

# Study Guide - Final Exam

Jeff Nguyen

April 28, 2021

## Properties of a Good ARIMA Model:

- Parsimonious: don't select large  $p$  and  $q$  (nothing more than order of 2).
- Has coefficients that are stationary and are invertible: recall stationary ( $|\alpha| < 1$ ) and invertibility ( $|\beta| < 1$ ).
- Fits data well (measures through AIC, SBC/BIC and select the best one).
- Has residuals that are white noise (No spikes in ACF and PACF. If residuals of ARIMA are not white noise, it means you have not filtered out information from your variable. The point of ARIMA is to filter out the information from variable  $Y$ ).
- Has coefficients that do not change over sample period: If split sample to different time periods, run 2 ARIMA models and test coefficients are statistically the same or not.
- Has good out-of-sample forecast.

**Stationary:** Means  $AR(p)$  to be stationary.

**Invertibility:** Means  $MA(q)$  to be stationary.

## Models of Volatility:

Many financial time series go through periods of tranquility and periods of volatility. Volatility and its measures are some of the most important concepts in finance.

## Historical Volatility:

Measures the variance or standard deviation of returns over some period and uses it as measure of forecast of future volatility. It's used in option valuation and option pricing.

$$\sigma_{historical}^2 = \frac{\sum (r - \bar{r})^2}{n - 1} \quad (1)$$

## Characteristics of Financial Data:

- Non-stationary (price). Most of the time, returns will be stationary.

- Leptokurtosis: sharp peaks, fat tails.
- Volatility clustering: Property of financial data to go through period of tranquility and volatility pooling/clustering.
- Leverage effect: Volatility rise more with price falls than price rises.
- Has ARIMA coefficients ( $p$  and  $q$ ) that do not change over sample period (F-test of split periods).
- Has plenty noises as well as signals.
- Volatility is mean reverting.

**Conditional Heteroscedasticity:** When forecasting, we are concerned more with conditional mean and conditional variance rather than historical mean and historical variance (long-run variance).

**Heteroscedasticity in Regression:** happens when the variance of the errors depends on an independent variable  $Var[\epsilon_i] = cX_i^2$ . For financial data, heteroscedasticity happens when variance of errors depends on a dependent variable (volatility pooling).

### Exponentially Weighted Moving Average Model (EWMA):

$$\sigma_t^2 = (1 - \lambda) \sum \lambda_i (r_{t-i} - \bar{r})^2$$

where:

$$\sigma_t^2 = \text{estimate variance for the period } t, \text{ also the forecast of future volatility for all period} \quad (2)$$

$\bar{r}$  = average returns

$\lambda$  = decay factor, usually .94

### Autoregressive Volatility Series:

If a time series observations on some volatility proxy are estimated, then ARIMA type model can be used to forecast future volatility. Some proxy series are:

- $r^2$ , square of returns.
- $\sigma_t^2 = \ln H/L$ .
- *ARCH* and *GARCH* models.

### ARCH and GARCH Models:

Volatility clustering property of financial data describes the current level of volatility to be positively correlated with the preceding period level. Volatility is “auto-correlated”.

#### General ARCH Model:

$$\sigma_{\epsilon t}^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \dots + \epsilon_t$$

or Engle’s simplified model:

$$\sigma_{\epsilon t}^2 = \alpha_0 + \alpha_1 (.4\epsilon_{t-1}^2 + .3\epsilon_{t-2}^2 + .2\epsilon_{t-3}^2 + .1\epsilon_{t-4}^2) + \epsilon_t \quad (3)$$

where

$$\text{weighted average} = .4\epsilon_{t-1}^2 + .3\epsilon_{t-2}^2 + .2\epsilon_{t-3}^2 + .1\epsilon_{t-4}^2$$

Note that most of the time, only 2 or 3 lags are used instead of 4.

The 2 most popular models are ARCH(1)–AR(1)–and GARCH(1)–Generalized ARCH(1), i.e. ARMA(1,1).

### ARCH(1) Model:

These 2 equations are estimated simultaneously:

$$\begin{aligned}
 Y_t &= \gamma_0 + \gamma_1 Y_{t-1} + \epsilon_t \\
 \epsilon_t^2 &= a_0 + a_1 \epsilon_{t-1}^2 \\
 \text{where} \\
 \gamma_1 &= \text{duration of volatility} \\
 a_1 &= \text{magnitutde of volatility}
 \end{aligned} \tag{4}$$

Estimations:

Conditional Mean of Y at t+1:

$$\mathbb{E}[Y_{t+1}|Y_t, Y_{t-1}, \dots] = \hat{\gamma}_0 + \hat{\gamma}_1 Y_t$$

Unconditional Mean of Y:

$$\begin{aligned}
 Y_{LR} &= \mathbb{E}[Y] \\
 &= \frac{\hat{\gamma}_0}{1 - \hat{\gamma}_1}
 \end{aligned}$$

