Homework2

Jeff Nguyen

06/11/2020

Pre-MFE Probability Seminar
Baruch College, Fall 2020

Homework 02
Student Name: Ngoc Son (Jeff) Nguyen

Question 1

Because X, Y are independent random variables:

$$P(Y > 11 \mid X > 12) = P(Y > 11) = \frac{15}{22} \tag{1}$$

Thus, based on the property of distribution function of a random variable:

$$\begin{aligned} P(Y \leqslant 11) &= 1 - P(Y > 11) \\ &= 1 - \frac{15}{22} \\ &= \frac{7}{22} \end{aligned} \tag{2}$$

Furthermore, based on the definition of the cumulative distribution fucntion of Y, we have:

$$P(Y \leqslant 11) = \sum_{y_0 < y} P(Y = y_0)$$

or in this case:

$$\begin{aligned} \frac{7}{22} &= P(Y = 10) + P(Y = 11) \\ &= P(Y = 10) + \frac{2}{11} \\ \Rightarrow P(Y = 10) &= \frac{7}{22} - \frac{2}{11} \\ &= \frac{3}{22} \end{aligned} \tag{3}$$

With the given set $\omega = \{10, 11, 12, ..., 110\}$, we have $P(X + Y = 20) = P(\{\omega : X(\omega) + Y(\omega) = 20\})$. We can see that the only possible scenario for $\{\omega : X(\omega) + Y(\omega) = 20\}$ is $\omega : \{X(\omega) = 10, Y(\omega) = 10\}$. Thus:

$$
\begin{aligned}
P(X+Y=20) &= P(X=10, Y=10) \\
&= P(X=10) * P(Y=10) \quad \text{(Properties of independence random variables)} \\
&= \frac{3}{11} * \frac{3}{22} \\
&= \frac{9}{242}
\end{aligned}
$$

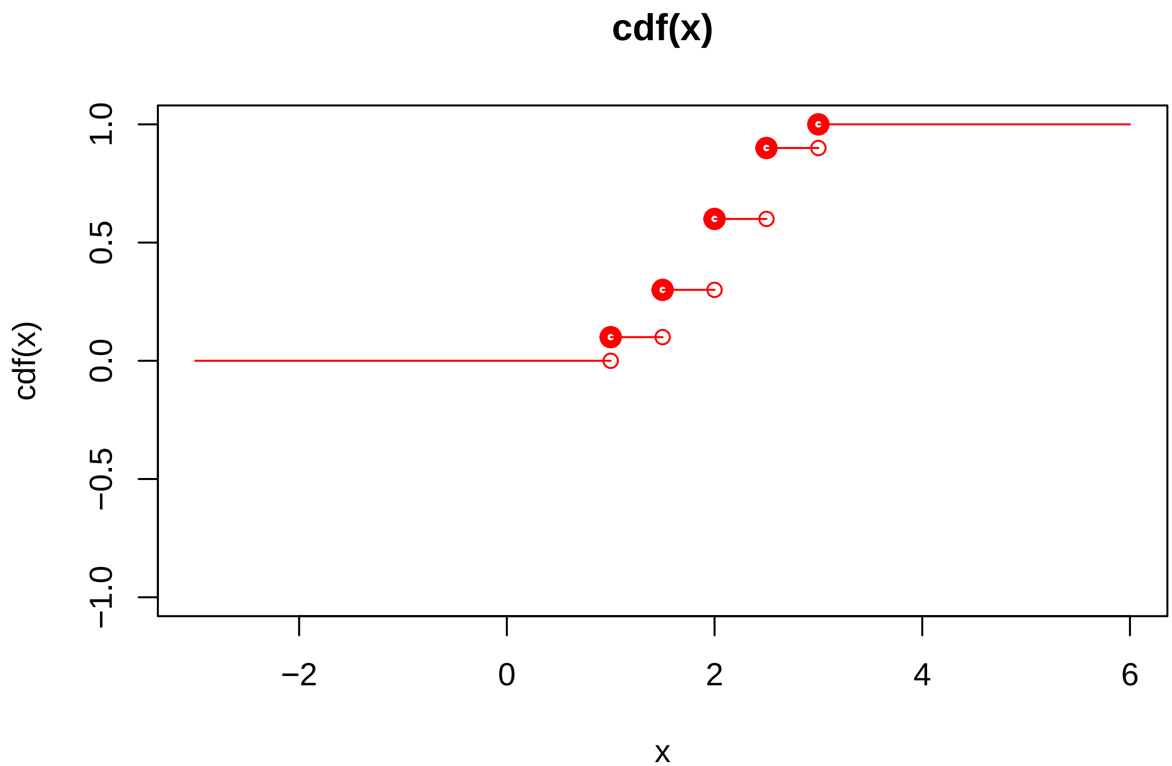
Therefore, we have the result we are looking for:

