A Thesis

entitled

Something About Swirling Modes with Asymmetric Price Information and No Goods

by

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Submitted to the Graduate Faculty as partial fulfillment of the requirements for the Masters of Science Degree in Mechanical Engineering

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The University of Toledo
May 2021

An Abstract of

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[Insert the abstract to your work here]

For my friends and family, who have always believed in my potential when I did not believe it myself.

Acknowledgments

This work is supported by the NASa Advanced Air Transport Technologies (AATT) Project. I would like to thank Edmane Envia(?) of the NASA Glenn Research Center, who is the technical monitor of this work. A very special thanks goes to Dr. Ray Hixon who supervised and guided me through out my course work and Master's Thesis. His rigor and tenacity in his profession has been the model example for an aspiring aeroacoustician. I would like to also thank all of my committee members, Dr. Chunhua Sheng and Dr. Sorin Cioc. Their contributions have been instrumental. Thanks to Dr. Clifford Brown for his programatic insights.

I would also like to thank my focus group peers, Zaid Sabri, and Matthew Gibbons for their patience and support over the years. I wish them the best in all of their endevours. I would also like to thank Copper Moon Coffee Shop for the endless supply of caffine and conversations.

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A .... mean flow speed of sound
A_T .... speed of sound at the duct radius
$\tilde{A}$ .... dimensionless speed of sound, $\frac{A}{A_T}$
D/Dt ...... material derivative, $\partial/\partial t + V \cdot \nabla$
D_N .... derivative matrix using N points
$\mathbf{e}_x$,$\mathbf{e}_\theta$ .... s

List of Abbreviations

ABBREV	This is where you provide a brief definition of the abbreviation "ABBREV"
BB	B.B. King
BSE	Bovine Spongiform Encephalopathy (Mad Cow Disease)
CB	L.D. Caskey and J.D. Beazley, *Attic Vase Paintings in the Museum of Fine Arts*, Boston (Oxford 1931–1963)
GLE	Gauss' law for electricity: $\nabla \cdot E = \frac{\rho}{\varepsilon_0} = 4\pi k \rho$
HHS	Department of Health and Human Services
IaR	I am root

Preface

[Insert your preface here]

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3.1 Divergence operations in new coordinate systems

The divergence, (∇), represents the operation of taking derivatives of a vector field. However, understanding the mathematical and physical representation of the divergence operator into new coordinate systems serves as a good prerequisite for the application of the Navier Stokes equations for the evaluation of aerodynamic models in unusual flow domains. Although there are many resources that will provide equations in varying coordinate systems, the derivation offers insight into the advantages and drawbacks of using a new reference frame for a flow domain. The divergence operator in Cartesian coordinates is,

$$\vec{\nabla} \equiv \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} = 0$$

The vectors, $\hat{e}_x, \hat{e}_y, \hat{e}_z$ (commonly denoted in literature as $\hat{i}$,$\hat{j}$,$\hat{k}$) are the basis vectors of the Cartesian coordinate system. The vector hat ($\vec{}$) reminds us that divergence operation includes a scalar product of the basis vectors and the individual derivative terms themselves. These basis vectors *scale* with the derivatives d/dx d/dy d/dz in the direction of these basis vectors themselves. This implicitly captures the coordinate system and assumptions that corresponds to the basis vectors themselves.

To relate the basis vectors of the cylindrical coordinate system to the Cartesian coordinate system, we use the following relations,

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\theta &= \tan^{-1}\left(\frac{y}{x}\right) \\
&= \cos^{-1}\left(\frac{x}{r}\right) \\
&= \sin^{-1}\left(\frac{y}{r}\right)
\end{aligned}
$$

Note that the equation above also establishes $x = r\cos\theta$ and $y = r\sin\theta$. The Cartesian basis vectors are related to the cylindrical basis vectors of the new coordinate system by,

$$
\begin{aligned}
&\hat{e}_r = \hat{e}_x \cos\theta + \hat{e}_y \sin\theta \\
&\hat{e}_\theta = -\hat{e}_x \sin\theta + \hat{e}_y \cos\theta \\
&\hat{e}_z = \hat{e}_z
\end{aligned}
$$

Defining these relationships, (they'll be useful later)

$$
\begin{aligned}
&\frac{\partial \hat{e}_r}{\partial r} = \frac{\partial \hat{e}_\theta}{\partial r} = \frac{\partial \hat{e}_z}{\partial r} = 0 \\
&\frac{\partial \hat{e}_r}{\partial \theta} = -\hat{e}_x \sin\theta + \hat{e}_y \cos\theta = \hat{e}_\theta \\
&\frac{\partial \hat{e}_\theta}{\partial \theta} = -\left(\hat{e}_x \cos\theta + \hat{e}_y \sin\theta\right) = -\hat{e}_r
\end{aligned}
$$

The multi-variable chain rule for differentiation is then used to express the Cartesian variables, $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, with respect to the cylindrical variable.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{dr}{dx} + \frac{\partial}{\partial \theta}\frac{d\theta}{dx} + \frac{\partial}{\partial z}\frac{dz}{dx}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{dr}{dy} + \frac{\partial}{\partial \theta}\frac{d\theta}{dy} + \frac{\partial}{\partial z}\frac{dz}{dy}$$

By,finding the derivatives of r & θ with respect to x and y, we can substitute terms in the Cartesian divergence definition. First, $\frac{dr}{dx}$ & $\frac{dr}{dy}$ is calculated,

$$\begin{aligned}
\frac{dr}{dx} &= \frac{d}{dx}\left(\left[x^2 + y^2\right]^{1/2}\right) \\
&= \frac{1}{2}\left[x^2 + y^2\right]^{-1/2}(2x) \\
&= \frac{x}{\sqrt{x^2 + y^2}} \\
&= \frac{rcos\theta}{r} \\
&\boxed{\frac{dr}{dx} = cos\theta}
\end{aligned}$$

$$\begin{aligned}
\frac{dr}{dy} &= \frac{d}{dy}\left(\left[x^2 + y^2\right]^{1/2}\right) \\
&= \frac{1}{2}\left[x^2 + y^2\right]^{-1/2}(2y) \\
&= \frac{y}{\sqrt{x^2 + y^2}} \\
&= \frac{rsin\theta}{r} \\
&\boxed{\frac{dr}{dy} = sin\theta}
\end{aligned}$$

Then, $\frac{d\theta}{dx}$ & $\frac{d\theta}{dy}$ is found.

$$\begin{aligned}
\frac{d\theta}{dx} &= \frac{d}{dx}\left(tan^{-1}\left(\frac{y}{x}\right)\right) \\
&= \frac{d}{du}tan^{-1}(u)\frac{d}{dx}\left(\frac{y}{x}\right) \\
&= \frac{1}{u^2+1}\frac{-y}{x^2} \\
&= -\frac{y}{y^2+x^2} \\
&\boxed{\frac{d\theta}{dx} = -\frac{sin\theta}{r}}
\end{aligned}$$

$$\begin{aligned}
\frac{d\theta}{dy} &= \frac{d}{dy}\left(tan^{-1}\left(\frac{y}{x}\right)\right) \\
&= \frac{d}{du}tan^{-1}(u)\frac{d}{dy}\left(\frac{y}{x}\right) \\
&= \frac{1}{u^2+1}\frac{1}{x} \\
&= \frac{x}{y^2+x^2} \\
&\boxed{\frac{d\theta}{dy} = \frac{cos\theta}{r}}
\end{aligned}$$

Through substitution back into the chain rule expansion,

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial r}\cos\theta - \frac{\partial}{\partial \theta}\frac{1}{r}\sin\theta \\
\frac{\partial}{\partial y} &= \frac{\partial}{\partial r}\sin\theta + \frac{\partial}{\partial \theta}\frac{1}{r}\cos\theta
\end{aligned}$$

We can now convert our divergence operator, $\vec{\nabla}$

$$\begin{aligned}
\vec{\nabla} &= \frac{\partial}{\partial x}\hat{e}_x + \frac{\partial}{\partial y}\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z = 0 \\
&= \left(\frac{\partial}{\partial r}\cos\theta - \frac{\partial}{\partial\theta}\frac{1}{r}\sin\theta\right)\hat{e}_x + \left(\frac{\partial}{\partial r}\sin\theta + \frac{\partial}{\partial\theta}\frac{1}{r}\cos\theta\right)\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z = 0
\end{aligned}$$

Rearranging like terms (containing cylindrical derivative variables), and factoring out $1/r$

$$\begin{aligned}
\vec{\nabla} &= \left(\frac{\partial}{\partial r}\cos\theta - \frac{\partial}{\partial\theta}\frac{1}{r}\sin\theta\right)\hat{e}_x + \left(\frac{\partial}{\partial r}\sin\theta + \frac{\partial}{\partial\theta}\frac{1}{r}\cos\theta\right)\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z = 0 \\
&= \left(\hat{e}_x\cos\theta + \hat{e}_y\sin\theta\right)\frac{\partial}{\partial r} + \frac{1}{r}\left(\hat{e}_y\cos\theta - \hat{e}_x\sin\theta\right)\frac{\partial}{\partial\theta} + \frac{\partial}{\partial z}\hat{e}_z = 0
\end{aligned}$$

Recalling the definitions for $\hat{e}_r$ and $\hat{e}_\theta$, we can use these expressions to rewrite ∇ in polar coordinates

$$\vec{\nabla} = \hat{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\hat{e}_\theta\frac{\partial}{\partial\theta} + \frac{\partial}{\partial z}\hat{e}_z = 0$$

$$\frac{DV}{dt} = \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma}$$

$$\boldsymbol{\sigma} = -p\boldsymbol{I_3} + \tau$$

where $[\boldsymbol{I_3}]$ is a 3 by 3 identity matrix and τ is the shear stress tensor. The velocity vector for a three dimensional flow.

$$\vec{V} = v_r(r,\theta,x,t)\hat{e}_r + v_\theta(r,\theta,x,t)\hat{e}_\theta + v_x(r,\theta,x,t)\hat{e}_x \tag{3.1}$$

In Kousen's work, a velocity vector is written as a function of radius, and the radial velocity component is neglected.

$$\vec{V} = v_\theta(r)\hat{e}_\theta + v_x(r)\hat{e}_x \tag{3.2}$$

We will go with the first definition and cancel out the radial velocity later on.

$$\frac{DV}{dt} = \frac{\partial\vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial\vec{V}}{\partial r}\frac{dr}{dt} + \frac{\partial\vec{V}}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\vec{V}}{\partial x}\frac{dx}{dt}$$