Conditional Variance of  $\epsilon$  at time t:

$$\mathbb{E}[\epsilon_{t+1}^2 | \epsilon_t, \epsilon_{t-1}, \dots] = \hat{a}_0 + \hat{a}_1 \epsilon_t^2$$

Unconditional Variance of  $\epsilon$ :

$$\begin{aligned}
 \epsilon_{LR} &= \mathbb{E}[\epsilon^2] \\
 &= \sigma_\epsilon^2 \\
 &= \frac{\hat{a}_0}{1 - \hat{a}_1}
 \end{aligned} \tag{5}$$

Unconditional Variance of Y:

$$\begin{aligned}
 Var[Y_{LR}] &= \frac{\mathbb{E}[\epsilon_t^2]}{1 - \gamma_1^2} \\
 &= \frac{\sigma_\epsilon^2}{1 - \gamma_1^2} \\
 &= \frac{\hat{a}_0}{1 - \hat{a}_1} \cdot \frac{1}{1 - \hat{\gamma}_1^2}
 \end{aligned}$$

The equation,  $Var[Y_{LR}] = \frac{\hat{a}_0}{1 - \hat{a}_1} \cdot \frac{1}{1 - \hat{\gamma}_1^2}$ , shows the mean reverting property of the variance in the long run. This means in the long-run, volatility is a constant.

### Interesting conclusions of the ARCH(1) model:

- $Var[Y]$  increases both with  $\gamma_1$  and  $a_1$ .
- $Var[Y_{LR}]$  is a constant, consistent with mean reverting property of the variance of financial data.
- Conditional variance of  $Y$   $Var[Y]$  changes with lagged values of  $\epsilon_t$ .
- $\gamma_1$  decides the duration and  $a_1$  decides the magnitude of volatility.

### Numerical Example: Calculations:

```
# First ARCH(1) Equation
## y_t = gamma_0 + gamma_1 y_tMinus1 + epsilon_t
gamma_0 = 10
gamma_1 = 0.6
y_tFunc <- function(y_tMinus1) gamma_0 + gamma_1*y_tMinus1

## Given parameters
y_t = 5
sigmaSq_epsilon = 9

# Second ARCH(1) Equation
## epsilon_t^2 = a_0 + a_1 * epsilon_tMinus1^2
a_0 = 3
a_1 = .4
epsilonSq_tFunc <- function(epsilonSq_tMinus1) a_0 + a_1*epsilonSq_tMinus1

## Given parameters
e_t = 1.1

# Mean Y
y_tPlus1 <- y_tFunc(y_t)
y_tPlus2 <- y_tFunc(y_tPlus1)
y_tPlus3 <- y_tFunc(y_tPlus2)
y_LR <- gamma_0 / (1-gamma_1)

# Variance epsilon
epsilonSq_tPlus1 <- epsilonSq_tFunc(sigmaSq_epsilon)
epsilonSq_tPlus2 <- epsilonSq_tFunc(epsilonSq_tPlus1)
epsilonSq_tPlus3 <- epsilonSq_tFunc(epsilonSq_tPlus2)
epsilonSq_LR <- a_0 / (1 - a_1)

# Variance Y
var_y_tPlus1 <- epsilonSq_tPlus1
var_y_tPlus2 <- (1+gamma_1^2) * epsilonSq_tPlus2
var_y_tPlus3 <- (1+gamma_1^2+gamma_1^4) * epsilonSq_tPlus3
var_y_LR <- sigmaSq_epsilon / (1-gamma_1^2)
var_y_meanReverting <- (a_0/(1-a_1))*(1/(1-gamma_1^2))
```

Given ARCH(1) model:

$$\begin{cases} Y_t = 10 + 0.6Y_{t-1} + \epsilon_t, Y_t = 5, \epsilon \sim iid(0, 9) \\ \epsilon_t^2 = 3 + 0.4\epsilon_{t-1}^2 \end{cases} \quad (6)$$

Var Desc	Forecast Var	95% Confidence Interval
Conditional Mean $E[Y_{t+1} \Omega]$	$E[Y_{t+1} \Omega] = 10 + 0.6Y_t = 13$	$E[Y_{t+1} \Omega] \pm 1.96\sqrt{Var[Y_{t+1} \Omega]} = 13 \pm 1.96\sqrt{6.6}$
Conditional Mean $E[Y_{t+2} \Omega]$	$E[Y_{t+2} \Omega] = 10 + 0.6Y_{t+1} = 17.8$	$E[Y_{t+2} \Omega] \pm 1.96\sqrt{Var[Y_{t+2} \Omega]} = 17.8 \pm 1.96\sqrt{7.6704}$
Conditional Mean $E[Y_{t+3} \Omega]$	$E[Y_{t+3} \Omega] = 10 + 0.6Y_{t+2} = 20.68$	$E[Y_{t+2} \Omega] \pm 1.96\sqrt{Var[Y_{t+3} \Omega]} = 17.8 \pm 1.96\sqrt{7.8293376}$
Unconditional Mean $Y_{LR}$	$Y_{LR} = \frac{\gamma_0}{1-\gamma_1} = 25$	$Y_{LR} \pm 1.96\sqrt{Var[Y_{LR}]} = 25 \pm 1.96\sqrt{14.0625}$

