LU-SGS

Ray Hixon
University of Toledo
Toledo, OH 43606
email: duane.hixon@utoledo.edu

1 Viscous extension of the LU-SGS (Yoon and Jameson, 1988) Scheme

The approach of interest is the inviscid LU-SGS method of Yoon and Jameson. For viscous flows, we add the effect of viscosity into the LHS operators.

To illustrate this method, we'll look at the unsteady viscous 1-D equation in Cartesian coordinates:

$$\frac{\partial Q}{\partial t} + F_x - R_{xx} \quad = \quad 0$$

We write the equation in implicit form (with $n+1$ denoting the new time level and i denoting the grid location):

$$\left(\frac{\partial Q}{\partial t}\right)_i^{n+1} + (F_x)_i^{n+1} - (R_{xx})_i^{n+1} \quad = \quad 0$$

and introduce an implicit iteration at the new time level (with $l+1$ denoting the new iteration level):

$$\left(\frac{\partial Q}{\partial t}\right)_i^{n+1,l+1} + (F_x)_i^{n+1,l+1} - (R_{xx})_i^{n+1,l+1} \quad = \quad 0$$

We define the flux Jacobians as:

$$\begin{aligned} A_i^{n+1,l} &= \left(\frac{\partial F}{\partial Q}\right)_i^{n+1,l} \\ B_i^{n+1,l} &= \left(\frac{\partial R}{\partial Q}\right)_i^{n+1,l} \end{aligned}$$

and define the change in Q between iteration levels as:

$$\Delta Q_i \quad = \quad \frac{Q_i^{n+1,l+1} - Q_i^{n+1,l}}{\Delta \tau_i}$$

where $\Delta\tau$ is the 'distance' between iteration levels in the iteration direction. Note that I'm allowing $\Delta\tau$ to have different values at different spatial locations.

At this point, I'm going to linearize about the old iteration level:

$$\begin{aligned} (F_x)_i^{n+1,l+1} &\approx (F_x)_i^{n+1,l} + \left(A_i^{n+1,l}\Delta Q_i\right)_x \\ (R_{xx})_i^{n+1,l+1} &\approx (R_{xx})_i^{n+1,l} + \left(B_i^{n+1,l}\Delta Q_i\right)_{xx} \end{aligned}$$

which gives the equation as:

$$\left\{I + \Delta\tau_i\delta_x A_i^{n+1,l} - \Delta\tau_i\delta_{xx}B_i^{n+1,l}\right\}\Delta Q_i + \Delta\tau_i\left\{RHS\right\}_i^{n+1,l} = 0$$

where δ_x is a central differencing operator:

$$(\delta_x A\Delta Q)_i^{n+1,l} = \frac{(A\Delta Q)_{i+1}^{n+1,l} - (A\Delta Q)_{i-1}^{n+1,l}}{2\Delta x}$$

and

$$(\delta_{xx} B\Delta Q)_i^{n+1,l} = \frac{(B\Delta Q)_{i+1}^{n+1,l} - 2\,(B\Delta Q)_i^{n+1,l} + (B\Delta Q)_{i-1}^{n+1,l}}{\Delta x^2}$$

We decompose the A matrix into an A^+ matrix with only positive eigenvalues and an A^- matrix with only negative eigenvalues:

$$A = A^+ + A^-$$

We define forward and backward differences as:

$$\begin{aligned} \left(\delta_x^+ A\Delta Q\right)_i^{n+1,l} &= \frac{(A\Delta Q)_{i+1}^{n+1,l} - (A\Delta Q)_i^{n+1,l}}{\Delta x} \\ \left(\delta_x^- A\Delta Q\right)_i^{n+1,l} &= \frac{(A\Delta Q)_i^{n+1,l} - (A\Delta Q)_{i-1}^{n+1,l}}{\Delta x} \end{aligned}$$

For ease of use, we can define:

$$
\begin{aligned}
A^{\pm} &= \frac{1}{2}\left(A \pm \epsilon I\right) \\
\epsilon &\geq max\left|\lambda\left(A\right)\right|
\end{aligned}
$$

where $\lambda\left(A\right)$ are the eigenvalues of the A matrix.
The equation can be rewritten as:

$$
\left\{I + \Delta\tau_i \delta_x^- A^+ + \Delta\tau_i \delta_x^+ A^- - \Delta\tau_i \delta_{xx} B\right\} \Delta Q_i + \Delta\tau_i \left\{RHS\right\}_i^{n+1,l} \quad = \quad 0
$$

Defining

$$
\begin{aligned}
\alpha_i &= \frac{\Delta\tau_i}{\Delta x} \\
\beta_i &= \frac{\Delta\tau_i}{\Delta x^2}
\end{aligned}
$$

and expanding it out,

$$
\begin{pmatrix}
\Delta Q_i + \alpha_i \left(A_{i+1}^- \Delta Q_{i+1} - A_i^- \Delta Q_i + A_i^+ \Delta Q_i - A_{i-1}^+ \Delta Q_{i-1}\right) \\
-\beta_i \left(B_{i+1} \Delta Q_{i+1} - 2B_i \Delta Q_i + B_{i-1} \Delta Q_{i-1}\right)
\end{pmatrix} + \Delta\tau_i \left\{RHS\right\}_i^{n+1,l} \quad = \quad 0
$$

A symmetric Gauss-Seidel scheme is:

$$
\begin{aligned}
\begin{pmatrix}
\Delta Q_i^{(1)} + \left(\alpha_i \left(A_i^+ - A_i^-\right) + 2\beta_i B_i\right) \Delta Q_i^{(1)} \\
- \left(\alpha_i A_{i-1}^+ + \beta_i B_{i-1}\right) \Delta Q_{i-1}^{(1)}
\end{pmatrix} + \Delta\tau_i \left\{RHS\right\}_i^{n+1,l} &= 0 \\
\begin{pmatrix}
\Delta Q_i^{(2)} + \left(\alpha_i \left(A_i^+ - A_i^-\right) + 2\beta_i B_i\right) \Delta Q_i^{(2)} \\
+ \left(\alpha_i A_{i+1}^- - \beta_i B_{i+1}\right) \Delta Q_{i+1}^{(2)} \\
- \left(\alpha_i A_{i-1}^+ + \beta_i B_{i-1}\right) \Delta Q_{i-1}^{(1)}
\end{pmatrix} + \Delta\tau_i \left\{RHS\right\}_i^{n+1,l} &= 0
\end{aligned}
$$

Subtracting the first equation from the second gives:

$$
\begin{pmatrix}
\left\{I + \alpha_i \left(A_i^+ - A_i^-\right) + 2\beta_i B_i\right\} \Delta Q_i^{(2)} \\
+ \left(\alpha_i A_{i+1}^- - \beta_i B_{i+1}\right) \Delta Q_{i+1}^{(2)}
\end{pmatrix} \quad = \quad \left\{I + \alpha_i \left(A_i^+ - A_i^-\right) + 2\beta_i B_i\right\} \Delta Q_i^{(1)}
$$

The first stage can be rewritten as:

$$
\begin{aligned}
\begin{pmatrix} \Delta Q_i^{(1)} + \left(\alpha_i \left(A_i^+ - A_i^-\right) + 2\beta_i B_i\right) \Delta Q_i^{(1)} \\ - \left(\alpha_i A_{i-1}^+ + \beta_i B_{i-1}\right) \Delta Q_{i-1}^{(1)} \end{pmatrix} + \Delta\tau_i \left\{RHS\right\}_i^{n+1,l} &= 0 \\
\begin{pmatrix} \Delta Q_i^{(1)} - \alpha_i A_i^- \Delta Q_i^{(1)} + \beta_i B_i \Delta Q_i^{(1)} \\ +\alpha_i \left(A_i^+ \Delta Q_i^{(1)} - A_{i-1}^+ \Delta Q_{i-1}^{(1)}\right) \\ +\beta_i \left(B_i \Delta Q_i^{(1)} - B_{i-1} \Delta Q_{i-1}^{(1)}\right) \end{pmatrix} + \Delta\tau_i \left\{RHS\right\}_i^{n+1,l} &= 0 \\
\left(\left(I - \alpha_i A^- + \beta_i B\right) + \alpha_i \Delta x \delta_x^- A_i^+ + \beta_i \Delta x \delta_x^- B\right) \Delta Q^{(1)} &= -\Delta\tau_i \left\{RHS\right\}
\end{aligned}
$$

