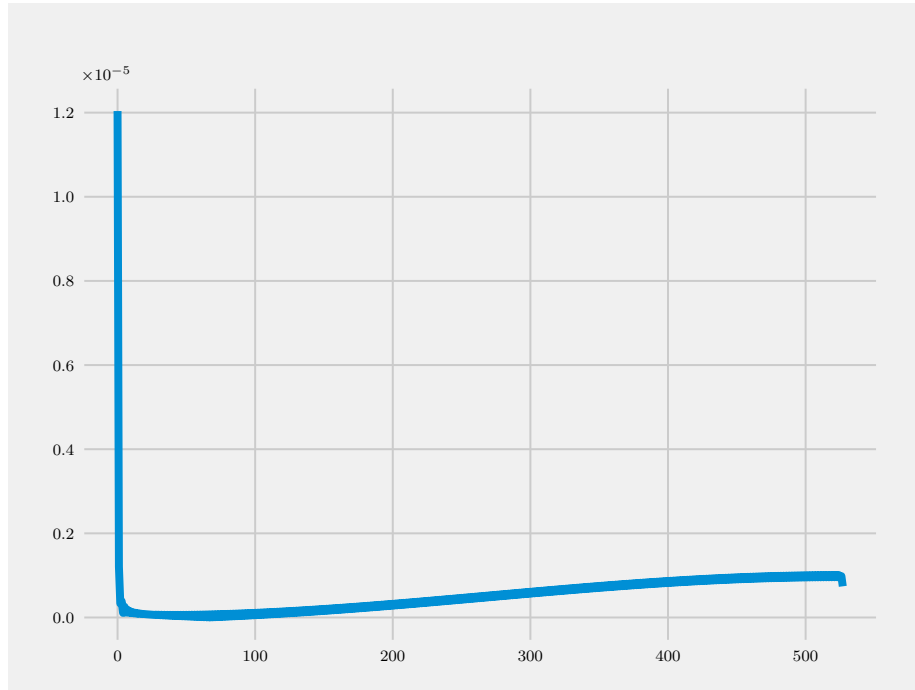


Figure 46: Fourth Order, Case number 0, 528points

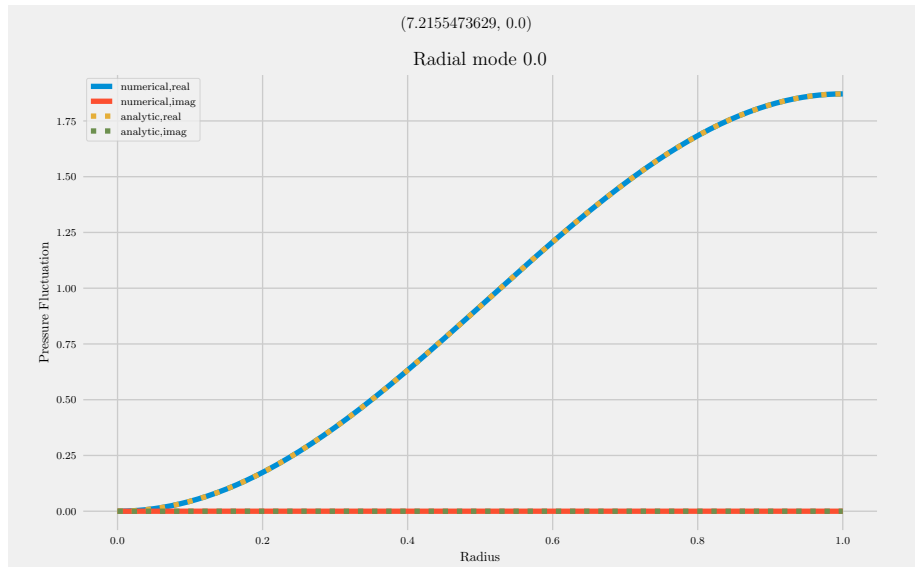


Figure 47: Fourth Order, Case number 0, 1056points

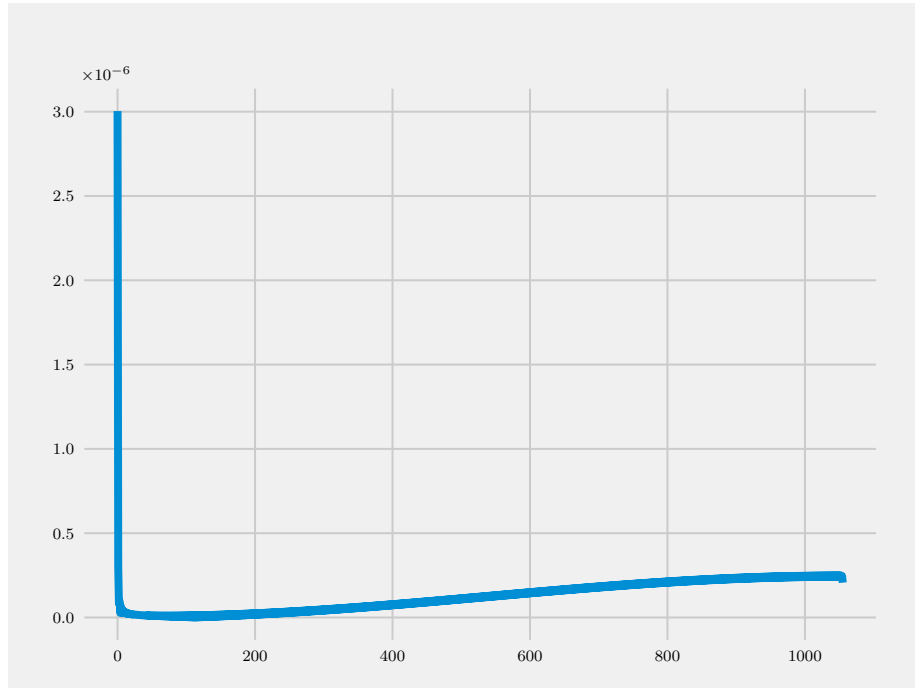


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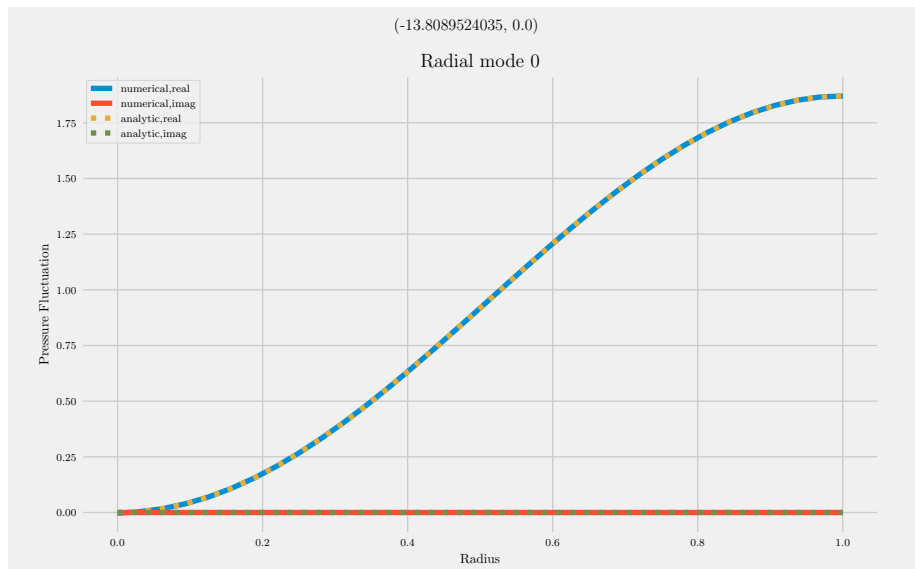


Figure 49: Fourth Order, Case number 1, 33points

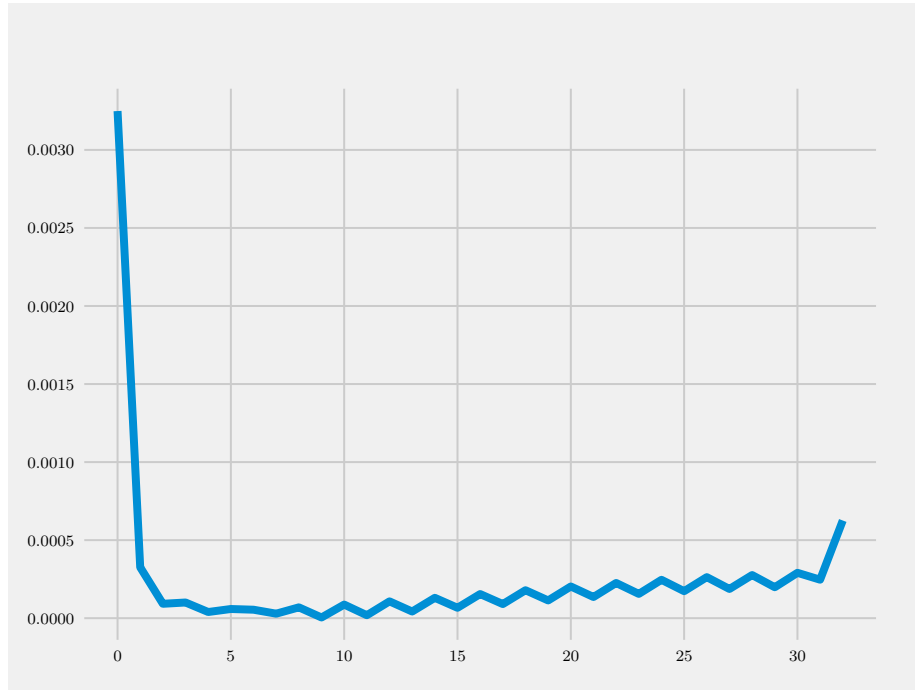


Figure 50: Fourth Order, Case number 1, 33points

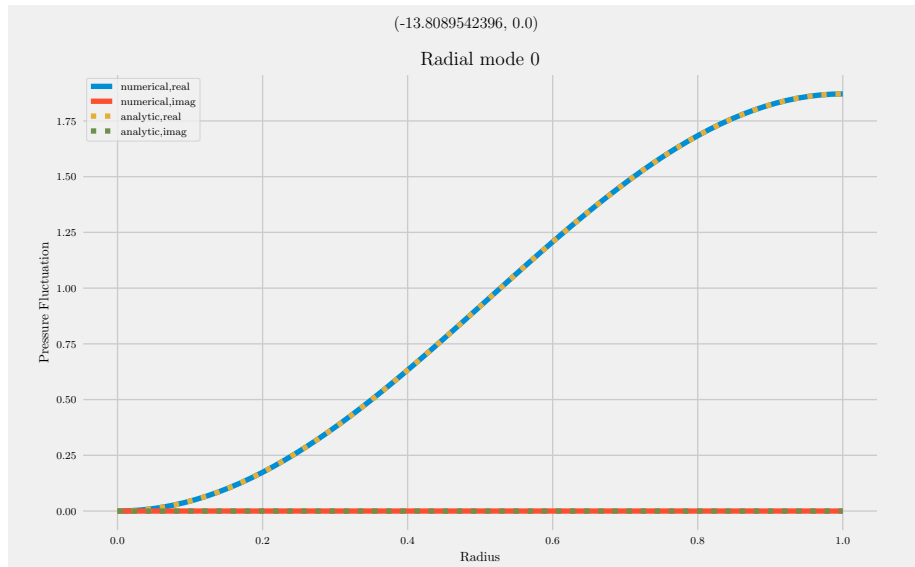


Figure 51: Fourth Order, Case number 1, 66points

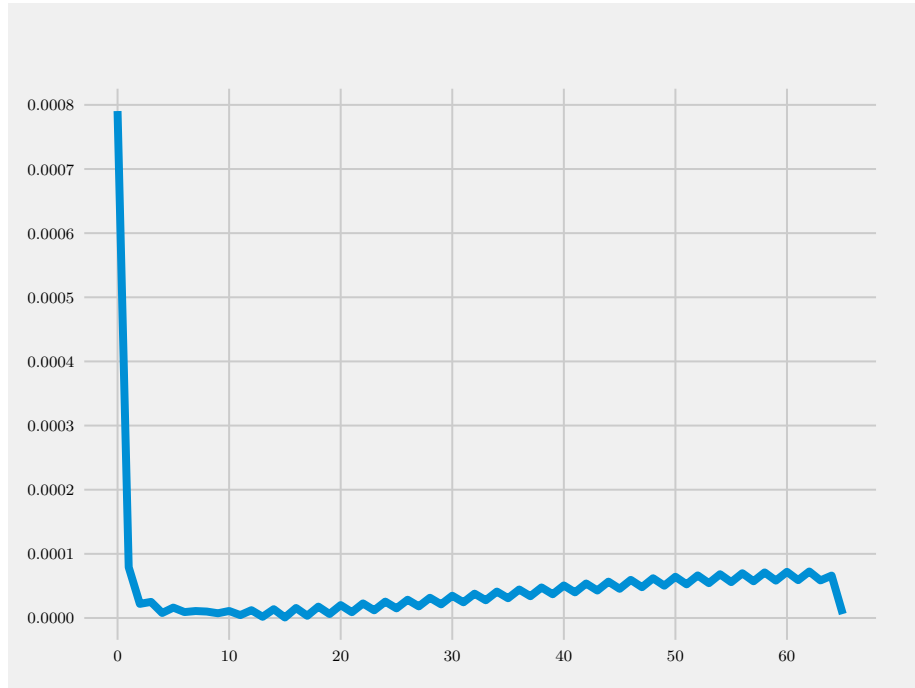


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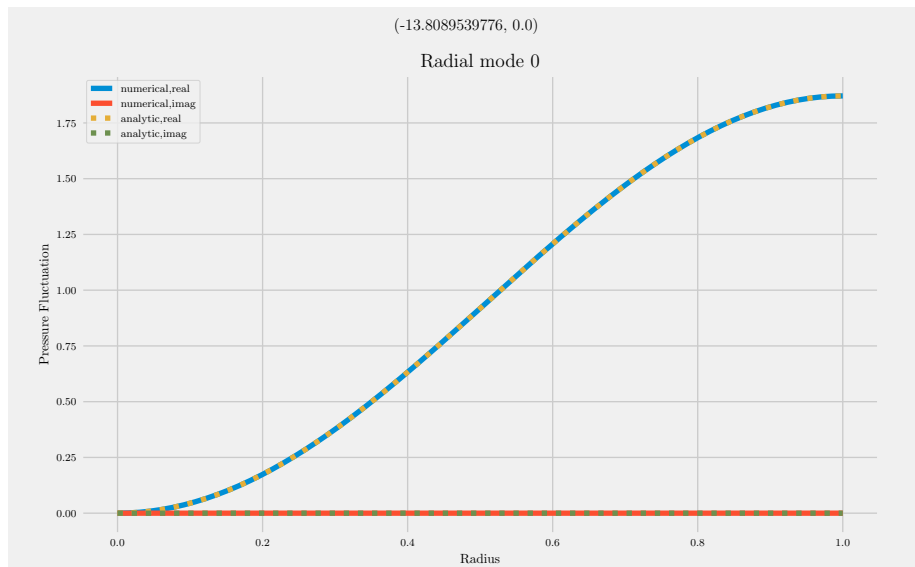


Figure 53: Fourth Order, Case number 1, 132points

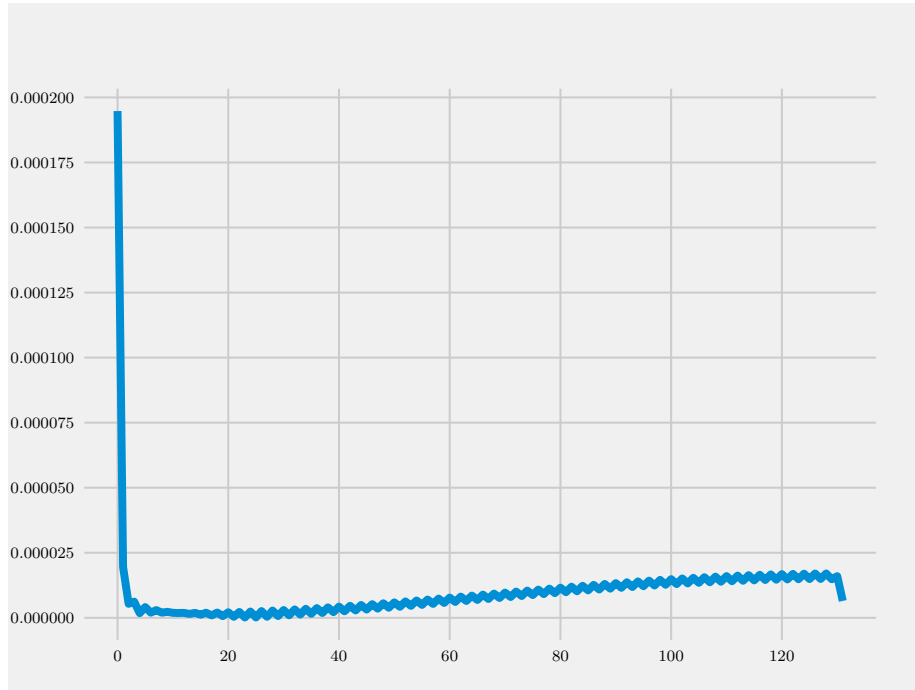


Figure 54: Fourth Order, Case number 1, 132points

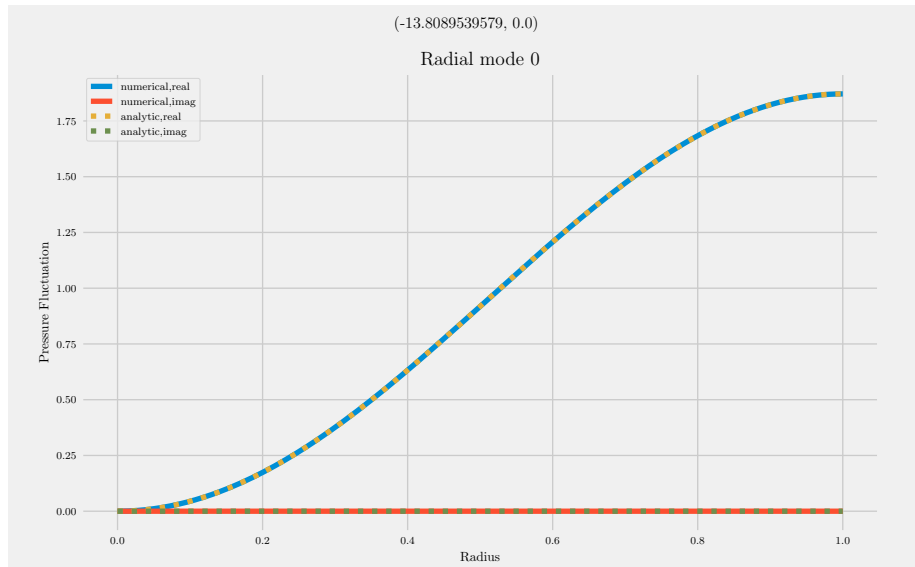


Figure 55: Fourth Order, Case number 1, 264points

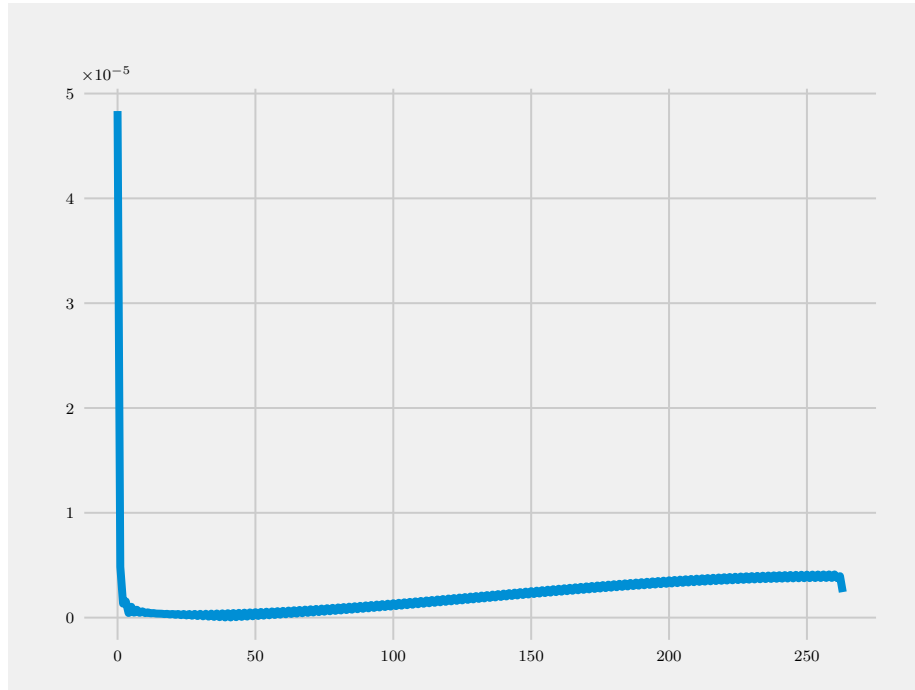


Figure 56: Fourth Order, Case number 1, 264points

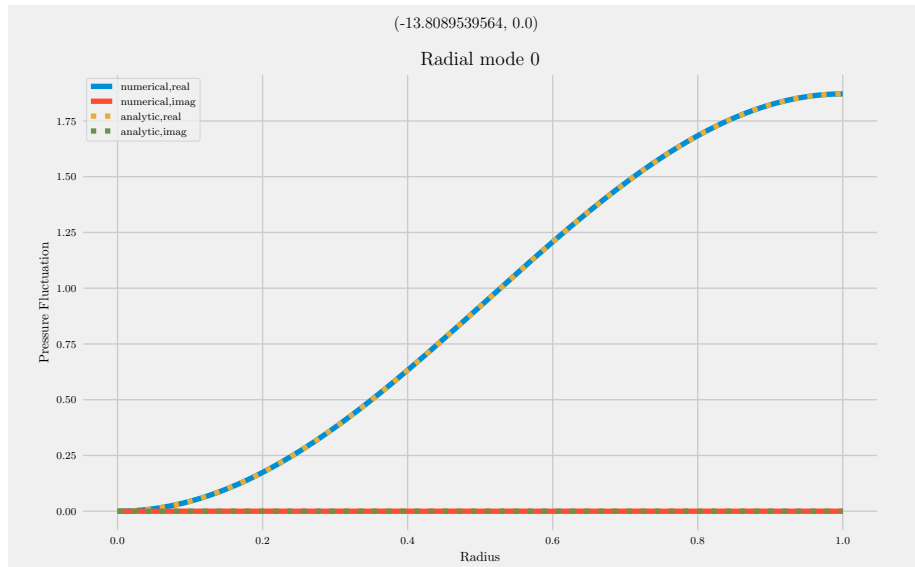


Figure 57: Fourth Order, Case number 1, 528points

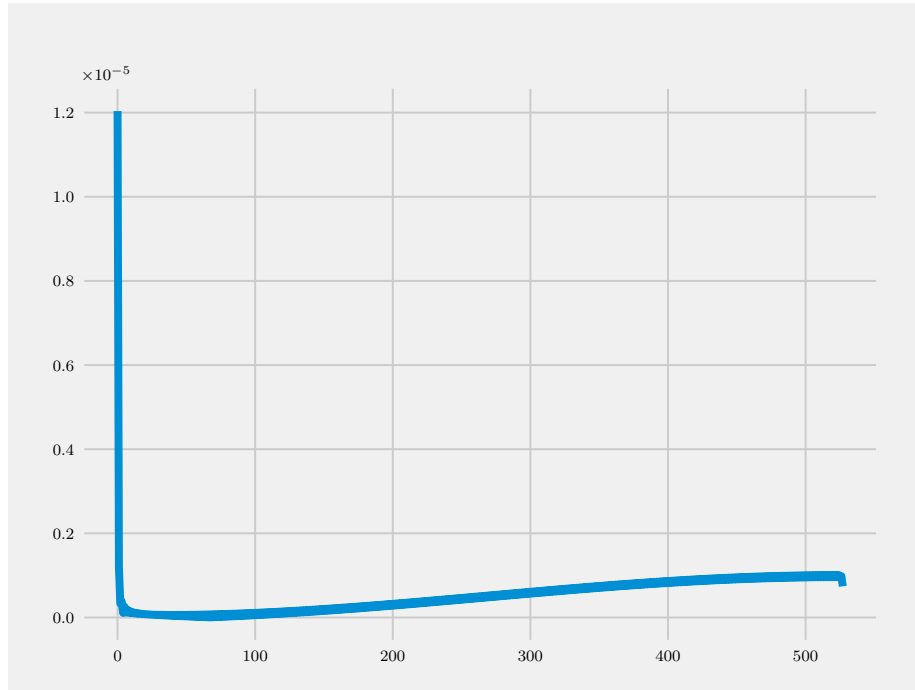


Figure 58: Fourth Order, Case number 1, 528points

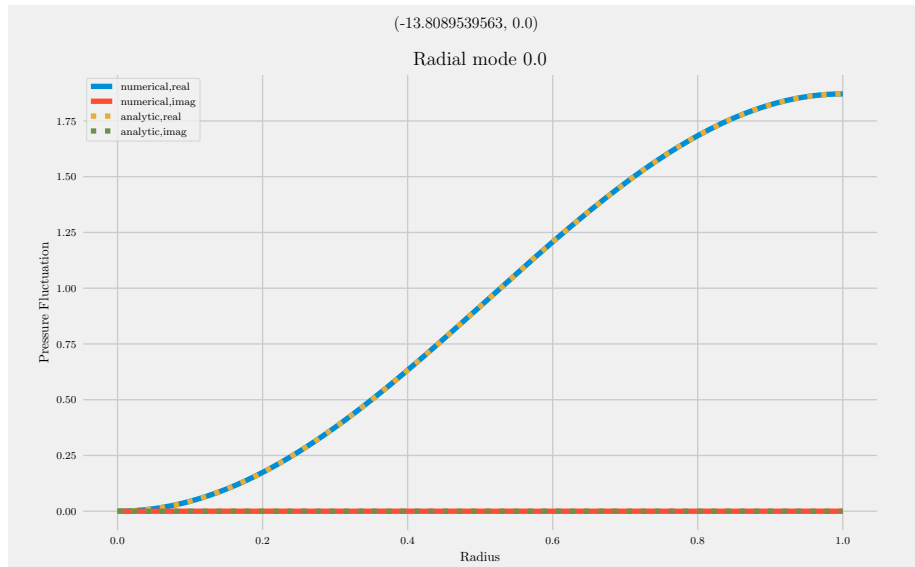


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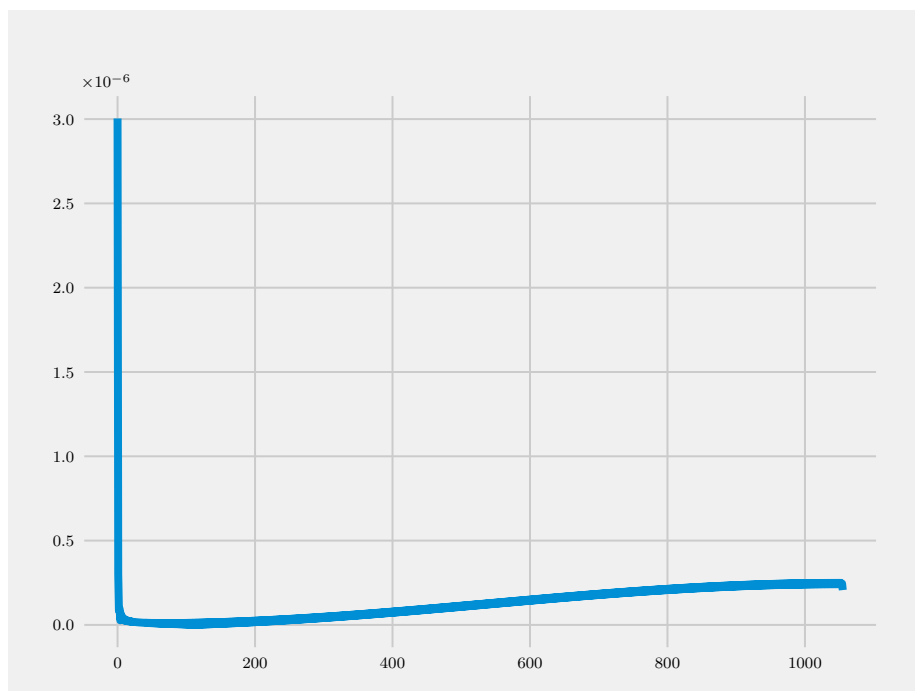


