Analytical Solution for Duct Mode Propagation in Uniform Flow

Swirl Validation

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This document shows the analytical duct mode solution as well as a numerical comparison.

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0.1 Introduction - Turbomachinery Noise

Turbomachinery noise generation occurs from pressure fluctuations from the series of fans within it's annular duct. While the jet that is produced from this stream of air freely radiates to the observer, the pressure fluctuations produced from the rotor may or may not propagate out of the inlet and exhaust and radiate to the observer. The production of this propagation can be characterized by standing waves referred to as modes, in particular, duct modes because the mode itself is dependent on the geometry of the column of air within the annular duct, as well as the speed of the flow moving through it

0.2 Duct Mode Theory

The pressure field within a duct is governed by the convective wave equation, a second order ODE as a function of radius.

The solution of the convective wave equation are eigenvalues and eigenvectors which may or may not correspond to acoustic disturbances fall into two groups. One group corresponding to the acoustics propagation and the other group corresponding to the convection speed of the flow. Both are modes that are a result from the pressure distribution from within the cylindrical domain.

0.3 Analytical Solution to Sound Propagation in ducted flows

0.3.1 Introduction

The steps to get up to this point are described in the appendix, but for the purposes of understanding this repository and its functions this document will start with the Pridmore-Brown equation. The general solution of this differential equation will be assumed to be harmonic and used to present a eigenvalue problem which is outlined here.

$$\frac{1}{A^2}\frac{D^2\tilde{p}}{Dt^2} - \nabla^2\tilde{p} = 2\bar{\rho}\frac{dV_x}{dx}\frac{\partial \tilde{v}_r}{\partial x} \tag{1}$$

Substituting the definitions for ∇ and ∇^2 and setting $\vec{V} = 0$ in cylindrical coordinates gives,

$$\frac{1}{A^2}\left(\frac{\partial^2\tilde{p}}{\partial t^2} + \vec{V}\cdot\left(\frac{\partial\tilde{p}}{\partial t} + \frac{1}{\tilde{r}}\frac{\partial\tilde{p}}{\partial\tilde{r}} + \frac{\partial\tilde{p}}{\partial\theta} + \frac{\partial\tilde{p}}{\partial x}\right)\right) - \left(\frac{\partial^2\tilde{p}}{\partial t^2} + \frac{1}{\tilde{r}}\frac{\partial\tilde{p}}{\partial r} + \frac{1}{\tilde{r}^2}\frac{\partial^2\tilde{p}}{\partial\theta^2} + \frac{\partial^2\tilde{p}}{\partial x^2}\right) = 0$$

Setting $\vec{V} = 0$,

$$\frac{1}{A^2}\left(\frac{\partial^2\tilde{p}}{\partial t^2}\right) - \left(\frac{\partial^2\tilde{p}}{\partial t^2} + \frac{1}{\tilde{r}}\frac{\partial\tilde{p}}{\partial r} + \frac{1}{\tilde{r}^2}\frac{\partial^2\tilde{p}}{\partial\theta^2} + \frac{\partial^2\tilde{p}}{\partial x^2}\right) = 0$$

Utilizing the relation, $\tilde{p} = p/\bar{\rho}A^2$,

$$\frac{1}{A^2}\left(\frac{\partial^2 p}{\partial t^2}\right)-\left(\frac{\partial^2 p}{\partial t^2}+\frac{1}{\tilde{r}}\frac{\partial p}{\partial r}+\frac{1}{\tilde{r}^2}\frac{\partial^2 p}{\partial \theta^2}+\frac{\partial^2 p}{\partial x^2}\right)=0$$

The method of separation of variables requires an assumed solution as well as initial and boundary conditions. For a partial differential equation, the assumed solution can be a linear combination of solutions to a system of ordinary differential equations that comprises the partial differential equation. Since p is a function of four variables, the solution is assumed to be a linear combination of four solutions. Each solution is assumed to be Euler's identity, a common ansant for linear partial differential equations and boundary conditions.

