1 Divergence and Vorticity?

The divergence and vorticity of the perturbation velocity in cylindrical coordinates is:

$$\begin{aligned}
\vec{\nabla}\cdot\vec{v'} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r v_r'\right) + \frac{1}{r}\frac{\partial v_\theta'}{\partial \theta} + \frac{\partial v_x'}{\partial x} \\
&= \frac{1}{r_T}\left(\frac{1}{\widetilde{r}}\frac{\partial}{\partial \widetilde{r}}\left(\widetilde{r} v_r'\right) + \frac{1}{\widetilde{r}}\frac{\partial v_\theta'}{\partial \theta} + \frac{\partial v_x'}{\partial \widetilde{x}}\right) \\
\vec{\nabla}\times\vec{v'} &= \begin{pmatrix} \left(\frac{1}{r}\frac{\partial v_x'}{\partial \theta} - \frac{\partial v_\theta'}{\partial x}\right)\vec{e}_r \\ +\left(\frac{\partial v_r'}{\partial x} - \frac{\partial v_x'}{\partial r}\right)\vec{e}_\theta \\ +\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r v_\theta'\right) - \frac{\partial v_r'}{\partial \theta}\right)\vec{e}_x \end{pmatrix} \\
&= \frac{1}{r_T}\begin{pmatrix} \left(\frac{1}{\widetilde{r}}\frac{\partial v_x'}{\partial \theta} - \frac{\partial v_\theta'}{\partial \widetilde{x}}\right)\vec{e}_r \\ +\left(\frac{\partial v_r'}{\partial \widetilde{x}} - \frac{\partial v_x'}{\partial \widetilde{r}}\right)\vec{e}_\theta \\ +\frac{1}{\widetilde{r}}\left(\frac{\partial}{\partial \widetilde{r}}\left(\widetilde{r} v_\theta'\right) - \frac{\partial v_r'}{\partial \theta}\right)\vec{e}_x \end{pmatrix}
\end{aligned}$$

Remembering the definitions:

$$\begin{aligned}
v_r' &= v_r\left(r\right)e^{i\left(k_x x + m\theta - \omega t\right)} \\
v_\theta' &= v_\theta\left(r\right)e^{i\left(k_x x + m\theta - \omega t\right)} \\
v_x' &= v_x\left(r\right)e^{i\left(k_x x + m\theta - \omega t\right)} \\
p' &= p\left(r\right)e^{i\left(k_x x + m\theta - \omega t\right)}
\end{aligned}$$

with the nondimensional counterparts:

$$\begin{aligned}
v_r' &= A\widetilde{v}_r\left(\widetilde{r}\right)e^{i\left(\overline{\gamma}\widetilde{x} + m\theta - \omega t\right)} \\
v_\theta' &= A\widetilde{v}_\theta\left(\widetilde{r}\right)e^{i\left(\overline{\gamma}\widetilde{x} + m\theta - \omega t\right)} \\
v_x' &= A\widetilde{v}_x\left(\widetilde{r}\right)e^{i\left(\overline{\gamma}\widetilde{x} + m\theta - \omega t\right)} \\
p' &= \overline{\rho}A^2\widetilde{p}\left(\widetilde{r}\right)e^{i\left(\overline{\gamma}\widetilde{x} + m\theta - \omega t\right)}
\end{aligned}$$

where:

$$\overline{\gamma} = k_x r_T$$

The axial derivatives are:

$$
\begin{aligned}
\frac{\partial v_r'}{\partial x} &= \frac{1}{r_T}\frac{\partial v_r'}{\partial \widetilde{x}} \\
&= i\overline{\gamma}\widetilde{v}_r\left(\frac{A}{r_T}e^{i\left(\overline{\gamma}\widetilde{x}+m\theta-\omega t\right)}\right) \\
\frac{\partial v_\theta'}{\partial x} &= i\overline{\gamma}\widetilde{v}_\theta\left(\frac{A}{r_T}e^{i\left(\overline{\gamma}\widetilde{x}+m\theta-\omega t\right)}\right) \\
\frac{\partial v_x'}{\partial x} &= i\overline{\gamma}\widetilde{v}_x\left(\frac{A}{r_T}e^{i\left(\overline{\gamma}\widetilde{x}+m\theta-\omega t\right)}\right)
\end{aligned}
$$