We then define:

$$
\Delta Q^* = \left(I + \alpha \left(A^+ - A^-\right) + 2\beta B\right) \Delta Q^{(1)}
$$

and the second stage becomes:

$$
\begin{aligned}
\begin{pmatrix} \left\{I + \alpha_i \left(A_i^+ - A_i^-\right) + 2\beta_i B_i\right\} \Delta Q_i^{(2)} \\ + \left(\alpha_i A_{i+1}^- - \beta_i B_{i+1}\right) \Delta Q_{i+1}^{(2)} \end{pmatrix} &= \left\{I + \alpha_i \left(A_i^+ - A_i^-\right) + 2\beta_i B_i\right\} \Delta Q_i^{(1)} \\
\begin{pmatrix} \left\{I + \alpha_i A_i^+ + \beta_i B_i\right\} \Delta Q_i^{(2)} \\ +\alpha_i \left(A_{i+1}^- \Delta Q_{i+1}^{(2)} - A_i^- \Delta Q_i^{(2)}\right) \\ -\beta_i \left(B_{i+1} \Delta Q_{i+1}^{(2)} - B_i \Delta Q_i^{(2)}\right) \end{pmatrix} &= \Delta Q_i^* \\
\left(\left\{I + \alpha A^+ + \beta B\right\} + \alpha \Delta x \delta_x^+ A^- - \beta \Delta x \delta_x^+ B\right) \Delta Q^{(2)} &= \Delta Q^*
\end{aligned}
$$

We can now write the scheme as:

$$
LD^{-1}U\Delta Q = -\Delta\tau \left\{RHS\right\}
$$

where

$$
\begin{aligned}
L &= I - \alpha A^- + \beta B + \alpha \Delta x \delta_x^- A^+ + \beta \Delta x \delta_x^- B \\
U &= I + \alpha A^+ + \beta B + \alpha \Delta x \delta_x^+ A^- - \beta \Delta x \delta_x^+ B \\
D &= I + \alpha \left(A^+ - A^-\right) + 2\beta B
\end{aligned}
$$

Note that L and U can each be solved by sweeping from one end of the grid to the other.

1.1 Just to be clear...

In 1D, the actual equation to solve on the first sweep is:

$$\left(I + \alpha\left(A^{+} - A^{-}\right)_i + 2\beta B_i\right)\Delta Q_i^{*} - \alpha_{i-1}A_{i-1}^{+}\Delta Q_{i-1}^{*} - \beta_{i-1}B_{i-1}\Delta Q_{i-1}^{*} \quad = \quad -\Delta\tau\left(RHS\right)_i$$

which becomes:

$$\left(I + \alpha\epsilon_i + 2\beta B_i\right)\Delta Q_i^{*} - \alpha_{i-1}A_{i-1}^{+}\Delta Q_{i-1}^{*} - \beta_{i-1}B_{i-1}\Delta Q_{i-1}^{*} \quad = \quad -\Delta\tau\left(RHS\right)_i$$

The second sweep is:

$$\left(\begin{array}{c} \left(I + \alpha\left(A^{+} - A^{-}\right)_i + 2\beta B_i\right)\Delta Q_i \\ +\alpha_{i+1}A_{i+1}^{-}\Delta Q_{i+1} - \beta_{i+1}B_{i+1}\Delta Q_{i+1} \end{array}\right) \quad = \quad \left(I + \alpha\epsilon_i + 2\beta B_i\right)\Delta Q_i^{*}$$

This is solved as:

$$\begin{aligned}
\Delta Q_i^{*} &= \left(I + \alpha\epsilon_i + 2\beta B_i\right)^{-1}\left(\begin{array}{c} -\Delta\tau\left(RHS\right)_i \\ +\left(\alpha_{i-1}A_{i-1}^{+} + \beta_{i-1}B_{i-1}\right)\Delta Q_{i-1}^{*} \end{array}\right) \\
\Delta Q_i &= \left(I + \alpha\epsilon_i + 2\beta B_i\right)^{-1}\left(\begin{array}{c} \left(I + \alpha\epsilon_i + 2\beta B_i\right)\Delta Q_i^{*} \\ +\left(-\alpha_{i+1}A_{i+1}^{-} + \beta_{i+1}B_{i+1}\right)\Delta Q_{i+1} \end{array}\right)
\end{aligned}$$

If we reverse the sweep directions, the scheme becomes:

$$\begin{aligned}
\Delta Q_i^{*} &= \left(I + \alpha\epsilon_i + 2\beta B_i\right)^{-1}\left(\begin{array}{c} -\Delta\tau\left(RHS\right)_i \\ +\left(-\alpha_{i+1}A_{i+1}^{-} + \beta_{i+1}B_{i+1}\right)\Delta Q_{i+1}^{*} \end{array}\right) \\
\Delta Q_i &= \left(I + \alpha\epsilon_i + 2\beta B_i\right)^{-1}\left(\begin{array}{c} \left(I + \alpha\epsilon_i + 2\beta B_i\right)\Delta Q_i^{*} \\ +\left(\alpha_{i-1}A_{i-1}^{+} + \beta_{i-1}B_{i-1}\right)\Delta Q_{i-1} \end{array}\right)
\end{aligned}$$

2 Implementation of the viscous extension of the LU-SGS (Yoon and Jameson, 1988) Scheme

The first stage of the LU-SGS scheme in curvilinear 4D is:

$$L \Delta Q^{(1)} = -\Delta\tau RHS$$

$$\left(\left(I + \Delta\tau \begin{pmatrix} -A^-_{i,j,k,l} - B^-_{i,j,k,l} \\ -C^-_{i,j,k,l} - D^-_{i,j,k,l} \\ +2R_{i,j,k,l} + 2S_{i,j,k,l} \\ +2T_{i,j,k,l} + 2U_{i,j,k,l} \\ +A^+_{i,j,k,l} + B^+_{i,j,k,l} \\ +C^+_{i,j,k,l} + D^+_{i,j,k,l} \end{pmatrix} \right) \Delta Q^{(1)}_{i,j,k,l} - \begin{pmatrix} \Delta\tau \begin{pmatrix} A^+_{i-1,j,k,l} \\ +R_{i-1,j,k,l} \end{pmatrix} \Delta Q^{(1)}_{i-1,j,k,l} \\ +\Delta\tau \begin{pmatrix} B^+_{i,j-1,k,l} \\ +S_{i,j-1,k,l} \end{pmatrix} \Delta Q^{(1)}_{i,j-1,k,l} \\ +\Delta\tau \begin{pmatrix} C^+_{i,j,k-1,l} \\ +T_{i,j,k-1,l} \end{pmatrix} \Delta Q^{(1)}_{i,j,k-1,l} \\ +\Delta\tau \begin{pmatrix} D^+_{i,j,k,l-1} \\ +U_{i,j,k,l-1} \end{pmatrix} \Delta Q^{(1)}_{i,j,k,l-1} \end{pmatrix} \right) = -\Delta\tau RHS_{i,j,k,l}$$

Note that:

$$\begin{aligned} A^\pm &= \frac{1}{2}\left(A \pm \epsilon_\xi I\right) \\ B^\pm &= \frac{1}{2}\left(B \pm \epsilon_\eta I\right) \\ C^\pm &= \frac{1}{2}\left(C \pm \epsilon_\zeta I\right) \\ D^\pm &= \frac{1}{2}\left(D \pm \epsilon_\tau I\right) \end{aligned}$$

which means that:

$$\begin{aligned} A^+ - A^- &= \epsilon_\xi I \\ B^+ - B^- &= \epsilon_\eta I \\ C^+ - C^- &= \epsilon_\zeta I \\ D^+ - D^- &= \epsilon_\tau I \end{aligned}$$