$$p(x,r,\theta,t)=X(x)R(r)\Theta(\theta)T(t) \tag{2}$$

where,

$$\begin{aligned}X(x)&=A_1e^{ik_xx}+B_1e^{-ik_xx}\\ \Theta(\theta)&=A_2e^{ik_\theta\theta}+B_2e^{-ik_\theta\theta}\\ T(t)&=A_3e^{i\omega t}+B_3e^{-i\omega t}\end{aligned}$$

and A_1,A_2,A_3,B_1,B_2,B_3 are arbitrary constants. The next step is to rewrite Equation 1 in terms of X, R, Θ, and T. To further simplify the result, each term is divided by p. Before the substitution, the derivatives of the assumed solutions need to be evaluated. The simplification of these derivatives are included in the Appendix.

$$\frac{1}{A^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2}=\frac{1}{R}\frac{\partial^2 R}{\partial r^2}+\frac{1}{r}\frac{1}{R}\frac{\partial R}{\partial r}+\frac{1}{r^2}\frac{1}{\Theta}\frac{\partial \Theta}{\partial \theta}+\frac{1}{X}\frac{\partial^2 X}{\partial x^2} \tag{3}$$

Notice that each term is only a function of its associated independent variable. So, if we vary the time, only the term on the left-hand side can vary. However, since none of the terms on the right-hand side depend on time, that means the right-hand side cannot vary, which means that the ratio of time with its second derivative is independent of time. The practical upshot is that each of these terms is constant, which has been shown. The wave numbers are the *separation constants* that allow the PDE to be split into four separate ODE's. Substituting the separation constants into Equation (8) gives,

$$\frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2}+\frac{1}{r}\frac{\partial R}{\partial r}\right)-\frac{k_\theta^2}{r^2}-k_x^2+k^2=0 \tag{4}$$

The remaining terms are manipulated to follow the same form as *Bessel's Differntial Equation* ,

The general solution to Bessel's differential equation is a linear combination of the Bessel functions of the first kind, $J_n(k_rr)$ and of the second kind, $Y_n(k_rr)$ [**?**]. The subscript n refers to the order of Bessel's equation.

$$R(r)=(AJ_n(k_rr)+BY_n(k_rr)) \tag{5}$$

where the coefficients A and B are found after applying radial boundary conditions.

By rearranging Equation (4), a comparison can be made to Equation (**??**) to show that the two equations are of the same form.

The first step is to revisit the radial derivatives that have not been addressed. As was done for the other derivative terms, the radial derivatives will also be set equal to a separation constant, $-k_r^2$.

$$\underbrace{\frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2}+\frac{1}{r}\frac{\partial R}{\partial r}\right)-\frac{k_\theta^2}{r^2}}_{-k_r^2}-k_x^2+k^2=0 \tag{6}$$

Hard Wall boundary condition

$$\begin{aligned}\frac{\partial p}{\partial r}|_{r=r_{min}}=\frac{\partial p}{\partial r}|_{r=r_{max}}=0\rightarrow\frac{\partial}{\partial r}\left(X\Theta TR\right)&=0\\ X\Theta T\frac{\partial R}{\partial r}&=0\\ \frac{\partial R}{\partial r}&=0\end{aligned}$$

where,

$$\frac{\partial R}{\partial r}|_{r_{min}}=AJ_n'(k_r r_{min})+BY_n'(k_r r_{min})=0\rightarrow B=-A\frac{J_n'(k_r r_{min})}{Y_n'(k_r r_{min})}$$

$$\begin{aligned}\frac{\partial R}{\partial r}&=AJ_n'(k_r r_{max})+BY_n'(k_r r_{max})=0\\ &=AJ_n'(k_r r_{max})-A\frac{J_n'(k_r r_{min})}{Y_n'(k_r r_{min})}Y_n'(k_r r_{max})=0\\ &=\frac{J_n'(k_r r_{min})}{J_n'(k_r r_{max})}-\frac{Y_n'(k_r r_{min})}{Y_n'(k_r r_{max})}=0\end{aligned}$$

where $k_r r$ are the zero crossings for the derivatives of the Bessel functions of the first and second kind.