Var Desc	Forecast Var
Conditional Variance $Var[Y_{t+1} \Omega]$	$Var[Y_{t+1} \Omega] = E[\epsilon_{t+1}^2 \Omega] = 6.6$
Conditional Variance $Var[Y_{t+2} \Omega]$	$Var[Y_{t+2} \Omega] = (1 + \gamma_1^2)E[\epsilon_{t+2}^2 \Omega] = 7.6704$
Conditional Variance $Var[Y_{t+3} \Omega]$	$Var[Y_{t+3} \Omega] = (1 + \gamma_1^2 + \gamma_1^4)E[\epsilon_{t+3}^2 \Omega] = 7.8293376$
Unconditional Variance $Var[Y_{LR}]$	$Var[Y_{LR}] = \frac{\sigma_\epsilon^2}{1-\gamma_1^2} = 14.0625$
Mean-Reverting Variance $Var[Y_{mr}]$	$Var[Y_{mr}] = \frac{a_0}{(1-a_1)(1-\gamma_1^2)} = 7.8125$
Conditional Variance $E[\epsilon_{t+1}^2 \Omega]$	$E[\epsilon_{t+1}^2 \Omega] = 3 + 0.4\sigma_\epsilon^2 = 6.6$
Conditional Variance $E[\epsilon_{t+2}^2 \Omega]$	$E[\epsilon_{t+2}^2 \Omega] = 3 + 0.4E[\epsilon_{t+1}^2 \Omega] = 5.64$
Conditional Variance $E[\epsilon_{t+3}^2 \Omega]$	$E[\epsilon_{t+3}^2 \Omega] = 3 + 0.4E[\epsilon_{t+1}^2 \Omega] = 5.256$

### GARCH(1,1) Model:

$$\sigma_{\epsilon t}^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta \sigma_{\epsilon t-1}^2$$

where

$$\epsilon_t^2 = \sigma_{\epsilon t}^2 + e_t$$

substituting this to the original model (7)

$$\begin{aligned}\sigma_{\epsilon t}^2 &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta \sigma_{\epsilon t-1}^2 \\ \epsilon_t^2 - e_t &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta \sigma_{\epsilon t-1}^2 \\ \epsilon_t^2 - e_t &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta (\epsilon_{t-1}^2 - e_{t-1}) \\ \epsilon_t^2 &= \alpha_0 + (\alpha_1 + \beta) \epsilon_{t-1}^2 + e_t - \beta e_{t-1}\end{aligned}$$

GARCH(1) is an ARMA(1,1) model in variance.

**GARCH vs ARMA:** ARMA models the levels of a time series, GARCH models the variance of a time series. GARCH models are used for the error variance of a time series. GARCH model can be combine with ARMA processes. The ARMA models assume constant conditional variance. However, not all time series processes have constant variance. For example, in modeling stock returns, if the observed returns today are usually volatile, we might expect the returns tomorrow to also be more volatile than usual. An ARMA model, with constant conditional variance, cannot capture this behavior.

**Strengths and Weaknesses of GARCH:** GARCH is more flexible than ARCH, as it can describe the volatility with fewer parameters. However, the baseline GARCH is characterized by a symmetric response of current volatility to positive and negative values of the lagged error  $\epsilon_{t-1}$ . It is common for the future volatility of financial time series to be much more affected by negative values of  $\epsilon_{t-1}$  than positive ones. This asymmetry is not captured by the GARCH model.

# Intervention Function Analysis

## Objective

Intervention Function Analysis is a statistical methodology to test the effect a shock to a time series variable. The objective of the test is to find whether the post-shock mean and variance of a time series has changed due to shock or not. If the test concludes that the pre-shock mean is different from the post-shock mean, the methodology then can measure the long-run effect of the shock on the variable as well as the time path of the adjustment of the variable to the shock using impulse response function.

## Traditional Way of Conducting the Test

The conventional method is to compare the mean and variance of the variable before and after the shock and then test for statistical significance of the change using z and F tests.

$$\begin{aligned} H_0 : \mu_1 &= \mu_2 \\ H_a : \mu_1 &\neq \mu_2 \\ Z &= \frac{\overline{X_1} - \overline{X_2}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \end{aligned} \tag{8}$$

$$\begin{aligned} H_0 : \sigma_1^2 &= \sigma_2^2 \\ H_a : \sigma_1^2 &\neq \sigma_2^2 \\ F &= \frac{S_1^2}{S_2^2} \end{aligned}$$

The conventional method is inappropriate in time-series analysis because of serial correlation nature of financial data and some of the effects of the pre-shock period may “carry over” to the next period. This makes any statistical inference based on conventional method of comparing means and variances biased.