$$
\begin{aligned}
242 * P(X+Y=20) &= 242 * \frac{9}{242} \\
&= 9
\end{aligned} \tag{4}
$$

Question 2

(a) Sketch the graph of F and show that F is the cdf for a discrete random variable.

```r
# R plotting of the given cdf function
x <- c(-3, 6)
y <- 0*x
plot(x, y, xlab="x", ylab="cdf(x)", type="n", main="cdf(x)")
par(new=TRUE)

points(1, 0, pch=21, col="red")
segments(x0=-3, y0=0, x1=1, y1=0, col="red")

points(1, 1/10, pch=21, lwd=4, col="red")
segments(x0=1, y0=1/10, x1=3/2, y1=1/10, col="red")
points(3/2, 1/10, pch=21, col="red")

points(3/2, 3/10, pch=21, lwd=4, col="red")
segments(x0=3/2, y0=3/10, x1=2, y1=3/10, col="red")
points(2, 3/10, pch=21, col="red")

points(2, 6/10, pch=21, lwd=4, col="red")
segments(x0=2, y0=6/10, x1=5/2, y1=6/10, col="red")
points(5/2, 6/10, pch=21, col="red")

points(5/2, 9/10, pch=21, lwd=4, col="red")
segments(x0=5/2, y0=9/10, x1=3, y1=9/10, col="red")
points(3, 9/10, pch=21, col="red")

points(3, 1, pch=21, lwd=4, col="red")
segments(x0=3, y0=1, x1=6, y1=1, col="red")
```



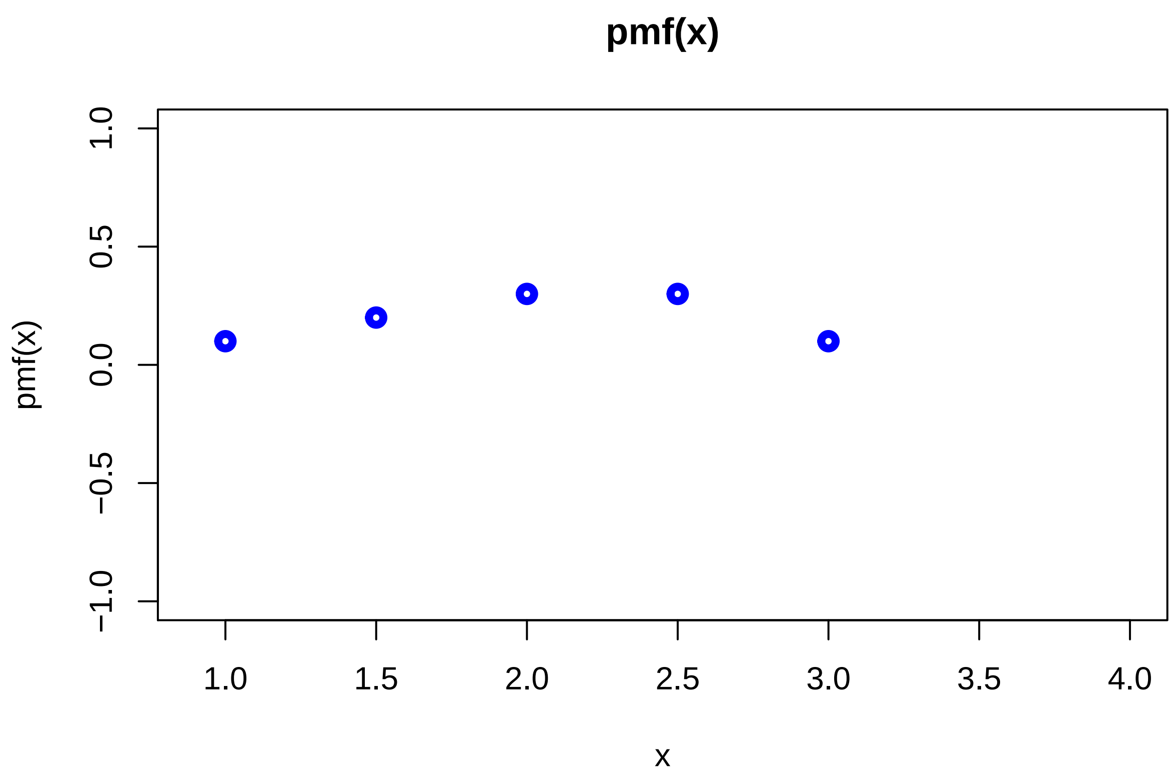
We can see that $F_X(x)$ is a cumulative distribution function of a discrete random variable X as $F_X(x)$ is defined for every x and its graph is mostly flat, except jumps. It jumps at the points in the support of X. We can clearly see $F_X(x)$ is a step-function with left-closed and right-open intervals, denoted by the filled red circles and un-filled red circles on the graph, respectively.