In summary, the wave equation for no flow in a hollow duct with hard walls is obtained from Equation (6).

$$k^2=k_r^2+k_x^2 \tag{7}$$

Following the same procedure, the axial wavenumber is,

$$\begin{aligned}k_x&=\frac{2M_x k\pm\sqrt{4M_x^2k^2+4\beta^2\left(k^2-k_r^2\right)}}{-2\beta^2}\\ &=\frac{-M_x k\pm\sqrt{k^2-k_r^2}}{\beta^2}\end{aligned}$$

0.4 Appendix

0.4.1 Simplification of the Assumed Solution

Temporal Derivatives

$$\begin{aligned}\frac{\partial p}{\partial t} &= \frac{\partial}{\partial t}\left(XR\Theta T\right)\\ &= XR\Theta\frac{\partial T}{\partial t}\end{aligned}$$

$$\begin{aligned}\frac{1}{p}\frac{\partial p}{\partial t} &= \frac{1}{XR\Theta T}\left(XR\Theta\frac{\partial T}{\partial t}\right)\\ &= \frac{1}{T}\frac{\partial T}{\partial t}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 p}{\partial t^2} &= \frac{\partial^2}{\partial t^2}\left(XR\Theta T\right)\\ &= XR\Theta\frac{\partial^2 T}{\partial t^2}\end{aligned}$$

$$\begin{aligned}\frac{1}{p}\frac{\partial^2 p}{\partial t^2} &= \frac{1}{XR\Theta T}\left(XR\Theta\frac{\partial^2 T}{\partial t^2}\right)\\ &= \frac{1}{T}\frac{\partial^2 T}{\partial t^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{\partial}{\partial t}\left(A_3e^{i\omega t} + B_3e^{-i\omega t}\right)\\ &= \frac{\partial}{\partial t}\left(A_3e^{i\omega t}\right) + \frac{\partial}{\partial t}\left(B_3e^{-i\omega t}\right)\\ &= i\omega A_3e^{i\omega t} - i\omega B_3e^{i\omega t}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 T}{\partial t^2} &= \frac{\partial^2}{\partial t^2}\left(i\omega A_3e^{i\omega t} + i\omega B_3e^{-i\omega t}\right)\\ &= \frac{\partial^2}{\partial t^2}\left(i\omega A_3e^{i\omega t}\right) + \frac{\partial^2}{\partial t^2}\left(-i\omega B_3e^{-i\omega t}\right)\\ &= (i\omega)^2A_3e^{i\omega t} - (i\omega)^2B_3e^{i\omega t}\end{aligned}$$

$$\begin{aligned}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} &= (i\omega)^2\\ &= -\omega^2\end{aligned}$$

Radial Derivatives

$$\begin{aligned}\frac{\partial p}{\partial r} &= \frac{\partial}{\partial r}\left(XR\Theta T\right)\\ &= X\Theta T\frac{\partial R}{\partial r}\end{aligned}$$

$$\begin{aligned}\frac{1}{p}\frac{\partial p}{\partial r} &= \frac{1}{XR\Theta T}\left(X\Theta T\frac{\partial R}{\partial r}\right)\\ &= \frac{1}{R}\frac{\partial R}{\partial r}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 p}{\partial r^2} &= \frac{\partial^2}{\partial r^2}\left(XR\Theta T\right)\\ &= X\Theta T\frac{\partial^2 R}{\partial r^2}\end{aligned}$$

$$\begin{aligned}\frac{1}{p}\frac{\partial^2 p}{\partial r^2} &= \frac{1}{XR\Theta T}\left(X\Theta T\frac{\partial^2 R}{\partial r^2}\right)\\ &= \frac{1}{R}\frac{\partial^2 R}{\partial r^2}\end{aligned}$$

The radial derivatives will be revisited once the remaining derivatives are evaluated,