Figure 60: Fourth Order, Case number 1, 1056points

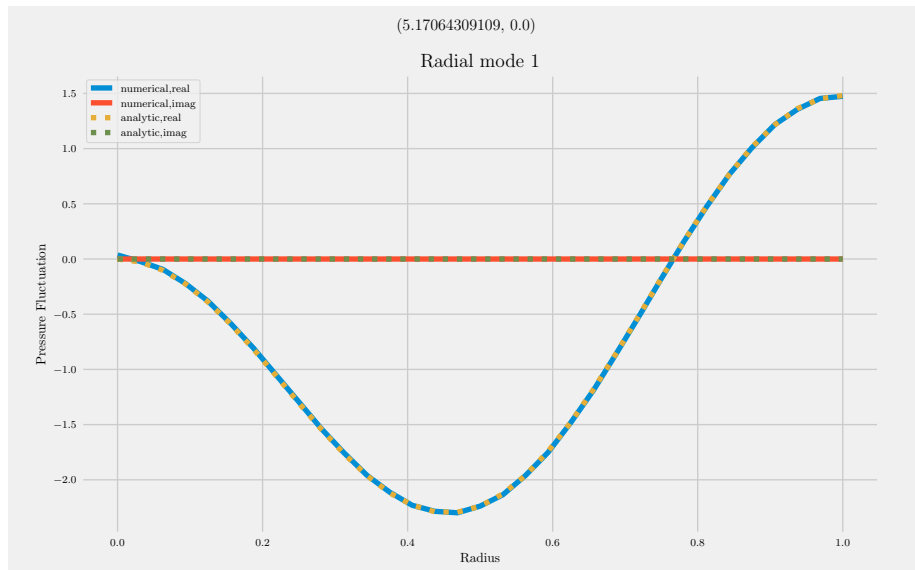


Figure 61: Second Order, Case number 0, 33points

Second Order, Radial Mode 1

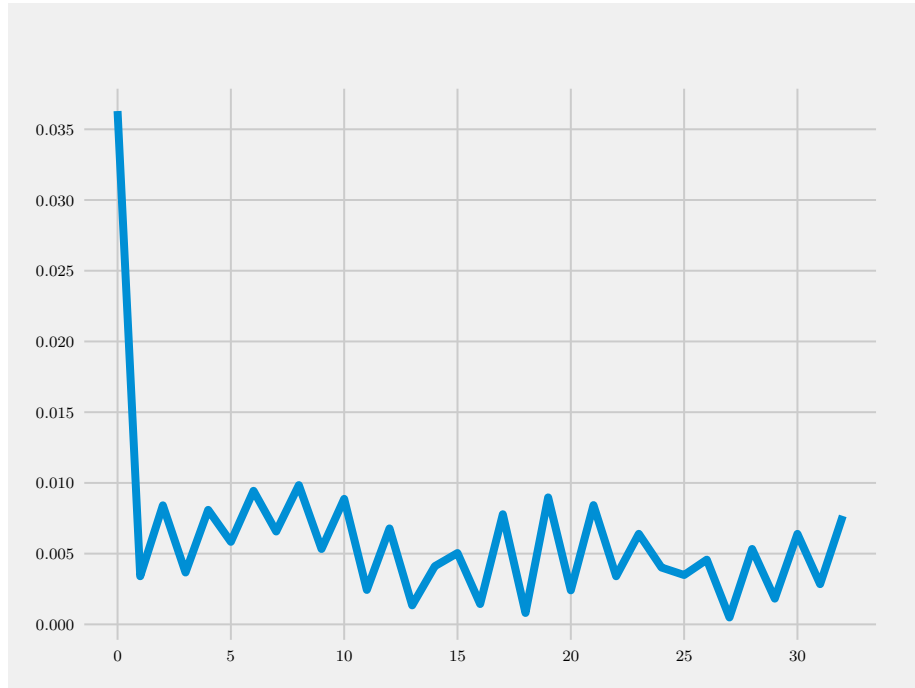


Figure 62: Second Order, Case number 0, 33points

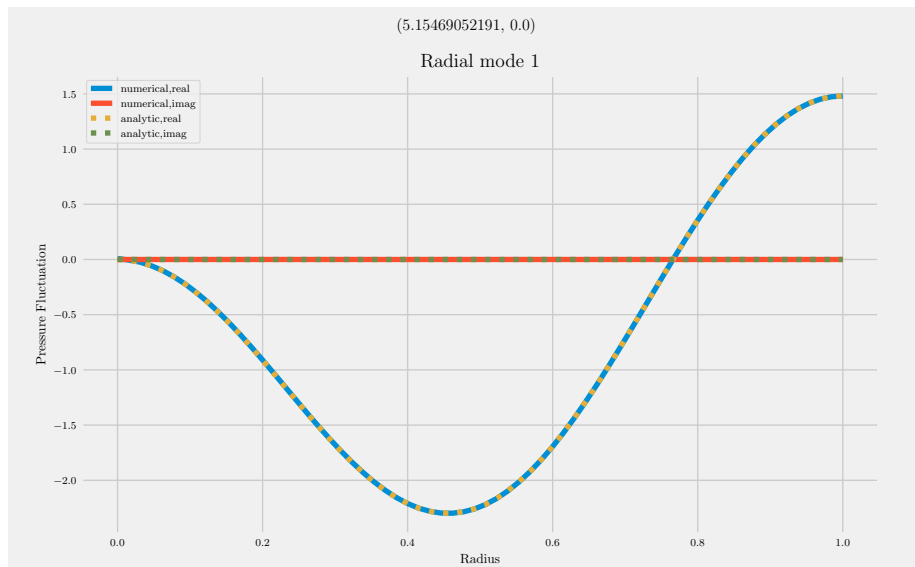


Figure 63: Second Order, Case number 0, 66points

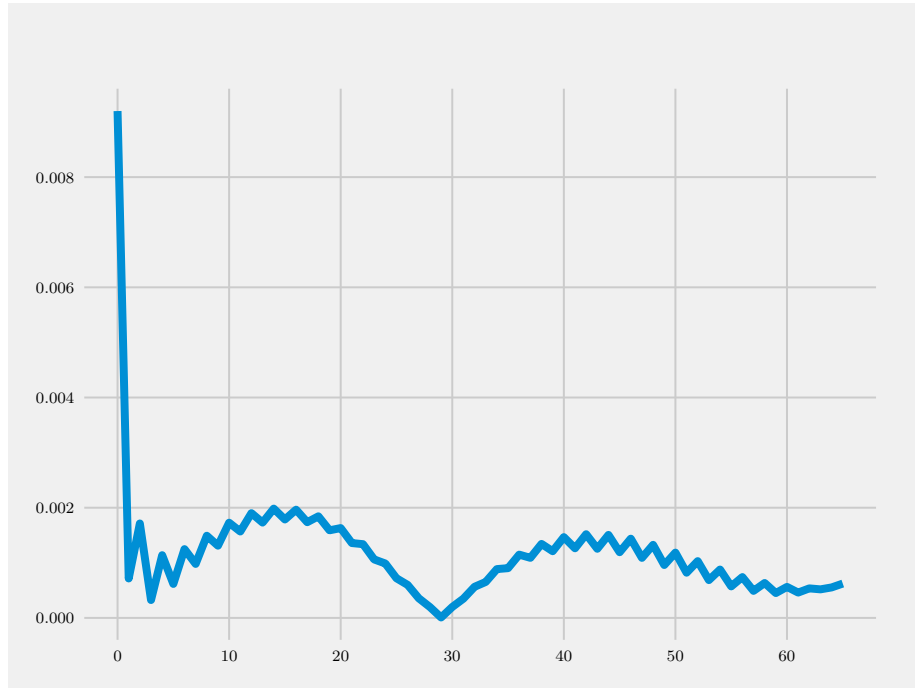


Figure 64: Second Order, Case number 0, 66points

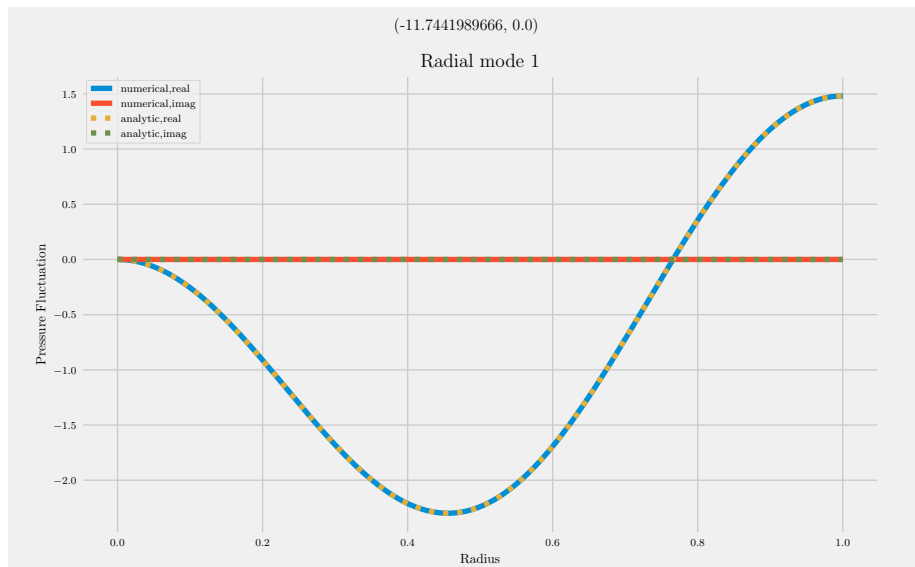


Figure 65: Second Order, Case number 0, 132points

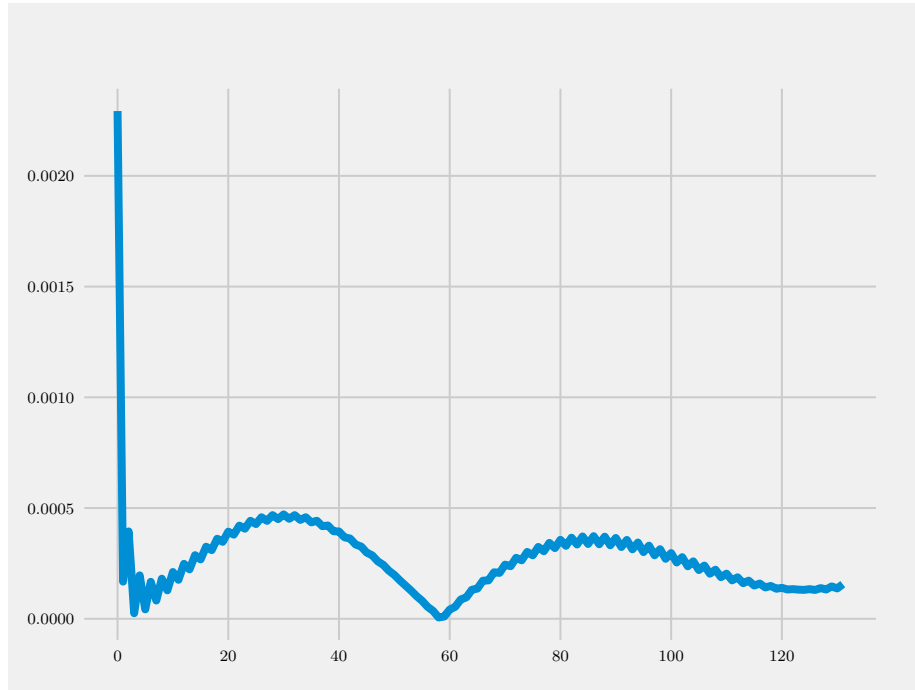


Figure 66: Second Order, Case number 0, 132points

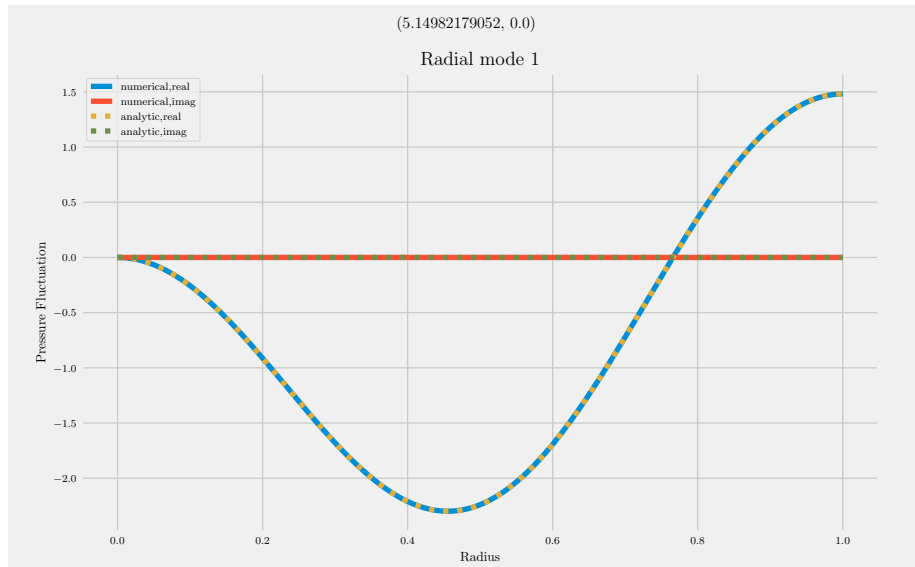


Figure 67: Second Order, Case number 0, 264points

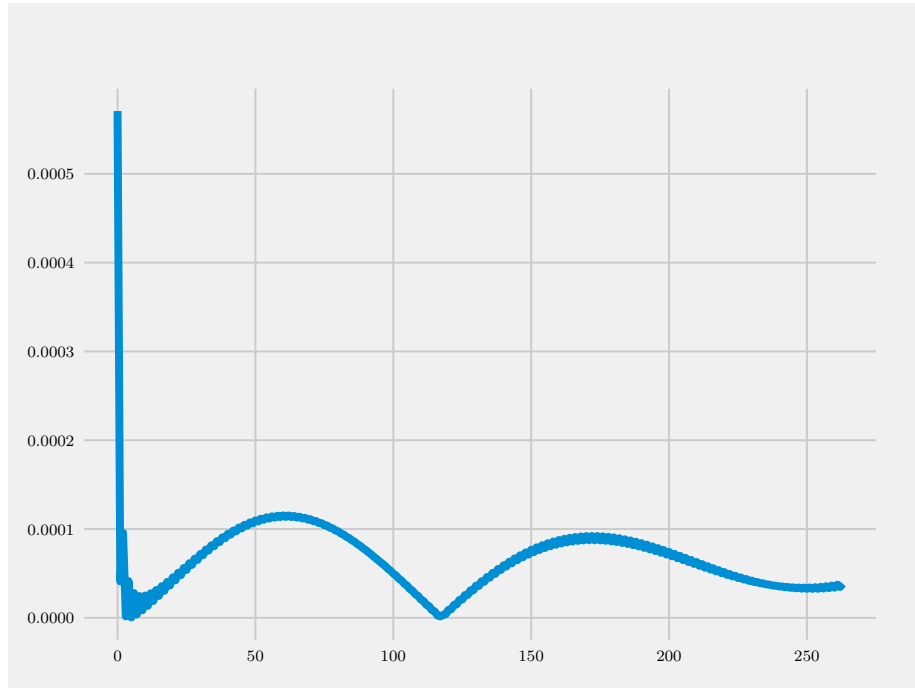


Figure 68: Second Order, Case number 0, 264points

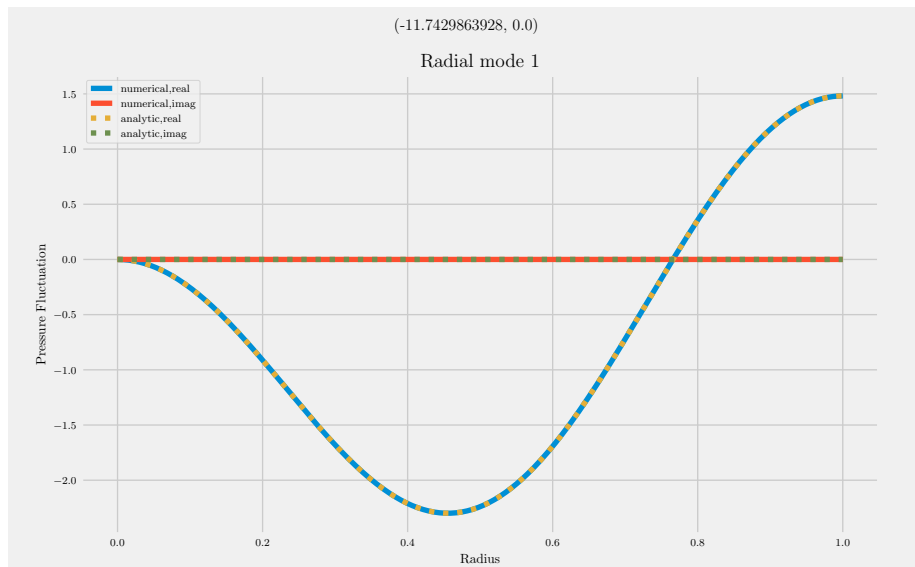


Figure 69: Second Order, Case number 0, 528points

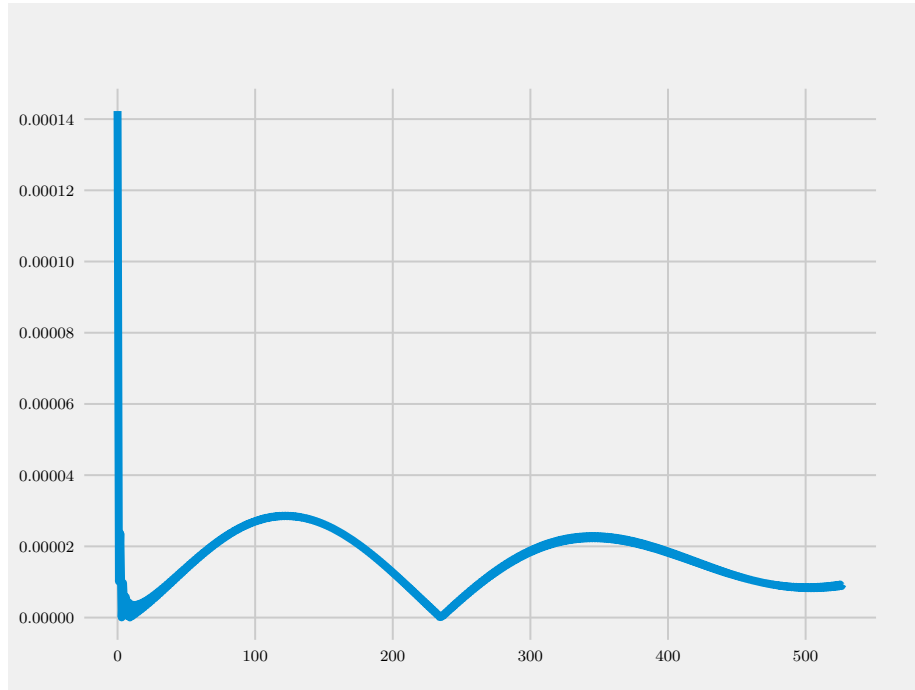


Figure 70: Second Order, Case number 0, 528points

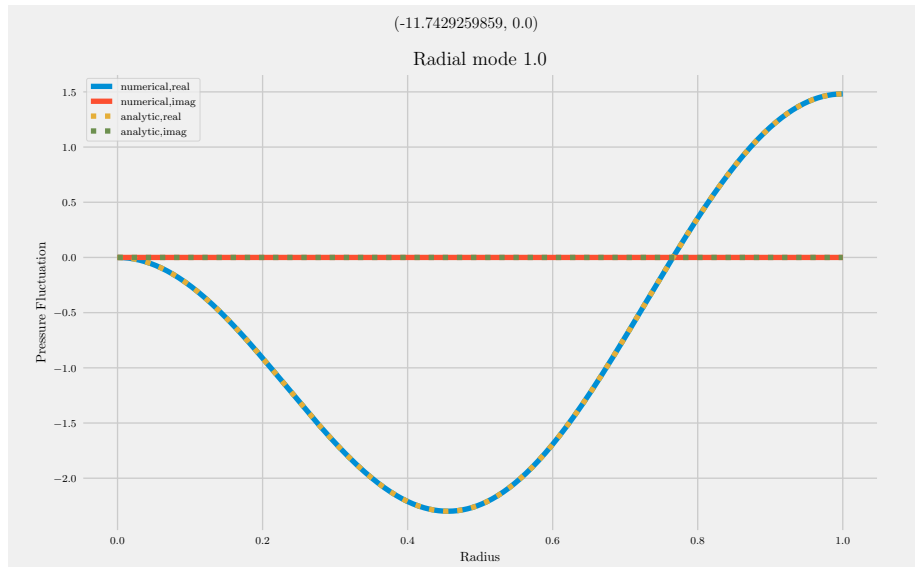


Figure 71: Second Order, Case number 0, 1056points

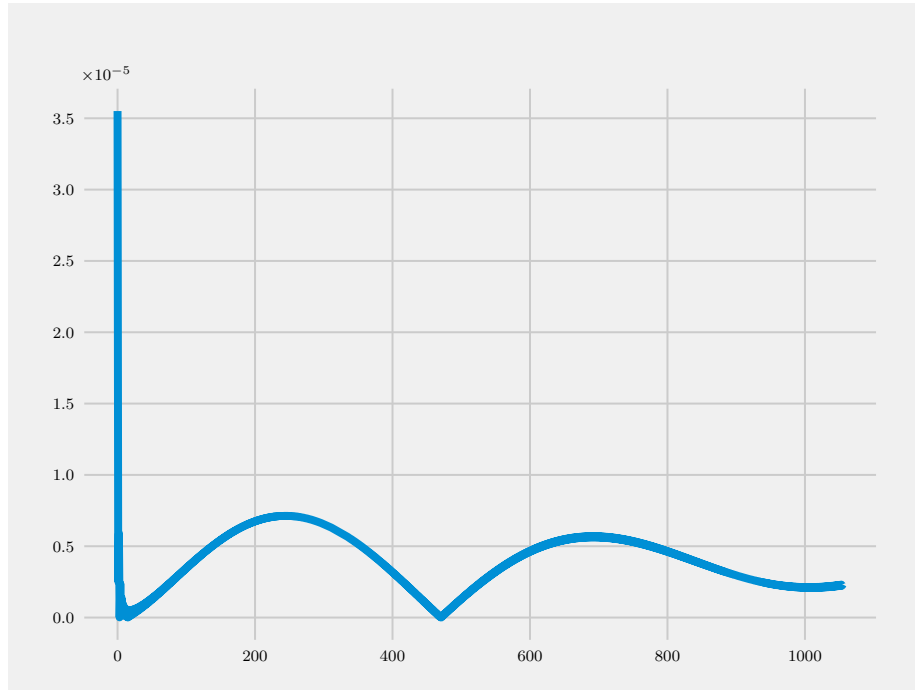


Figure 72: Second Order, Case number 0, 1056points

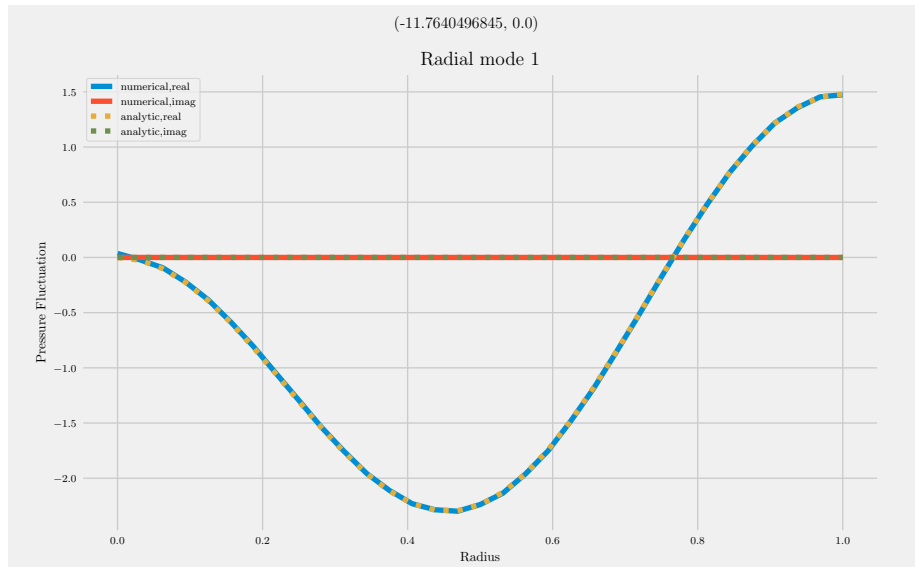


Figure 73: Second Order, Case number 1, 33points

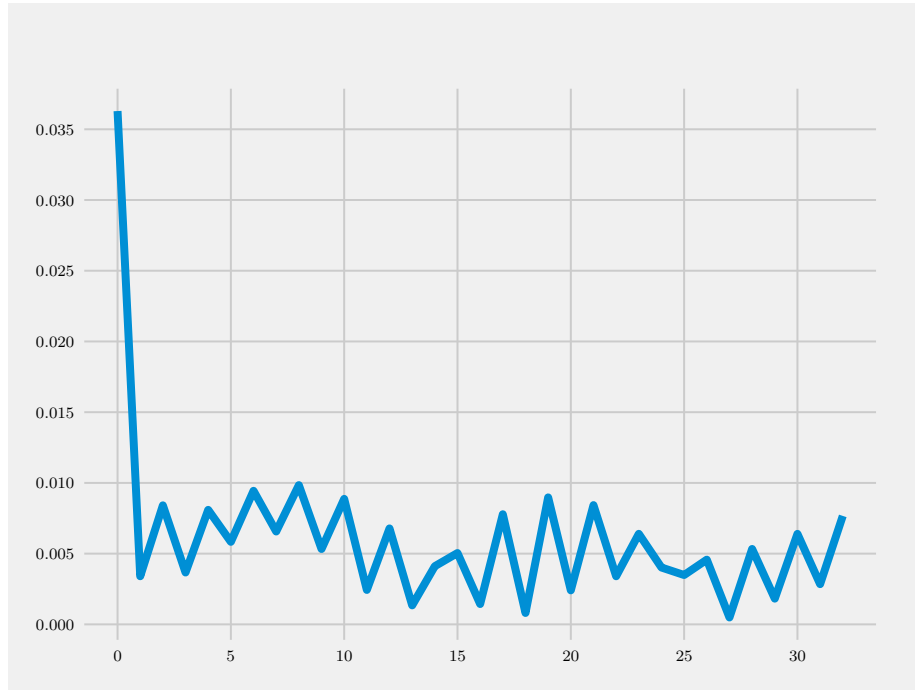


Figure 74: Second Order, Case number 1, 33points

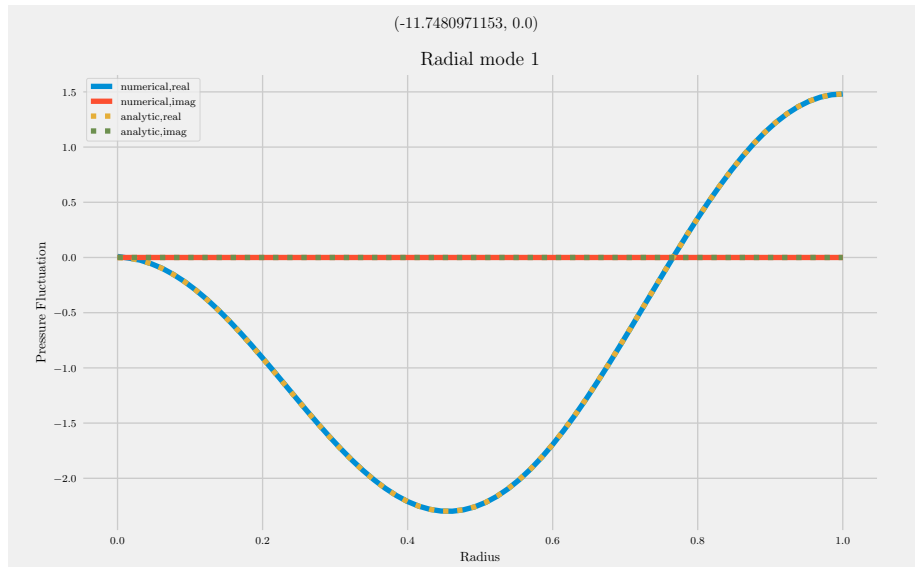


Figure 75: Second Order, Case number 1, 66points

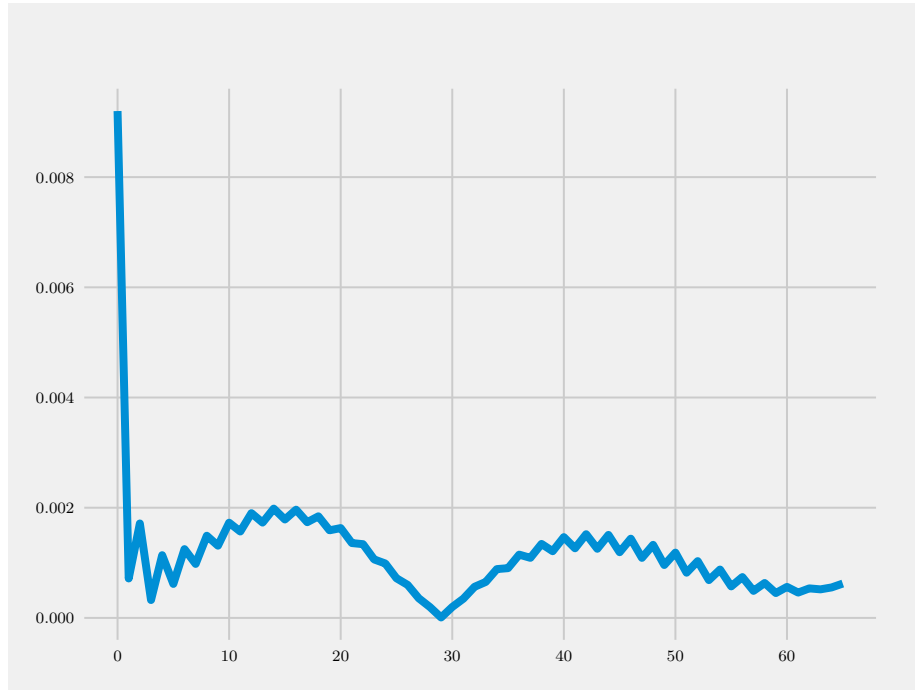


Figure 76: Second Order, Case number 1, 66points

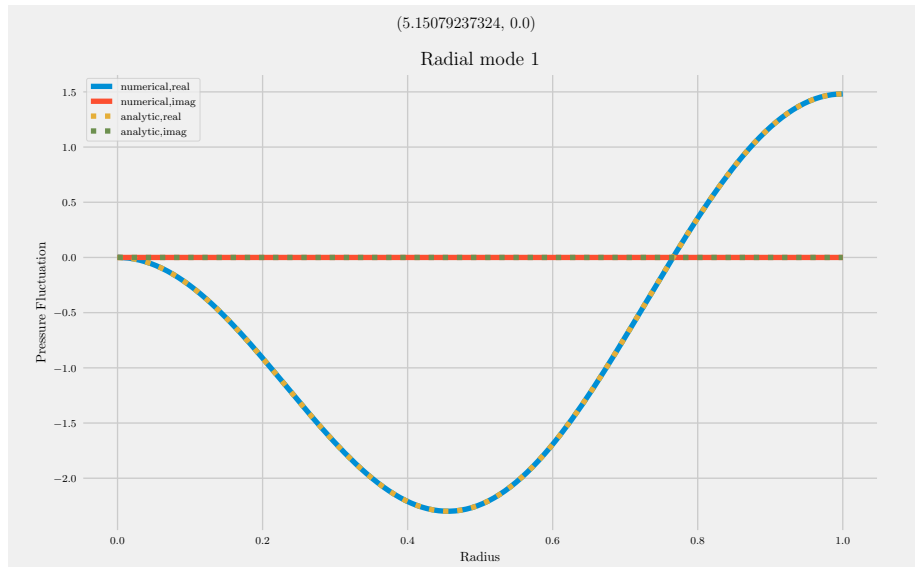


Figure 77: Second Order, Case number 1, 132points

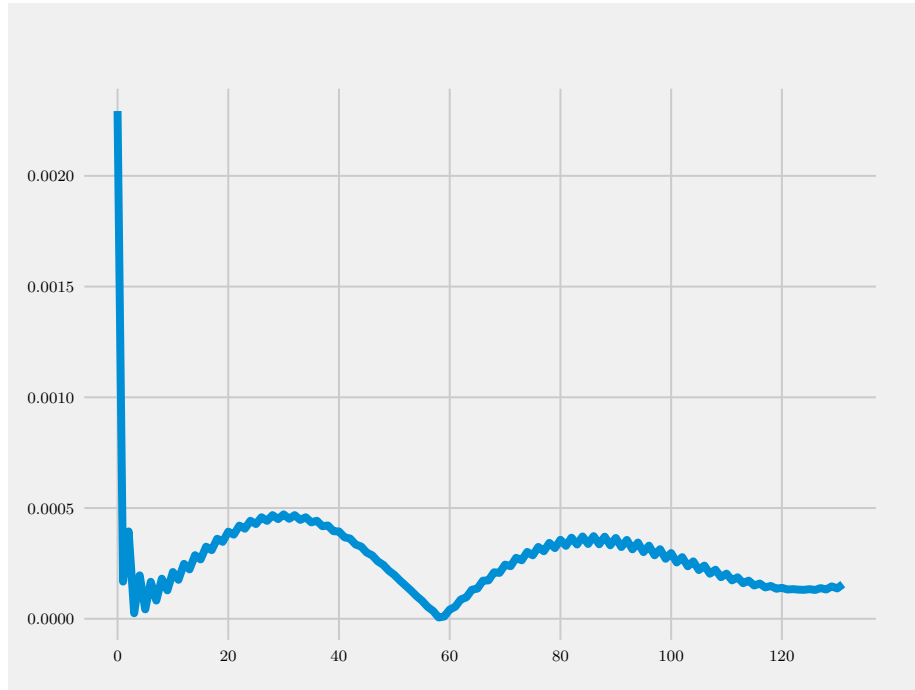


Figure 78: Second Order, Case number 1, 132points

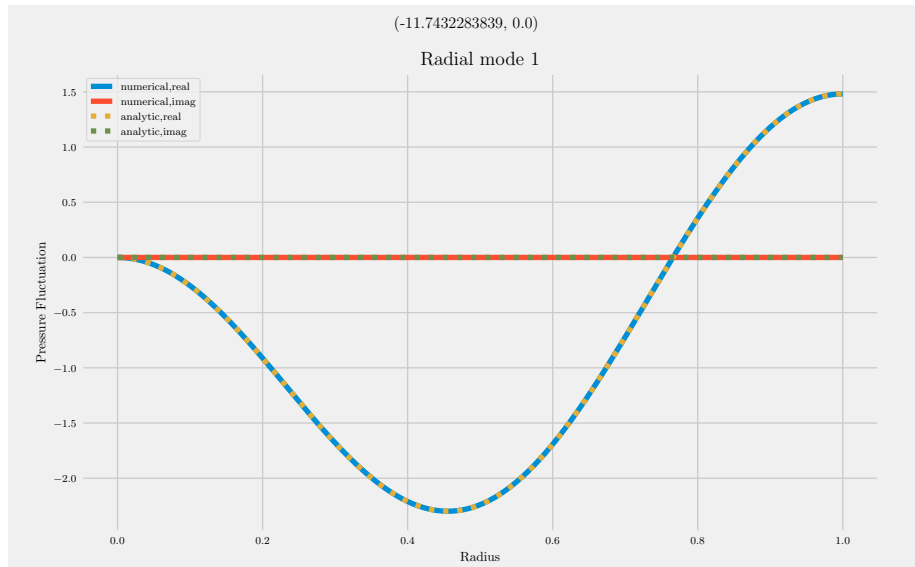


Figure 79: Second Order, Case number 1, 264points

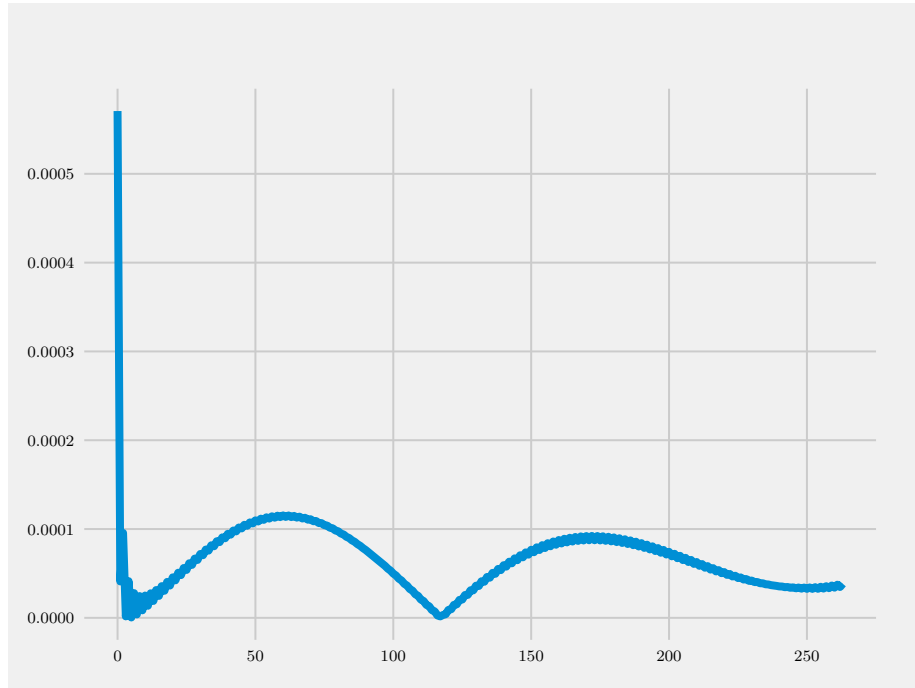


Figure 80: Second Order, Case number 1, 264points

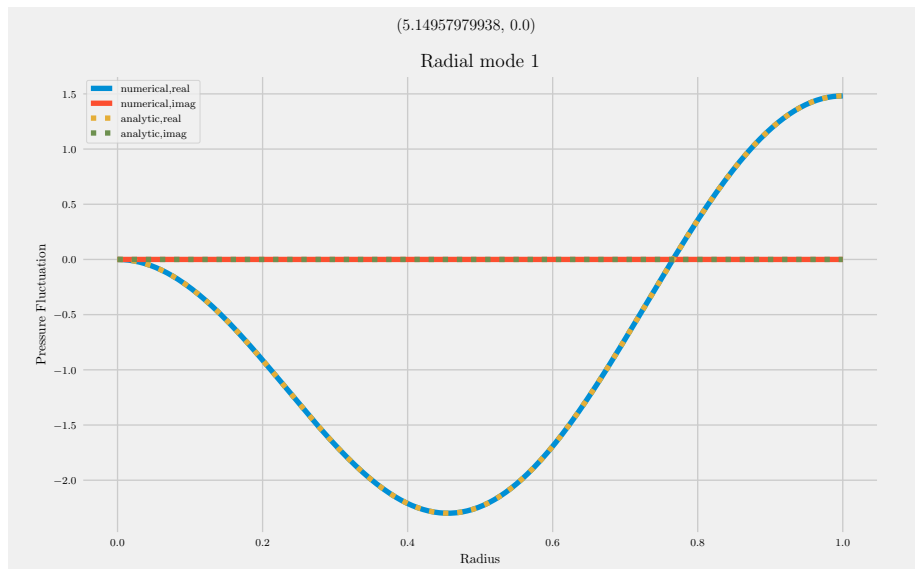


Figure 81: Second Order, Case number 1, 528points

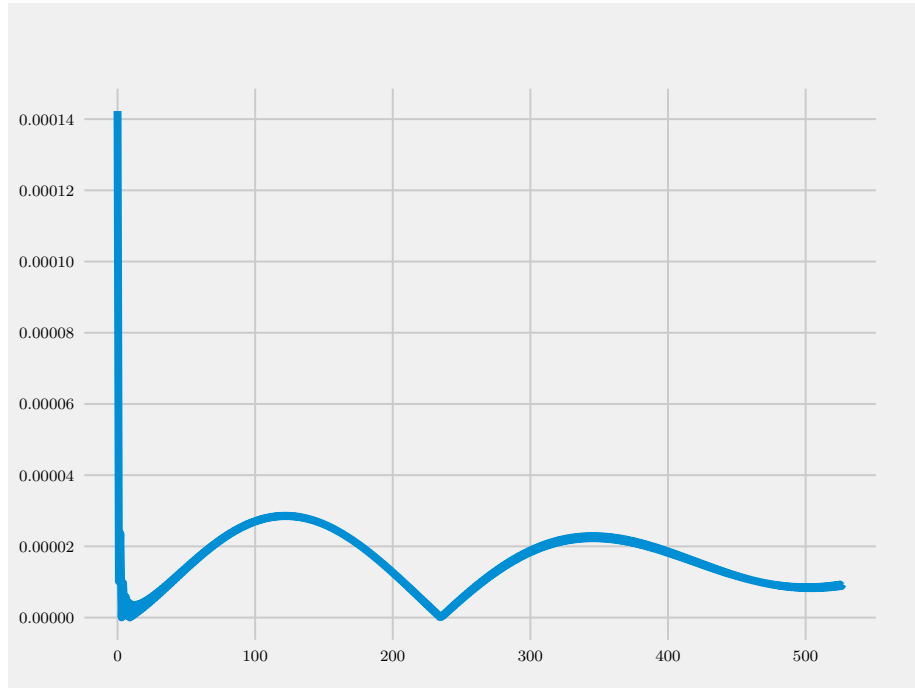


Figure 82: Second Order, Case number 1, 528points

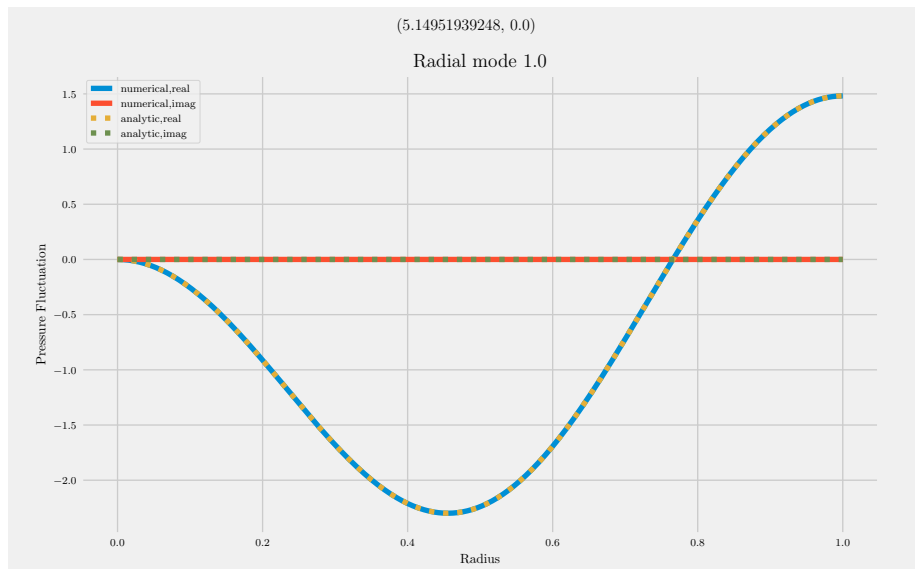


Figure 83: Second Order, Case number 1, 1056points

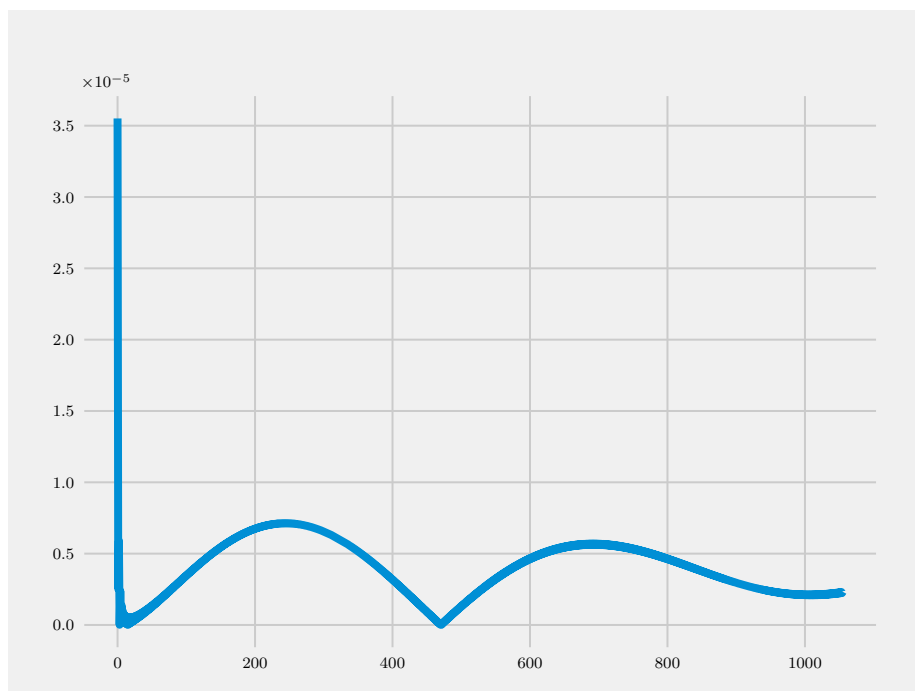


Figure 84: Second Order, Case number 1, 1056points

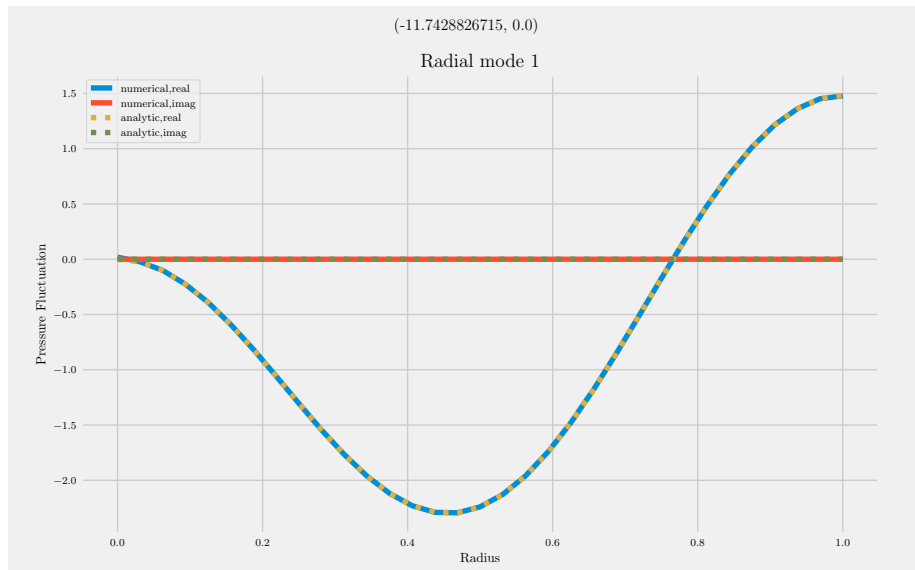


Figure 85: Fourth Order, Case number 0, 33points

Fourth Order, Radial Mode 1

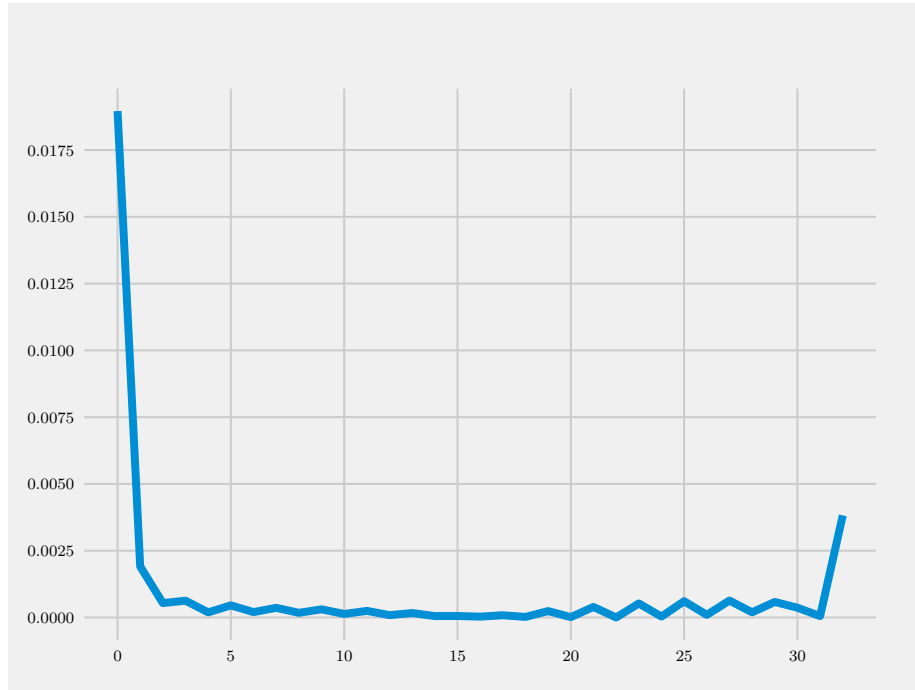


Figure 86: Fourth Order, Case number 0, 33points

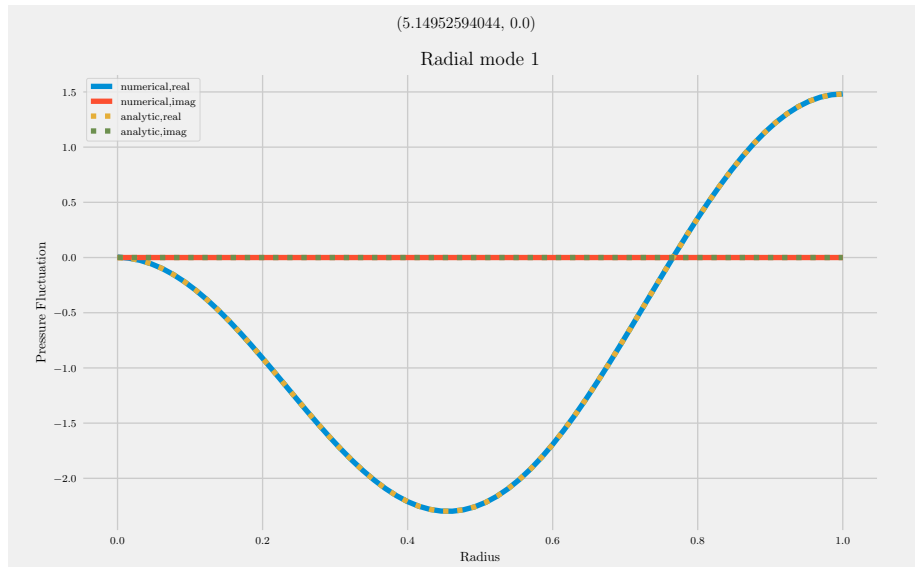


Figure 87: Fourth Order, Case number 0, 66points