Defining:

$$\Gamma_{i,j,k,l} = \left(\begin{array}{c} \left(1 + \Delta\tau\left(\epsilon_\xi + \epsilon_\eta + \epsilon_\zeta + \epsilon_\tau\right)\right) I \\ +2\Delta\tau \left(\begin{array}{r} R_{i,j,k,l} \\ +S_{i,j,k,l} \\ +T_{i,j,k,l} \\ +U_{i,j,k,l} \end{array} \right) \end{array} \right)$$

gives the first stage of the LU-SGS as:

$$\Gamma_{i,j,k,l}\Delta Q^{(1)}_{i,j,k,l} = \Delta\tau \left(\begin{array}{r} \left(\begin{array}{r} A^+_{i-1,j,k,l} \\ +R_{i-1,j,k,l} \end{array} \right) \Delta Q^{(1)}_{i-1,j,k,l} \\ + \left(\begin{array}{r} B^+_{i,j-1,k,l} \\ +S_{i,j-1,k,l} \end{array} \right) \Delta Q^{(1)}_{i,j-1,k,l} \\ + \left(\begin{array}{r} C^+_{i,j,k-1,l} \\ +T_{i,j,k-1,l} \end{array} \right) \Delta Q^{(1)}_{i,j,k-1,l} \\ + \left(\begin{array}{r} D^+_{i,j,k,l-1} \\ +U_{i,j,k,l-1} \end{array} \right) \Delta Q^{(1)}_{i,j,k,l-1} \\ -RHS_{i,j,k,l} \end{array} \right)$$

The second stage is:

$$U\Delta Q^{(2)} = D\Delta Q^{(1)}$$

which becomes:

$$\Gamma_{i,j,k,l}\Delta Q^{(2)}_{i,j,k,l} = \Delta\tau \left(\begin{array}{r} \left(\begin{array}{r} -A^-_{i+1,j,k,l} \\ +R_{i+1,j,k,l} \end{array} \right) \Delta Q^{(2)}_{i+1,j,k,l} \\ + \left(\begin{array}{r} -B^-_{i,j+1,k,l} \\ +S_{i,j+1,k,l} \end{array} \right) \Delta Q^{(2)}_{i,j+1,k,l} \\ + \left(\begin{array}{r} -C^-_{i,j,k+1,l} \\ +T_{i,j,k+1,l} \end{array} \right) \Delta Q^{(2)}_{i,j,k+1,l} \\ + \left(\begin{array}{r} -D^-_{i,j,k,l+1} \\ +U_{i,j,k,l+1} \end{array} \right) \Delta Q^{(2)}_{i,j,k,l+1} \\ +\Gamma_{i,j,k,l}\Delta Q^{(1)}_{i,j,k,l} \end{array} \right)$$

Note that the Γ matrix must be inverted at every grid point in both stages. In the inviscid form of the LU-SGS, this was trivial because Γ was a diagonal matrix. However, if the viscous terms are retained, Γ has nondiagonal terms and the inversion process is more expensive.

2.1 Flux Jacobian matrices

The inviscid flux Jacobians have the form:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix}$$

The viscous flux Jacobians have the form:

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ R_{21} & R_{22} & R_{23} & R_{24} & 0 \\ R_{31} & R_{32} & R_{33} & R_{34} & 0 \\ R_{41} & R_{42} & R_{43} & R_{44} & 0 \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} \end{bmatrix}$$

Defining:

$$\begin{aligned} V_{ab} &= 2\Delta\tau \left(R_{ab} + S_{ab} + T_{ab} + U_{ab}\right) \\ \kappa &= 1 + \Delta\tau \left(\epsilon_\xi + \epsilon_\eta + \epsilon_\zeta + \epsilon_\tau\right) \end{aligned}$$

gives the form of the Γ matrix as:

$$\Gamma = \begin{bmatrix} \kappa & 0 & 0 & 0 & 0 \\ V_{21} & V_{22}+\kappa & V_{23} & V_{24} & 0 \\ V_{31} & V_{32} & V_{33}+\kappa & V_{34} & 0 \\ V_{41} & V_{42} & V_{43} & V_{44}+\kappa & 0 \\ V_{51} & V_{52} & V_{53} & V_{54} & V_{55}+\kappa \end{bmatrix}$$

$$= \begin{bmatrix} \Gamma_{11} & 0 & 0 & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & 0 \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \Gamma_{34} & 0 \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} & 0 \\ \Gamma_{51} & \Gamma_{52} & \Gamma_{53} & \Gamma_{54} & \Gamma_{55} \end{bmatrix}$$

Maxima gives the determinant of the matrix as:

$$\sigma = \Gamma_{11}\Gamma_{55} \begin{pmatrix} \Gamma_{22}\left(\Gamma_{33}\Gamma_{44} - \Gamma_{34}\Gamma_{43}\right) \\ -\Gamma_{23}\left(\Gamma_{32}\Gamma_{44} - \Gamma_{34}\Gamma_{42}\right) \\ +\Gamma_{24}\left(\Gamma_{32}\Gamma_{43} - \Gamma_{33}\Gamma_{42}\right) \end{pmatrix}$$

Then,

$$\Gamma^{-1} = \begin{bmatrix} \phi_{11} & 0 & 0 & 0 & 0 \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} & 0 \\ \phi_{31} & \phi_{32} & \phi_{33} & \phi_{34} & 0 \\ \phi_{41} & \phi_{42} & \phi_{43} & \phi_{44} & 0 \\ \phi_{51} & \phi_{52} & \phi_{53} & \phi_{54} & \phi_{55} \end{bmatrix}$$

where

$$\begin{aligned}
\phi_{11} &= \frac{1}{\Gamma_{11}} \\
\phi_{21} &= \frac{\Gamma_{55}}{\sigma} \begin{pmatrix} -\Gamma_{21}\left(\Gamma_{33}\Gamma_{44} - \Gamma_{34}\Gamma_{43}\right) \\ +\Gamma_{23}\left(\Gamma_{31}\Gamma_{44} - \Gamma_{34}\Gamma_{41}\right) \\ -\Gamma_{24}\left(\Gamma_{31}\Gamma_{43} - \Gamma_{33}\Gamma_{41}\right) \end{pmatrix} \\
\phi_{22} &= \frac{\Gamma_{11}\Gamma_{55}}{\sigma}\left(\Gamma_{33}\Gamma_{44} - \Gamma_{34}\Gamma_{43}\right) \\
\phi_{23} &= \frac{\Gamma_{11}\Gamma_{55}}{\sigma}\left(\Gamma_{23}\Gamma_{44} - \Gamma_{24}\Gamma_{43}\right) \\
\phi_{24} &= \frac{-\Gamma_{11}\Gamma_{55}}{\sigma}\left(\Gamma_{23}\Gamma_{34} - \Gamma_{24}\Gamma_{33}\right) \\
\phi_{31} &= \frac{\Gamma_{55}}{\sigma} \begin{pmatrix} \Gamma_{21}\left(\Gamma_{32}\Gamma_{44} - \Gamma_{34}\Gamma_{42}\right) \\ -\Gamma_{22}\left(\Gamma_{31}\Gamma_{44} - \Gamma_{34}\Gamma_{41}\right) \\ +\Gamma_{24}\left(\Gamma_{31}\Gamma_{42} - \Gamma_{32}\Gamma_{41}\right) \end{pmatrix} \\
\phi_{32} &= \frac{-\Gamma_{11}\Gamma_{55}}{\sigma}\left(\Gamma_{32}\Gamma_{44} - \Gamma_{34}\Gamma_{42}\right)
\end{aligned}$$