Tangential Derivatives

$$\begin{aligned}\frac{\partial p}{\partial \theta} &= \frac{\partial}{\partial t}\left(XR\Theta T\right)\\ &= XRT\frac{\partial \Theta}{\partial \theta}\end{aligned}$$

$$\begin{aligned}\frac{1}{p}\frac{\partial p}{\partial \theta} &= \frac{1}{XR\Theta T}\left(XR\Theta\frac{\partial T}{\partial \theta}\right)\\ &= \frac{1}{\Theta}\frac{\partial \Theta}{\partial \theta}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 p}{\partial \theta^2} &= \frac{\partial^2}{\partial \theta^2}\left(XR\Theta T\right)\\ &= XRT\frac{\partial^2 \Theta}{\partial \theta^2}\end{aligned}$$

$$\begin{aligned}\frac{1}{p}\frac{\partial^2 p}{\partial \theta^2} &= \frac{1}{XR\Theta T}\left(XRT\frac{\partial^2 \Theta}{\partial \theta^2}\right)\\ &= \frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2}\end{aligned}$$

$$\begin{aligned}
\frac{\partial \Theta}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(A_2 e^{ik_\theta \theta} + B_2 e^{-ik_\theta \theta}\right) \\
&= \frac{\partial}{\partial \theta}\left(A_2 e^{ik_\theta \theta}\right) + \frac{\partial}{\partial \theta}\left(B_2 e^{-ik_\theta \theta}\right) \\
&= ik_\theta A_2 e^{ik_\theta \theta} - ik_\theta B_2 e^{ik_\theta \theta}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \Theta}{\partial \theta^2} &= \frac{\partial^2}{\partial \theta^2}\left(ik_\theta A_2 e^{ik_\theta \theta} - ik_\theta B_2 e^{ik_\theta \theta}\right) \\
&= \frac{\partial^2}{\partial \theta^2}\left(ik_\theta A_2 e^{ik_\theta \theta}\right) + \frac{\partial^2}{\partial \theta^2}\left(-ik_\theta B_2 e^{-ik_\theta \theta}\right) \\
&= (ik_\theta)^2 A_2 e^{ik_\theta \theta} - (ik_\theta)^2 B_2 e^{ik_\theta \theta}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} &= (ik_\theta)^2 \\
&= -k_\theta^2
\end{aligned}$$

Axial Derivatives

$$\begin{aligned}
\frac{\partial p}{\partial x} &= \frac{\partial}{\partial x}(XR\Theta T) \\
&= R\Theta T \frac{\partial X}{\partial x}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{p}\frac{\partial p}{\partial x} &= \frac{1}{XR\Theta T}\left(R\Theta \frac{\partial X}{\partial x}\right) \\
&= \frac{1}{X}\frac{\partial X}{\partial x}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 p}{\partial x^2} &= \frac{\partial^2}{\partial x^2}(XR\Theta T) \\
&= R\Theta T \frac{\partial^2 X}{\partial x^2}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{p}\frac{\partial^2 p}{\partial x^2} &= \frac{1}{XR\Theta T}\left(R\Theta T \frac{\partial^2 X}{\partial x^2}\right) \\
&= \frac{1}{X}\frac{\partial^2 X}{\partial x^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial X}{\partial x} &= \frac{\partial}{\partial t}\left(A_3 e^{ik_x t} + B_3 e^{-i\omega t}\right) \\
&= \frac{\partial}{\partial t}\left(A_1 e^{ik_x x}\right) + \frac{\partial}{\partial t}\left(B_1 e^{-ik_x x}\right) \\
&= ik_x A_1 e^{ik_x x} - ik_x B_1 e^{ik_x x}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 X}{\partial x^2} &= \frac{\partial^2}{\partial x^2}\left(ik_x A_1 e^{ik_x x} + ik_x B_1 e^{-ik_x x}\right)\\
&= \frac{\partial^2}{\partial x^2}\left(ik_x A_1 e^{ik_x x}\right) + \frac{\partial^2}{\partial x^2}\left(-ik_x B_1 e^{-ik_x x}\right)\\
&= (ik_x)^2 A_1 e^{ik_x x} - (ik_x)^2 B_1 e^{ik_x x}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{X}\frac{\partial^2 X}{\partial x^2} &= (ik_x)^2\\
&= -k_x^2
\end{aligned}
$$

Substituting this back into the Equation 1 yields ,

$$
\frac{1}{A^2}\left(\frac{\partial^2 p}{\partial t^2}\right) = \left(\frac{\partial^2 p}{\partial t^2} + \frac{1}{\tilde{r}}\frac{\partial p}{\partial r} + \frac{1}{\tilde{r}^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial x^2}\right)
$$

$$
\frac{1}{A^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{1}{R}\frac{\partial R}{\partial r} + \frac{1}{r^2}\frac{1}{\Theta}\frac{\partial \Theta}{\partial \theta} + \frac{1}{X}\frac{\partial^2 X}{\partial x^2} \tag{8}
$$