The azimuthal derivatives are:

$$
\begin{aligned}
\frac{1}{r}\frac{\partial v_r'}{\partial \theta} &= \frac{im\widetilde{v}_r}{\widetilde{r}}\left(\frac{A}{r_T}e^{i\left(\overline{\gamma}\widetilde{x}+m\theta-\omega t\right)}\right) \\
\frac{1}{r}\frac{\partial v_\theta'}{\partial \theta} &= \frac{im\widetilde{v}_\theta}{\widetilde{r}}\left(\frac{A}{r_T}e^{i\left(\overline{\gamma}\widetilde{x}+m\theta-\omega t\right)}\right) \\
\frac{1}{r}\frac{\partial v_x'}{\partial \theta} &= \frac{im\widetilde{v}_x}{\widetilde{r}}\left(\frac{A}{r_T}e^{i\left(\overline{\gamma}\widetilde{x}+m\theta-\omega t\right)}\right)
\end{aligned}
$$

To do the radial derivatives, note that:

$$
\frac{\partial A}{\partial r} = \frac{A}{r_T}\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}
$$

The radial derivatives (remembering that $A = A\,(r)!$) are:

$$
\begin{aligned}
\frac{\partial v_r'}{\partial r} &= \left(\widetilde{v_r'}\frac{\partial A}{\partial r} + \frac{A}{r_T}\frac{\partial v_r'}{\partial \widetilde{r}}\right)e^{i\left(\overline{\gamma}\widetilde{x}+m\theta-\omega t\right)} \\
&= \left(\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}\widetilde{v_r} + \frac{\partial \widetilde{v}_r}{\partial \widetilde{r}}\right)\left(\frac{A}{r_T}e^{i\left(\overline{\gamma}\widetilde{x}+m\theta-\omega t\right)}\right) \\
\frac{\partial v_\theta'}{\partial r} &= \left(\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}\widetilde{v}_\theta + \frac{\partial \widetilde{v}_\theta}{\partial \widetilde{r}}\right)\left(\frac{A}{r_T}e^{i\left(\overline{\gamma}\widetilde{x}+m\theta-\omega t\right)}\right) \\
\frac{\partial v_x'}{\partial r} &= \left(\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}\widetilde{v}_x + \frac{\partial \widetilde{v}_x}{\partial \widetilde{r}}\right)\left(\frac{A}{r_T}e^{i\left(\overline{\gamma}\widetilde{x}+m\theta-\omega t\right)}\right)
\end{aligned}
$$

gives:

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{v'} &= \left(\frac{\widetilde{v}_r}{\widetilde{r}} + \left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2\widetilde{v}_r}{\widetilde{r}} + \frac{\partial\widetilde{v}_r}{\partial\widetilde{r}} + \frac{im}{\widetilde{r}}\widetilde{v}_\theta + i\overline{\gamma}\widetilde{v}_x\right)\left(\frac{A}{r_T}e^{i\left(\overline{\gamma}\widetilde{x}+m\theta-\omega t\right)}\right)\\
\vec{\nabla}\times\vec{v'} &= \begin{pmatrix} \left(\frac{im\widetilde{v}_x}{\widetilde{r}} - i\overline{\gamma}\widetilde{v}_\theta\right)\vec{e}_r \\ +\left(i\overline{\gamma}\widetilde{v}_r - \left(\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}\widetilde{v}_x + \frac{\partial\widetilde{v}_x}{\partial\widetilde{r}}\right)\right)\vec{e}_\theta \\ +\left(\frac{\widetilde{v}_\theta}{\widetilde{r}} + \left(\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}\widetilde{v}_\theta + \frac{\partial\widetilde{v}_\theta}{\partial\widetilde{r}}\right) - \frac{im\widetilde{v}_r}{\widetilde{r}}\right)\vec{e}_x \end{pmatrix}\left(\frac{A}{r_T}e^{i\left(\overline{\gamma}\widetilde{x}+m\theta-\omega t\right)}\right)
\end{aligned}
$$

If all three vorticity components are zero, the perturbations will be irrotational. From the first two velocity components, this gives:

$$
\begin{aligned}
\widetilde{v}_\theta &= \frac{m}{\overline{\gamma}\widetilde{r}}\widetilde{v}_x\\
\widetilde{v}_r &= \frac{-i}{\overline{\gamma}}\left(\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}\widetilde{v}_x + \frac{\partial\widetilde{v}_x}{\partial\widetilde{r}}\right)
\end{aligned}
$$