$$
\begin{aligned}
\phi_{33} &= \frac{\Gamma_{11}\Gamma_{55}}{\sigma}\left(\Gamma_{22}\Gamma_{44}-\Gamma_{24}\Gamma_{42}\right)\\
\phi_{34} &= \frac{-\Gamma_{11}\Gamma_{55}}{\sigma}\left(\Gamma_{22}\Gamma_{34}-\Gamma_{24}\Gamma_{32}\right)\\
\phi_{41} &= \frac{\Gamma_{55}}{\sigma}\left(\begin{array}{c} -\Gamma_{21}\left(\Gamma_{32}\Gamma_{43}-\Gamma_{33}\Gamma_{42}\right)\\ +\Gamma_{22}\left(\Gamma_{31}\Gamma_{43}-\Gamma_{33}\Gamma_{41}\right)\\ -\Gamma_{23}\left(\Gamma_{31}\Gamma_{42}-\Gamma_{32}\Gamma_{41}\right)\end{array}\right)\\
\phi_{42} &= \frac{\Gamma_{11}\Gamma_{55}}{\sigma}\left(\Gamma_{32}\Gamma_{43}-\Gamma_{33}\Gamma_{42}\right)\\
\phi_{43} &= \frac{-\Gamma_{11}\Gamma_{55}}{\sigma}\left(\Gamma_{22}\Gamma_{43}-\Gamma_{23}\Gamma_{42}\right)\\
\phi_{44} &= \frac{\Gamma_{11}\Gamma_{55}}{\sigma}\left(\Gamma_{22}\Gamma_{33}-\Gamma_{23}\Gamma_{32}\right)\\
\phi_{51} &= \frac{1}{\sigma}\left(\begin{array}{r} \Gamma_{21}\left(\begin{array}{c} \Gamma_{32}\left(\Gamma_{43}\Gamma_{54}-\Gamma_{44}\Gamma_{53}\right)\\ -\Gamma_{33}\left(\Gamma_{42}\Gamma_{54}-\Gamma_{44}\Gamma_{52}\right)\\ +\Gamma_{34}\left(\Gamma_{42}\Gamma_{53}-\Gamma_{43}\Gamma_{52}\right)\end{array}\right)\\ -\Gamma_{22}\left(\begin{array}{c} \Gamma_{31}\left(\Gamma_{43}\Gamma_{54}-\Gamma_{44}\Gamma_{53}\right)\\ -\Gamma_{33}\left(\Gamma_{41}\Gamma_{54}-\Gamma_{44}\Gamma_{51}\right)\\ +\Gamma_{34}\left(\Gamma_{41}\Gamma_{53}-\Gamma_{43}\Gamma_{51}\right)\end{array}\right)\\ +\Gamma_{23}\left(\begin{array}{c} \Gamma_{31}\left(\Gamma_{42}\Gamma_{54}-\Gamma_{44}\Gamma_{52}\right)\\ -\Gamma_{32}\left(\Gamma_{41}\Gamma_{54}-\Gamma_{44}\Gamma_{51}\right)\\ +\Gamma_{34}\left(\Gamma_{41}\Gamma_{52}-\Gamma_{42}\Gamma_{51}\right)\end{array}\right)\\ -\Gamma_{24}\left(\begin{array}{c} \Gamma_{31}\left(\Gamma_{42}\Gamma_{53}-\Gamma_{43}\Gamma_{52}\right)\\ -\Gamma_{32}\left(\Gamma_{41}\Gamma_{53}-\Gamma_{43}\Gamma_{51}\right)\\ +\Gamma_{33}\left(\Gamma_{41}\Gamma_{52}-\Gamma_{42}\Gamma_{51}\right)\end{array}\right)\end{array}\right)\\
\phi_{52} &= \frac{-\Gamma_{11}}{\sigma}\left(\begin{array}{c} \Gamma_{32}\left(\Gamma_{43}\Gamma_{54}-\Gamma_{44}\Gamma_{53}\right)\\ -\Gamma_{33}\left(\Gamma_{42}\Gamma_{54}-\Gamma_{44}\Gamma_{52}\right)\\ +\Gamma_{34}\left(\Gamma_{42}\Gamma_{53}-\Gamma_{43}\Gamma_{52}\right)\end{array}\right)\\
\phi_{53} &= \frac{\Gamma_{11}}{\sigma}\left(\begin{array}{c} \Gamma_{22}\left(\Gamma_{43}\Gamma_{54}-\Gamma_{44}\Gamma_{53}\right)\\ -\Gamma_{23}\left(\Gamma_{42}\Gamma_{54}-\Gamma_{44}\Gamma_{52}\right)\\ +\Gamma_{24}\left(\Gamma_{42}\Gamma_{53}-\Gamma_{43}\Gamma_{52}\right)\end{array}\right)\\
\phi_{54} &= \frac{-\Gamma_{11}}{\sigma}\left(\begin{array}{c} \Gamma_{22}\left(\Gamma_{33}\Gamma_{54}-\Gamma_{34}\Gamma_{53}\right)\\ -\Gamma_{23}\left(\Gamma_{32}\Gamma_{54}-\Gamma_{34}\Gamma_{52}\right)\\ +\Gamma_{24}\left(\Gamma_{32}\Gamma_{53}-\Gamma_{33}\Gamma_{52}\right)\end{array}\right)\\
\phi_{55} &= \frac{\Gamma_{11}}{\sigma}\left(\begin{array}{c} \Gamma_{22}\left(\Gamma_{33}\Gamma_{44}-\Gamma_{34}\Gamma_{43}\right)\\ -\Gamma_{23}\left(\Gamma_{32}\Gamma_{44}-\Gamma_{34}\Gamma_{42}\right)\\ +\Gamma_{24}\left(\Gamma_{32}\Gamma_{43}-\Gamma_{33}\Gamma_{42}\right)\end{array}\right)
\end{aligned}
$$

It's probably much faster to reduce the matrix. At a grid point, the block equation is:

$$[\Gamma]\{\Delta Q\} = \{R\}$$

Expanding this out gives:

$$\begin{bmatrix} \Gamma_{11} & 0 & 0 & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & 0 \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \Gamma_{34} & 0 \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} & 0 \\ \Gamma_{51} & \Gamma_{52} & \Gamma_{53} & \Gamma_{54} & \Gamma_{55} \end{bmatrix} \begin{Bmatrix} \Delta Q_1 \\ \Delta Q_2 \\ \Delta Q_3 \\ \Delta Q_4 \\ \Delta Q_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix}$$

becomes

$$\begin{bmatrix} \Gamma_{11} & 0 & 0 & 0 & 0 \\ \Gamma'_{21} & \Gamma'_{22} & \Gamma'_{23} & 0 & 0 \\ \Gamma'_{31} & \Gamma'_{32} & \Gamma'_{33} & 0 & 0 \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} & 0 \\ \Gamma_{51} & \Gamma_{52} & \Gamma_{53} & \Gamma_{54} & \Gamma_{55} \end{bmatrix} \begin{Bmatrix} \Delta Q_1 \\ \Delta Q_2 \\ \Delta Q_3 \\ \Delta Q_4 \\ \Delta Q_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R'_2 \\ R'_3 \\ R_4 \\ R_5 \end{Bmatrix}$$

where

$$\begin{aligned} \Gamma'_{21} &= \Gamma_{21} - \frac{\Gamma_{24}}{\Gamma_{44}}\Gamma_{41} \\ \Gamma'_{22} &= \Gamma_{22} - \frac{\Gamma_{24}}{\Gamma_{44}}\Gamma_{42} \\ \Gamma'_{23} &= \Gamma_{23} - \frac{\Gamma_{24}}{\Gamma_{44}}\Gamma_{43} \\ R'_2 &= R_2 - \frac{\Gamma_{24}}{\Gamma_{44}}R_4 \\ \Gamma'_{31} &= \Gamma_{31} - \frac{\Gamma_{34}}{\Gamma_{44}}\Gamma_{41} \\ \Gamma'_{32} &= \Gamma_{32} - \frac{\Gamma_{34}}{\Gamma_{44}}\Gamma_{42} \\ \Gamma'_{33} &= \Gamma_{33} - \frac{\Gamma_{34}}{\Gamma_{44}}\Gamma_{43} \\ R'_3 &= R_3 - \frac{\Gamma_{34}}{\Gamma_{44}}R_4 \end{aligned}$$

The equation then becomes:

$$\begin{bmatrix} \Gamma_{11} & 0 & 0 & 0 & 0 \\ \Gamma''_{21} & \Gamma''_{22} & 0 & 0 & 0 \\ \Gamma'_{31} & \Gamma'_{32} & \Gamma'_{33} & 0 & 0 \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} & 0 \\ \Gamma_{51} & \Gamma_{52} & \Gamma_{53} & \Gamma_{54} & \Gamma_{55} \end{bmatrix} \begin{Bmatrix} \Delta Q_1 \\ \Delta Q_2 \\ \Delta Q_3 \\ \Delta Q_4 \\ \Delta Q_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R''_2 \\ R'_3 \\ R_4 \\ R_5 \end{Bmatrix}$$

where

$$\begin{aligned} \Gamma''_{21} &= \Gamma'_{21} - \frac{\Gamma'_{23}}{\Gamma'_{33}}\Gamma'_{31} \\ \Gamma''_{22} &= \Gamma'_{22} - \frac{\Gamma'_{23}}{\Gamma'_{33}}\Gamma'_{32} \\ R''_{2} &= R'_{2} - \frac{\Gamma'_{23}}{\Gamma'_{33}}R'_{3} \end{aligned}$$

and, at this point, it can be solved by backsubstitution.