0.4.2 Analytic Solution: Axial wavenumbers and Pressure Modes

Modes can be categorized based on the sign of the axial wavenumber and if it is complex in value. For example, for the uniform axial flow case, propagating modes are defined by axial wavenumbers, k_x, that have a real-part only, yielding the assumed fluctuation to resemble Euler's Formula ($e^{ik_x x}$). On the other hand, if the k_x is complex, then the mode will resemble an exponentially decaying function since the imaginary number cancels, leaving a minus sign in front of the axial wavenumber. These two distinctions are referred to as "cut-on" and "cut-off" in the field of ducted sound propagation. Furthermore, the sign of the imaginary part will change the direction of the mode's decay. If k_x is positive, the decay rate occurs in the negative direction. Conversely, if k_x is negative, the decay occurs in the positive direction. The axial wavenumber for uniform axial flow in a hollow duct is,

$$
k_x = \frac{-M_x k \pm \sqrt{k^2 - (1 - M_x^2)k'^2_{r,m,n}}}{(1 - M_x^2)}. \tag{9}
$$

where M_x is the axial Mach number, k is the temporal (referred to as reduced) frequency, and $J'_{m,n}$ is the derivative of the Bessel function of the first kind. The $\pm$ accounts for both upstream and downstream modes. See Appendix for detailed derivation.

The condition for propagation is such that the axial wavenumber is larger than a "cut-off" value

$$
k_{x,real} = \frac{\pm M_x k}{(M_x^2 - 1)}. \tag{10}
$$

Every term that is being raised to the one half i.e. square rooted must be larger than zero to keep axial wavenumber from being imaginary. The mode will propagate or decay based on this condition.

0.4.3 Methods

Bessel's Function,

$$R(r) = AJ_0(k_r r) + BY_0(k_r r) \tag{11}$$

where J_0 and Y_0 are Bessel's functions of the first and second kind, and A and B are arbitrary constants. Both functions reduce as $k_r r$ gets large. The Bessel function of the second kind is unbounded as $k_r r$ goes to zero.

For a hard-walled duct, the radial velocity component is zero at the boundaries r_{min} and r_{max}.

$$\frac{dR(r)}{dr}\bigg|_{r=r_{max}} = \frac{dR(r)}{dr}\bigg|_{r=r_{min}} = 0 \tag{12}$$

In the case of a hollow duct, there is no minimum radius, therefore the wall boundary condition only applies at r_{max}.

Since the solution must be finite as $k_r r$ approaches zero, it can be observed that Y_0 approaches infinity. Since this would yield in a trivial solution, the coefficient $B = 0$, which reduces 11 to,

$$R(r) = AJ_0(k_r r) \tag{13}$$

Taking the derivative with respect to r yields,

$$\frac{dR(r)}{dr}\bigg|_{r} = AJ_0'(k_r r) = 0 \tag{14}$$

The boundary condition requires the Bessel function be zero at a hard wall. The terms inside of the Bessel function would then correspond to values along the domain, $k_r r$, which satisfy our equation. Let $k_r r = \alpha_{m,n}$ where, *alpha* represents the zeros of the Bessel function, and m corresponds to the azimuthal mode order and n represents the radial mode order, i.e. the index for the number of zero crossings in the derivative of the Bessel function of the first kind.

Therefore,

$$\frac{dR}{dr}\bigg|_{r=r_{max}} = AJ_0'(k_r r_{max}) = 0 \tag{15}$$

$$(k_r r_{max}) = 0 \tag{16}$$

Recalling $\alpha_{m,n}$

$$(k_r r_{max}) = \alpha_{m,n} \tag{17}$$

$$k_r = \alpha_{m,n}/r_{max} \tag{18}$$

For annular ducts, r_{min} is no longer zero, therefore B cannot be removed since Y_0' has finite values as $k_r r$ increases.

0.5 Results and Discussion