As a test, the last vorticity component is:

$$
\begin{aligned}
\frac{\widetilde{v}_\theta}{\widetilde{r}} + \left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}\widetilde{v}_\theta + \frac{\partial\widetilde{v}_\theta}{\partial\widetilde{r}} - \frac{im}{\widetilde{r}}\widetilde{v}_r &= \begin{pmatrix} \frac{m}{\overline{\gamma}\widetilde{r}}\left(\frac{1}{\widetilde{r}} + \left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}\right)\widetilde{v}_x \\ +\frac{\partial}{\partial\widetilde{r}}\left(\frac{m}{\overline{\gamma}\widetilde{r}}\widetilde{v}_x\right) \\ -\frac{m}{\overline{\gamma}\widetilde{r}}\left(\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}\widetilde{v}_x + \frac{\partial\widetilde{v}_x}{\partial\widetilde{r}}\right) \end{pmatrix}\\
&= \begin{pmatrix} \frac{m}{\overline{\gamma}\widetilde{r}}\left(\frac{1}{\widetilde{r}} + \left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}\right)\widetilde{v}_x \\ -\frac{m}{\overline{\gamma}\widetilde{r}^2}\widetilde{v}_x \\ +\frac{m}{\overline{\gamma}\widetilde{r}}\frac{\partial\widetilde{v}_x}{\partial\widetilde{r}} \\ -\frac{m}{\overline{\gamma}\widetilde{r}}\left(\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}\widetilde{v}_x + \frac{\partial\widetilde{v}_x}{\partial\widetilde{r}}\right) \end{pmatrix}\\
&= 0
\end{aligned}
$$

showing that these are the irrotational relations between the perturbation velocities.

1.1 Velocity decomposition

Let's decompose the perturbation velocity field into irrotational and divergence-free components:

$$
\begin{aligned}
\widetilde{v}_x &= \widetilde{v}_{x,\omega} + \widetilde{v}_{x,\phi} \\
\widetilde{v}_r &= \widetilde{v}_{r,\omega} + \widetilde{v}_{r,\phi} \\
\widetilde{v}_\theta &= \widetilde{v}_{\theta,\omega} + \widetilde{v}_{\theta,\phi}
\end{aligned}
$$

with the relations:

$$
\begin{aligned}
\widetilde{v}_{\theta,\phi} &= \frac{m}{\overline{\gamma}\widetilde{r}}\widetilde{v}_{x,\phi} \\
\widetilde{v}_{r,\phi} &= \frac{-i}{\overline{\gamma}}\left(\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}\widetilde{v}_{x,\phi} + \frac{\partial \widetilde{v}_{x,\phi}}{\partial \widetilde{r}}\right)
\end{aligned}
$$

The velocity divergence gives one relation for the rotational components of the perturbation velocities:

$$
\begin{aligned}
\widetilde{v}_{x,\omega} &= \frac{i}{\overline{\gamma}}\left(\frac{\widetilde{v}_{r,\omega}}{\widetilde{r}} + \left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2\widetilde{v}_{r,\omega}}{\widetilde{r}} + \frac{\partial \widetilde{v}_{r,\omega}}{\partial \widetilde{r}}\right) - \frac{m}{\overline{\gamma}\widetilde{r}}\widetilde{v}_{\theta,\omega} \\
&= i\left(\frac{1+\Gamma}{\overline{\gamma}\widetilde{r}}\right)\widetilde{v}_{r,\omega} + \frac{i}{\overline{\gamma}}\frac{\partial \widetilde{v}_{r,\omega}}{\partial \widetilde{r}} - \frac{m}{\overline{\gamma}\widetilde{r}}\widetilde{v}_{\theta,\omega}
\end{aligned}
$$

As a check, the velocity perturbations are put into the velocity divergence equation:

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{v'} &= \left(\frac{\widetilde{v}_r}{\widetilde{r}} + \left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2\widetilde{v}_r}{\widetilde{r}} + \frac{\partial \widetilde{v}_r}{\partial \widetilde{r}} + \frac{im}{\widetilde{r}}\widetilde{v}_\theta + i\overline{\gamma}\widetilde{v}_x\right) \\
&= \left(\begin{array}{c} \frac{\widetilde{v}_{r,\phi}}{\widetilde{r}} + \left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2\widetilde{v}_{r,\phi}}{\widetilde{r}} + \frac{\partial \widetilde{v}_{r,\phi}}{\partial \widetilde{r}} + \frac{im}{\widetilde{r}}\widetilde{v}_{\theta,\phi} + i\overline{\gamma}\widetilde{v}_{x,\phi} \\ + \frac{\widetilde{v}_{r,\omega}}{\widetilde{r}} + \left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2\widetilde{v}_{r,\omega}}{\widetilde{r}} + \frac{\partial \widetilde{v}_{r,\omega}}{\partial \widetilde{r}} + \frac{im}{\widetilde{r}}\widetilde{v}_{\theta,\omega} + i\overline{\gamma}\widetilde{v}_{x,\omega} \end{array}\right)
\end{aligned}
$$

$$
= \left(\begin{array}{r}
\frac{1}{\widetilde{r}}\left(1+\left(\frac{\gamma-1}{2}\right)M_\theta^2\right)\frac{-i}{\overline{\gamma}}\left(\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}\widetilde{v}_{x,\phi}+\frac{\partial\widetilde{v}_{x,\phi}}{\partial\widetilde{r}}\right) \\
+\frac{\partial}{\partial\widetilde{r}}\left(\frac{-i}{\overline{\gamma}}\left(\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}\widetilde{v}_{x,\phi}+\frac{\partial\widetilde{v}_{x,\phi}}{\partial\widetilde{r}}\right)\right) \\
+\frac{im^2}{\overline{\gamma}\widetilde{r}^2}\widetilde{v}_{x,\phi} \\
+i\overline{\gamma}\widetilde{v}_{x,\phi} \\
+\frac{\widetilde{v}_{r,\omega}}{\widetilde{r}}+\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2\widetilde{v}_{r,\omega}}{\widetilde{r}}+\frac{\partial\widetilde{v}_{r,\omega}}{\partial\widetilde{r}}+\frac{im}{\widetilde{r}}\widetilde{v}_{\theta,\omega} \\
+i\overline{\gamma}\left(\frac{i}{\overline{\gamma}}\left(\frac{\widetilde{v}_{r,\omega}}{\widetilde{r}}+\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2\widetilde{v}_{r,\omega}}{\widetilde{r}}+\frac{\partial\widetilde{v}_{r,\omega}}{\partial\widetilde{r}}\right)-\frac{m}{\overline{\gamma}\widetilde{r}}\widetilde{v}_{\theta,\omega}\right)
\end{array}\right)
$$

$$
= \left(\begin{array}{r}
\frac{-i}{\overline{\gamma}\widetilde{r}^2}\left(1+\left(\frac{\gamma-1}{2}\right)M_\theta^2\right)\left(\left(\frac{\gamma-1}{2}\right)M_\theta^2\right)\widetilde{v}_{x,\phi} \\
-\frac{i}{\overline{\gamma}\widetilde{r}}\left(1+\left(\frac{\gamma-1}{2}\right)M_\theta^2\right)\left(\frac{\partial\widetilde{v}_{x,\phi}}{\partial\widetilde{r}}\right) \\
-\frac{i}{\overline{\gamma}}\frac{\partial}{\partial\widetilde{r}}\left(\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}\widetilde{v}_{x,\phi}\right) \\
-\frac{i}{\overline{\gamma}}\frac{\partial^2\widetilde{v}_{x,\phi}}{\partial\widetilde{r}^2} \\
+i\overline{\gamma}\left(1+\frac{m^2}{\overline{\gamma}^2\widetilde{r}^2}\right)\widetilde{v}_{x,\phi} \\
+\frac{\widetilde{v}_{r,\omega}}{\widetilde{r}}+\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2\widetilde{v}_{r,\omega}}{\widetilde{r}}+\frac{\partial\widetilde{v}_{r,\omega}}{\partial\widetilde{r}}+\frac{im}{\widetilde{r}}\widetilde{v}_{\theta,\omega} \\
-\left(\frac{\widetilde{v}_{r,\omega}}{\widetilde{r}}+\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2\widetilde{v}_{r,\omega}}{\widetilde{r}}+\frac{\partial\widetilde{v}_{r,\omega}}{\partial\widetilde{r}}\right)-\frac{im}{\widetilde{r}}\widetilde{v}_{\theta,\omega}
\end{array}\right)
$$