3 LU-SGS for the Grid Generation Equations

The 4D grid generation equations have this form:

$$
RHS_x = \left(\begin{array}{r} \alpha_{11}\widehat{\phi}_1 x_\xi \\ +\alpha_{22}\widehat{\phi}_2 x_\eta \\ +\alpha_{33}\widehat{\phi}_3 x_\zeta \\ +\alpha_{44}\widehat{\phi}_4 x_\tau \end{array} \right) + \left(\begin{array}{r} \left(\alpha_{11} + x_\xi^2\beta_1\right) x_{\xi\xi} \\ +x_\xi y_\xi \beta_1 y_{\xi\xi} \\ +x_\xi z_\xi \beta_1 z_{\xi\xi} \\ +x_\xi t_\xi \beta_1 t_{\xi\xi} \\ + \left(\alpha_{22} + x_\eta^2\beta_2\right) x_{\eta\eta} \\ +x_\eta y_\eta \beta_2 y_{\eta\eta} \\ +x_\eta z_\eta \beta_2 z_{\eta\eta} \\ +x_\eta t_\eta \beta_2 t_{\eta\eta} \\ + \left(\alpha_{33} + x_\zeta^2\beta_3\right) x_{\zeta\zeta} \\ +x_\zeta y_\zeta \beta_3 y_{\zeta\zeta} \\ +x_\zeta z_\zeta \beta_3 z_{\zeta\zeta} \\ +x_\zeta t_\zeta \beta_3 t_{\zeta\zeta} \\ + \left(\alpha_{44} + x_\tau^2\beta_4\right) x_{\tau\tau} \\ +x_\tau y_\tau \beta_4 y_{\tau\tau} \\ +x_\tau z_\tau \beta_4 z_{\tau\tau} \\ +x_\tau t_\tau \beta_4 t_{\tau\tau} \\ +2\alpha_{12}x_{\xi\eta} + 2\alpha_{13}x_{\xi\zeta} + 2\alpha_{14}x_{\xi\tau} \\ +2\alpha_{23}x_{\eta\zeta} + 2\alpha_{24}x_{\eta\tau} + 2\alpha_{34}x_{\zeta\tau} \end{array} \right)
$$

$$
RHS_y = \begin{pmatrix} \alpha_{11}\widehat{\phi}_1 y_\xi \\ +\alpha_{22}\widehat{\phi}_2 y_\eta \\ +\alpha_{33}\widehat{\phi}_3 y_\zeta \\ +\alpha_{44}\widehat{\phi}_4 y_\tau \end{pmatrix} + \begin{pmatrix} \left(\alpha_{11} + y_\xi^2 \beta_1\right) y_{\xi\xi} \\ +x_\xi y_\xi \beta_1 x_{\xi\xi} \\ +y_\xi z_\xi \beta_1 z_{\xi\xi} \\ +y_\xi t_\xi \beta_1 t_{\xi\xi} \\ + \left(\alpha_{22} + y_\eta^2 \beta_2\right) y_{\eta\eta} \\ +x_\eta y_\eta \beta_2 x_{\eta\eta} \\ +y_\eta z_\eta \beta_2 z_{\eta\eta} \\ +y_\eta t_\eta \beta_2 t_{\eta\eta} \\ + \left(\alpha_{33} + y_\zeta^2 \beta_3\right) y_{\zeta\zeta} \\ +x_\zeta y_\zeta \beta_3 x_{\zeta\zeta} \\ +y_\zeta z_\zeta \beta_3 z_{\zeta\zeta} \\ +y_\zeta t_\zeta \beta_3 t_{\zeta\zeta} \\ + (\alpha_{44} + y_\tau^2 \beta_4) y_{\tau\tau} \\ +x_\tau y_\tau \beta_4 x_{\tau\tau} \\ +y_\tau z_\tau \beta_4 z_{\tau\tau} \\ +y_\tau t_\tau \beta_4 t_{\tau\tau} \\ +2\alpha_{12} y_{\xi\eta} + 2\alpha_{13} y_{\xi\zeta} + 2\alpha_{14} y_{\xi\tau} \\ +2\alpha_{23} y_{\eta\zeta} + 2\alpha_{24} y_{\eta\tau} + 2\alpha_{34} y_{\zeta\tau} \end{pmatrix}
$$

$$
RHS_z = \begin{pmatrix} \alpha_{11}\widehat{\phi}_1 z_\xi \\ +\alpha_{22}\widehat{\phi}_2 z_\eta \\ +\alpha_{33}\widehat{\phi}_3 z_\zeta \\ +\alpha_{44}\widehat{\phi}_4 z_\tau \end{pmatrix} + \begin{pmatrix} \left(\alpha_{11} + z_\xi^2\beta_1\right) z_{\xi\xi} \\ +x_\xi z_\xi \beta_1 x_{\xi\xi} \\ +y_\xi z_\xi \beta_1 y_{\xi\xi} \\ +z_\xi t_\xi \beta_1 t_{\xi\xi} \\ +\left(\alpha_{22} + z_\eta^2\beta_2\right) z_{\eta\eta} \\ +x_\eta z_\eta \beta_2 x_{\eta\eta} \\ +y_\eta z_\eta \beta_2 y_{\eta\eta} \\ +z_\eta t_\eta \beta_2 t_{\eta\eta} \\ +\left(\alpha_{33} + z_\zeta^2\beta_3\right) z_{\zeta\zeta} \\ +x_\zeta z_\zeta \beta_3 x_{\zeta\zeta} \\ +y_\zeta z_\zeta \beta_3 y_{\zeta\zeta} \\ +z_\zeta t_\zeta \beta_3 t_{\zeta\zeta} \\ +\left(\alpha_{44} + z_\tau^2\beta_4\right) z_{\tau\tau} \\ +x_\tau z_\tau \beta_4 x_{\tau\tau} \\ +y_\tau z_\tau \beta_4 y_{\tau\tau} \\ +z_\tau t_\tau \beta_4 t_{\tau\tau} \\ +2\alpha_{12} z_{\xi\eta} + 2\alpha_{13} z_{\xi\zeta} + 2\alpha_{14} z_{\xi\tau} \\ +2\alpha_{23} z_{\eta\zeta} + 2\alpha_{24} z_{\eta\tau} + 2\alpha_{34} z_{\zeta\tau} \end{pmatrix}
$$

$$
RHS_t = \begin{pmatrix} \alpha_{11}\widehat{\phi}_1 t_\xi \\ +\alpha_{22}\widehat{\phi}_2 t_\eta \\ +\alpha_{33}\widehat{\phi}_3 t_\zeta \\ +\alpha_{44}\widehat{\phi}_4 t_\tau \end{pmatrix} + \begin{pmatrix} \left(\alpha_{11} + t_\xi^2\beta_1\right) t_{\xi\xi} \\ +x_\xi t_\xi \beta_1 x_{\xi\xi} \\ +y_\xi t_\xi \beta_1 y_{\xi\xi} \\ +z_\xi t_\xi \beta_1 z_{\xi\xi} \\ +\left(\alpha_{22} + t_\eta^2\beta_2\right) t_{\eta\eta} \\ +x_\eta t_\eta \beta_2 x_{\eta\eta} \\ +y_\eta t_\eta \beta_2 y_{\eta\eta} \\ +z_\eta t_\eta \beta_2 z_{\eta\eta} \\ +\left(\alpha_{33} + t_\zeta^2\beta_3\right) t_{\zeta\zeta} \\ +x_\zeta t_\zeta \beta_3 x_{\zeta\zeta} \\ +y_\zeta t_\zeta \beta_3 y_{\zeta\zeta} \\ +z_\zeta t_\zeta \beta_3 z_{\zeta\zeta} \\ +\left(\alpha_{44} + t_\tau^2\beta_4\right) t_{\tau\tau} \\ +x_\tau t_\tau \beta_4 x_{\tau\tau} \\ +y_\tau t_\tau \beta_4 y_{\tau\tau} \\ +z_\tau t_\tau \beta_4 z_{\tau\tau} \\ +2\alpha_{12}t_{\xi\eta} + 2\alpha_{13}t_{\xi\zeta} + 2\alpha_{14}t_{\xi\tau} \\ +2\alpha_{23}t_{\eta\zeta} + 2\alpha_{24}t_{\eta\tau} + 2\alpha_{34}t_{\zeta\tau} \end{pmatrix}
$$