To include such “carry over” effects, the mean values of two different periods are compared using the Intervention Function Analysis. Intervention Function Analysis allows for a formal test of a change in the mean of a variable with time series characteristic.

## Methodology:

Intervention Function Analysis requires running the best fitting ARIMA on the variable for the longest span of data before or after intervention (whichever is longer), to find the order of the ARIMA, and then running the same order of ARIMA for the entire period by including the intervention variable and testing for the significance of the coefficient of the intervention variable.

## The Model:

First, use the whole span of data to estimate AR(1) model including impulse dummy which will have a value of 1 at time  $\tau$  and 0 elsewhere.

$$Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \lambda Z_t + \epsilon_t \tag{9}$$

If  $\lambda$  is statistically significant, then intervention has changed the mean. If not, intervention did not have any effect on  $Y_t$ .

```
# Given  $Y_t = 20 + .5 Y_{t-1} - 2.5Z$ 
alpha_0 <- 20
alpha_1 <- .5
lambda <- -2.5
```

Day	Immediate Effect	Cumulative Effect
0	$\lambda = -2.5$	$\lambda = -2.5$
1	$\lambda\alpha_1 = -1.25$	$\lambda(1 + \alpha_1) = -3.75$
2	$\lambda\alpha_1^2 = -0.625$	$\lambda(1 + \alpha_1 + \alpha_1^2) = -4.375$

Statistics	Value
LR effect of shock	$\frac{\lambda}{1-\alpha_1} = -5$
LR effect of no shock	$\frac{\alpha_0}{1-\alpha_1} = 40$
LR cumulative effect	$\frac{\lambda+\alpha_0}{1-\alpha_1} = 35$

To interpret the Orthogonal Impulse Response graph: if there is fluctuation after the shock, then the shock is significant.

## Reading ACF and PACF

### Non-stationary:

- If PACF has a single spike at  $r_0 \approx 1$  and the rest is insignificant and ACF mostly significant and gradually decreasing  $\Rightarrow$  non-stationary model.

### $MA(\infty)$ and $AR(\infty)$

Several spikes in ACF (significant coefficients)  $\Rightarrow MA(\infty) \Rightarrow AR(1)$ .

Several spikes in PACF (significant coefficients)  $\Rightarrow AR(\infty) \Rightarrow MA(1)$ .

## (A)DF Test

- Failed to reject the null hypothesis: there is a unit root  $\Rightarrow$  non-stationary.
- Reject the null hypothesis: stationary.

## Test for White Noise

### Box-Pierce Test

Null hypothesis: Residuals are White Noise.

$$Q = T \sum r^2 \quad (10)$$

## Box-Ljung Test

Null hypothesis: Residuals are White Noise.

$$Q = T(T+2) \sum (T-K)^{-r^2} \quad (11)$$

Recall AR(2)

$$Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2}$$

Regressing  $\epsilon$ :

$$b_i = \alpha_1 b_{i-1} + \alpha_2 b_{i-2}$$

We have:

$$\begin{aligned} \epsilon_{Y_{t+1}} &= \sum_{i=0}^1 b_i \epsilon_{t-1} \\ &= b_0 \epsilon_t + b_1 \epsilon_{t-1} \\ &= \epsilon_t + \alpha_1 \epsilon_{t-1} \end{aligned} \quad (12)$$

Thus:

$$Var[Y_{t+1}] = (1 + \alpha_1^2) \sigma_\epsilon^2$$

Similarly:

$$\begin{aligned} \epsilon_{Y_{t+2}} &= \sum_{i=0}^2 b_i \epsilon_{t-1} \\ &= b_0 \epsilon_t + b_1 \epsilon_{t-1} + b_2 \epsilon_{t-2} \end{aligned}$$

Recall that:  $b_i = \alpha_1 b_{i-1} + \alpha_2 b_{i-2}$

$$\begin{aligned} \text{We have: } \epsilon_{Y_{t+2}} &= \epsilon_t + \alpha_1 \epsilon_{t-1} + (\alpha_1 b_1 + \alpha_2 b_0) \epsilon_{t-2} \\ &= \epsilon_t + \alpha_1 \epsilon_{t-1} + (\alpha_1^2 + \alpha_2) \epsilon_{t-2} \end{aligned} \quad (13)$$

Thus:

$$Var[Y_{t+2}] = [1 + \alpha_1^2 + (\alpha_1^2 + \alpha_2)^2] \sigma_\epsilon^2$$