$$
= \left(\begin{array}{r}
i\overline{\gamma}\left(\left(1+\frac{m^2}{\overline{\gamma}^2\widetilde{r}^2}\right)-\frac{1}{\overline{\gamma}^2\widetilde{r}^2}\left(1+\left(\frac{\gamma-1}{2}\right)M_\theta^2\right)\left(\left(\frac{\gamma-1}{2}\right)M_\theta^2\right)\right)\widetilde{v}_{x,\phi} \\
-\frac{i}{\overline{\gamma}\widetilde{r}}\left(1+\left(\frac{\gamma-1}{2}\right)M_\theta^2\right)\left(\frac{\partial\widetilde{v}_{x,\phi}}{\partial\widetilde{r}}\right) \\
+\frac{i}{\overline{\gamma}}\left(\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2}{\widetilde{r}^2}\widetilde{v}_{x,\phi}\right) \\
-\frac{i}{\overline{\gamma}}\frac{\partial M_\theta}{\partial\widetilde{r}}\left(\left(\frac{\gamma-1}{2}\right)\frac{2M_\theta}{\widetilde{r}}\widetilde{v}_{x,\phi}\right) \\
-\frac{i}{\overline{\gamma}\widetilde{r}}\left(\left(\frac{\gamma-1}{2}\right)M_\theta^2\frac{\partial\widetilde{v}_{x,\phi}}{\partial\widetilde{r}}\right) \\
-\frac{i}{\overline{\gamma}}\frac{\partial^2\widetilde{v}_{x,\phi}}{\partial\widetilde{r}^2}
\end{array}\right)
$$

$$
= \left(\begin{array}{r}
i\left(\overline{\gamma}+\frac{1}{\overline{\gamma}\widetilde{r}^2}\left(m^2-\Gamma^2-\widetilde{r}\frac{\partial\Gamma}{\partial\widetilde{r}}\right)\right)\widetilde{v}_{x,\phi} \\
-\frac{i}{\overline{\gamma}\widetilde{r}}\left(1+2\Gamma\right)\frac{\partial\widetilde{v}_{x,\phi}}{\partial\widetilde{r}} \\
-\frac{i}{\overline{\gamma}}\frac{\partial^2\widetilde{v}_{x,\phi}}{\partial\widetilde{r}^2}
\end{array}\right)
$$

where

$$
\Gamma = \left(\frac{\gamma-1}{2}\right)M_\theta^2
$$

Note that the divergence of velocity can be written solely in terms of $\widetilde{v}_{x,\phi}$:

$$\left(\frac{\widetilde{v}_r}{\widetilde{r}}+\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^2\widetilde{v}_r}{\widetilde{r}}+\frac{\partial\widetilde{v}_r}{\partial\widetilde{r}}+\frac{im}{\widetilde{r}}\widetilde{v}_\theta+i\overline{\gamma}\widetilde{v}_x\right) = \Phi_0\widetilde{v}_{x,\phi}+\Phi_1\frac{\partial\widetilde{v}_{x,\phi}}{\partial\widetilde{r}}+\Phi_2\frac{\partial^2\widetilde{v}_{x,\phi}}{\partial\widetilde{r}^2}$$

where

$$\begin{aligned}
\Phi_0 &= i\left(\overline{\gamma}+\frac{1}{\overline{\gamma}\widetilde{r}^2}\left(m^2-\Gamma^2-\widetilde{r}\frac{\partial\Gamma}{\partial\widetilde{r}}\right)\right)\\
\Phi_1 &= -\frac{i}{\overline{\gamma}\widetilde{r}}\left(1+2\Gamma\right)\\
\Phi_2 &= -\frac{i}{\overline{\gamma}}
\end{aligned}$$

1.2 Back to business...

The SWIRL code equations are:

$$\begin{aligned}
-i\left(\frac{k}{\widetilde{A}}-\frac{mM_\theta}{\widetilde{r}}-\overline{\gamma}M_x\right)\widetilde{v}_r-\frac{2M_\theta}{\widetilde{r}}\widetilde{v}_\theta = -\frac{\partial\widetilde{p}}{\partial\widetilde{r}}-\frac{(\gamma-1)\,M_\theta^2}{\widetilde{r}}\widetilde{p}\\
-i\left(\frac{k}{\widetilde{A}}-\frac{mM_\theta}{\widetilde{r}}-\overline{\gamma}M_x\right)\widetilde{v}_\theta+\left(\frac{M_\theta}{\widetilde{r}}+\frac{\partial M_\theta}{\partial\widetilde{r}}+\left(\frac{\gamma-1}{2}\right)\frac{M_\theta^3}{\widetilde{r}}\right)\widetilde{v}_r = -\frac{im}{\widetilde{r}}\widetilde{p}\\
-i\left(\frac{k}{\widetilde{A}}-\frac{mM_\theta}{\widetilde{r}}-\overline{\gamma}M_x\right)\widetilde{v}_x+\left(\frac{\partial M_x}{\partial\widetilde{r}}+\left(\frac{\gamma-1}{2}\right)\frac{M_xM_\theta^2}{\widetilde{r}}\right)\widetilde{v}_r = -i\overline{\gamma}\widetilde{p}\\
-i\left(\frac{k}{\widetilde{A}}-\frac{mM_\theta}{\widetilde{r}}-\overline{\gamma}M_x\right)\widetilde{p}+\frac{\partial\widetilde{v}_r}{\partial\widetilde{r}}+\left(\left(\frac{\gamma+1}{2}\right)\frac{M_\theta^2}{\widetilde{r}}+\frac{1}{\widetilde{r}}\right)\widetilde{v}_r+\frac{im}{\widetilde{r}}\widetilde{v}_\theta+i\overline{\gamma}\widetilde{v}_x = 0
\end{aligned}$$

Defining:

$$\alpha = \frac{k}{\widetilde{A}}-\frac{mM_\theta}{\widetilde{r}}-\overline{\gamma}M_x$$

and expanding the pressure equation to isolate the divergence of velocity term gives:

$$-i\alpha\widetilde{v}_r-\frac{2M_\theta}{\widetilde{r}}\widetilde{v}_\theta = -\frac{\partial\widetilde{p}}{\partial\widetilde{r}}-\frac{(\gamma-1)\,M_\theta^2}{\widetilde{r}}\widetilde{p}$$

$$
\begin{aligned}
-i\alpha\widetilde{v}_\theta + \left(\frac{M_\theta}{\widetilde{r}} + \frac{\partial M_\theta}{\partial \widetilde{r}} + \left(\frac{\gamma-1}{2}\right)\frac{M_\theta^3}{\widetilde{r}}\right)\widetilde{v}_r &= -\frac{im}{\widetilde{r}}\widetilde{p} \\
-i\alpha\widetilde{v}_x + \left(\frac{\partial M_x}{\partial \widetilde{r}} + \left(\frac{\gamma-1}{2}\right)\frac{M_x M_\theta^2}{\widetilde{r}}\right)\widetilde{v}_r &= -i\overline{\gamma}\widetilde{p} \\
-i\alpha\widetilde{p} + \frac{M_\theta^2}{\widetilde{r}}\widetilde{v}_r + \left(\Phi_0\widetilde{v}_{x,\phi} + \Phi_1\frac{\partial \widetilde{v}_{x,\phi}}{\partial \widetilde{r}} + \Phi_2\frac{\partial^2 \widetilde{v}_{x,\phi}}{\partial \widetilde{r}^2}\right) &= 0
\end{aligned}
$$

Rewriting this in terms of the rotational and irrotational velocities,

$$
\begin{aligned}
-i\alpha\left(\widetilde{v}_{r,\phi} + \widetilde{v}_{r,\omega}\right) - \frac{2M_\theta}{\widetilde{r}}\left(\widetilde{v}_{\theta,\phi} + \widetilde{v}_{\theta,\omega}\right) &= -\frac{\partial \widetilde{p}}{\partial \widetilde{r}} - \frac{2\Gamma}{\widetilde{r}}\widetilde{p} \\
-i\alpha\left(\widetilde{v}_{\theta,\phi} + \widetilde{v}_{\theta,\omega}\right) + \left((1+\Gamma)\frac{M_\theta}{\widetilde{r}} + \frac{\partial M_\theta}{\partial \widetilde{r}}\right)\left(\widetilde{v}_{r,\phi} + \widetilde{v}_{r,\omega}\right) &= -\frac{im}{\widetilde{r}}\widetilde{p} \\
-i\alpha\left(\widetilde{v}_{x,\phi} + \widetilde{v}_{x,\omega}\right) + \left(\frac{\partial M_x}{\partial \widetilde{r}} + \Gamma\frac{M_x}{\widetilde{r}}\right)\left(\widetilde{v}_{r,\phi} + \widetilde{v}_{r,\omega}\right) &= -i\overline{\gamma}\widetilde{p} \\
-i\alpha\widetilde{p} + \frac{M_\theta^2}{\widetilde{r}}\left(\widetilde{v}_{r,\phi} + \widetilde{v}_{r,\omega}\right) + \left(\Phi_0\widetilde{v}_{x,\phi} + \Phi_1\frac{\partial \widetilde{v}_{x,\phi}}{\partial \widetilde{r}} + \Phi_2\frac{\partial^2 \widetilde{v}_{x,\phi}}{\partial \widetilde{r}^2}\right) &= 0
\end{aligned}
$$