Starting with the first term,

$$\begin{aligned}
\frac{\partial\vec{V}}{\partial t}\frac{dt}{dt} &= \frac{\partial}{\partial t}\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_x\hat{e}_x\right) * 1 \\
&= \frac{\partial v_r}{\partial t}\hat{e}_r + \cancel{\frac{\partial\hat{e}_r}{\partial t}}v_r + \frac{\partial v_\theta}{\partial t}\hat{e}_\theta + \cancel{\frac{\partial\hat{e}_\theta}{\partial t}}v_\theta + \frac{\partial v_x}{\partial t}\hat{e}_x + \cancel{\frac{\partial\hat{e}_x}{\partial t}}v_x \\
&\boxed{\frac{\partial\vec{V}}{\partial t} = \frac{\partial v_r}{\partial t}\hat{e}_r + \frac{\partial v_\theta}{\partial t}\hat{e}_\theta + \frac{\partial v_x}{\partial t}\hat{e}_x}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \vec{V}}{\partial r}\frac{dr}{dt} &= \frac{\partial \vec{V}}{\partial r} v_r \\
&= \frac{\partial}{\partial r}\left[v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_x \hat{e}_x\right] v_r \\
&= \left(\frac{\partial v_r}{\partial r}\hat{e}_r + \cancel{\frac{\partial \hat{e}_r}{\partial r}} v_r + \frac{\partial v_\theta}{\partial r}\hat{e}_\theta + \cancel{\frac{\partial \hat{e}_\theta}{\partial r}} v_\theta + \frac{\partial v_x}{\partial r}\hat{e}_x + \cancel{\frac{\partial \hat{e}_x}{\partial r}} v_\theta\right) v_r
\end{aligned}
$$

$$
\boxed{\frac{\partial \vec{V}}{\partial r}\frac{dr}{dt} = \left[\frac{\partial v_r}{\partial r}\hat{e}_r + \frac{\partial v_\theta}{\partial r}\hat{e}_\theta + \frac{\partial v_x}{\partial r}\hat{e}_x\right] v_r}
$$

Recalling that arc length is $ds = r d\theta$, and angular velocity is $d\theta/dt = v_\theta/r$

$$
\begin{aligned}
\frac{\partial \vec{V}}{\partial \theta}\frac{d\theta}{dt} &= \frac{\partial \vec{V}}{\partial \theta}\frac{v_\theta}{r} \\
&= \frac{\partial}{\partial \theta}\left[v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_x \hat{e}_x\right] \frac{v_\theta}{r} \\
&= \left[\frac{\partial v_r}{\partial \theta}\hat{e}_r + \underbrace{\frac{\partial \hat{e}_r}{\partial \theta}}_{\hat{e}_\theta} v_r + \frac{\partial v_\theta}{\partial \theta}\hat{e}_\theta + \underbrace{\frac{\partial \hat{e}_\theta}{\partial \theta}}_{-\hat{e}_r} v_\theta + \frac{\partial v_x}{\partial \theta}\hat{e}_x\right] \frac{v_\theta}{r}
\end{aligned}
$$

$$
\boxed{\frac{\partial \vec{V}}{\partial \theta}\frac{dr}{dt} = \left[\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\hat{e}_r + \left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\hat{e}_\theta + \frac{\partial v_x}{\partial \theta}\hat{e}_x\right] \frac{v_\theta}{r}}
$$

$$
\begin{aligned}
\frac{\partial \vec{V}}{\partial x}\frac{dx}{dt} &= \frac{\partial \vec{V}}{\partial x} v_x \\
&= \frac{\partial}{\partial x}\left[v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_x\hat{e}_x\right] v_r \\
&= \left(\frac{\partial v_r}{\partial x}\hat{e}_r + \cancel{\frac{\partial \hat{e}_r}{\partial x}} v_r + \frac{\partial v_\theta}{\partial x}\hat{e}_\theta + \cancel{\frac{\partial \hat{e}_\theta}{\partial x}} v_\theta + \frac{\partial v_x}{\partial r}\hat{e}_x + \cancel{\frac{\partial \hat{e}_x}{\partial x}} v_\theta\right) v_x \\
&\boxed{\frac{\partial \vec{V}}{\partial x}\frac{dx}{dt} = \left[\frac{\partial v_r}{\partial x}\hat{e}_r + \frac{\partial v_\theta}{\partial x}\hat{e}_\theta + \frac{\partial v_x}{\partial x}\hat{e}_x\right] v_x}
\end{aligned}
$$

Putting these terms together,

$$
\begin{aligned}
\frac{DV}{dt} = &\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta^2}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} + v_x\frac{\partial v_r}{\partial x}\right]\hat{e}_r + \\
&\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_x\frac{\partial v_\theta}{\partial x}\right]\hat{e}_\theta + \\
&\left[\frac{\partial v_x}{\partial t} + v_r\frac{\partial v_x}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_x}{\partial \theta} + v_x\frac{\partial v_x}{\partial x}\right]\hat{e}_x
\end{aligned}
$$

If we neglect viscosity on the right hand side, we will arrive at the linearized Euler equations

$$
\begin{aligned}
\nabla\sigma &= -\nabla p[\mathbf{I_3}] \\
&= -\frac{1}{\rho}\left\{\begin{matrix} \frac{\partial p}{\partial r} & 0 & 0 \\ 0 & \frac{1}{r}\frac{\partial p}{\partial \theta} & 0 \\ 0 & 0 & \frac{\partial p}{\partial x} \end{matrix}\right\}
\end{aligned}
$$

3.1.1 Setting up SWIRL's Aerodynamic Model

The Euler Equations in Cylindrical Form are,

$$\frac{\partial \rho}{\partial t} + v_r\frac{\partial \rho}{\partial r} + \frac{v_\theta}{r}\frac{\partial \rho}{\partial \theta} + v_x\frac{\partial \rho}{\partial \theta} + \rho\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_x}{\partial x}\right) = 0$$

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_x\frac{\partial v_r}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_x\frac{\partial v_\theta}{\partial x} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

$$\frac{\partial v_x}{\partial t} + v_r\frac{\partial v_x}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_x}{\partial \theta} + v_x\frac{\partial v_x}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial p}{\partial t} + v_r\frac{\partial p}{\partial r} + \frac{v_\theta}{r}\frac{\partial p}{\partial \theta} + v_x\frac{\partial p}{\partial \theta} + \gamma p\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{v_\theta}{\partial \theta} + \frac{\partial v_x}{\partial x}\right) = 0$$

SWIRL utilizes the following assumptions to simplify the aerodynamic model

- No flow in the radial direction. Consequentially, the flow is axisymmetric along the downstream direction.
- No surface or body forces
- Isentropic conditions

For steady flow, the continuity, momentum and entropy equations are

$$\nabla(\vec{V}\bar{\rho}) = 0$$

$$(\vec{V}\cdot\nabla)\vec{V}$$

$$\nabla S = 0$$

If we neglect radial velocity, the velocity vector in cylindrical coordinates are

$$\vec{V}(r,\theta,x) = V_x(r)\hat{e}_x + V_\theta(r)\hat{e}_\theta$$

3.2 Applying model to various flows

Kousen studied three specific flow configuration.

- axial shear flow
- solid body swirl
- free vortex swirl

3.2.1 Axial Shear Flow

In Kousen's paper, axial sheared flows through a constant area duct was also investigated.he only effect on the velocity gradient occurs along the x axis. All other primitive variables (pressure and density which is $\propto$ speed of sound) are constant. As a result, the only changes that occur are in the x direction. This implies that $\partial/\partial\theta = 0$. For the conservation of mass,

$$\nabla(\vec{V}\bar{\rho}) = \left(\underbrace{\cancel{\frac{\partial(\bar{\rho}v_r)}{\partial r}}}_{v_r=0} + \underbrace{\cancel{\frac{1}{r}\frac{\partial\bar{\rho}v_\theta}{\partial\theta}}}_{\frac{\partial}{\partial\theta}} + \frac{\partial\bar{\rho}v_x}{\partial x} \right) = \frac{\partial\bar{\rho}v_x}{\partial x}$$

Conservation of Momentum in the radial direction becomes:

$$(\vec{V}\cdot\nabla)\vec{V} = \cancel{v_r\frac{\partial v_r}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}} - \frac{v_\theta^2}{r} + \cancel{v_x\frac{\partial v_r}{\partial x}} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{v_\theta^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{\rho v_\theta^2}{r} = \frac{\partial p}{\partial r}$$

θ direction

$$(\vec{V}\cdot\nabla)\vec{V} = \cancel{v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r v_\theta}{r}} + v_x\frac{\partial v_\theta}{\partial x} = \cancel{-\frac{1}{\rho r}\frac{\partial p}{\partial\theta}}$$

Dividing v_x to the other side,

$$\frac{\partial v_\theta}{\partial x} = 0$$

Similarly for the x direction,

$$\frac{\partial v_x}{\partial x} = 0$$

In regards to the entropy equation, having an isentropic flow, $\nabla S = 0$ implies $a^2 = \frac{\nabla \bar{p}}{\nabla \bar{\rho}}$

3.3 Accounting for solid body swirl

If the flow contains a swirling component, then the primitive variables are nonuniform through the flow, and mean flow assumptions are not valid. To account to this, we integrate the momentum equation in the radial direction with respect to the radius.

Equation (2.5) in [**?**] is

$$P = \int_{\tilde{r}}^{1} \frac{\bar{\rho} V_\theta^2}{\tilde{r}} d\tilde{r}$$

where $\tilde{r}$ is the radius dimensional radius normalized by the tip diameter $r_t = r_{max}$

To show the work, we will start with the dimensional form of the equation,

$$\frac{\bar{\rho} v_\theta^2}{r} = \frac{\partial p}{\partial r}$$

Applying separation of variables

$$\int_{r}^{r_{max}} \frac{\bar{\rho} v_\theta^2}{r} \partial r = - \int_{P(r)}^{P(r_{max})} \partial p$$

Since $\tilde{r} = r/r_{max}$

$$r = \tilde{r} r_{max}$$

taking total derivatives(applying chain rule)

$$dr = d(\tilde{r} r_{max}) = d(\tilde{r}) r_{max}$$

Substituting these back in and evaluating the right hand side,

$$\int_{\tilde{r}}^{1} \frac{\bar{\rho} v_\theta^2}{\tilde{r}} \partial \tilde{r} = P(1) - P(\tilde{r})$$

For reference the minimum value of $\tilde{r}$ is

$$\sigma = \frac{r_{max}}{r_{min}}$$

For

$$\frac{\partial a^2}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\gamma P}{\rho}\right)$$

Using the quotient rule, we can extract the definition of the speed of sound.

$$\begin{aligned}
&= \frac{\partial P}{\partial r}\frac{\gamma \bar{\rho}}{\bar{\rho}^2} - \left(\frac{\gamma P}{\bar{\rho}^2}\right)\frac{\partial \bar{\rho}}{\partial r} \\
&= \frac{\partial P}{\partial r}\frac{\gamma}{\bar{\rho}} - \left(\frac{a^2}{\bar{\rho}}\right)\frac{\partial \bar{\rho}}{\partial r} \\
\text{Using } \partial P / a^2 = \partial \rho \rightarrow &= \frac{\partial P}{\partial r}\frac{\gamma}{\bar{\rho}} - \left(\frac{1}{\bar{\rho}}\right)\frac{\partial \bar{P}}{\partial r} \\
\frac{\partial a^2}{\partial r} &= \frac{\partial P}{\partial r}\frac{\gamma - 1}{\bar{\rho}} \\
\text{or..} \frac{\bar{\rho}}{\gamma - 1}\frac{\partial a^2}{\partial r} &= \frac{\partial P}{\partial r}
\end{aligned}$$