where

$$
\begin{aligned}
\alpha_{11} &= \xi_x^2 + \xi_y^2 + \xi_z^2 + \xi_t^2 \\
\alpha_{12} &= \xi_x\eta_x + \xi_y\eta_y + \xi_z\eta_z + \xi_t\eta_t \\
\alpha_{13} &= \xi_x\zeta_x + \xi_y\zeta_y + \xi_z\zeta_z + \xi_t\zeta_t \\
\alpha_{14} &= \xi_x\tau_x + \xi_y\tau_y + \xi_z\tau_z + \xi_t\tau_t \\
\alpha_{22} &= \eta_x^2 + \eta_y^2 + \eta_z^2 + \eta_t^2 \\
\alpha_{23} &= \eta_x\zeta_x + \eta_y\zeta_y + \eta_z\zeta_z + \eta_t\zeta_t \\
\alpha_{24} &= \eta_x\tau_x + \eta_y\tau_y + \eta_z\tau_z + \eta_t\tau_t \\
\alpha_{33} &= \zeta_x^2 + \zeta_y^2 + \zeta_z^2 + \zeta_t^2 \\
\alpha_{34} &= \zeta_x\tau_x + \zeta_y\tau_y + \zeta_z\tau_z + \zeta_t\tau_t \\
\alpha_{44} &= \tau_x^2 + \tau_y^2 + \tau_z^2 + \tau_t^2
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_1 &= \frac{\alpha_{11}}{x_\xi^2 + y_\xi^2 + z_\xi^2 + t_\xi^2} \\
\beta_2 &= \frac{\alpha_{22}}{x_\eta^2 + y_\eta^2 + z_\eta^2 + t_\eta^2} \\
\beta_3 &= \frac{\alpha_{33}}{x_\zeta^2 + y_\zeta^2 + z_\zeta^2 + t_\zeta^2} \\
\beta_4 &= \frac{\alpha_{44}}{x_\tau^2 + y_\tau^2 + z_\tau^2 + t_\tau^2}
\end{aligned}
$$

$\widehat{\phi}_{1-4}$ are source terms that are used to obtain orthogonal grid at the boundaries, and to cluster grid points as necessary in the domain. These source terms are determined iteratively as the solution converges. They begin with a zero value.

3.1 Flux Jacobian Matrices

Using σ ($\sigma = \xi, \eta, \zeta, \tau$) as the n ($n = 1, 2, 3, 4$) curvilinear direction, the inviscid flux Jacobian matrix is:

$$
[A]_\sigma = \begin{bmatrix} \alpha_{nn}\widehat{\phi}_n & 0 & 0 & 0 \\ 0 & \alpha_{nn}\widehat{\phi}_n & 0 & 0 \\ 0 & 0 & \alpha_{nn}\widehat{\phi}_n & 0 \\ 0 & 0 & 0 & \alpha_{nn}\widehat{\phi}_n \end{bmatrix}
$$

and it's pretty clear that the magnitude of the maximum eigenvalue is:

$$
\epsilon_\sigma = \left| \alpha_{nn}\widehat{\phi}_n \right|
$$

Note that the A matrix is very likely zero through a large portion of the domain!

The viscous flux Jacobian matrix is:

$$
[B]_\sigma = \begin{bmatrix} \alpha_{nn} + x_\sigma^2\beta_n & x_\sigma y_\sigma \beta_n & x_\sigma z_\sigma \beta_n & x_\sigma t_\sigma \beta_n \\ x_\sigma y_\sigma \beta_n & \alpha_{nn} + y_\sigma^2\beta_n & y_\sigma z_\sigma \beta_n & y_\sigma t_\sigma \beta_n \\ x_\sigma z_\sigma \beta_n & y_\sigma z_\sigma \beta_n & \alpha_{nn} + z_\sigma^2\beta_n & z_\sigma t_\sigma \beta_n \\ x_\sigma t_\sigma \beta_n & y_\sigma t_\sigma \beta_n & z_\sigma t_\sigma \beta_n & \alpha_{nn} + t_\sigma^2\beta_n \end{bmatrix}
$$

Some notes here: The B matrix is full, so I'll need to use a matrix solver to get the ΔQ's during the LU-SGS sweeps.

The same formulation of the LU-SGS scheme will be used for both the flow solver and the grid generator, trusting it works in multidimensions.

If the LU-SGS scheme performs well, I'm planning to use a multigrid version of the LU-SGS method as the default scheme for our group – until there's something clearly better.

4 Verification Method for LU-SGS Scheme using the Method of Manufactured Solutions

First, some initial definitions:

$$
\begin{aligned}
\Delta\xi\delta_\xi^+ Q &= \left(Q\left(\xi+\Delta\xi\right)-Q\left(\xi\right)\right)\\
&\approx \left(\left(Q\left(\xi\right)+\left(\Delta\xi\right)\frac{\partial Q}{\partial\xi}+\frac{\left(\Delta\xi\right)^2}{2}\frac{\partial^2 Q}{\partial\xi^2}+\ldots\right)-Q\left(\xi\right)\right)\\
&\approx \Delta\xi\frac{\partial Q}{\partial\xi}+\frac{\Delta\xi^2}{2}\frac{\partial^2 Q}{\partial\xi^2}+\ldots\\
\Delta\xi\delta_\xi^- Q &= \left(Q\left(\xi\right)-Q\left(\xi-\Delta\xi\right)\right)\\
&\approx \left(Q\left(\xi\right)-\left(Q\left(\xi\right)+\left(-\Delta\xi\right)\frac{\partial Q}{\partial\xi}+\frac{\left(-\Delta\xi\right)^2}{2}\frac{\partial^2 Q}{\partial\xi^2}+\ldots\right)\right)\\
&\approx \Delta\xi\frac{\partial Q}{\partial\xi}-\frac{\Delta\xi^2}{2}\frac{\partial^2 Q}{\partial\xi^2}+\ldots
\end{aligned}
$$

and (for the moment) defining:

$$
\begin{aligned}
\alpha^{(1)} &= \frac{\Delta\sigma}{\Delta\xi}\\
\alpha^{(2)} &= \frac{\Delta\sigma}{\Delta\eta}\\
\alpha^{(3)} &= \frac{\Delta\sigma}{\Delta\zeta}\\
\alpha^{(4)} &= \frac{\Delta\sigma}{\Delta\tau}
\end{aligned}
$$

and

$$\begin{aligned}
\beta^{(1)} &= \frac{\Delta\sigma}{\Delta\xi^2} \\
\beta^{(2)} &= \frac{\Delta\sigma}{\Delta\eta^2} \\
\beta^{(3)} &= \frac{\Delta\sigma}{\Delta\zeta^2} \\
\beta^{(4)} &= \frac{\Delta\sigma}{\Delta\tau^2}
\end{aligned}$$

4.1 Initial LU-SGS sweep

The first sweep of the LU-SGS equations in curvilinear coordinates is:

$$\left(\begin{array}{c} \left(I - \left(\begin{array}{r} \alpha^{(1)}A^- - \beta^{(1)}R \\ +\alpha^{(1)}B^- - \beta^{(1)}S \\ +\alpha^{(1)}C^- - \beta^{(1)}T \\ +\alpha^{(1)}D^- - \beta^{(1)}U \end{array} \right) \right) \Delta Q^* \\ + \left(\begin{array}{c} \Delta\xi\delta_\xi^- \left(\left(\alpha^{(1)}A^+ + \beta^{(1)}R \right) \Delta Q^* \right) \\ +\Delta\eta\delta_\eta^- \left(\left(\alpha^{(2)}B^+ + \beta^{(2)}S \right) \Delta Q^* \right) \\ +\Delta\zeta\delta_\zeta^- \left(\left(\alpha^{(3)}C^+ + \beta^{(3)}T \right) \Delta Q^* \right) \\ +\Delta\tau\delta_\tau^- \left(\left(\alpha^{(4)}D^+ + \beta^{(4)}U \right) \Delta Q^* \right) \end{array} \right) \end{array} \right) = -\Delta\sigma\,(RHS)$$

which analytically becomes:

$$\left(\begin{array}{c} \left(I - \left(\begin{array}{r} \alpha^{(1)}A^- - \beta^{(1)}R \\ +\alpha^{(1)}B^- - \beta^{(1)}S \\ +\alpha^{(1)}C^- - \beta^{(1)}T \\ +\alpha^{(1)}D^- - \beta^{(1)}U \end{array} \right) \right) \Delta Q^* \\ + \left(\begin{array}{c} \left(\begin{array}{c} \Delta\xi \frac{\partial}{\partial\xi} \left(\left(\alpha^{(1)}A^+ + \beta^{(1)}R \right) \Delta Q^* \right) \\ -\frac{\Delta\xi^2}{2} \frac{\partial^2}{\partial\xi^2} \left(\left(\alpha^{(1)}A^+ + \beta^{(1)}R \right) \Delta Q^* \right) \end{array} \right) \\ + \left(\begin{array}{c} \Delta\eta \frac{\partial}{\partial\eta} \left(\left(\alpha^{(2)}B^+ + \beta^{(2)}S \right) \Delta Q^* \right) \\ -\frac{\Delta\eta^2}{2} \frac{\partial^2}{\partial\eta^2} \left(\left(\alpha^{(2)}B^+ + \beta^{(2)}S \right) \Delta Q^* \right) \end{array} \right) \\ + \left(\begin{array}{c} \Delta\zeta \frac{\partial}{\partial\zeta} \left(\left(\alpha^{(3)}C^+ + \beta^{(3)}T \right) \Delta Q^* \right) \\ -\frac{\Delta\zeta^2}{2} \frac{\partial^2}{\partial\zeta^2} \left(\left(\alpha^{(3)}C^+ + \beta^{(3)}T \right) \Delta Q^* \right) \end{array} \right) \\ + \left(\begin{array}{c} \Delta\tau \frac{\partial}{\partial\tau} \left(\left(\alpha^{(4)}D^+ + \beta^{(4)}U \right) \Delta Q^* \right) \\ -\frac{\Delta\tau^2}{2} \frac{\partial^2}{\partial\tau^2} \left(\left(\alpha^{(4)}D^+ + \beta^{(4)}U \right) \Delta Q^* \right) \end{array} \right) \end{array} \right) \end{array} \right) = -\Delta\sigma \left(RHS \right)$$

This is rearranged as:

$$\left(RHS \right) = \frac{-1}{\Delta\sigma} \left(\begin{array}{c} \left(I - \left(\begin{array}{r} \alpha^{(1)}A^- - \beta^{(1)}R \\ +\alpha^{(1)}B^- - \beta^{(1)}S \\ +\alpha^{(1)}C^- - \beta^{(1)}T \\ +\alpha^{(1)}D^- - \beta^{(1)}U \end{array} \right) \right) \Delta Q^* \\ + \left(\begin{array}{c} \left(\begin{array}{c} \Delta\xi \frac{\partial}{\partial\xi} \left(\left(\alpha^{(1)}A^+ + \beta^{(1)}R \right) \Delta Q^* \right) \\ -\frac{\Delta\xi^2}{2} \frac{\partial^2}{\partial\xi^2} \left(\left(\alpha^{(1)}A^+ + \beta^{(1)}R \right) \Delta Q^* \right) \end{array} \right) \\ + \left(\begin{array}{c} \Delta\eta \frac{\partial}{\partial\eta} \left(\left(\alpha^{(2)}B^+ + \beta^{(2)}S \right) \Delta Q^* \right) \\ -\frac{\Delta\eta^2}{2} \frac{\partial^2}{\partial\eta^2} \left(\left(\alpha^{(2)}B^+ + \beta^{(2)}S \right) \Delta Q^* \right) \end{array} \right) \\ + \left(\begin{array}{c} \Delta\zeta \frac{\partial}{\partial\zeta} \left(\left(\alpha^{(3)}C^+ + \beta^{(3)}T \right) \Delta Q^* \right) \\ -\frac{\Delta\zeta^2}{2} \frac{\partial^2}{\partial\zeta^2} \left(\left(\alpha^{(3)}C^+ + \beta^{(3)}T \right) \Delta Q^* \right) \end{array} \right) \\ + \left(\begin{array}{c} \Delta\tau \frac{\partial}{\partial\tau} \left(\left(\alpha^{(4)}D^+ + \beta^{(4)}U \right) \Delta Q^* \right) \\ -\frac{\Delta\tau^2}{2} \frac{\partial^2}{\partial\tau^2} \left(\left(\alpha^{(4)}D^+ + \beta^{(4)}U \right) \Delta Q^* \right) \end{array} \right) \end{array} \right) \end{array} \right)$$

All terms on the right hand side of the equation are specified analytically, giving a function for RHS. This data is then given to the first sweep of the LU-SGS equation, which returns values for ΔQ^*. As the grid is refined, the solution should converge at second order accuracy.

4.1.1 Reverse form of the initial sweep

The reverse sweep is given as:

$$\left(\begin{array}{c} \left(I+\left(\begin{array}{r} \alpha^{(1)}A^{+}+\beta^{(1)}R \\ +\alpha^{(1)}B^{+}+\beta^{(1)}S \\ +\alpha^{(1)}C^{+}+\beta^{(1)}T \\ +\alpha^{(1)}D^{+}+\beta^{(1)}U \end{array}\right)\right)\Delta Q^{*} \\ +\left(\begin{array}{c} \Delta\xi\delta_{\xi}^{+}\left(\left(\alpha^{(1)}A^{-}-\beta^{(1)}R\right)\Delta Q^{*}\right) \\ +\Delta\eta\delta_{\eta}^{+}\left(\left(\alpha^{(2)}B^{-}-\beta^{(2)}S\right)\Delta Q^{*}\right) \\ +\Delta\zeta\delta_{\zeta}^{+}\left(\left(\alpha^{(3)}C^{-}-\beta^{(3)}T\right)\Delta Q^{*}\right) \\ +\Delta\tau\delta_{\tau}^{+}\left(\left(\alpha^{(4)}D^{-}-\beta^{(4)}U\right)\Delta Q^{*}\right) \end{array}\right) \end{array}\right) = -\Delta\sigma\,(RHS)$$

which analytically becomes:

$$(RHS) = \frac{-1}{\Delta\sigma}\left(\begin{array}{c} \left(I+\left(\begin{array}{r} \alpha^{(1)}A^{+}+\beta^{(1)}R \\ +\alpha^{(1)}B^{+}+\beta^{(1)}S \\ +\alpha^{(1)}C^{+}+\beta^{(1)}T \\ +\alpha^{(1)}D^{+}+\beta^{(1)}U \end{array}\right)\right)\Delta Q^{*} \\ +\left(\begin{array}{c} \left(\begin{array}{c} \Delta\xi\frac{\partial}{\partial\xi}\left(\left(\alpha^{(1)}A^{-}-\beta^{(1)}R\right)\Delta Q^{*}\right) \\ +\frac{\Delta\xi^{2}}{2}\frac{\partial^{2}}{\partial\xi^{2}}\left(\left(\alpha^{(1)}A^{-}-\beta^{(1)}R\right)\Delta Q^{*}\right) \end{array}\right) \\ +\left(\begin{array}{c} \Delta\eta\frac{\partial}{\partial\eta}\left(\left(\alpha^{(2)}B^{-}-\beta^{(2)}S\right)\Delta Q^{*}\right) \\ +\frac{\Delta\eta^{2}}{2}\frac{\partial^{2}}{\partial\eta^{2}}\left(\left(\alpha^{(2)}B^{-}-\beta^{(2)}S\right)\Delta Q^{*}\right) \end{array}\right) \\ +\left(\begin{array}{c} \Delta\zeta\frac{\partial}{\partial\zeta}\left(\left(\alpha^{(3)}C^{-}-\beta^{(3)}T\right)\Delta Q^{*}\right) \\ +\frac{\Delta\zeta^{2}}{2}\frac{\partial^{2}}{\partial\zeta^{2}}\left(\left(\alpha^{(3)}C^{-}-\beta^{(3)}T\right)\Delta Q^{*}\right) \end{array}\right) \\ +\left(\begin{array}{c} \Delta\tau\frac{\partial}{\partial\tau}\left(\left(\alpha^{(4)}D^{-}-\beta^{(4)}U\right)\Delta Q^{*}\right) \\ +\frac{\Delta\tau^{2}}{2}\frac{\partial^{2}}{\partial\tau^{2}}\left(\left(\alpha^{(4)}D^{-}-\beta^{(4)}U\right)\Delta Q^{*}\right) \end{array}\right) \end{array}\right) \end{array}\right)$$