Substituting in:

$$
\begin{aligned}
\left(\begin{array}{r} -i\alpha\left(\frac{-i}{\overline{\gamma}}\left(\frac{\Gamma}{\widetilde{r}}\widetilde{v}_{x,\phi} + \frac{\partial \widetilde{v}_{x,\phi}}{\partial \widetilde{r}}\right) + \widetilde{v}_{r,\omega}\right) \\ -\frac{2M_\theta}{\widetilde{r}}\left(\frac{m}{\overline{\gamma}\widetilde{r}}\widetilde{v}_{x,\phi} + \widetilde{v}_{\theta,\omega}\right) \end{array}\right) &= -\frac{\partial \widetilde{p}}{\partial \widetilde{r}} - \frac{2\Gamma}{\widetilde{r}}\widetilde{p} \\
\left(\begin{array}{r} -i\alpha\left(\frac{m}{\overline{\gamma}\widetilde{r}}\widetilde{v}_{x,\phi} + \widetilde{v}_{\theta,\omega}\right) \\ +\left((1+\Gamma)\frac{M_\theta}{\widetilde{r}} + \frac{\partial M_\theta}{\partial \widetilde{r}}\right)\left(\frac{-i}{\overline{\gamma}}\left(\frac{\Gamma}{\widetilde{r}}\widetilde{v}_{x,\phi} + \frac{\partial \widetilde{v}_{x,\phi}}{\partial \widetilde{r}}\right) + \widetilde{v}_{r,\omega}\right) \end{array}\right) &= -\frac{im}{\widetilde{r}}\widetilde{p} \\
\left(\begin{array}{r} -i\alpha\left(\widetilde{v}_{x,\phi} + i\left(\frac{1+\Gamma}{\overline{\gamma}\widetilde{r}}\right)\widetilde{v}_{r,\omega} + \frac{i}{\overline{\gamma}}\frac{\partial \widetilde{v}_{r,\omega}}{\partial \widetilde{r}} - \frac{m}{\overline{\gamma}\widetilde{r}}\widetilde{v}_{\theta,\omega}\right) \\ +\left(\frac{\partial M_x}{\partial \widetilde{r}} + \Gamma\frac{M_x}{\widetilde{r}}\right)\left(\frac{-i}{\overline{\gamma}}\left(\frac{\Gamma}{\widetilde{r}}\widetilde{v}_{x,\phi} + \frac{\partial \widetilde{v}_{x,\phi}}{\partial \widetilde{r}}\right) + \widetilde{v}_{r,\omega}\right) \end{array}\right) &= -i\overline{\gamma}\widetilde{p} \\
\left(\begin{array}{r} -i\alpha\widetilde{p} \\ +\frac{M_\theta^2}{\widetilde{r}}\left(\frac{-i}{\overline{\gamma}}\left(\frac{\Gamma}{\widetilde{r}}\widetilde{v}_{x,\phi} + \frac{\partial \widetilde{v}_{x,\phi}}{\partial \widetilde{r}}\right) + \widetilde{v}_{r,\omega}\right) \\ +\left(\Phi_0\widetilde{v}_{x,\phi} + \Phi_1\frac{\partial \widetilde{v}_{x,\phi}}{\partial \widetilde{r}} + \Phi_2\frac{\partial^2 \widetilde{v}_{x,\phi}}{\partial \widetilde{r}^2}\right) \end{array}\right) &= 0
\end{aligned}
$$