Going back to the radial momentum equation, and rearranging the

$$\frac{\bar{\rho} v_\theta^2}{r} = \frac{\partial P}{\partial r}$$
$$\frac{\cancel{\bar{\rho}} v_\theta^2}{r} = \frac{\cancel{\bar{\rho}}}{\gamma - 1} \frac{\partial a^2}{\partial r}$$
$$\frac{v_\theta^2}{r} (\gamma - 1) = \frac{\partial a^2}{\partial r}$$
$$\text{Dividing both sides by } a^2 \rightarrow \frac{M_\theta}{r} (\gamma - 1) = \frac{\partial a^2}{\partial r} \frac{1}{a^2}$$

$$\text{Integrating both sides } \int_r^{r_{max}} \frac{M_\theta}{r} (\gamma - 1) \, \partial r = \int_{a^2(r)}^{a^2(r_{max})} \frac{1}{a^2} \partial a^2$$
$$\int_r^{r_{max}} \frac{M_\theta^2}{r} (\gamma - 1) \, \partial r = ln(a^2(r_{max})) - ln(a^2(r))$$
$$\int_r^{r_{max}} \frac{M_\theta^2}{r} (\gamma - 1) \, \partial r = ln \left(\frac{a^2(r_{max})}{a^2(r)} \right)$$

Defining non dimensional speed of sound $\tilde{a} = \frac{a(r)}{a(r_{max})}$

$$\int_r^{r_{max}} \frac{M_\theta}{r} (\gamma - 1) \, \partial r = ln \left(\frac{1}{\tilde{a}^2} \right)$$
$$= -2ln(\tilde{a})$$
$$\tilde{a}(r) = exp \left[- \int_r^{r_{max}} \frac{M_\theta}{r} \frac{(\gamma - 1)}{2} \partial r \right]$$
$$\text{replacing r with } \tilde{r} \rightarrow \tilde{a}(r) = exp \left[- \int_r^{r_{max}} \frac{M_\theta}{r} \frac{(\gamma - 1)}{2} \partial r \right]$$
$$\tilde{a}(\tilde{r}) = exp \left[\left(\frac{1 - \gamma}{2} \right) \int_{\tilde{r}}^{1} \frac{M_\theta}{\tilde{r}} \partial \tilde{r} \right]$$

3.3.1 Linearizing the governing equations

3.3.1.1 Linearizing Conservation of Mass

To linearize the Euler equations, we substitute each flow variable with its equivalent mean and perturbation components. Note that the mean term is only a function of space whereas the perturbation component is a dependent on both space and time (functional dependence is not explicity written with each variable). Assuming that we can divide the variable into a known laminar flow solution to the Navier-Stokes equations and a small amplitude perturbation solution:

$$v_r = V_r(x) + v_r' \tag{3.3}$$

$$v_\theta = V_\theta + v_\theta' \tag{3.4}$$

$$v_x = V_x + v_x' \tag{3.5}$$

$$p = \bar{p} + p' \tag{3.6}$$

$$\rho = \bar{\rho} + \rho' \tag{3.7}$$

Starting with continuity,

$$
\begin{aligned}
\frac{\partial\rho}{\partial t}+v_r\frac{\partial\rho}{\partial r}+\frac{v_\theta}{r}\frac{\partial\rho}{\partial\theta}+v_x\frac{\partial\rho}{\partial\theta}+\rho\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_x}{\partial x}\right)&=0\\
\frac{\partial\bar{\rho}+\rho'}{\partial t}+(V_r+v_r')\frac{\partial\bar{\rho}+\rho'}{\partial r}+\frac{V_\theta+v_\theta'}{r}\frac{\partial\bar{\rho}+\rho'}{\partial\theta}+(V_x+v_x')\frac{\partial\bar{\rho}+\rho'}{\partial\theta}+&\\
(\bar{\rho}+\rho')\left(\frac{1}{r}\frac{\partial(r(V_r+v_r'))}{\partial r}+\frac{1}{r}\frac{\partial V_\theta+v_\theta'}{\partial\theta}+\frac{\partial(V_x+v_x')}{\partial x}\right)&=0\\
\frac{\partial\bar{\rho}}{\partial t}+\frac{\partial\rho'}{\partial t}+&\\
V_r\frac{\partial\bar{\rho}}{\partial r}+v_r'\frac{\partial\bar{\rho}}{\partial r}+V_r\frac{\partial\rho'}{\partial r}+v_r'\frac{\partial\rho'}{\partial r}+&\\
\frac{1}{r}\left(V_\theta\frac{\partial\bar{\rho}}{\partial\theta}+v_\theta'\frac{\partial\bar{\rho}}{\partial\theta}+V_\theta\frac{\partial\rho'}{\partial\theta}+v_\theta'\frac{\partial\rho'}{\partial\theta}\right)+&\\
V_x\frac{\partial\bar{\rho}}{\partial x}+v_x'\frac{\partial\bar{\rho}}{\partial x}+V_x\frac{\partial\rho'}{\partial x}+v_x'\frac{\partial\rho'}{\partial x}+&\\
\bar{\rho}\left(\frac{1}{r}\left(\frac{\partial(rV_r)}{\partial r}+\frac{\partial(r(v_r'))}{\partial r}+\frac{\partial V_\theta}{\partial\theta}+\frac{\partial v_\theta'}{\partial\theta}\right)+\frac{\partial V_x}{\partial x}+\frac{\partial v_x'}{\partial x}\right)+&\\
\rho'\left(\frac{1}{r}\left(\frac{\partial(rV_r)}{\partial r}+\frac{\partial(r(v_r'))}{\partial r}+\frac{\partial V_\theta}{\partial\theta}+\frac{\partial v_\theta'}{\partial\theta}\right)+\frac{\partial V_x}{\partial x}+\frac{\partial v_x'}{\partial x}\right)&=0
\end{aligned}
$$

Important things to note

- The small disturbances are infinitesimal (thus linearized)
- Neglect second order terms.
- The continuity equation is comprised of mean velocity components. This is subtracted off in each of the governing equations

Blue will be used for terms that are removed after subtracting in the original continuity equation, green will be used to cancel higher(2nd) order terms.Red will be used if we take the radial velocity to be zero.

$$
\begin{aligned}
&= \cancel{\frac{\partial \bar{\rho}}{\partial t}} + \frac{\partial \rho'}{\partial t} + \\
&\cancel{V_r \frac{\partial \bar{\rho}}{\partial r}} + \cancel{v_r' \frac{\partial \bar{\rho}}{\partial r}} + \cancel{V_r \frac{\partial \rho'}{\partial r}} + \cancel{v_r' \frac{\partial \rho'}{\partial r}} + \\
&\frac{1}{r}\left(\cancel{V_\theta \frac{\partial \bar{\rho}}{\partial \theta}} + \cancel{v_\theta' \frac{\partial \bar{\rho}}{\partial \theta}} + V_\theta \frac{\partial \rho'}{\partial \theta} + \cancel{v_\theta' \frac{\partial \rho'}{\partial \theta}}\right) + \\
&\cancel{V_x \frac{\partial \bar{\rho}}{\partial x}} + \cancel{v_x' \frac{\partial \bar{\rho}}{\partial x}} + V_x \frac{\partial \rho'}{\partial \theta} + \cancel{v_x' \frac{\partial \rho'}{\partial x}} + \\
&\bar{\rho}\left(\frac{1}{r}\left(\cancel{\frac{\partial (r V_r)}{\partial r}} + \frac{\partial (r (v_r')}{\partial r} + \cancel{\frac{\partial V_\theta}{\partial \theta}} + \frac{\partial v_\theta'}{\partial \theta}\right) + \cancel{\frac{\partial V_x}{\partial x}} + \frac{\partial v_x'}{\partial x}\right) + \\
&\rho'\left(\frac{1}{r}\left(\cancel{\frac{\partial (r V_r)}{\partial r}} + \cancel{\frac{\partial (r (v_r')}{\partial r}} + \cancel{\frac{\partial V_\theta}{\partial \theta}} + \cancel{\frac{\partial v_\theta'}{\partial \theta}}\right) + \cancel{\frac{\partial V_x}{\partial x}} + \cancel{\frac{\partial v_x'}{\partial x}} = 0\right)
\end{aligned}
$$

$$
\boxed{\frac{\partial \rho'}{\partial t} + \frac{V_\theta}{r}\frac{\partial \rho'}{\partial \theta} + V_x \frac{\partial \rho'}{\partial x} + \bar{\rho}\left(\frac{1}{r}\left(\frac{\partial r v_r'}{\partial r} + \frac{\partial v_\theta'}{\partial \theta}\right) + \frac{\partial v_x'}{\partial x}\right) = 0}
$$

3.3.1.2 Linearizing the Conservation of Momentum in the r direction

Starting with the mean momentum equation

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_x \frac{\partial v_r}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

Looking at the left hand side first

$$\frac{\partial (V_r + v_r')}{\partial t} + (V_r + v_r')\frac{\partial (V_r + v_r')}{\partial r} + \frac{V_\theta + v_\theta'}{r}\frac{\partial (V_r + v_r')}{\partial \theta} - \frac{(V_\theta + v_\theta')^2}{r} + (V_x + v_x')\frac{\partial (V_r + v_r')}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\begin{aligned}
\cancel{\frac{\partial V_r}{\partial t}} + \frac{\partial v_r'}{\partial t} &+ \\
\cancel{V_r \frac{\partial V_r}{\partial r}} + \cancel{v_r' \frac{\partial V_r}{\partial r}} + \cancel{V_r \frac{\partial v_r'}{\partial r}} + \cancel{v_r' \frac{\partial v_r'}{\partial r}} &+ \\
\frac{1}{r}\left(\cancel{V_\theta \frac{\partial V_r}{\partial \theta}} + \cancel{v_\theta' \frac{\partial V_r}{\partial \theta}} + V_\theta \frac{\partial v_r'}{\partial \theta} + \cancel{v_\theta' \frac{\partial v_r'}{\partial \theta}}\right) &- \\
\frac{1}{r}\left(\cancel{V_\theta^2} + 2V_\theta v_\theta' + \cancel{v_\theta'^2}\right) &+ \\
\cancel{V_x \frac{\partial V_r}{\partial x}} + \cancel{v_x' \frac{\partial V_r}{\partial x}} + V_x \frac{\partial v_r'}{\partial x} + \cancel{v_x' \frac{\partial v_r'}{\partial x}} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} \\
\frac{\partial v_r'}{\partial t} + V_r \frac{\partial v_r'}{\partial r} + \frac{V_\theta}{r}\frac{\partial v_r'}{\partial \theta} - \frac{2V_\theta v_\theta'}{r} + V_x \frac{\partial v_r'}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial r}
\end{aligned}$$