4.2 Second LU-SGS sweep

The second sweep of the LU-SGS equations in curvilinear coordinates are:

$$
\left(\begin{array}{c}
\left(I+\left(\begin{array}{r}\alpha^{(1)}A^{+}+\beta^{(1)}R\\+\alpha^{(1)}B^{+}+\beta^{(1)}S\\+\alpha^{(1)}C^{+}+\beta^{(1)}T\\+\alpha^{(1)}D^{+}+\beta^{(1)}U\end{array}\right)\right)\Delta Q\\
+\left(\begin{array}{c}\Delta\xi\delta_{\xi}^{+}\left(\left(\alpha^{(1)}A^{-}-\beta^{(1)}R\right)\Delta Q\right)\\+\Delta\eta\delta_{\eta}^{+}\left(\left(\alpha^{(2)}B^{-}-\beta^{(2)}S\right)\Delta Q\right)\\+\Delta\zeta\delta_{\zeta}^{+}\left(\left(\alpha^{(3)}C^{-}-\beta^{(3)}T\right)\Delta Q^{*}\right)\\+\Delta\tau\delta_{\tau}^{+}\left(\left(\alpha^{(4)}D^{-}-\beta^{(4)}U\right)\Delta Q\right)\end{array}\right)
\end{array}\right)
=\left(I+\left(\begin{array}{r}\alpha^{(1)}A^{+}+\beta^{(1)}R\\+\alpha^{(1)}B^{+}+\beta^{(1)}S\\+\alpha^{(1)}C^{+}+\beta^{(1)}T\\+\alpha^{(1)}D^{+}+\beta^{(1)}U\end{array}\right)\right)\Delta Q^{*}
$$

which analytically becomes:

$$
\Psi=\left(\begin{array}{c}
\left(I+\left(\begin{array}{r}\alpha^{(1)}A^{+}+\beta^{(1)}R\\+\alpha^{(1)}B^{+}+\beta^{(1)}S\\+\alpha^{(1)}C^{+}+\beta^{(1)}T\\+\alpha^{(1)}D^{+}+\beta^{(1)}U\end{array}\right)\right)\Delta Q\\
+\left(\begin{array}{c}
\left(\begin{array}{c}\Delta\xi\frac{\partial}{\partial\xi}\left(\left(\alpha^{(1)}A^{-}-\beta^{(1)}R\right)\Delta Q\right)\\+\frac{\Delta\xi^{2}}{2}\frac{\partial^{2}}{\partial\xi^{2}}\left(\left(\alpha^{(1)}A^{-}-\beta^{(1)}R\right)\Delta Q\right)\end{array}\right)\\
+\left(\begin{array}{c}\Delta\eta\frac{\partial}{\partial\eta}\left(\left(\alpha^{(2)}B^{-}-\beta^{(2)}S\right)\Delta Q\right)\\+\frac{\Delta\eta^{2}}{2}\frac{\partial^{2}}{\partial\eta^{2}}\left(\left(\alpha^{(2)}B^{-}-\beta^{(2)}S\right)\Delta Q\right)\end{array}\right)\\
+\left(\begin{array}{c}\Delta\zeta\frac{\partial}{\partial\zeta}\left(\left(\alpha^{(3)}C^{-}-\beta^{(3)}T\right)\Delta Q\right)\\+\frac{\Delta\zeta^{2}}{2}\frac{\partial^{2}}{\partial\zeta^{2}}\left(\left(\alpha^{(3)}C^{-}-\beta^{(3)}T\right)\Delta Q\right)\end{array}\right)\\
+\left(\begin{array}{c}\Delta\tau\frac{\partial}{\partial\tau}\left(\left(\alpha^{(4)}D^{-}-\beta^{(4)}U\right)\Delta Q\right)\\+\frac{\Delta\tau^{2}}{2}\frac{\partial^{2}}{\partial\tau^{2}}\left(\left(\alpha^{(4)}D^{-}-\beta^{(4)}U\right)\Delta Q\right)\end{array}\right)
\end{array}\right)
\end{array}\right)
$$

where

$$
\Psi=\left(I+\left(\begin{array}{r}\alpha^{(1)}A^{+}+\beta^{(1)}R\\+\alpha^{(1)}B^{+}+\beta^{(1)}S\\+\alpha^{(1)}C^{+}+\beta^{(1)}T\\+\alpha^{(1)}D^{+}+\beta^{(1)}U\end{array}\right)\right)\Delta Q^{*}
$$

4.2.1 Reverse sweep

The reverse sweep is:

$$\begin{pmatrix} \left(I - \begin{pmatrix} \alpha^{(1)}A^- - \beta^{(1)}R \\ +\alpha^{(1)}B^- - \beta^{(1)}S \\ +\alpha^{(1)}C^- - \beta^{(1)}T \\ +\alpha^{(1)}D^- - \beta^{(1)}U \end{pmatrix}\right)\Delta Q \\ + \begin{pmatrix} \Delta\xi\delta_\xi^- \left(\left(\alpha^{(1)}A^+ + \beta^{(1)}R\right)\Delta Q\right) \\ +\Delta\eta\delta_\eta^- \left(\left(\alpha^{(2)}B^+ + \beta^{(2)}S\right)\Delta Q\right) \\ +\Delta\zeta\delta_\zeta^- \left(\left(\alpha^{(3)}C^+ + \beta^{(3)}T\right)\Delta Q\right) \\ +\Delta\tau\delta_\tau^- \left(\left(\alpha^{(4)}D^+ + \beta^{(4)}U\right)\Delta Q\right) \end{pmatrix} \end{pmatrix} = \left(I - \begin{pmatrix} \alpha^{(1)}A^- - \beta^{(1)}R \\ +\alpha^{(1)}B^- - \beta^{(1)}S \\ +\alpha^{(1)}C^- - \beta^{(1)}T \\ +\alpha^{(1)}D^- - \beta^{(1)}U \end{pmatrix}\right)\Delta Q^*$$

which analytically becomes:

$$\Phi = \begin{pmatrix} \left(I - \begin{pmatrix} \alpha^{(1)}A^- - \beta^{(1)}R \\ +\alpha^{(1)}B^- - \beta^{(1)}S \\ +\alpha^{(1)}C^- - \beta^{(1)}T \\ +\alpha^{(1)}D^- - \beta^{(1)}U \end{pmatrix}\right)\Delta Q \\ + \begin{pmatrix} \begin{pmatrix} \Delta\xi\frac{\partial}{\partial\xi}\left(\left(\alpha^{(1)}A^+ + \beta^{(1)}R\right)\Delta Q\right) \\ -\frac{\Delta\xi^2}{2}\frac{\partial^2}{\partial\xi^2}\left(\left(\alpha^{(1)}A^+ + \beta^{(1)}R\right)\Delta Q\right) \end{pmatrix} \\ + \begin{pmatrix} \Delta\eta\frac{\partial}{\partial\eta}\left(\left(\alpha^{(2)}B^+ + \beta^{(2)}S\right)\Delta Q\right) \\ -\frac{\Delta\eta^2}{2}\frac{\partial^2}{\partial\eta^2}\left(\left(\alpha^{(2)}B^+ + \beta^{(2)}S\right)\Delta Q\right) \end{pmatrix} \\ + \begin{pmatrix} \Delta\zeta\frac{\partial}{\partial\zeta}\left(\left(\alpha^{(3)}C^+ + \beta^{(3)}T\right)\Delta Q\right) \\ -\frac{\Delta\zeta^2}{2}\frac{\partial^2}{\partial\zeta^2}\left(\left(\alpha^{(3)}C^+ + \beta^{(3)}T\right)\Delta Q\right) \end{pmatrix} \\ + \begin{pmatrix} \Delta\tau\frac{\partial}{\partial\tau}\left(\left(\alpha^{(4)}D^+ + \beta^{(4)}U\right)\Delta Q\right) \\ -\frac{\Delta\tau^2}{2}\frac{\partial^2}{\partial\tau^2}\left(\left(\alpha^{(4)}D^+ + \beta^{(4)}U\right)\Delta Q\right) \end{pmatrix} \end{pmatrix} \end{pmatrix}$$

where

$$\Phi = \left(I - \begin{pmatrix} \alpha^{(1)}A^- - \beta^{(1)}R \\ +\alpha^{(1)}B^- - \beta^{(1)}S \\ +\alpha^{(1)}C^- - \beta^{(1)}T \\ +\alpha^{(1)}D^- - \beta^{(1)}U \end{pmatrix}\right)\Delta Q^*$$

As before, the right hand side terms in the equation are analytically specified, allowing Φ to be determined analytically. The values of Φ are then used to initialize the data for the second sweep.