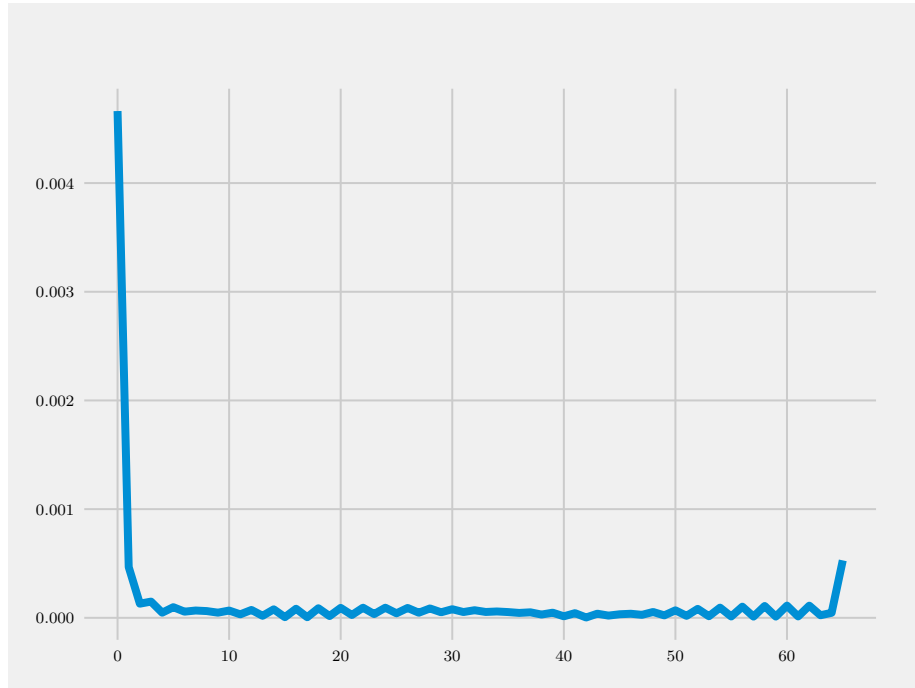


Figure 88: Fourth Order, Case number 0, 66points

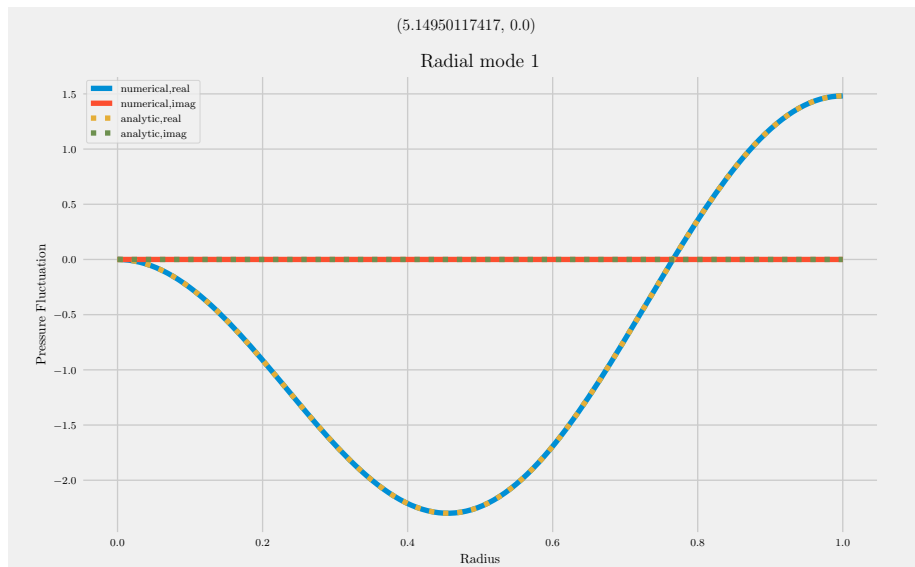


Figure 89: Fourth Order, Case number 0, 132points

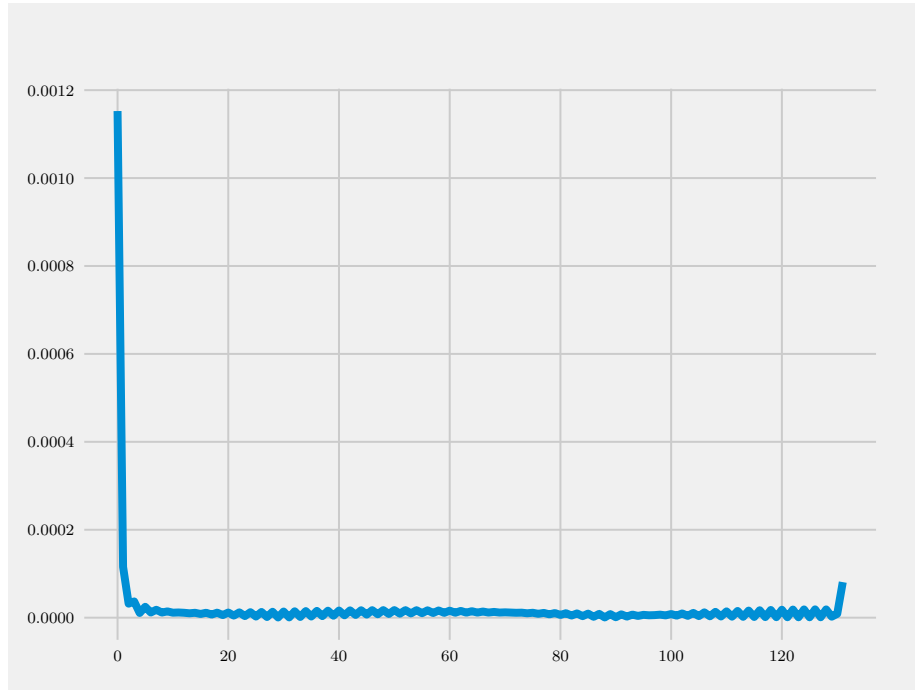


Figure 90: Fourth Order, Case number 0, 132points

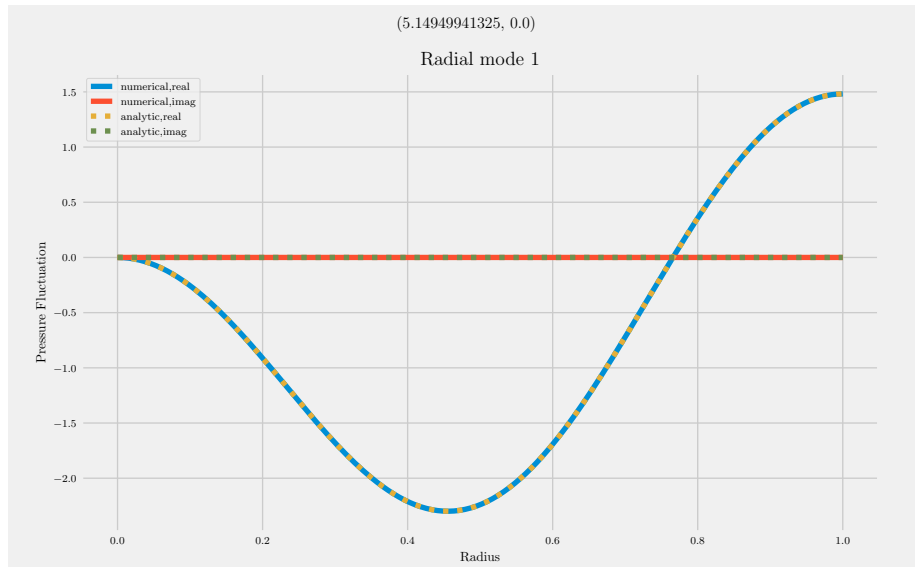


Figure 91: Fourth Order, Case number 0, 264points

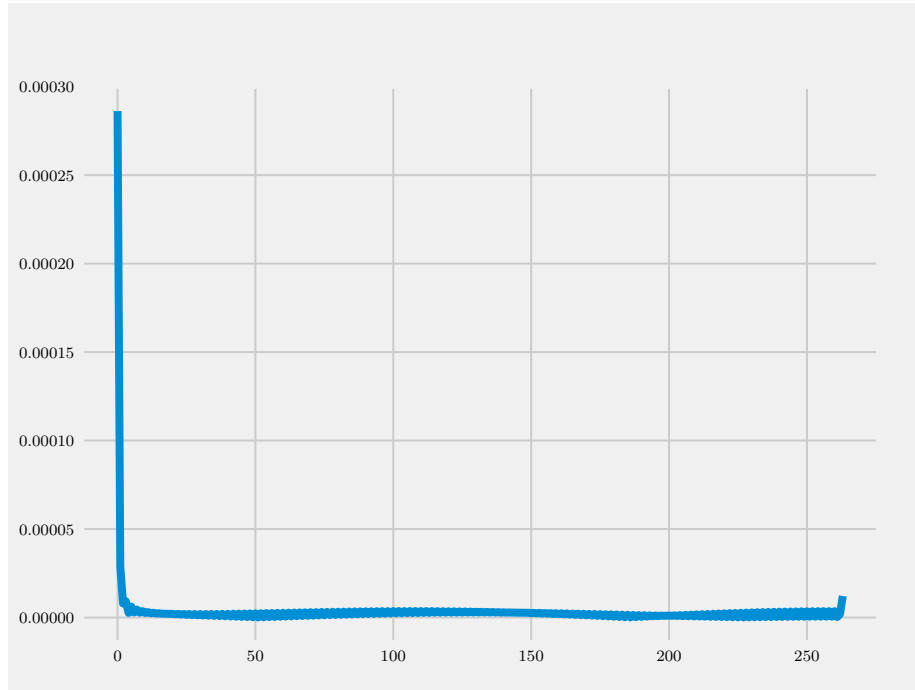


Figure 92: Fourth Order, Case number 0, 264points

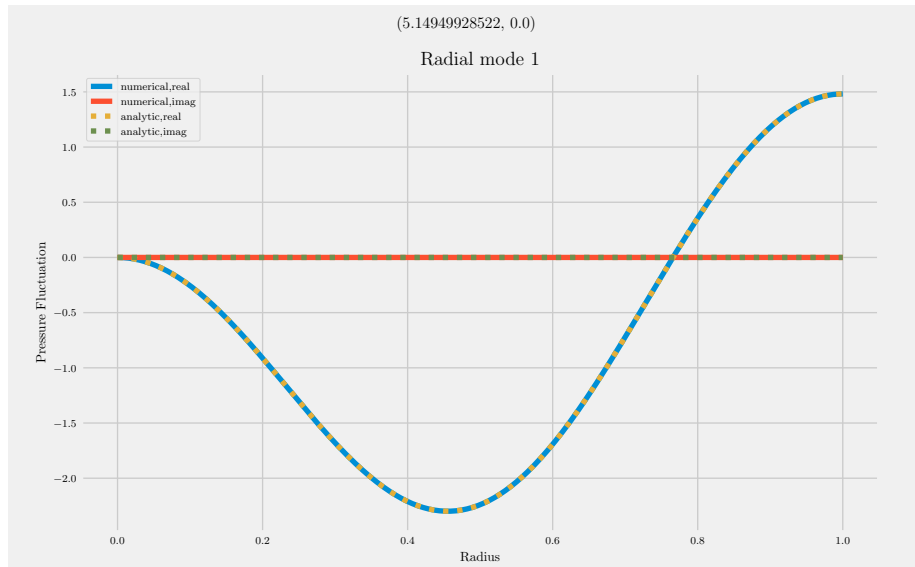


Figure 93: Fourth Order, Case number 0, 528points

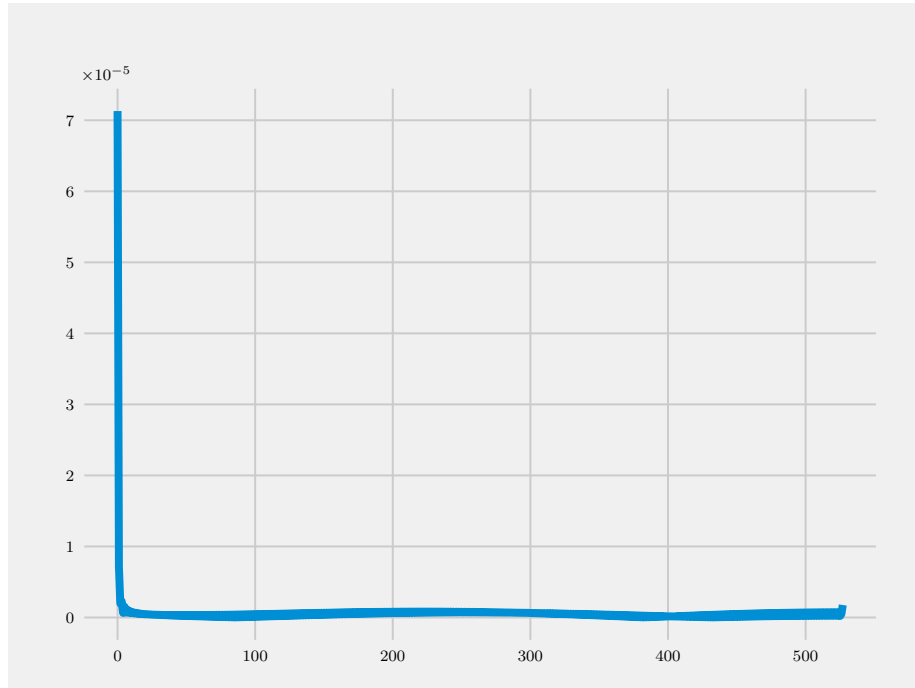


Figure 94: Fourth Order, Case number 0, 528points

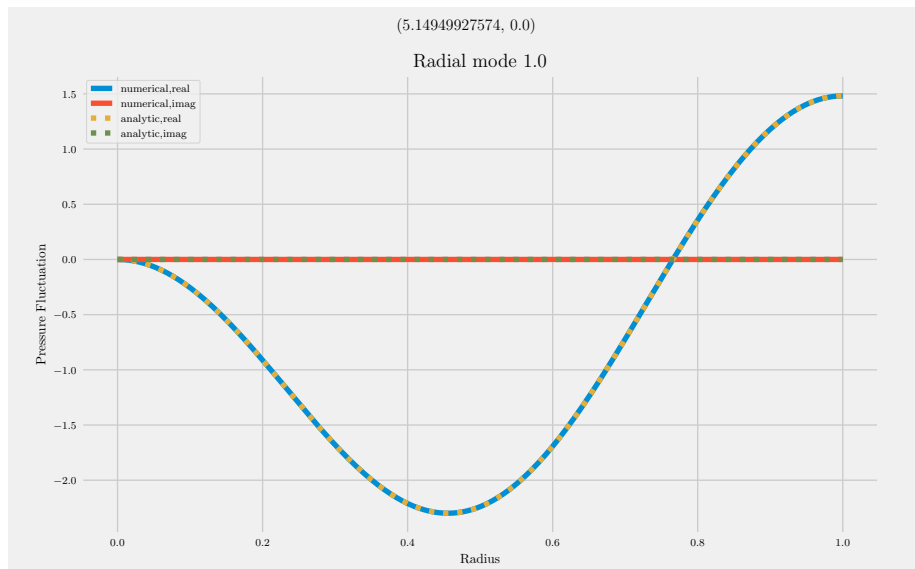


Figure 95: Fourth Order, Case number 0, 1056points

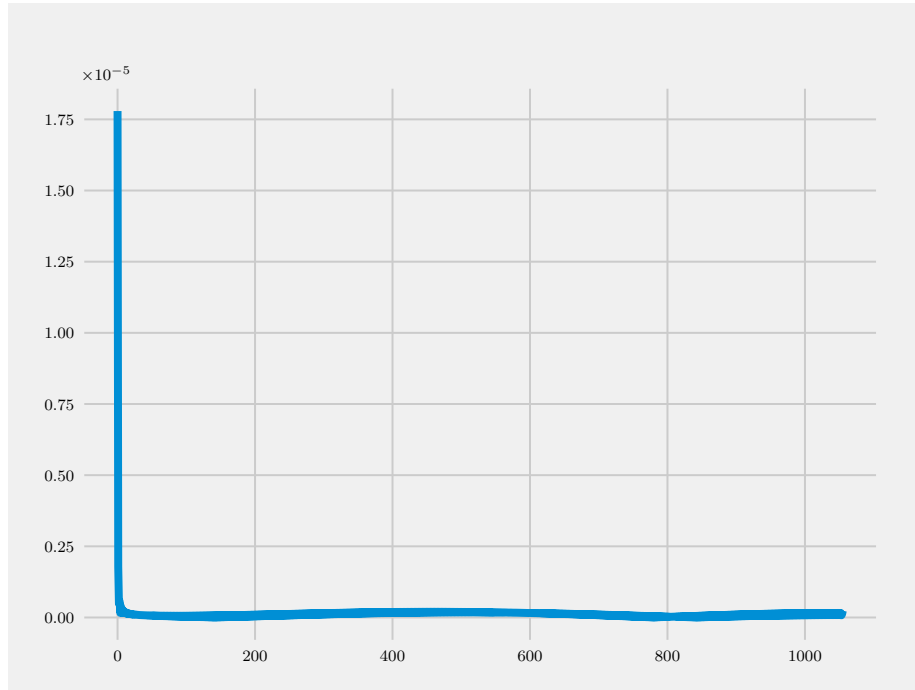


Figure 96: Fourth Order, Case number 0, 1056points

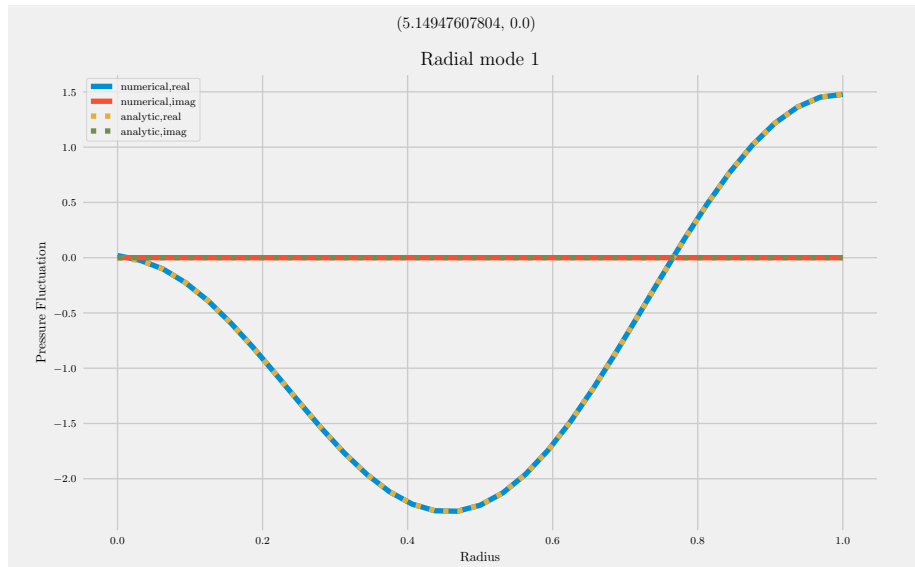


Figure 97: Fourth Order, Case number 1, 33points

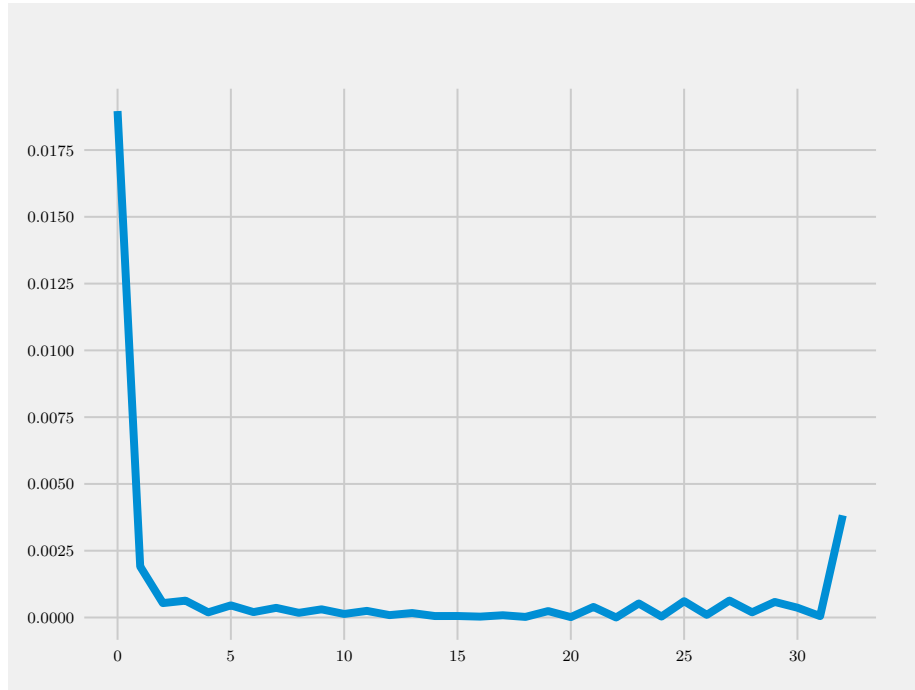


Figure 98: Fourth Order, Case number 1, 33points

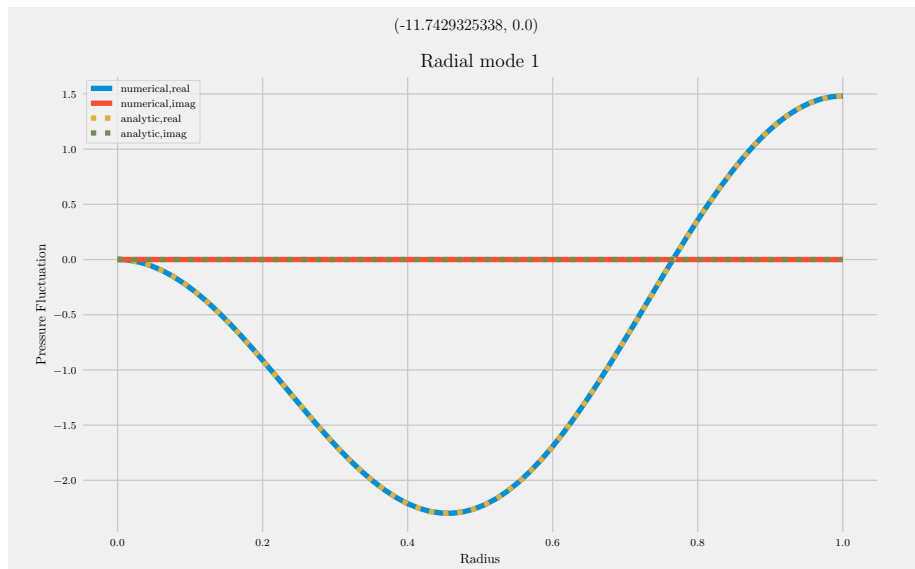


Figure 99: Fourth Order, Case number 1, 66points

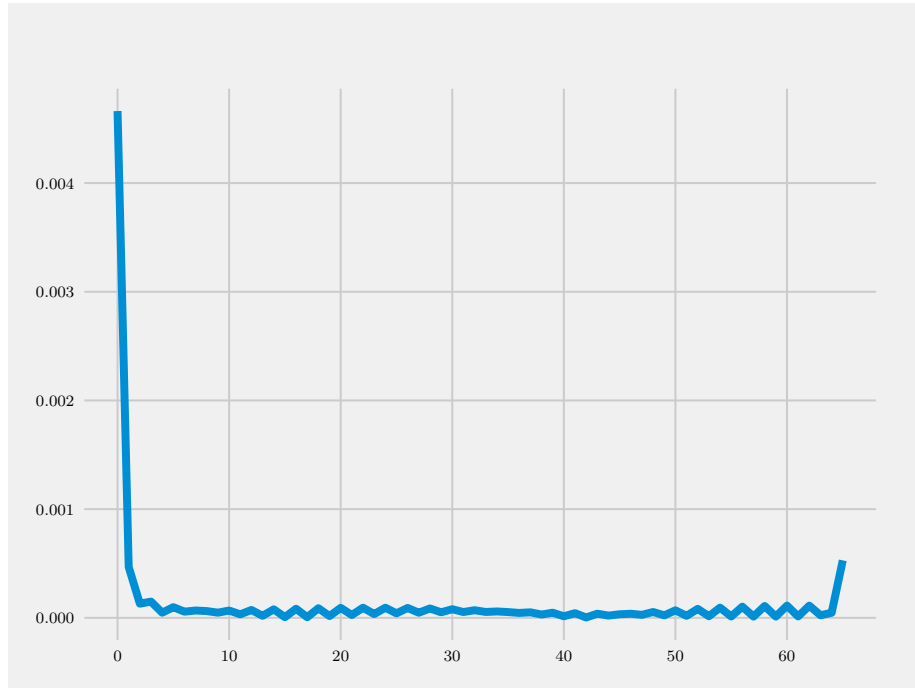


Figure 100: Fourth Order, Case number 1, 66points

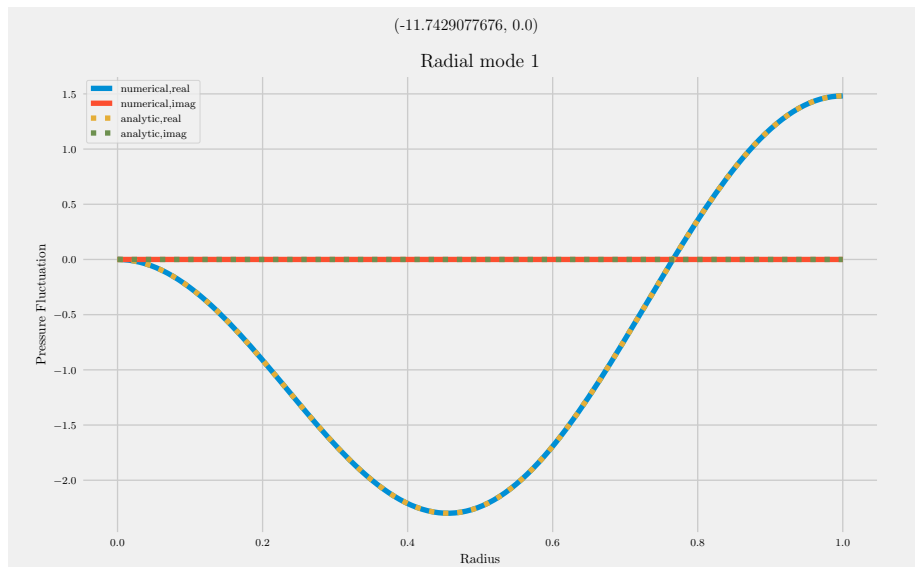


Figure 101: Fourth Order, Case number 1, 132points

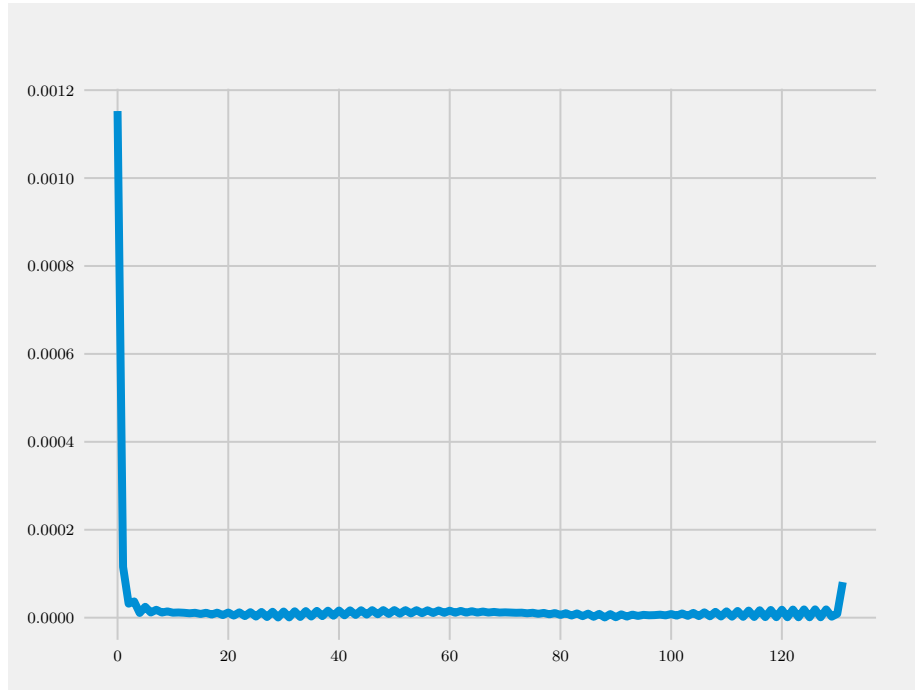


Figure 102: Fourth Order, Case number 1, 132points

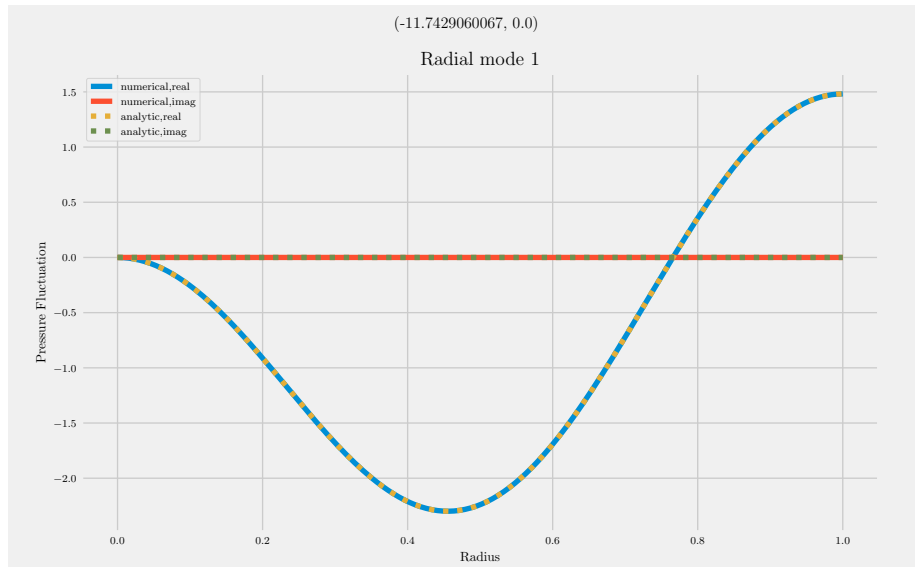


Figure 103: Fourth Order, Case number 1, 264points

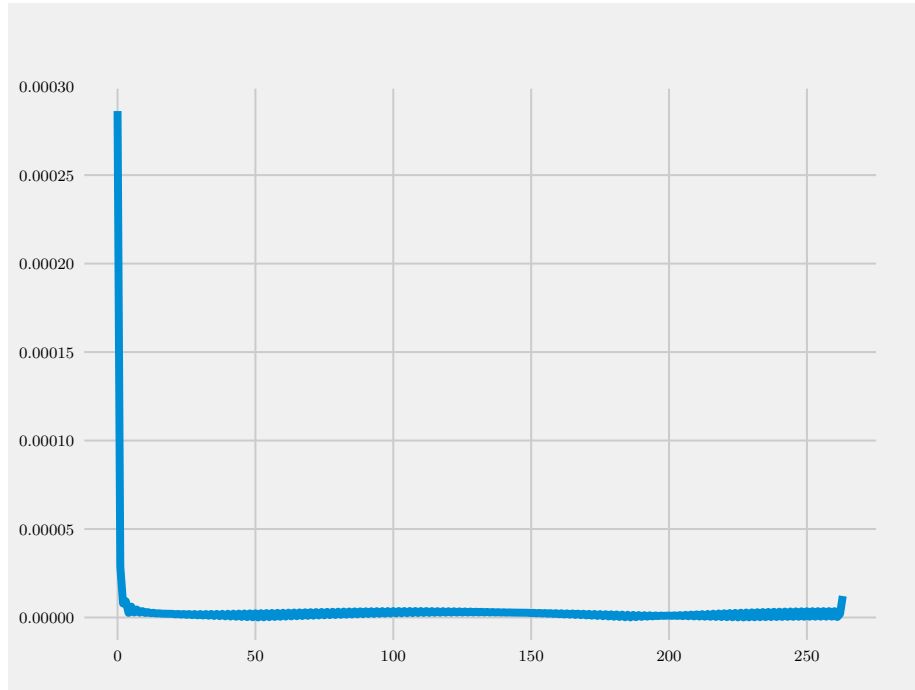


Figure 104: Fourth Order, Case number 1, 264points

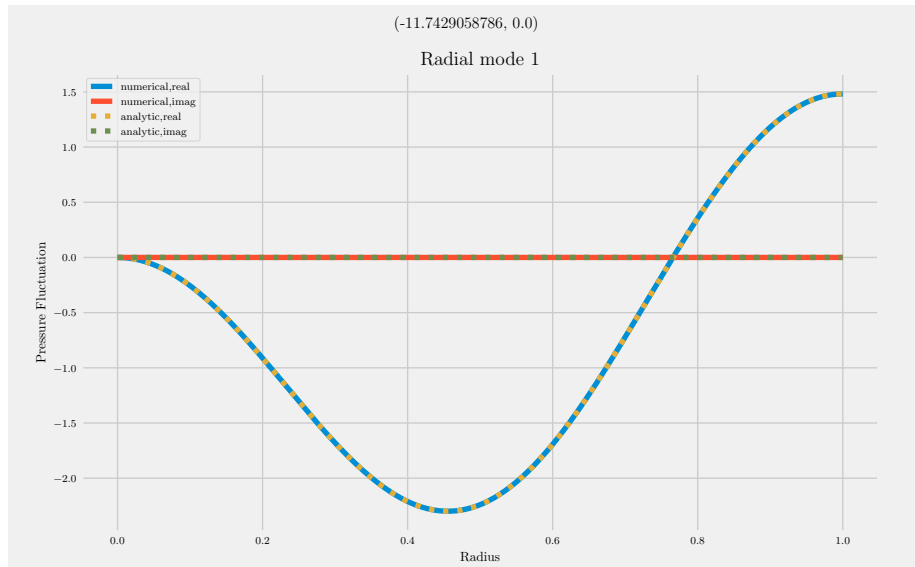


Figure 105: Fourth Order, Case number 1, 528points

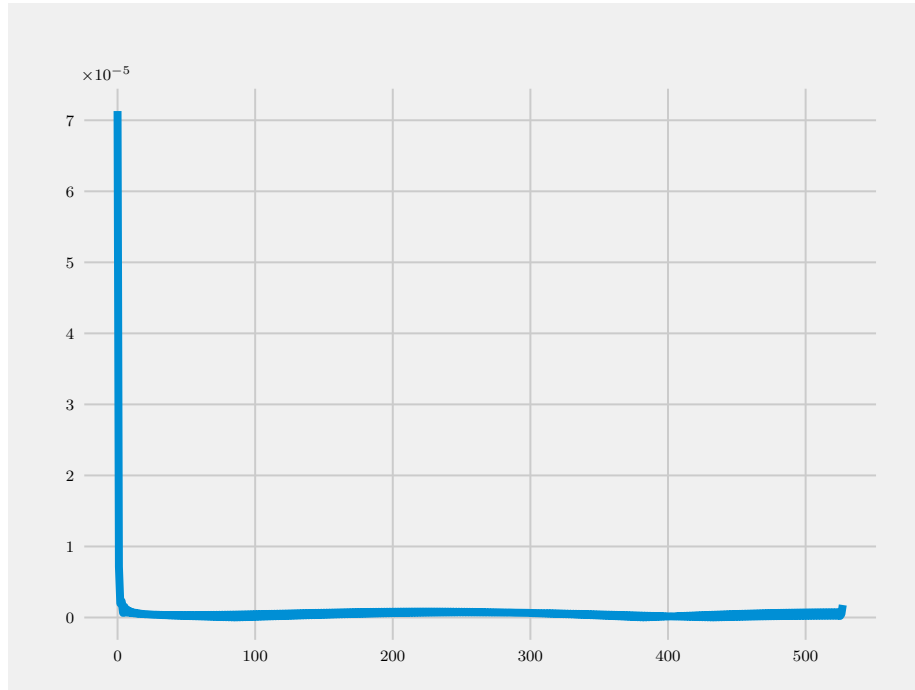


Figure 106: Fourth Order, Case number 1, 528points

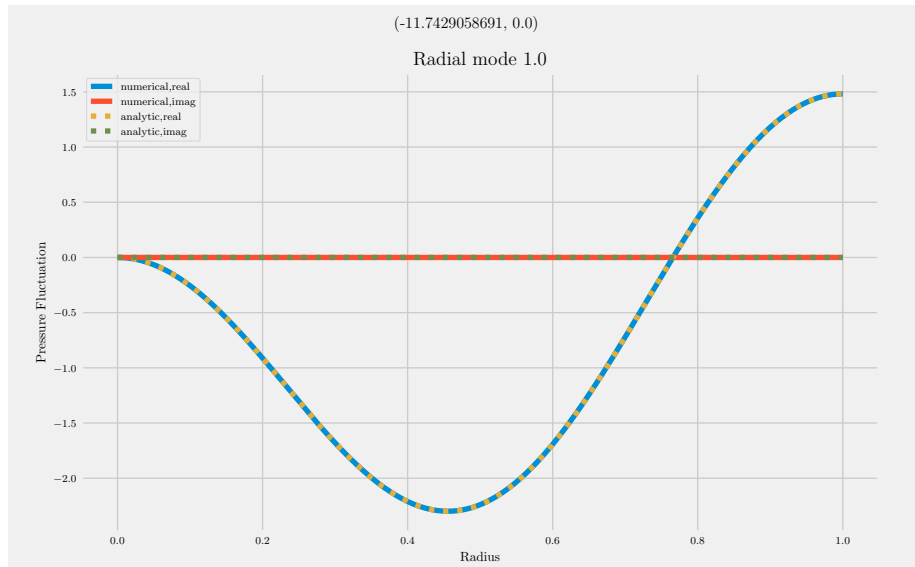


Figure 107: Fourth Order, Case number 1, 1056points

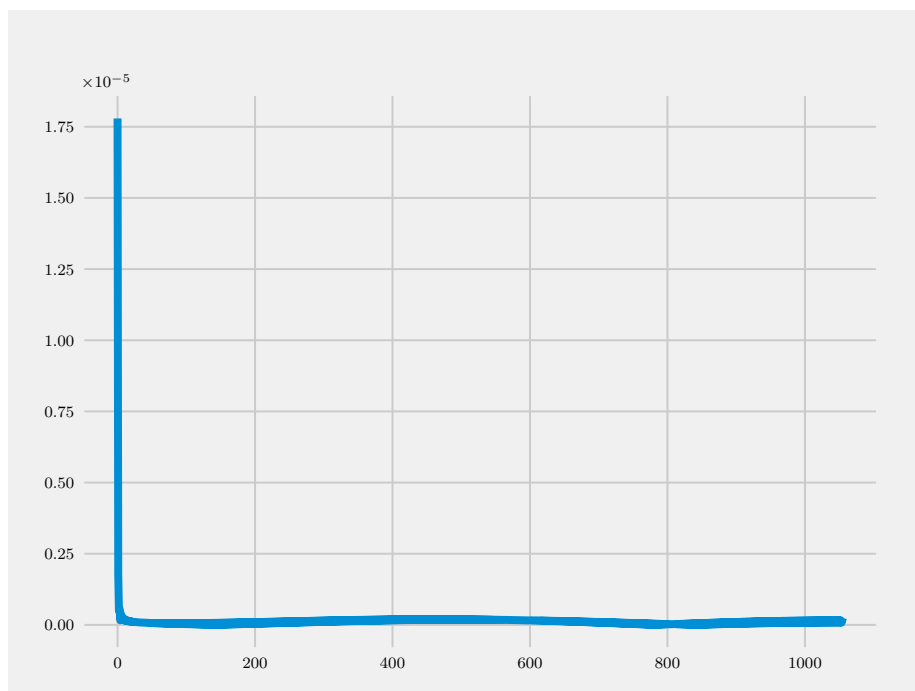


Figure 108: Fourth Order, Case number 1, 1056points

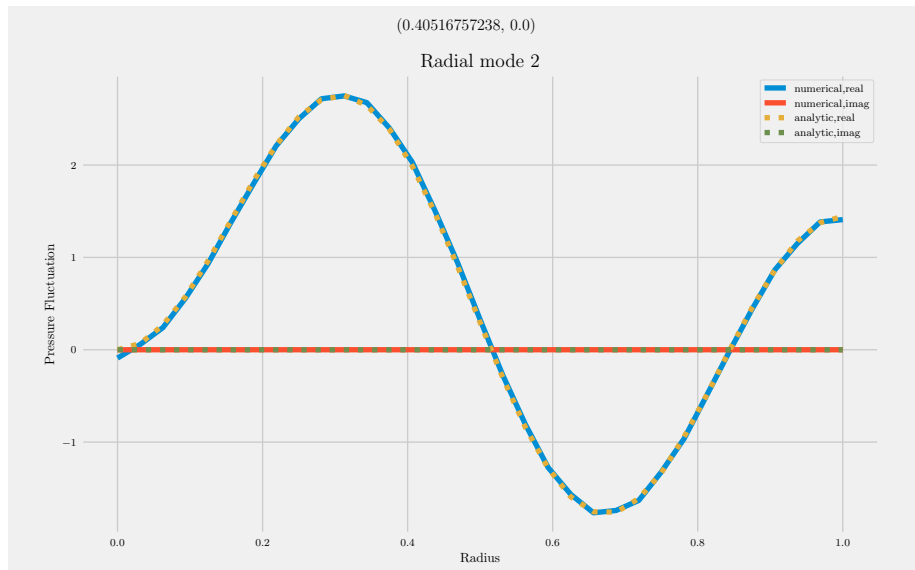


Figure 109: Second Order, Case number 0, 33points

Second Order, Radial Mode 2

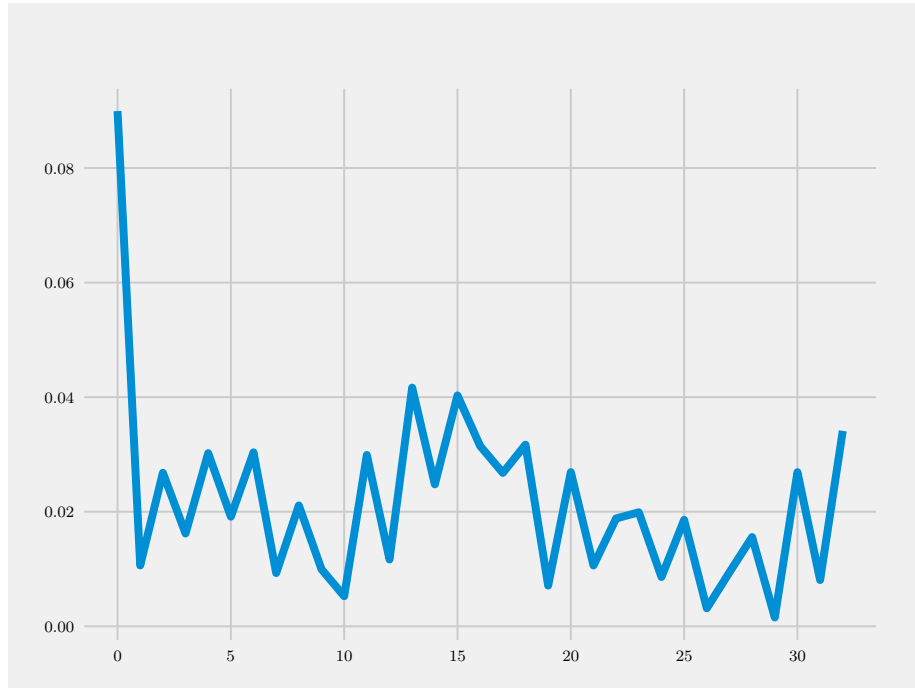


Figure 110: Second Order, Case number 0, 33points

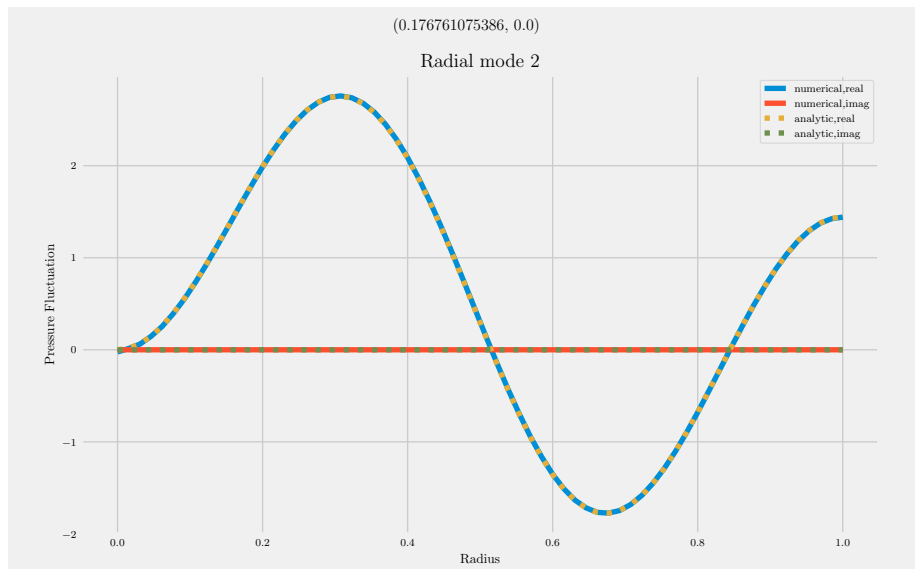


Figure 111: Second Order, Case number 0, 66points

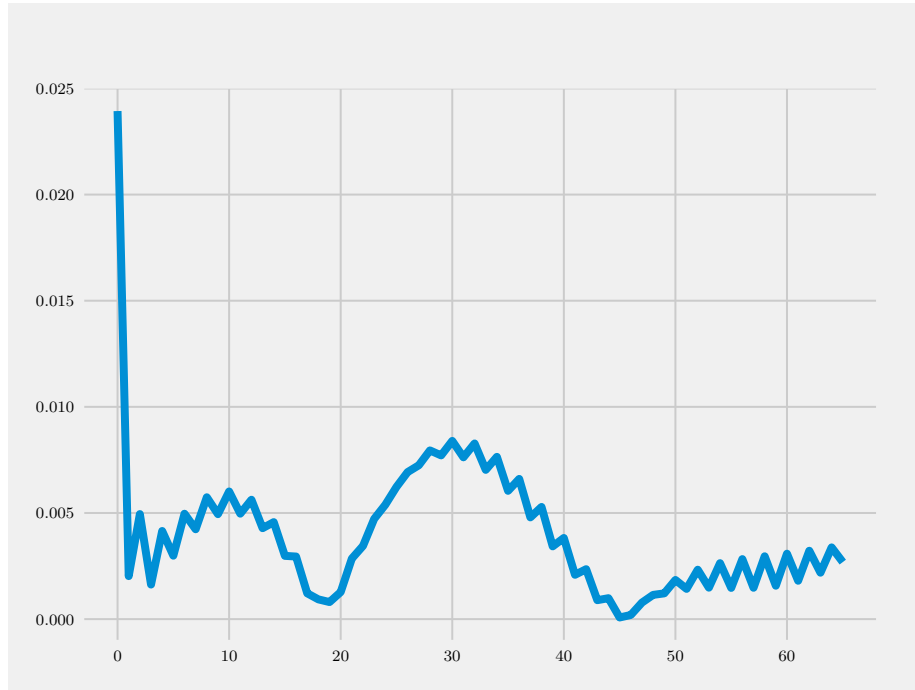


Figure 112: Second Order, Case number 0, 66points

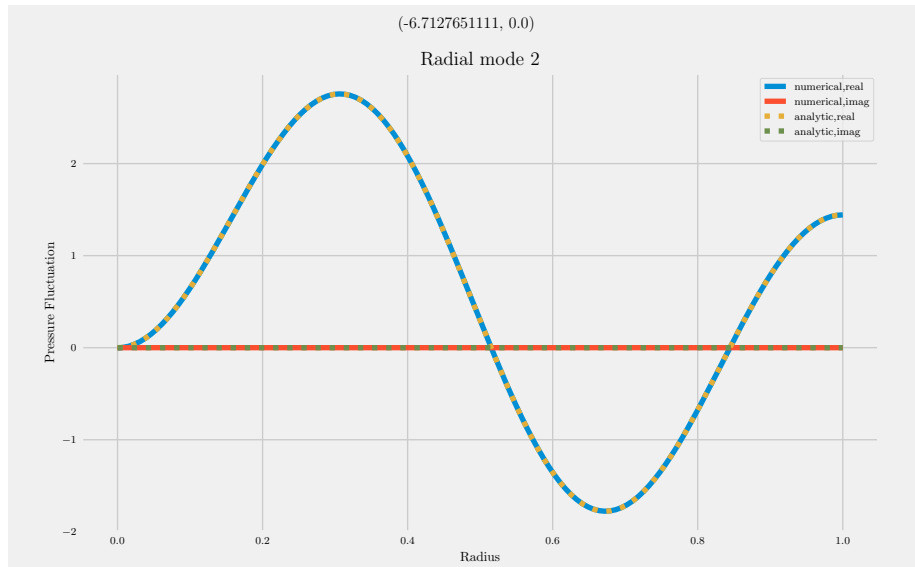


Figure 113: Second Order, Case number 0, 132points

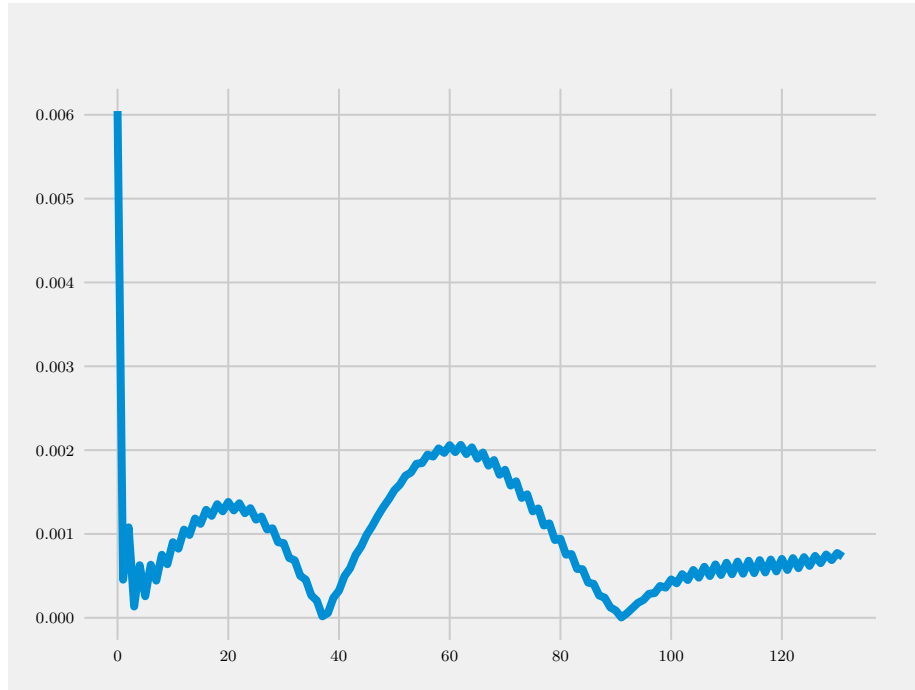


Figure 114: Second Order, Case number 0, 132points

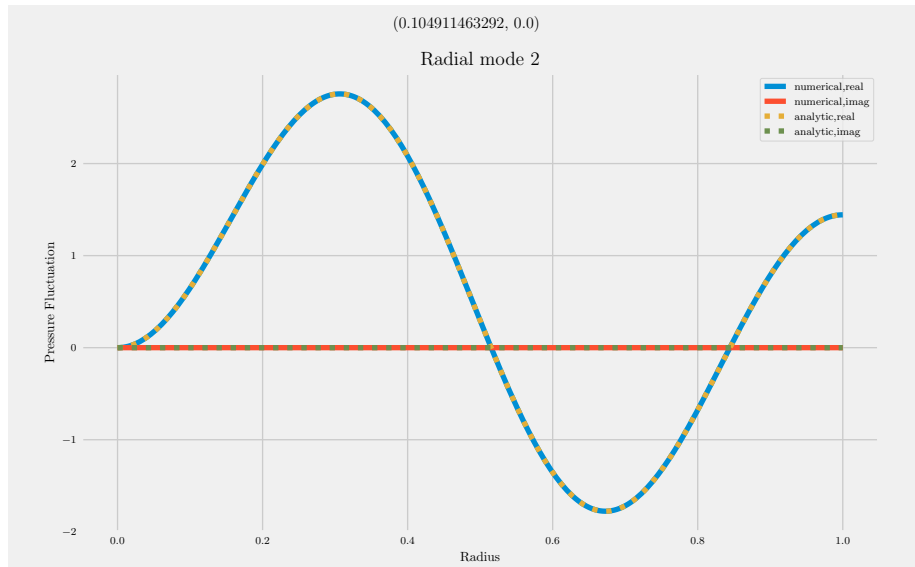