Now looking at the right side, Expanding the $1/\rho$ using a Taylor series approximation

$$
\begin{aligned}
\frac{1}{\bar{\rho}+\rho'} &= \frac{1}{\bar{\rho}} + \left(\frac{1}{\bar{\rho}+\rho'} - \frac{1}{\rho}\right) \\
&= \frac{1}{\bar{\rho}} + \left(\frac{\bar{\rho}}{\bar{\rho}(\bar{\rho}+\rho')} - \frac{1}{\rho}\frac{\bar{\rho}+\rho'}{\bar{\rho}+\rho'}\right) \\
&= \frac{1}{\bar{\rho}} - \left(\frac{\bar{\rho}-\bar{\rho}+\rho'}{\bar{\rho}(\bar{\rho}+\rho')}\right) \\
&= \frac{1}{\bar{\rho}} - \frac{\rho'}{\bar{\rho}} \underbrace{\left(\frac{1}{\bar{\rho}+\rho}\right)}_{\text{This is what we're solving for!}} \\
&= \frac{1}{\bar{\rho}} - \frac{\rho'}{\bar{\rho}} \underbrace{\left[\frac{1}{\bar{\rho}} + \left(\frac{1}{\bar{\rho}+\rho'} - \frac{1}{\rho}\right)\right]}_{\text{This is from step 1}} \\
&= \frac{1}{\bar{\rho}} - \frac{\rho'}{\bar{\rho}^2} + \underbrace{\left[\left(\frac{\rho'}{\bar{\rho}}\right)^2 \frac{1}{\bar{\rho}+\rho'}\right]}_{\text{These are higher order terms that will go to } \infty}
\end{aligned}
$$

$$
\frac{1}{\rho}\frac{\partial p}{\partial r} = \left(-\frac{1}{\bar{\rho}} + \frac{\rho'}{\bar{\rho}^2}\right)\left(\frac{\partial\bar{p}+p'}{\partial r}\right)
$$

$$
\frac{1}{\rho}\frac{\partial p}{\partial r} = -\cancel{\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial r}} - \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial r} + \frac{\rho'}{\bar{\rho}^2}\frac{\partial\bar{p}}{\partial r} + \cancel{\frac{\rho'}{\bar{\rho}^2}\frac{\partial p'}{\partial r}}
$$

$$
\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial r} + \frac{\rho'}{\bar{\rho}^2}\frac{\partial\bar{p}}{\partial r}
$$

$$
\boxed{\frac{\partial v_r'}{\partial t} + \frac{V_\theta}{r}\frac{\partial v_r'}{\partial\theta} - \frac{2V_\theta v_\theta'}{r} + V_x\frac{\partial v_r'}{\partial x} = \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial r} + \frac{\rho'}{\bar{\rho}^2}\frac{\partial\bar{p}}{\partial r}}
$$

3.3.1.3 Linearizing the Conservation of Momentum in the θ direction

Starting with the mean momentum equation

$$\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_x\frac{\partial v_\theta}{\partial x} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

Looking at the left hand side first

$$\begin{aligned}
\frac{\partial (V_\theta + v'_\theta)}{\partial t} + (V_r + v'_r)\frac{\partial (V_\theta + v'_\theta)}{\partial r} + \\
\frac{V_\theta + v'_\theta}{r}\frac{\partial (V_\theta + v'_\theta)}{\partial \theta} + \frac{(V_r + v'_r)(V_\theta + v'_\theta)}{r} + (V_x + v'_x)\frac{\partial (V_\theta + v'_\theta)}{\partial x} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \\
\cancel{\frac{\partial V_\theta}{\partial t}} + \frac{\partial v'_\theta}{\partial t} + \\
\cancel{V_r\frac{\partial V_\theta}{\partial r}} + \underbrace{v'_r\frac{\partial V_\theta}{\partial r}}_{v'_r=0} + \cancel{V_r\frac{\partial v'_\theta}{\partial r}} + \cancel{v'_r\frac{\partial v'_\theta}{\partial r}} + \\
\frac{1}{r}\left(\cancel{V_\theta\frac{\partial V_\theta}{\partial \theta}} + \cancel{v'_\theta\frac{\partial V_\theta}{\partial \theta}} + V_\theta\frac{\partial v'_\theta}{\partial \theta} + \cancel{v'_\theta\frac{\partial v'_\theta}{\partial \theta}}\right) + \\
\frac{1}{r}\left(\cancel{V_r V_\theta} + v'_r V_\theta + \cancel{V_r v'_\theta} + \cancel{v'_r v'_\theta}\right) + \\
\cancel{V_x\frac{\partial V_\theta}{\partial x}} + \cancel{v'_x\frac{\partial V_\theta}{\partial x}} + V_x\frac{\partial v'_\theta}{\partial x} + \cancel{v'_x\frac{\partial v'_\theta}{\partial x}} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \\
\frac{\partial v'_\theta}{\partial t} + v'_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial v'_\theta}{\partial \theta} + \frac{v'_r V_\theta}{r} + V_x\frac{\partial v'_r}{\partial x} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}
\end{aligned}$$

Now looking at the right side, Expanding the $1/\rho$ using a Taylor series approximation

$$\begin{aligned}
-\frac{1}{\rho r}\frac{\partial p}{\partial \theta} &= \left(-\frac{1}{\bar\rho} + \frac{\rho'}{\bar\rho^2}\right)\left(\frac{\partial \bar p + p'}{\partial \theta}\right) \\
\frac{1}{\rho r}\frac{\partial p}{\partial \theta} &= -\cancel{\frac{1}{\bar\rho}\frac{\partial \bar p}{\partial \theta}} - \frac{1}{\bar\rho r}\frac{\partial p'}{\partial \theta} + \cancel{\frac{\rho'}{\bar\rho^2 r}\frac{\partial \bar p}{\partial \theta}} + \cancel{\frac{\rho'}{\bar\rho^2 r}\frac{\partial p'}{\partial \theta}} \\
\frac{1}{\rho r}\frac{\partial p}{\partial \theta} &= -\frac{1}{\bar\rho}\frac{\partial p'}{\partial \theta}
\end{aligned}$$

$$\boxed{\frac{\partial v'_\theta}{\partial t} + v'_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial v'_\theta}{\partial \theta} + \frac{v'_r V_\theta}{r} + V_x \frac{\partial v'_r}{\partial x} = -\frac{1}{\bar{\rho} r}\frac{\partial p'}{\partial \theta}}$$

3.3.1.4 Linearizing the Conservation of Momentum in the x direction

Starting with the mean momentum equation

$$\frac{\partial v_x}{\partial t} + v_r \frac{\partial v_x}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_x}{\partial \theta} + v_x \frac{\partial v_x}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\begin{aligned}
\frac{\partial (V_x + v'_x)}{\partial t} + (V_r + v'_r)\frac{\partial (V_x + v'_x)}{\partial r} + \frac{V_\theta + v'_\theta}{r}\frac{\partial (V_x + v'_x)}{\partial \theta} + (V_x + v'_x)\frac{\partial (V_x + v'_x)}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
\cancel{\frac{\partial V_x}{\partial t}} + \frac{\partial v'_x}{\partial t} + & \\
\cancel{V_r \frac{\partial V_x}{\partial r}} + v'_r \frac{\partial V_x}{\partial r} + \cancel{V_r \frac{\partial v'_x}{\partial r}} + \cancel{v'_r \frac{\partial v'_x}{\partial r}} + & \\
\frac{1}{r}\left(\cancel{V_\theta \frac{\partial V_x}{\partial \theta}} + \cancel{v'_\theta \frac{\partial V_x}{\partial \theta}} + V_\theta \frac{\partial v'_x}{\partial \theta} + \cancel{v'_\theta \frac{\partial v'_x}{\partial \theta}}\right) + & \\
\cancel{V_x \frac{\partial V_x}{\partial x}} + \cancel{v'_x \frac{\partial V_x}{\partial x}} + V_x \frac{\partial v'_x}{\partial x} + \cancel{v'_x \frac{\partial v'_x}{\partial x}} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
\boxed{\frac{\partial v'_x}{\partial t} + v'_r \frac{\partial V_x}{\partial r} + \frac{V_\theta}{r}\frac{\partial v'_x}{\partial \theta} + V_x \frac{\partial v'_x}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}} &
\end{aligned}$$

$$
\begin{aligned}
\rightarrow -\frac{1}{\rho}\frac{\partial p}{\partial x} &= \left(-\frac{1}{\bar{\rho}} + \frac{\rho'}{\bar{\rho}^2}\right)\left(\frac{\partial \bar{p} + p'}{\partial x}\right) \\
\frac{1}{\rho}\frac{\partial p}{\partial x} &= -\cancel{\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x}} - \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} + \cancel{\frac{\rho'}{\bar{\rho}^2 r}\frac{\partial \bar{p}}{\partial x}} + \cancel{\frac{\rho'}{\bar{\rho}^2 r}\frac{\partial p'}{\partial x}} \\
\frac{1}{\rho}\frac{\partial p}{\partial x} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}
\end{aligned}
$$

$$
\boxed{\frac{\partial v_x'}{\partial t} + v_r'\frac{\partial V_x}{\partial r} + \frac{V_\theta}{r}\frac{\partial v_x'}{\partial \theta} + V_x\frac{\partial v_x'}{\partial x} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}}
$$

3.3.1.5 Linearizing the Energy Equation

$$\frac{\partial p}{\partial t} + v_r\frac{\partial p}{\partial r} + \frac{v_\theta}{r}\frac{\partial p}{\partial x} + v_x\frac{\partial p}{\partial x} + \gamma p\left(\frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_x}{\partial x}\right) = 0$$

$$\begin{aligned}\frac{\partial(\bar{P}+P')}{\partial t} + (V_r + v_r')\frac{\partial(\bar{P}+P')}{\partial r} + \frac{(V_\theta + v_\theta')}{r}\frac{\partial(\bar{P}+p')}{\partial x} + (V_x + v_x')\frac{\partial(\bar{P}+P')}{\partial x} + ...\\ \gamma(\bar{P}+P')\left(\frac{1}{r}\frac{\partial r(V_r+v_r')}{\partial r} + \frac{1}{r}\frac{\partial(V_\theta + v_\theta')}{\partial \theta} + \frac{\partial(V_x+v_x')}{\partial x}\right) = 0\end{aligned}$$

$$\begin{aligned}\frac{\partial \bar{P}}{\partial t} + \frac{\partial P'}{\partial t} +\\ V_r\frac{\partial \bar{P}}{\partial r} + V_r\frac{\partial P'}{\partial r} + V_r'\frac{\partial \bar{P}}{\partial r} + V_r'\frac{\partial P'}{\partial r} +\\ \frac{V_\theta}{r}\frac{\partial \bar{P}}{\partial x} + \frac{V_\theta}{r}\frac{\partial P'}{\partial x} + \frac{v_\theta'}{r}\frac{\partial \bar{P}}{\partial x} + \frac{v_\theta'}{r}\frac{\partial P'}{\partial x} +\\ V_x\frac{\partial \bar{P}}{\partial x} + V_x\frac{\partial P'}{\partial x} + v_x'\frac{\partial \bar{P}}{\partial x} + v_x'\frac{\partial P'}{\partial x} +\\ \gamma\bar{P}\left(\frac{1}{r}\frac{\partial r V_r}{\partial r} + \frac{1}{r}\frac{\partial r v_r'}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial v_\theta'}{\partial \theta} + \frac{\partial V_x}{\partial x} + \frac{\partial v_x'}{\partial x}\right)\\ \gamma P'\left(\frac{1}{r}\frac{\partial r V_r}{\partial r} + \frac{1}{r}\frac{\partial r v_r'}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial v_\theta'}{\partial \theta} + \frac{\partial V_x}{\partial x} + \frac{\partial v_x'}{\partial x}\right)\end{aligned}$$

$$\boxed{\frac{\partial p'}{\partial t} + v_r'\frac{\partial P}{\partial r} + \frac{V_\theta}{r}\frac{\partial p'}{\partial \theta} + V_x\frac{\partial p'}{x} + \gamma P\left(\frac{1}{r}\frac{\partial(r v_r')}{\partial r} + \frac{1}{r}\frac{\partial v_\theta'}{\partial \theta} + \frac{\partial v_x'}{\partial x}\right) = 0}$$

