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Scale Invariance in Landscape Evolution Models Using Stream Power Laws

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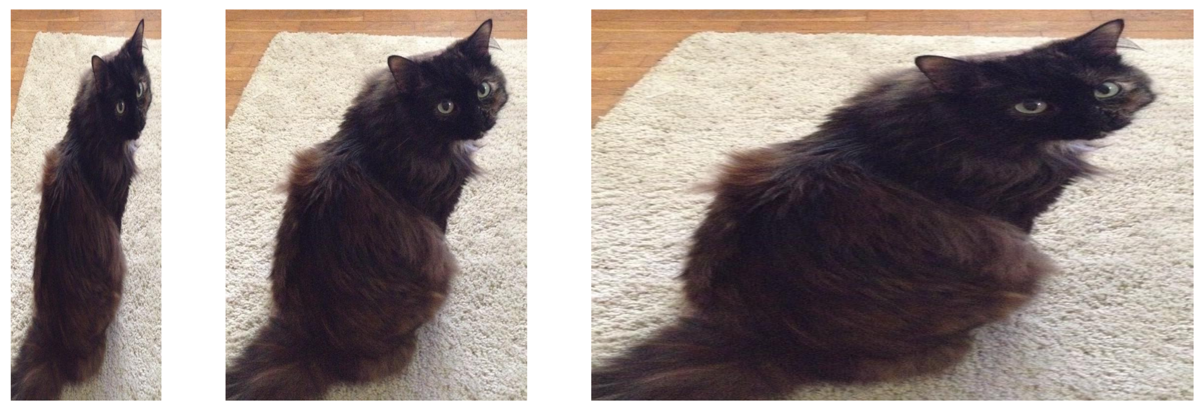
Scale Invariance in Domestic Cats

It is well documented that domestic cats (*Felis catus*) increase in horizontal scale without change in vertical scale. Cats get fatter but never taller.



In this poster, we see that numerical landscapes can exhibit the very same phenomenon.

2D Landscape Evolution Model (LEM)

Our landscape evolution model is a detachment limited model (*Figure 1*), written as:

$$\frac{\partial \eta}{\partial t} = \upsilon - \varepsilon$$

where η [L] is elevation, t [T] is time, υ [L/T] is uplift rate, and ε [L/T] is incision rate. Incision is modeled using a steam power incision model where it is assumed to be dependent on bed shear stress:

$$\varepsilon = \alpha_e A^m S^n$$

where α_e [$T^{-1}L^{1-2m}$] is an erodibility constant, A [L^2] is the drainage area, S [-] is slope, m [-] and n [-] are positive exponents. The 2D model is written as:

$$\frac{\partial \eta}{\partial t} = \upsilon - \alpha_e A^m \left[\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 \right]^{\frac{n}{2}}$$

where x and y are the horizontal coordinates. When the horizontal variables are made dimensionless by the length of one side of the basin, L,

$$x = L\hat{x} \quad y = L\hat{y} \quad A = L^2 \hat{A}$$

our governing equation with dimensionless horizontal variables is

$$\frac{\partial \eta}{\partial t} = \upsilon - \alpha_e L^{2m-n} \hat{A}^m \left[\left(\frac{\partial \eta}{\partial \hat{x}}\right)^2 + \left(\frac{\partial \eta}{\partial \hat{y}}\right)^2 \right]^{\frac{n}{2}}$$

The term, L^{2m-n}, is the only term that accounts for the horizontal scale in equation above. The solution dependency of the governing equations on the horizontal scale is set by the relationship between the exponents, m and n. When $2m < n$, the relief of the landscape increases as L increases; when $2m = n$, the relief is invariant to L; and when $2m > n$, the relief decreases as L increases (*Figure 2*).

1D Landscape Evolution Model (LEM)

Our 1D numerical model is similarly a detachment limited model (*Figure 1*), written as:

$$\frac{\partial \eta}{\partial t} = \upsilon - \alpha_e A^m S^n$$

Unlike the 2D model, the 1D model has an analytical form for the drainage area function,

$$A = Bx$$

where B [L] is the profile width. The governing equation with boundary conditions where the ridge is at $x = 0$ and the outlet ($\eta = 0$) is at $x = L$, is

$$\frac{\partial \eta}{\partial t} = \upsilon - \alpha_e B^m x^m \left(-\frac{\partial \eta}{\partial x}\right)^n$$

At steady state ($\partial\eta/\partial t = 0$), the elevation profile has an analytical solution, which is shown below with the horizontal scale non-dimensionalized by L (*Figure 3*).

$$\eta = \begin{cases} -\left(\dfrac{\upsilon}{\alpha_e B^m}\right)^{\frac{1}{n}} \ln(\hat{x}) & \text{for } m = n \\ \dfrac{1}{\frac{m}{n} - 1}\left(\dfrac{\upsilon}{\alpha_e L^{m-n} B^m}\right)^{\frac{1}{n}} \left(\hat{x}^{-\frac{m}{n}+1} - 1\right) & \text{for } m \neq n \end{cases}$$

In the 1D model, the elevation profile becomes invariant of the horizontal scale when $m = n$. When $m < n$, the relief of the landscape increases as L increases; and when $m > n$, the relief decreases as L increases.

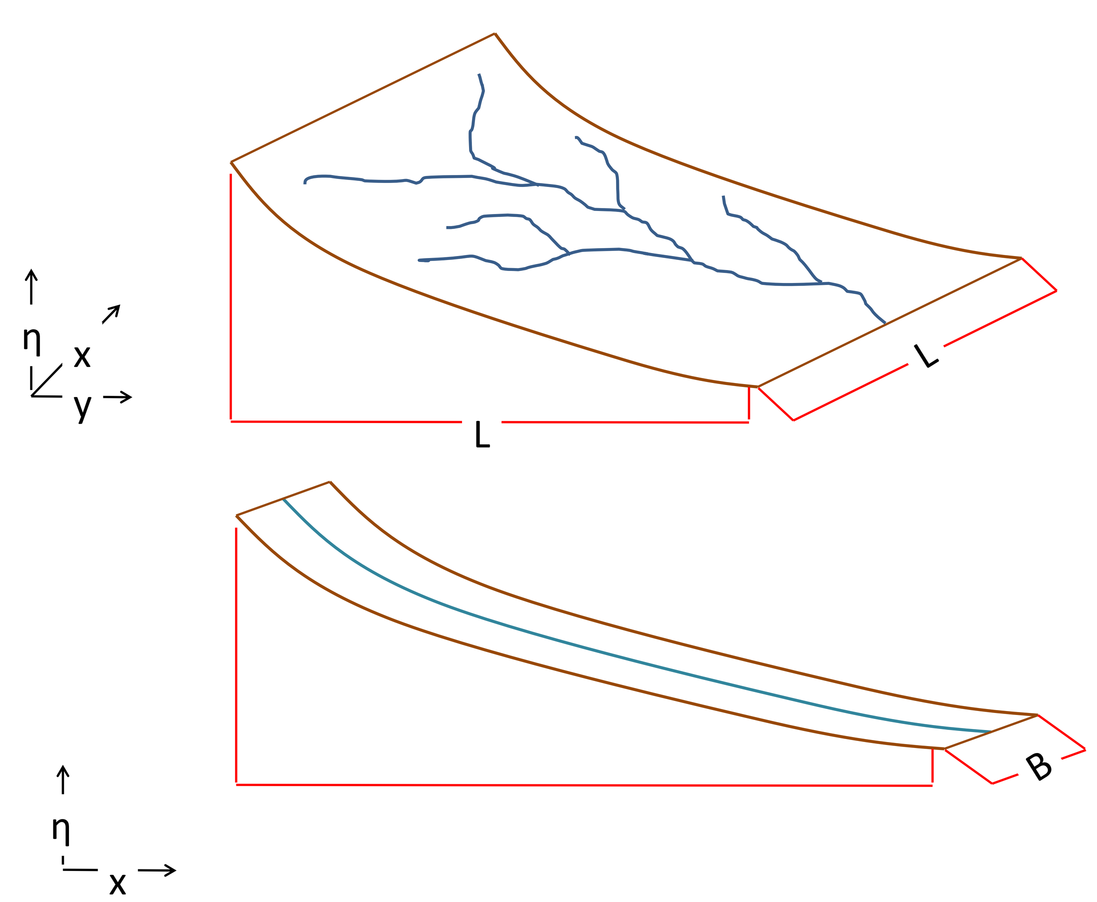


Figure 1: Schematic of the 2d numerical (top) and 1D analytical (top) landscape evolution model. Coordinate system is shown to the bottom left of each schematic.

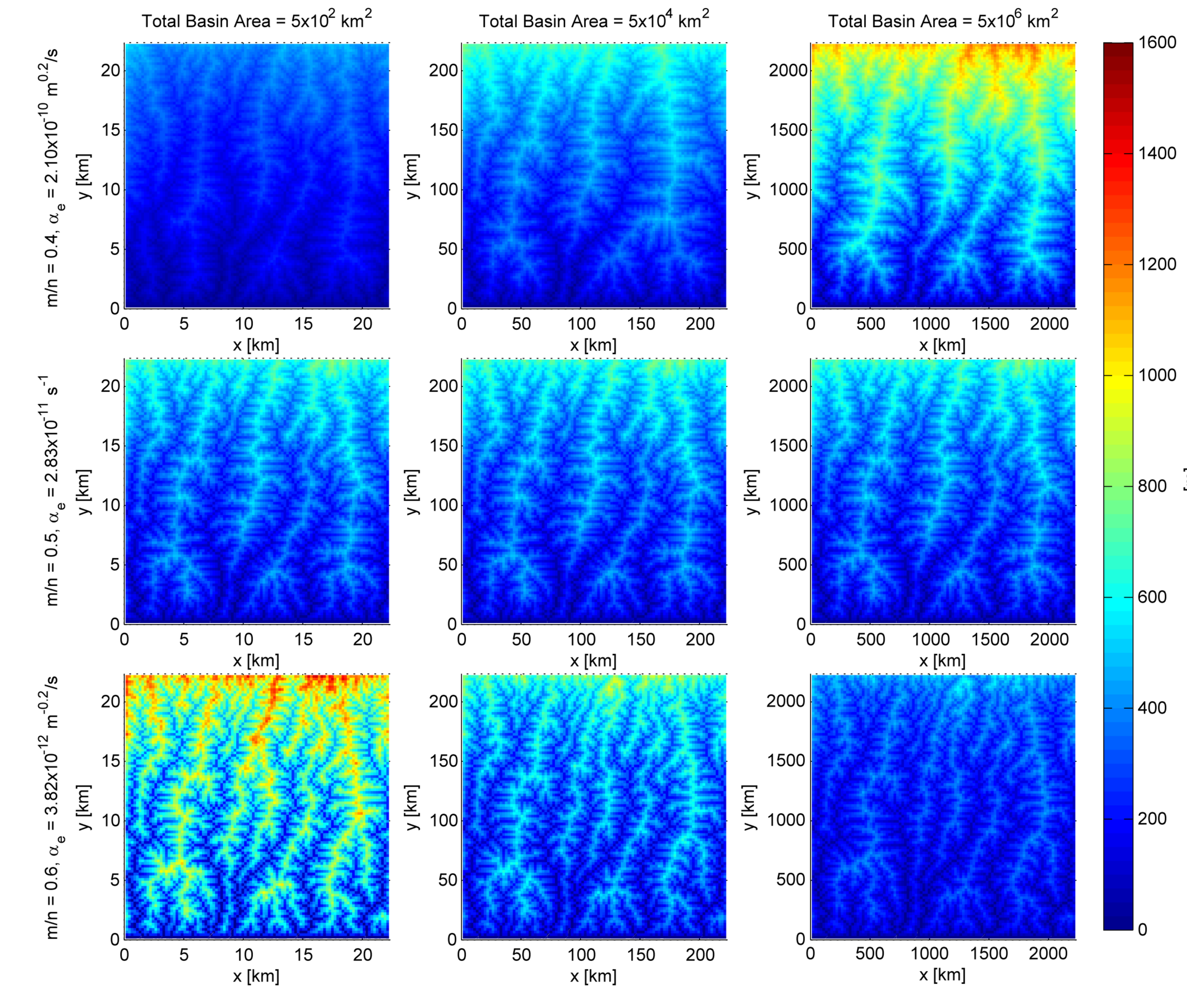


Figure 2: 2d numerical model results with m/n ratios equal to 0.4 (top row), 0.5 (middle row), and 0.6 (bottom row) and basin sizes equal to 500 km² (left column), 50,000 km² (middle column), and 5,000,000 km² (right column). The vertical scale is the same for all cases (see color bar).

Singularities

The 1D analytical steady state solution shows a singularity in elevation at the ridge ($x = 0$) when $m \geq n$ (*Figure 3*). For the 2D model, we can rewrite the model as:

$$\frac{\partial \eta}{\partial t} = \upsilon - \alpha_e A^m \frac{\partial \eta}{\partial s}^n$$

where s is the streamwise direction from the ridge. When approaching the ridge ($s \to 0$), $A \propto s^2$, and therefore, at the ridge, the equation becomes

$$\frac{\partial \eta}{\partial t} = \upsilon - \alpha_e s^{2m} \frac{\partial \eta}{\partial s}^n$$

If we non-dimensionalize s by L, the analytical solution at the ridge at equilibrium ($\partial\eta/\partial t = 0$) is

$$\eta = \begin{cases} -\left(\dfrac{\upsilon}{\alpha_e B^m}\right)^{\frac{1}{n}} \ln(\hat{s}) & \text{for } 2m = n \\ \dfrac{1}{\frac{2m}{n} - 1}\left(\dfrac{\upsilon}{\alpha_e L^{2m-n} B^m}\right)^{\frac{1}{n}} \left(\hat{s}^{1-\frac{2m}{n}} - 1\right) & \text{for } 2m \neq n \end{cases}$$

This equation is in a similar form as the 1D analytical solution, and therefore, the 2D numerical model also shows a singularity in elevation at the ridge, but in this case when $2m \geq n$ (*Figure 4 & 5*).

Conclusions

In landscape evolution models that incorporate stream power laws to model incision, we see:

1. The values m and n are responsible for the shape of the river profile and the dependency of the elevations on the horizontal scale.
2. When $m = n$ (1D) and $2m = n$ (2D), the landscape's elevations are invariant to the horizontal scale.
3. There are elevation singularities at the ridges when $m \geq n$ (1D) and $2m \geq n$ (2D).

Parameters

2D model (Figure 2)	1D model (Figure 3)
υ = 4 mm/yr	υ = 4 mm/yr
Grid = 100 x 100 cells	α_e = 1.0 x 10^{-10} m^{1-2m}/s
2D model (Figure 4 & 5)	L = 1000 m
υ = 3 mm/yr	B = 1 m
L^2 = 500 km²	

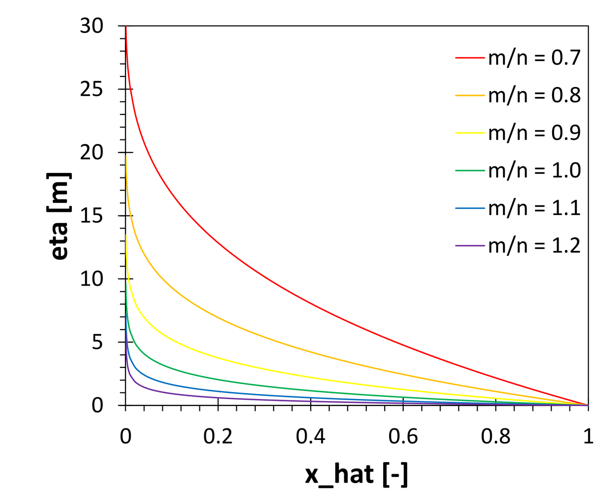


Figure 3: 1D analytical model results with varying ratios of m/n (0.7 to 1.2) plotted with a dimensionless horizontal scale, x_hat. The warm colors signify smaller ratios.

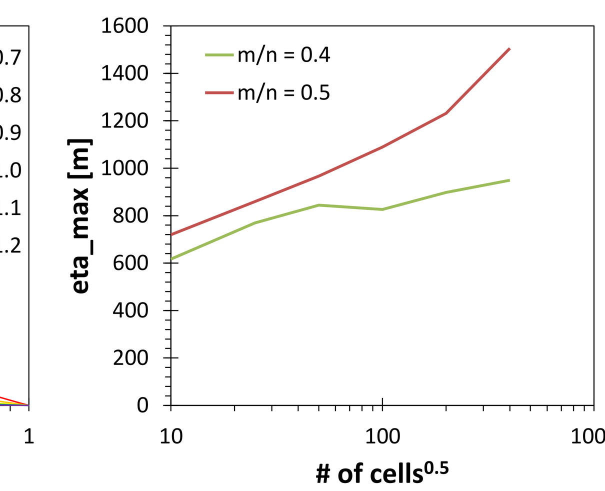


Figure 4: Maximum elevation values for ratios , $m/n = 0.4$ (green) & 0.5 (red), for different resolutions in the 2D numerical model.

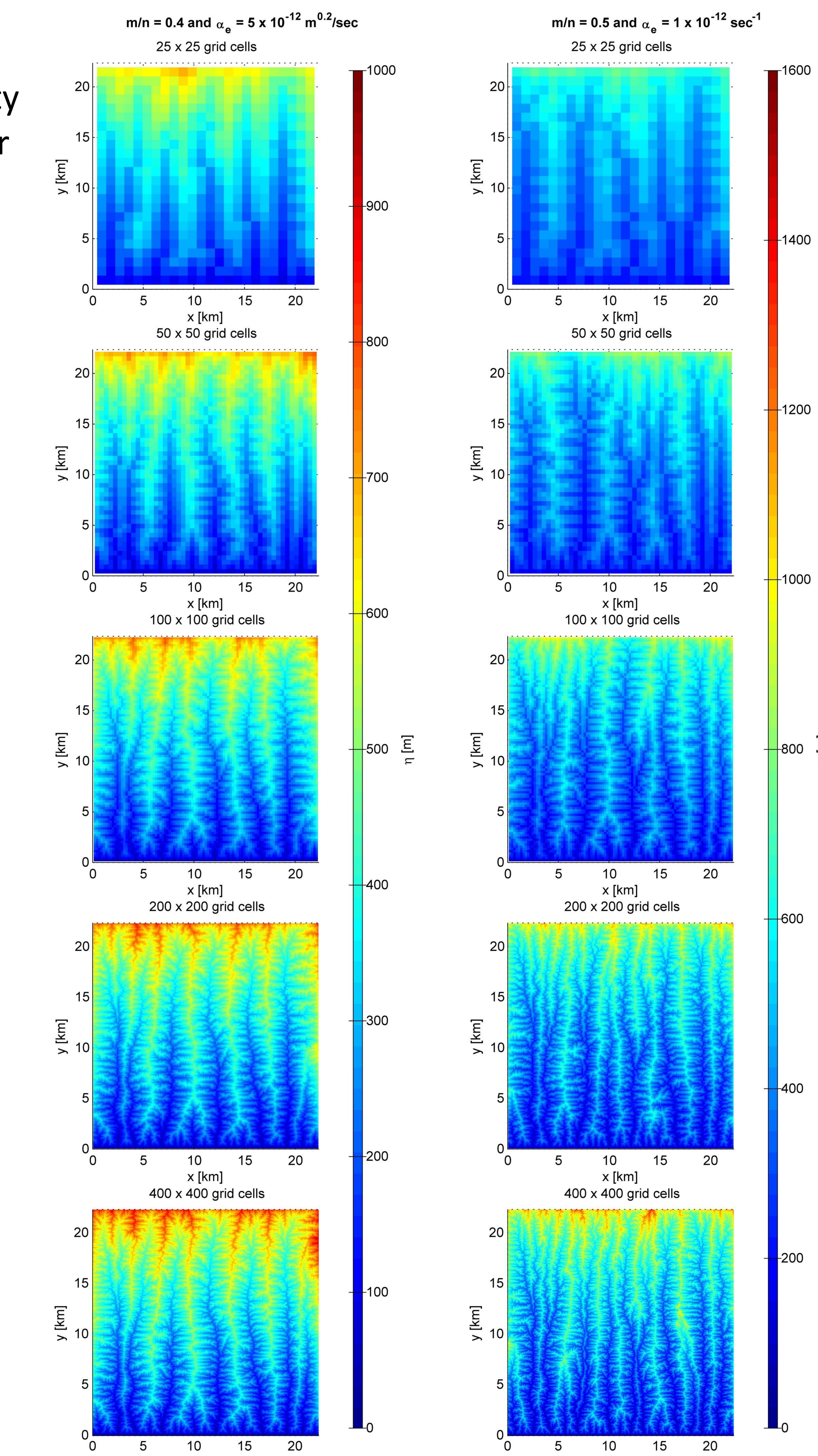


Figure 5: 2D numerical model results with m/n ratios equal to 0.4 (left column) and 0.5 (right column) and number of grid cells in ascending order. The vertical scales for the $m/n = 0.4$ case and $m/n = 0.5$ case are different.

Acknowledgments

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