1 Setting Boundary Condition Values Using a Fairing Function

Given a specified analytical function, $f(\widetilde{r})$, where

$$\widetilde{r} = \frac{r - r_{min}}{r_{max} - r_{min}}$$

Substituting r_{min} r_{max} for r gives,

$$\widetilde{r}_{min} = \frac{r_{min} - r_{min}}{r_{max} - r_{min}} = 0$$
$$\widetilde{r}_{max} = \frac{r_{max} - r_{min}}{r_{max} - r_{min}} = 1$$

The goal is to set desired values at the boundaries of the specified analytical function. First we define the values at the boundaries, i.e.

$$f(\widetilde{r} = \widetilde{r}_{min}) = f_{min}$$
$$f(\widetilde{r} = \widetilde{r}_{max}) = f_{max}$$

Then, the change between our desired boundary condition value and the actual is, To do so, a desired change in the boundary condition must be defined.

$$\Delta f_{min} = (f_{min}) - (f_{min})_{desired}$$
$$\Delta f_{max} = (f_{max}) - (f_{max})_{desired}$$

To ensure that the desired changes are imposed *smoothly.* The smoothness of a function is measured by the number of continuous derivatives the desired function has over the domain of the function. At the very minimum, a smooth function will be continuous and hence differentiable everywhere. When generating manufactured solutions, smoothness of the solution is often times assumed but is not guarenteed [**oberkampf2002verification**]. (transition sentance)

Defining the faring function:

$$f_{imposed}(\widetilde{r}) = f(\widetilde{r}) + A_{min}(\widetilde{r})\Delta f_{min} + A_{max}(\widetilde{r})\Delta f_{max}$$

In order for the imposed boundary conditons to work, the desired values must be such that,

$$A_{min}(\widetilde{r}_{min}) = 1 \qquad A_{min}(\widetilde{r}_{max}) = 0$$
$$A_{max}(\widetilde{r}_{max}) = 1 \qquad A_{max}(\widetilde{r}_{min}) = 1$$

This assured that the opposite boundaries are not affected.

For simplicity lets define:

$$A_{min}(\widetilde{r}) = 1 - A_{max}(\widetilde{r})$$

so now only A_{max} needs to be defined.

As mentioned, the desired boundary condition need to allow the analytical function to be differentiable, and as a consequence, it would be wise to also set those. In addition, different types of boundary conditions (such as Neumann) that would require this.

$$\frac{\partial A_{max}}{\partial \widetilde{r}}|_{\widetilde{r}_{min}} = 0$$
$$\frac{\partial A_{max}}{\partial \widetilde{r}}|_{\widetilde{r}_{max}} = 0$$

A straight forward choice would be

$$A_{max}(\widetilde{r}) = 3\widetilde{r}^2 - 2\widetilde{r}^3$$

Note that the correction is carried from boundary to boundary, as opposed to applying the correction to only to a region near the boundaries. This ensures smooth derivatives in the interior domain¿