The linearized Euler equations are,

$$
\begin{aligned}
\frac{\partial \rho'}{\partial t} + \frac{V_\theta}{r}\frac{\partial \rho'}{\partial \theta} + V_x\frac{\partial \rho'}{\partial x} + \bar{\rho}\left(\frac{1}{r}\left(\frac{\partial r v_r'}{\partial r} + \frac{\partial v_\theta'}{\partial \theta}\right) + \frac{\partial v_x'}{\partial x}\right) &= 0 \\
\frac{\partial v_r'}{\partial t} + \frac{V_\theta}{r}\frac{\partial v_\theta'}{\partial \theta} - \frac{2V_\theta v_\theta'}{r} + V_x\frac{\partial v_r'}{\partial x} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial r} + \frac{\rho'}{\bar{\rho}^2}\frac{\partial \bar{p}}{\partial r} \\
\frac{\partial v_\theta'}{\partial t} + v_r'\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial v_\theta'}{\partial \theta} + \frac{v_r' V_\theta}{r} + V_x\frac{\partial v_r'}{\partial x} &= -\frac{1}{\bar{\rho} r}\frac{\partial p'}{\partial \theta} \\
\frac{\partial v_x'}{\partial t} + v_r'\frac{\partial V_x}{\partial r} + \frac{V_\theta}{r}\frac{\partial v_x'}{\partial \theta} + V_x\frac{\partial v_x'}{\partial x} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} \\
\frac{\partial p'}{\partial t} + v_r'\frac{\partial P}{\partial r} + \frac{V_\theta}{r}\frac{\partial p'}{\partial \theta} + V_x\frac{\partial p'}{x} + \gamma P\left(\frac{1}{r}\frac{\partial (r v_r')}{\partial r} + \frac{1}{r}\frac{\partial v_\theta'}{\partial \theta} + \frac{\partial v_x'}{\partial x}\right) &= 0
\end{aligned}
$$

Recalling:

$$\frac{\partial P}{\partial r} = \frac{\bar{\rho} V_\theta^2}{r}$$

$$\gamma P = \bar{\rho} A^2$$

$$\rho' = \frac{1}{A^2} p'$$

We can rearrange the equations to reflect Equations 2.33-2.36. Note that the momentum equation in the θ and x directions remain unchanged. The term $\frac{\partial (r v_r')}{\partial r} = \frac{\partial (r)}{\partial r} v_r' + \frac{\partial v_r'}{\partial r} r$ in the Energy equation

$$
\begin{aligned}
\frac{1}{\bar{\rho} A^2}\left(\frac{\partial p'}{\partial t} + \frac{V_\theta}{r}\frac{\partial p'}{\partial \theta} + V_x\frac{\partial p'}{\partial x}\right) + \frac{V_\theta^2}{A^2 r} v_r' + \frac{\partial v_r'}{\partial r} + \frac{v_r'}{r} + \frac{1}{r}\frac{\partial v_\theta'}{\partial \theta} + \frac{\partial v_x'}{\partial x} &= 0 \\
\frac{\partial v_r'}{\partial t} + \frac{V_\theta}{r}\frac{\partial v_r'}{\partial \theta} - \frac{2V_\theta v_\theta'}{r} + V_x\frac{\partial v_r'}{\partial x} &= \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial r} + \frac{V_\theta}{\bar{\rho} r A^2} p' \\
\frac{\partial v_\theta'}{\partial t} + v_r'\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial v_\theta'}{\partial \theta} + \frac{v_r' V_\theta}{r} + V_x\frac{\partial v_\theta'}{\partial x} &= -\frac{1}{\bar{\rho} r}\frac{\partial p'}{\partial \theta} \\
\frac{\partial v_x'}{\partial t} + v_r'\frac{\partial V_x}{\partial r} + \frac{V_\theta}{r}\frac{\partial v_x'}{\partial \theta} + V_x\frac{\partial v_x'}{\partial x} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}
\end{aligned}
$$

3.4 Substituting Pertubation Variables

One key assumption is that the perturbation quantites, $\tilde{p}$, $\tilde{v}_r$,$\tilde{v}_\theta$, and $\tilde{v}_x$, are all exponential and that they are solely a function of radius,

$$v'_r = v_r(r)e^{i(k_x x+m\theta-\omega t)}$$

$$v'_\theta = v_\theta(r)e^{i(k_x x+m\theta-\omega t)}$$

$$v'_x = v_x(r)e^{i(k_x x+m\theta-\omega t)}$$

$$p' = p(r)e^{i(k_x x+m\theta-\omega t)}$$

Substituting the quantities into the linearized equations will give us the final governing equations. Starting with the Conservation of Momentum in the r direction,

$$\frac{\partial v_r'}{\partial t} + \frac{V_\theta}{r}\frac{\partial v_r'}{\partial \theta} - \frac{2V_\theta}{r}v_r' + V_x\frac{\partial v_r'}{\partial x} = \frac{\partial p'}{\partial r} + \frac{\rho'}{\bar{\rho}^2}\frac{\partial P}{\partial r}$$

Looking at the left hand side (LHS) of the equation, the derivatives are:

$$\frac{\partial v_r'}{\partial t} = \underbrace{\frac{\partial v_r(r)}{\partial t}}_{0} e^{i(k_x x + m\theta - \omega t)} + v_r(r)\left(-i\omega e^{i(k_x x + m\theta - \omega t)}\right)$$

$$\frac{\partial v_r'}{\partial \theta} = \underbrace{\frac{\partial v_r(r)}{\partial \theta}}_{0} e^{i(k_x x + m\theta - \omega t)} + v_r(r)\left(im e^{i(k_x x + m\theta - \omega t)}\right)$$

$$\frac{\partial v_r'}{\partial x} = \underbrace{\frac{\partial v_r(r)}{\partial x}}_{0} e^{i(k_x x + m\theta - \omega t)} + v_r(r)\left(ik_x e^{i(k_x x + m\theta - \omega t)}\right)$$

Similarly for the right hand side (RHS),

$$\frac{\partial p'}{\partial r} = \frac{\partial P(r)}{\partial r} e^{i(k_x x + m\theta - \omega t)} + P(r)\underbrace{\frac{\partial}{\partial r} e^{i(k_x x + m\theta - \omega t)}}_{0}$$

Recalling $p'/\rho' = A^2 \rightarrow \rho' = \frac{1}{A^2}p'$ $\frac{\partial \bar{P}}{\partial r} = \frac{\bar{\rho} v_\theta^2}{r}$

$$\frac{\rho'}{\bar{\rho}^2}\frac{\partial \bar{P}}{\partial r} = \frac{V_\theta^2}{A^2}\frac{1}{\bar{\rho} r}p'$$

After substituting and canceling common terms,

$$v_r\left(-i\omega \cancel{e^{i(k_x x + m\theta - \omega t)}}\right) + \frac{V_\theta}{r}v_r\left(im\cancel{e^{i(k_x x + m\theta - \omega t)}}\right) - \frac{2V_\theta}{r}v_r\cancel{e^{i(k_x x + m\theta - \omega t)}} + V_x\left(v_r\left(ik_x\cancel{e^{i(k_x x + m\theta - \omega t)}}\right)\right)$$
$$= \left(-\frac{1}{\bar{\rho}}\frac{\partial P(r)}{\partial r}\cancel{e^{i(k_x x + m\theta - \omega t)}} + \frac{V_\theta^2}{A^2}\frac{1}{\bar{\rho} r}P(r)\cancel{e^{i(k_x x + m\theta - \omega t)}}\right)$$

$$\boxed{\left(-i\omega + \frac{imV_\theta}{r} + ik_x V_x\right) v_r - \frac{2V_\theta}{r} v_\theta = -\frac{1}{\bar{\rho}}\frac{\partial P}{\partial r} + \frac{V_{\theta^2}}{A^2}\frac{1}{\bar{\rho} r} p}$$

Continuing with conservation of momentum in the θ direction,

$$\frac{\partial v'_\theta}{\partial t} + v'_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial v'_\theta}{\partial \theta} + \frac{v'_r V_\theta}{r} + V_x \frac{\partial v'_\theta}{\partial x} = -\frac{1}{\bar{\rho} r}\frac{\partial p'}{\partial \theta}$$

$$\frac{\partial v'_\theta}{\partial t} = \underbrace{\frac{\partial v_\theta(r)}{\partial t}}_{0} e^{i(k_x x + m\theta - \omega t)} + v_\theta(r)\left(-i\omega e^{i(k_x x + m\theta - \omega t)}\right)$$

$$\frac{\partial v'_\theta}{\partial \theta} = \underbrace{\frac{\partial v'_\theta(r)}{\partial \theta}}_{0} e^{i(k_x x + m\theta - \omega t)} + v_\theta(r)\left(ime^{i(k_x x + m\theta - \omega t)}\right)$$

$$\frac{\partial v'_\theta}{\partial x} = \underbrace{\frac{\partial v_\theta(r)}{\partial x}}_{0} e^{i(k_x x + m\theta - \omega t)} + v_\theta(r)\left(ik_x e^{i(k_x x + m\theta - \omega t)}\right)$$

$$\frac{\partial p'}{\partial \theta} = \underbrace{\frac{\partial P(r)}{\partial \theta}}_{0} e^{i(k_x x + m\theta - \omega t)} + P(r)\underbrace{\frac{\partial}{\partial \theta} e^{i(k_x x + m\theta - \omega t)}}_{me^{i(k_x x + m\theta - \omega t)}}$$

After substituting and canceling common terms

$$\boxed{\left(-i\omega + \frac{imV_\theta}{r} + ik_x V_x\right) v_\theta + \left(\frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r}\right) v_\theta = -\frac{m}{\bar{\rho} r} p}$$

Next, the conservation of momentum in the x direction,

$$\frac{\partial v'_x}{\partial t} + v'_r \frac{\partial V_x}{\partial r} + \frac{V_\theta}{r}\frac{\partial v'_x}{\partial \theta} + V_x \frac{\partial v'_x}{\partial x} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v'_x}{\partial t} = \underbrace{\frac{\partial v_x(r)}{\partial t}}_{0} e^{i(k_x x + m\theta - \omega t)} + v_x(r)\left(-i\omega e^{i(k_x x + m\theta - \omega t)}\right)$$

$$\frac{\partial v_x'}{\partial \theta} = \underbrace{\frac{\partial v_x(r)}{\partial \theta}}_{0} e^{i(k_x x + m\theta - \omega t)} + v_x(r) \left(ime^{i(k_x x + m\theta - \omega t)} \right)$$

$$\frac{\partial v_x'}{\partial x} = \frac{\partial v_x(r)}{\partial x} e^{i(k_x x + m\theta - \omega t)} + v_x(r) \left(ik_x e^{i(k_x x + m\theta - \omega t)} \right)$$

$$\frac{\partial p'}{\partial x} = 0 + ik_x p e^{i(k_x x + m\theta - \omega t)}$$

$$\boxed{\left(-i\omega + \frac{imV_\theta}{r} + ik_x V_x \right) v_x + \frac{\partial V_x}{\partial r} v_r = -\frac{ik_x}{\bar{\rho}} p}$$

Continuing with the Conservation of Energy,

$$\frac{1}{\bar{\rho} A^2} \left(\frac{\partial p'}{\partial t} + \frac{V_\theta}{r} \frac{\partial p'}{\partial \theta} + V_x \frac{\partial p'}{\partial x} \right) + \frac{V_\theta^2}{A^2 r} v_r' + \frac{\partial v_r'}{\partial r} + \frac{1}{r} \frac{\partial v_\theta'}{\partial \theta} + \frac{\partial v_x'}{\partial x} = 0$$

Left hand side (LHS) derivatives:

$$\frac{\partial p'}{\partial t} = \underbrace{\frac{\partial p(r)}{\partial t}}_{0} e^{i(k_x x + m\theta - \omega t)} + p(r) \left(-i\omega e^{i(k_x x + m\theta - \omega t)} \right)$$

$$\frac{\partial p'}{\partial \theta} = \underbrace{\frac{\partial p(r)}{\partial \theta}}_{0} e^{i(k_x x + m\theta - \omega t)} + p(r) \left(ime^{i(k_x x + m\theta - \omega t)} \right)$$

$$\frac{\partial p'}{\partial x} = \underbrace{\frac{\partial p(r)}{\partial x}}_{0} e^{i(k_x x + m\theta - \omega t)} + p(r) \left(ik_x e^{i(k_x x + m\theta - \omega t)} \right)$$

$$\frac{\partial v_r'}{\partial r} = \frac{\partial v_r(r)}{\partial r} e^{i(k_x x + m\theta - \omega t)} + v_r(r) \underbrace{\frac{\partial}{\partial r} \left(e^{i(k_x x + m\theta - \omega t)} \right)}_{0}$$

$$\frac{\partial v_\theta'}{\partial \theta} = \frac{\partial v_\theta(r)}{\partial \theta} e^{i(k_x x + m\theta - \omega t)} + v_\theta(r) \left(ime^{i(k_x x + m\theta - \omega t)} \right)$$

$$\frac{\partial v_x'}{\partial x} = \underbrace{\frac{\partial v_x(r)}{\partial x}}_{0} e^{i(k_x x + m\theta - \omega t)} + v_x(r)\left(ik_x e^{i(k_x x + m\theta - \omega t)}\right)$$

After substituting and canceling common terms,

$$\frac{1}{\bar{\rho}A^2}\left(p(r)\left(-i\omega e^{i(k_x x + m\theta - \omega t)}\right) + \frac{V_\theta}{r}p(r)\left(ime^{i(k_x x + m\theta - \omega t)}\right) + V_x p(r)\left(ik_x e^{i(k_x x + m\theta - \omega t)}\right)\right) +$$
$$\frac{V_\theta^2}{A^2 r}v_r' + \frac{\partial v_r(r)}{\partial r}e^{i(k_x x + m\theta - \omega t)} + \frac{1}{r}\left(v_\theta(r)\left(ime^{i(k_x x + m\theta - \omega t)}\right)\right) + v_x(r)\left(ik_x e^{i(k_x x + m\theta - \omega t)}\right) = 0$$

$$\boxed{\frac{1}{\bar{\rho}A^2}\left(-i\omega + \frac{imV_\theta}{r} + ik_x V_x +\right)p(r) + \frac{V_\theta^2}{A^2 r}v_r + \frac{v_r}{r} + \frac{\partial v_r(r)}{\partial r} + \frac{im}{r}v_\theta(r) + ik_x v_x(r) = 0}$$

The Linearized Euler equations now become

$$\begin{aligned}
r\text{-direction: } & i\left(-\omega + \frac{m}{r} + k_x V_x\right)v_r - \frac{2\bar{v}_\theta}{r}v_\theta = -\frac{1}{\bar{\rho}}\frac{\partial P}{\partial r} + \frac{V_{\theta^2}}{A^2}\frac{1}{\bar{\rho}r}p \\
\theta\text{-direction: } & i\left(-\omega + \frac{m}{r} + k_x V_x\right)v_\theta + \left(\frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r}\right)v_\theta = -\frac{m}{\bar{\rho}r}p \\
x\text{-direction: } & i\left(-\omega + \frac{mV_\theta}{r} + k_x Vx\right)v_x + \frac{\partial V_x}{\partial r}v_r = -\frac{ik_x}{\bar{\rho}}p \\
\text{Energy: } & \frac{1}{\bar{\rho}A^2}\left(-i\omega + \frac{imV_\theta}{r} + ik_x V_x +\right)p(r) + \frac{V_\theta^2}{A^2 r}v_r + \frac{v_r}{r} + \frac{\partial v_r(r)}{\partial r} + \frac{im}{r}v_\theta(r) + ik_x v_x(r) \quad = 0
\end{aligned}$$

3.5 Non-Dimensionalization

Defining

$$r_T = r_{max}$$

$$A_T = A(r_{max})$$

$$k = \frac{\omega r_T}{A_T}$$

$$\bar{\gamma} = k_x r_T$$

$$\tilde{r} = \frac{r}{r_T}$$

$$\frac{\partial}{\partial r} = \frac{\partial \tilde{r}}{\partial r}\frac{\partial}{\partial \tilde{r}} = \frac{1}{r_T}\frac{\partial}{\partial \tilde{r}}$$

$$V_\theta = M_\theta A$$

$$V_x = M_x A$$

$$\tilde{A} = \frac{A}{A_T}$$

$$v_x = \tilde{v}_x A$$

$$v_r = \tilde{v}_r A$$

$$v_\theta = \tilde{v}_\theta A$$

$$p = \tilde{p}\bar{\rho}A^2$$

$$r\text{-direction: } i\left(-\omega+\frac{mV_\theta}{r}+k_xV_x\right)v_r-\frac{2\bar{v}_\theta}{r}v_\theta=-\frac{1}{\bar{\rho}}\frac{\partial P}{\partial r}+\frac{V_{\theta^2}}{A^2}\frac{1}{\bar{\rho}r}p$$

$$\theta\text{-direction: } i\left(-\omega+\frac{m}{r}+k_xV_x\right)v_\theta+\left(\frac{V_\theta}{r}+\frac{\partial V_\theta}{\partial r}\right)v_\theta=-\frac{m}{\bar{\rho}r}p$$

$$x\text{-direction: } i\left(-\omega+\frac{mV_\theta}{r}+k_xVx\right)v_x+\frac{\partial V_x}{\partial r}v_r=-\frac{ik_x}{\bar{\rho}}p$$

$$\text{Energy: } \frac{1}{\bar{\rho}A^2}i\left(-\omega+\frac{mV_\theta}{r}+k_xV_x\right)p(r)+\frac{V_\theta^2}{A^2r}v_r+\frac{v_r}{r}+\frac{\partial v_r(r)}{\partial r}+\frac{im}{r}v_\theta(r)+ik_xv_x(r)=0$$

Substituting the non dimensional quantities, and noting r_T and A^2 in each term, the radial momentum equation becomes,

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{v}_r-\frac{2M_\theta\tilde{v}_\theta}{\tilde{r}}=-\frac{1}{\bar{\rho}A^2}\frac{\partial\tilde{p}\bar{\rho}A^2}{\partial\tilde{r}}+M_\theta\frac{\tilde{p}}{\tilde{r}}}$$

Similarly for the $\theta$and x directions:

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{v}_\theta+\left(\frac{M_\theta}{\tilde{r}}+\frac{1}{A}\frac{\partial M_\theta A}{\partial\tilde{r}}\right)\tilde{v}_r=\frac{im}{\tilde{r}}\tilde{P}}$$

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{v}_x+\frac{1}{A}\frac{\partial M_xA}{\partial\tilde{r}}\tilde{v}_r=-i\bar{\gamma}\tilde{P}}$$

and for the Energy equation:

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{p}+\frac{M_\theta^2}{\tilde{r}}\tilde{v}_r+\frac{1}{A}\frac{\partial(\tilde{v}_rA)}{\partial\tilde{r}}+\frac{\tilde{v}_r}{\tilde{r}}+\frac{im}{\tilde{r}}\tilde{v}_\theta+i\bar{\gamma}\tilde{v}_x=0}$$

Expanding mean derivatives (using product rule) $\frac{\partial\tilde{p}\bar{\rho}A^2}{\partial\tilde{r}}$

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{v}_r-\frac{2M_\theta\tilde{v}_\theta}{\tilde{r}}=-\left(\frac{\partial\tilde{p}}{\partial\tilde{r}}+\frac{\tilde{p}}{\bar{\rho}A^2}\frac{\partial\bar{\rho}A^2}{\partial\tilde{r}}\right)+M_\theta\frac{\tilde{p}}{\tilde{r}}}$$

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{v}_\theta+\left(\frac{M_\theta}{\tilde{r}}+\frac{1}{A}\frac{\partial M_\theta A}{\partial\tilde{r}}\right)\tilde{v}_r=\frac{im}{\tilde{r}}\tilde{P}}$$

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{v}_x+\frac{1}{A}\frac{\partial M_x A}{\partial\tilde{r}}\tilde{v}_r=-i\bar{\gamma}\tilde{P}}$$

$$\boxed{i\left[-\frac{k}{\tilde{A}}+\frac{mM_\theta}{\tilde{r}}+\bar{\gamma}M_x\right]\tilde{p}+\frac{M_\theta^2}{\tilde{r}}\tilde{v}_r+\frac{\partial\tilde{v}_r}{\partial\tilde{r}}+\frac{1}{A}\frac{\partial A}{\partial\tilde{r}}v_r+\frac{\tilde{v}_r}{\tilde{r}}+\frac{im}{\tilde{r}}\tilde{v}_\theta+i\bar{\gamma}\tilde{v}_x=0}$$

$$\frac{1}{A}\frac{\partial A}{\partial \tilde{r}}$$

Recall, $\partial/\partial r = (1/r_T)(\partial/\partial \tilde{r}$,

$$\begin{aligned}\frac{1}{A}\frac{\partial A}{\partial \tilde{r}} &= r_T\left(\frac{1}{A}\frac{\partial A}{\partial r}\right)\\ &= \frac{r_T}{A^2}\left(A\frac{\partial A}{\partial r}\right)\end{aligned}$$

Using the trick, $\frac{\partial}{\partial r}\left(\frac{A^2}{2}\right) = A\frac{\partial A}{\partial r}$

$$\begin{aligned}&= \frac{r_T}{A^2}\left(\frac{\partial}{\partial r}\left(\frac{A^2}{2}\right)\right)\\ &= \frac{r_T}{2A^2}\frac{\partial A^2}{\partial r}\end{aligned}$$

Using the definition derived earlier (Needs eqn reference) $\frac{\partial A^2}{\partial r} = \frac{\gamma-1}{2}\frac{v_\theta^2}{r}$

$$\begin{aligned}&= \frac{r_T}{2A^2}\frac{\gamma-1}{2}\frac{v_\theta^2}{r}\\ &= \frac{\gamma-1}{2}\frac{M_\theta^2}{\tilde{r}}\end{aligned}$$

$$\begin{aligned}\frac{\partial \rho A^2}{\partial \tilde{r}} &= \gamma\frac{\partial p}{\partial \tilde{r}}\\ &= \gamma\frac{\rho v_\theta^2}{\tilde{r}}\\ &= r_T\gamma\frac{\rho v_\theta^2}{r}\end{aligned}$$

$$\frac{1}{\rho A^2}\frac{\partial \rho A^2}{\partial r} = \frac{\gamma M_\theta^2}{\tilde{r}}$$

Substituting in yields ,

$$\boxed{i\left[-\frac{k}{\tilde{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma}M_x\right]\tilde{v}_r - \frac{2M_\theta \tilde{v}_\theta}{\tilde{r}} = -\frac{\partial \tilde{p}}{\partial \tilde{r}} - (\gamma - 1)\frac{\gamma M_\theta}{\tilde{r}}\tilde{p}}$$

$$\boxed{i\left[-\frac{k}{\tilde{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma}M_x\right]\tilde{v}_\theta + \left(\frac{M_\theta}{\tilde{r}} + \frac{1}{A}\frac{\partial M_\theta A}{\partial \tilde{r}}\right)\tilde{v}_r = \frac{im}{\tilde{r}}\tilde{p}}$$

$$\boxed{i\left[-\frac{k}{\tilde{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma}M_x\right]\tilde{v}_x + \frac{1}{A}\frac{\partial M_x A}{\partial \tilde{r}}\tilde{v}_r = -i\bar{\gamma}\tilde{p}}$$

$$\boxed{i\left[-\frac{k}{\tilde{A}} + \frac{mM_\theta}{\tilde{r}} + \bar{\gamma}M_x\right]\tilde{p} + \frac{M_\theta^2}{\tilde{r}}\tilde{v}_r + \frac{\partial \tilde{v}_r}{\partial \tilde{r}} + \frac{1}{A}\frac{\partial A}{\partial \tilde{r}}v_r + \frac{\tilde{v}_r}{\tilde{r}} + \frac{im}{\tilde{r}}\tilde{v}_\theta + i\bar{\gamma}\tilde{v}_x = 0}$$

Defining, $\lambda = -i\bar{\gamma}$

and defining

$$\{\bar{x}\} = \begin{Bmatrix} \tilde{v}_r \\ \tilde{v}_\theta \\ \tilde{v}_x \\ \tilde{p} \end{Bmatrix}$$

The governing equations can be written in the form of $[A]x - \lambda[B]x$

$$\begin{bmatrix} -i\left(\frac{k}{\bar{A}}-\frac{mM_\theta}{\tilde{r}}\right)-\lambda M_x & -\frac{2M_\theta}{\tilde{r}} & 0 & \frac{\partial}{\partial\tilde{r}}+\frac{\gamma-1}{\tilde{r}} \\ \frac{M_\theta}{\tilde{r}}+\frac{\partial M_\theta}{\partial\tilde{r}}+\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^3}{\tilde{r}} & -i\left(\frac{k}{\bar{A}}-\frac{mM_\theta}{\tilde{r}}\right)-\lambda M_x & 0 & \frac{im}{\tilde{r}} \\ \frac{\partial M_x}{\partial\tilde{r}}+\left(\frac{\gamma-1}{2}\frac{M_x M_\theta^2}{\tilde{r}}\right) & 0 & -i\left(\frac{k}{\bar{A}}-\frac{mM_\theta}{\tilde{r}}\right)-\lambda M_x & -\lambda \\ \frac{\partial}{\partial\tilde{r}}+\frac{\gamma+1}{2}\frac{M_\theta^2}{\tilde{r}}+\frac{1}{\tilde{r}} & \frac{im}{\tilde{r}} & -\lambda & -i\left(\frac{k}{\bar{A}}-\frac{mM_\theta}{\tilde{r}}\right)-\lambda M_x \end{bmatrix}\bar{x}=$$

Chapter 4

Chapter 4: Numerical Models

4.1 Introduction

The Method of Manufactured Solutions (MMS) is a process for generating an analytical solution for a code that provides the numerical solution for a given domain. The goal of MMS is to establish a manufactured solution that can be used to establish the accuracy of the code within question. For this study, SWIRL, a code used to calculate the radial modes within an infinitely long duct is being validated through code verification. SWIRL accepts a given mean flow and uses numerical integration to obtain the speed of sound. The integration technique is found to be the composite trapezoidal rule through asymptotic error analysis.

For SWIRL, the absolute bare minimum requirement is to define the corresponding flow components for the domain of interest. SWIRL assumes no flow in the radial direction, leaving only two other components, axial and tangential for a 3D cylindrical domain. Since SWIRL is also non dimensionalized, the mean flow components are defined using the Mach number. SWIRL uses the tangential mach number to obtain the speed of sound using numerical integration. The speed of sound is then used to find the rest of the primative variables for the given flow.

4.2 Methods

SWIRL is a linearized Euler equations of motion code that calculates the axial wavenumber and radial mode shapes from small unsteady disturbances in a mean flow. The mean flow varies along the axial and tangetial directons as a function of radius. The flow domain can either be a circular or annular duct, with or without acoustic liner. SWIRL was originally written by Kousen [insert ref].

The SWIRL code requires two mean flow parameters as a function of radius, M_x , and M_θ. Afterwards, the speed of sound, $\widetilde{A}$ is calculated by integrating M_θ with respect to r. To verify that SWIRL is handling and returning the accompanying mean flow parameters, the error between the mean flow input and output variables are computed. Since the trapezoidal rule is used to numerically integrate M_θ, the discretization error and order of accuracy is computed. Since finite differencing schemes are to be used on the result of this integration, it is crucial to accompany the integration with methods of equal or less order of accuracy. This will be determined by applying another MMS on the eigenproblem which will also have an order of accuracy.

4.2.1 Theory

To relate the speed of sound to a given flow, the radial momentum equation is used. If the flow contains a swirling component,then the primitive variables are nonuniform through the flow, and mean flow assumptions are not valid.

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} - \frac{v_\theta^2}{r} v_x \frac{\partial v_r}{\partial x} = \frac{1}{\rho}\frac{\partial P}{\partial r}$$

To account to for this, the radial momentum is simplified by assuming the flow is steady, the flow has no radial component. In addition, the viscous and body forces are neglected. Then the radial pressure derivative term is set equal to the dynamic pressure term. Seperation of variables is applied.

$$\frac{v_\theta^2}{r} = \frac{1}{\rho}\frac{\partial P}{\partial r}$$
$$P = \int_r^{r_{max}} \frac{\rho V_\theta^2}{r}$$

To show the work, we will start with the dimensional form of the equation and differentiate both sides. Applying separation of variables,

$$\int_r^{r_{max}} \frac{\bar{\rho} v_\theta^2}{r} \partial r = -\int_{P(r)}^{P(r_{max})} \partial p.$$

Since $\tilde{r} = r/r_{max}$,

$$r = \tilde{r} r_{max}.$$

Taking total derivatives (i.e. applying chain rule),

$$dr = d(\tilde{r} r_{max}) = d(\tilde{r}) r_{max},$$

Substituting these back in and evaluating the right hand side,

$$\int_{\tilde{r}}^{1} \frac{\bar{\rho} v_\theta^2}{\tilde{r}} \partial \tilde{r} = P(1) - P(\tilde{r})$$

For reference the minimum value of $\tilde{r}$ is,

$$\sigma = \frac{r_{max}}{r_{min}}$$

For the radial derivative, the definition of the speed of sound is utilized,

$$\frac{\partial A^2}{\partial r} = \frac{\partial}{\partial r} \left(\frac{\gamma P}{\rho} \right).$$

Using the quotient rule, the definition of the speed of sound is extracted,

$$\begin{aligned} &= \frac{\partial P}{\partial r} \frac{\gamma \bar{\rho}}{\bar{\rho}^2} - \left(\frac{\gamma P}{\bar{\rho}^2} \right) \frac{\partial \bar{\rho}}{\partial r} \\ &= \frac{\partial P}{\partial r} \frac{\gamma}{\bar{\rho}} - \left(\frac{A^2}{\bar{\rho}} \right) \frac{\partial \bar{\rho}}{\partial r} \end{aligned}$$

Using isentropic condition $\partial P / A^2 = \partial \rho$,

$$\begin{aligned} &= \frac{\partial P}{\partial r} \frac{\gamma}{\bar{\rho}} - \left(\frac{1}{\bar{\rho}} \right) \frac{\partial P}{\partial r} \\ \frac{\partial A^2}{\partial r} &= \frac{\partial P}{\partial r} \frac{\gamma - 1}{\bar{\rho}} \end{aligned}$$

$$\frac{\bar{\rho}}{\gamma - 1} \frac{\partial A^2}{\partial r} = \frac{\partial P}{\partial r}$$

Going back to the radial momentum equation, and rearranging the terms will simplify the expression. The following terms are defined to start the nondimensionalization.

$$
\begin{aligned}
M_\theta &= \frac{V_\theta}{A} \\
\widetilde{r} &= \frac{r}{r_{max}} \\
\widetilde{A} &= \frac{A}{A_{r,max}} \\
A &= \widetilde{A} A_{r,max} \\
r &= \widetilde{r} r_{max} \\
\frac{\partial}{\partial r} &= \frac{\partial \widetilde{r}}{\partial r} \frac{\partial}{\partial \widetilde{r}} \\
&= \frac{1}{r_{max}} \frac{\partial}{\partial \widetilde{r}}
\end{aligned}
$$

Dividing by A,

$$\frac{M_\theta^2}{r} (\gamma - 1) = \frac{\partial A^2}{\partial r} \frac{1}{A^2}$$

Now there is two options, either find the derivative of $\bar{A}$ or the integral of M_θ with respect to r.

1. Defining non dimensional speed of sound $\tilde{A} = \frac{A(r)}{A(r_{max})}$

$$\int_r^{r_{max}} \frac{M_\theta}{r} (\gamma - 1) \partial r = ln\left(\frac{1}{\tilde{A}^2}\right)$$
$$= -2ln(\tilde{A})$$
$$\tilde{A}(r) = exp\left[-\int_r^{r_{max}} \frac{M_\theta}{r} \frac{(\gamma-1)}{2} \partial r\right]$$
$$\text{replacing r with } \tilde{r} \rightarrow \tilde{A}(r) = exp\left[-\int_r^{r_{max}} \frac{M_\theta}{r} \frac{(\gamma-1)}{2} \partial r\right]$$
$$\tilde{A}(\tilde{r}) = exp\left[\left(\frac{1-\gamma}{2}\right) \int_{\tilde{r}}^1 \frac{M_\theta}{\tilde{r}} \partial \tilde{r}\right]$$

2. Or we can differentiate

Solving for M_θ ,

$$M_\theta^2 = \frac{\partial A^2}{\partial r} \frac{r}{A^2 (\gamma - 1)}$$

Nondimensionalizing and substituting,

$$M_\theta^2 \frac{(\gamma-1)}{\widetilde{r} r_{max}} = \frac{1}{(\widetilde{A} A_{r,max})^2} \frac{A_{r,max}^2}{r_{max}} \frac{\partial \widetilde{A}^2}{\partial \widetilde{r}}$$
$$M_\theta^2 \frac{(\gamma-1)}{\widetilde{r}} = \frac{1}{\widetilde{A}^2} \frac{\partial \widetilde{A}^2}{\partial \widetilde{r}}$$
$$M_\theta = \sqrt{\frac{\widetilde{r}}{(\gamma-1)\widetilde{A}^2} \frac{\partial \widetilde{A}^2}{\partial \widetilde{r}}} \tag{4.1}$$

4.2.2 Calculation of Observed Order-of-Accuracy

The numerical scheme used to perform the integration of the tangential velocity will have a theoretical order-of-accuracy. To find the theoretical order-of-accuracy, the discretization error must first be defined. The error, ϵ, is a function of id spacing, Δr

$$\epsilon = \epsilon(\Delta r)$$

The discretization error in the solution should be proportional to $(\Delta r)^{\alpha}$ where $\alpha > 0$ is the theoretical order for the computational method. The error for each grid is expressed as

$$\epsilon_{M_\theta}(\Delta r) = |M_{\theta,analytic} - M_{\theta,calc}|$$

where $M_{\theta,analytic}$ is the tangential mach number that is defined from the speed of sound we also defined and the $M_{\theta,calc}$ is the result from SWIRL. The Δr is to indicate that this is a discretization error for a specific grid spacing. Applying the same concept to to the speed of sound,

If we define this error on various grid sizes and compute ϵ for each grid, the observed order of accuracy can be estimated and compared to the theoretical order of accuracy. For instance, if the numerical soution is second-order accurate and the error is converging to a value, the L2 norm of the error will decrease by a factor of 4 for every halving of the grid cell size.