(b) Find the corresponding pmf f and sketch the graph.

x	$F_X(x)$	$p_X(x)$
0	0	
1	$\frac{1}{10}$	$F_X(1) - F_X(1-) = \frac{1}{10} - 0 = \frac{1}{10}$
$\frac{3}{2}$	$\frac{3}{10}$	$F_X(\frac{3}{2}) - F_X(\frac{3}{2}-) = \frac{3}{10} - \frac{1}{10} = \frac{1}{5}$
2	$\frac{6}{10}$	$F_X(2) - F_X(2-) = \frac{6}{10} - \frac{3}{10} = \frac{3}{10}$
$\frac{5}{2}$	$\frac{9}{10}$	$F_X(\frac{5}{2}) - F_X(\frac{5}{2}-) = \frac{9}{10} - \frac{6}{10} = \frac{3}{10}$
3	1	$F_X(3) - F_X(3-) = 1 - \frac{9}{10} = \frac{1}{10}$

```
# R plotting of the pmf function
x <- c(9/10, 4)
y <- 0*x
plot(x, y, xlab="x", ylab="pmf(x)", type="n", main="pmf(x)")
par(new=TRUE)

points(1, 1/10, pch=21, lwd=4, col="blue")
```

```r
points(3/2, 2/10, pch=21, lwd=4, col="blue")

points(2, 3/10, pch=21, lwd=4, col="blue")

points(5/2, 3/10, pch=21, lwd=4, col="blue")

points(3, 1/10, pch=21, lwd=4, col="blue")
```



(c) Find the probability $P(2 \leqslant X < 3)$

According to the property of the distribution function $F_X(x)$ of a random variable X with $a < b$ and $a, b \in \Re$:

$$
\begin{aligned}
P(a \leqslant X < b) &= F(b-) - F(a-) \\
\text{we can easily prove this:} & \\
\{a \leqslant X < b\} &= \{X < b\} \backslash \{X < a\} \\
\text{thus, based on properties of probabilityt measure:} & \\
P(a \leqslant X < b) &= P(X < b) - P(X < a) \\
&= F(b-) - F(a-) \\
\text{substituting a=2 and b=3:} & \\
P(2 \leqslant X < 3) &= F(3-) - F(2-) \\
&= \frac{9}{10} - \frac{3}{10} \\
&= \frac{6}{10} \\
&= \frac{3}{5}
\end{aligned} \tag{5}
$$

Question 3

(a)

By definition of a cdf of Y, denoted by $F_Y(y)$, we have:

$$
\begin{aligned}
F_Y(y) &= P(Y \leqslant y) \\
&= P(aX + b \leqslant y) \\
&= P(aX \leqslant (y - b)) \\
&= P\Big(X \leqslant \frac{y-b}{b}\Big) \\
&= F_X\Big(\frac{y-b}{a}\Big)
\end{aligned} \tag{6}
$$

(b)

By definition of the cdf of X, denoted by $F_X(x)$, we have:

$$F_X(x) = P(X \leqslant x) \tag{7}$$

Since $Q_X(p)$ is the inverse of $F_X(x)$, by definition, we have:

$$
\begin{aligned}
F_X(Q_X(p)) &= P(X \leqslant Q_X(p)) \\
p &= P\Big(\frac{Y-b}{a} \leqslant Q_X(p)\Big) \quad \text{linear transformation of X to Y} \\
p &= P(Y \leqslant aQ_X(p) + b) \\
\text{thus, by definition of the quantile function:} & \\
Q_Y(p) &= aQ_X(x) + b
\end{aligned} \tag{8}
$$

Question 4

Let X be the number of heads in n coin tosses. X follows a binomial distribution with $p = \frac{1}{2}$. We have X as a continuous random variable with finite expectation.
We have $X = I_1 + I_2 + ... + I_n$ with I_i be the number of heads on the i^{th} toss.
Thus: $E(X) = \sum_{i=1}^{n} I_{E_i} = np$
In addition: $Var(X) = \sum_{i=1}^{n} Var(I_{E_i}) = \sum_{i=1}^{n} p(1-p) = np(1-p)$

With X as the number of heads in n coin tosses, we have $n - X$ as the number of tails in those n coin tosses. The expected product of the number of heads and the number of tails is then:

$$E(X(n-X)) = E(nX - X^2) \tag{9}$$

Because X as a continuous random variable with finite expectation:

$$\begin{aligned} E(X(n-X)) &= E(nX) - E(X^2) \\ &= nE(X) - E(X^2) \quad \text{(property of expectation)} \\ &= nE(X) - [(E(X)^2) + Var(X)] \\ &= nnp - (np)^2 - np(1-p) \\ &= n^2p - n^2p^2 - np(1-p) \\ &= \frac{n^2}{2} - \frac{n^2}{4} - \frac{n}{4} \quad \left(\text{substituting} \quad p = \frac{1}{2}\right) \\ &= \frac{n^2}{4} - \frac{n}{4} \end{aligned} \tag{10}$$

Question 5

We have the same space $\Omega = \{BBBBBBBBGGGGGGG, BBBBBBBGBGGGGGG, ...\}$
We are looking for expected number of ordered pair $s = \{BG\}$ or $s = \{GB\}$

Let B and G denoting the type, corresponding to "Boy" and "Girl" respectively.
Consider the first pair of kids. We have the probability that this ordered pair of two kids has different gender is:

$$\begin{aligned} P &= P(BG) + P(GB) \\ &= 2 * P(BG) \end{aligned} \tag{11}$$

The above holds true because the process of establishing the ordered pair is indifferent to the outcome, i.e. the ordered inside the pair.

To find $P(BG)$, we note that for a pair of two kids, there are 15 ways to arrange seating for 1 kid. Consequently, there are 14 ways to arrange seating for the immediate following kid. Thus, there are $15 * 14 = 210$ ways to arrange seating for this pair of kid.

Further note that there $8 * 7 = 56$ ways to arrange seating for a pair of B and G.

Therefore, we have $P(\{BG\}) = \frac{56}{210} = \frac{4}{15}$. Consequently, $P = 2 * \frac{4}{15} = \frac{8}{15}$. In other world, the probability that an pair of kids will have different "types" is $\frac{8}{15}$.

Let X be the binomial random variable whose value of 1 representing the kids in the event the kids at position X_k and X_{k+1} having different "type": "B" or "G". Based on the addition properties of expectation, we have: $E^* = \sum\limits_{i=1}^{n=14} E(X) = 14 * E(X)$.

Moreover, $E(X) = P(\{BG\})$.

Thus: $E^* = 14 * \frac{8}{15} = \frac{212}{15}$