Figure 115: Second Order, Case number 0, 264points

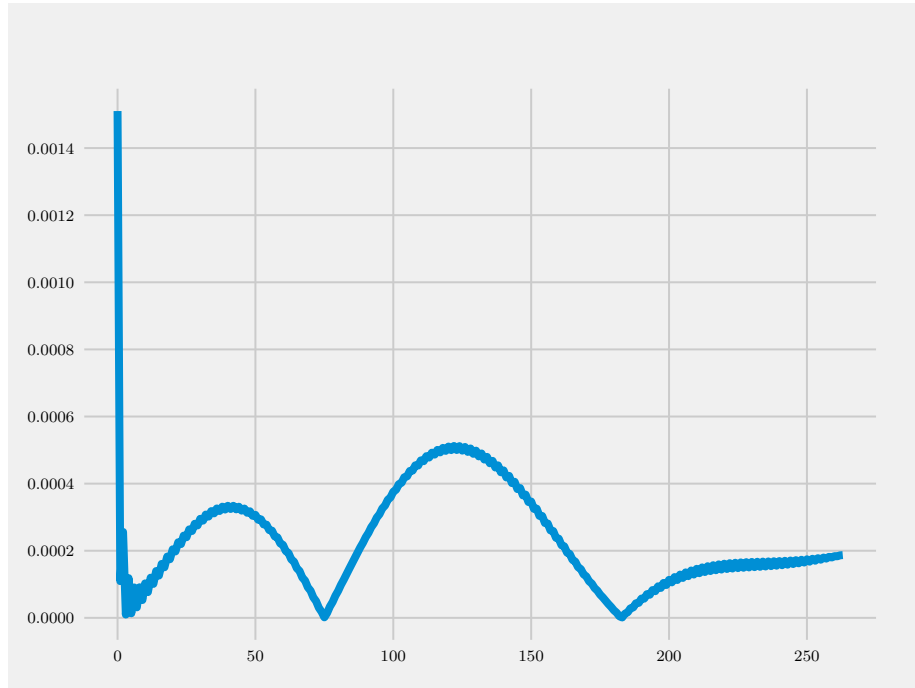


Figure 116: Second Order, Case number 0, 264points

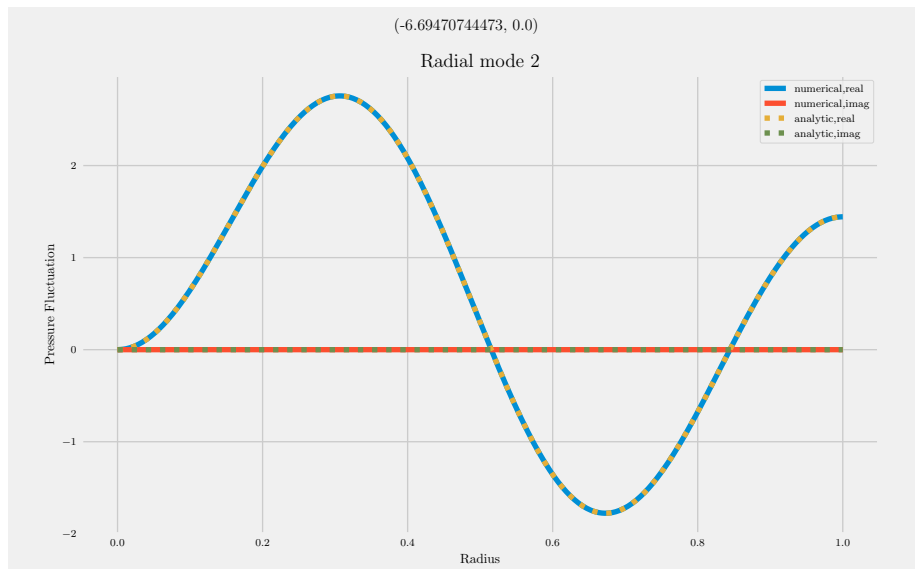


Figure 117: Second Order, Case number 0, 528points

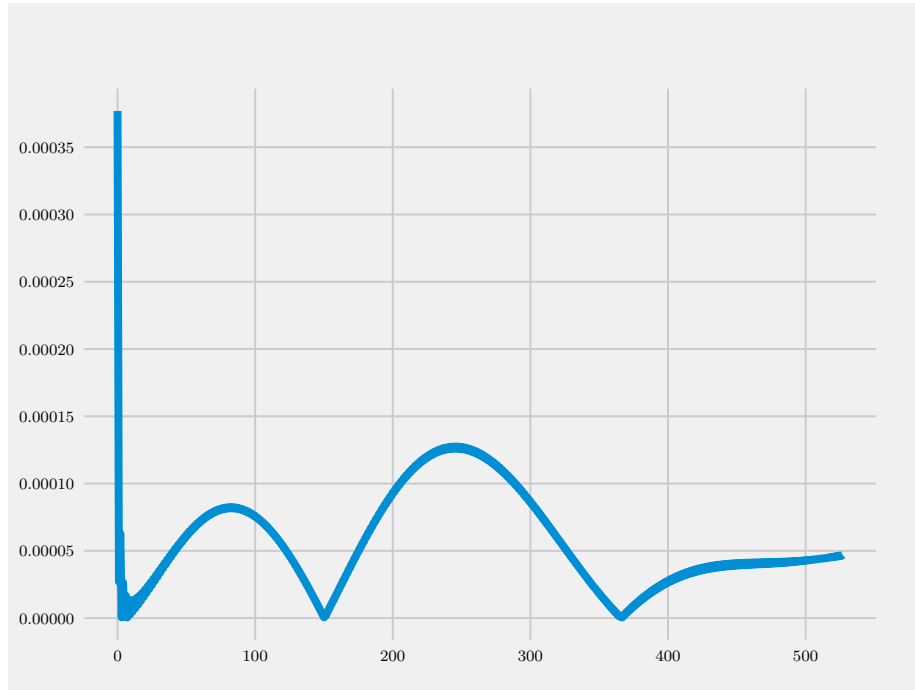


Figure 118: Second Order, Case number 0, 528points

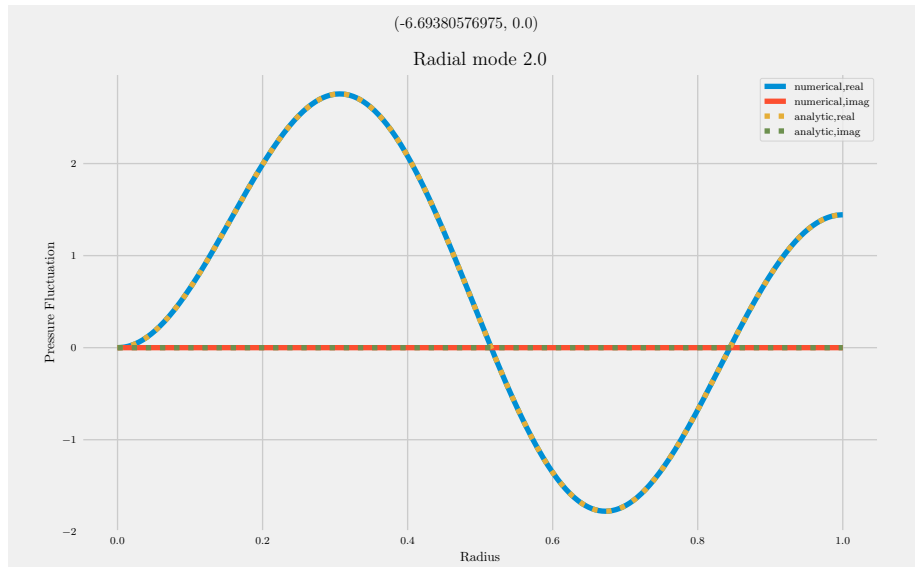


Figure 119: Second Order, Case number 0, 1056points

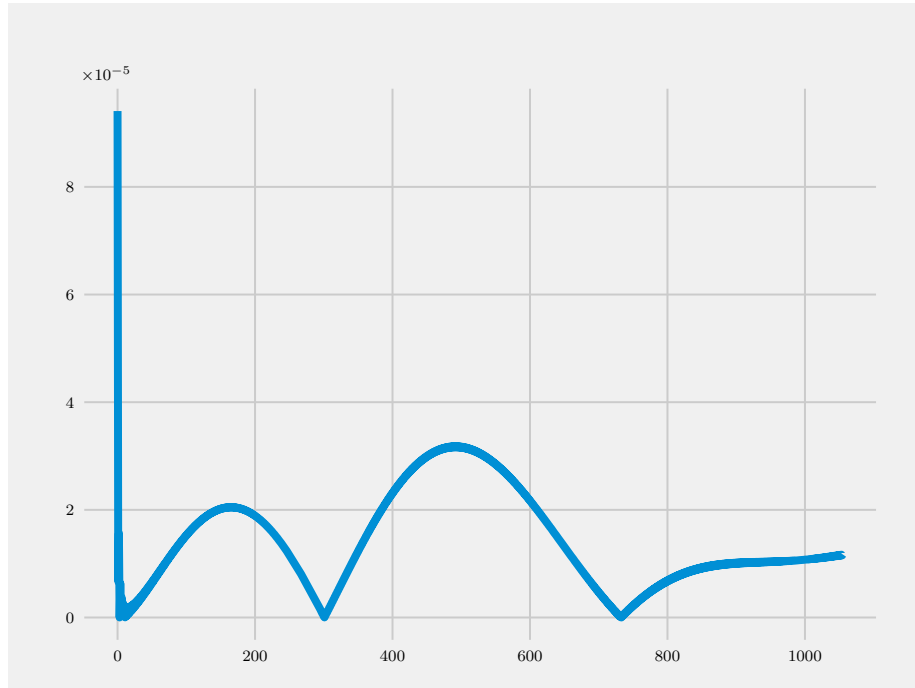


Figure 120: Second Order, Case number 0, 1056points

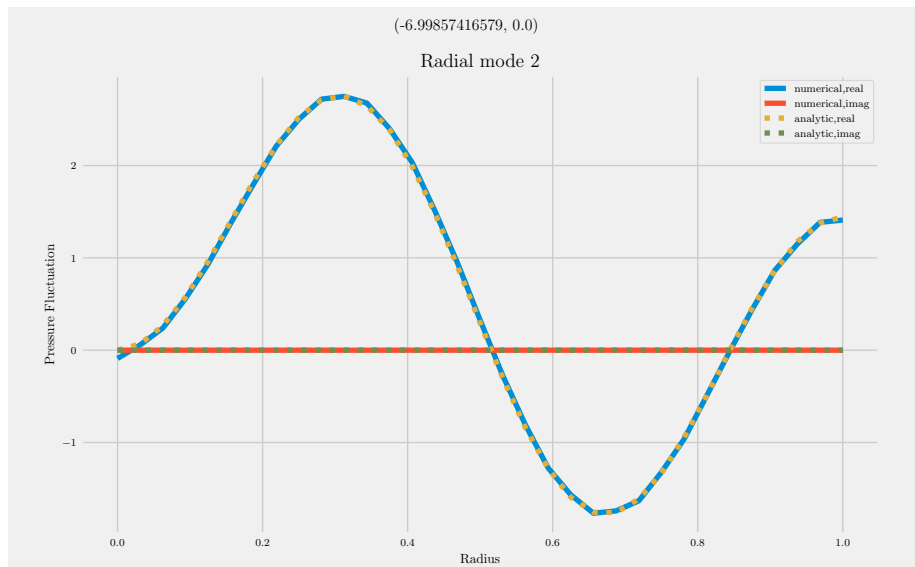


Figure 121: Second Order, Case number 1, 33points

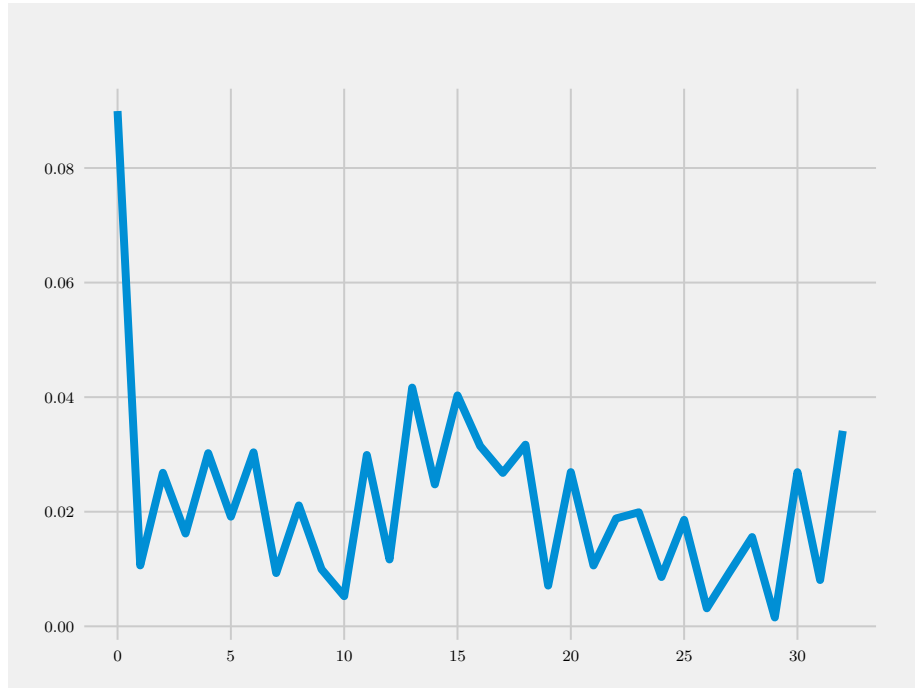


Figure 122: Second Order, Case number 1, 33points

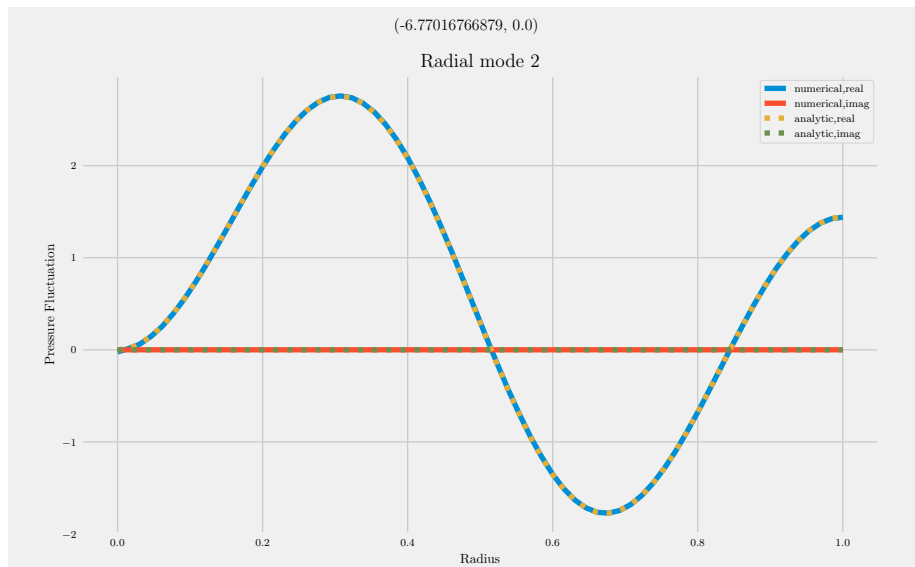


Figure 123: Second Order, Case number 1, 66points

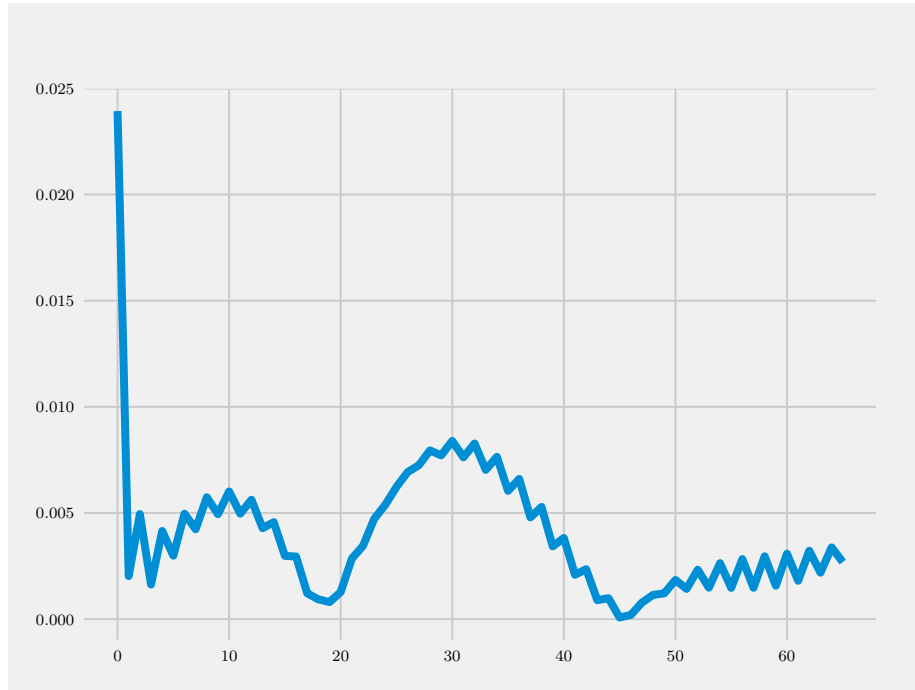


Figure 124: Second Order, Case number 1, 66points

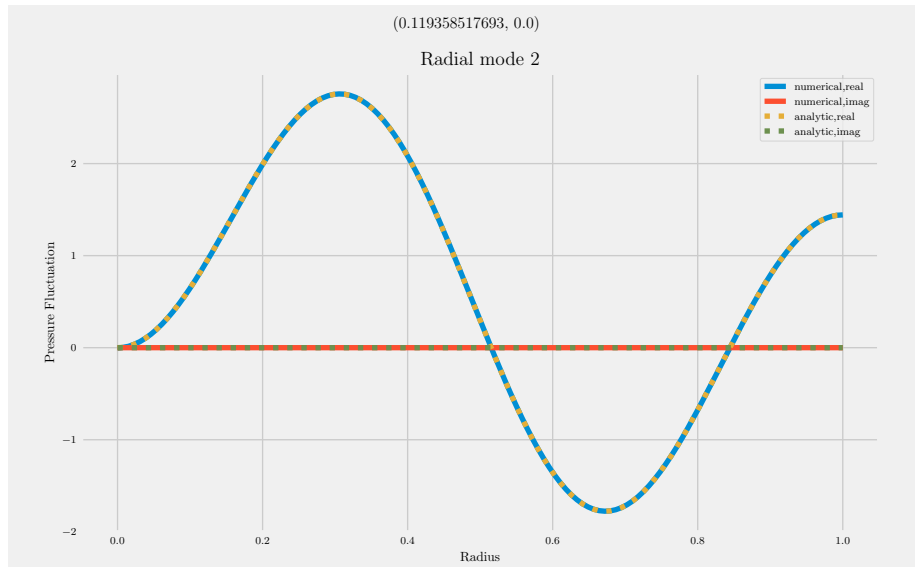


Figure 125: Second Order, Case number 1, 132points

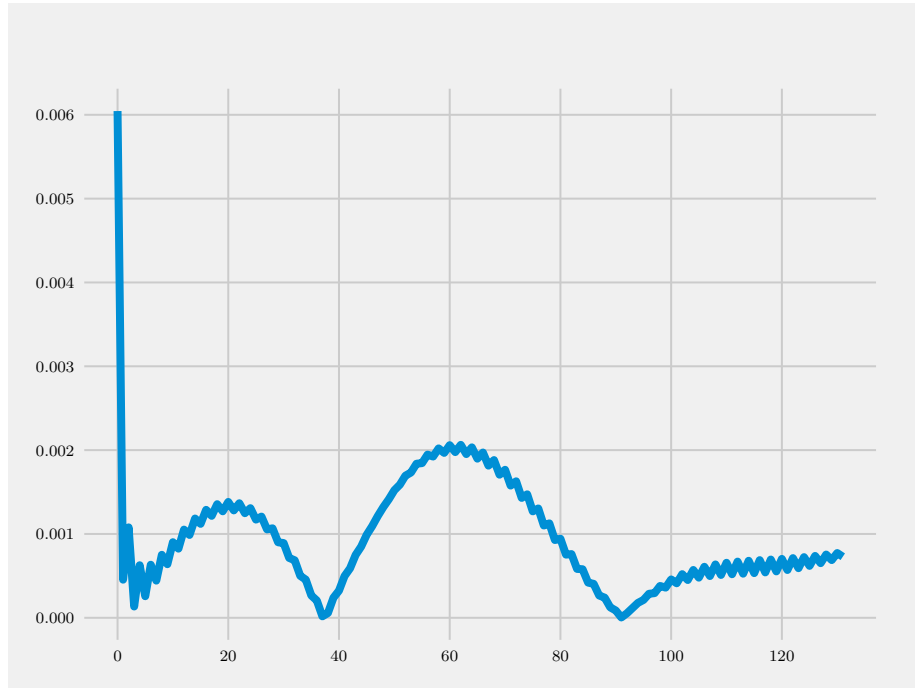


Figure 126: Second Order, Case number 1, 132points

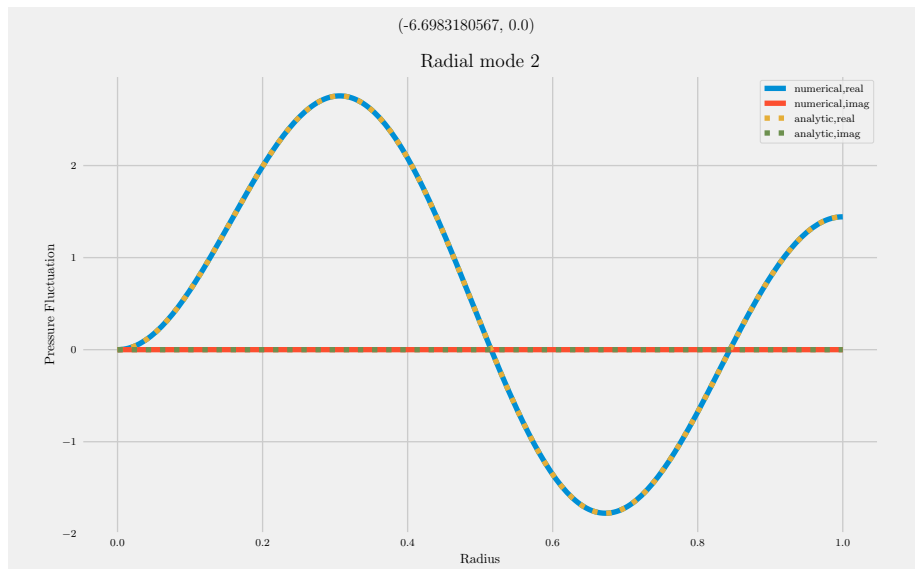


Figure 127: Second Order, Case number 1, 264points

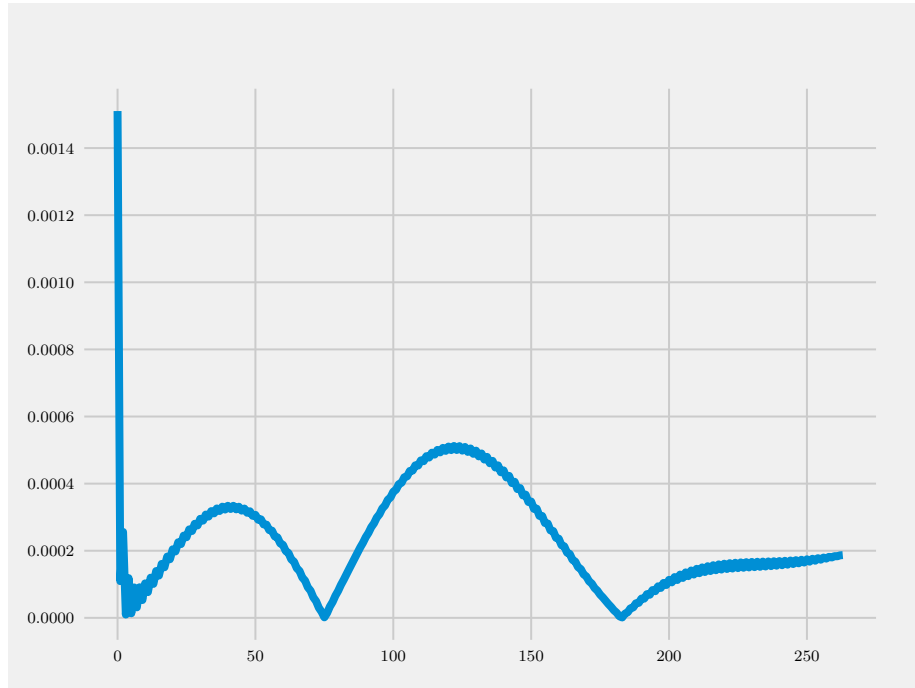


Figure 128: Second Order, Case number 1, 264points

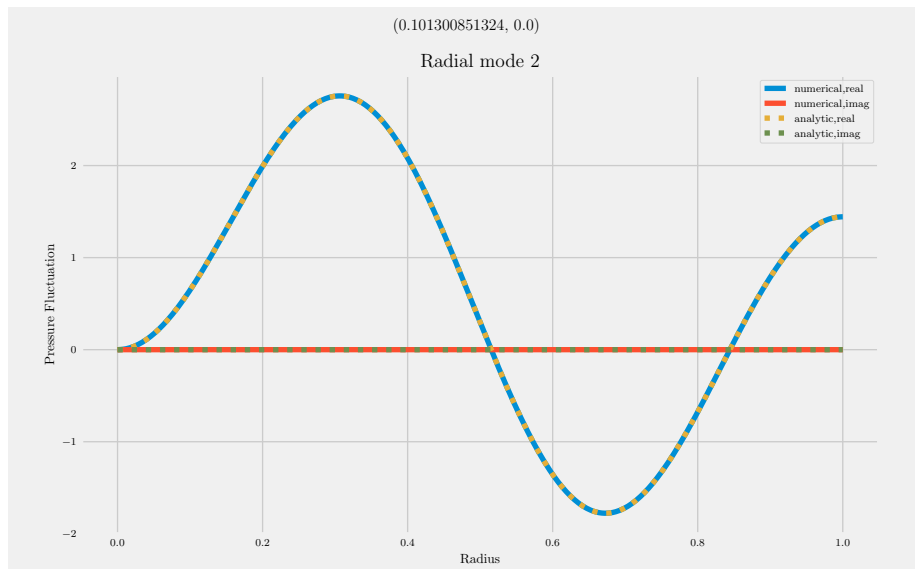


Figure 129: Second Order, Case number 1, 528points

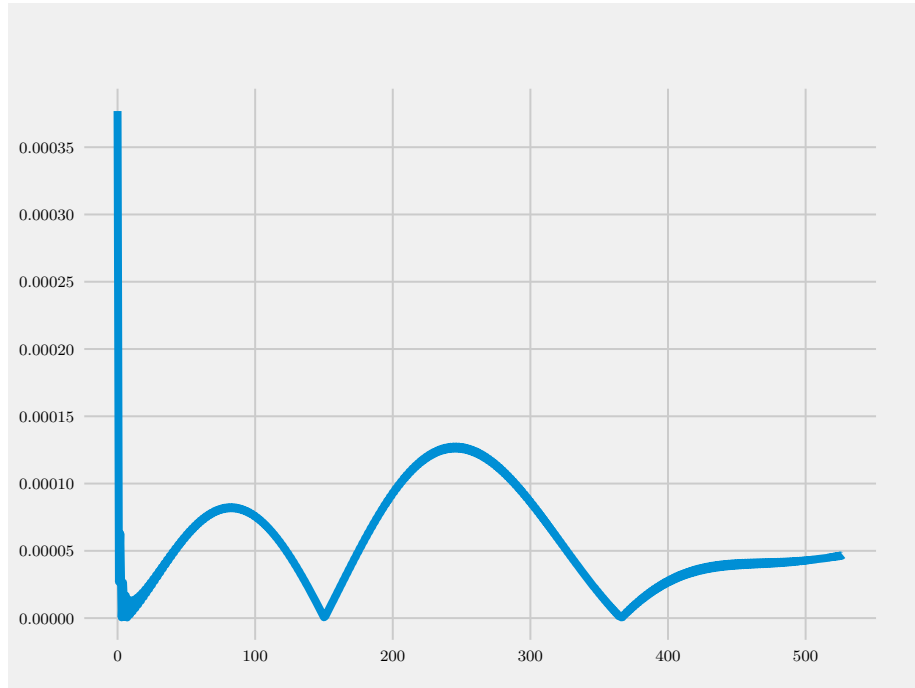


Figure 130: Second Order, Case number 1, 528points

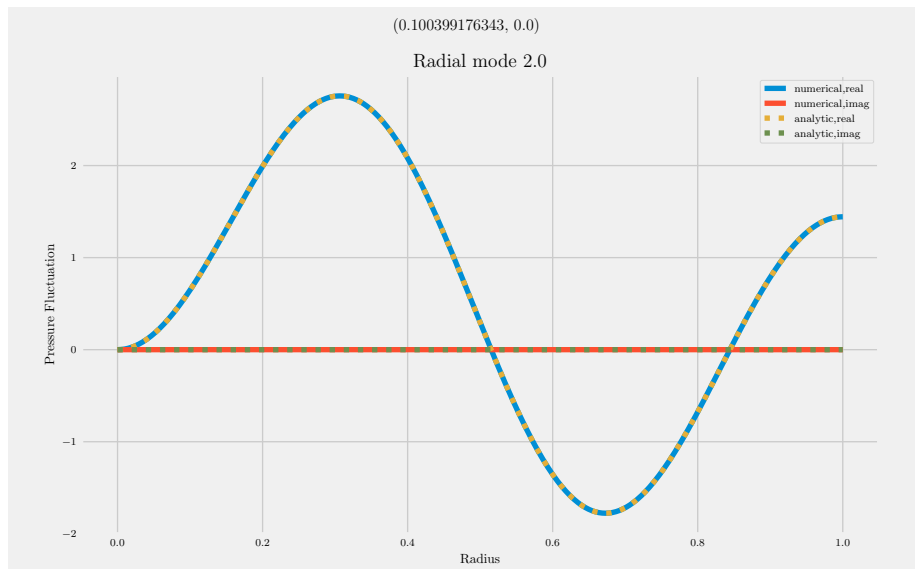


Figure 131: Second Order, Case number 1, 1056points

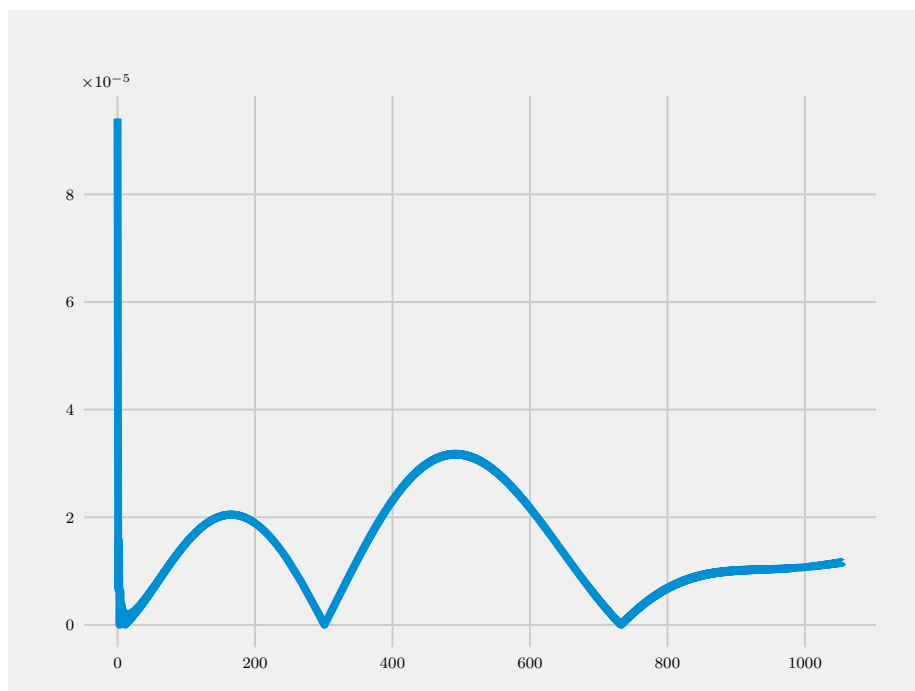


Figure 132: Second Order, Case number 1, 1056points

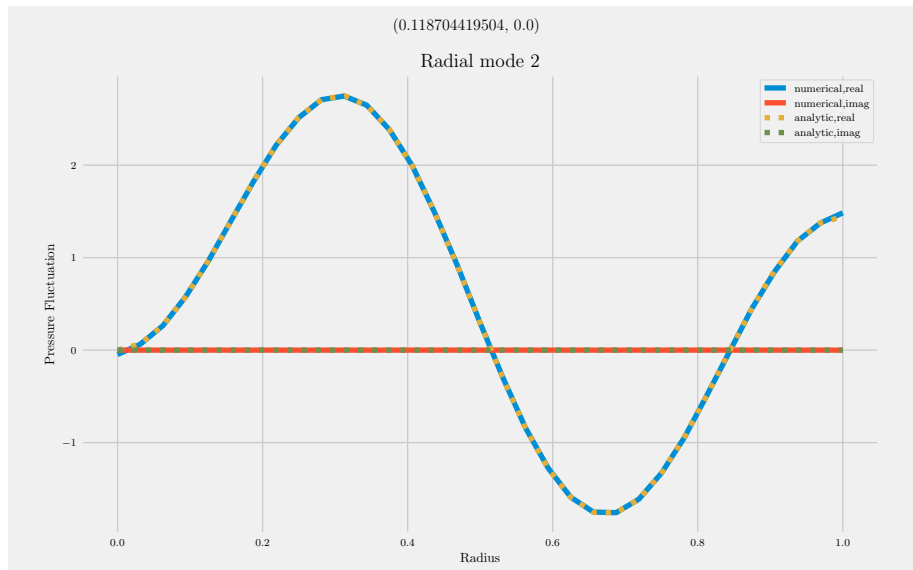


Figure 133: Fourth Order, Case number 0, 33points

Fourth Order, Radial Mode 2

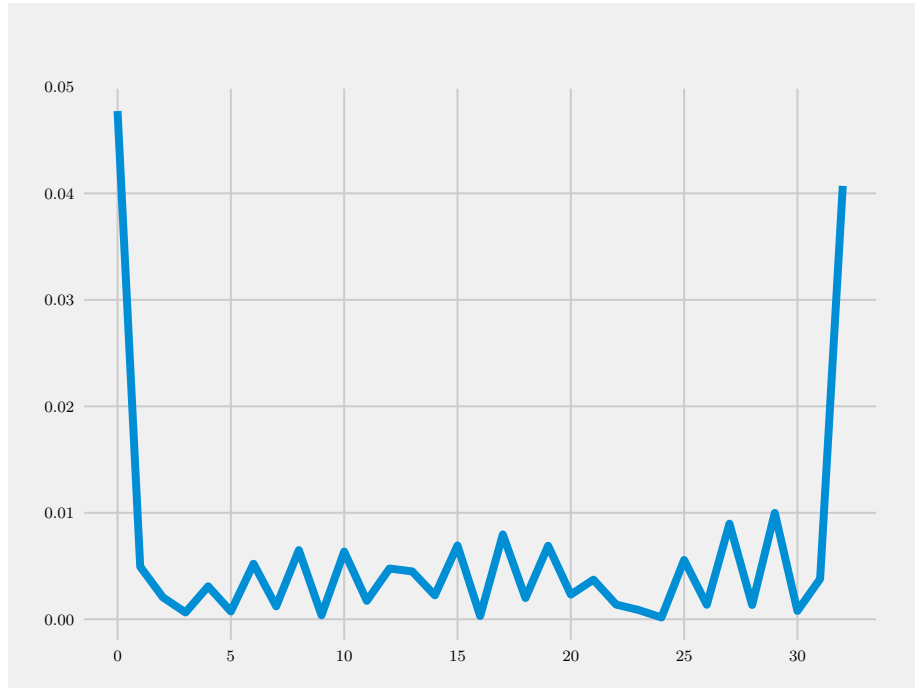


Figure 134: Fourth Order, Case number 0, 33points

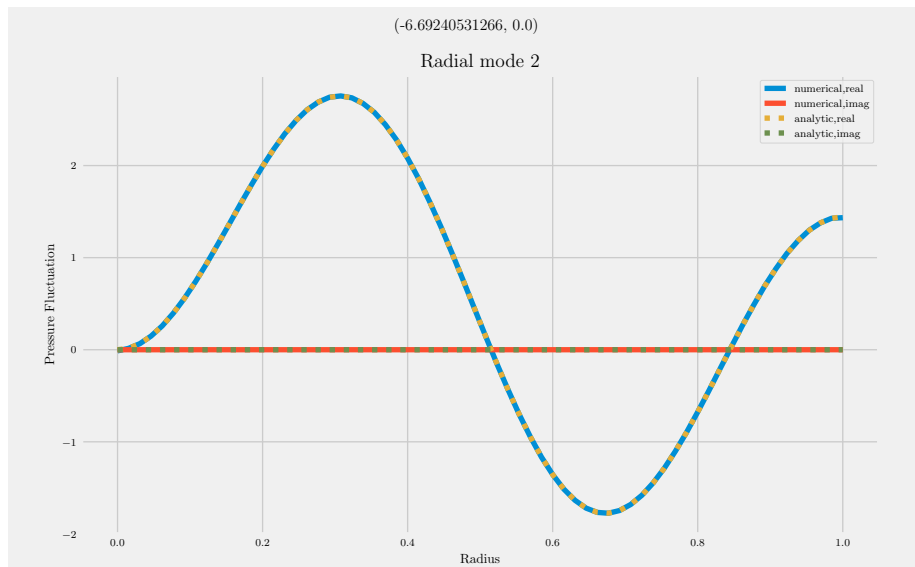


Figure 135: Fourth Order, Case number 0, 66points

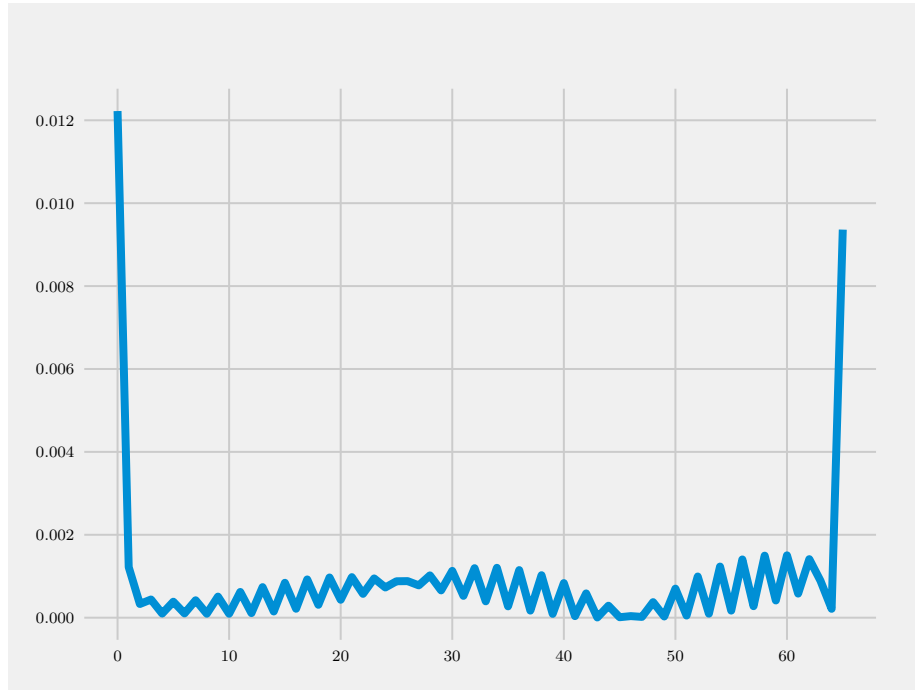


Figure 136: Fourth Order, Case number 0, 66points

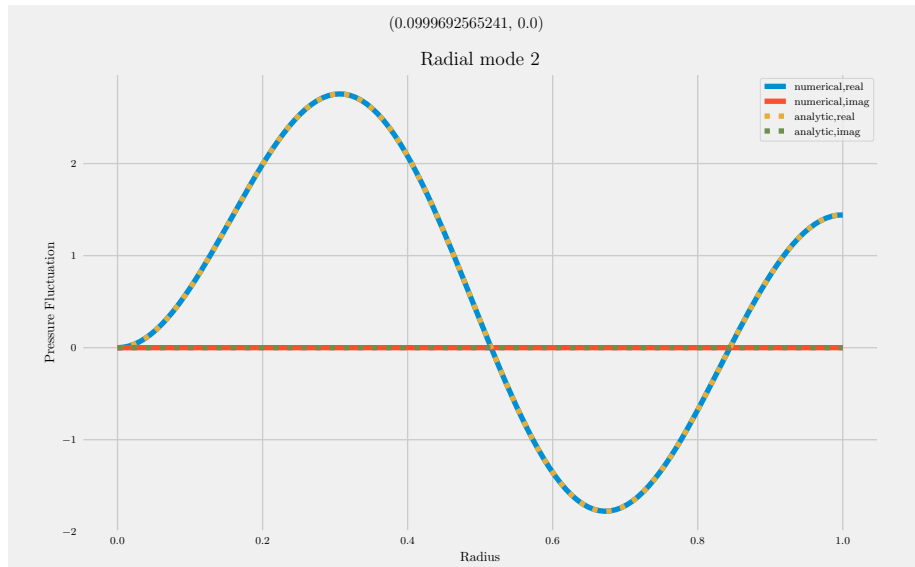


Figure 137: Fourth Order, Case number 0, 132points

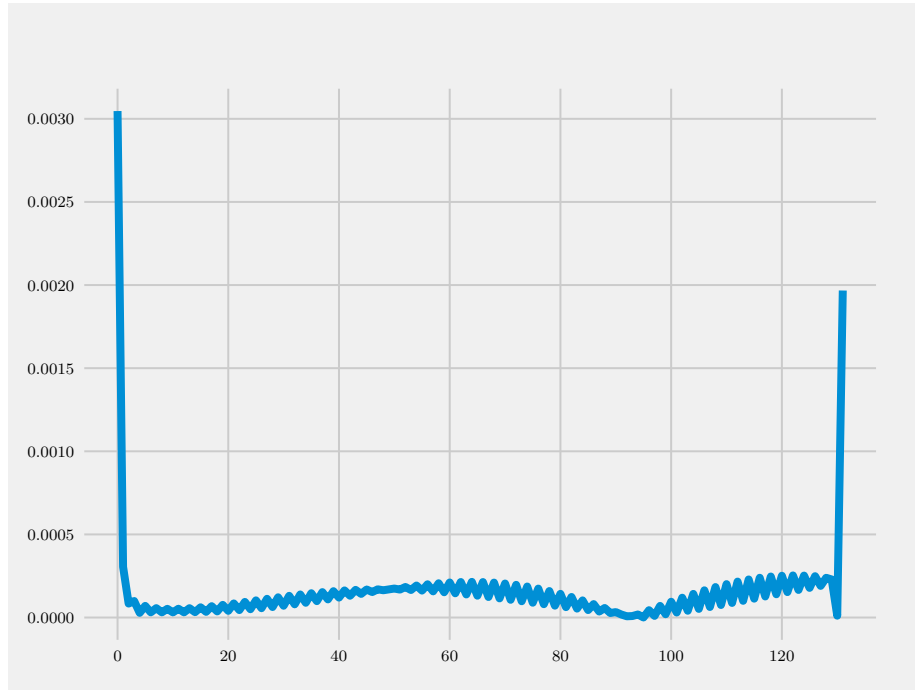


Figure 138: Fourth Order, Case number 0, 132points

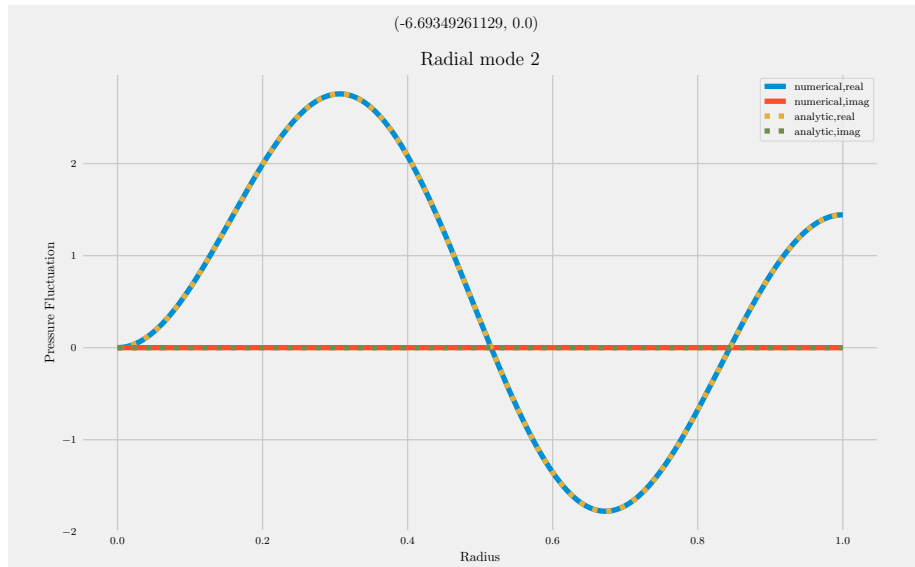


Figure 139: Fourth Order, Case number 0, 264points

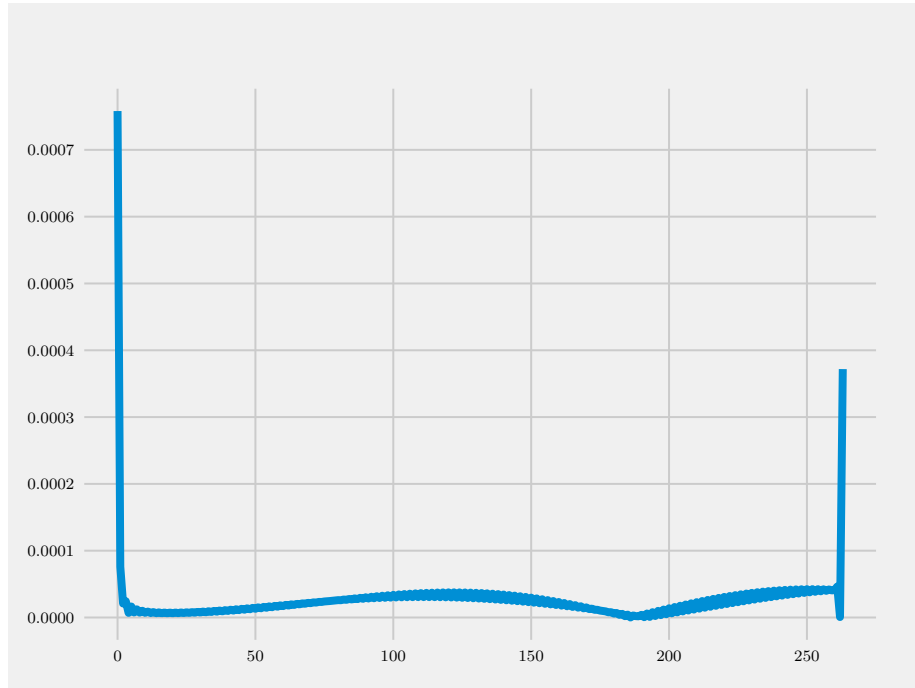


Figure 140: Fourth Order, Case number 0, 264points

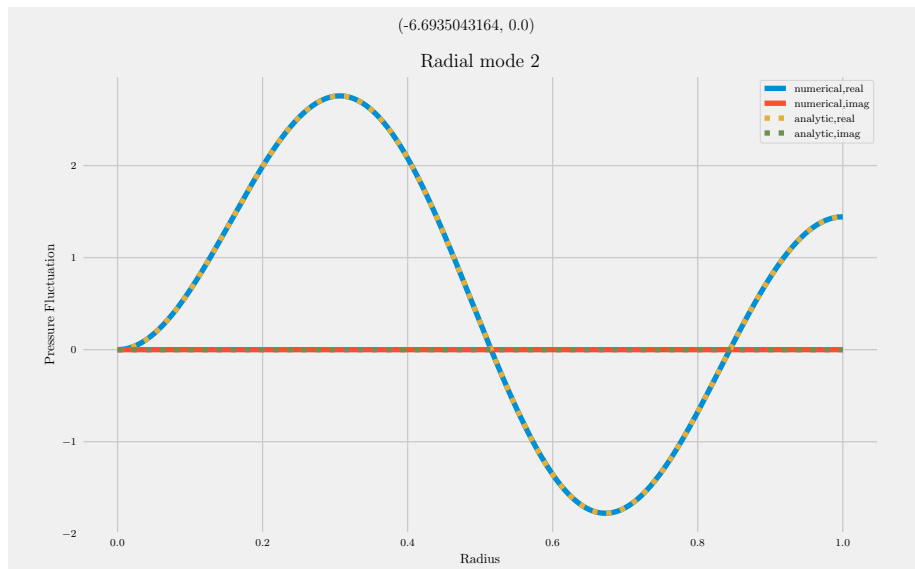


Figure 141: Fourth Order, Case number 0, 528points

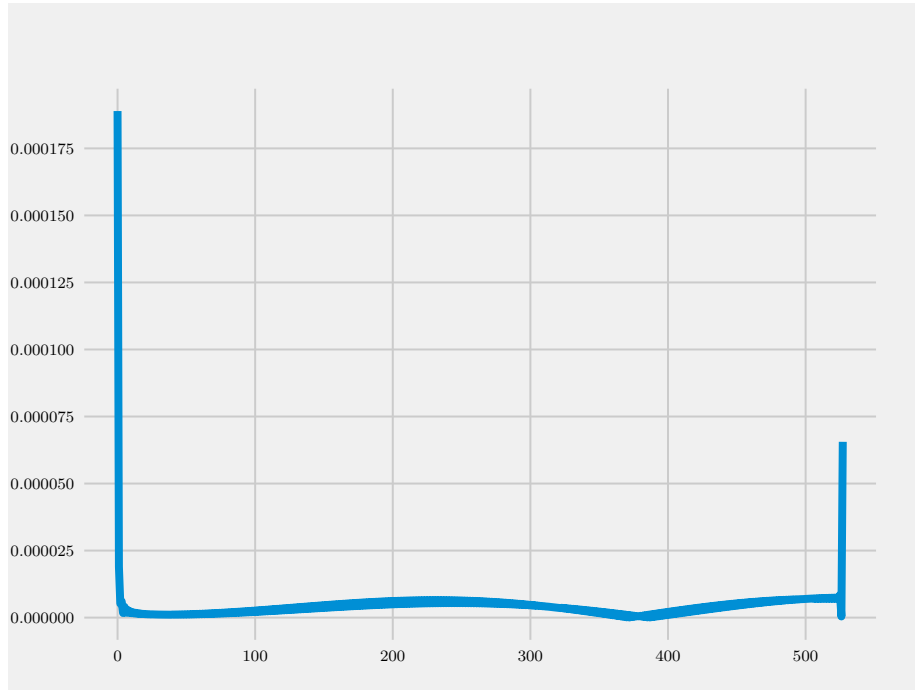


Figure 142: Fourth Order, Case number 0, 528points

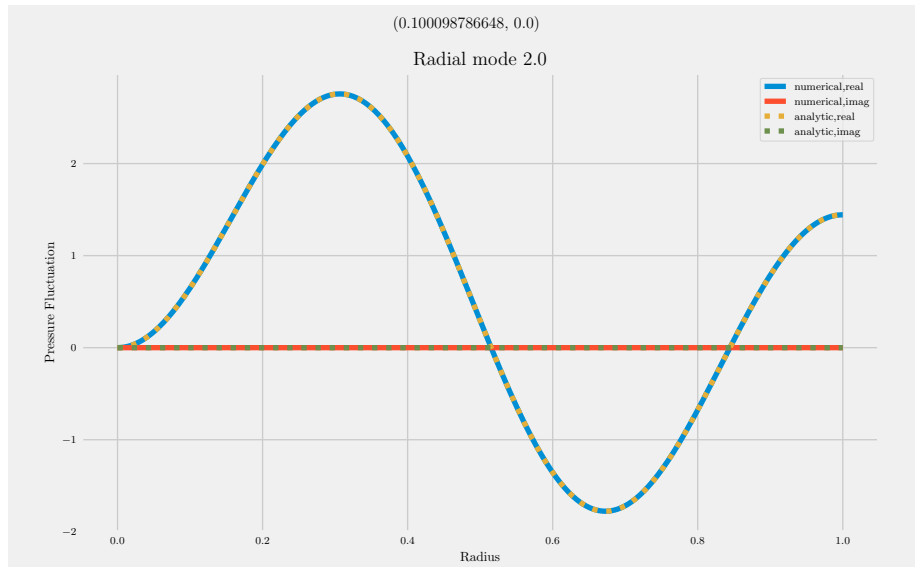


Figure 143: Fourth Order, Case number 0, 1056points

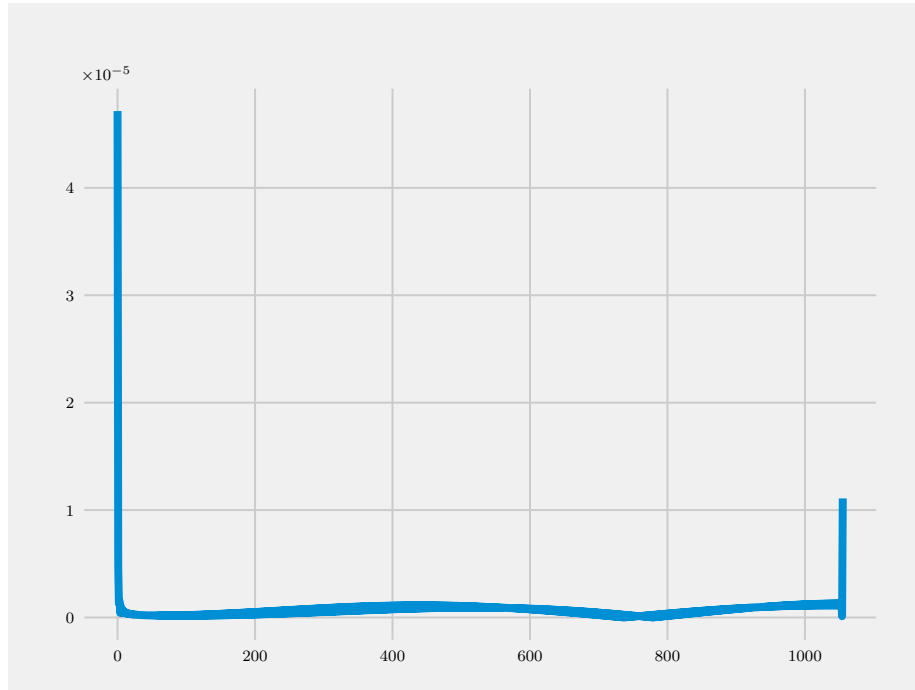


Figure 144: Fourth Order, Case number 0, 1056points

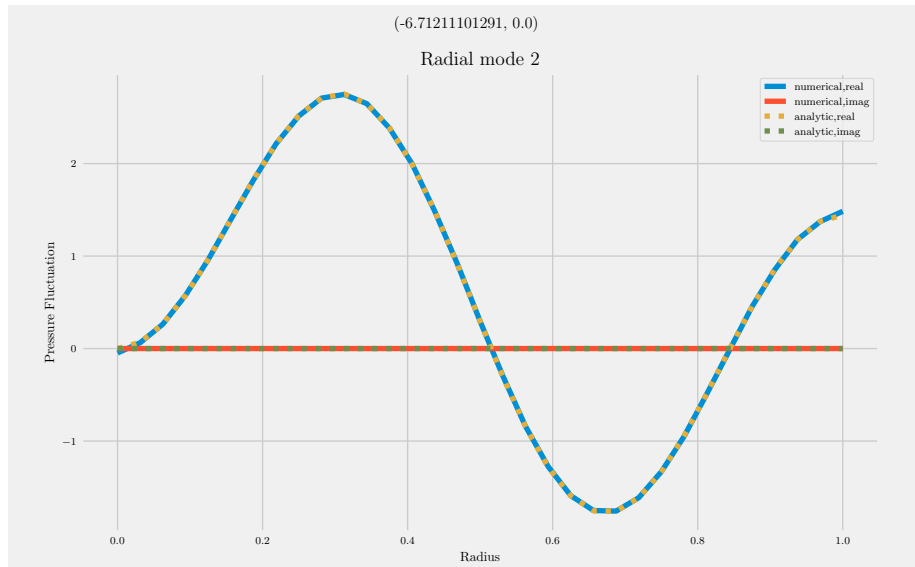


Figure 145: Fourth Order, Case number 1, 33points

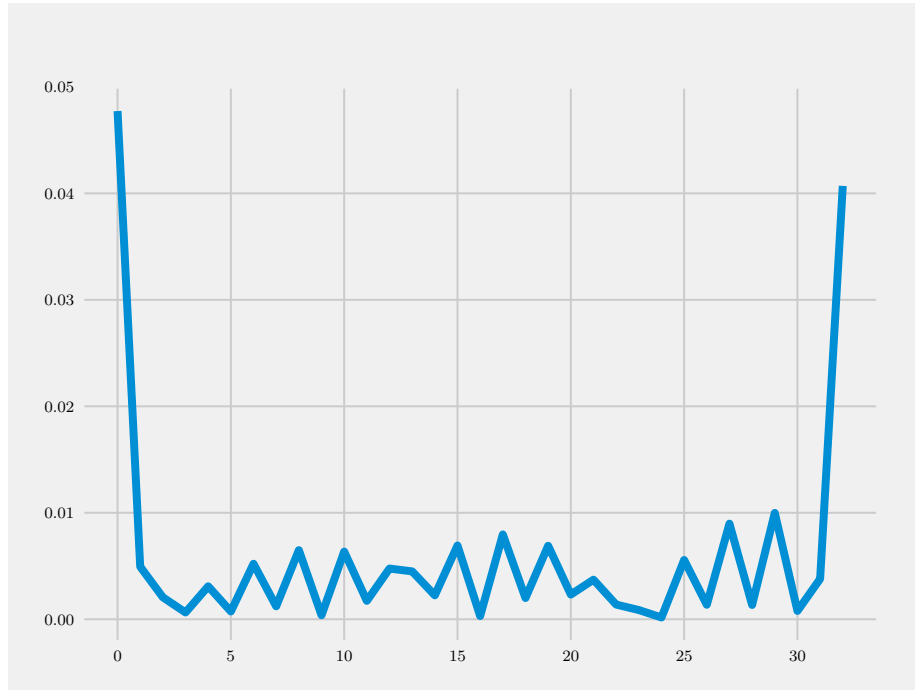


Figure 146: Fourth Order, Case number 1, 33points

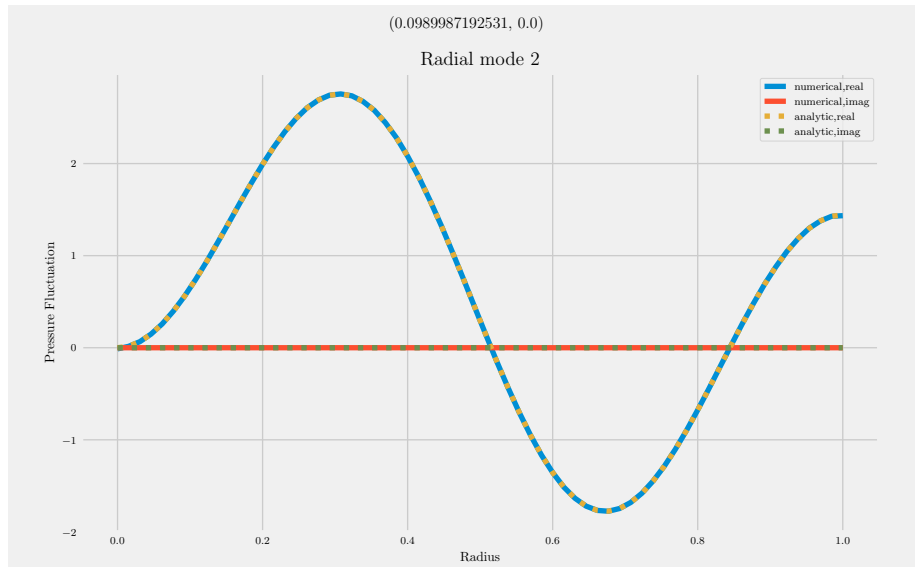


Figure 147: Fourth Order, Case number 1, 66points

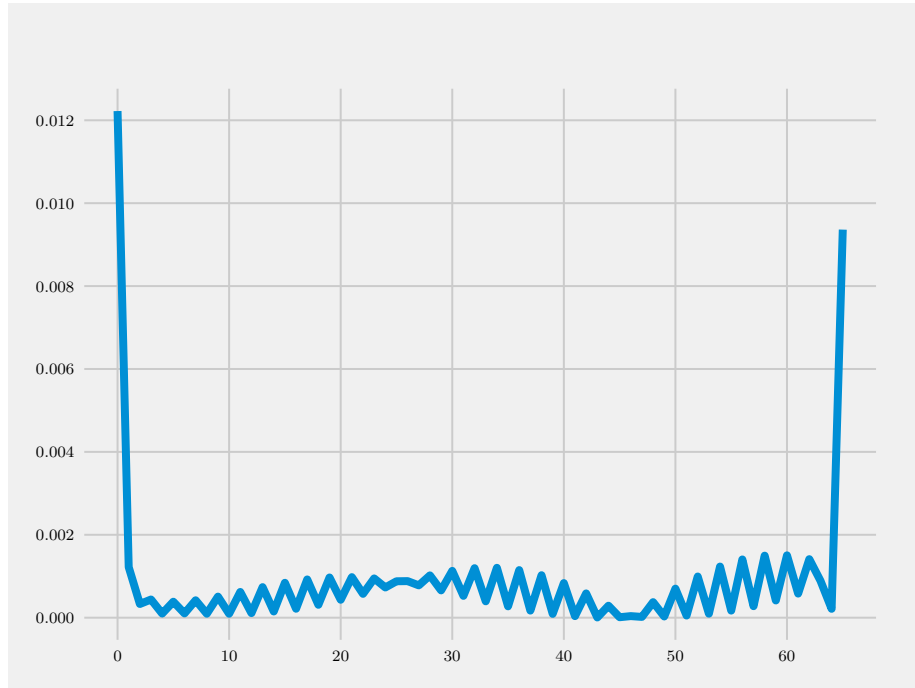


Figure 148: Fourth Order, Case number 1, 66points

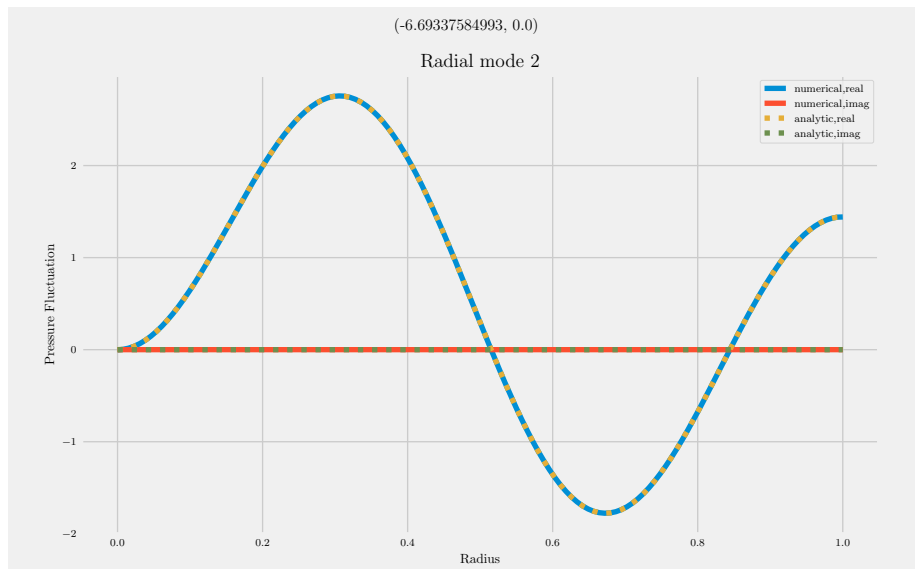


Figure 149: Fourth Order, Case number 1, 132points

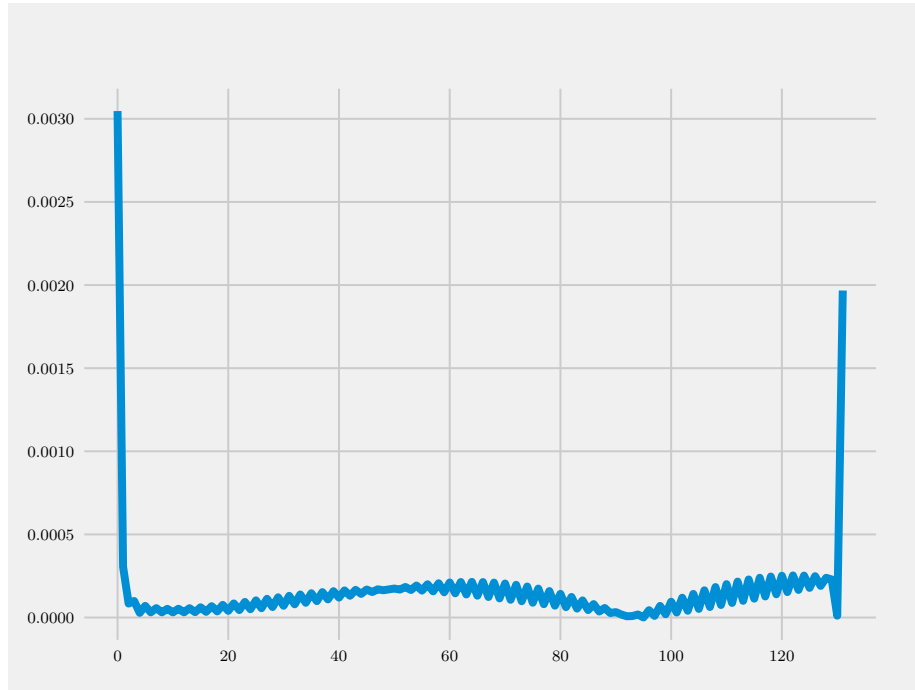


Figure 150: Fourth Order, Case number 1, 132points

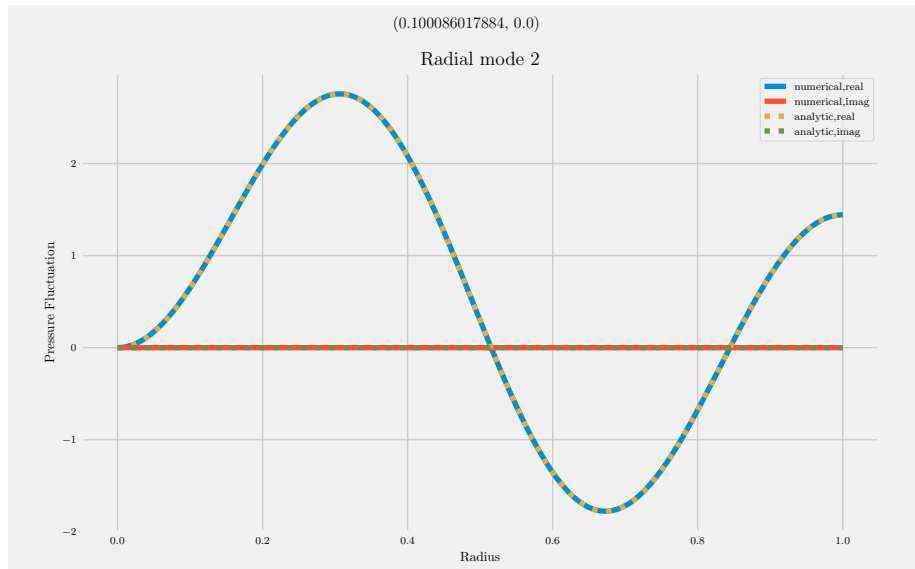


Figure 151: Fourth Order, Case number 1, 264points

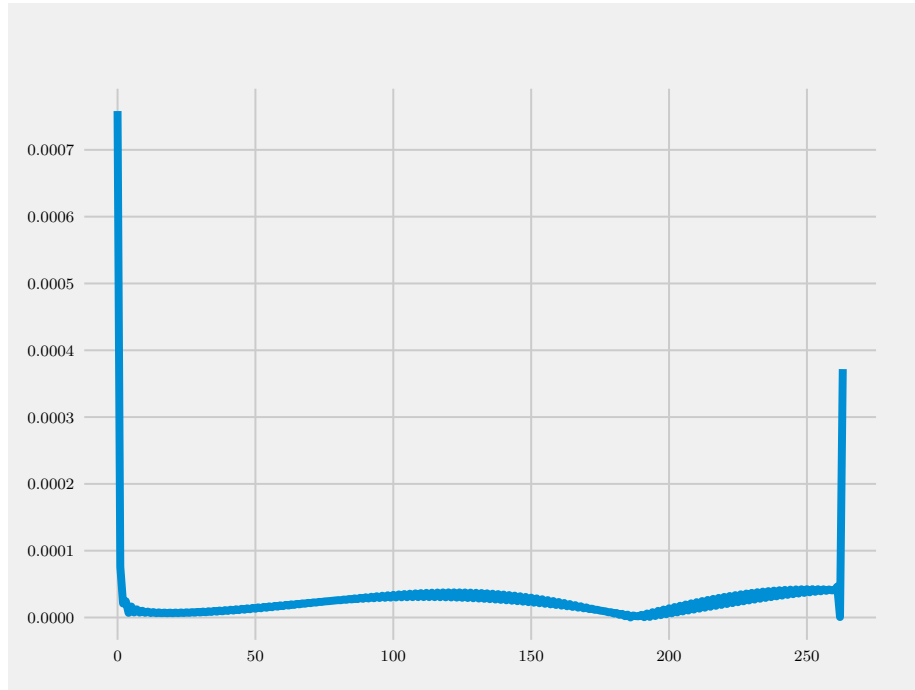


Figure 152: Fourth Order, Case number 1, 264points

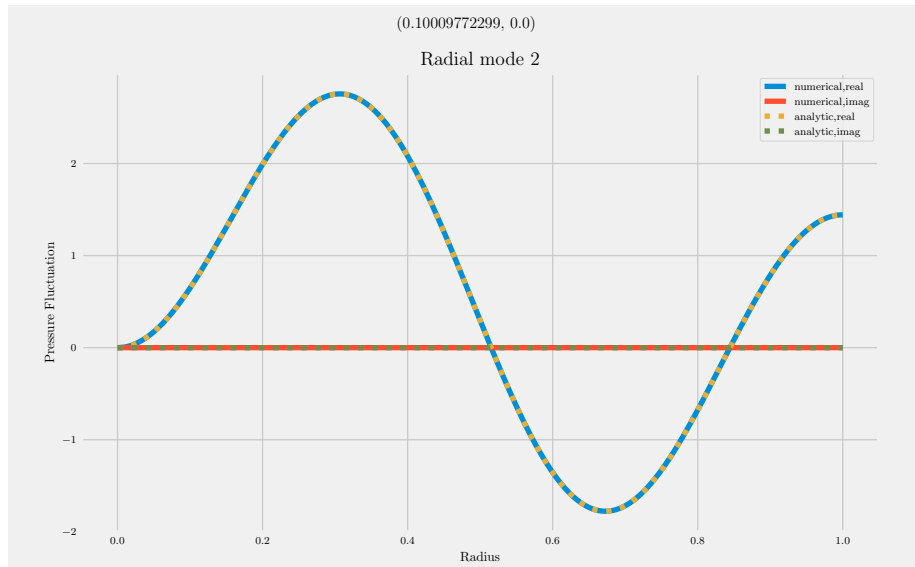


Figure 153: Fourth Order, Case number 1, 528points

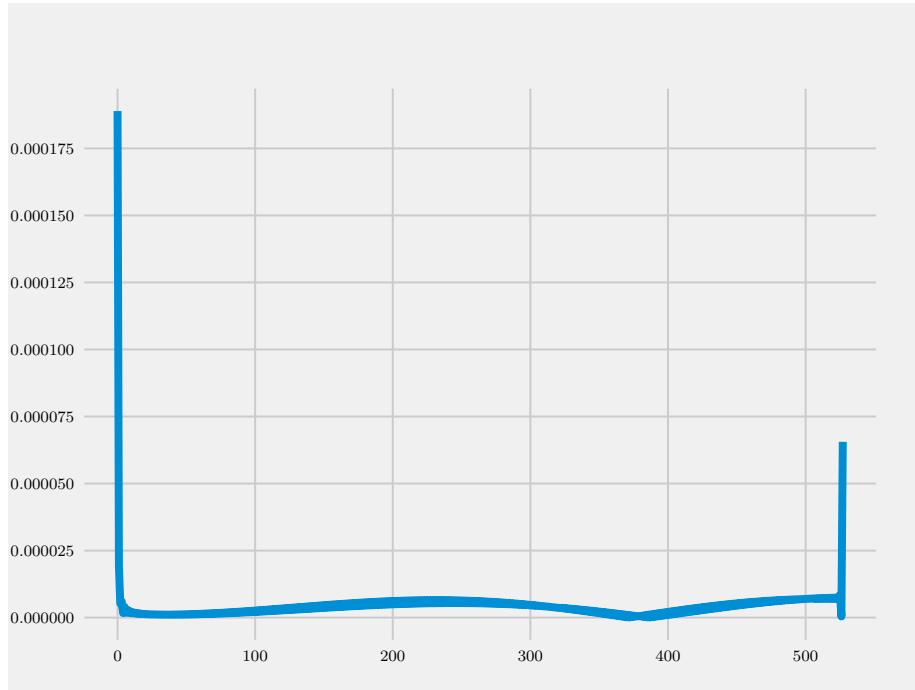


Figure 154: Fourth Order, Case number 1, 528points

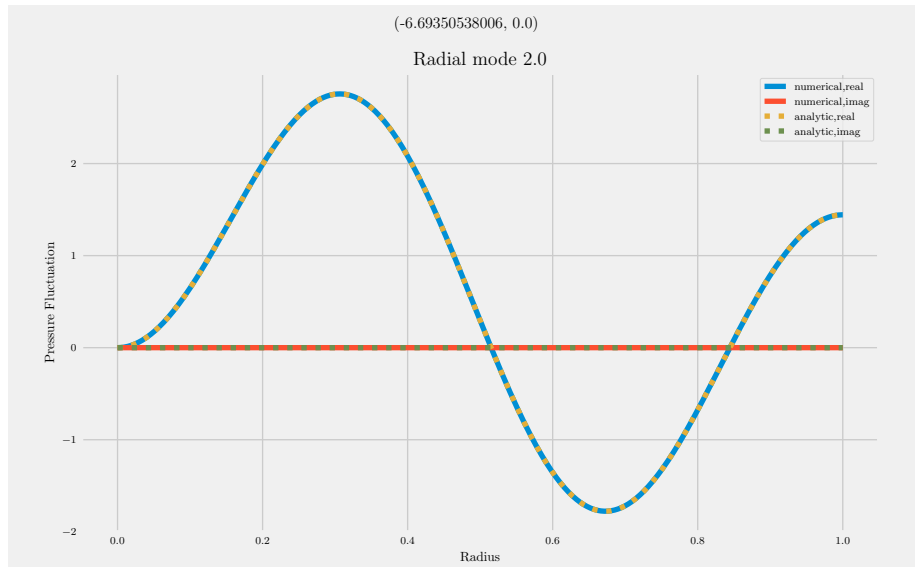


Figure 155: Fourth Order, Case number 1, 1056points

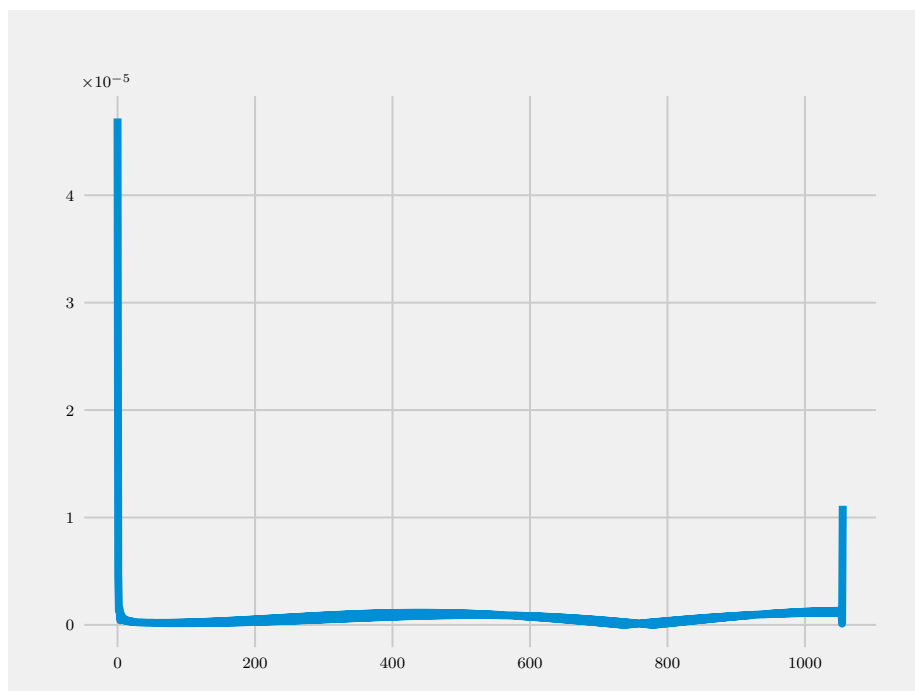


Figure 156: Fourth Order, Case number 1, 1056points

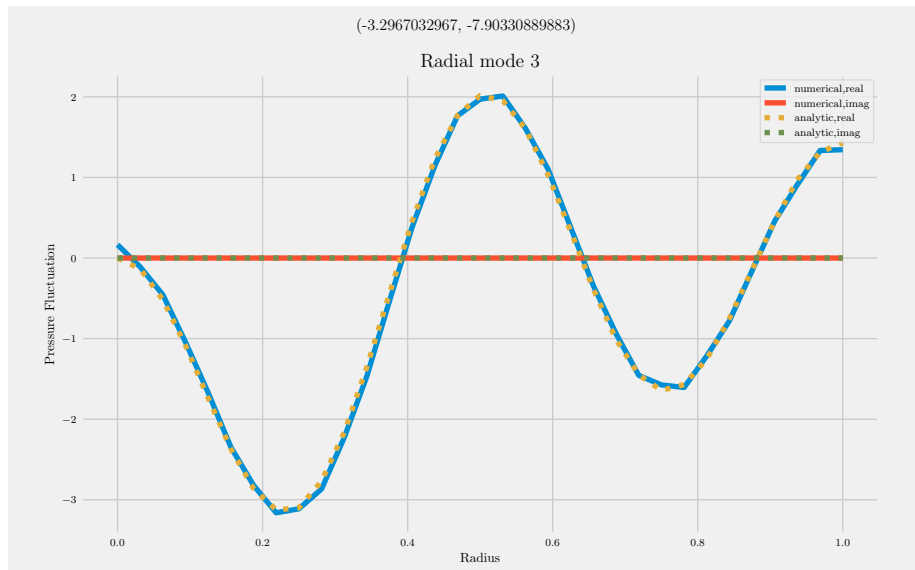


Figure 157: Second Order, Case number 0, 33points

Second Order, Radial Mode 3

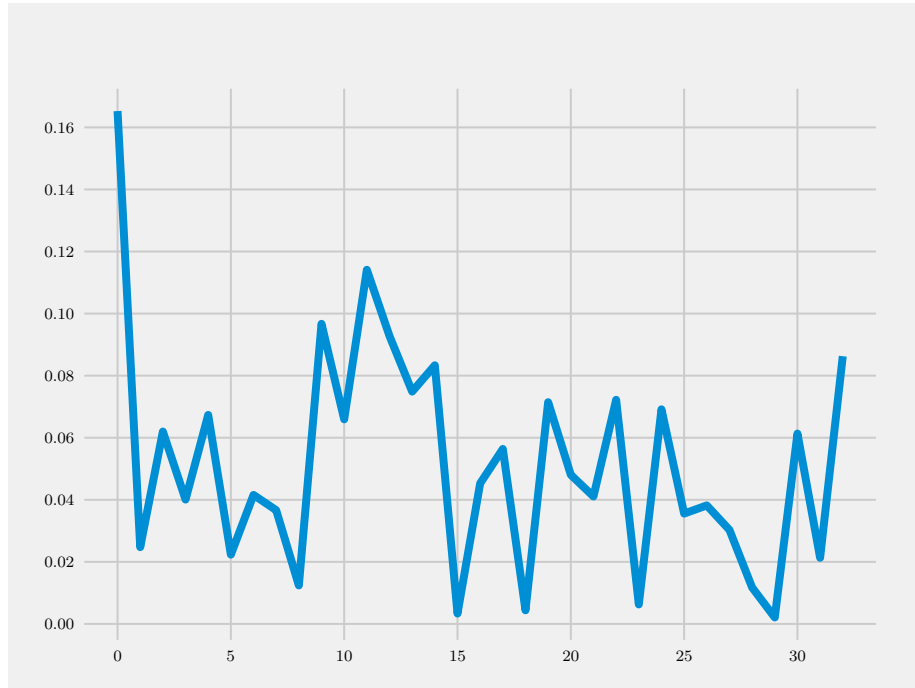


Figure 158: Second Order, Case number 0, 33points

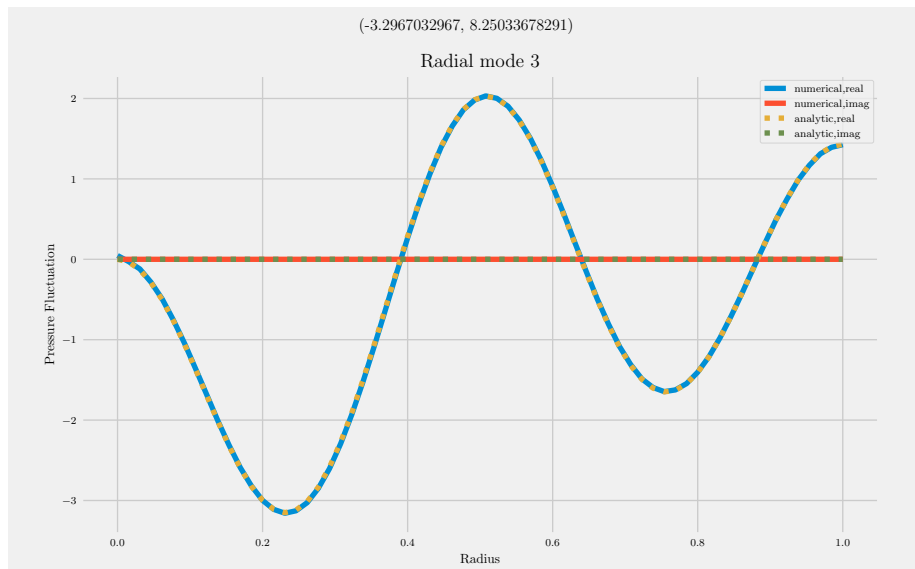


Figure 159: Second Order, Case number 0, 66points

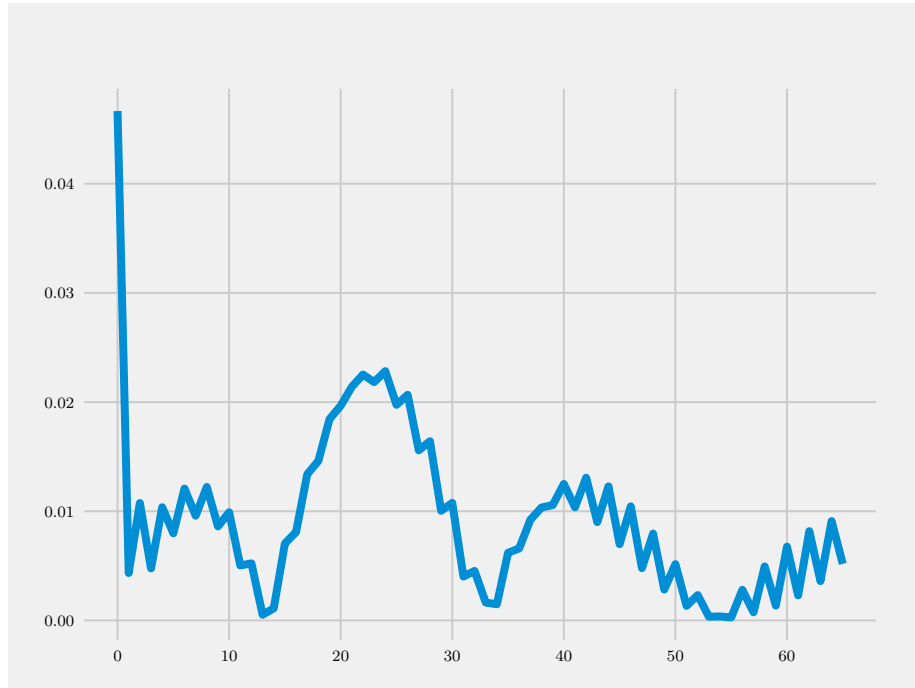


Figure 160: Second Order, Case number 0, 66points

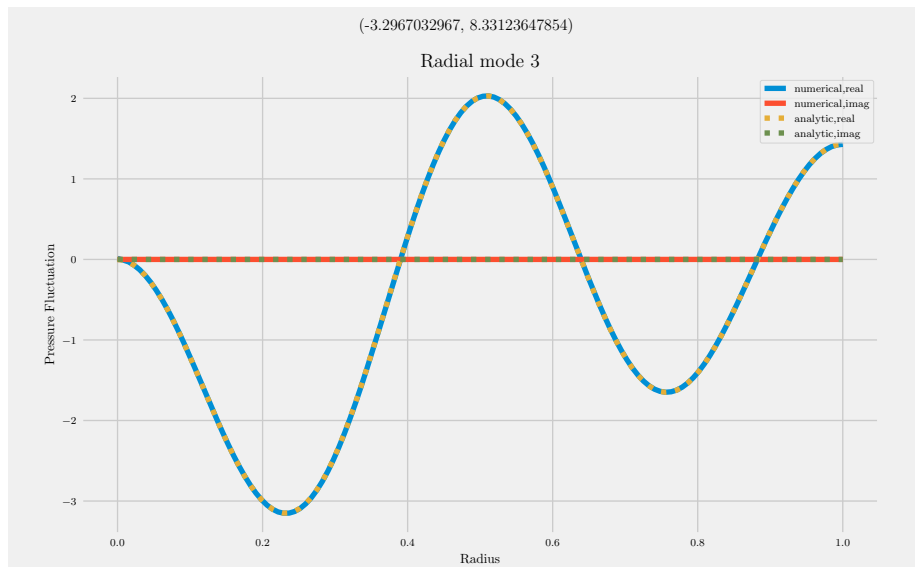


Figure 161: Second Order, Case number 0, 132points

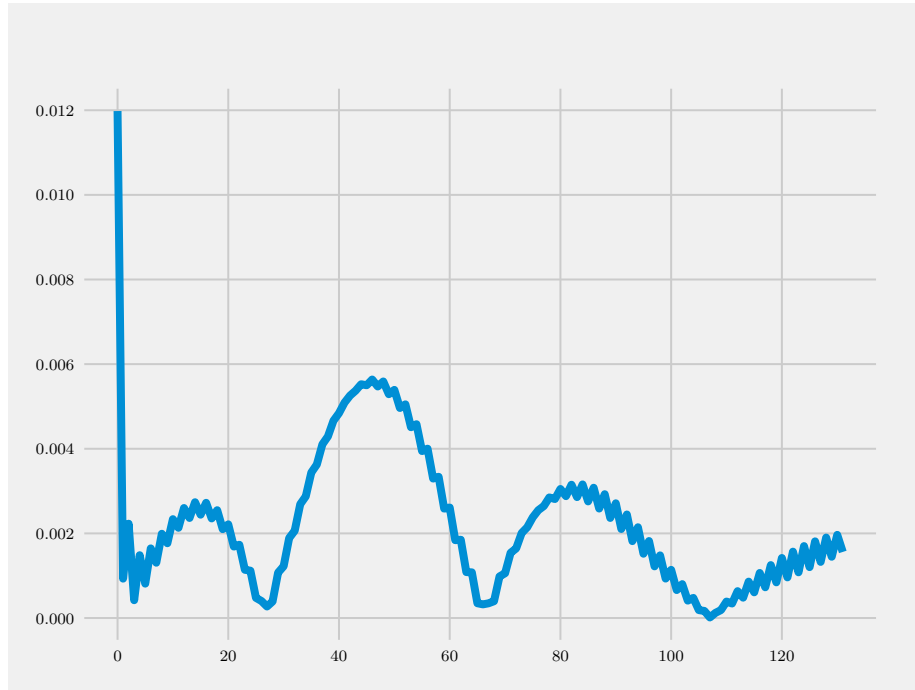


Figure 162: Second Order, Case number 0, 132points

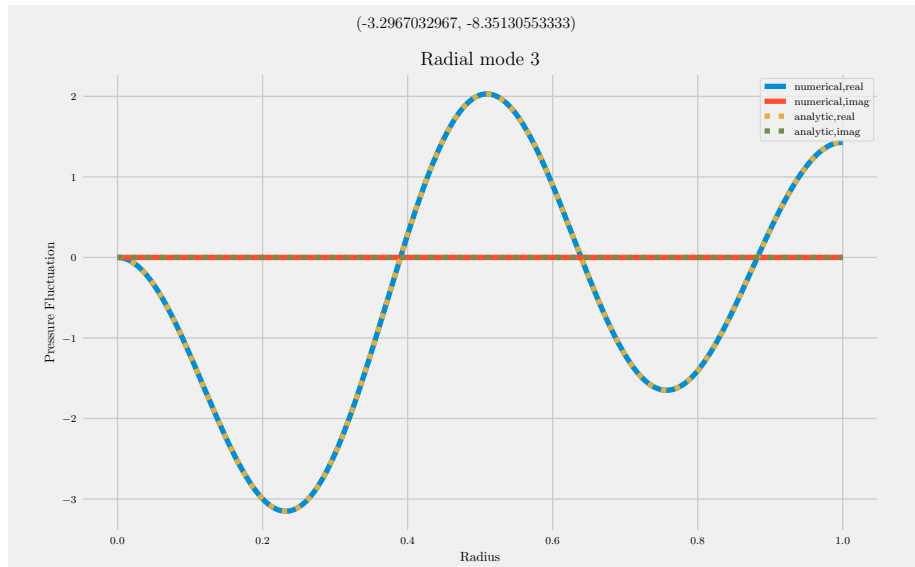


Figure 163: Second Order, Case number 0, 264points

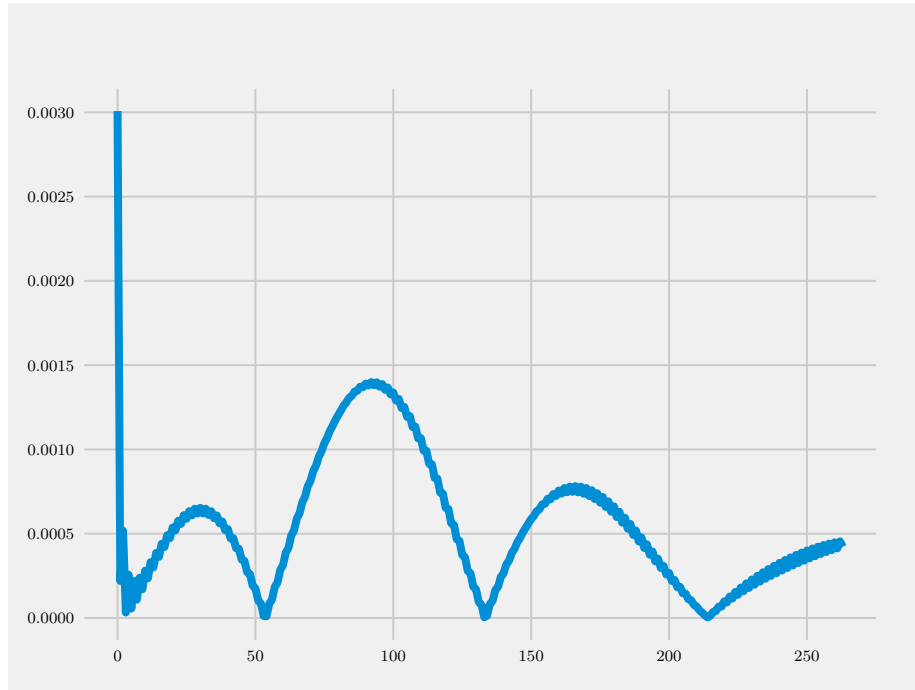


Figure 164: Second Order, Case number 0, 264points

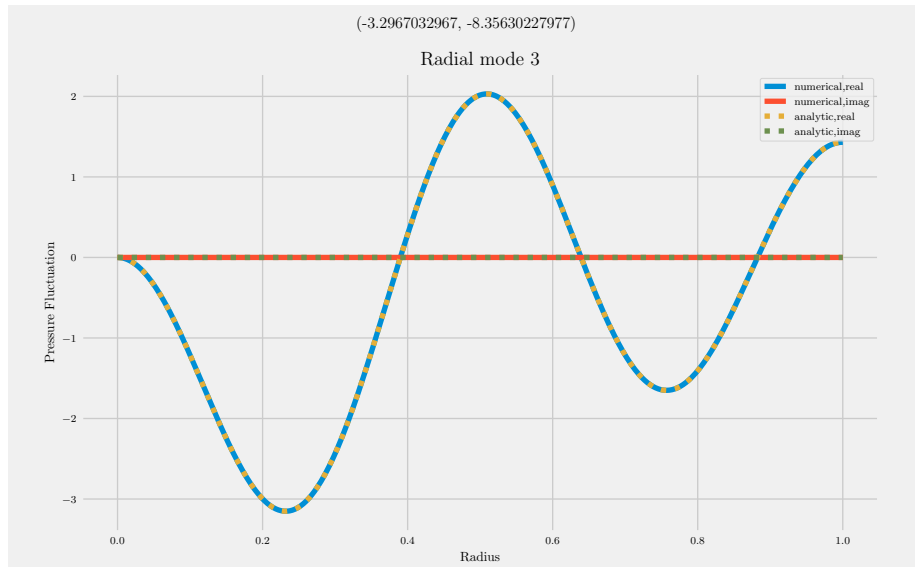


Figure 165: Second Order, Case number 0, 528points

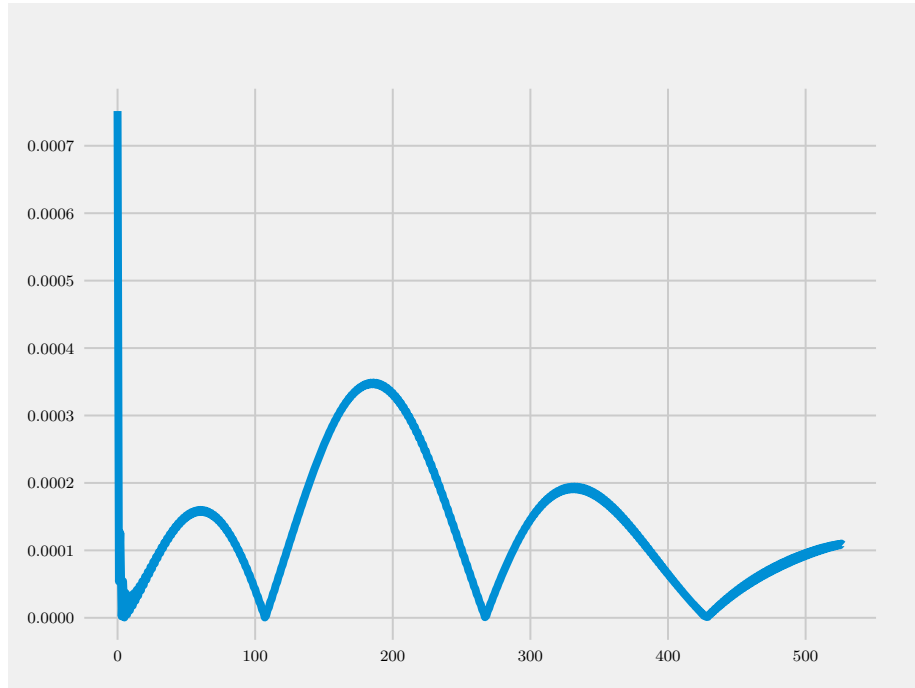


Figure 166: Second Order, Case number 0, 528points

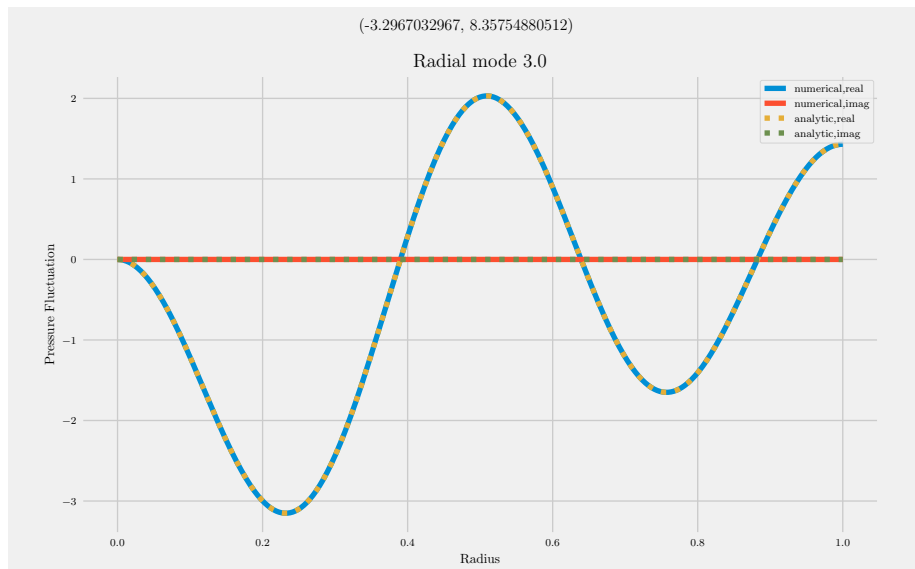


Figure 167: Second Order, Case number 0, 1056points

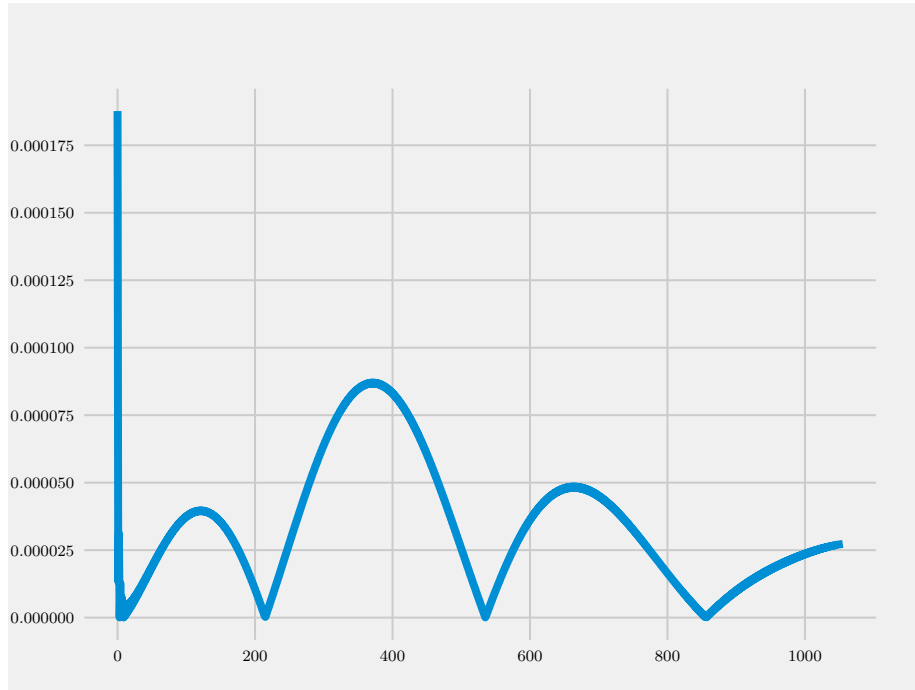


Figure 168: Second Order, Case number 0, 1056points

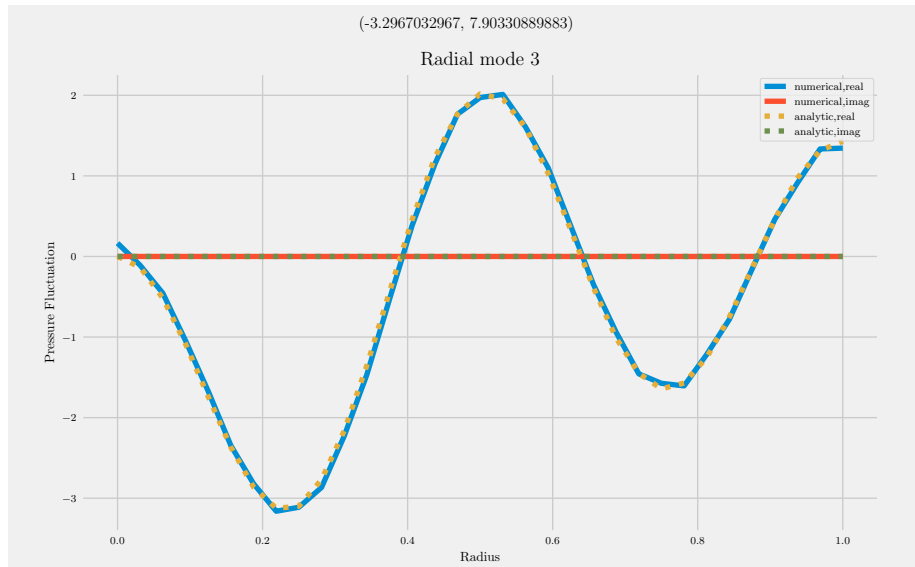


Figure 169: Second Order, Case number 1, 33points

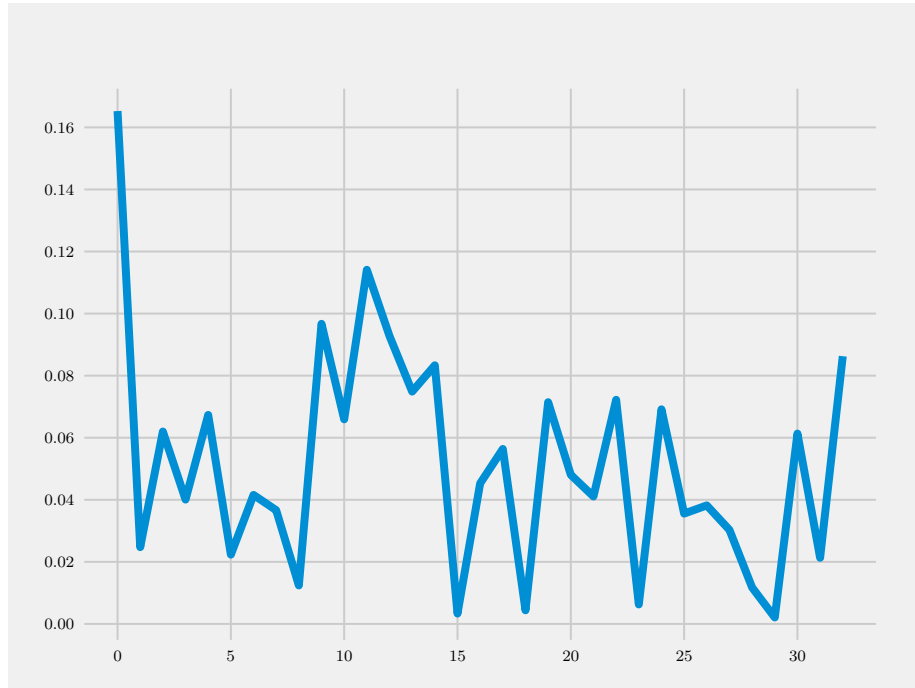


Figure 170: Second Order, Case number 1, 33points

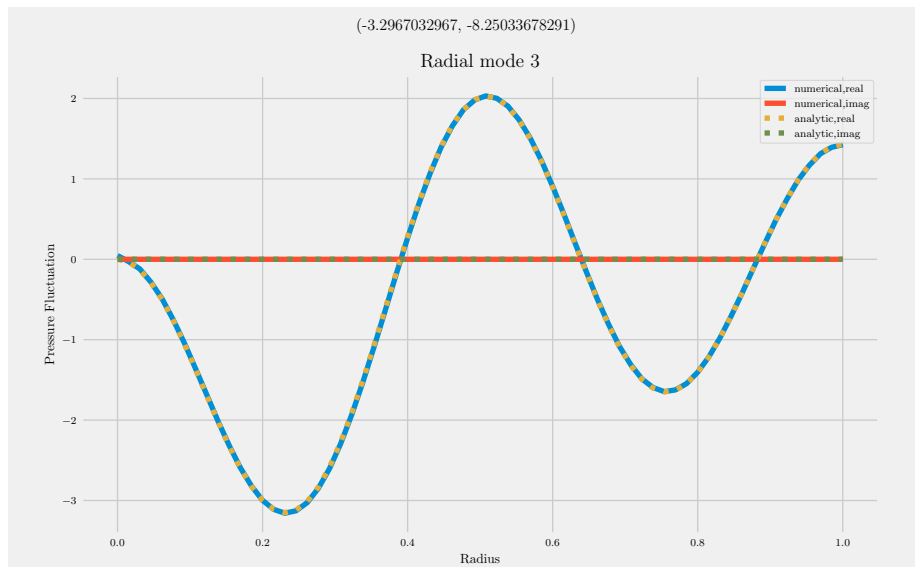


Figure 171: Second Order, Case number 1, 66points

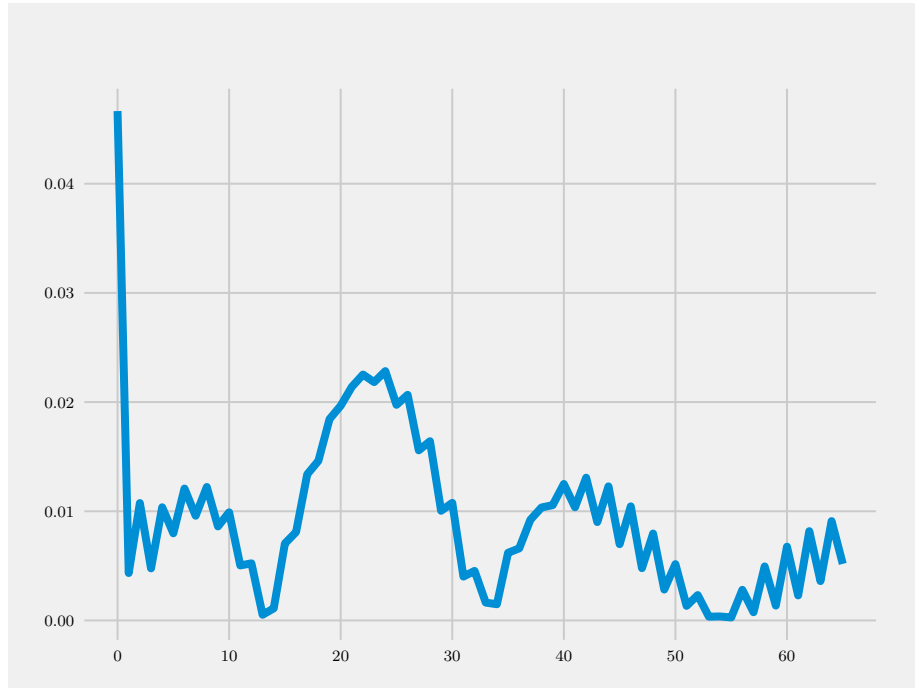


Figure 172: Second Order, Case number 1, 66points

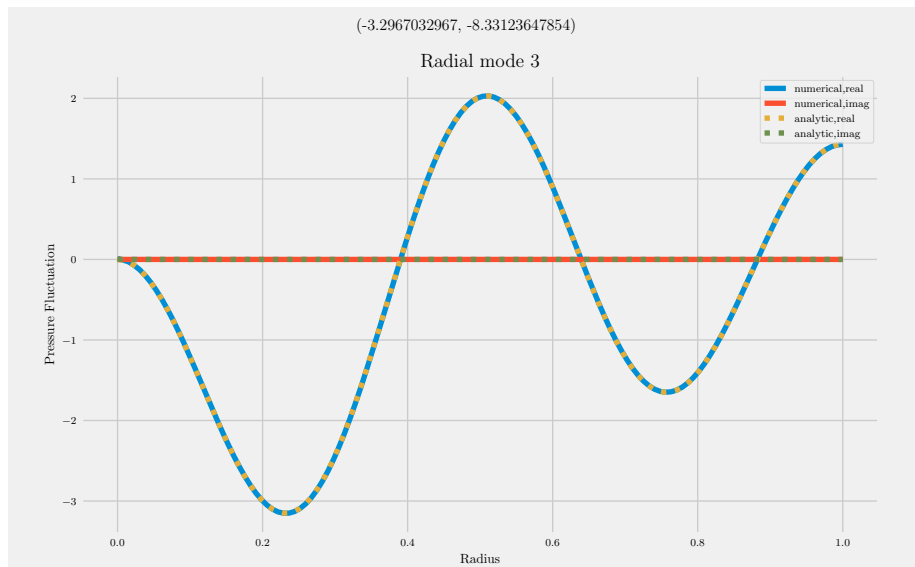


Figure 173: Second Order, Case number 1, 132points

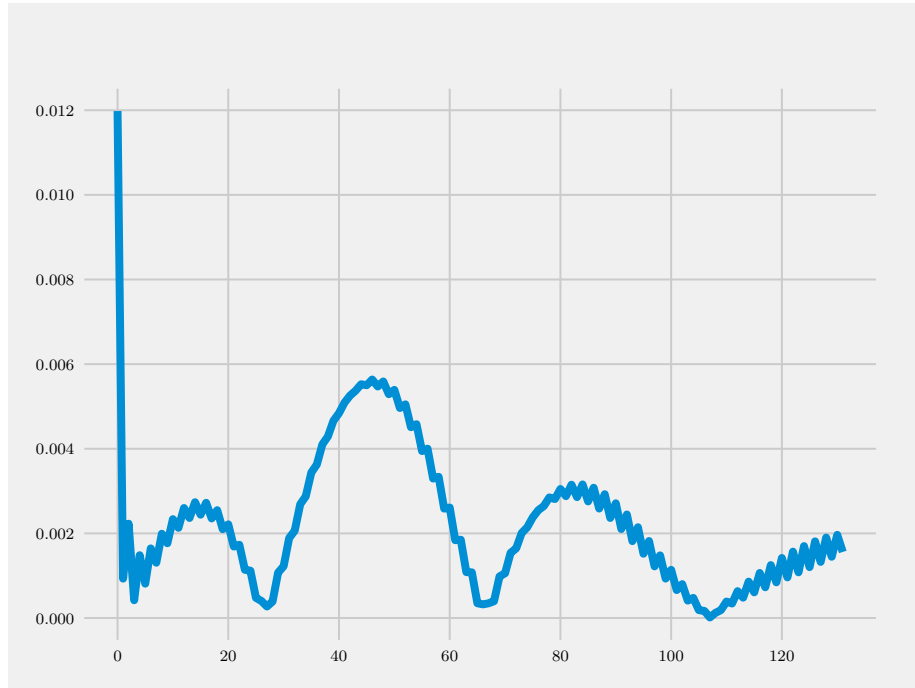


Figure 174: Second Order, Case number 1, 132points

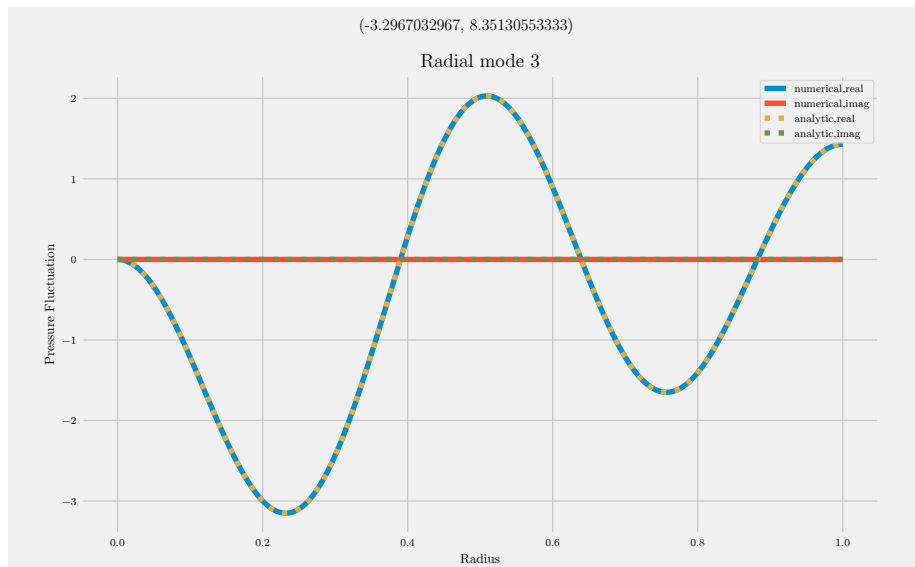


Figure 175: Second Order, Case number 1, 264points

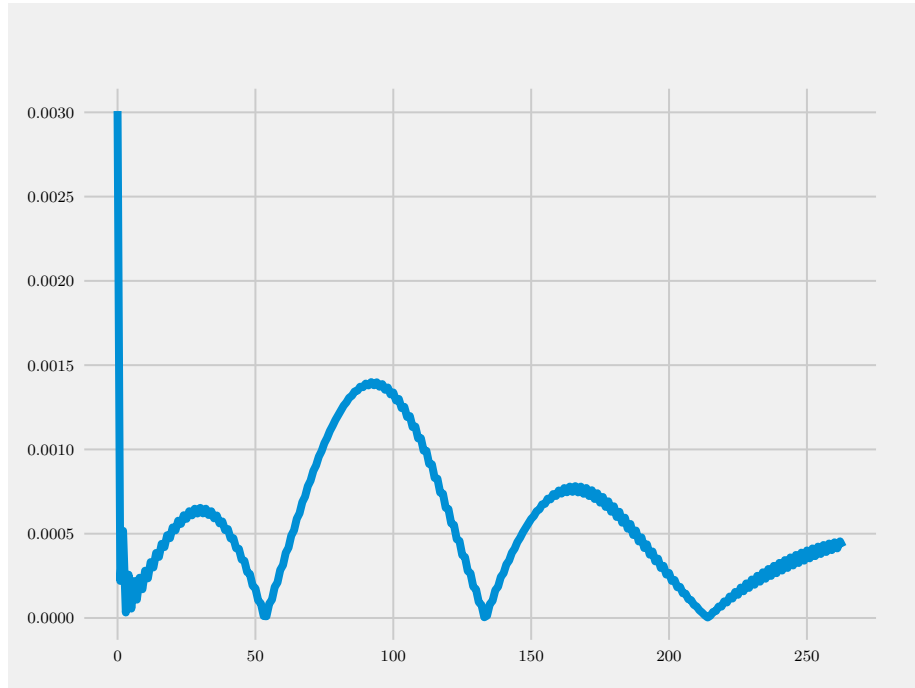


Figure 176: Second Order, Case number 1, 264points

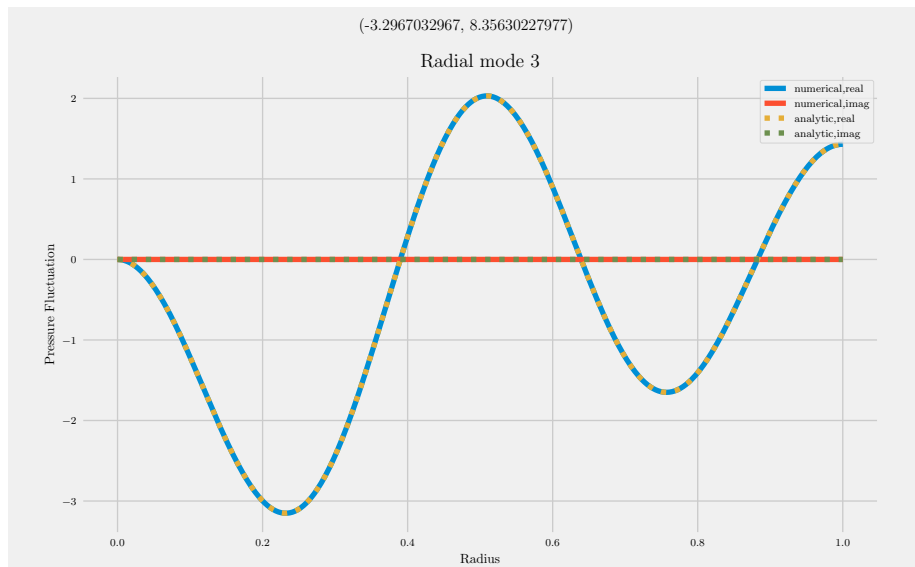


Figure 177: Second Order, Case number 1, 528points

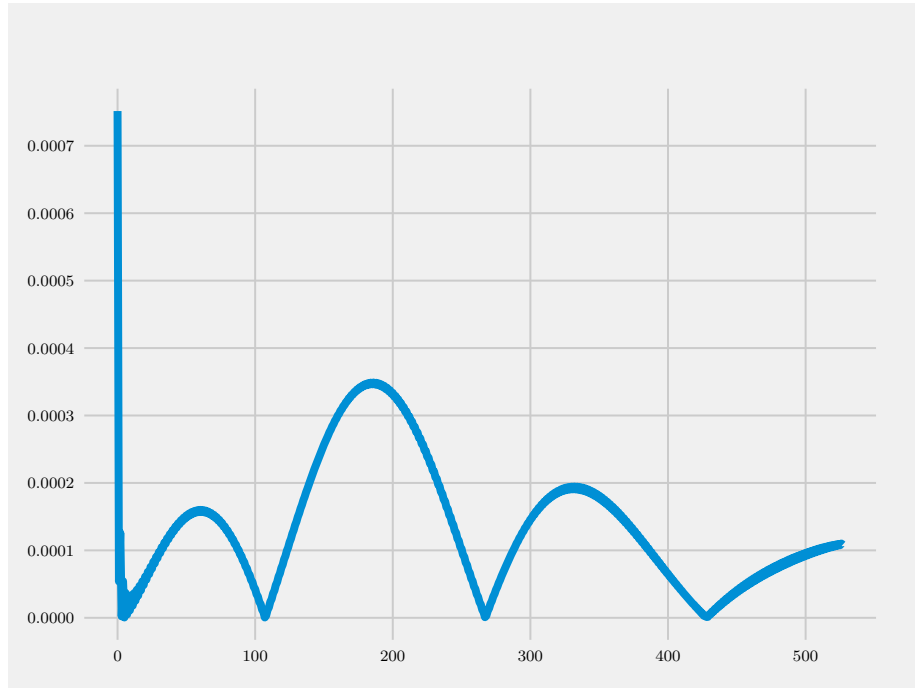


Figure 178: Second Order, Case number 1, 528points

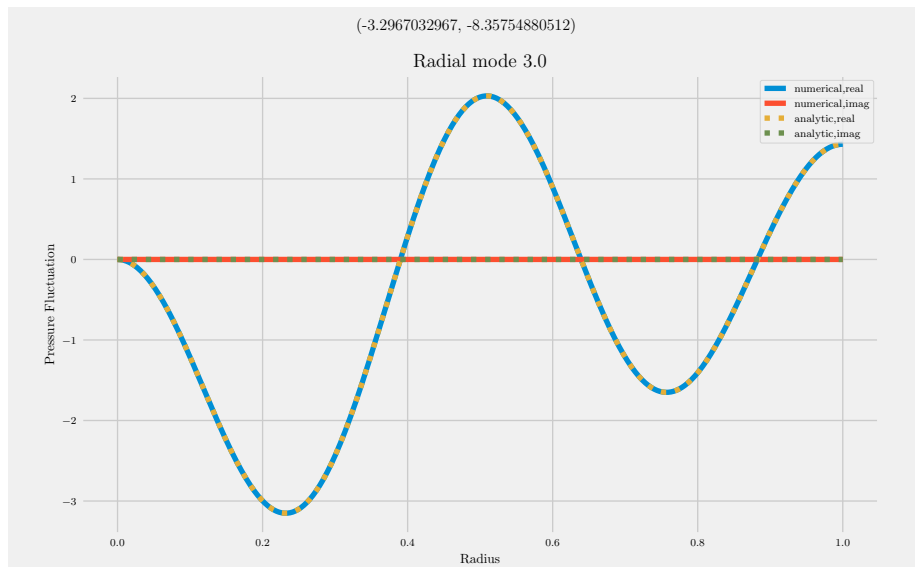


Figure 179: Second Order, Case number 1, 1056points

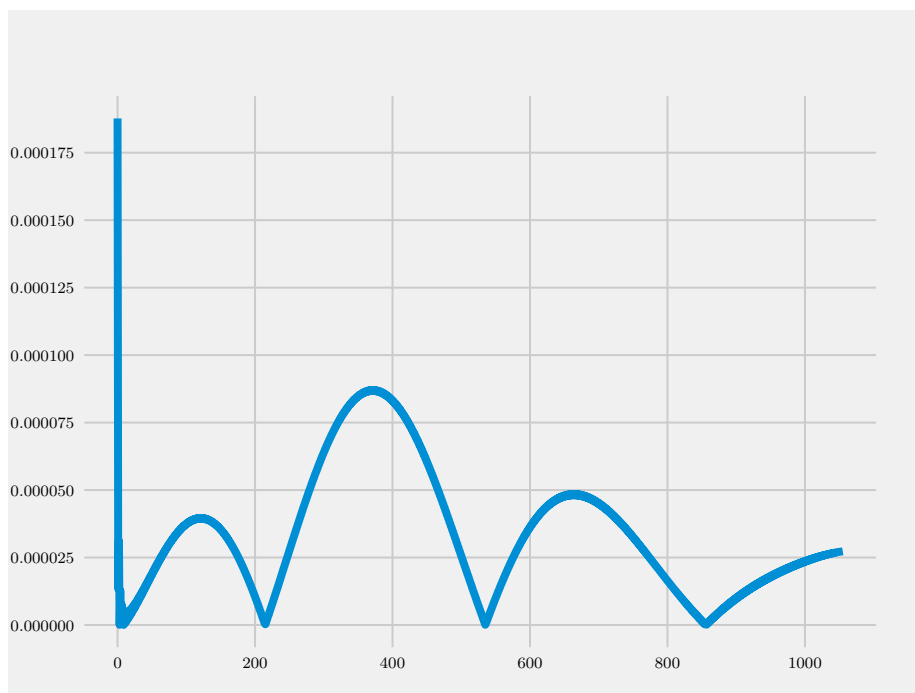


Figure 180: Second Order, Case number 1, 1056points

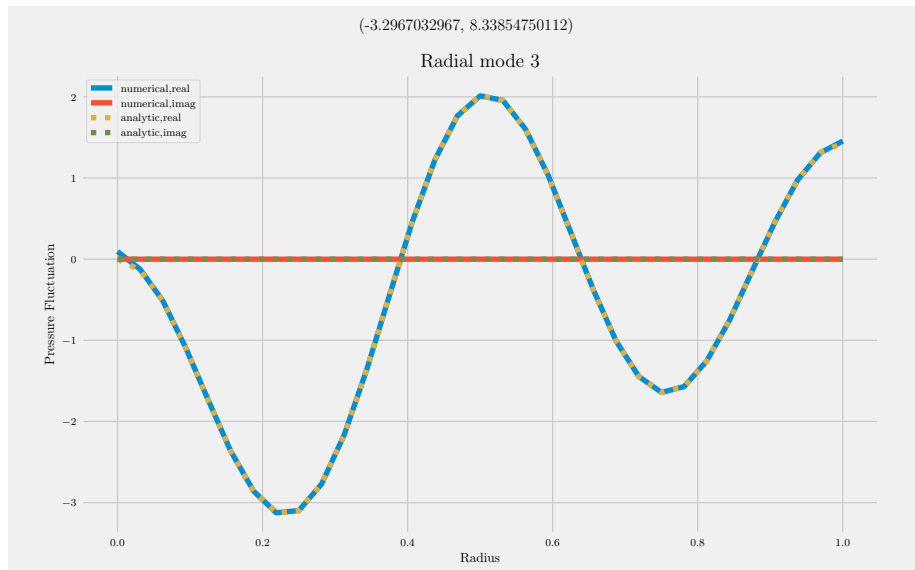


Figure 181: Fourth Order, Case number 0, 33points

Fourth Order, Radial Mode 3

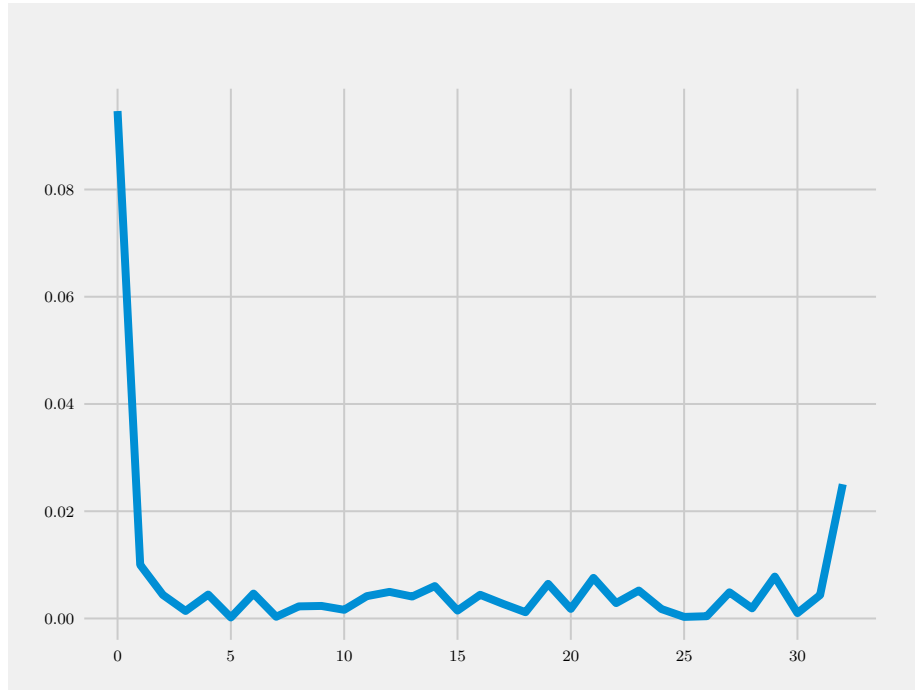


Figure 182: Fourth Order, Case number 0, 33points

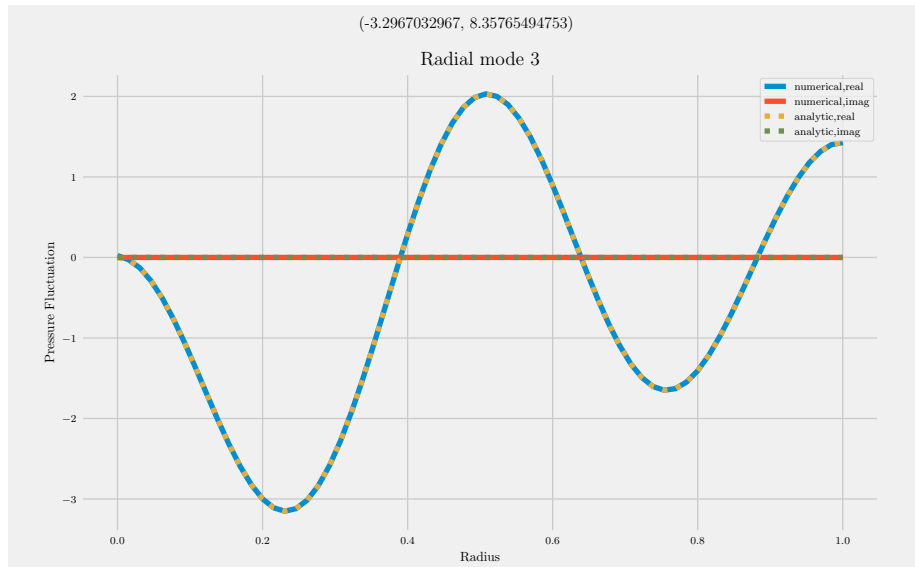


Figure 183: Fourth Order, Case number 0, 66points

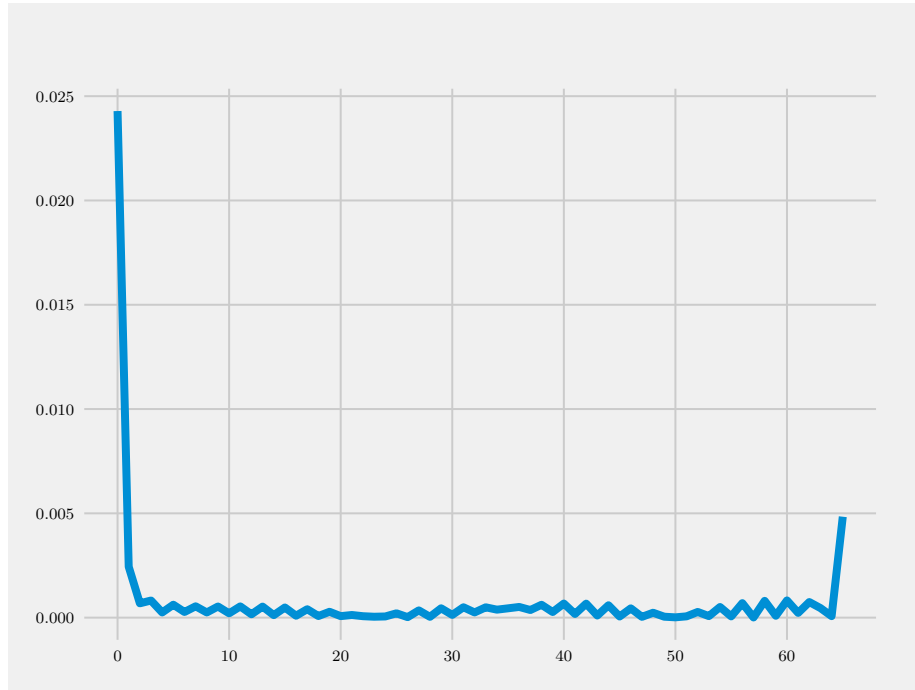


Figure 184: Fourth Order, Case number 0, 66points

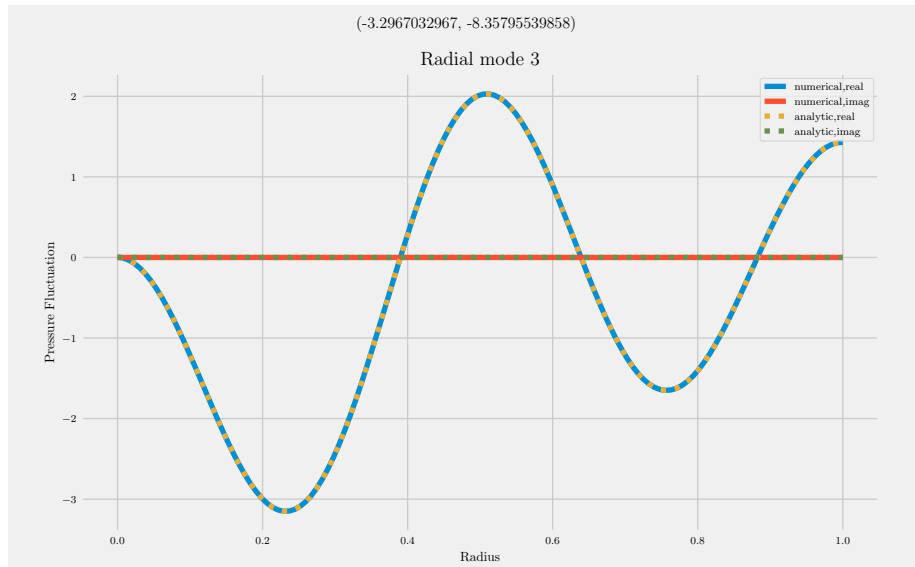


Figure 185: Fourth Order, Case number 0, 132points

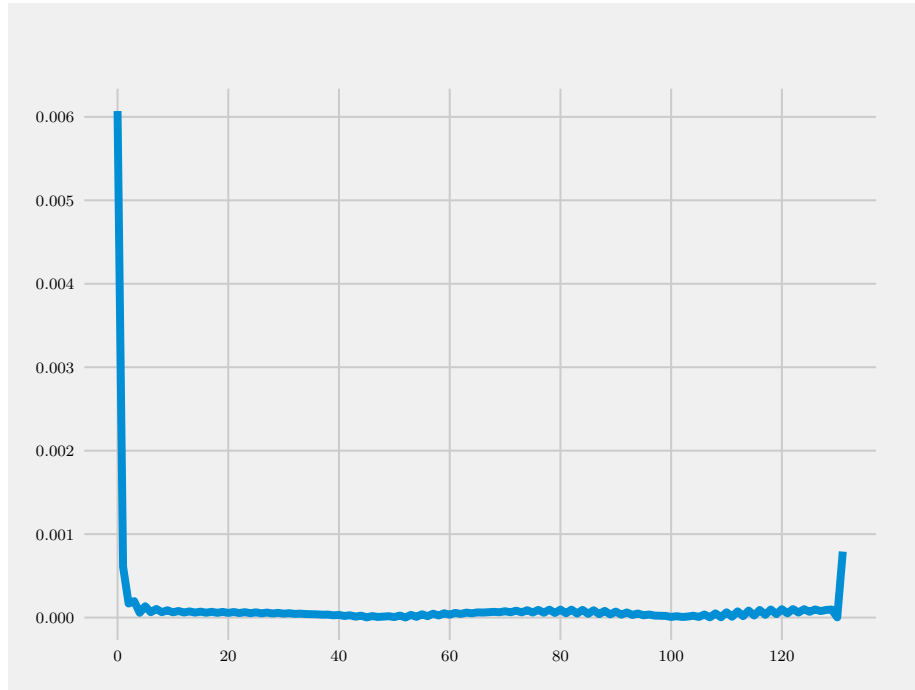


Figure 186: Fourth Order, Case number 0, 132points

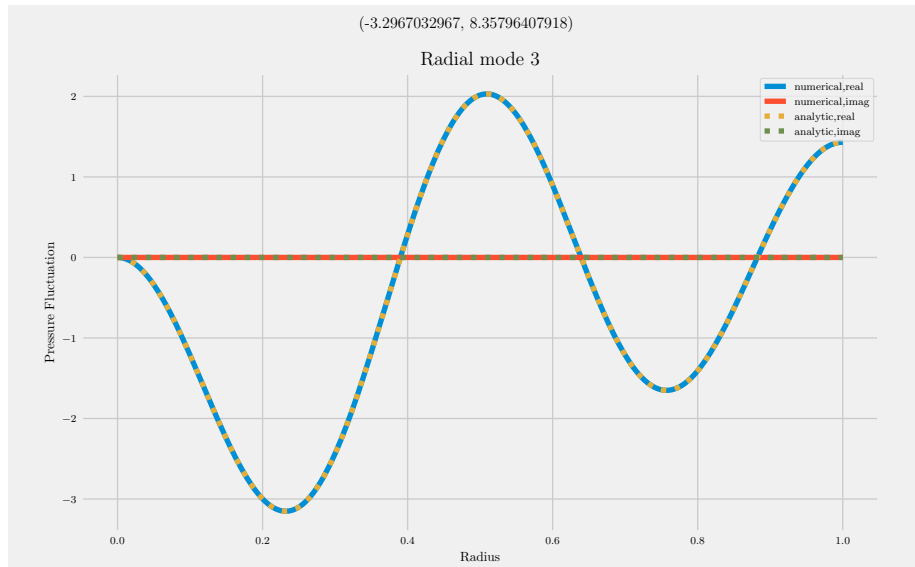


Figure 187: Fourth Order, Case number 0, 264points

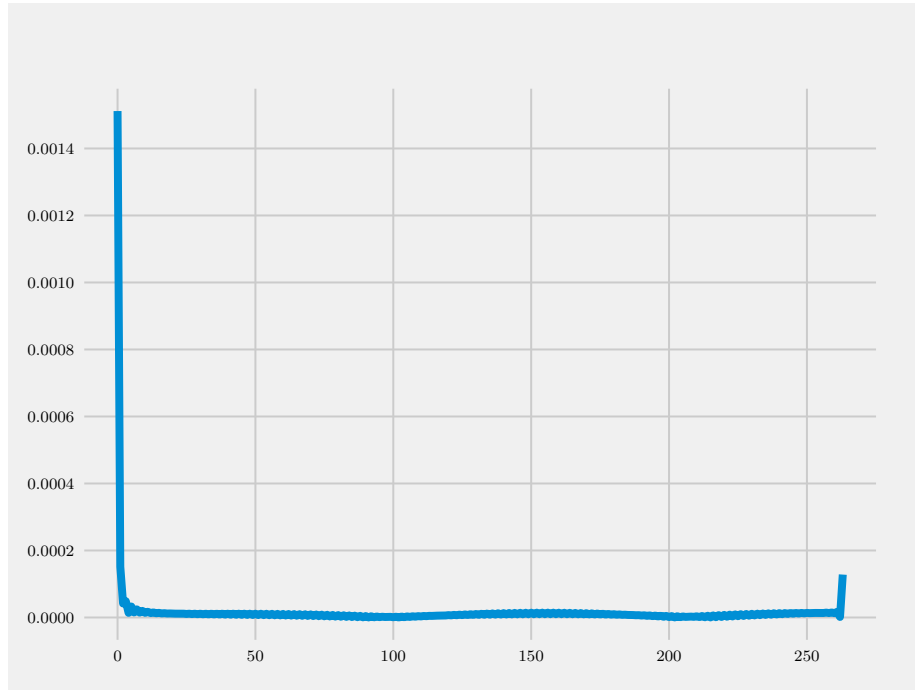


Figure 188: Fourth Order, Case number 0, 264points

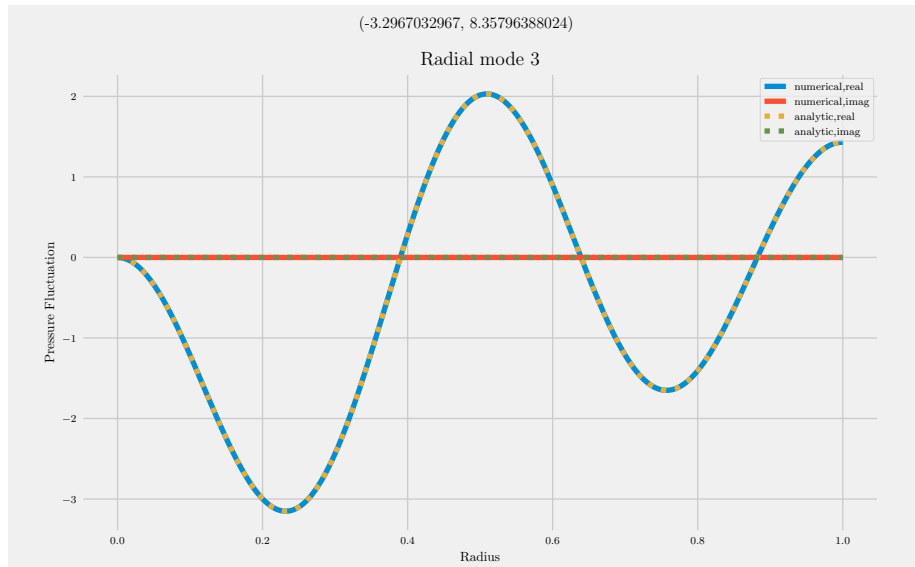


Figure 189: Fourth Order, Case number 0, 528points

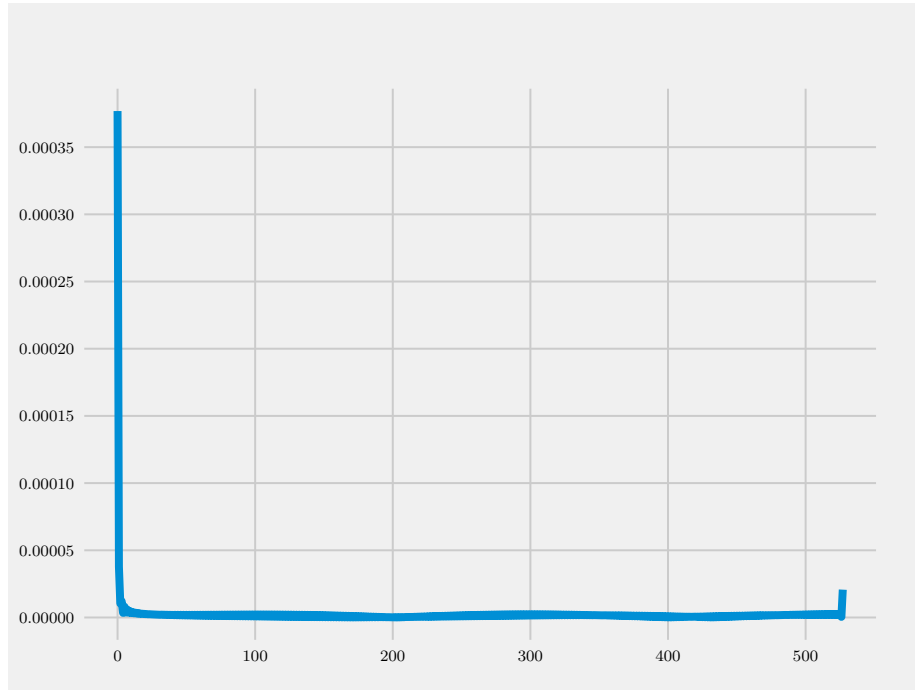


Figure 190: Fourth Order, Case number 0, 528points

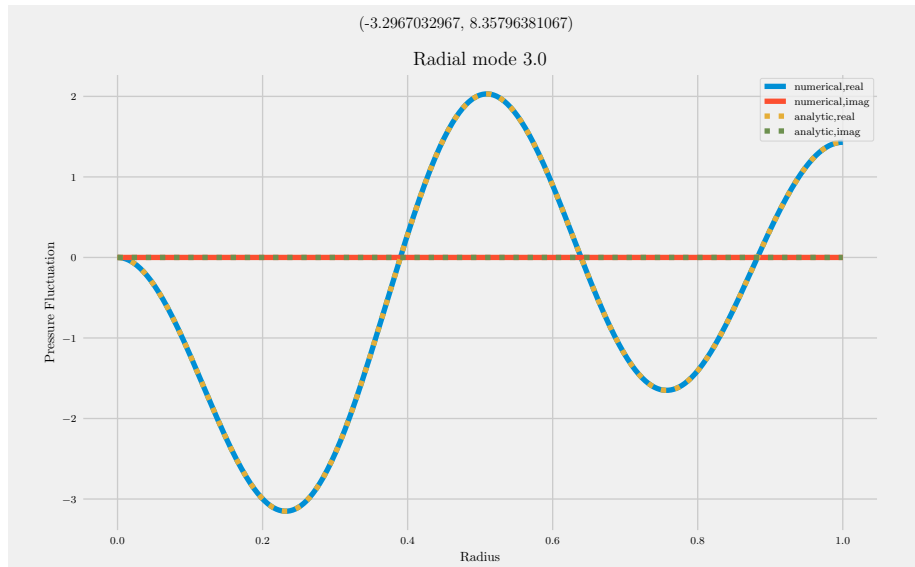


Figure 191: Fourth Order, Case number 0, 1056points

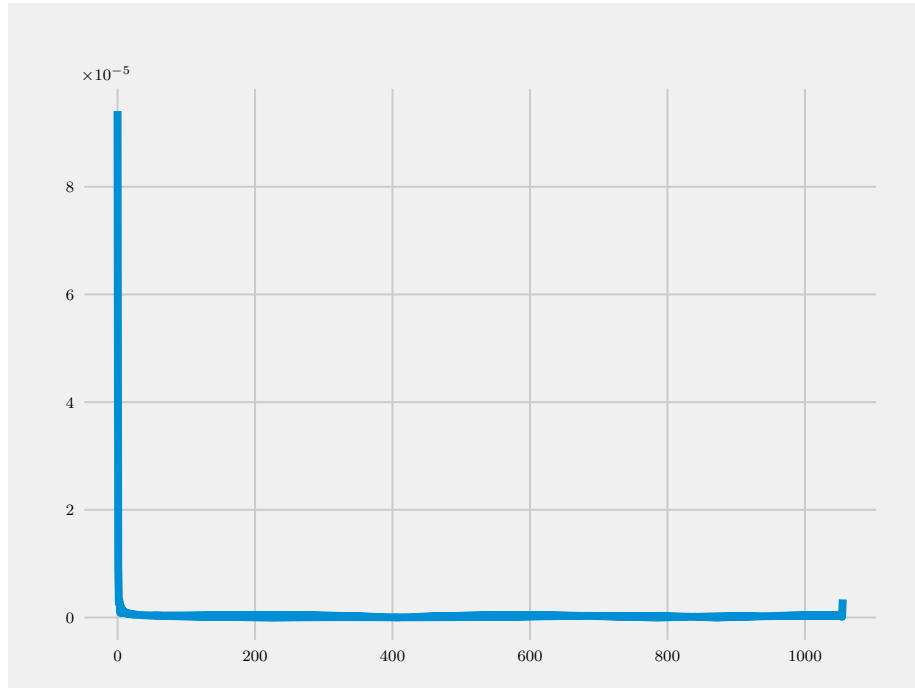


Figure 192: Fourth Order, Case number 0, 1056points

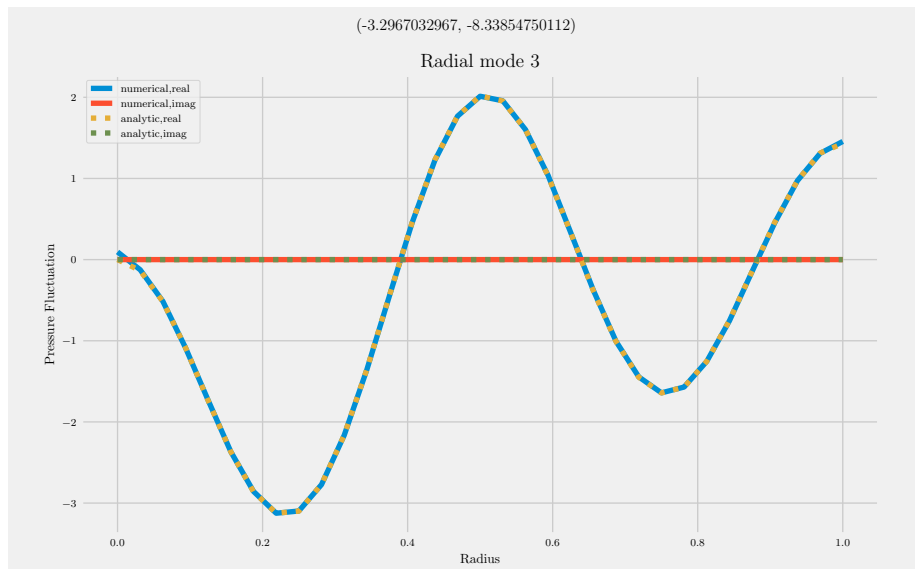


Figure 193: Fourth Order, Case number 1, 33points

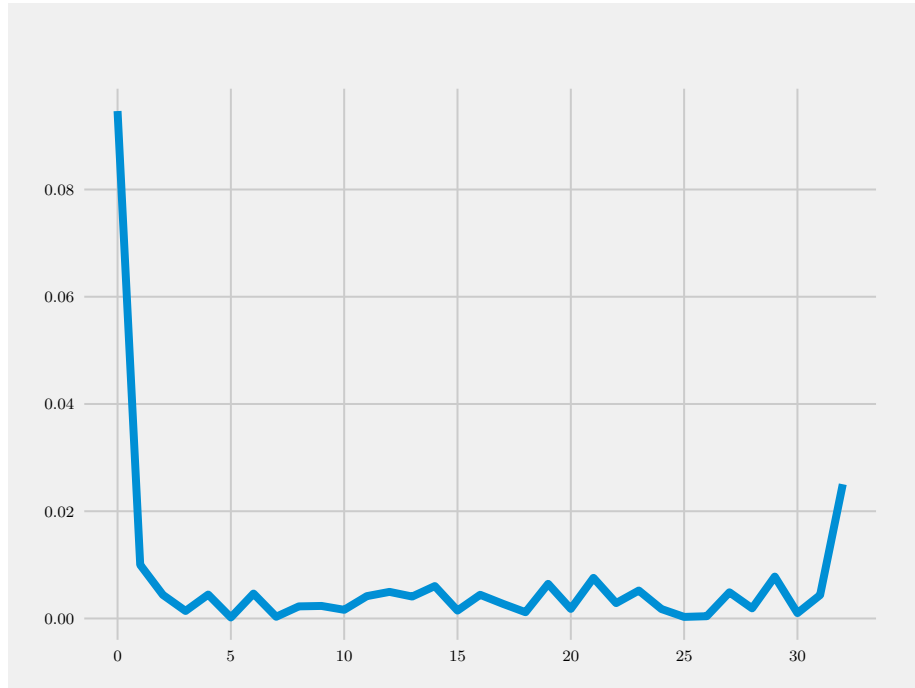


Figure 194: Fourth Order, Case number 1, 33points

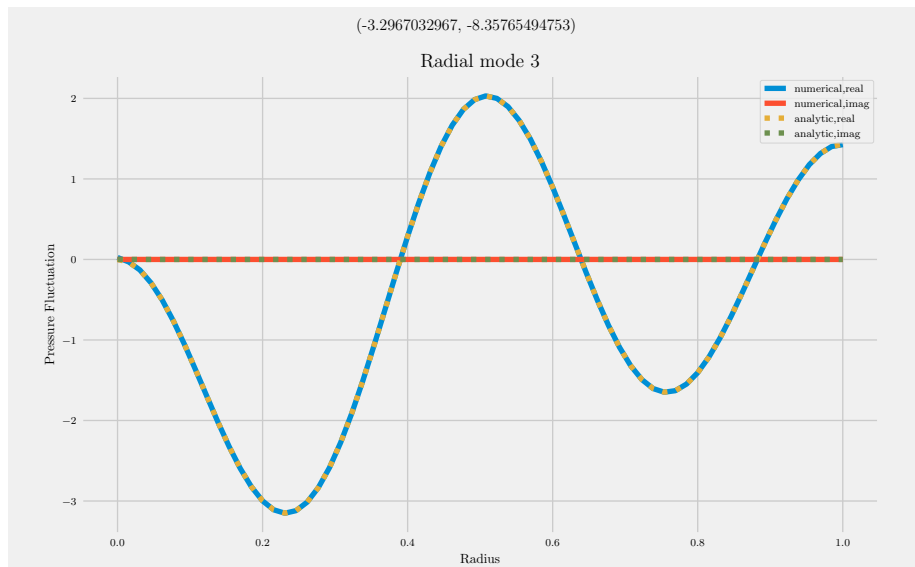


Figure 195: Fourth Order, Case number 1, 66points

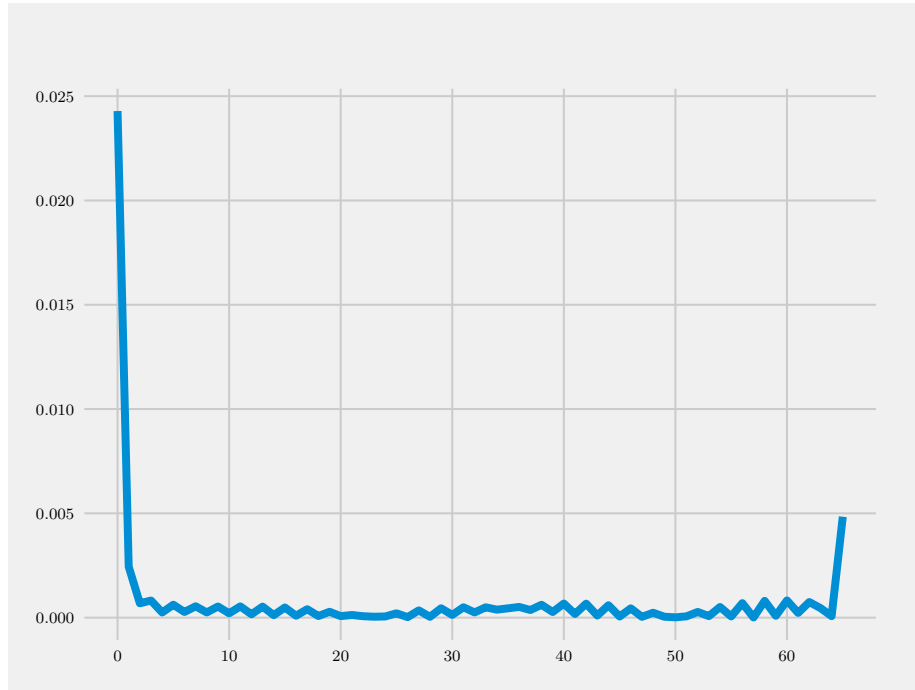


Figure 196: Fourth Order, Case number 1, 66points

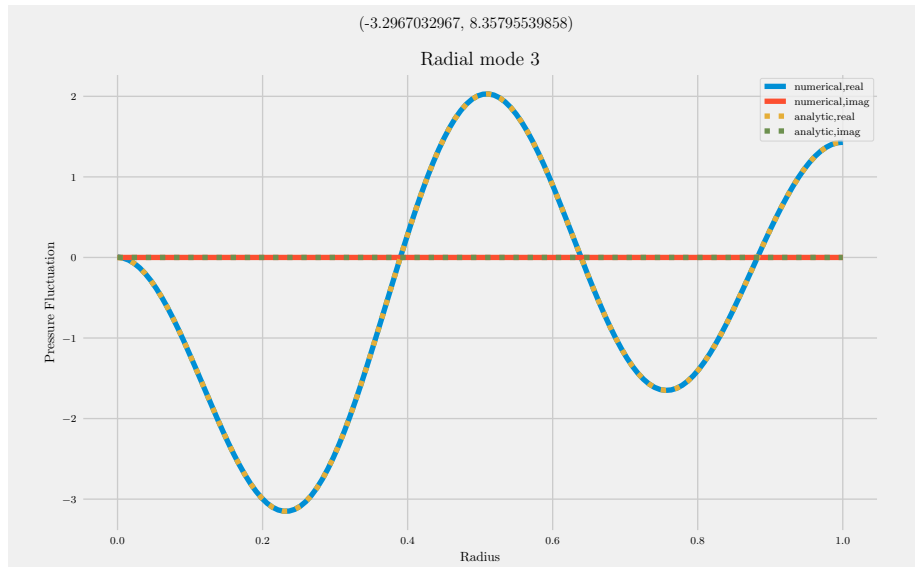


Figure 197: Fourth Order, Case number 1, 132points

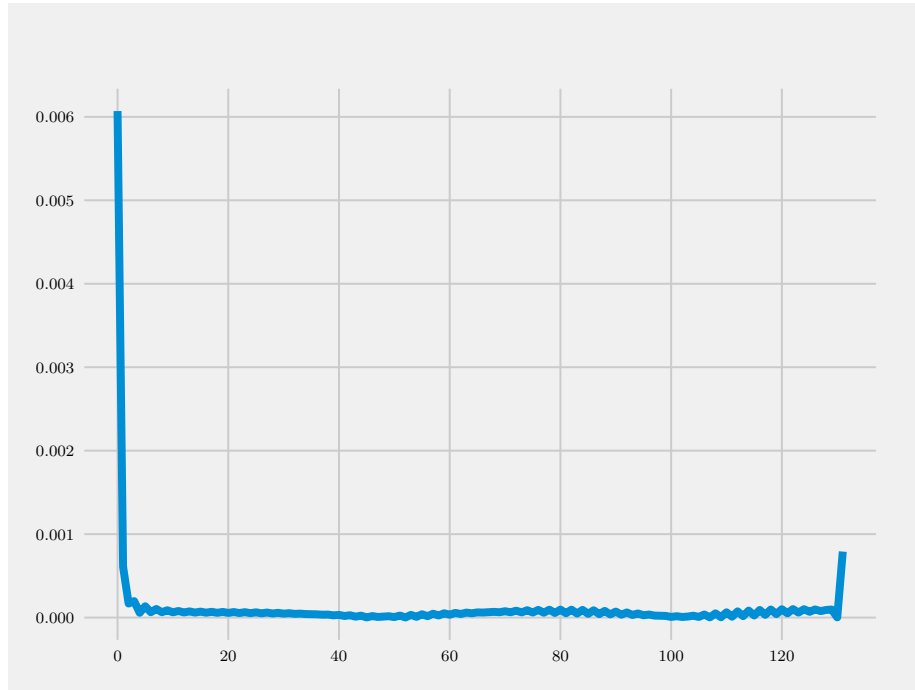


Figure 198: Fourth Order, Case number 1, 132points

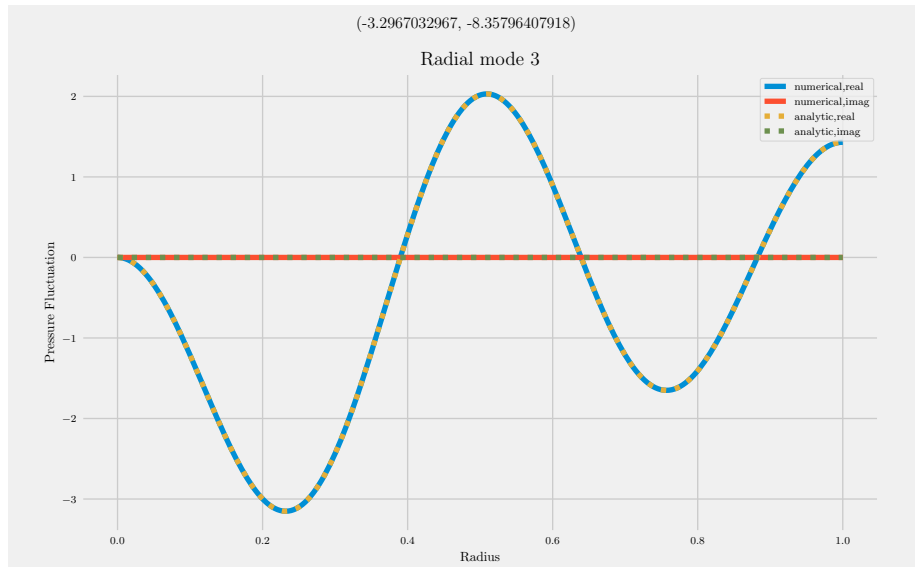


Figure 199: Fourth Order, Case number 1, 264points

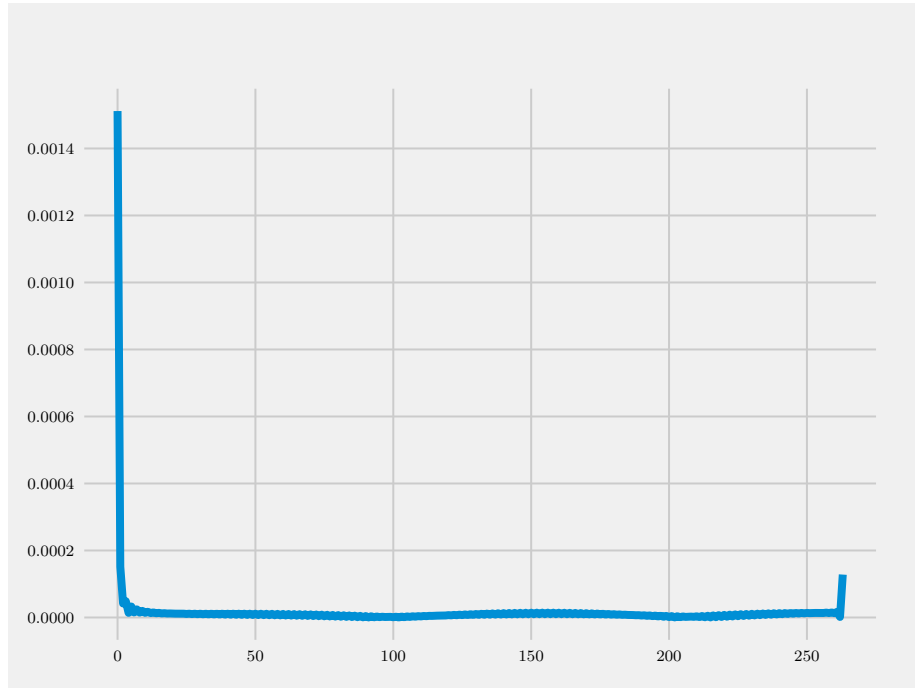


Figure 200: Fourth Order, Case number 1, 264points

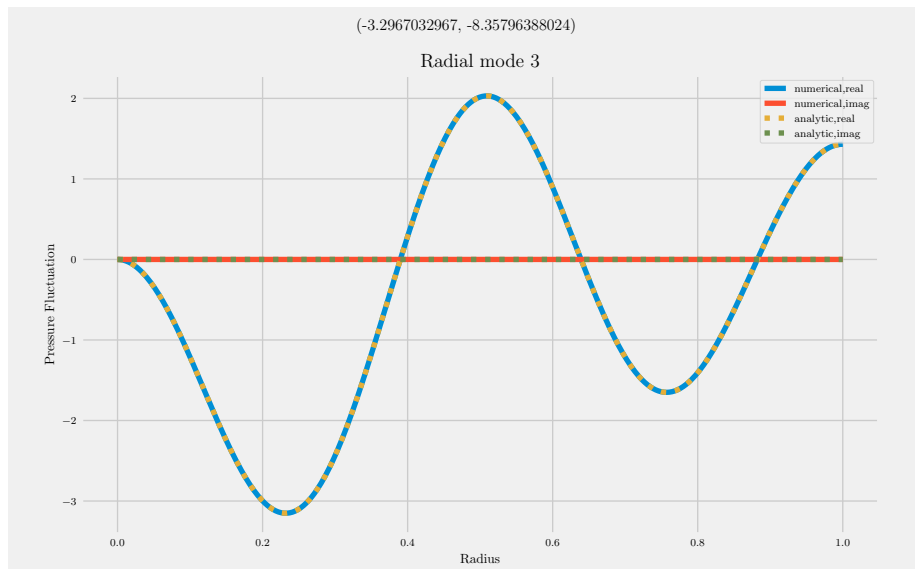


Figure 201: Fourth Order, Case number 1, 528points

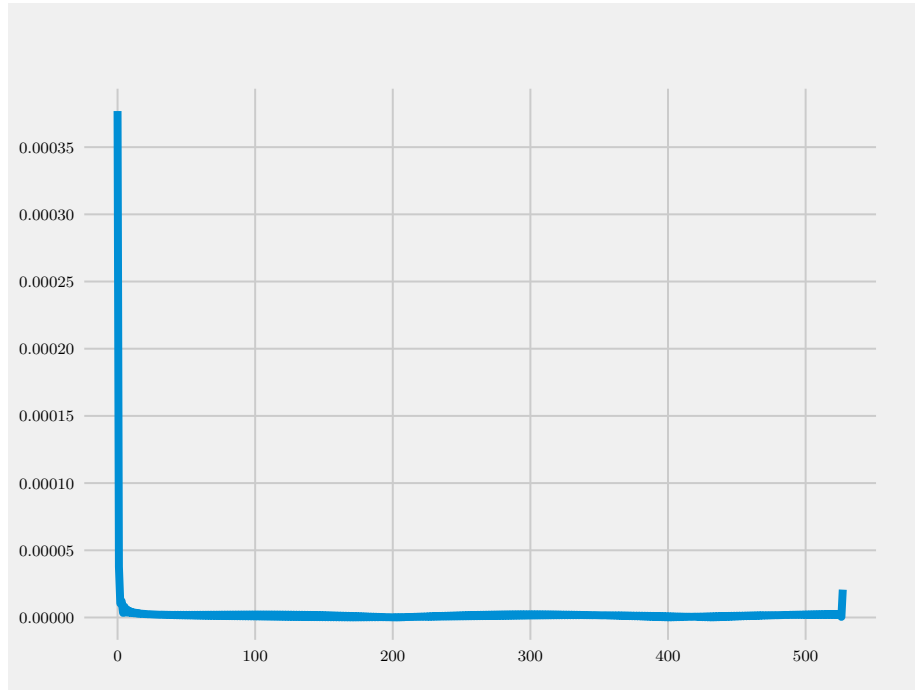


Figure 202: Fourth Order, Case number 1, 528points

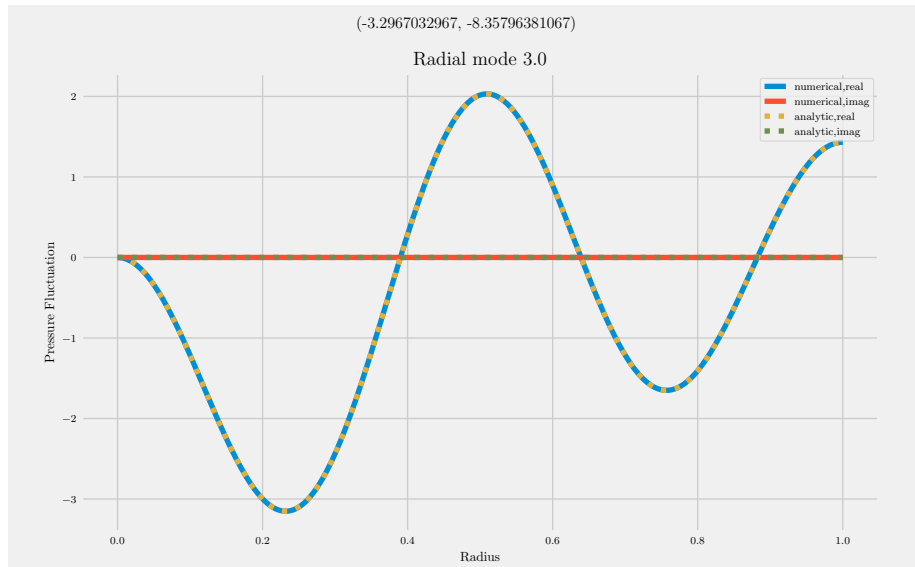


Figure 203: Fourth Order, Case number 1, 1056points

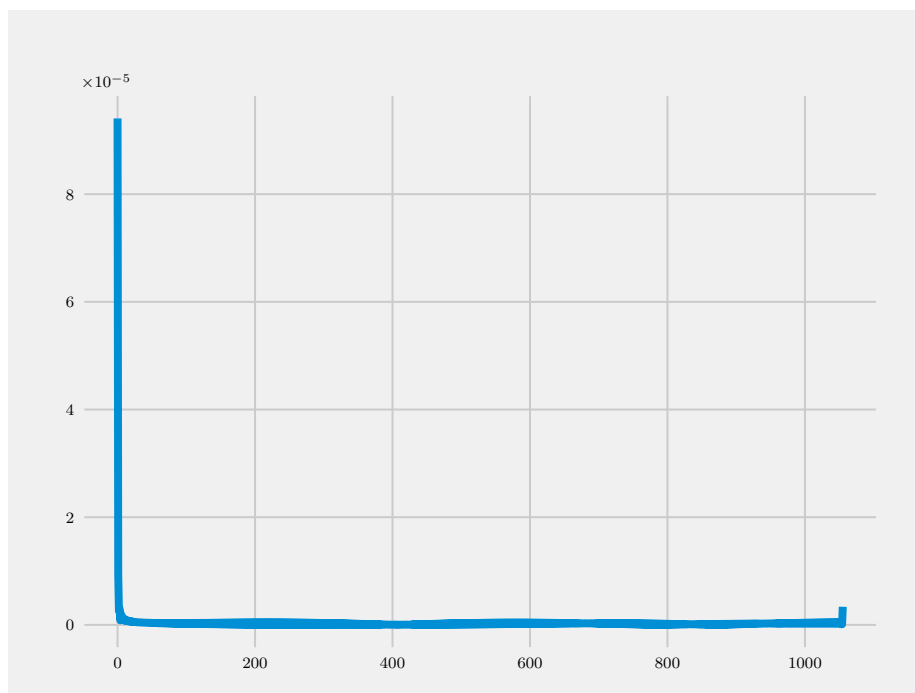


Figure 204: Fourth Order, Case number 1, 1056points

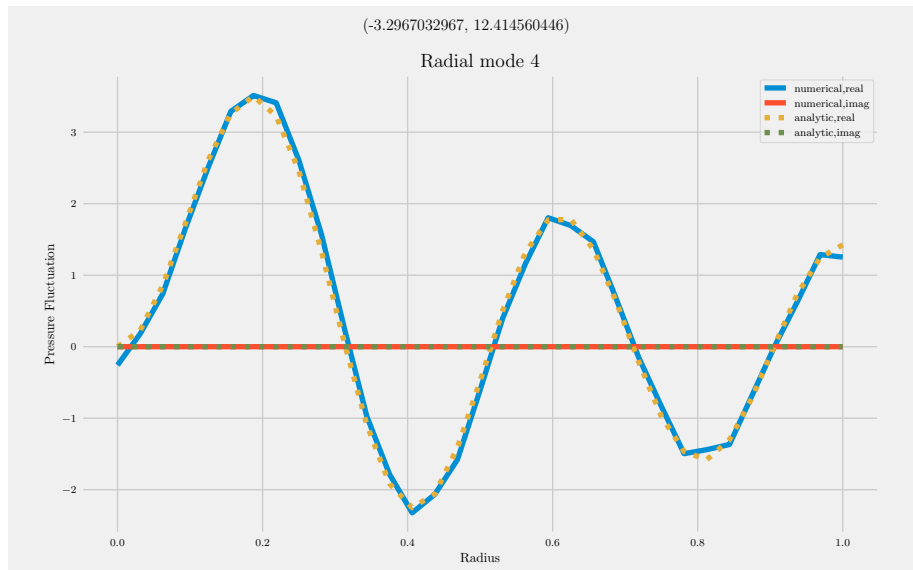


Figure 205: Second Order, Case number 0, 33points

Second Order, Radial Mode 4

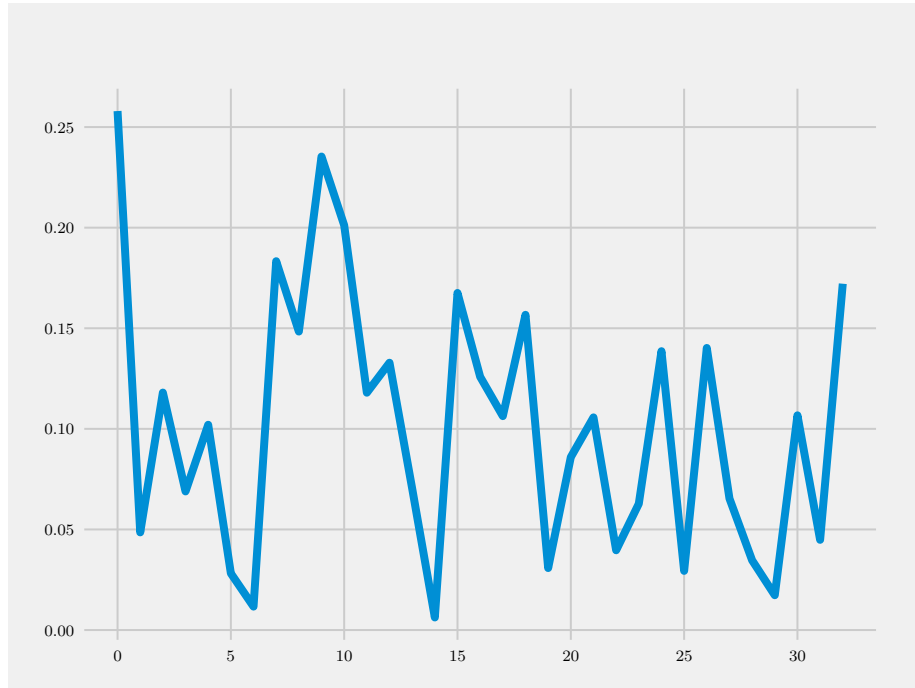


Figure 206: Second Order, Case number 0, 33points

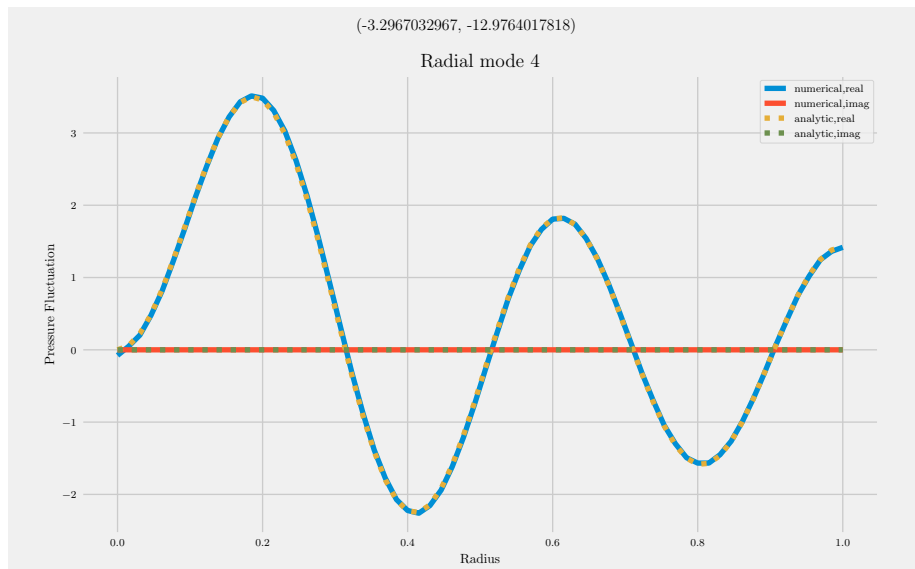


Figure 207: Second Order, Case number 0, 66points

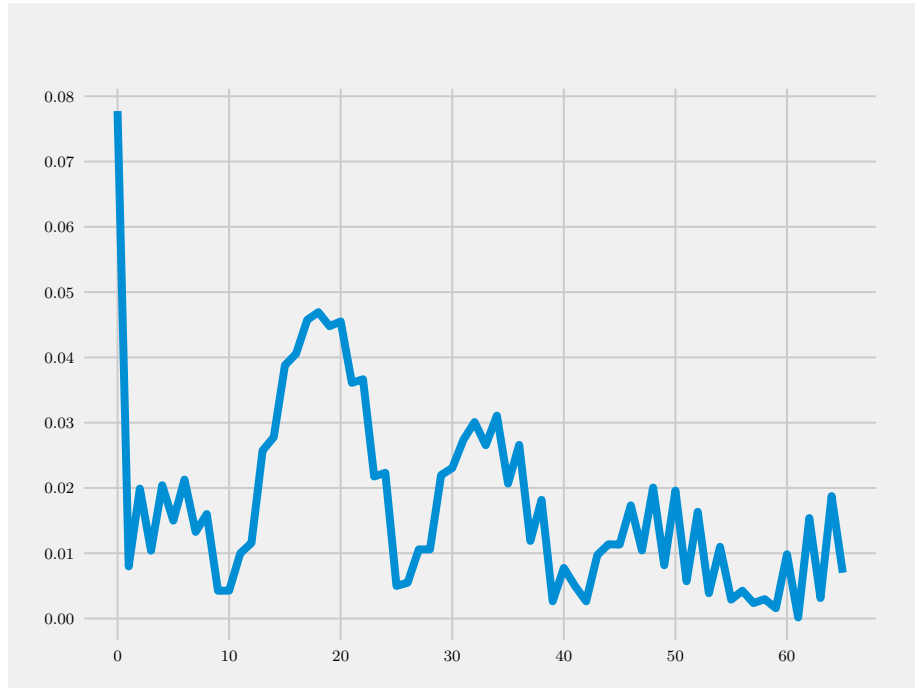


Figure 208: Second Order, Case number 0, 66points

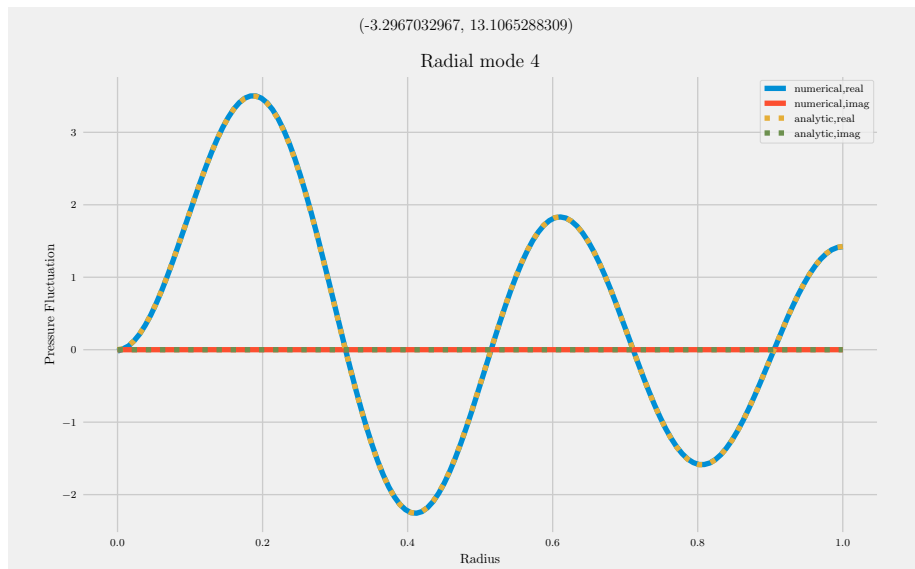


Figure 209: Second Order, Case number 0, 132points

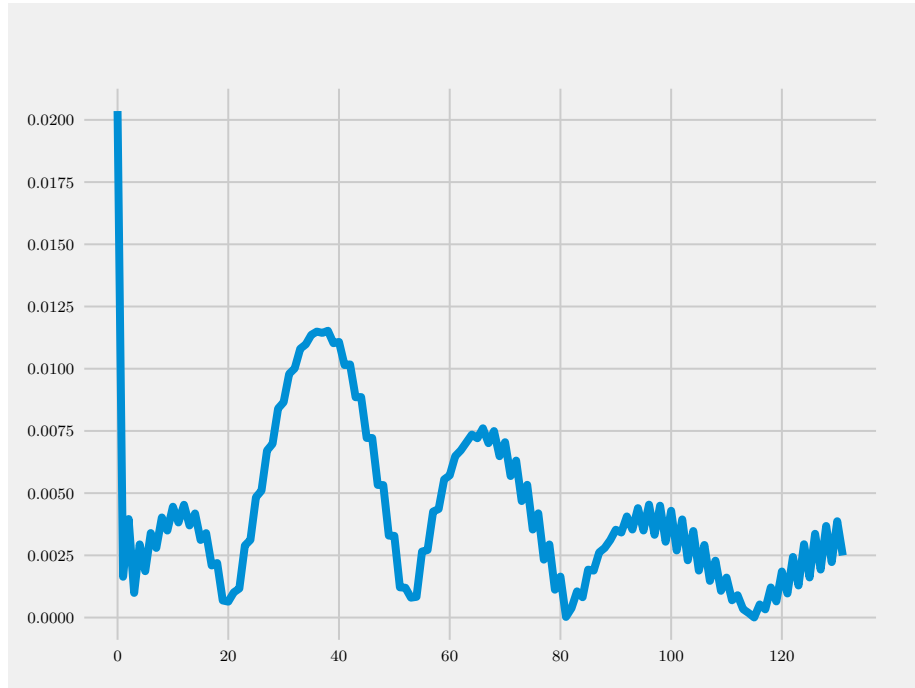


Figure 210: Second Order, Case number 0, 132points

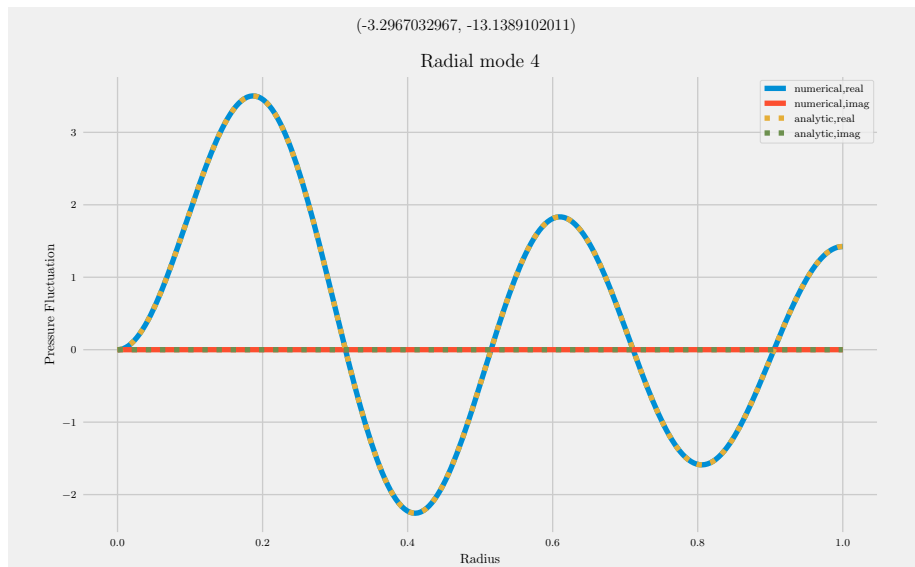


Figure 211: Second Order, Case number 0, 264points

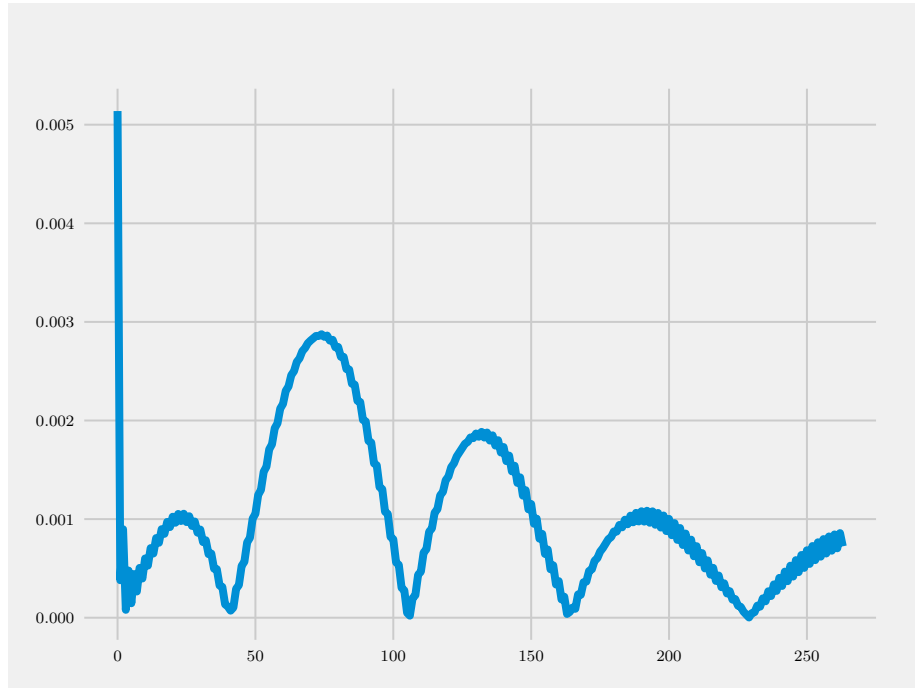


Figure 212: Second Order, Case number 0, 264points

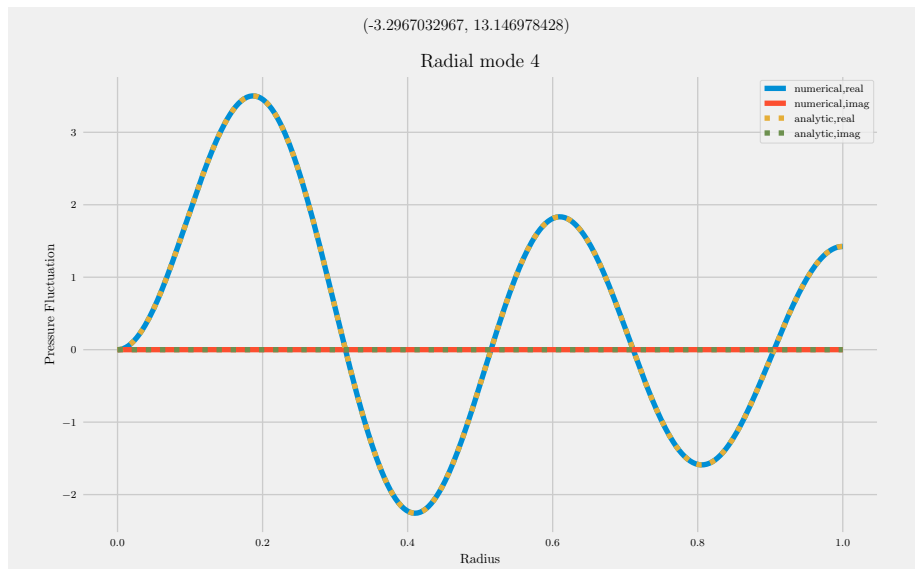


Figure 213: Second Order, Case number 0, 528points

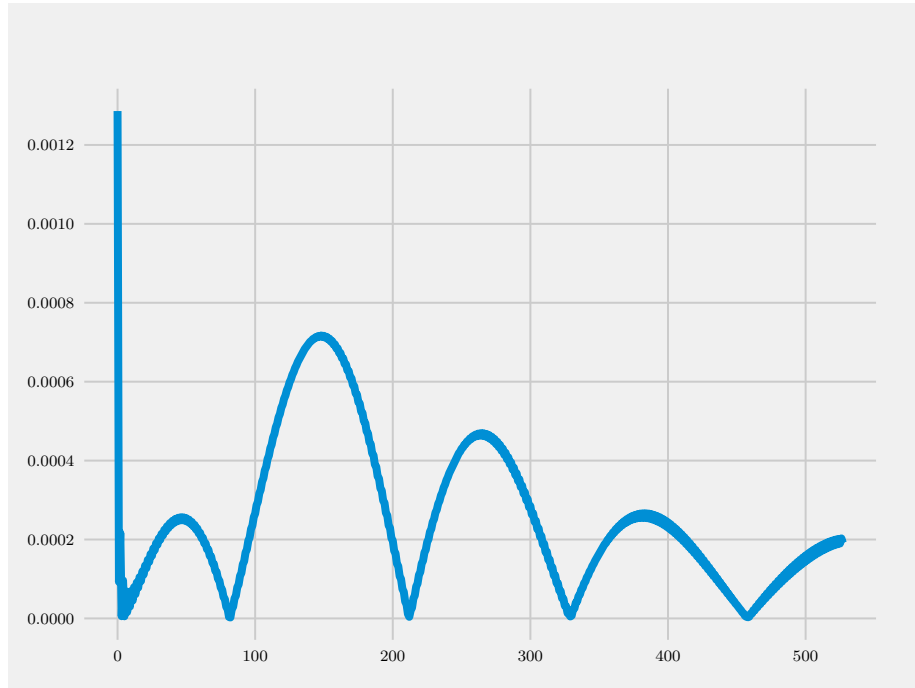


Figure 214: Second Order, Case number 0, 528points

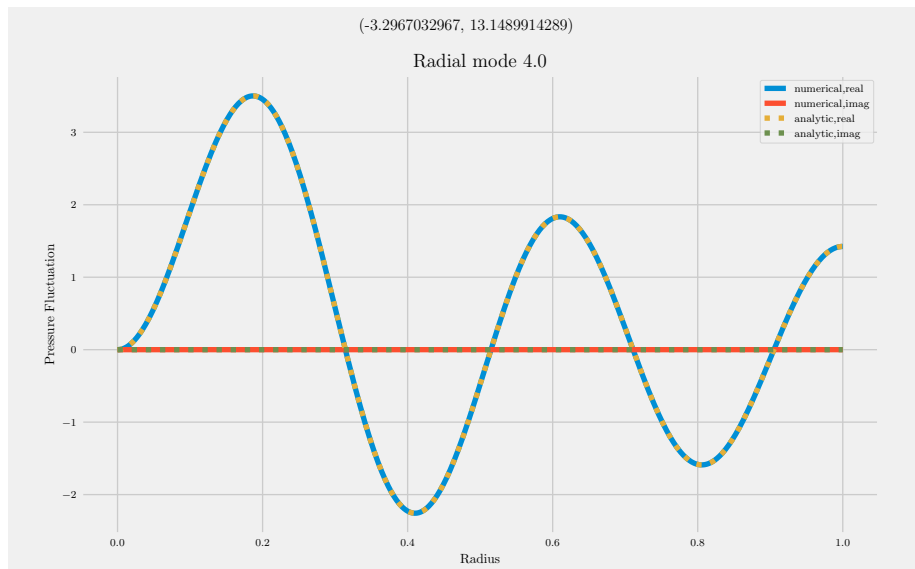


Figure 215: Second Order, Case number 0, 1056points

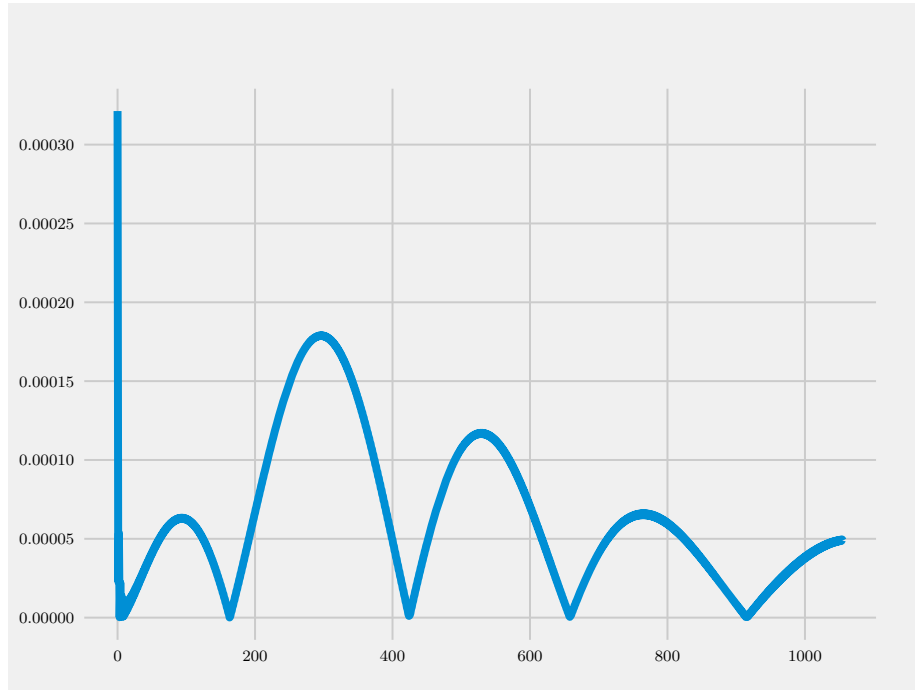


Figure 216: Second Order, Case number 0, 1056points

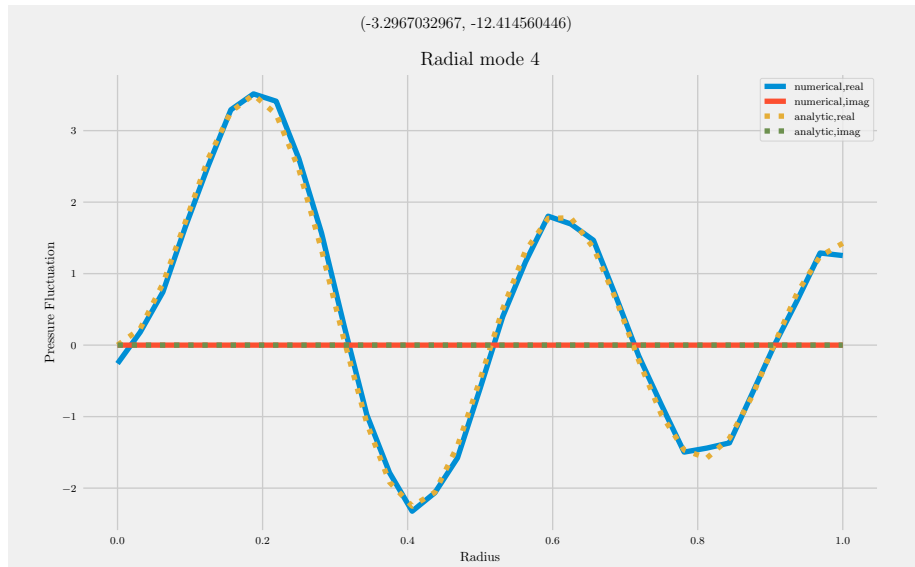


Figure 217: Second Order, Case number 1, 33points

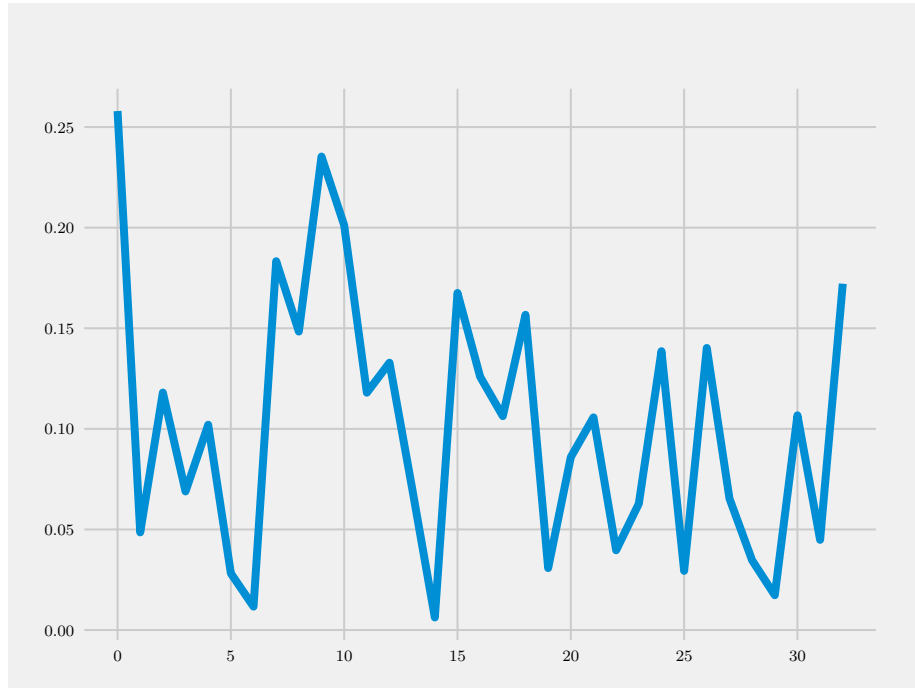


Figure 218: Second Order, Case number 1, 33points

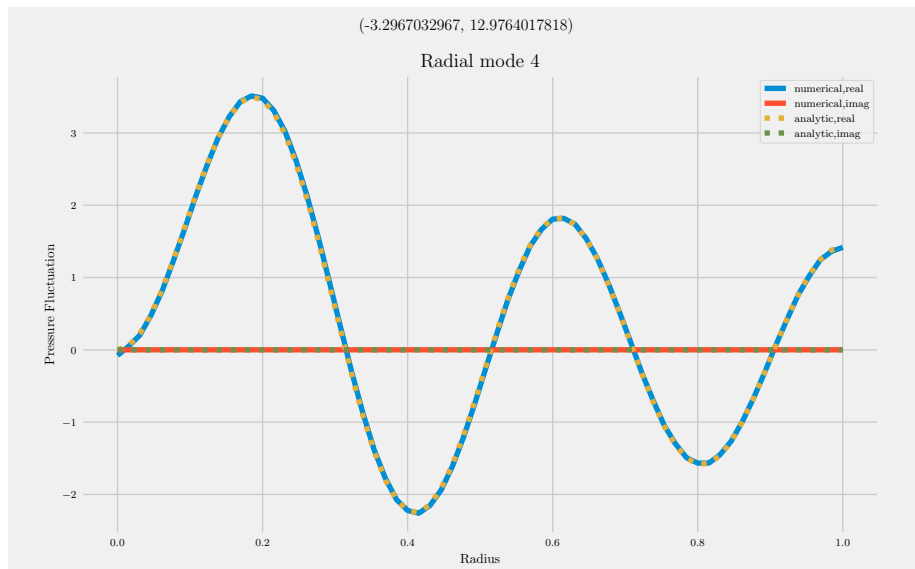


Figure 219: Second Order, Case number 1, 66points

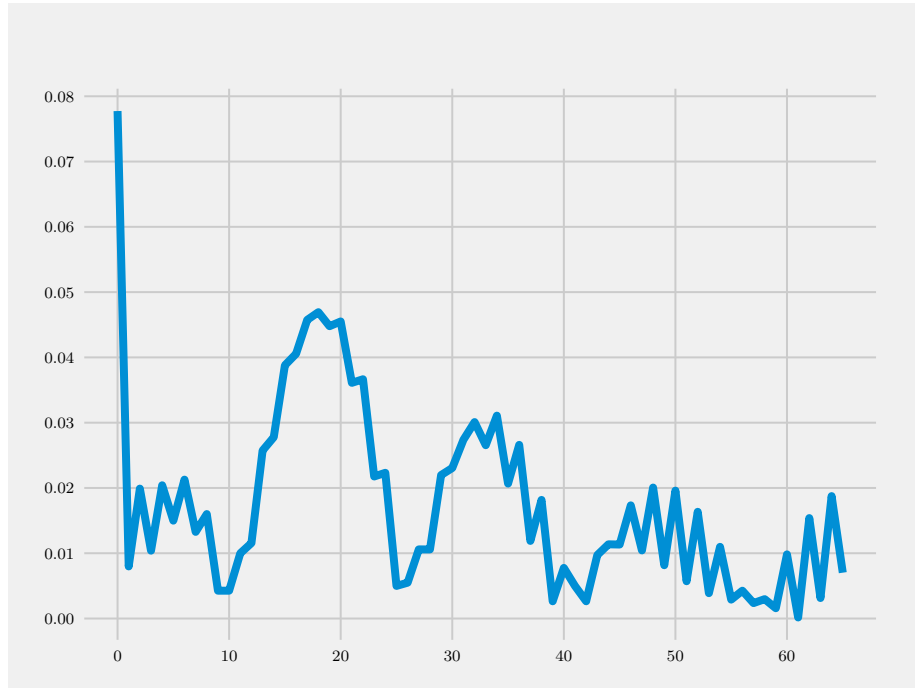


Figure 220: Second Order, Case number 1, 66points

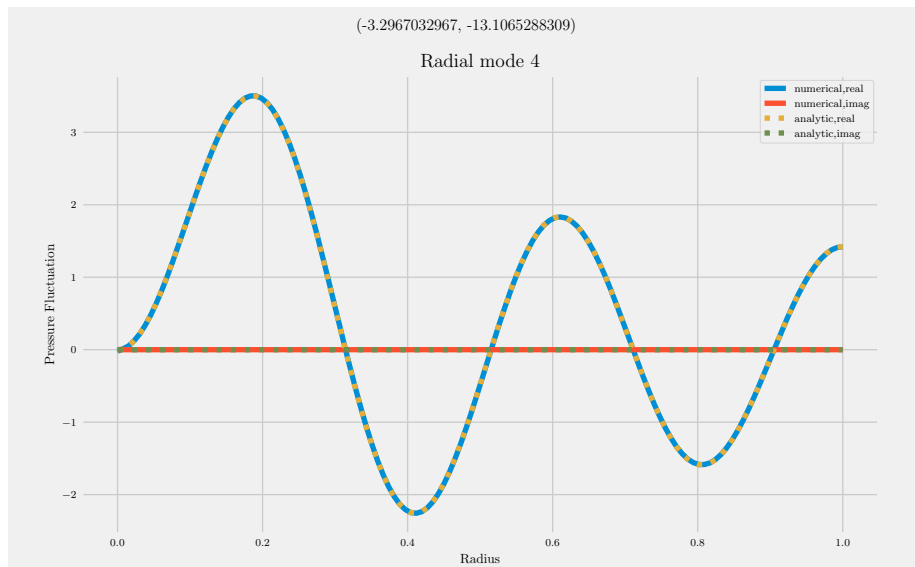


Figure 221: Second Order, Case number 1, 132points

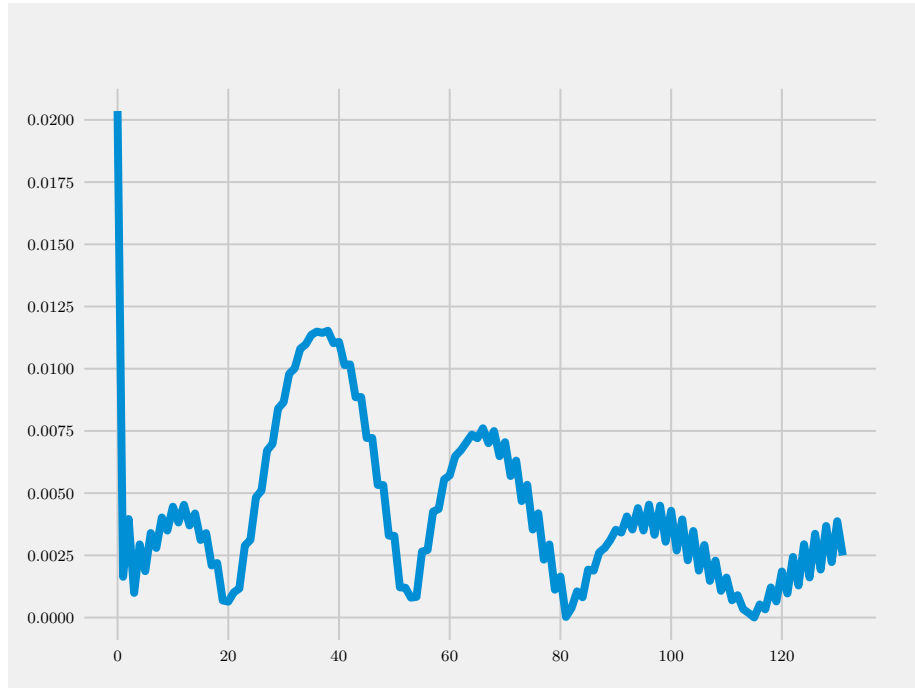


Figure 222: Second Order, Case number 1, 132points

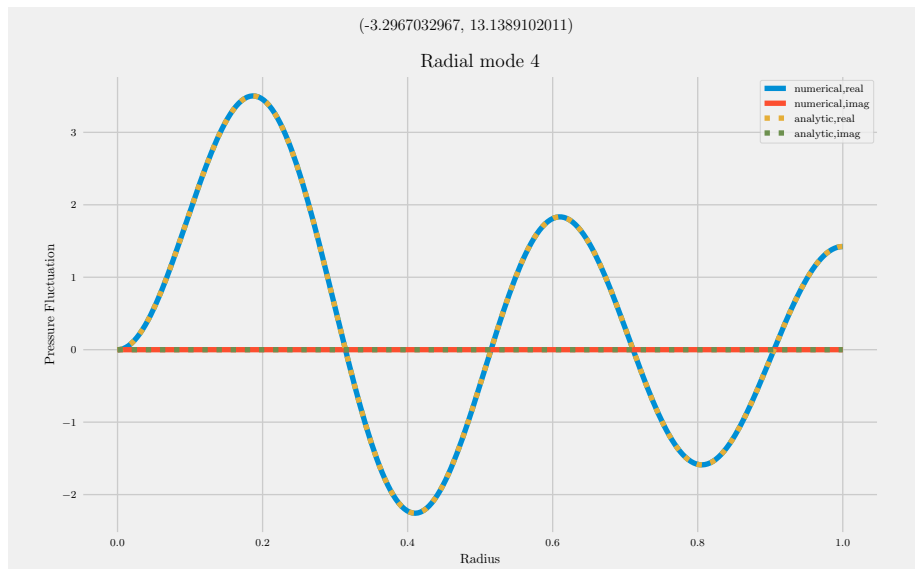


Figure 223: Second Order, Case number 1, 264points

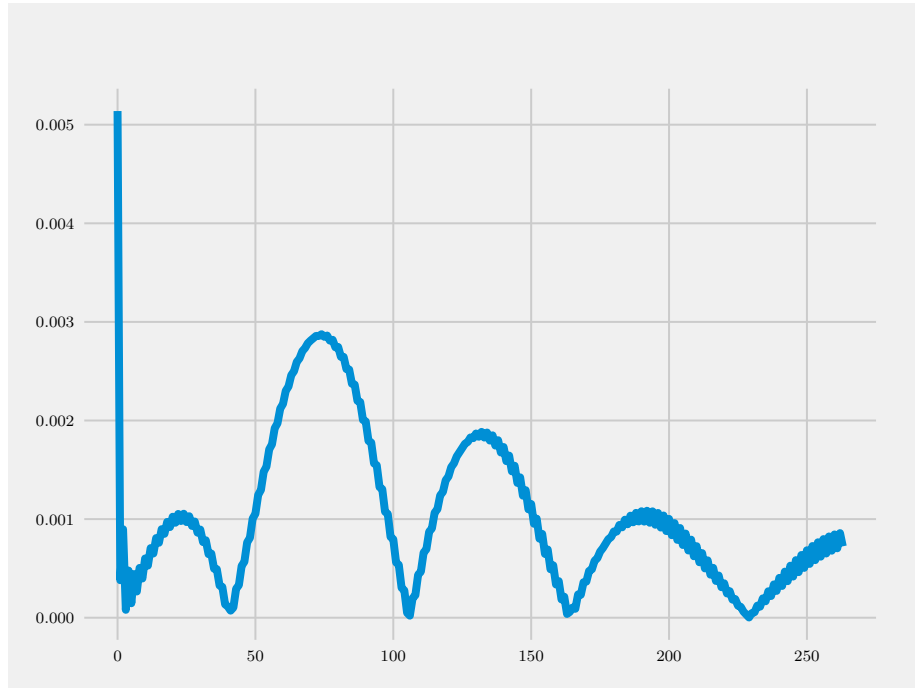


Figure 224: Second Order, Case number 1, 264points

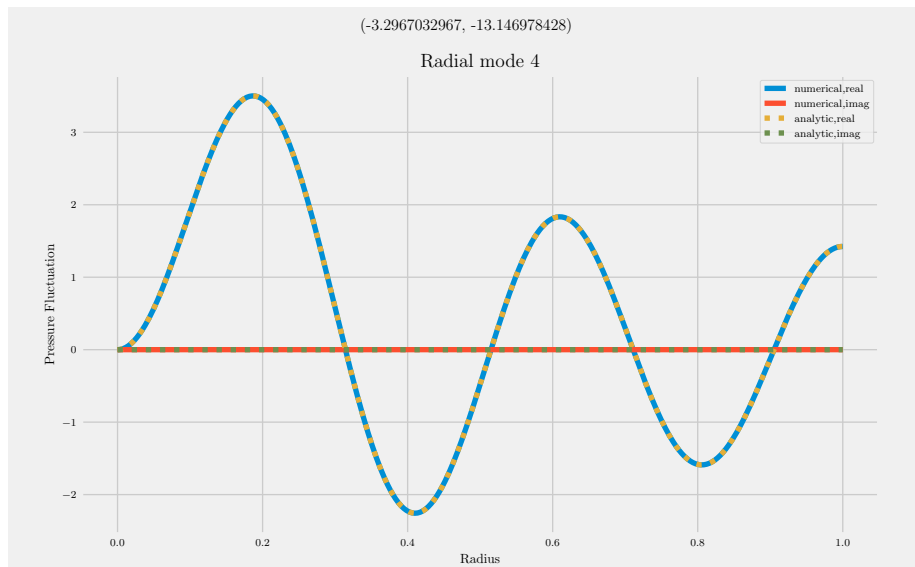