Defining:

$$
\begin{aligned}
\Theta &= (1+\Gamma)\frac{M_\theta}{\widetilde{r}} + \frac{\partial M_\theta}{\partial \widetilde{r}} \\
\tau_\phi &= \frac{\Gamma}{\widetilde{r}}\widetilde{v}_{x,\phi} + \frac{\partial \widetilde{v}_{x,\phi}}{\partial \widetilde{r}} \\
\tau_\omega &= \frac{\Gamma}{\widetilde{r}}\widetilde{v}_{r,\omega} + \frac{\partial \widetilde{v}_{r,\omega}}{\partial \widetilde{r}} \\
S &= \Gamma\frac{M_x}{\widetilde{r}} + \frac{\partial M_x}{\partial \widetilde{r}}
\end{aligned}
$$

and gathering the rotational and irrotational components together:

$$
\begin{aligned}
\left(\begin{array}{r} \frac{\alpha}{\overline{\gamma}}\tau_\phi - \frac{2mM_\theta}{\overline{\gamma}\widetilde{r}^2}\widetilde{v}_{x,\phi} \\ -i\alpha\widetilde{v}_{r,\omega} - \frac{2M_\theta}{\widetilde{r}}\widetilde{v}_{\theta,\omega} \end{array}\right) &= -\frac{\partial \widetilde{p}}{\partial \widetilde{r}} - \frac{2\Gamma}{\widetilde{r}}\widetilde{p} \\
\left(\begin{array}{r} -\frac{i\Theta}{\overline{\gamma}}\tau_\phi - \frac{im\alpha}{\overline{\gamma}\widetilde{r}}\widetilde{v}_{x,\phi} \\ +\Theta\widetilde{v}_{r,\omega} - i\alpha\widetilde{v}_{\theta,\omega} \end{array}\right) &= -\frac{im}{\widetilde{r}}\widetilde{p} \\
\left(\begin{array}{r} -i\alpha\widetilde{v}_{x,\phi} + \frac{\alpha}{\overline{\gamma}}\tau_\omega + \frac{\alpha}{\overline{\gamma}\widetilde{r}}\widetilde{v}_{r,\omega} + i\frac{m\alpha}{\overline{\gamma}\widetilde{r}}\widetilde{v}_{\theta,\omega} \\ -i\frac{S}{\overline{\gamma}}\tau_\phi + S\widetilde{v}_{r,\omega} \end{array}\right) &= -i\overline{\gamma}\widetilde{p} \\
\left(\begin{array}{r} -i\alpha\widetilde{p} \\ -i\frac{M_\theta^2}{\overline{\gamma}\widetilde{r}}\tau_\phi \\ +\Phi_0\widetilde{v}_{x,\phi} + \Phi_1\frac{\partial\widetilde{v}_{x,\phi}}{\partial\widetilde{r}} + \Phi_2\frac{\partial^2\widetilde{v}_{x,\phi}}{\partial\widetilde{r}^2} \\ -i\frac{M_\theta^2}{\widetilde{r}}\widetilde{v}_{r,\omega} \end{array}\right) &= 0
\end{aligned}
$$

I can't help but notice these groupings:

$$
\begin{aligned}
A &= \frac{1}{\overline{\gamma}}\tau_\phi - i\widetilde{v}_{r,\omega} \\
B &= \frac{m}{\overline{\gamma}\widetilde{r}}\widetilde{v}_{x,\phi} + \widetilde{v}_{\theta,\omega}
\end{aligned}
$$

which gives:

$$
\begin{aligned}
\alpha A - \frac{2M_\theta}{\widetilde{r}}B &= -\frac{\partial \widetilde{p}}{\partial \widetilde{r}} - \frac{2\Gamma}{\widetilde{r}}\widetilde{p} \\
i\Theta A - i\alpha B &= -\frac{im}{\widetilde{r}}\widetilde{p}
\end{aligned}
$$

$$\begin{pmatrix} -i\alpha\widetilde{v}_{x,\phi} + \frac{\alpha}{\overline{\gamma}}\tau_\omega + \frac{\alpha}{\overline{\gamma}\widetilde{r}}\widetilde{v}_{r,\omega} + i\frac{m\alpha}{\overline{\gamma}\widetilde{r}}\widetilde{v}_{\theta,\omega} \\ -iSA \end{pmatrix} = -i\overline{\gamma}\widetilde{p}$$

$$\begin{pmatrix} -i\alpha\widetilde{p} \\ -i\frac{M_\theta^2}{\widetilde{r}}A \\ +\Phi_0\widetilde{v}_{x,\phi} + \Phi_1\frac{\partial\widetilde{v}_{x,\phi}}{\partial\widetilde{r}} + \Phi_2\frac{\partial^2\widetilde{v}_{x,\phi}}{\partial\widetilde{r}^2} \end{pmatrix} = 0$$

I can't help but notice these groupings: