Report on Analytic Solution for Annular Ducts

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1 Current Research Direction

The goal is to currently compute the coefficients A and B, the weighting factors for the Bessel Functions for the first and second kind. V072 contains FORTRAN subroutines that compute these along with the Bessel functions.

2 Numerical Computation of Annular Duct Modes

3 Research Performed This Week

3.1 Theoretical Background

The analytic radial mode shape is of the form,

$$R_m(k_{r,mn}r) = AJ_m(k_{r,mn}r) + BY_m(k_{r,nmn}r) \tag{1}$$

The key to the numerical procedure is the following "transcendental" equation,

$$\begin{vmatrix} J'_m(k_{r,mn}r_H) & Y'_m(k_{r,mn}r_H) \\ J'_m(k_{r,mn}r_T) & Y'_m(k_{r,mn}r_T) \end{vmatrix} = 0 \tag{2}$$

The non-dimensional roots $k_{r,mn}r_T$ are found using initial guess and then incrementing from there

$$k_{r,mn} = \begin{cases} m & \text{if } n = 1 \\ k_{r,m(n-1)}r_T + \pi, & \text{if } n > 1. \end{cases} \tag{3}$$

The estimate is refined by incrementing the value of $k_{r,mn}$ by $\pi/10$ until the determinant above changes sign. The step size is then halved and also changes sign. This iterative process continues until the absolute value of the determinant is reduced to a preassigned value (error tolerance?). The non dimensional versions of these equations are used in the FORTRAN 77 Code.

Once $k_{r,mn}$ has been computed, the weighting factors A and B are assigned to one of the following two sets of values (`eigen.f`)

The remainder of the procedure is being documented along with directions on how to pass in the correct inputs to the F77 calls The estimate is refined by incrementing the value of $k_{r,mn}$ by $\pi/10$ until the determinant above changes sign. The step size is then halved and also changes sign. This iterative process continues until the absolute value of the determinant is reduced to a preassigned value (error tolerance?). The non dimensional versions of these equations are used in the FORTRAN 77 Code, `anrt.f`, which calls `anfu.f`.

Once $k_{r,mn}$ has been computed, the weighting factors A and B are assigned to one of the following two sets of values (`eigen.f`)

$$\begin{cases} A = 1 \\ B = -\frac{J'_m(k_{r,mn} r_H)}{Y'_m(k_{r,mn} r_H)} \end{cases} \quad \text{or,} \begin{cases} A = -\frac{Y'_m(k_{r,mn} r_H)}{J'_m(k_{r,mn} r_H)} \\ B = 1 \end{cases}$$

Of these two values, which ever has the smaller $A^2 + B^2$ is chosen.

The desired normalization,

$$\int_{r_T}^{r_H} R_m^2(k_{r,mn} r) r dr = \frac{1}{2}\left(r_T^2 - r_H^2\right) \tag{4}$$

is obtained by computing the value on the left-hand side of Equation (4) using the expression,

$$\int_{r_H}^{r_T} R_m^2(k_{r,mn} r) r dr = \frac{1}{2}\left(r^2 - \frac{m^2}{k_{r,mn}^2}\right) R_m^2(k_{r,mn} r)\Big|_{r=r_H}^{r_T} \equiv C \tag{5}$$

The constants A and B are divided by,

$$\sqrt{\frac{2C}{(r_T^2 - r_H^2)}} \tag{6}$$

to give the normalization needed in Equation (4)

The non-dimensionalization in this code uses the following,

$$X_{mn} = k_{r,mn} r_T$$

$$x = r/r_T$$

$$\sigma = r_H/r_T$$

The non-dimensional equivalent expressions are

$$\int_{\sigma}^{1} R_m^2(X_{mn} x) x dr = \frac{1}{2}\left(1 - \sigma^2\right) \tag{7}$$

$$\int_{\sigma}^{1} R_m^2(X_{mn}x)xdr = \frac{1}{2}\left(x^2 - \frac{m^2}{X_{mn}^2}\right) R_m^2(X_{mn}x)\Big|_{x=\sigma}^{1} \tag{8}$$

and the constant that A and B are divided by becomes

$$\sqrt{\frac{\left(1 - \frac{m^2}{X_{mn}^2}\right) R_m^2(X_{mn}) - \left(\sigma^2 - \frac{m^2}{X_{mn}^2}\right) R_m^2(X_{mn}\sigma)}{(1-\sigma^2)}} \tag{9}$$

- `eigen.f`,

  Computes the weighting factors, A and B for the radial mode shpe

- `besj.f`,

  Computes the bessel functions of the first kind and their derivatives of positive or zero order, n, and zero or positive argument x

- `besy.f`, and

  Computes the bessel function of the second kind and their derivatives of positive or zero order, n, and zero or positive argument x

- `rmode.f`.

  Calculate radial mode shape psi for the (m,n) radial mode of an annular duct

4 Issues and Concerns

The mode shapes produced do not have zero slope at the boundaries.

Currently a subdirectory within the `v070_nasalib` directory has a testing code to obtain the annular duct modes.