Figure 225: Second Order, Case number 1, 528points

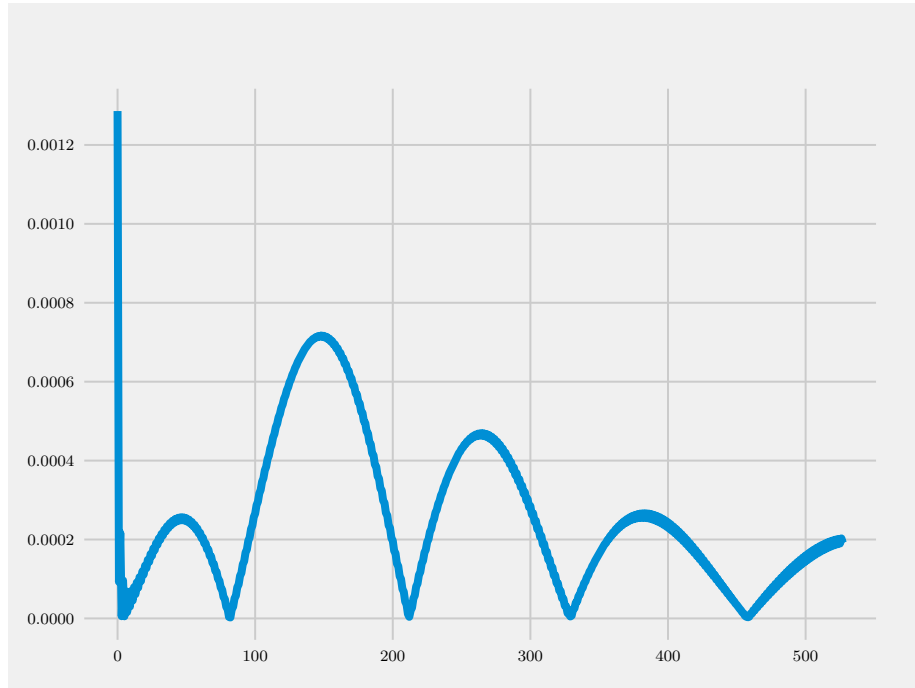


Figure 226: Second Order, Case number 1, 528points

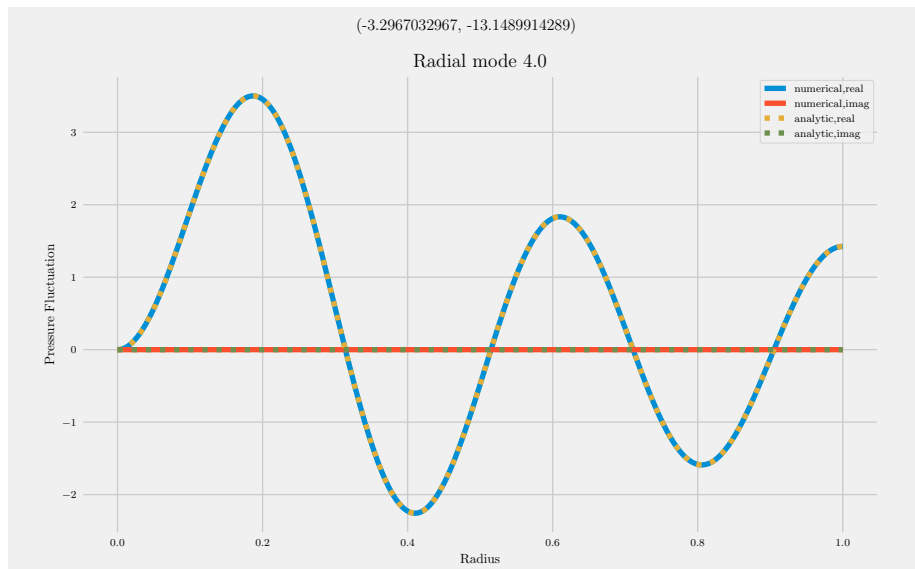


Figure 227: Second Order, Case number 1, 1056points

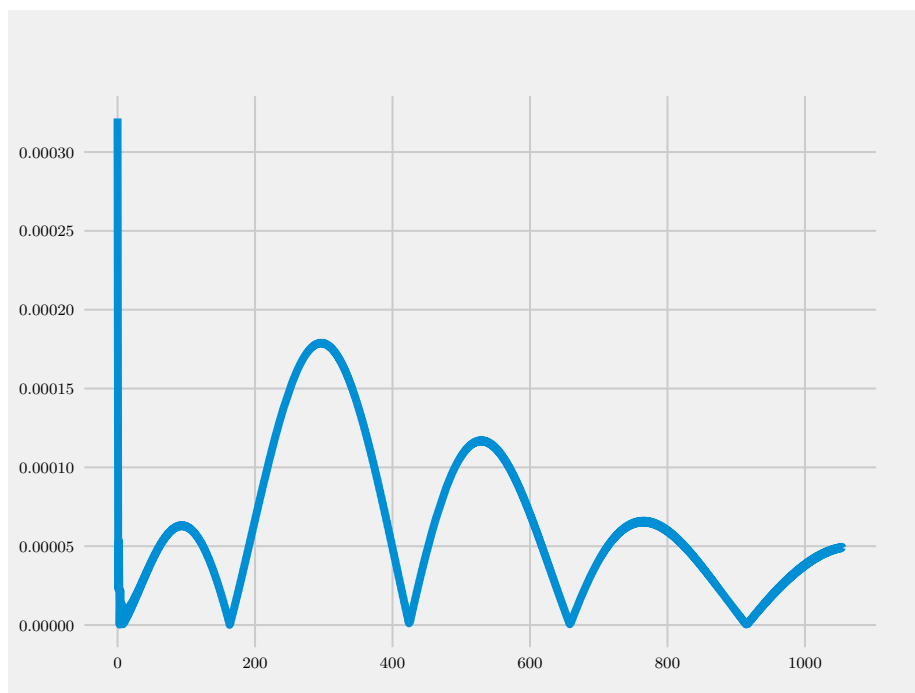


Figure 228: Second Order, Case number 1, 1056points

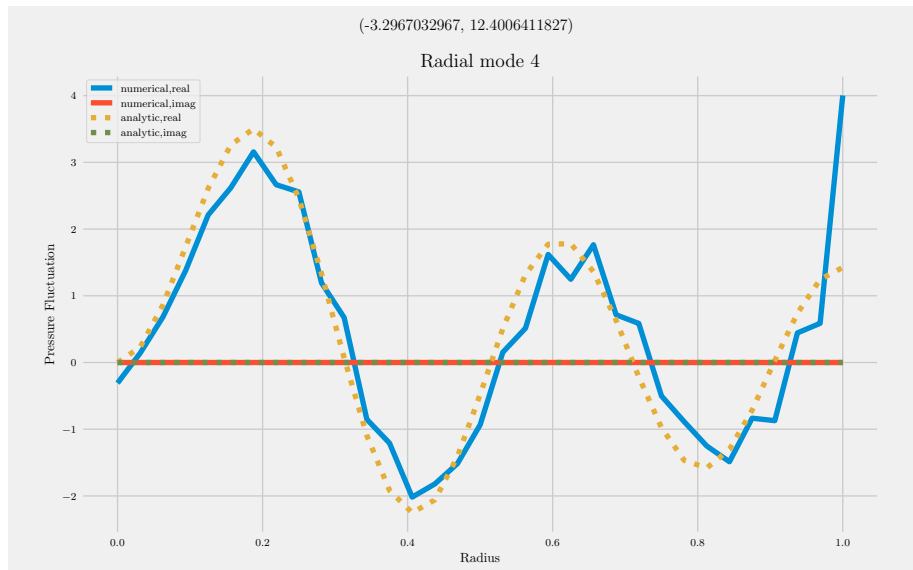


Figure 229: Fourth Order, Case number 0, 33points

Fourth Order, Radial Mode 4

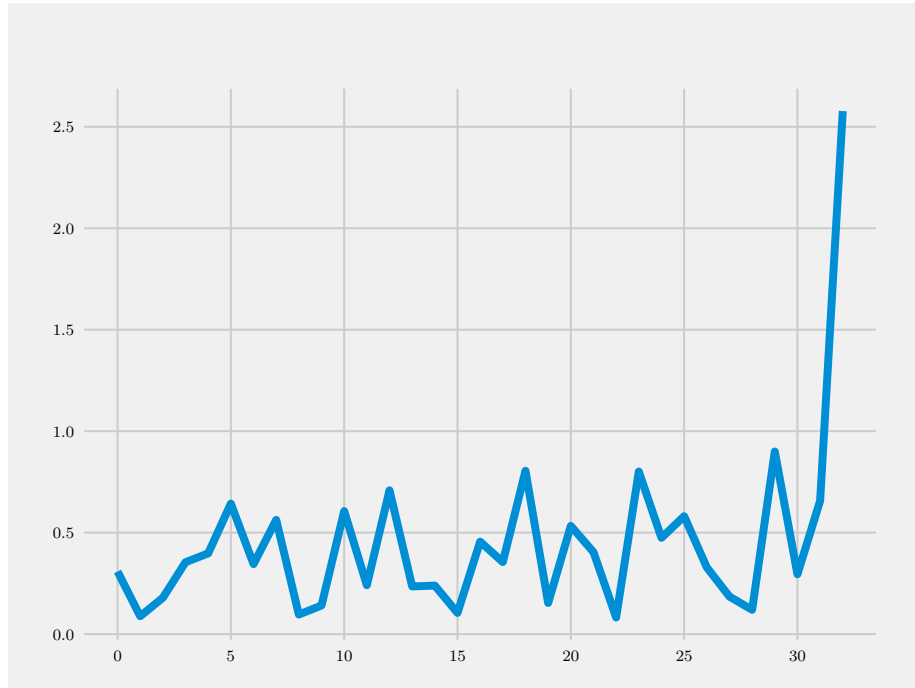


Figure 230: Fourth Order, Case number 0, 33points

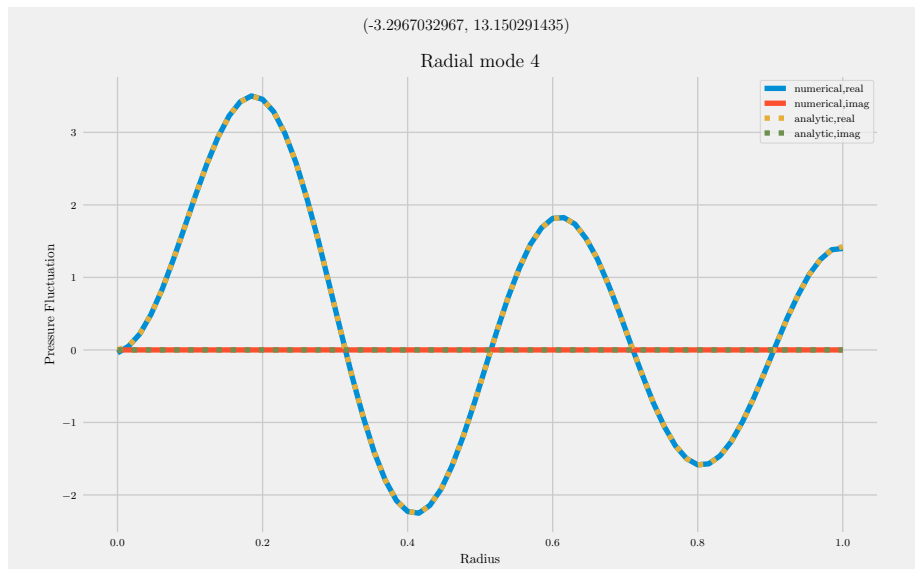


Figure 231: Fourth Order, Case number 0, 66points

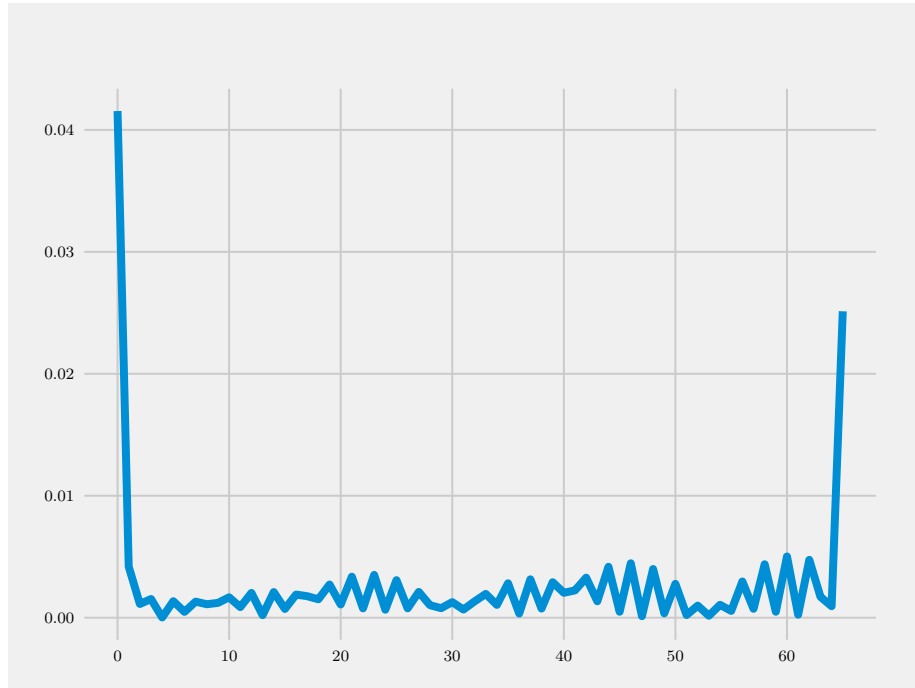


Figure 232: Fourth Order, Case number 0, 66points

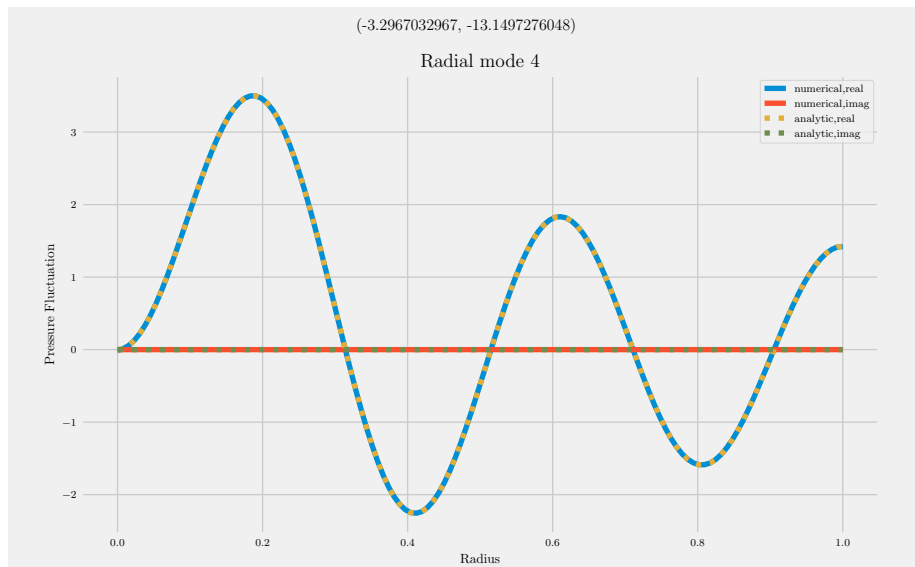


Figure 233: Fourth Order, Case number 0, 132points

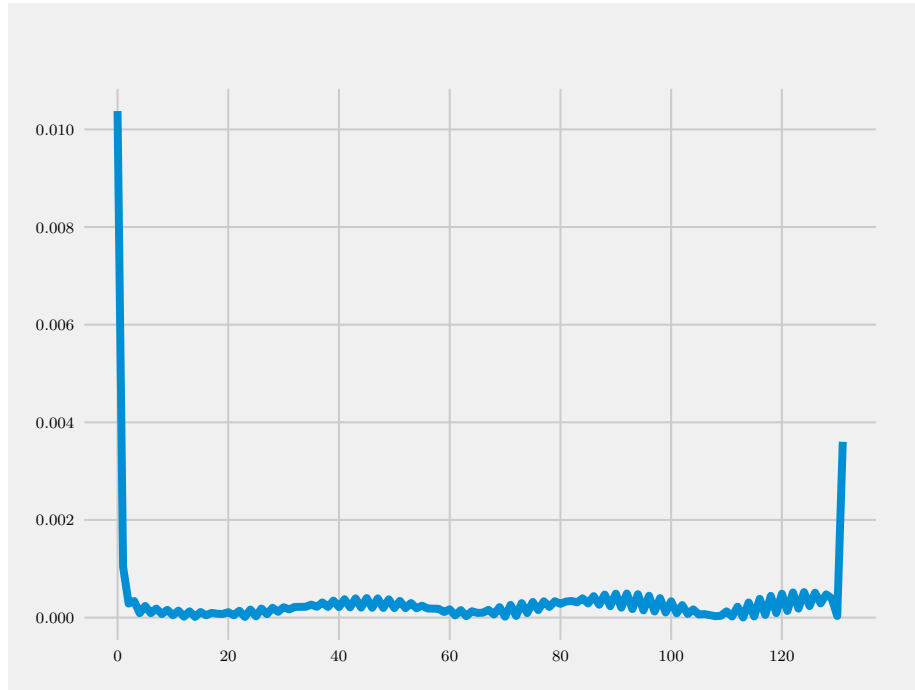


Figure 234: Fourth Order, Case number 0, 132points

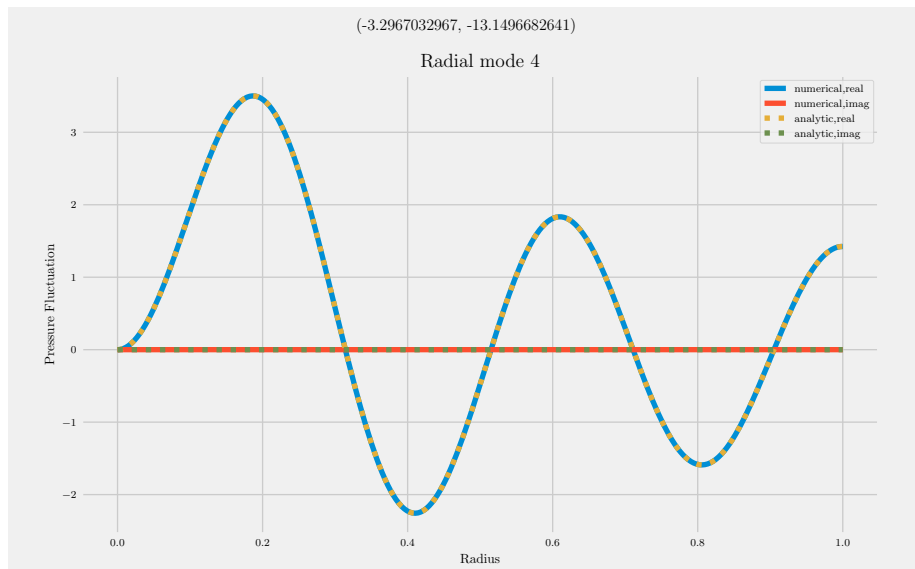


Figure 235: Fourth Order, Case number 0, 264points

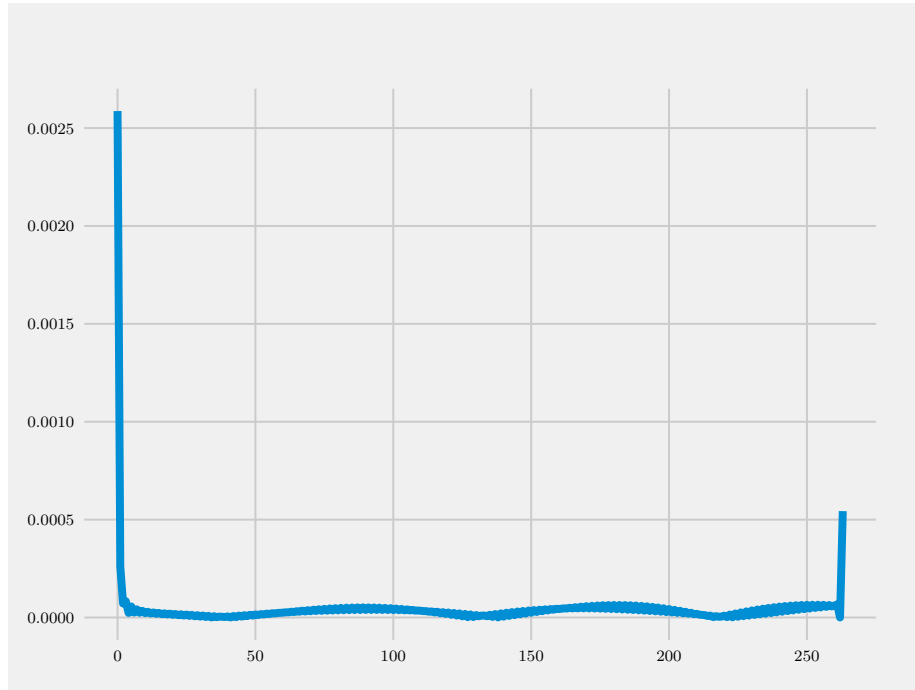


Figure 236: Fourth Order, Case number 0, 264points

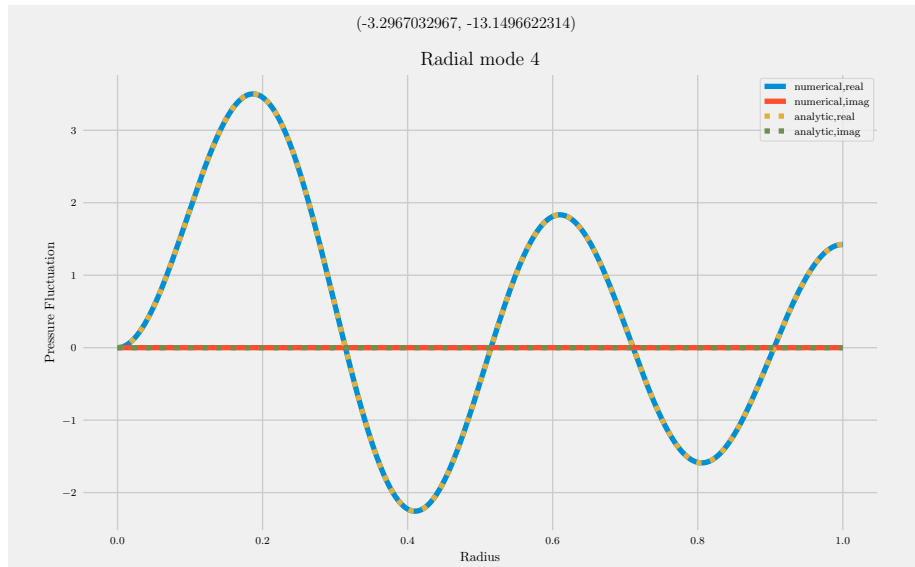


Figure 237: Fourth Order, Case number 0, 528points

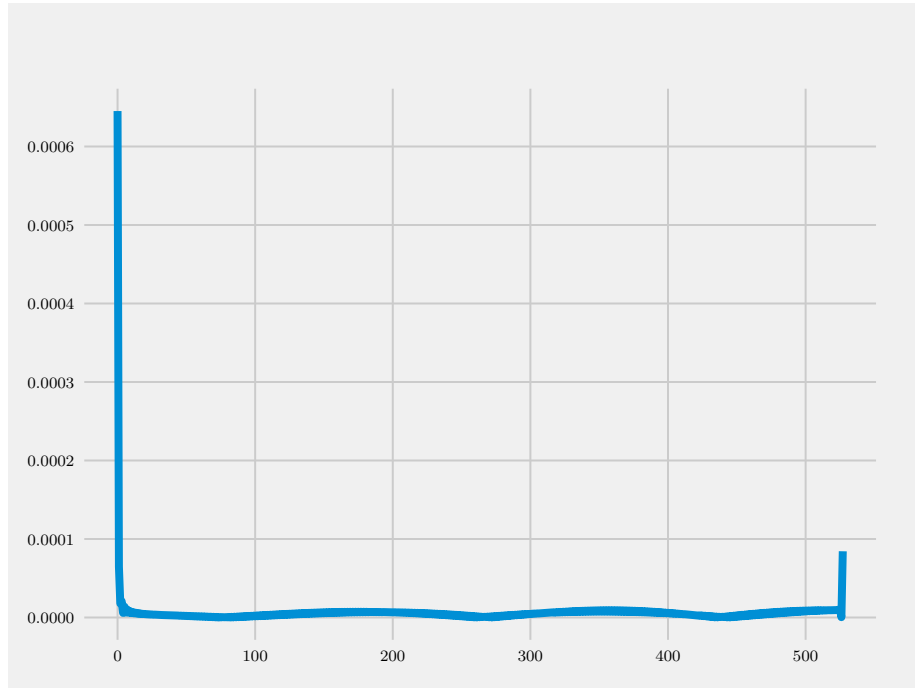


Figure 238: Fourth Order, Case number 0, 528points

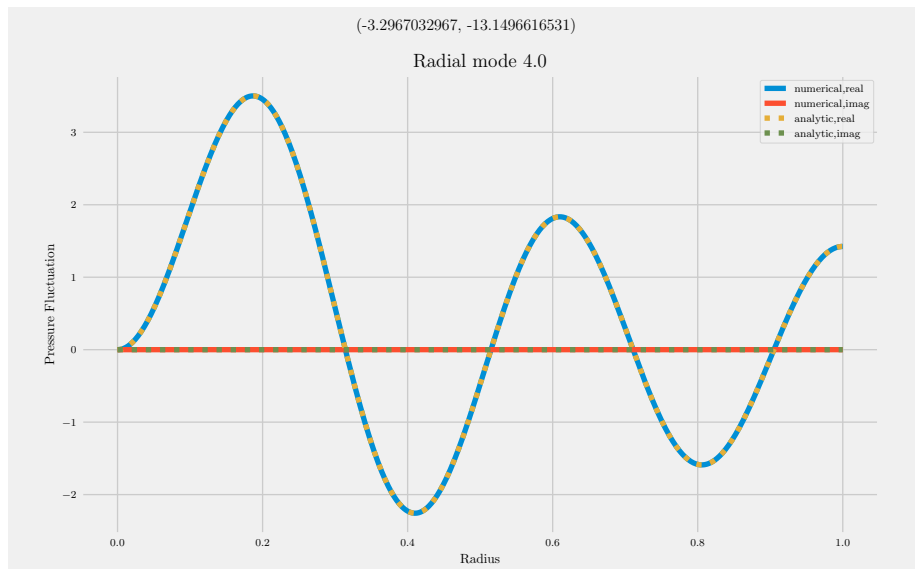


Figure 239: Fourth Order, Case number 0, 1056points

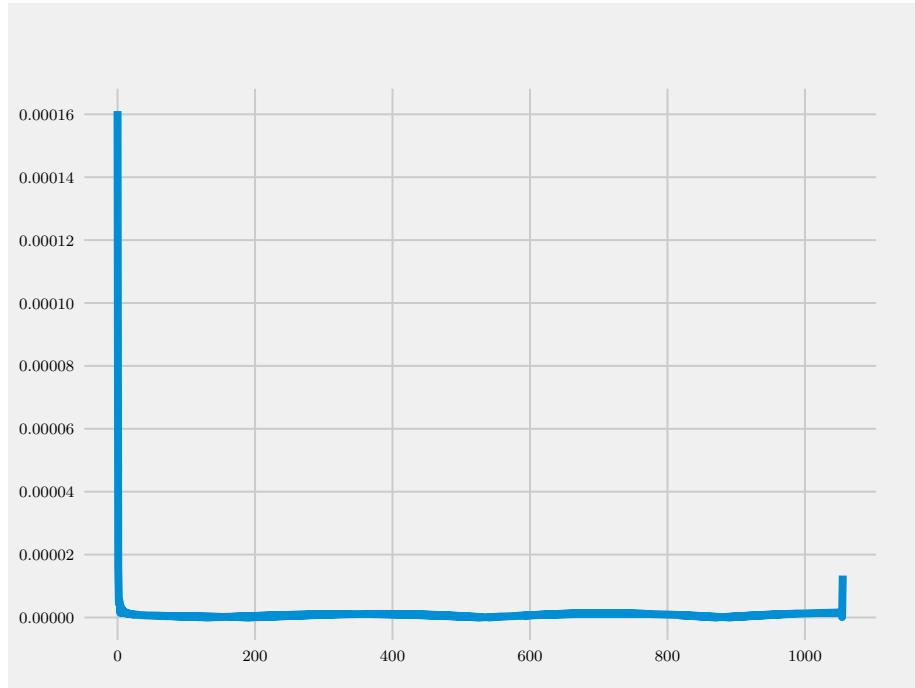


Figure 240: Fourth Order, Case number 0, 1056points

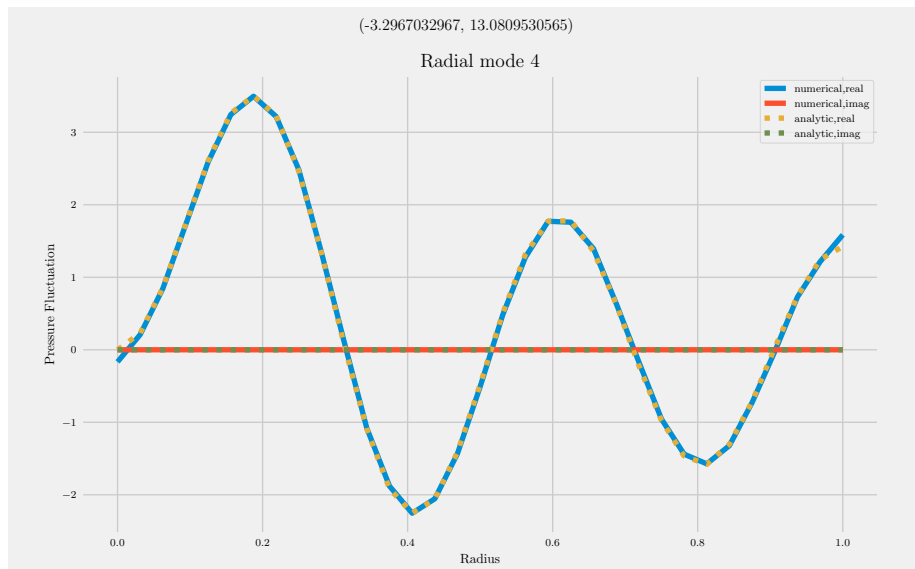


Figure 241: Fourth Order, Case number 1, 33points

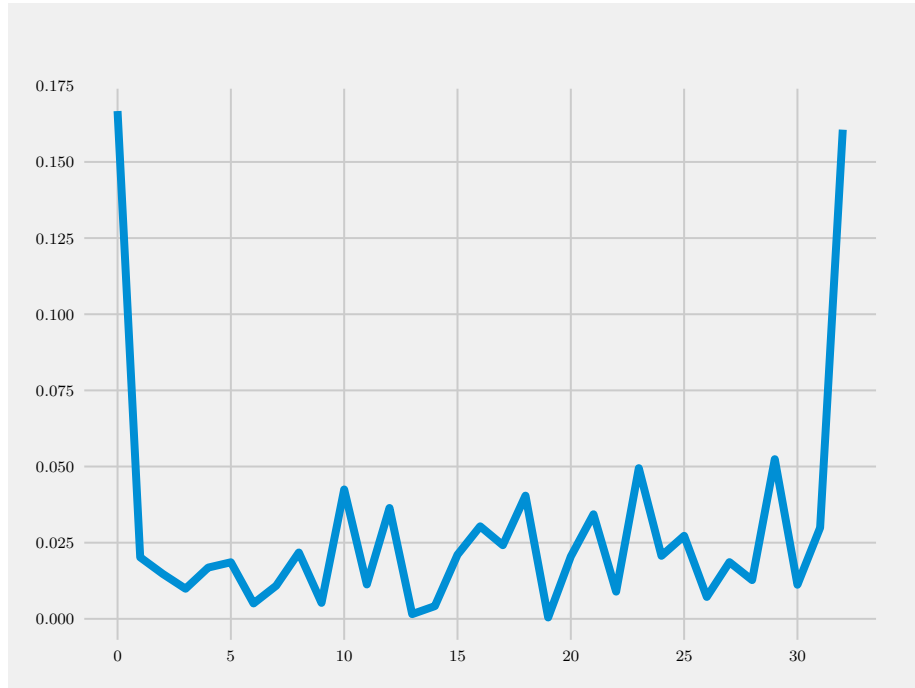


Figure 242: Fourth Order, Case number 1, 33points

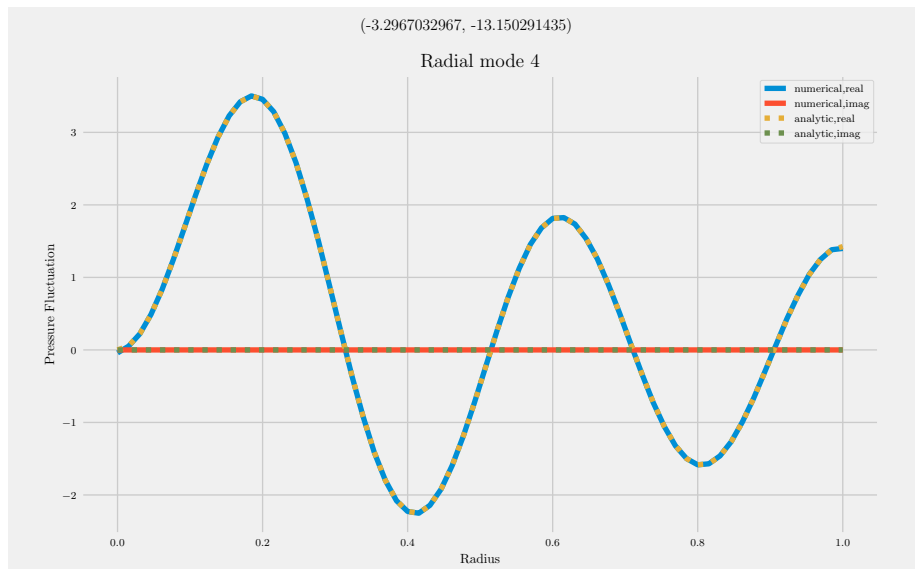


Figure 243: Fourth Order, Case number 1, 66points

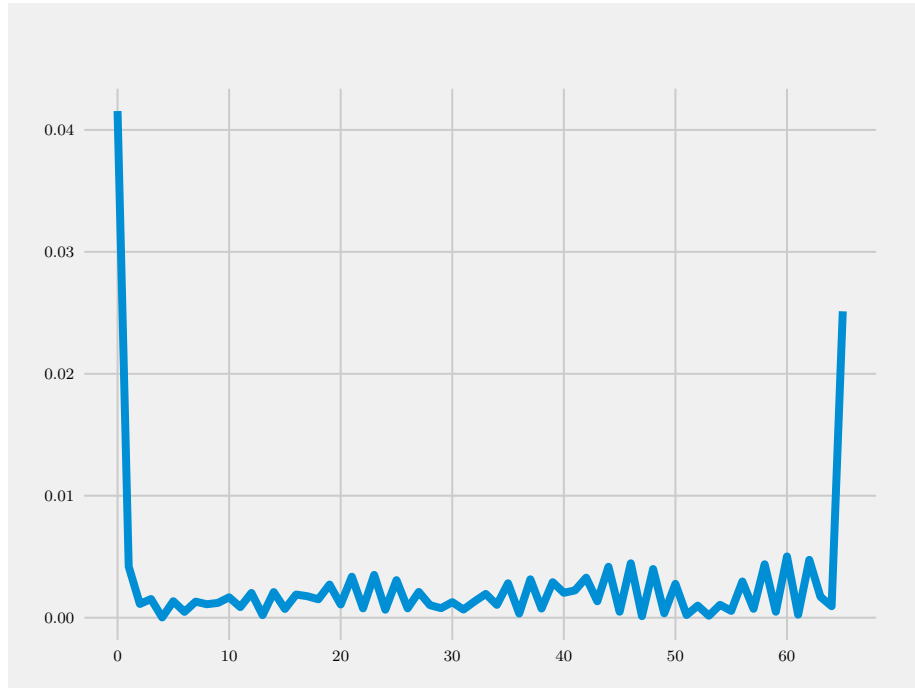


Figure 244: Fourth Order, Case number 1, 66points

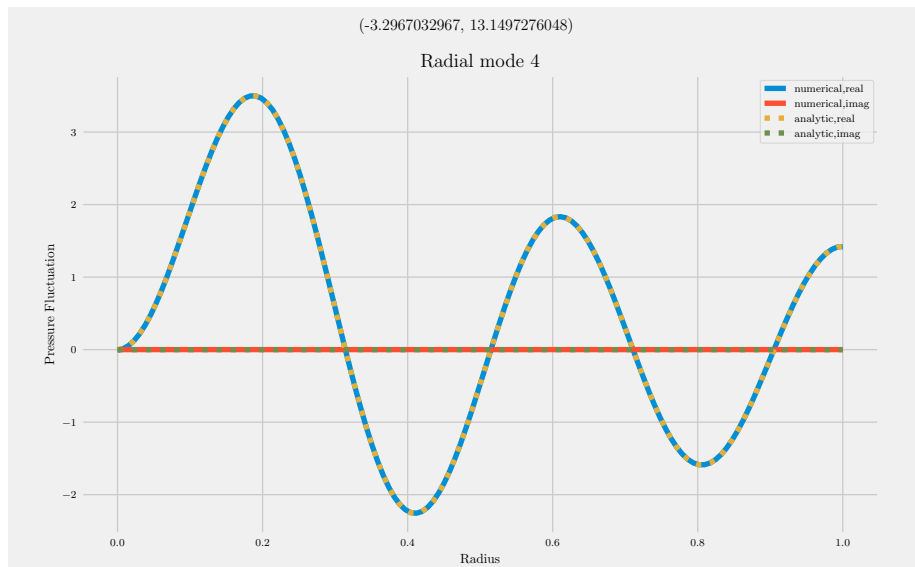


Figure 245: Fourth Order, Case number 1, 132points

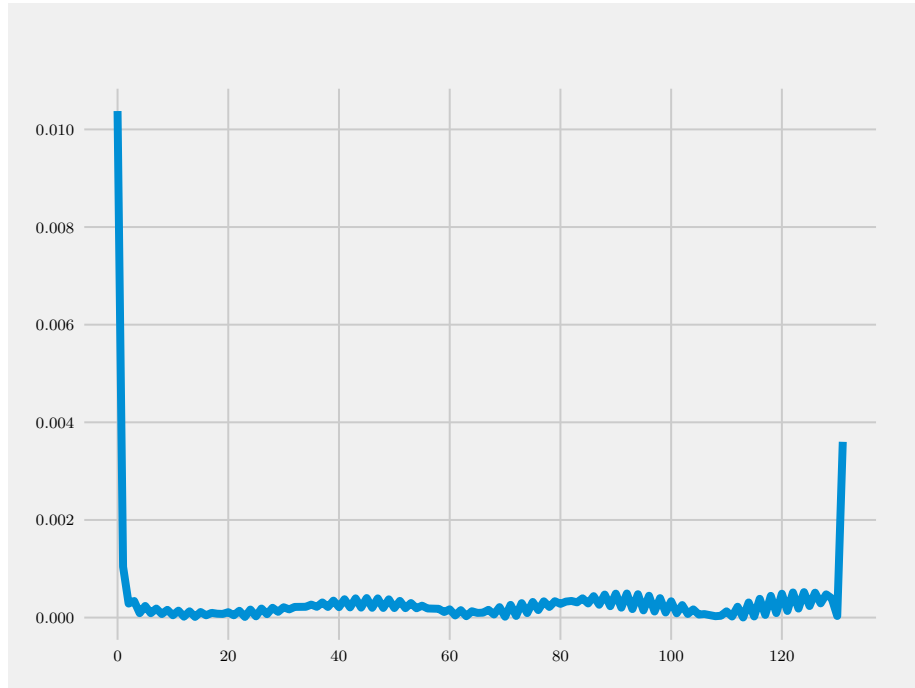


Figure 246: Fourth Order, Case number 1, 132points

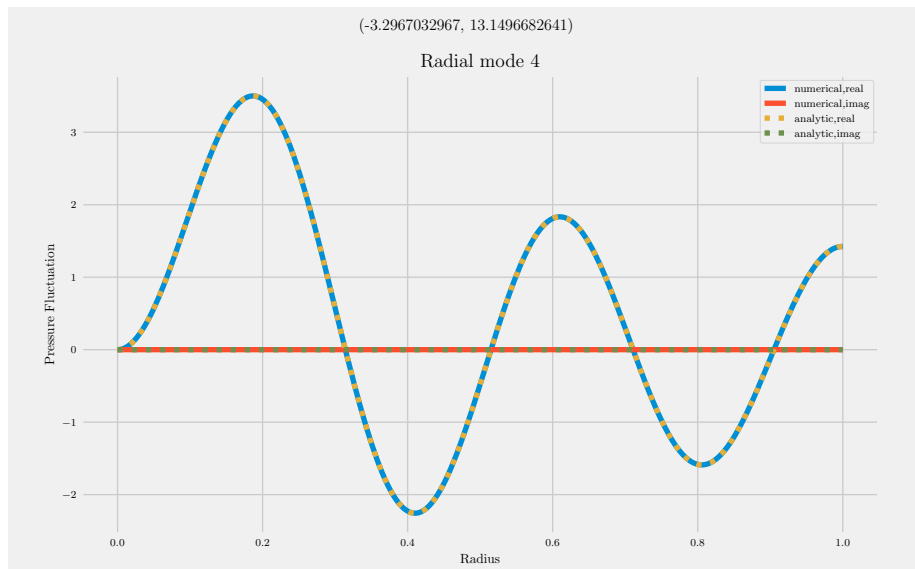


Figure 247: Fourth Order, Case number 1, 264points

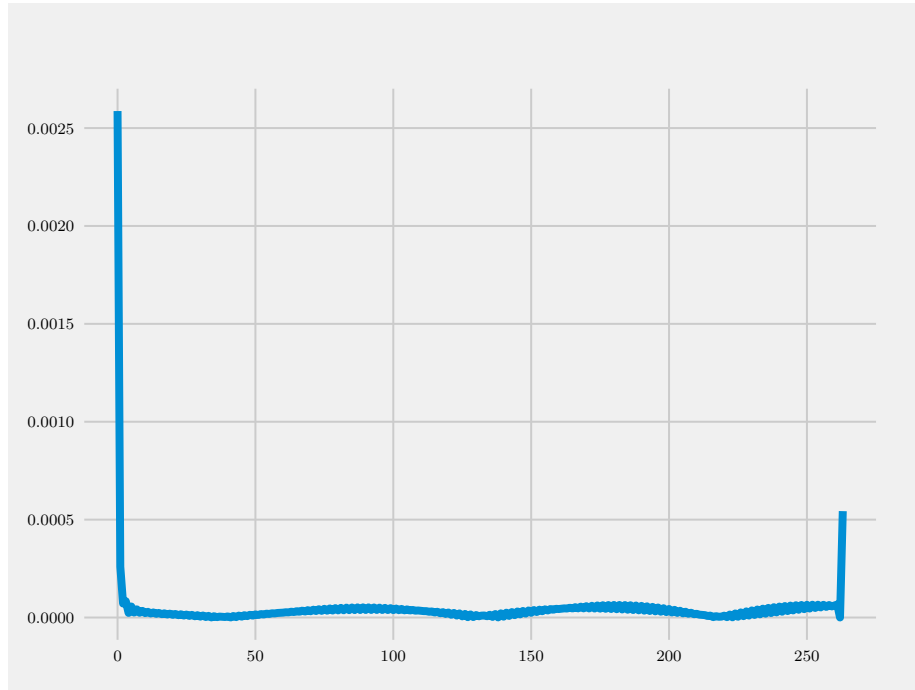


Figure 248: Fourth Order, Case number 1, 264points

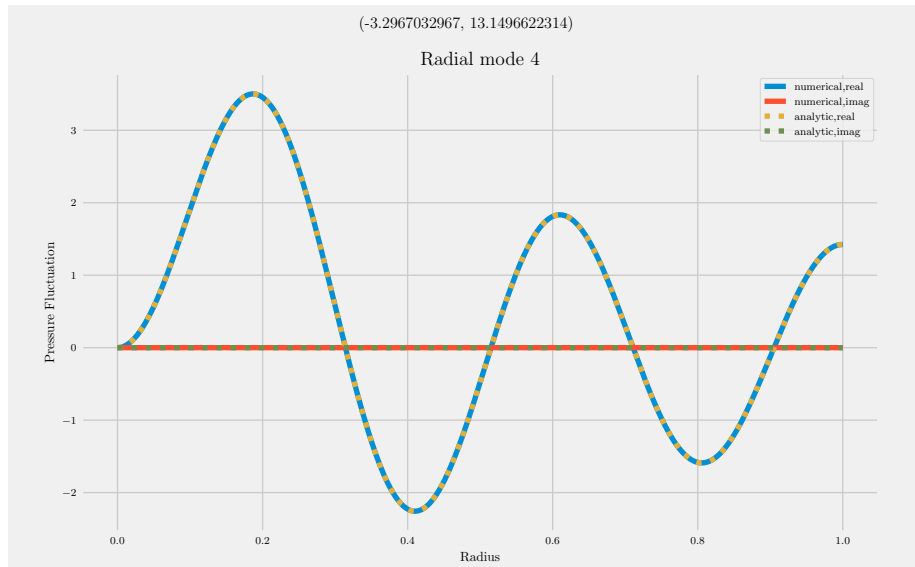


Figure 249: Fourth Order, Case number 1, 528points

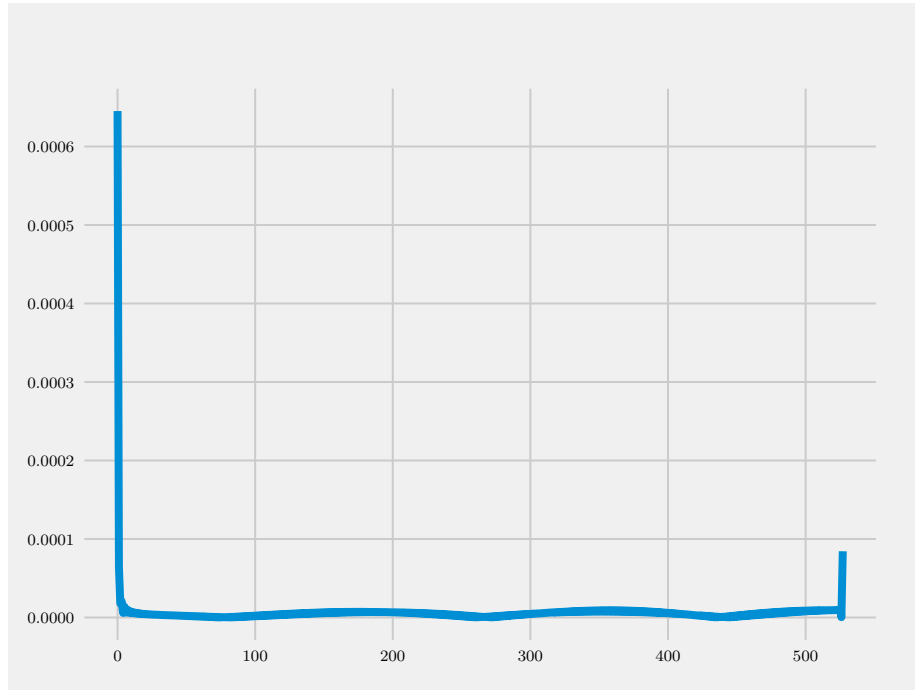


Figure 250: Fourth Order, Case number 1, 528points

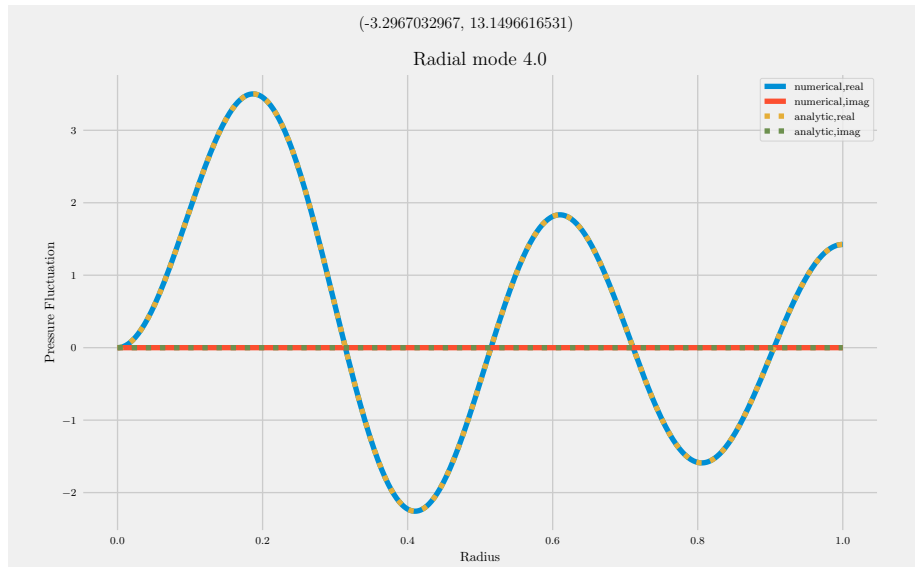


Figure 251: Fourth Order, Case number 1, 1056points

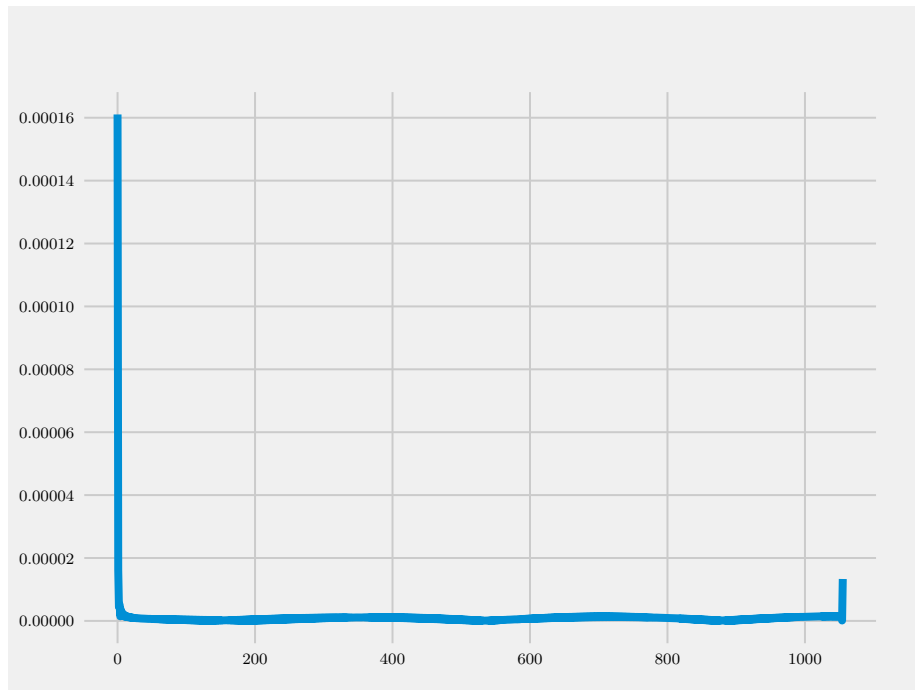


Figure 252: Fourth Order, Case number 1, 1056points

0.4 Discussion