Since the input variables should remain unchanged (except from minor changes from the Akima interpolation), the error for the axial and tangential mach number should be zero. As for the speed of sound, since we are using an analytic expression for the tangential mach number, we know what the theoretical result would be from the numerical integration technique as shown above. Similarly we define the discretization error for the speed of sound.

$$\epsilon_A(\Delta r) = |A_{analytic} - A_{calc}|$$

For a perfect answer, we expect ϵ to be zero. Since a Taylor series can be used to derive the numerical schemes, we know that the truncation of higher order terms is what indicates the error we expect from using a scheme that is constructed with such truncated Taylor series.

The error at each grid point j is expected to satisfy the following,

$$0 = |A_{analytic}(r_j) - A_{calc}(r_j)|$$
$$\widetilde{A}_{analytic}(r_j) = \widetilde{A}_{calc}(r_j) + (\Delta r)^{\alpha}\beta(r_j) + H.O.T$$

where the value of $\beta(r_j)$ does not change with grid spacing, and α is the asymptotic order of accuracy of the method. It is important to note that the numerical method recovers the original equations as the grid spacing approached zero. It is important to note that β represents the first derivative of the Taylor Series. Subtracting $A_{analytic}$ from both sides gives,

$$A_{calc}(r_j) - A_{analytic}(r_j) = A_{analytic}(r_j) - A_{analytic}(r_j) + \beta(r_j)(\Delta r)^{\alpha}$$
$$\epsilon_A(r_j)(\Delta r) = \beta(r_j)(\Delta r)^{\alpha}$$

To estimate the order of accuracy of the accuracy, we define the global errors by calculating the L2 Norm of the error which is denoted as $\hat{\epsilon}_A$

$$\hat{\epsilon}_A = \sqrt{\frac{1}{N}\sum_{j=1}^{N} \epsilon(r_j)^2}$$

$$\hat{\beta}_A(r_j) = \sqrt{\frac{1}{N}\sum_{j=1}^{N}\beta(r_j)^2}$$

As the grid density increases, $\hat{\beta}$ should asymptote to a constant value. Given two grid densities, Δr and $\sigma\Delta r$, and assuming that the leading error term is much larger than any other error term,

$$\begin{aligned}
\hat{\epsilon}_{grid1} &= \hat{\epsilon}(\Delta r) = \hat{\beta}(\Delta r)^{\alpha} \\
\hat{\epsilon}_{grid2} &= \hat{\epsilon}(\sigma\Delta r) = \hat{\beta}(\sigma\Delta r)^{\alpha} \\
&= \hat{\beta}(\Delta r)^{\alpha}\sigma^{\alpha}
\end{aligned}$$

The ratio of two errors is given by,

$$\begin{aligned}
\frac{\hat{\epsilon}_{grid2}}{\hat{\epsilon}_{grid1}} &= \frac{\hat{\beta}(\Delta r)^{\alpha}}{\hat{\beta}(\Delta r)^{\alpha}}\sigma^{\alpha} \\
&= \sigma^{\alpha}
\end{aligned}$$

Thus, α,the asymptotic rate of convergence is computed as follows

$$\alpha = \frac{\ln\frac{\hat{\epsilon}_{grid2}}{\hat{\epsilon}_{grid1}}}{\ln(\sigma)}$$

Defining for a doubling of grid points , Similarly for the eigenvalue problem,

$$[A]x = \lambda[B]x$$

[Insert your text to chapter 2 here. A pretend example of a silly figure is provided below (i.e., Figure **??**).]

References

First example: references generated by the “single” option

Friedman, Milton, “The Role of Monetary Policy,” *American Economic Review*, March 1968, 58(1), 1–17.

Keynes, John Maynard, *The General Theory of Employment, Interest, and Money*, New York: Harcourt Brace Jovanovic, 1936.

Smith, Adam, *An Inquiry into the Nature and Causes of the Wealth of Nations*, Edwin Cannan, ed., London: Methuen & Co., Ltd. 1904.

Tobin, James, “A Dynamic Aggregative Model,” *Journal of Political Economy,* April 1955, 63(2), 103–115.

Second example: references generated by the “double” option

Friedman, Milton, “The Role of Monetary Policy,” *American Economic Review*, March 1968, 58(1), 1–17.

Keynes, John Maynard, *The General Theory of Employment, Interest, and Money*, New York: Harcourt Brace Jovanovic, 1936.

Smith, Adam, *An Inquiry into the Nature and Causes of the Wealth of Nations*, Edwin Cannan, ed., London: Methuen & Co., Ltd. 1904.

Tobin, James, “A Dynamic Aggregative Model,” *Journal of Political Economy,* April 1955, 63(2), 103–115.

Appendix A

Chapter 3 Appendix

A.1 Appendix A: Speed of Sound

Sound wave is a pressure disturbance that moves with at a speed a By applying a rectangular control volume around this pressure wave, we can apply our conservation equations. We are assuming that these properities are increasing by a small increment. This is why each variable is added by a infinitesimally small term.

Recalling the conservation of mass (continuity equation), $\dot{m} = constant$

$$\dot{m}_{left} = \dot{m}_{right}$$

Recalling he definition of density, $\rho = m/\bar{V}$ and rewriting $\bar{V} = uA$ (check the units)

$$\rho a \cancel{A} = (\rho + d\rho)(a + da)\cancel{A}$$

Futher expanding gives,

$$\rho a = (\rho a + \rho da + ad\rho + dad\rho)$$

We say that $dad\rho$ is so small, we can assume it is zero. This is often referred to as "neglecting higher order terms (H.O.T)". The expression then becomes

$$\frac{da}{a} = -\frac{d\rho}{\rho}$$

For the momentum equation $P + \rho u^2 = constant$

$$P + \rho a^2 = P + dP + (\rho + d\rho)(a + da)(a + da)$$

But we just said that $\rho a = (\rho + d\rho)(a + da)$

$$P + \rho a^2 = P + dP + \rho a(a + da)$$

$$dP + \rho a da = a$$

Multiplying the second term by a and divide by a, this is essentially multiplying the second term by one.

$$dP + \rho a^2 \frac{da}{a} = a$$

recalling the relation $\frac{da}{a} = -\frac{d\rho}{\rho}$

$$dp - a^2 d\rho = 0$$

$$a^2 = \frac{dp}{d\rho}$$

Since a sound wave is a very weak wave, when it travels through a medium, it only increases the pressure and density, etc. slightly. The effect of this is that friction and heat transfer can be neglected. Since friction cannot be undone, we call this an irreversible process. Whenever there is no transfer of heat, it is called this adiabatic. Thus, the propagation of sound is an adiabatic, reversible process, otherwise called isentropic. Isentropic implies no increase in entropy, which is *not* true in the presence of shock waves.

In the case of a thermally perfect gas, we can say $P = \rho RT$

For a callorically perfect gas we can say $pv^{\gamma} = constant$, where v is volume per unit mass, or specific volume

Differentiating and recalling that $v = 1/\rho$

$$a = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\gamma RT}$$

$$dm = (\rho u)_2 - (\rho u)_1$$

$$D(mV) = \left(\rho u^2 + P\right)_2$$

For Steady flow,

$$\cancel{dm} = (\rho u)_2 - (\rho u)_1$$

$$(\rho u)_1 = (\rho u)_2$$

Similarly, for the Momentum equation,

$$\left(\rho u^2 + P\right)_1 = \left(\rho u^2 + P\right)_2$$

Let us change the coordinate system motion for the traveling wave be independent of time, and thus corresponds to *steady state* wave propagation.

$$u_1 = \bar{u} + a - \frac{1}{2}\partial u$$

$$u_2 = \bar{u} + a + \frac{1}{2}\partial u$$

where, $\bar{u}$ is the average flow velocity and a is the wave speed.

Substituting this back into the conservation of mass

$$(\rho u)_1 = (\rho u)_2$$

$$\left(\rho - \frac{1}{2}\partial\rho\right)\left(\bar{u} - a - \frac{1}{2}\partial u\right) = \left(\rho + \frac{1}{2}\partial\rho\right)\left(\bar{u} - a + \frac{1}{2}\partial u\right)$$

Further expanding

$$\cancel{\rho\bar{u} - \rho a} + \frac{1}{2}\left(-\rho\partial u - \bar{u} - \bar{u}\partial\rho + a\partial\rho\right) + \cancel{\frac{1}{4}\partial\rho\partial u} = \cancel{\rho\bar{u} - \rho a} + \frac{1}{2}\left(\rho\partial u + \bar{u} - \bar{u}\partial\rho - a\partial\rho\right) + \cancel{\frac{1}{4}\partial\rho\partial u}$$

$$\frac{1}{2}\left(-\rho\partial u - \bar{u}\partial\rho + a\partial\rho\right) = \frac{1}{2}\left(\rho\partial u + \bar{u}\partial\rho - a\partial\rho\right)$$

$$\rho\partial u + u\partial\rho - a\partial\rho = 0$$

$$\rho\partial u + \left(u - a\right)\partial\rho = 0$$

Momentum Equation

$$\left(\rho u^2 + P\right)_1 = \left(\rho u^2 + P\right)_2$$

A.2 Appendix B: Isentropic Waves

$$dU = dW + dQ$$

For adiabatic, reversible processes, the work done by a system with constant pressure and a change in volume is $-pdV$ and the change in heat energy is zero. Hence,

$$dU = -pdV$$

.

The change in enthalpy of such a system can be found by taking the derivative of its expression for a thermodynamic process

$$H = U + pV$$

$$dH = dU + pdV + vdP$$

$$dH = -pdV + pdV + vdP$$

$$dH = vdP$$

The specific heats at constant pressure and constant volume

$$\left(\frac{\partial U}{\partial T}\right)_v = C_v$$

$$\left(\frac{\partial H}{\partial T}\right)_p = C_p$$

$$\gamma = \frac{C_p}{C_v} = \frac{dH}{dU} = -\frac{VdP}{pdV} = -\frac{V}{dV}\frac{dP}{p}$$

Integrating both sides

$$\gamma\frac{dV}{V} = \frac{-dP}{P} \to \gamma\int\frac{1}{V}dV = -\int\frac{1}{P}dP$$

$$\gamma ln(V) + ln(P) = C$$

Using log rules

$$ln(V^\gamma) + ln(P) = C$$

$$ln(pV^\gamma) = C$$

$$pV^\gamma = e^C = C$$

$$\frac{p}{\rho^\gamma}$$

Appendix B

Insert the Heading to Appendix B

[Insert text to Appendix B (if appendix is needed)]