The current code is,

```
! Notes:
! Pros of f90 for V072:
!   better management of inputs and outputs
!       - explicit interfaces
PROGRAM BesselFunctionCode
      USE, INTRINSIC :: ISO_FORTRAN_ENV
```

```
IMPLICIT NONE
INTEGER, PARAMETER :: &
    rDef = REAL64, &
    numberOfGridPoints = 1000

INTEGER :: &
    UNIT                  ,&
    azimuthal_mode_number ,&
    radial_mode_number    ,&
    i


REAL(KIND=rDef) :: &
    r_min, &
    r_max, &
    dr,  &
    AMN, &
    BMN, &
    hubTipRatio,&
    convergence_criteria ,&
    PSI

REAL(KIND=rDef), DIMENSION(:), ALLOCATABLE :: X

REAL(KIND=rDef),DIMENSION(2) :: IERROR1, IERROR2
REAL(KIND=rDef),DIMENSION(4) :: anfu_bessel_function_error

REAL(KIND=rDef) ::  anrt_convergence_flag

REAL(KIND=rDef) :: non_dimensional_roots


ALLOCATE(X(numberOfGridPoints))

! Notes:
! r_min cant be zero

r_min = 0.20_rDef
```

```
r_max = 3.0_rDef

hubTipRatio = r_min/r_max

dr    = (r_max-r_min)/REAL(numberOfGridPoints-1, rDef)

DO i =1,numberOfGridPoints

X(i)  = (r_min+REAL(i-1, rDef)*dr)/r_max !radial grid


ENDDO


azimuthal_mode_number = 2
radial_mode_number = 1
convergence_criteria = 1.0E-5_rDef


CALL ANRT(&
    azimuthal_mode_number,&
    radial_mode_number,&
    hubTipRatio,&
    convergence_criteria,&
    non_dimensional_roots,&
    anfu_bessel_function_error,&
    anrt_convergence_flag)

WRITE(0,*) 'k_mn=' ,non_dimensional_roots

IF (anrt_convergence_flag .gt. 0.0_rDef) THEN
    WRITE(0,*) 'ERROR: ANRT DID NOT CONVERGE ',anrt_convergence_flag
ELSE
ENDIF

! STOP
! Obtaining A and B coefficients for radial mode shape
CALL EIGEN(&
```

```
        azimuthal_mode_number, &
        hubTipRatio               , &
        non_dimensional_roots                  , &
        AMN                  , &
        BMN                  , &
        IERROR1)

    ! in the case of cylindrical ducts...
    ! AMN = 1.0_rDef
    ! BMN = 0.0_rDef
    WRITE(0,*) 'A =' , AMN
    WRITE(0,*) 'B =' , BMN
    WRITE(0,*) 'non_dimensional_roots =' , non_dimensional_roots
    OPEN(NEWUNIT=UNIT,FILE='radial_mode_data.dat')
    WRITE(UNIT,*) 'radius ', 'pressure '

    DO i = 1,numberOfGridPoints

    CALL RMODE(&
    azimuthal_mode_number,&
    X(i),&
    AMN,&
    BMN,&
    PSI,&
    IERROR2)

    WRITE(UNIT,*) X(i), PSI

    ENDDO
    CLOSE(UNIT)

END PROGRAM
```

This produces the following:

Figure 1 does not have zero slope boundaries as expected for a hard wall.

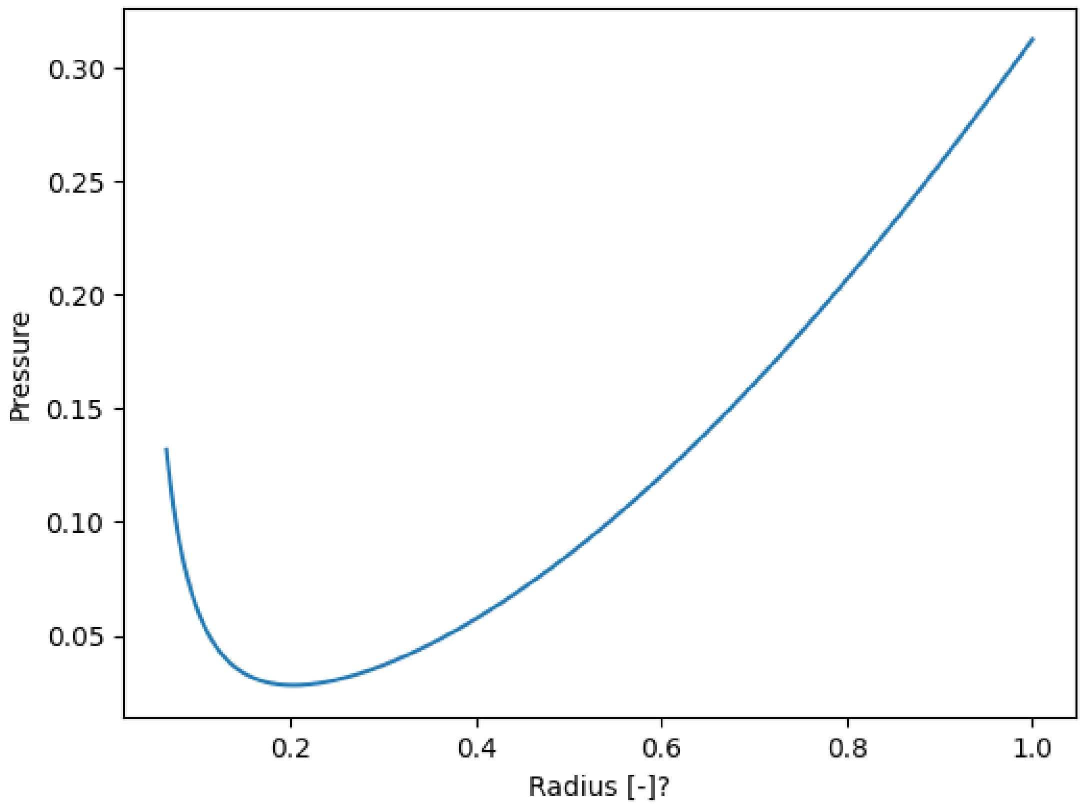

Figure 1: initial result

5 Planned Research

Come up with sanity checks to test the functionality of this testing code.

I can compare bessel function plots in python to the output in F77 to see if they are identical. . . .